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VLADMIR SICCA GONÇALVES

Cauchy Horizons as Dynamical Systems

Horizontes de Cauchy como Sistemas Dinâmicos

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Vladmir Sicca Gonçalves

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Prof(a). Dr(a). ALBERTO VAZQUEZ SAA

Prof(a). Dr(a). RAFAEL DE FREITAS LEÃO

Prof(a). Dr(a). CARLOS HENRIQUE GROSSI FERREIRA

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Resumo

Neste trabalho é discutida a possibilidade do uso da teoria de Sistemas Dinâmicos como ferramenta para um melhor entendimento de horizontes de Cauchy em espaços-tempos. É feita uma revisão da relação entre o comportamento dos geradores tipo-luz do horizonte e sua diferenciabilidade como subvariedade Lipschitz do espaço-tempo ambiente para analisar a aplicabilidade dos resultados sobre estabilidade de Sistemas Dinâmicos à distribuição de tangentes aos geradores. Em seguida é apresentado um modo de construir espaços-tempos com horizontes de Cauchy a partir de uma variedade compacta dotada de um campo de vetores que não se anula. Os requisitos de Geometria Lorentziana necessários são apresentados ou referências são dadas no texto.

Palavras-chave: Horizontes de Cauchy. Relatividade Matemática. Sistemas Dinâmicos.

Abstract

In this work we discuss the possibility of using Dynamical Systems as a tool to better understand Cauchy horizons in spacetimes. We make a review of the relationship between the behavior of the lightlike generators of horizons and their differentiability as a Lipschitz submanifold of the ambient spacetime in order to analyze the applicability of results in the stability of Dynamical Systems to the distribution of tangents to the generators. In the sequence we present a way to construct spacetimes with Cauchy horizons from a given compact manifold with a non-vanishing vector field. The needed requisites from Lorentzian Geometry are either presented or references are given in the text.

Keywords: Cauchy horizons. Mathematical Relativity. Dynamical Systems.

List of symbols

$\gamma * \alpha$	Concatenation of the curves γ and α, with γ being transversed first
$\lvert . \rvert$	Euclidean norm in $\mathbb{R}^n$
$\nabla_V W$	Covariant derivative of the vector field W in the direction V
$D_p f$	Differential of the function f at point p
f_*	Pushforward by the function f
$\#$ S	Cardinality of the set S

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Introduction

In his paper (HAWKING, 1992), Stephen Hawking makes some claims concerning the stability of certain kinds of behavior of the lightlike curves that foliate Cauchy horizons, called generators. He claims in the paper that, if horizons are compact, the absence of closed generators should be an unstable condition, while the presence of closed generators which are "fountain-like" should be stable. In the paper, Hawking does not give very rigorous details either to back the claims or to make clear the precise meaning of the notion of "stability" in this case.

After that, (CHRUŜCIEL; ISENBERG, 1994) presented a construction that allowed one to find a spacetime with an horizon diffeomorphic to a prescribed manifold, with a prescribed set of generators. Such generators were the orbits of the flow associated to a vector field tangent to the manifold. With this tool, a myriad of examples of spacetimes becomes available. In particular, the paper brings a counterexample to Hawking's claim that "fountain-like" generators should be stable. The construction from (CHRUŜCIEL; ISENBERG, 1994) is presented in this work in chapter 4, followed by some remarks on the well-posedness of the problem of stability questions concerning horizons.

The presentation of (CHRUŜCIEL; ISENBERG, 1994), maybe for focusing on the elaboration of counterexamples through the study of generators themselves, does not enter deeply into the details that guarantee the equivalence built between generators of horizons and dynamical systems. Particularly, it avoids any intricacies of the passage from the distribution of tangents to the generators of the horizon to a dynamical system.

In order to fill this gap, in this work we make a review of results - presented in (CHRUŜCIEL; GALLOWAY, 1998) and (BEEM; KRÓLAK, 1998) - concerning the relationship between the behavior of generators of an horizon and the differentiability of said horizon. In doing so we can show that the possibility of globally assigning a dynamical system with orbits that are the generators of the horizon is equivalent to the horizon being C^1 at all of its points. Also, we prove that the dynamical system so obtained is continuous. This development is put in place in chapter 3.

Our hope in building the bridge between the generators of the horizons and dynamical systems is that the broad theory of the latter's may be applied to the former in order to obtain new results in the theory of Cauchy horizons. In particular, we expect that information about the stability of Cauchy horizons may be derived from imposing that the generators give form to a stable dynamical system. In order to give it substance, we describe the main result in the structural stability of C^1 dynamical systems on compact manifolds, the Pallis-Smale Stability Conjecture, in chapter 2.

In order to make the presentation as self-contained as possible, we begin the text by chapter 1, in which basic definitions from Lorentzian Geometry are given and fundamental results that lead to the proof that horizons are indeed foliated by lightlike geodesics - generators - are presented. Some results from Lorentzian Geometry that are used in the text but that would not fit well along the construction of the horizons are also proved in a separate section, while others come only with a reference to external sources, usually texts (HAWKING; ELLIS, 1973) and (PENROSE, 1972), which cover widely the tools of Lorentzian Geometry needed in the study of spacetimes.

1 Preliminaries from Lorentzian Geometry

When first approaching the literature on topological aspects of General Relativity, one finds concepts used loosely through texts without explicit definitions and some slight variations in terms and conventions, what can bring some difficulties to those entering the field. In order to settle our ground and increase self-containment of this work, we begin by presenting an introduction to the concepts of Lorentzian geometry demanded for our goal. The basic references for this chapter are (HAWKING; ELLIS, 1973) and (PENROSE, 1972).

1.1 Spacetimes

The objects of study in Lorentzian Geometry are manifolds endowed with Lorentzian metrics, which are non-degenerate symmetric bilinear forms of signature $(-, +, ..., +)$. General Relativity describes the universe as a Lorentzian manifold that obeys some restrictions, ranging from some topological and differential regularity to its geometrical nature, such as Einstein's equation. In this section we present the tools necessary to define the notion of *spacetime* that shall be used in our work.

Throughout this section, unless stated otherwise, we shall denote (M, g) the Lorentzian manifold, with M being a smooth manifold and g a Lorentzian metric. In general, there are questions to be made concerning the regularity of the manifold and of the metric, but we will set those aside in this work and assume both M and g to be smooth.

The fact that g is not positive definite offers a useful way to classify a vector V tangent to M, using the signal of $g(V, V)$. To be precise, we say a vector V tangent at a point $p \in M$ is called:

- *spacelike* if $g_p(V, V) > 0$ or $V = 0$;
- *lightlike* if $g_p(V, V) = 0$ and $V \neq 0$;
- *timelike* if $g_p(V, V) < 0$;
- *causal* if V is either timelike or lightlike.

The naming comes from Physics. The trajectory of a particle, according to General Relativity, is a curve in M with an everywhere timelike tangent vector, unless said particle is massless, such as a photon, in which case the tangent vector is everywhere lightlike.

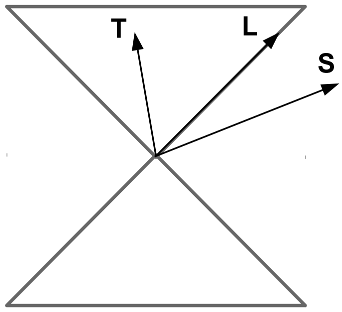


Figure 1 – In each tangent plane timelike vectors (T), lightlike vectors (L) and spacelike vectors (S) can be classified with regard to their position relative to a cone centered in the origin, respectively if they are in the interior, the boundary or exterior to the cone.

We say that the property of being spacelike, lightlike or timelike is the *causal character* of the vector, since it is the cornerstone of the theory of *causality* in Lorentzian Geometry, which will be the focus of the next sections. We introduced it here only to define what a *time-oriented* Lorentzian manifold is: it is a Lorentzian manifold that admits a global continuous timelike vector field.

Clearly, given a global timelike vector field T tangent to M, $-T$ is also a global timelike vector field. Akin to the notion of orientability, we have a choice of time-orientation to be made in each time-oriented manifold[1]. When a time-orientation is chosen, say the vector field T, we can refine the classification of causal vectors. A causal tangent vector V at $p \in M$ is called

- *future-directed* if $g_p(V, T) < 0$;
- *past-directed* if $g_p(V, T) > 0$;

All this said, we can define precisely our working notion of a spacetime.

Definition 1.1 (Spacetime)**.** *A spacetime* (M, g) *is a connected, oriented and time-oriented smooth Lorentzian manifold.*

We must remark that, for physical reasons, it is usual to set the dimension of the spacetime to be four, but in this work we will not restrict the dimension of M. Our only results that demand M to be of dimension four are those related to the stability of dynamical systems, given in section 2. In some proofs we assume M to have dimension four only for simplicity of notation, while many examples present spacetimes of smaller dimension which are easier to deal with.

From this point on (M, g) will always satisfy the conditions of definition 1.1.

[1] That there are only two options from which to choose from is one of the basic results of Lorentzian geometry. See (PENROSE, 1972), remark 1.4.

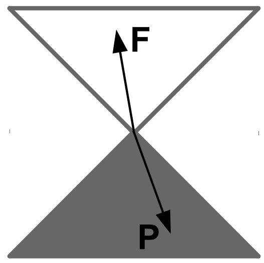


Figure 2 – The interior of the light cone in each tangent space has two connected components, one of future-directed vectors (the white area in the drawing) and one of the past-directed vectors (the gray area).

1.2 Causality

In physical terms, causality concerns the study of whether or not some subsets of M can be reached by physical "signals" sent by other subsets. In order to define this notion mathematically, we must first extend the causality character of tangent vectors to piecewise smooth curves in M. A piecewise smooth curve γ is said to be:

- *spacelike* if its tangent vector is spacelike at all points in the image of γ;
- *lightlike* if its tangent vector is lightlike at all points in the image of γ;
- *timelike* if its tangent vector is timelike at all points in the image of γ;
- *causal* if its tangent vector is either lightlike or timelike at all points in the image of γ.

With some adaptation, this definition can be extended to submanifolds of M of arbitrary dimension, as we will discuss in section 1.4.1.

A causal curve is called *past-directed* (*future-directed*) if its tangent vector is *past-directed* (*future-directed*) at all its points. At singular points of γ we demand the causal character of both directional derivatives, before and after the singular point, to be the same of the one at the non-singular points in order to decide the causal character of the curve as a whole.

It may be good to clarify that, different from tangent vectors, there are (plenty of) piecewise smooth curves which are neither causal nor spacelike, but that won't be a concern in this work.

Now we can give a precise meaning to the notion of "reachable by signals" through the following relations defined between points on M.

Definition 1.2. *Given two points $x, y \in M$ we say:*

i. $x \ll y$, said "x precedes y chronologically", iff there is a future-directed timelike curve from x to y;

ii. $x \prec y$, said "x precedes y causally", iff there is a future-directed causal curve from x to y.

Since the concatenation of two curves of the same causal character has itself the same causal character as its components, both relations defined above are transitive.

With this definition we can speak of past-inextensible and future-inextensible causal curves.

Definition 1.3. *A timelike (lightlike) curve γ is said past-inextensible if there is no $x \in M$ such that $x \ll y$ ($x \prec y, x \not\ll y$), $\forall y \in \gamma$.*

To define future-inextensible curves one changes the order of x and y in the causality relations.

These definitions allow the introduction of useful notation to denote the domains of influence of subsets of M. These sets will be essential to the definition of Cauchy horizons in the next section.

Definition 1.4. *Let $A \subset M$ we define:*

- $I^+(A) := \{y \in M | \exists x \in A$ *such that* $x \ll y\}$, *called the chronological future of A;*
- $I^-(A) := \{y \in M | \exists x \in A$ *such that* $y \ll x\}$, *called the chronological past of A;*
- $J^+(A) := \{y \in M | \exists x \in A$ *such that* $x \prec y\}$, *called the causal future of A;*
- $J^-(A) := \{y \in M | \exists x \in A$ *such that* $y \prec x\}$, *called the causal past of A;*
- $D^+(A) := \{p \in M |$ *every past-directed, past-inextensible timelike curve with future endpoint at p intersects* $A\}$, *called the future Cauchy development of A.*
- $D^-(A) := \{p \in M |$ *every future-directed, future-inextensible timelike curve with past endpoint at p intersects* $A\}$, *called the past Cauchy development of A.*

Now we prove a proposition with respect to the nature of the chronological future of points, for we will use it in the future. In order to do so, we introduce an important kind of neighborhood that appears in many proofs:

Definition 1.5. *A convex normal neighborhood $N \subset M$ is an open set such that $(N, \exp_p^{-1})$ is a coordinate chart for all $p \in N$.*

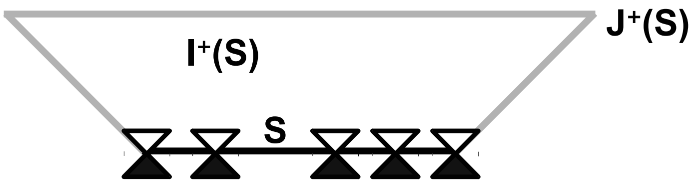


Figure 3 – The interior of the white area in the drawing is the chronological future $I^+(S)$ of the set S, its closure is the causal future $J^+(S)$.

(a) Future Cauchy development of S

(b) Future Cauchy horizon of S

Figure 4 – The so-called future boundary of the Cauchy development of a set (presented on the left) is called future Cauchy horizon of the set, highlighted in the right figure, that we shall define at definition 1.8

It is proven in section 9.3 from (HICKS, 1965) that the topology of any manifold with a connection admits a basis of convex normal neighborhoods.

Now we can proceed to the proposition.

Proposition 1.6. *For any $p \in M$, $I^+(p)$ is open.*

Proof. Take $p \in M$ and $q \in I^+(p)$. Let N_q be a convex normal neighborhood around q and $\alpha : [0,1] \to M$ be a future-directed timelike curve from p to q. Then, there is $\epsilon > 0$ such that $r := \gamma(1-\epsilon) \in N_p$. As will be proved in proposition 1.27, since $r \ll q$, $\exp_r^{-1}(q)$ is in the interior of the future Lorentzian lightcone with vertex at $0 \in T_rM$. Call $\tilde{U} := [\exp_r^{-1}(N_p) \cap I^+(0)] \subset T_rM$. Then $\tilde{U}$ is open and $q \in \exp_r(\tilde{U})$, which is open in M.

Now, again as a result of proposition 1.27, $0 \ll s$ and $s \in \exp_r^{-1}(N_p)$ implies that $r \ll \exp_r(s)$. Hence $\exp_r(\tilde{U}) \subset I^+(r) \subset I^+(p)$, since $\ll$ is transitive. Thus, $I^+(p)$ is open. □

This result has an immediate equivalent for the chronological past of a point, achieved by simply changing time-orientations. Also, it implies that the future of any set is open, since it is the union of the futures of its points.

From the sets presented in definition 1.4, one that deserves deeper comment is the future Cauchy development of a set. The fact that all timelike curves through p

intersect A when extended to the past conceals the information not only that A influences p, but that the state of p is **completely** determined by initial conditions in A. In particular, if we have some physically meaningful differential equation[2], the solution's value at p should be fully determined by an initial condition in A. It should be stressed also that $D^+(A) \subset \big(I^+(A) \cup A\big)$, but the converse is usually not true.

The previous definitions are meaningful independently of the nature of A. To define and prove properties of horizons we will refer to specific kinds of sets, which we present below.

Definition 1.7. *A set $A \subset M$ is called:*

- *achronal if $\forall x \in A,\ I^+(x) \cap A = \varnothing$;*
- *acausal if $\forall x \in A,\ J^+(x) \cap A = \varnothing$;*
- *future if $\forall x \in A, I^+(x) \subset A$;*
- *past if $\forall x \in A, I^-(x) \subset A$;*

It is useful to keep in mind that, since $\ll$ is a transitive relation, an equivalent way to characterize A as an achronal set is to say $I^+(A) \cap I^-(A) = \varnothing$.

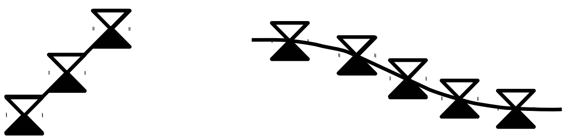


Figure 5 – Examples of achronal and acausal sets. Note that every acausal set is achronal, but the reverse is not true, for achronal sets may contain lightlike curves.

Now we can finally define our main object of study in this work.

Definition 1.8 (Cauchy Horizon)**.** *If S is a closed achronal set, we call the future Cauchy horizon*[3] *of S the set*

$$H^+(S) := D^+(S)\backslash[I^-(D^+(S))].$$

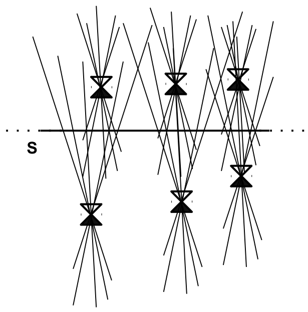
The past Cauchy horizon of S is defined in the same way interchanging the + and - signs in the definition.

2 That is, one which solution propagates at most at light speed.

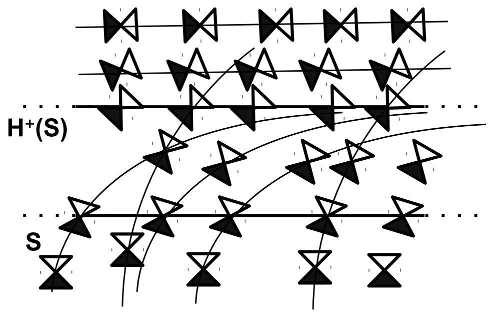
3 This is NOT the definition of *event horizon* that usually appears in discussions concerning black holes. Mathematically, event horizons are the boundaries of black holes, which are regions of some class of spacetimes defined with reference to the asymptotic behavior of causal curves. Cauchy horizons are properly defined in any spacetime. For a discussion of the definition of black holes and event horizons check (CHRUŠCIEL, 2002).

Since there is a choice of time orientation to be made, all results for future Cauchy horizons have equivalents for past Cauchy horizons. In this work we will restrict ourselves to the future ones and we say "horizon" for short, instead of "future Cauchy horizon".

Simply from the definition above it is meaningless to discuss whether or not a spacetime admits a Cauchy horizon, since it is actually a property of each achronal set. But there is a useful meaning to the claim that a spacetime has a Cauchy horizon, through the concept of *partial Cauchy surfaces*, which are acausal hypersurfaces without boundary. If there is a partial Cauchy surface S such that $M = [D^+(S) \cup D^-(S)]$, M is said to be *globally hyperbolic*[4]. If that is not the case, M is said to have an horizon and the future Cauchy horizons of partial Cauchy surfaces are regarded simply as horizons on M.



(a) Partial Cauchy surface in Minkowski space.



(b) Taub-NUT type spacetime.

Figure 6 – Minkowski space (on the left) is a globally hyperbolic space, while the spacetime on the right, with the rotating light cones, every timelike curve below the horizon, even those that cross it, intersect S, while there are timelike curves above the horizon that never cross it when extended to the past.

1.3 Horizons

In order to specifically study their regularity and dynamics we will need some more general results on horizons, which we present in this section. The goal is to prove that every horizon is generated by lightlike geodesics. In other words, given $\mathcal{H}$ an horizon, $p \in \mathcal{H}$, there is a lightlike curve through p that stays in $\mathcal{H}$. A more precise meaning for "stay" in the last sentence shall also be given in proposition 1.16 and corollary 1.17. This section draws its results from chapter 6 of (HAWKING; ELLIS, 1973).

From the definition alone there is not much we can say about the topological nature of horizons, but, as they are defined by taking off the past of a set from the set itself, that they are achronal sets. In order to learn more about them we shall make a small detour and talk about *achronal boundaries*, which are the boundaries of either past or future sets (the reason for the name is given in proposition 1.11).

[4] This is not the only possible definition of global hyperbolicity in spacetimes. For a broader discussion see section 3.11 in (MINGUZZI; SÁNCHEZ, 2008)

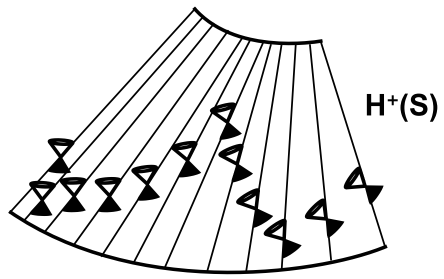


Figure 7 – A horizon $H^+(S)$ is foliated by lightlike curves, called generators as in the straight lines generators of a ruled surface.

To understand the relation between horizons and achronal boundaries we should first note that, taking S a closed achronal set, $D^+(S)$ is **not** a past set in general, since points in the past of S are not in $D^+(S)$[5]. On the other hand, S is trivially in $D^+(S)$, so every past-directed curve with future-endpoint inside $I^-(S)$ may be concatenated to a timelike curve that crosses $D^+(S)$. With this in mind we may define $\mathcal{F}(S) := D^+(S) \cup I^-(S)$, which is a past set with $H^+(S)$ in its boundary, as we show in the next proposition.

Proposition 1.9. *If S is a closed achronal set:*

i. $\mathcal{F}(S) = H^+(S)\dot{\cup} I^-(D^+(S))$;

ii. $\mathcal{F}(S)$ *is a past set and*

iii. $H^+(S) \subset \partial[\mathcal{F}(S)]$.

Proof. i. Since $S \subset D^+(S)$, $I^-(S) \subset I^-(D^+(S))$, so:

$$\mathcal{F}(S) = [D^+(S) \cup I^-(S)] \subset [D^+(S) \cup I^-(D^+(S))]$$

But from the definition of the horizon, $D^+(S) \cup I^-(D^+(S)) = H^+(S) \cup I^-(D^+(S))$ hence $\mathcal{F}(S) \subset [H^+(S)\dot{\cup} I^-(D^+(S))]$.

On the other hand, take $x \in [H^+(S)\dot{\cup} I^-(D^+(S))]$. If $x \notin D^+(S)$, $x \in I^-(D^+(S))$. Let $y \in D^+(S)$ be such that $x \ll y$. Then there is a timelike past-directed curve γ from y to x. Therefore, each α past-directed, past-inextensible timelike curve with x as future endpoint may be concatenated to γ to yield a past-directed, past-inextensible timelike curve with y as future endpoint.

Now, since $y \in D^+(S)$, $(\gamma * \alpha)$ intersects S. Pick $p_\alpha \in [(\gamma * \alpha) \cap S]$. If there is α such that $x \ll p_\alpha$, $x \in I^-(S)$. Otherwise, $\alpha \cap S \neq \emptyset$ for each α satisfying the given conditions, thus $x \in D^+(S)$. In both cases, $x \in \mathcal{F}(S)$.

[5] This is a reason for asking S to be achronal, since $x \in D^+(S) \cap I^-(S)$ would result in a past-directed timelike curve starting in S, passing through x and crossing S again. Both intersection points would be chronologically related. That would violate the well-posedness of initial conditions on S mentioned in the end of last section.

ii. Follows directly from the fact that $H^+(S) \subset D^+(S)$.

iii. If $x \in H^+(S)$ and $U \subset M$ is an open set such that $x \in U$, $I^+(x) \cap U \neq \varnothing$, as will be proved in proposition 1.27. Since $H^+(S)$ is achronal, $I^+(x) \cap H^+(S) = \varnothing$ and, as a consequence, $I^-(D^+(S)) \cap I^+(x) \neq \varnothing$ would imply $x \in I^-(D^+(S))$, which contradicts the definition of $H^+(S)$. So, from item i, $I^+(x) \subset [\mathcal{F}(S)]^c$ and, since $x \in \mathcal{F}(S)$, $x \in \partial[\mathcal{F}(S)]$. □

It should be noted that the last part of the proof above implies also that $H^+(S) \subset \partial[D^+(S)]$. In fact, we can have a better picture of $D^+(S)$ with respect to its boundary in the following proposition.

Proposition 1.10. *Take S a closed achronal set. Then:*

i. $D^+(S)$ is closed in M;

ii. $\partial[D^+(S)] = H^+(S) \cup S$;

iii. $[D^+(S)]^o = I^+(S) \cap I^-(D^+(S))$.

Proof. i. Let $x \in [D^+(S)]^c$, in particular $x \notin S$. Then, there is a convex normal neighborhood U around x such that $U \subset S^c$. Also, there is a timelike past-directed, past-inextensible curve γ with future endpoint x that does not intersect S. Take $y \in (U \cap \gamma)$. Then $(I^+(y) \cap U) \cap D^+(S) = \varnothing$: if $y \ll z$, $z \notin S$, there is a past-directed timelike geodesic α in U from z to y and the concatenation $\alpha * \gamma$ does not intersect S, therefore $z \notin D^+(S)$. But $I^+(y)$ is open, as proved in proposition 1.6. That yields that $[D^+(S)]^c$ is open.

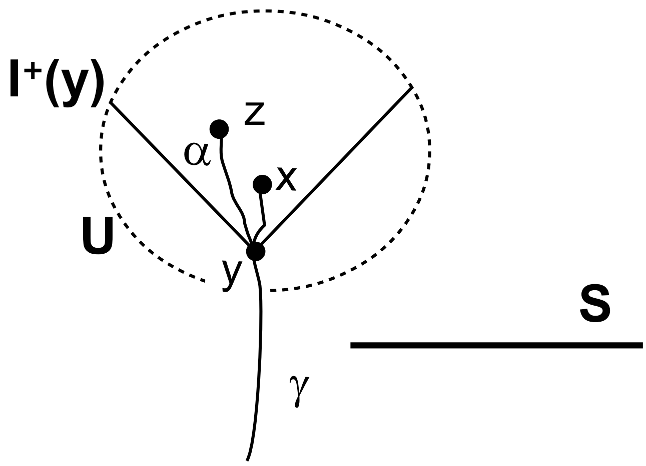


Figure 8 – Just as we concatenate α with γ, any point future to a timelike curve that does not cross S can be shown not to be in $D^+(S)$.

ii and iii. We have already seen that $H^+(S) \subset \partial[D^+(S)] \subset D^+(S)$, the last from the previous item. Also, given $x \in S \subset D^+(S)$, U open such that $x \in U$, $I^-(x) \cap U \neq \varnothing$. But, since S is achronal and $D^+(S) \subset I^+(S)$, $I^-(S) \subset [D^+(S)]^c$, so $S \subset \partial[D^+(S)]$.

Now, from the definitions of the Cauchy development and of the Cauchy horizon:

$$D^+(S) \subset [[S \cup I^+(S)] \cap [H^+(S) \cup I^-(D^+(S))]] \ \ [1]$$

If we prove $[I^+(S) \cap I^-(D^+(S))] \subset D^+(S)$ we wil have proved that [1] is actually an equality. In fact, if $x \in I^+(S) \cap I^-(D^+(S))$, there is $y \in D^+(S)$ such that $x \ll y$. Thus, there is γ a past-directed timelike curve from y to x. Now, given α a past-directed, past-inextensible timelike curve with future endpoint x, the concatenation $\gamma * \alpha$ is a past-directed, past-inextensible timelike curve with future endpoint y, hence $(\gamma * \alpha) \cap S \neq \emptyset$. But $x \in I^+(S)$, and S is achronal, so $\gamma \cap S = \emptyset$. It follows that $\alpha \cap S \neq \emptyset$, so $x \in D^+(S)$.

Now, since $S \cap I^+(S) = \emptyset$ and $H^+(S) \cap I^-(D^+(S)) = \emptyset$ we can rewrite [1] as (now with the equality just proved):

$$D^+(S) = [S \cup H^+(S)] \dot{\cup} [I^-(D^+(S)) \cap I^+(S)] \ \ [1']$$

But $I^-(D^+(S)) \cap I^+(S)$ is open and $[D^+(S)]^o \cap \partial[D^+(S)] = \emptyset$, as is valid for any set. That proves both itens ii and iii. □

Now that we have a better picture of $D^+(S)$, we resource to the fact that $H^+(S) \subset \partial[\mathcal{F}(S)]$ in order to have further information on the horizon by means of theorems on achronal boundaries.

Proposition 1.11. *If $\mathcal{W}$ is either a past or a future set in the spacetime (M, g), $\partial\mathcal{W}$ is an achronal, Lipschitz, 3-manifold without boundary.*

Proof. We only need to prove for the case in which $\mathcal{W}$ is a past set, as changing the time orientation of the spacetime makes $\mathcal{W}$ a future set without changes in topology.

1. First, $\partial\mathcal{W}$ is an achronal set: Let $p \in \partial\mathcal{W}$. $I^-(p)$ is open, as proved in proposition 1.6. Assume then that $q \in [I^-(p) \cap \partial\mathcal{W}]$. Then $p \in I^+(q)$, which is open, thus $\exists r \in I^+(q) \cap \mathcal{W} \Rightarrow q \in I^-(r)$, which is, also, open. Since $q \in \partial\mathcal{W}$, it follows that $I^-(r) \cap (\mathcal{W})^c \neq \emptyset$, contradicting the fact that $\mathcal{W}$ is a past set. Summing up, $\forall p \in \partial\mathcal{W}, \nexists q \in \partial\mathcal{W}$ s.t. $q \ll p$. So, $\partial\mathcal{W}$ is achronal.

2. Take $p \in \partial\mathcal{W}$ and $\{e_0, e_1, e_2, e_3\}$ an orthonormal basis for T_pM such that e_0 is timelike. Let $\exp_p : V_p \to M$ be the exponential map, with $\exp_p(V_p)$ a normal neighborhood around p. Also, call $\tilde{\mathcal{W}} := \exp_p^{-1}(\mathcal{W} \cap \exp_p(V_p))$. Since $0 \in V_p$, we call $\tilde{V} := \{(0, x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\} \cap V_p \neq \emptyset$.

Now, $\exists \delta > 0$ such that $\left(\overline{B^3_{\delta/2}(0)} \times [-\delta, \delta]\right) \subset V_p$, for $B^3_r(p) := \{0\} \times B_r(p_1, p_2, p_3)$ in $\{(0, x_1, x_2, x_3)\}$. Let $W_p := B^3_{\delta/2}(0) \times (-\delta, \delta)$. Then, for each $(x, y, z) \in B^3_{\delta/2}(0)$, the curve $\gamma_{(x,y,z)}(t) := (t, x, y, z)$, $t \in (-\delta, \delta)$ intersects both $\exp_p^{-1}(I^-(p))$ and $\exp_p^{-1}(I^+(p))$, for proposition 1.27 holds and, since we chose an orthonormal basis for T_pM, the expression of the metric on T_pM in this basis is the diagonal matrix with diagonal $[-1, 1, 1, 1]$.

Now, $I^-(p) \subset \mathcal{W}$, since if $r \in [I^-(p) \backslash \mathcal{W}]$, $p \in I^+(r)$, hence $\exists s \in [I^+(r) \cap \mathcal{W}]$. In this case, $r \in [I^-(s) \backslash \mathcal{W}]$, contradicting the fact that $\mathcal{W}$ is a past set.

On the other hand, $I^+(p) \subset \mathcal{W}^c$, since if there was $r \in [I^+(p) \cap \mathcal{W}]$ we would have $p \in I^-(r) \subset \mathcal{W}$, which contradicts $p \in \partial \mathcal{W}$, for $I^-(r)$ is open.

From continuity, $\gamma_{(x,y,z)} \cap \partial \tilde{\mathcal{W}} \neq \varnothing$ and, since $\gamma_{(x,y,z)}$ is timelike and $\partial \mathcal{W}$ is achronal, $\#\left(\gamma_{(x,y,z) \cap \partial \tilde{\mathcal{W}}}\right) \leqslant 1$.

Define $f : B^3_{\delta/2}(0) \to \mathbb{R}$ by $f(x,y,z) = \pi_4(\gamma_{(x,y,z)} \cap \partial \tilde{\mathcal{W}})$. Then f is a function and $\partial \tilde{\mathcal{W}}$ is the graphic of f.

3. f is Lipschitz, with Lipschitz constant 1: Assume there are $x, \tilde{x} \in B^3_{\delta/2}(0)$ such that $\dfrac{|f(x) - f(\tilde{x})|}{||x - \tilde{x}||} > 1$. Then the segment $(1-t)(x, f(x)) + t(\tilde{x}, f(\tilde{x}))$ in T_pM is a timelike curve, which contradicts the fact that ∂W is achronal. Therefore, since $\exp_p$ is a diffeomorphism, ∂S is a Lipschitz manifold without boundary. □

Since we have seen that $H^+(S)$ is part of an achronal boundary, it is part of a Lipschitz manifold. In fact, given proposition 1.10, $[H^+(S) \backslash S] \subset I^+(S)$, therefore it is an open subset of $\partial \mathcal{F}(S)$, being a Lipschitz manifold itself. On the other hand, keep in mind that $H^+(S)$ is a closed subset of $D^+(S)$, which is closed itself, so $H^+(S)$ is a closed subset of M, hence $H^+(S)$ is closed in $\partial \mathcal{F}(S)$.

It should be noted also that, in terms of regularity, in general one can't expect a spacetime to be more than Lipschitz. In fact, there are horizons that are indeed Lipschitz manifolds but which are not C^1. Take for example $M = \mathbb{R}^3$ and g to be the usual plane Lorentzian metric of signature $(-,+,+)$. If $S = \{0\} \times B^3_1(0)$, $H^+(S)$ is the section of the cone $(t-1)^2 - x^2 - y^2 = 0$ with $t \in [0,1]$. Hence, it is not differentiable at the cone apex $(1,0,0)$, although it is Lipschitz. A more thorough discussion on the regularity of Cauchy horizons is presented in chapter 3.

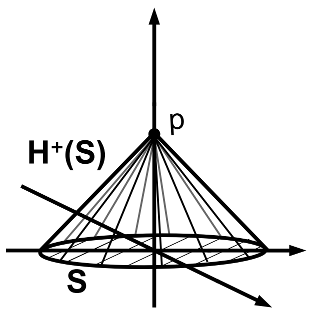


Figure 9 – If M is Minkowski 3-space and $S = \{0\} \times B^3_1(0)$, $H^+(S)$ is the upper part of a cone and fails to be differentiable at the apex $p = (0,0,1)$.

To proceed we will use a convenient way of partitioning an achronal set with respect to whether each point is contained in the interior of a lightlike curve or not. The full classification has four cases, presented in the following definition:

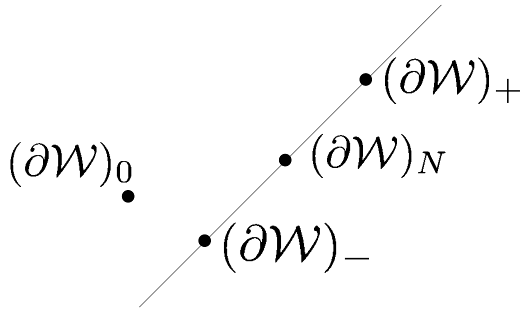


Figure 10 – In a very artificial example of an achronal set consisting of four points in Minkowski space we may see the different possible conditions of points in the set with relation to lightlike curves (the straight line in the drawing).

Definition 1.12. *If $\partial\mathcal{W}$ is an achronal set and $q \in \partial\mathcal{W}$, we have two cases divided in two subcases:*

i. $\exists p \in [\partial\mathcal{W} \cap [J^-(q)\backslash(\{q\} \cup I^-(q))]]$, i.e., $p \neq q, p \prec q$ and $p \not\ll q$, since $\partial\mathcal{W}$ is achronal and

i.a. $\exists r \in [\partial\mathcal{W} \cap [J^+(q)\backslash(\{q\} \cup I^+(q))]]$, i.e., $r \neq q, q \prec r$ and $q \not\ll r$, since $\partial\mathcal{W}$ is achronal, then we say $q \in (\partial\mathcal{W})_N$ (q is interior to a null curve from p to r);

i.b. $\partial\mathcal{W} \cap (J^+(q)\backslash(\{q\} \cup I^+(q))) = \varnothing$ then we say $q \in (\partial\mathcal{W})_+$ (q is the future endpoint of a null curve crossing $\partial\mathcal{W}$ at least twice but i.a. does not occur).

ii. $\partial\mathcal{W} \cap (J^-(q)\backslash(\{q\} \cup I^-(q))) = \varnothing$ and

ii.a. $\exists r \in [\partial\mathcal{W} \cap [J^+(q)\backslash(\{q\} \cup I^+(q))]]$, then we say $q \in (\partial\mathcal{W})_-$ (q is the past endpoint of a null curve crossing $\partial\mathcal{W}$ at least twice);

ii.b. $\partial\mathcal{W} \cap (J^+(q)\backslash(\{q\} \cup I^+(q))) = \varnothing$ then we say $q \in (\partial\mathcal{W})_0$ (there is no null curve crossing $\partial\mathcal{W}$ twice that contains q).

There is a useful condition to recognize in which of these partitions a given point is located, and its proof demands a technical result which we only state here. It is theorem 3.1 from (MINGUZZI, 2008).

Lemma 1.13 (Limit curve theorem)**.** *Take (M, g) a spacetime and h a complete Riemannian metric on M.*

Let $U \subset M$ be open and $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of continuous future-directed causal curves in U. Assume there is $y \in U$ an accumulation point for $\{\lambda_n\}_{n\in\mathbb{N}}$, i.e., $\exists\{\lambda_{n_k}\}_{k\in\mathbb{N}} \subset \{\lambda_n\}_{n\in\mathbb{N}}$ such that $\forall V \subset U$ open with $y \in V$, $\exists k_V \in \mathbb{N}$ satisfying the condition $k > k_V \Rightarrow \lambda_{n_k} \cap V \neq \varnothing$.

Then, there is a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}} \subset \{\lambda_n\}_{n\in\mathbb{N}}$ such that, if $\lambda_{n_j} : [a_j, b_j] \to M$ is parametrized by h-arc length with $0 \in [a_j, b_j]$ and $\lambda_{n_j}(0) \to y$:

i. $\exists a \leqslant 0, b \geqslant 0$ such that $a_j \to a$ and $b_j \to b$;

ii. If $\exists W$ *open such that* $y \in W$ *and* $\lambda_{n_j} \cap W^c \neq \varnothing, \forall j \in [\mathbb{N}\backslash I], I$ *finite set,* $\exists \lambda : [a, b] \to M$ *causal curve, continuous, such that* $0 \in [a, b], \lambda(0) = y$ *and* $\lambda_n \to \lambda$ *uniformly in each compact set of* M*.*

The aforementioned condition to recognize the nature of the points in the achronal set is the following:

Proposition 1.14. *If* $q \in \partial\mathcal{W}$*,* $\partial\mathcal{W}$ *the achronal boundary of a past set* $\mathcal{W}$*, and* U *is an open set in* M *such that* $q \in U$*:*

i. $I^-(q) \subset I^-(\mathcal{W}\backslash U) \Rightarrow q \in [(\partial\mathcal{W})_N \cup (\partial\mathcal{W})_-]$*;*

ii. $I^+(q) \subset I^+((M\backslash\mathcal{W})\backslash U) \Rightarrow q \in [(\partial\mathcal{W})_N \cup (\partial\mathcal{W})_+]$*.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset [I^-(q) \cap U]$ such that $x_n \to q$ (there is such a sequence because there are timelike curves in $I^-(q) \cap \mathcal{W}$ with future endpoint q).

i. If $I^-(q) \subset I^-(\mathcal{W}\backslash U), \forall n \in \mathbb{N}, x_n \in I^-(\mathcal{W}\backslash U) \Rightarrow \exists \lambda_n : [0, 1] \to M$ future-directed timelike curve such that $\lambda_n(1) \in [\mathcal{W}\backslash U]$ and $\lambda_n(0) = x_n$.

Since $x_n \to q$, q is an accumulation point for $\{\lambda_n\}_{n\in\mathbb{N}}$. Then, as $\lambda_n \cap U^c \neq \varnothing$, $\forall n$, $\exists \lambda : [0, 1] \to M$ past-directed causal curve such that $\exists\{\lambda_{n_k}\}_{k\in\mathbb{N}} \subset \{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_{n_k} \to \lambda$ uniformly in any complete Riemannian metric one attributes M. Therefore, $\lambda_{n_k}(1) \to \lambda(1) \in \overline{\mathcal{W}\backslash U}$. ($\lambda(1) \neq q$ because $q \in U$, which is open).

It remains to prove that λ is not timelike. Assume it is. Then if $q = \lambda(0) \ll \lambda(1)$, $\lambda(1) \in I^+(q)$. Now, if $I^+(q) \cap \mathcal{W} \neq \varnothing$, say $q \ll r$ and $r \in \mathcal{W}$. Then, since $\mathcal{W}$ is a past set, $I^-(r) \subset \mathcal{W}$. But $I^-(r)$ is open and $q \in I^-(r)$, which contradicts the fact that $q \in \partial\mathcal{W}$. Thus, $I^+(q) \cap \mathcal{W} = \varnothing$. But $\lambda(1) \in [I^+(q) \cap \overline{\mathcal{W}\backslash U}]$. Then, since $I^+(q)$ is open, $I^+(q) \cap (\mathcal{W}\backslash U) \neq \varnothing$. Absurd. Therefore, λ is not timelike. But λ is causal, therefore λ is lightlike $\Rightarrow q \in ((\partial\mathcal{W})_N \cup (\partial\mathcal{W})_-)$.

ii. If we take the reversed time orientation, $\mathcal{W}$ becomes a future set, $M\backslash\mathcal{W}$ turns into a past set and the hypothesis becomes $I^-(q) \subset I^-((M\backslash\mathcal{W})\backslash U)$. Therefore, $q \in ((\partial\mathcal{W})_N \cup (\partial\mathcal{W})_-)$. But, going back to the original time orientation, the conclusion becomes $q \in ((\partial\mathcal{W})_N \cup (\partial\mathcal{W})_+)$. □

Now we can prove the fact that horizons are generated by lightlike curves.

Theorem 1.15. *Let* S *be an achronal set and* $\mathcal{F}(S) = D^+(S) \cup I^-(S)$*. Then we have that* $[H^+(S)\backslash edge(S)] \subset [\partial\mathcal{F}(S)_N \cup \partial\mathcal{F}(S)_+]$*, i.e., given* $p \in [H^+(S)\backslash edge(S)]$ *there is a lightlike curve through* p *that intersects* $H^+(S)$ *in at least one other point. (For a definition and further discussion on the concept of "edge", check section 1.4.2).*

Proof. We want to show that any $p \in H^+(\mathcal{S})$ satisfies condition ii of proposition 1.14. Let us divide it in two cases:

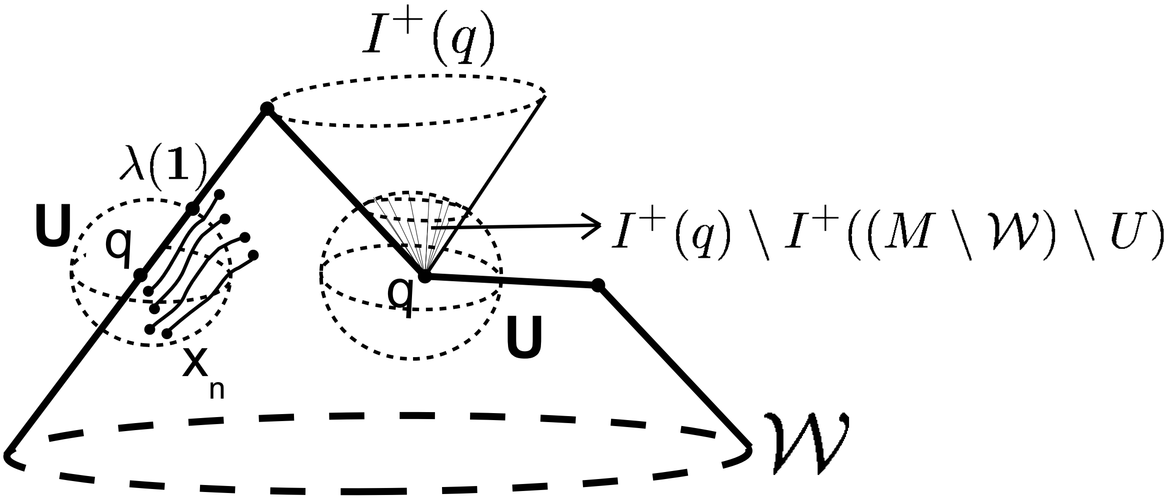


Figure 11 – In the drawing we see the difference between a point satisfying the condition of item i of the proof, across which we can construct a lightlike curve as a limit of lightlike cuves in $\mathcal{W}$, and one which does not satisfy said condition, and is in $(\partial\mathcal{W})_-$ in this case.

I. $p \in [H^+(S)\backslash S] \subset I^+(S)$: Let N_p be a convex normal neighborhood around p and $\epsilon > 0$ be such that $\exp_p(B_{2\epsilon}(0)) \subset [N_p \cap I^+(S)]$. Define $U = \exp_p(B_\epsilon(0))$. Let $q \in I^+(p)$. Now, if $q \notin I^+((M\backslash\mathcal{F}(S))\backslash U)$, $\gamma : [a, b) \to M$ being a past-directed, past-inextensible timelike curve with future endpoint $\gamma(a) = q$ implies $\gamma((a, b)) \subset [\mathcal{F}(S) \cup U]$.

But no past-inextensible curve stays in U. In fact, let $\gamma(c) \in U$. Since N_p is a convex neighborhood, it is a normal neighborhood also for $\gamma(c)$. But $\overline{U}$ is compact, because $\exp_p$ is a diffeomorphism in $\exp_p^{-1}(N_p)$. Hence, $\exp_{\gamma(c)}^{-1}(U)$ is precompact in $T_{\gamma(c)}M$. Thus, as will be shown inside the proof of proposition 1.27, any inextensible timelike curve through $\gamma(c)$ leaves $\overline{U}$.

As a consequence, $\exists t_0 > a$ such that $t > t_0 \Rightarrow \gamma(t) \notin U \Rightarrow \gamma(t) \in \mathcal{F}(S)$. If $\gamma(t) \in D^+(S)$, γ intersects S in some time after t_0. On the other hand, since $\gamma([a, b))$ is connected and $q \in I^+(S)$, for $p \in I^+(S)$, it is impossible to have $\gamma((a, b)) \subset [U \cup I^-(S)]$. Hence, γ intersects S. Since that is true for all γ, $q \in D^+(S)$, which contradicts the fact that $p \in H^+(S)$. As a consequence, $q \in I^+((M\backslash\mathcal{F}(S))\backslash U)$.

II. $p \in [H^+(S) \cap (S\backslash edge(S))]$: Let U be a convex normal neighborhood of M around p such that $q \in [I^+(p) \cap U]$ and $r \in [I^-(p) \cap U]$ implies that any timelike curve from r to q intersects S (such N_p exists because $p \notin edge(S)$). Fix $q \in I^+(p)$ and $\gamma : [a, b) \to M$ a past-directed, past-inextensible timelike curve with future-endpoint q. If $\gamma((a, b))$ intersects $I^-(S)$, γ crosses S and we may apply the same reasoning as in the previous case to prove that $q \in I^+((M\backslash\mathcal{F}(S))\backslash U)$. □

The term “generator” comes from the name given to the generating straight lines in ruled surfaces, meaning that each point of the surface is inside some generator and that they all lie inside the surface. For horizons that is partially true. In fact, generators may leave the horizon when extended to the future. In this case there is one last point in the intersection of the generator with the horizon, for $\mathcal{F}(S)$ is closed, which we call future endpoint of the generator in the horizon. On the other hand, when extended to the past the generator does not leave the horizon unless $S \cap H^+(S) \neq \varnothing$, in which case there may be a past endpoint of the generator in the horizon in $S \cap H^+(S)$.

None of that has been proven in theorem 1.15, only that given a point in the horizon there is another point in $\partial[\mathcal{F}(S)]$ in its causal past, which is not in its chronological past since $\partial[\mathcal{F}(S)]$ is achronal. Now we prove that the generator does not leave the horizon in-between.

Proposition 1.16. *If $p, r \in H^+(S)$ are such that $r \prec p$, $r \prec q \prec p \Rightarrow q \in H^+(S)$.*

Proof. Let U be an open set in M such that $q \in U$. Then there are $q_+, q_- \in U$ such that $q \ll q_+$ and $q_- \ll q$.

Now, as will be proven in proposition 1.30, $a \ll b \prec c \Rightarrow a \ll c$ and also $a \prec b \ll c \Rightarrow a \ll c$. Applying this relation to our problem gives $r \ll q^+ \Rightarrow q^+ \in \mathcal{F}(S)^c$ (check item i of proposition 1.9) and $q_- \ll p \Rightarrow q_- \in \mathcal{F}(S)$, since $\mathcal{F}(S)$ is a past set and $p, r \in \partial[\mathcal{F}(S)]$. Therefore $q \in \partial[\mathcal{F}(S)]$. But $r \prec q$ and $r \in [I^+(S) \cap S]$, so $q \in I^+(S)$, and hence $q \in H^+(S)$. □

Notice that the proof would be the same, but for the last sentence, if we replaced $H^+(S)$ with any achronal boundary. In particular it would still work for $\partial[\mathcal{F}(S)]$.

The final result we are aiming in this section comes in the following corollary:

Corollary 1.17. *Take $r, p \in H^+(S), r \prec p$ and $\gamma : [0, a) \to M$ a past-directed, past-inextensible lightlike curve with $\gamma(0) = p$ that crosses r. Then:*

i. $\exists b > 0, b \leqslant a$ such that $\gamma([0, b)) \subset \partial[\mathcal{F}(S)]$ and $\gamma(b) \notin [H^+(S) \backslash edge(S)]$, if $\gamma(b)$ exists;

ii. γ is a lightlike geodesic in the interval $[0, b)$.

Proof. i. From proposition 1.16, if $\gamma(t) \in \partial[\mathcal{F}(S)]$ for some $t > 0$, $\gamma([0, t]) \subset \partial[\mathcal{F}(S)]$. Now, assume there is a supremum for the set of such t. Since, as we have proved in theorem 1.15, $[H^+(S) \backslash edge(S)] \subset [\partial[\mathcal{F}(S)]_N \cup \partial[\mathcal{F}(S)]_+]$, said supremum is not realized in $H^+(S) \backslash edge(S)$.

ii. Assume there is a time $t_0 \in [0, b)$ such that γ fails to be a geodesic in $\gamma((t_0, t_0 + \epsilon))$ for some $\epsilon > 0, b - t_0 > \epsilon$. From the discussion that will follow proposition 1.27, that means that $\gamma(t_0 + \epsilon) \in I^-(\gamma(t_0))$. But $\gamma(t_0) \in \mathcal{F}(S)$, which is a past set. Hence, $\gamma(t_0 + \epsilon) \in [\mathcal{F}(S)]^o$, which contradicts item i. □

1.4 Further Technical Details

In this section we present some technical results from Lorentzian Geometry used in proofs throughout the work, above and below, that would not fit smoothly intertwined with other parts of the text.

1.4.1 Causal Character of Vector Spaces

In section 1.2 we defined the causal character of tangent vectors and curves. The concept may be extended to any vector subspace of a Lorentzian vector space in the following way:

Definition 1.18. *Given W a subspace of a Lorentzian vector space we say that:*

i. W is spacelike iff $v \in W \Rightarrow v$ is spacelike;

ii. W is lightlike iff $\exists v \in W$ such that v is lightlike but $w \in W \Rightarrow g(w, w) \geqslant 0$;

iii. W is timelike iff W is neither spacelike nor lightlike iff $\exists v \in W$ timelike.

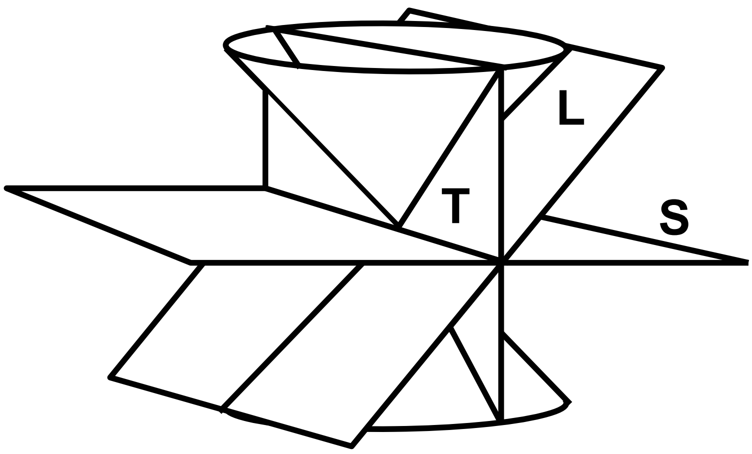


Figure 12 – In Minkowski space one can check the causal character of a plane looking at its position with respect to a lightcone. If it is tangent to the lightcone along a generator it is lightlike (as is L in the picture). If it touches the cone only at its vertex it is spacelike (as is S shown). If it intersects the cone at a pair of straight lines it is timelike (as T in the picture).

If $N \subset M$ is a submanifold, it is called respectively spacelike, lightlike or timelike iff its tangent plane is spacelike, lightlike or timelike at each of its points. The definition given for curves in section 1.2 is a special case of this one.

The concept of causal character for vector subspaces is useful because the causal character of vector subspaces may be related to the causal character of their normal subspaces.

Proposition 1.19. *Let W be a vector subspace of V, a Lorentzian vector space.*

i. W is timelike iff $W^\perp$ is spacelike;

ii. W is spacelike iff $W^\perp$ is timelike;

iii. W is lightlike iff $W^\perp$ is lightlike iff $W \cap W^\perp \neq \{0\}$.

Proof. i. If W is timelike, there is $v \in W$ a timelike vector and $W^\perp \subset v^\perp$. Now, we can use the Gram-Schmidt process to find an orthonormal basis $\left\{ \frac{v}{\sqrt{-g(v,v)}}, e_1, ..., e_{n-1} \right\}$ for V. Then $v^\perp = span\langle e_1, ..., e_{n-1}\rangle$, which is spacelike. Hence W is spacelike.

On the other hand, assume $W^\perp$ is spacelike and choose $v \in V$ timelike. Since $V = W \bigoplus W^\perp$, $\exists w \in W, \bar{w} \in W^\perp$ such that $v = w + \bar{w}$. Then $0 < g(v,v) = g(w,w) + g(\bar{w},\bar{w})$. But $g(\bar{w},\bar{w}) \geqslant 0$, hence w is timelike and $w \in W$, so W is timelike.

ii. It follows from item i, since $W^{\perp\perp} = W$.

iii. W lightlike iff $W^\perp$ lightlike follows from the contrapositive of items i and ii. Also, if W is lightlike there is $w \in W$ lightlike vector. Assume there is $v \in W$ such that $g(v,w) \neq 0$. Then

$$g\left(v - \frac{\alpha w}{2g(v,w)}, v - \frac{\alpha w}{2g(v,w)}\right) \quad = \quad -\alpha + \frac{\alpha^2 g(w,w)}{4g(v,w)^2}$$

So, if $g(w,w) = 0$, take $\alpha = 1$ and we shall have $v - \frac{w}{2g(v,w)} \in W$ timelike, contradiction. On the other hand, if $g(w,w) > 0$, take $\alpha = \frac{2g(v,w)^2}{g(w,w)} > 0$. Then $v - \frac{\alpha w}{2g(v,v)} \in W$ is timelike, again a contradiction. Hence, $v \in W \cap W^\perp$.

On the other hand, if $W \cap W^\perp \neq \{0\}$, $v \neq 0$ such that $v \in W \cap W^\perp$ implies that v is lightlike. But if W is timelike, from i, $W^\perp$ is spacelike, thus $v \notin W^\perp$, which is a contradiction. Hence, W is lightlike. □

Notice that, as will be seen in the proof of proposition 3.3, W lightlike implies there is only one lightlike direction in W, otherwise W would have a timelike vector. In particular, that means that if v is lightlike, the only lightlike vectors in $v^\perp$ are those parallel to v.

1.4.2 The Edge

Before proving that Cauchy horizons are generated by lightlike geodesics, (HAWKING; ELLIS, 1973) defines briefly the *edge* of a closed achronal set in the following way:

Definition 1.20. *If S is a closed achronal set, the edge of S - denoted edge(S) - is the set of points $p \in S$ such that $\forall U \subset M$, U open, and $p \in U$, $\exists q, r, \gamma$ such that:*

i. $q \in (I^+(p) \cap U)$;

ii. $r \in (I^-(p) \cap U)$;

iii. γ *is a future-directed timelike curve from* r *to* q *such that* $\gamma \subset (U \backslash S)$.

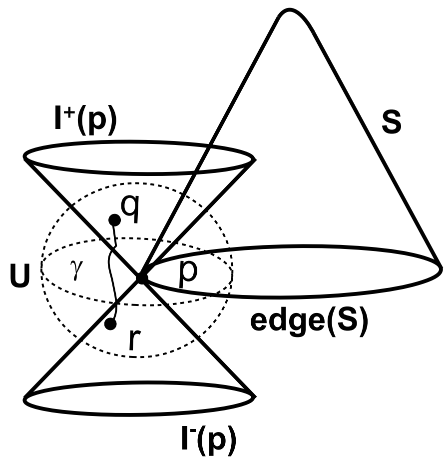


Figure 13 – For points on the edge of s, there are q, r satisfying the conditions of definition 1.20 for any neighborhood U.

Such definition is necessary to avoid the problem that the proof of theorem 1.15 demands neighborhoods which do not allow timelike curves going around the horizon. More than a technicality, the fact effectively proven in the theorem, that points in the horizon are in $\partial\mathcal{F}(S)_N \cup \partial\mathcal{F}(S)_+$, does **not** hold in general for points on $edge(S)$. In the examples below we show how our claims concerning the generators on horizons may break at edge points.

Example 1.21. *Let* $(\mathbb{R}^3, g)$ *be Minkowski space, with* g *the Lorentzian plane metric of signature* $(-,+,+)$, *and* $S = \{0\}$. *Then* $H^+(S) = S = edge(S)$ *and there is no generator at the horizon.*

Although this example is quite trivial, when we consider

$$\mathcal{F}(S) = \{0\} \cup \{(t,x,y)|x^2 + y^2 < t, t < 0\}$$

We still have that $0 \in \partial[\mathcal{F}(S)]_+$. In the next example even that is broken.

Example 1.22. *Let* $(\mathbb{R}^2 \backslash A, g)$ *be the Minkowski plane, with* g *the Lorentzian plane metric of signature* $(+,-)$, *from which we take the set* $A := \{(x,-1/2)|x \in [-3/2, 1/2]\}$. *Consider* $S := \{(x,0)|x \leqslant 0\} \cup \{(x,-1)|x \in [-1,1]\}$. *Then:*

$$(-1,-1) \in [edge(S) \cap \partial[H^+(S)] \cap \partial[\mathcal{F}(S)_-]].$$

Besides being an important limitation to theorem 1.15, in case S is a hypersurface, $edge(S)$ is the boundary of S regarded as a manifold with boundary (this is not true

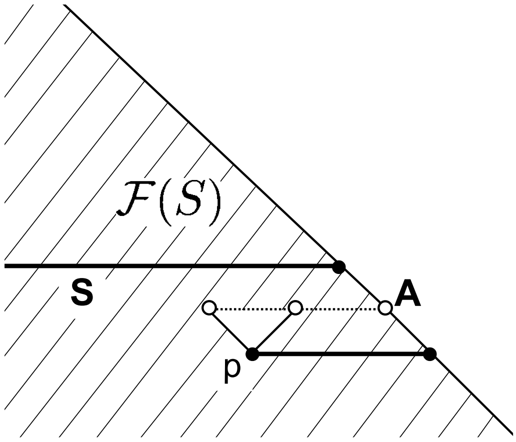


Figure 14 – In the drawing of example 1.22 one can see clearly that $\partial[\mathcal{F}(S)]$ bifurcates to the future at $p = (-1, -1)$ - to the left as part of the boundary of $I^-(S)$ and to the right as part of $H^+(S)$ - hence the generators of $\partial[\mathcal{F}(S)]$ leave the boundary at p if extended to the past.

for submanifolds of smaller dimension). This fact explains the interest in partial Cauchy surfaces, since in this case “without boundary” translates to “edgeless”. So, the future Cauchy horizons of partial Cauchy surfaces do not have pathological behavior like the ones from the examples given above, and are ruled by generators through all its points that remain in the horizon when extended to the past.

Horizons of achronal sets with non-empty edge, on the other hand, are specially easy to be used as counterexamples. This is true particularly when (M, g) is Minkowski space, for they can be easily manipulated to respect desired constraints. Also, they become useful tools to study the behavior of horizons in general. Besides, from a physical point of view, sets with edge might be a better representation for compact regions of the space, so the study of their horizons is also important from a practical viewpoint.

1.4.3 Gauss Lemma and Local Causal Structure

A usual step in our proofs regarding horizons is to take a normal neighborhood of a given point p in the manifold, perform the calculations in the preimage of the neighborhood with respect to the exponential map - which is easier since it is a copy of Minkowski space - and push the result back to the manifold through the exponential map.

The effectiveness of this maneuver for our objectives depends on some level of equivalence between the neighborhood of p itself and its preimage with respect to the exponential map in two senses. First, the regularity of subsets must be maintained, what is guaranteed by the fact that the exponential map is a diffeomorphism in the normal neighborhood. Second, the causal structure should be preserved by the exponential map in an appropriate sense. The goal of this section is to see if some causal structure is actually preserved and to what extent.

To begin, we compare proofs for two different statements of the Gauss Lemma, that are intertwined in the Riemannian case. The first is stated in (O'NEIL, 1983), page 127, and the second in (CARMO, 1979), page 59.

Throughout this section, (M, g) is a semi-Riemannian n-dimensional manifold (the signature of g is irrelevant unless explicitly stated), $p \in M$ and $N_p \subset M$ is a normal neighborhood around p with exponential map $\exp_p : U_p \subset T_pM \to N_p$ a diffeomorphism. Denote $\exp_p(v) := \tilde{v}$. For $v \in T_pM$, define $\phi_v : T_pM \to T_vT_pM$ by $\phi_v(w) = \frac{d}{ds}|_{s=0}\gamma_w^v(s)$, with $\gamma_w^v(s) = v + sw$, and call $\phi_v(w) := w_v$.

Lemma 1.23 (Symmetry Lemma)**.** *If $F : (a, b)\times(c, d) \to M$ is a smooth map, $F = F(t, s)$, and we call $\partial_t(t, s) := F_t(t, s) = D_{(t,s)}F(e_1)$ and $\partial_s(t, s) := F_s(t, s) = D_{(t,s)}F(e_2)$ the vector fields over F induced by the map*[6]*, then we have:*

$$\nabla_{\partial_s}\partial_t \quad = \quad \nabla_{\partial_t}\partial_s.$$

Proof. Let $\psi : V \subset M \to \mathbb{R}^n$, given by $q \mapsto (x^1(q), ..., x^n(q))$, be a coordinate chart for M around a point p in the image of F with $\exp_p$ well-defined in V and let $\{e_1, ..., e_n\}$ be the coordinate basis associated to ψ. Then:

$$\begin{aligned}
\nabla_{\partial_s}\partial_t &= D_{F_s^i e_i} F_t^j e_j \\
&= \left(\partial_s F_t^j\right) e_j + F_s^i F_t^j \nabla_{e_i} e_j \\
&= \left(\partial_s F_t^j\right) e_j + F_s^i F_t^j \nabla_{e_j} e_i \\
&= \left(\partial_s F_t^j\right) e_j + F_s^i \nabla_{\partial_t} e_i \\
&\overset{(1)}{=} \left(\partial_t F_s^j\right) e_j + F_s^i \nabla_{\partial_t} e_i \\
&= \nabla_{\partial_t}\partial_s
\end{aligned}$$

(1) is due to the fact that $F_s^j(t, s)$ is a function on $\mathbb{R}^n$, so the commutativity of the derivatives hold. ☐

We can proceed to O'Neil's version of the Gauss Lemma:

Proposition 1.24 (Gauss Lemma)**.** *If $v, w \in T_pM$:*

$$g_p(v, w) = g_{\tilde{v}}(D_v \exp_p(v_v), D_v \exp_p(w_v))$$

Proof. Using the notation from lemma 1.23, define $F : (-\delta, \delta) \times (-\epsilon, 1 + \epsilon) \to M$ by $F(t, s) = \exp_p(t\gamma_w^v(s))$. Then:

$$\begin{cases} F_t(t, s) = D_{t\gamma_w^v(s)} \exp_p(\gamma_w^v(s)_{t\gamma_w^v(s)}) \\ F_s(t, s) = D_{t\gamma_w^v(s)} \exp_p(t\dot{\gamma}_w^v(s)) \end{cases} \quad [1]$$

6 To avoid the trouble that might be caused by lack of injectivity by F, we use the notions of connection and vector field over a map presented in sections 2.0.1 and 2.0.2 of (SACHS; WU, 1977).

And, taking $(t, s) = (1, 0)$:

$$\begin{cases} F_t(1,0) = D_{\gamma_w^v(0)} \exp_p(\gamma_w^v(0)_{\gamma_w^v(0)}) = D_v \exp_p(v_v) \\ F_s(1,0) = D_{\gamma_w^v(0)} \exp_p(\dot{\gamma}_w^v(0)) = D_v \exp_p(w_v) \end{cases}$$

Therefore we can write:

$$g_{\tilde{v}}(D_v \exp_p(v_v), D_v \exp_p(w_v)) = g_{\tilde{v}}(F_t(1,0), F_s(1,0))$$

Now we can use some derivations to bring to light the equality we are looking for arise:

$$\begin{aligned} F_t g(F_t(t,s), F_s(t,s)) &= g(\nabla_{F_t} F_t(t,s), F_s(t,s)) + g(F_t(t,s), \nabla_{F_t} F_s(t,s)) \\ &\overset{(1)}{=} g(F_t(t,s), \nabla_{F_t} F_s(t,s)) \\ &\overset{(2)}{=} g(F_t(t,s), \nabla_{F_s} F_t(t,s)) \\ &= \frac{1}{2} F_s g(F_t(t,s), F_t(t,s)) \quad [2] \end{aligned}$$

(1) For the curve $F(t, s)$, with s fixed, is a geodesic, $\nabla_{F_t} F_t = 0$.

(2) By lemma 1.23.

Now, if we fix s, $F(t, s) = \lambda(1)$, with λ a geodesic such that $\lambda(0) = p$ and $\dot{\lambda}(0) = t\gamma_w^v(s)$. So, since the norm of the tangent vector to a geodesic remains constant:

$$\begin{aligned} g(F_t(t,s), F_t(t,s)) &\equiv g(F_t(0,s), F_t(0,s)) \\ &\overset{[1]}{=} g_p(D_0 \exp_p(\gamma_w^v(s)_0), D_0 \exp_p(\gamma_w^v(s)_0)) \end{aligned}$$

Replacing in [2]:

$$\begin{aligned} F_t g(F_t(t,s), F_s(t,s)) &= \frac{1}{2} F_s g_p(D_0 \exp_p(\gamma_w^v(s)_0), D_0 \exp_p(\gamma_w^v(s)_0)) \end{aligned}$$

§ (This will be referred to in the next proof.)

$$\begin{aligned} &= \frac{1}{2} F_s g_p(D_0 \exp_p(v_0 + s w_0), D_0 \exp_p(v_0 + s w_0)) \\ &= \frac{1}{2} F_s[2s g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) + \\ &\quad + s^2 g_p(D_0 \exp_p(w_0), D_0 \exp_p(w_0))] \\ &= g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) \\ &\quad + s g_p(D_0 \exp_p(w_0), D_0 \exp_p(w_0)) \end{aligned}$$

Now, taking $s = 0$:

$$\begin{aligned}
F_t g(F_t(t,0), F_s(t,0)) &= g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) \\
\Rightarrow \quad g(F_t(t,0), F_s(t,0)) &= t.g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) + g_p(F_t(0,0), F_s(0,0)) \\
&\overset{(1)}{=} t.g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) \\
\Rightarrow \quad g_{\tilde{v}}(F_t(1,0), F_s(1,0)) &\overset{t=1}{=} g_p(D_0 \exp_p(v_0), D_0 \exp_p(w_0)) \\
&= g_p(v, w)
\end{aligned}$$

(1) Looking at [1] we can see that $F_s(0,0) = 0$. □

It is interesting to compare this version of the Gauss Lemma with the one from (CARMO, 1979), which, although it has a very similar proof, focuses on the "geodesic spheres". Those are topologically spherical only in the Riemannian case, but the lemma may be rephrased in a way that reveals how it extends to the general semi-Riemannian case.

Proposition 1.25 (Gauss Lemma)**.** *Let $\gamma_w^v(s)$ be a curve in T_pM such that $g_p(\gamma_w^v(s), \gamma_w^v(s))$ is constant. Call $\gamma_w^v(0) := v$ and $\dot{\gamma}_w^v(0) := w_v$. Then*

$$g_{\tilde{v}}(D_v \exp_p(v_v), D_v \exp_p(w_v)) = 0$$

Proof. If we assume only here, in contrast to the definition used in rest of the section, that γ_w^v is any curve satisfying the hypothesis of the proposition, we may follow the proof from the previous version of the Gauss Lemma verbatim until §. Following on:

$$\begin{aligned}
F_t g(F_t(t,s), F_s(t,s)) &= \frac{1}{2} F_s g_p(D_0 \exp_p(\gamma_w^v(s)_0), D_0 \exp_p(\gamma_w^v(s)_0)) \\
&= 0
\end{aligned}$$

Given the hypothesis on the norm of $\gamma_w^v(s)$. It follows that:

$$\begin{aligned}
g(F_t(t,s), F_s(t,s)) &\equiv g(F_t(0,s), F_s(0,s)) \\
&\overset{[1]}{=} g_p(D_0 \exp_p(\gamma_w^v(s)_0), D_0 \exp_p(0)) \\
&= 0
\end{aligned}$$

□

Putting both versions of the Gauss Lemma together we may confirm that the radial geodesics are indeed normal to the surfaces at a constant distance to the center of the normal neighborhood. Such surfaces, though, are not actually spheres in the non-Riemannian cases, but may be even singular submanifolds, such as the Lorentzian light cones. We will call such surfaces *metric hyperquadrics* for short.

Since we are aiming the local equivalence between a Lorentzian manifold and Minkowski space, it is important to remark that the previous paragraph can be translated to a more general version of Lemma 4.5.2 in (HAWKING; ELLIS, 1973), that the radial timelike geodesics are normal to spacelike hypersurfaces of fixed distance to the center of the normal neighborhood. Our result shows that this is not specific to the timelike geodesics except for the causal character of the surface normal to the geodesics.

Now, a quick look at the thesis of proposition 1.25 could lead to the false conclusion that the local correspondence has been solved, since the exponential map looks like an isometry and should preserve the causal character of curves. The fact that it is not obvious highlights an important detail of the Gauss lemma, the subscript $\tilde{v}$ on $g_{\tilde{v}}$, which restricts the isometric character of $\exp_p$ to the radial direction, i.e., to the geodesics. A general curve $\tilde{\alpha}(t)$ in N_p could have a causal character different from its counterpart $\exp_p^{-1}(\tilde{\alpha}(t))$ in T_pM. To show there is an equivalence at the center of the coordinate chart we show that a causal curve in N_p which starts at p does not leave the lightcone of p, a claim that we will make a little more formal ahead.

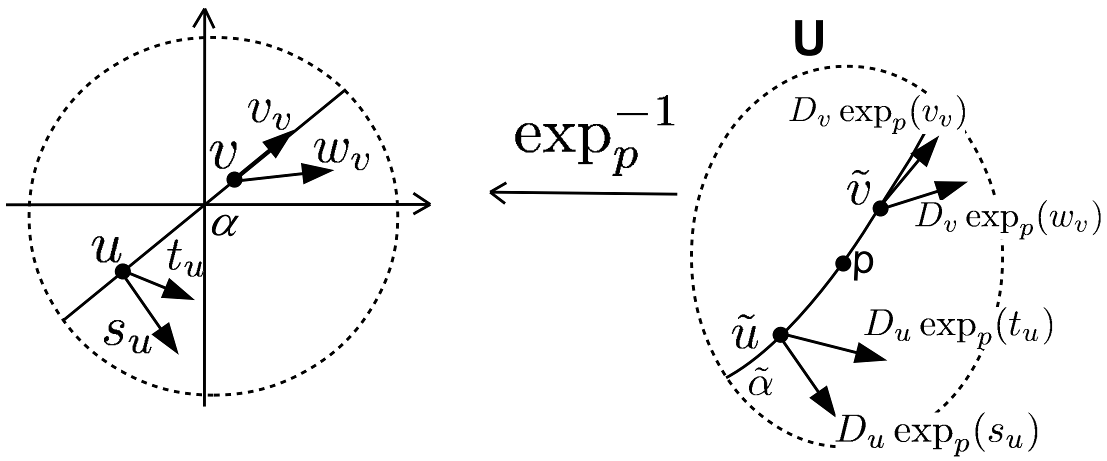


Figure 15 – While Gauss Lemma states that $g_p(v, w) = g_{\tilde{v}}(D_v \exp_p(v_v), D_v \exp_p(w_v))$, it does not guarantee that the same holds for $g_p(t, s)$ and $g_{\tilde{u}}(D_u \exp_p(t_u), D_u \exp_p(s_u))$, for neither t_u nor s_u are tangent to α at u.

Call $\tilde{\mathcal{T}}_p^+ \subset T_pM$ the set of future-directed timelike vectors in T_pM and call $\mathcal{T}_p^+ := \exp_p(\tilde{\mathcal{T}}_p^+)$. Also, define $G : T_pM \to \mathbb{R}$ by $G(v) = g_p(v, v)$ and call $\tilde{G} := G \circ \exp_p^{-1}$.

We need the following technical lemma:

Lemma 1.26. $\operatorname{grad} \tilde{G}_{\tilde{v}} = 2D_v \exp_p(v_v)$

Proof. Since the metric hyperquadrics are the level sets of $\tilde{G}$ and the radial geodesics are normal to them, $\operatorname{grad} \tilde{G}_{\tilde{v}}$ is parallel to $D_v \exp_p(v_v)$, the vector tangent to the radial geodesics, i.e., $\exists a(\tilde{v}) \in \mathbb{R}$ such that $\operatorname{grad} \tilde{G}_{\tilde{v}} = a(\tilde{v}) D_v \exp_p(v_v)$. Now

$$\begin{aligned}
a(\tilde{v}) g_{\tilde{v}}(D_v \exp_p(v_v), D_v \exp_p(v_v)) &= g_{\tilde{v}}(\operatorname{grad} \tilde{G}_{\tilde{v}}, D_v \exp_p(v_v)) \\
&= D_v \exp_p(v_v)(\tilde{G}) \\
&= v_v(\tilde{G} \circ \exp_p) \\
&= v_v(G) \\
&= \frac{d}{ds}|_{s=0} G(\gamma_v^v(s)) \\
&= \frac{d}{ds}|_{s=0} g_p(\gamma_v^v(s), \gamma_v^v(s)) \\
&= 2g_p(\dot{\gamma}_v^v(0), \gamma_v^v(0)) \\
&= 2g_p(v_v, v) \\
&= 2g_p(v, v)
\end{aligned}$$

The last three equalities are slight abuses of notation, that make sense if we consider G as the expression for g_p in a coordinate basis.

It follows from the Gauss Lemma that, for v non-lightlike, $a(\tilde{v}) = 2$. Then, since $\tilde{G}$ is smooth, we may extend the result to have $a(\tilde{v}) \equiv 2$ in N_p. □

Now we can show the main proposition, which is usually referenced as proposition 4.5.1 in page 103 of (HAWKING; ELLIS, 1973). This proof, though, using the technology of the Gauss Lemma for semi-Riemannian manifolds developed to this point, comes from (O'NEIL, 1983).

Proposition 1.27. *If $\tilde{\alpha} : [0, b) \to N_p$ is a piecewise smooth future-directed causal curve in N_p such that $\tilde{\alpha}(0) = p$, $\tilde{\alpha}(t)$ is in the closure of $\mathcal{T}_p^+, \forall t > 0$.*

Proof. Since N_p is a regular space, given $b > \epsilon > 0, \exists U_\epsilon \subsetneq N_p$, $\overline{U_\epsilon} \cap \overline{N_p^c} = \varnothing$, such that $\alpha(t) \in U_\epsilon$, $\forall t \in [0, b - \epsilon]$.

If $\alpha(t)$ is the lift of $\tilde{\alpha}(t)$ to T_pM, let $v \in \tilde{\mathcal{T}}_p^+$ and define the continuous function $H : [0, b - \epsilon] \times [0, \delta_\epsilon) \to T_pM$ given by

$$H(t, s) = \alpha(t) + sv$$

Again by regularity, we can choose δ_ϵ such that $H(t, s) \in \exp_p^{-1}(N_p), \forall (s, t)$. We will denote the curve $H(t, s) := H_s(t)$ where convenient.

Now, using the notation from lemma 1.26, define $\tilde{F}_s(t) = \tilde{G}(\tilde{H}_s(t))$. Then $\tilde{F}_s(0) < 0$ for $s > 0$. If we take the derivative (in the directional sense at singular points):

$$\begin{aligned}
\dot{\tilde{F}}_s(t) &= g_{\tilde{H}_s(t)}\left(\operatorname{grad} \tilde{G}_{\tilde{H}_s(t)}, \dot{\tilde{H}}_s(t)\right) \\
&= 2g_{\tilde{H}_s(t)}\left(D_{H_s(t)} \exp_p(H_s(t)_{H_s(t)}), D_{H_s(t)} \exp_p(\dot{H}_s(t))\right) \\
&= 2g_p\left(H_s(t), \phi_{\alpha(t)}^{-1}(\dot{\alpha}(t))\right)
\end{aligned}$$

The last passage is due to the Gauss Lemma and because $\dot{H}_s(t) = \dot{\alpha}(t)$.

From this result, since $\tilde{\alpha}$ is future-directed causal, we have that $\dot{\tilde{F}}_s(t_0) > 0$ if, and only if, $H_s(t_0) \notin \overline{\tilde{\mathcal{T}}_p^+}$.

The first consequence is that if $H_s(t) \in \tilde{\mathcal{T}}_p^+$ for some t_0, $H_s(t) \in \tilde{\mathcal{T}}_p^+$ for all $t > t_0$, since in that case both $\tilde{F}_s(t)$ and $\dot{\tilde{F}}_s(t)$ will be negative for $t > t_0$. In other words, this trajectory never leaves the interior of the light cone (in fact, since $\tilde{F}_s(t)$ will be strictly decreasing, this translates to the natural fact that time will never stop and the trajectory will keep moving further away into the future). Note that it will happen even if $H_s(t)$ has singular points for $t > t_0$, since the relevant derivative of $\tilde{F}_s(t)$ may be taken in a directional sense.

Now, since $\tilde{F}_s(0) < 0, \forall s > 0$, the reasoning of the last paragraph holds and $\tilde{H}(s,t) \in \tilde{\mathcal{T}}_p^+, \forall t \in [0, b-\epsilon], s > 0$. Since $\tilde{H}$ is continuous, taking $s \to 0$ along the curves $\tilde{H}(s,t_0), t_0$ constant, gives $\tilde{H}(0,t_0) = \tilde{\alpha}(t_0) \in \overline{\tilde{\mathcal{T}}}_p^+$. Since the construction works $\forall \epsilon \in (0,b)$, the result follows. $\square$

Clearly, all this reasoning has an equivalent for past-directed causal curves.

It is worth mentioning that if $\alpha(t_0) \in \partial\tilde{\mathcal{T}}_p^+$ and $\dot{\alpha}(t_0) \in \overline{\tilde{\mathcal{T}}_p^+}$ is not parallel to $\alpha(t_0)$, $\dot{\tilde{F}}_0(t_0) < 0$, which means that $\exists \epsilon > 0$ such that $G(\alpha(t)) < 0, \forall t \in (t_0, t_0 + \epsilon)$. In other words, if the curve is in the surface of the lightcone but the tangent points inwards, the curve enters the lightcone and, as seen in the proof of proposition 1.27, never leaves.

Proposition 1.27, which is a slight variation of proposition 4.5.1 from (HAWKING; ELLIS, 1973), is as far as we can go in terms of a local equivalence between an arbitrary manifold and the Minkowski space of same dimension through the exponential map. Interestingly, a spacetime may be locally conformal to Minkowski space (which yields causal equivalence) but its causal structure not be preserved by the exponential map (the local conformal map is some map other than the exponential). In order to present an example where it happens, let us first present an interesting theorem about spacetimes of dimension 2. It is stated and proved in (WEINSTEIN, 1996) as lemma 2, in page 13.

Theorem 1.28. *If M has dimension two, given $p \in M, \exists(\psi_p := (x(q), t(q)), U_p)$ coordinate chart around p such that the expression of g in the coordinates (x,t) may be written as $g = \Omega_g(dx \otimes dt + dt \otimes dx)$ for $\Omega_g : U_p \to \mathbb{R}_{>0}$ smooth.*

In particular, the previous result yields immediately that all Lorentzian manifolds of dimension two are locally conformal, even disregarding the fact that the expression $dx \otimes dt + dt \otimes dx$ is the one assumed by the usual Minkowski metric in the basis $\{(\sqrt{2}^{-1}, \sqrt{2}^{-1}), (\sqrt{2}^{-1}, -\sqrt{2}^{-1})\}$.

Now an example where the exponential map is **not** the chart that satisfies the hypothesis of theorem 1.28:

Example 1.29. *Let* $M = (0, +\infty) \times \mathbb{R}$ *parametrized with coordinates* (r, t) *and take* $g = -\frac{1}{r} dt \otimes dt + dr \otimes dr$.

Making the usual calculations of the Christoffel symbols and solving the equation of the geodesics, one finds out that the equation of the geodesic with the initial point $\gamma(0) = (r(0), t(0))$ *and initial velocity* $\dot\gamma(0) = (\dot r(0), \dot t(0))$ *is:*

$$\gamma(s) = \begin{pmatrix} \dfrac{\dot t(0)^2 s^2}{4r(0)^2} + \dot r(0)s + r(0) \\ \\ \dfrac{\dot t(0)^3 s^3}{12 r(0)^3} + \dfrac{\dot t(0)\dot r(0) s^2}{2r(0)} + \dot t(0)s + t(0) \end{pmatrix}$$

Taking $s = 1$ *we have that* $\exp_{(r(0),t(0))}(\dot r(0), \dot t(0)) = \gamma(1)$ *gives an expression for the exponential map. Let us show that the exponential map does not preserve the causal character of vectors around any neighborhood around the point* $(1, 0)$.

We begin by writing the expression for the exponential map at $(1, 0)$ *and its derivative at* $(a, b) = (a, 0)$ *for* $a > 0$*:*

$$\exp_{(1,0)}(a, b) = \begin{pmatrix} \dfrac{b^2}{4} + a + 1 \\ \\ \dfrac{b^3}{12} + \dfrac{ba}{2} + b \end{pmatrix}; D_{(a,0)} \exp_{(1,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \dfrac{a}{2} \end{bmatrix}$$

Now we shall need the metrics:

$$\begin{cases} g_{(1,0)} &= -dt \otimes dt + dr \otimes dr \\ g_{(a+1,0)} &= -(a+1)^{-1} dt \otimes dt + dr \otimes dr \end{cases}$$

Now, since $g_{(1,0)}$ *is the usual Minkowski flat metric, the vector* $(1, 1)$ *is lightlike. On the other hand,* $D_{(a,0)} \exp_{(1,0)}(1, 1) = (1, 1 + a/2)$ *and:*

$$g_{(a+1,0)}((1, 1 + a/2), (1, 1 + a/2)) = -\frac{1 + a + \frac{a^2}{4}}{a + 1} + 1 = -\frac{a^2}{4(a + 1)} < 0$$

Hence $D_{(a,0)} \exp_{(1,0)}(1, 1)$ *is always timelike for* $a > 0$ *and the causal character of* $(1, 1)$ *is changed by the exponential map.*

Now, since $\exp_{(1,0)}$ *and its derivative are continuous and* $(1, 1)$ *is in the boundary of the lightcones in* $T_{(a,0)}(T_{(1,0)}M)$ *while* $D_{(a,0)} \exp_{(1,0)}(1, 1)$ *is in the interior of the lightcones in* $T_{(a+1,0)}M$*, there is a spacelike vector in a neighborhood of* $(1, 1)$ *that is sent inside the lightcones in* $T_{(a+1,0)}M$*. Therefore,* $\exp_{(1,0)}$ *is not a causal equivalence between* $T_{(1,0)}M$ *and any neighborhood of* $(1, 0) \in M$.

To finish this section we apply proposition 1.27 to prove proposition 2.18 from (PENROSE, 1972), because it is used in our work.

Proposition 1.30. *If $p, q, r \in M$, $p \ll q$ and $q \prec r$ implies $p \ll r$. In the same way, $p \prec q$ and $q \ll r$ implies $p \ll r$.*

Proof. We will only prove $p \prec q \ll r \Rightarrow p \ll r$, since the first result may be obtained from that by changing the time orientation of M.

Let γ be a future-directed lightlike curve from p to q and α be a future-directed timelike curve from q to r. Since γ is compact, there are $N_1, ..., N_k$ normal convex neighborhoods in M such that $\gamma \subset \bigcup_{i=1}^{k} N_i$. Assume, without loss of generality, that $q \in N_1$.

Now, if $x \in \gamma \cap [N_1 \backslash \{q\}]$ and $y \in \alpha \cap [N_1 \backslash \{q, r\}]$, as discussed in the proof of proposition 1.27, the curve $\gamma * \alpha$ enters the interior of the lightcone of x at latest at q. Thus, $x \ll y \ll r$. If $p \in N_1$, the result is proved.

If that is not the case, there is $i_2 \in \{1, .., k\}$ such that $N_{i_2} \cap N_1 \neq \varnothing$ and $N_{i_2} \not\subset N_1$, since γ is connected. Pick $q_2 \in \gamma \cap (N_{i_2} \cap N_1)$ and α_2 a future-directed timelike curve from q_2 to r and repeat the last paragraph changing q for q_2, α for α_2 and N_1 for N_{i_2}. If $p \in N_{i_2}$, the proof finishes here.

If it is not, we can repeat this last step taking $N_{i_3} \cap (N_1 \cup N_{i_2}) \neq \varnothing$, also $N_{i_3} \not\subset [N_1 \cup N_{i_2}]$ and so on. Since we have a finite number of N_i's, eventually we will have proved that $p \ll r$. $\square$

2 Preliminaries from Dynamical Systems

In this work we propose to take a geometrical property of the horizons, their generators, and translate it to a dynamical system to the possible extent, with details to be discussed in the next chapters. The hope behind this approach is that, for some notion of “stability”, the stability of the horizon could be related to the so called *structural stability* of the dynamical system obtained that way.

This chapter is devoted to briefly presenting the notion of stability we are considering for dynamical systems and the main result on the stability of C^1 dynamical systems on compact 3-manifolds, the Palis-Smale Stability Conjecture, which was ultimately confirmed in (HU, 1994). A good guide for the general discussion on the stability of dynamical systems is (PUGH; PEIXOTO, 2008). Here we present only the result and the elements necessary for its statement.

In this section, let N be a manifold, $\mathcal{X}(N)$ the set of tangent vector fields on N, $S \in \mathcal{X}(N)$ and $\phi^S : \mathbb{R} \times N \to N$ the flow generated by S on N. Many different topologies may be attributed to $\mathcal{X}(N)$, and our final result will be dependent on the specific topology chosen, but the definition of structural stability demands simply that some topology has been fixed.

Definition 2.1 (Structural stability)**.** *S is called structurally stable iff $\exists U \subset \mathcal{X}(N)$ open such that $S \in U$ and, $\forall V \in U, \exists h_V : N \to N$ an homeomorphism such that*

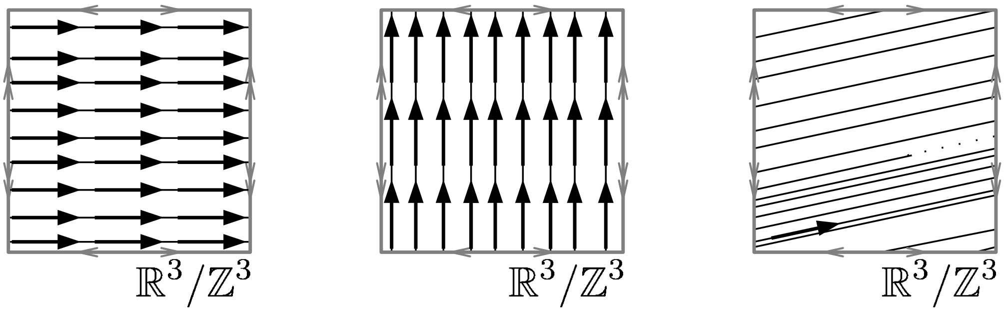
$$h_V(\phi^S(\mathbb{R}, p)) = \phi^V(\mathbb{R}, h_V(p)), \forall p \in N.$$

Notice that the equality is an equality of sets, meaning that orbits of S correspond to orbits of V regardless of the time parameter for each distinct trajectory. This will not be a problem to the analysis we intend to make.

Let us present examples of dynamical systems that are and that are not conjugated in the way of definition 2.1.

Example 2.2. *Let $N := \mathbb{R}^3/\mathbb{Z}^3$, the flat torus, and define $S(x) := (1, 0, 0)$, $V(x) :=$ $(0, 1, 0)$ and $W(x) = (1, e/4, 0)$ for all $x \in N$. Then the orbit any point with respect to ϕ^S is a horizontal circle in the plane torus, the orbit with respect to ϕ^V is a vertical circle and the orbit with respect to ϕ^W is a dense curve in a torus $z =$ constant which is not closed.*

Since any homeomorphism from the torus into itself preserves closed curves, ϕ^W is not conjugated to any of the other flows. On the other hand, if we take $h_V : N \to N$ as $h_V(x, y, z) = (y, -x, z)$, the map induced by the counterclockwise rotation of angle $\pi/2$ in the x, y-plane, $h_V(\phi^S(\mathbb{R}, p)) = \phi^V(\mathbb{R}, h_V(p)), \forall p \in N$.



(a) $S(x)$ (b) $V(x)$ (c) $W(x)$

Figure 16 – The orbits of the three flows defined in example 2.2 represented at any section of $\mathbb{R}^3/\mathbb{Z}^3$ of constant z-coordinate. Picture (c) depicts a single trajectory that is dense in the two-dimensional section of the torus.

Now we bring definitions needed to state the Stability Conjecture.

Definition 2.3. *The non-wandering set of the flow is*

$$\Omega := \{p \in N | p \in U \subset N \textit{ open} \Rightarrow \exists t \in \mathbb{R}, |t| \geqslant 1, \textit{ such that } \phi^S(t, U) \cap U \neq \varnothing\}$$

Two relevant remarks should be made concerning Ω to be reminded when we discuss the horizons. First, if N is compact, $\Omega \neq \varnothing$. Second, all closed orbits of ϕ^S are in Ω.

From the second observation we see that in the cases of S and V in example 2.2, $\Omega = N$. That is also true for W, since the orbit of any point is dense. Hence, given $p \in N$ and $U \subset N$ open such that $p \in U$, there is $t \in \mathbb{R}$ such that $\phi^W(t, p) \in U \backslash (\phi^W([-1, 1], p))$ and $p \in \Omega$. Also, if we lift any of the those three dynamical systems to $\mathbb{R}^3$, the orbits will become parallel straight lines, hence in that case $\Omega = \varnothing$ for each of the resulting flows.

We present also an example of a flow which has a non-wandering set which is neither empty nor the whole manifold:

Example 2.4. *Let N be the torus as in example 2.2 and define the field*

$$X(x, y, z) = \begin{pmatrix} \sin(2\pi x)\sin^2(2\pi y) \\ 1 \\ 0 \end{pmatrix}$$

Note that S is well-defined and smooth in N, when N is identified with the cube $[0, 1]^3$ with the usual identifications. To find the corresponding flow we must solve the differential equation

$$\begin{cases} (\dot{x}(y), \dot{y}(t)) = X(x(t), y(t), z(t)) \\ (x(0), y(0), z(0)) = (x_0, y_0, z_0) \end{cases}$$

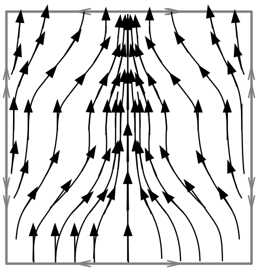


Figure 17 – A plot of field $X(x, y, z)$ in a section $z \equiv C$ and the orbits of the flow. Notice that all orbits converge towards the closed orbit $x \equiv 1/2$ while distancing from the orbit $x \equiv 0 \sim x \equiv 1$. That is why the non-wandering set of the field is the union of the orbits $x \equiv 1/2, z \equiv C$, $C \in [0, 1]$.

The system is separable and we may find that, calling $p := (x_0, y_0, z_0)$:

$$\phi^X(t,p) = \begin{cases} \left(\frac{1}{\pi}\text{cotg}^{-1}\left[\text{cotg}\,(\pi x_0)\,e^{\frac{1}{4}(\sin(4\pi(t+y_0))-\sin(4\pi y_0))-\pi t}\right], t+y_0, z_0\right), x_0 \in (0,1/2) \\ \\ (x_0, t+y_0, z_0), x_0 \in \{0, 1/2\} \end{cases}$$

Since the x component of X is even with respect to the plane $x = 1/2$, the flow has mirror symmetry with respect to this plane. Of course the solution should inherit such symmetry and be obtained by reflecting the solution above around the plane $x = 1/2$.

Now, since the x-component of $X(x, y, z)$ is strictly positive in $(0, 1/2) \times [0, 1]^2$, every orbit with $x_0 \in (0, 1/2)$ converges to the circle $\{1/2\} \times [0, 1] \times \{z_0\}$, which is a closed orbit. From this and the mirror symmetry, $\Omega = \{0, 1/2\} \times [0, 1]^2$.

Definition 2.5 (Hyperbolic Set). *A set $X \subset N$ such that $\phi^S(X) \subset X$ is called hyperbolic iff $\forall p \in X$:*

$$T_pN = E^s(p) \oplus span\langle S(p)\rangle \oplus F^u(p)$$

with $D\phi^S(E^s) \subset E^s$, $D\phi^S(F^s) \subset F^s$ and the following holds:

$\exists C_1, C_2, \lambda, \mu > 0$ *such that* $\lambda < 1 < \mu$ *and* $\forall t > 0$

i. $||D_p\phi^S(t,p)|_{E^s}|| \leqslant C_1\lambda^t$;

ii. $||D_p\phi^S(-t,p)|_{F^s}|| \leqslant C_2\mu^{-t}$;

iii.Both projections $\begin{cases} \pi^s : TX \to TX \text{ with } \pi^s(p,v) = (x, \pi_{E^s}(v)) \\ \pi^u : TX \to TX \text{ with } \pi^u(p,v) = (p, \pi_{F^u}(v)) \end{cases}$ *are continuous.*

We might want to check our examples to look for hyperbolic sets in them. The flows of the constant vector fields from example 2.2 and their lifts to $\mathbb{R}^3$ have constant derivatives with respect to the initial condition, while conditions i and ii of definition 2.5

yields the norm of the derivatives mentioned should go to 0 as t goes to infinity. Hence, there are no hyperbolic sets for those flows.

To evaluate the behavior of the field in example 2.4, let us look at the derivative of the flow in $(0, 1/2) \times [0, 1]^2$:

$$D_{(x,y,z)}\phi^X(t,(x,y,z)) = \begin{bmatrix} f & \frac{1}{2}fg & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with:

$$\begin{cases} f(t,x,y) = \dfrac{e^{\frac{1}{4}(\sin(4\pi(t+y)) - \sin 4\pi y) - \pi t}}{\sin^2(\pi x) + \cos^2(\pi x) e^{\frac{1}{2}(\sin(4\pi(t+y)) - \sin 4\pi y) - 2\pi t}} \\ \\ g(t,x,y) = \sin(2\pi x)(\cos(4\pi(t+y)) - 4\cos(4\pi y)) \end{cases}$$

So the eigenvalues of $D_{(x,y,z)}\phi^X(t(x,y,z))$ are f and 1, the latter with multiplicity 2. So, given an orbit $\phi^X(\mathbb{R},(p))$, any decomposition of T_pM in invariant subspaces will result in a subspace E where $||D_p\phi^X(t,p)|_E|| \geqslant 1, \forall t$ and X does not admit hyperbolic subspaces.

We may modify example 2.4 a little, though, to construct a dynamical system with a hyperbolic non-wandering set.

Example 2.6. *Again, let N be the plane torus and define the field*

$$Y(x,y,z) = \begin{pmatrix} \sin(2\pi x)\sin^2(2\pi y) \\ 1 \\ \sin(2\pi z)\sin^2(2\pi y) \end{pmatrix}$$

And the flow may be calculated in the same way as the one in example 2.4, calling $p := (x_0, y_0, z_0)$:

$$\phi^Y(t,p) = \begin{cases} \Big(h(t,x_0,y_0), t + y_0, h(t,z_0,y_0)\Big), (x_0,z_0) \in (0,1/2)^2 \\ \\ \Big(x_0, t + y_0, z_0\Big), (x_0,z_0) \in \{0,1/2\}^2 \end{cases}$$

with $h(t,x,y) = \dfrac{1}{\pi}\text{cotg}^{-1}\left[\text{cotg}\,(\pi x_0)\, e^{\frac{1}{4}(\sin(4\pi(t+y_0)) - \sin(4\pi y_0)) - \pi t}\right]$.

In $(x_0,z_0) \in \{0\} \times (0,1/2)$, $(x_0,z_0) \in \{1/2\} \times (0,1/2)$, $(x_0,z_0) \in (0,1/2) \times \{0\}$ and $(x_0,z_0) \in (0,1/2) \times \{1/2\}$ the behavior is the same as in example 2.4. Also, the field has mirror symmetry around the planes $x_0 = 1/2$ and $z_0 = 1/2$. So the solution is determined completely by the behavior on the section $[0,1/2] \times [0,1] \times [0,1/2]$ of the torus.

For this vector field $\Omega = \{(x,y,z)|x,z \in \{0,1/2\}\}$, the closed orbits. To check it is hyperbolic let us look at the derivative of the flow in $(0,1/2)\times[0,1]\times(0,1/2)$:

$$D_{(x,y,z)}\phi^Y(t,(x,y,z)) = \begin{bmatrix} f(t,x,y) & \frac{1}{2}f(t,x,y)g(t,x,y) & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2}f(t,z,y)g(t,z,y) & f(t,z,y) \end{bmatrix}$$

With $f(t,x,y)$ and $g(t,x,y)$ the same as the ones in the derivative of ϕ^X. At each $p \in \Omega$, then, $Y(p) = e_2$ and there is the decomposition:

$$T_pM = span < e_1 > \oplus span < Y(p) > \oplus < e_3 >$$

Since e_1 and e_3 are eigenvectors corresponding to the eigenvalues $f(t,x,y)$ and $f(t,z,y)$, respectively, the components of the decomposition are ϕ^Y invariant. Also:

$$\begin{cases} f(-t,0,y) = e^{-\frac{1}{2}(\sin(4\pi(-t+y))-\sin 4\pi y)-\pi t} \leqslant e.(e^{\pi})^{-t} \\ f(t,1/2,y) = e^{\frac{1}{4}(\sin(4\pi(t+y))-\sin 4\pi y)-\pi t} \leqslant e^{\frac{1}{2}}(e^{-\pi})^{t} \end{cases}$$

Thus, the four closed orbits of Ω satisfy definition 2.5 if we make the following choices:

i. For $x = z = 0$, call $F^u := span < e_1, e_3 >$;

ii. For $x = 0$ and $z = 1/2$, call $F^u := span < e_1 >$ and $E^s := span < e_3 >$;

iii. For $x = 1/2$ and $z = 0$, call $F^u := span < e_3 >$ and $E^s := span < e_1 >$;

iv. For $x = z = 1/2$, call $E^s := span < e_1, e_3 >$.

And Ω is seen to be hyperbolic. Notice that the definition of hyperbolic set is satisfied even if either E^s or F^u is not defined, as long as the other is span $< Y >^c$.

With all that set we can define the first condition for a dynamical system to be structurally stable.

Definition 2.7 (Axiom A)**.** *A flow is said to satisfy axiom A iff:*

1. $\{p \in N|\phi^S(t,p)$ is periodic$\}$ is dense in Ω;

2. Ω is hyperbolic.

Looking back at all the examples we have presented, all the vector fields from example 2.2 have $\Omega = N$ but while S (hence V, since both are conjugated) has all points within periodic orbits and satisfy condition 1 of Axiom A, W has no closed orbits, thus not satisfying condition 1 of axiom A. Also, none of those vector fields has hyperbolic sets, so they do not satisfy Axiom A.

Also, the field from example 2.4 satisfies condition 1 of Axiom A, since Ω in this case is a single closed orbit, but not 2, for it is not hyperbolic. The only flow satisfying Axiom A is ϕ^Y, from example 2.6.

The second condition concerns transversality of stable and unstable manifolds.

Definition 2.8 (Transversality)**.** *Two submanifolds $P, Q \subset N$ are said to intersect transversally iff $P \cap V = \varnothing$ or $p \in P \cap Q \Rightarrow T_pP + T_pQ = T_pN$.*

Definition 2.9. *Given $p \in N$ we call the stable and unstable manifolds of p, respectively:*

i. $W^s(p) := \left\{x \in N | \lim_{t\to\infty} \phi^S(t,x) = p\right\}$;

ii. $W^u(p) := \left\{x \in N | \lim_{t\to-\infty} \phi^S(t,x) = p\right\}$.

Note that in general not all points of a dynamical system have non-empty stable or unstable manifolds. For example, points on closed orbits cannot be part of stable or unstable manifolds, since their orbits do not converge. Hence the flow ϕ^S (hence ϕ^V) from example 2.2 do not have any stable or unstable manifold for any of its points. Also, since the orbits of ϕ^W are dense on N, they do not converge to any point and ϕ^W does not have any stable or unstable manifold for any of its points.

The same thing is true for the flows presented on examples 2.4 and 2.6. Since all the orbits on each of the examples is periodic on the y-coordinate, they do not converge to any point. Hence, no point in N has stable or unstable manifolds for the flows ϕ^X or ϕ^Y.

The aforementioned condition is the following:

Definition 2.10 (Strong Transversality Condition)**.** *S is said to satisfy the strong transversality condition iff $\forall x, y \in \Omega, W^s(x)$ intersects $W^u(y)$ transversally.*

Finally, the Palis-Smale Stability Conjecture, proved by Hu in (HU, 1994), is the following:

Theorem 2.11. *If N is compact and $\mathcal{X}(N)$ is restricted to the set of C^1 vector fields with the C^1 topology, S is structurally stable $\iff$ S satisfies axiom A and the strong transversality condition.*

In examples 2.2, 2.4 and 2.6 N is compact and the fields S, V, W, X and Y are C^1. Since all fields vacuously satisfy the strong transversality condition but only field Y satisfy Axiom A, field Y is the only structurally stable amog the fields presented.

It is not hard to see how the fields on example 2.2 may be slightly perturbed to other fields with different orbit behavior. If we take the constant field $(1, a, 0)$ tangent to N, its orbits are closed if $a \in \mathbb{Q}$ and are not-closed (and dense in a slice $z = constant$) otherwise. Hence, no field of this kind is structurally stable, since adding a field $(0, \epsilon, 0)$, $\epsilon > 0$, is a perturbation on the C^1 topology (remember N is compact) that may change the orbits behavior.

Also, for the field in example 2.4, adding a field $(0, 0, \epsilon)$ tilts the closed orbits and they become open, so the resulting field is not conjugate to the first since it does not have closed orbits. The hyperbolicity of Ω in example 2.6 more or less stabilizes the behavior around the closed orbits, so if a perturbation destroys one of them, another closed orbit should appear nearby.

3 From Horizons to Dynamical Systems

This work's kickoff was the paper (CHRUŜCIEL; ISENBERG, 1994), where a method to go back and forth between the generators of an horizon and a dynamical system was presented as a tool to build counterexamples to Hawking's claims on (HAWKING, 1992) concerning the genericity of a specific type of generators, called "fountains", in compact horizons.

Maybe because the main goal of the paper was to build an horizon from a dynamical system, through a construction we will present in the next chapter, there is only a brief mention to how a dynamical system could emerge from the generators of an horizon, namely by taking the vectors tangent to the generators. In the paper, though, there are no considerations regarding whether or not this process is always feasible and what kind of dynamical systems emerges from that.

We devote this chapter to discuss the questions of when we can bring a dynamical system off from a horizon by normalizing the tangents to the generators and what is the regularity of said dynamical system. To do so we recount the results from (BEEM; KRÓLAK, 1998) that relate the endpoints of a horizon with its differentiability and find that the horizon being C^1 is equivalent to the mentioned vector field being well-defined, but in general we can only hope for the field to be C^0.

3.1 Proposed Vector Field

There is an ideal situation to pass from the horizon to the dynamical system. Assume $H^+(S)$ is a smooth manifold without boundary and that for each $x \in H^+(S)$ there is only one lightlike direction in $T_xH^+(S)$. In this case we may assign a Riemannian metric h to M and define the vector field V on $T_xH^+(S)$ by saying that $V(x)$ is the future-directed lightlike vector on $T_xH^+(S)$ that is unitary with respect to h. Note that the notion of a future-directed vector can be attributed to $H^+(S)$ by saying that V_x is future-directed in $T_xH^+(S)$ if it is future-directed in T_xM, although $H^+(S)$ with the induced metric is not a Lorentzian manifold, as we shall see in proposition 3.3 ahead.

Since horizons are Lipschitz manifolds, as seen in proposition 1.11, the hypothesis presented may break if the boundary of $H^+(S)$ is non-empty, if $H^+(S)$ fails to be differentiable or if there is more than one lighlike direction at any of its points. Fortunately, as we will show in the subsequent discussion, the last two conditions happen to be the same, since the differentiability of the horizon is closely related to the behavior of the generators. Actually, even stronger assertions may be extracted from the behavior of the generators, as we will see, but first we need a convenient definition:

Definition 3.1 (Multiplicity of a point)**.** *If $p \in H^+(S)$, we say that the multiplicity of p is the number of distinct transverse generators of $H^+(S)$ through p.*

As a remark, it should be stressed that we mean "distinct" in a local level, that is, if a generator self-crosses transversely it should be counted twice (or as many times as the self-crossing happens). Note also that saying there is only one lightlike direction in $T_xH^+(S)$ is equivalent to saying that the multiplicity of x is one.

3.2 Differentiability

This whole section, and the core of the chapter, may be summed up in proposition 3.6 from (BEEM; KRÓLAK, 1998), that relates the behavior of the generators and the regularity of the horizon:

Theorem 3.2. *If W is open in $H^+(S)$ the following are equivalent:*

I. $H^+(S)$ is differentiable in W;

II. $H^+(S)$ is of class C^1 (at least);

III. $H^+(S)$ has no endpoint of a generator in W;

IV. $p \in W \Rightarrow p$ has multiplicity one.

In this section, along with a broader discussion, we shall prove the many equivalences of this theorem, and refer back to it while doing so. Notice that the implication $II \Rightarrow I$ is obvious.

We open up with the proof of a proposition from (CHRUŜCIEL; GALLOWAY, 1998), which effectively says that if the number of lightlike directions tangent to a point in the horizon is greater than one, in other words, if the multiplicity of a point is greater than one, the horizon fails to be differentiable at that point.

Proposition 3.3. *Let $p \in H^+(S)$. If the multiplicity of $p > 1$, $H^+(S)$ is NOT differentiable at p.*

Proof. This proves 3.2 $I \Rightarrow IV$.

If $H^+(S)$ is differentiable at p, $T_pH^+(S) \subset T_pM$ is well-defined. Assume there are two distinct generators through p, with tangent vectors $V_1, V_2 \in T_pH^+(S)$. Since the generators are lightlike curves, V_1 and V_2 are lightlike and, since the generators are distinct, $\{V_1, V_2\}$ is L. I. Thus, $C := g_p(V_1, V_2) \neq 0$.

Now, the vector $V := V_1 - \frac{1}{C}V_2 \in T_pH^+(S)$ and:

$$\begin{aligned} g_p(V,V) &= g_p(V_1,V_1) - \frac{1}{C}g_p(V_1,V_2) - \frac{1}{C}g_p(V_2,V_1) + \frac{1}{C^2}g_p(V_2,V_2) \\ &= -2 \end{aligned}$$

So V is timelike.

On the other hand, let $\exp_p^{-1} : U \subset M \to T_pM$ be the exponential map at p, $p \in U$ a normal neighborhood for M around p and $\exp_p^{-1}(H^+(S) \cap U)$ be the graph of a function $x^0 = f(x^1, ..., x^n)$, as proved in proposition 1.11. f is differentiable at the image of p since $H^+(S)$ is and, since $D_p \exp_p^{-1}$ is the identity map (in the usual parametrization of T_pM), $V \in T_0 \exp_p^{-1}(H^+(S))$. Let $\tilde{V}$ be the projection of V in the $x^0 = 0$ hyperplane. Then, since $\exp_p^{-1}(H^+(S))$ is the graph of f:

$$\frac{d}{dt}|_{t=0}(f(t\tilde{V}), t\tilde{V}) = (\langle \nabla f, \tilde{V}\rangle, \tilde{V}) \propto V$$

Then, since $\tilde{V}$ is the projection of V itself:

$$\frac{d}{dt}|_{t=0}(f(t\tilde{V}), t\tilde{V}) = (\langle \nabla f, \tilde{V}\rangle, \tilde{V}) = V$$

Always identifying T_pM and $T_0(T_pM)$ in the natural way (see section 1.4.3) it follows that:

$$g_p((\langle \nabla f, \tilde{V}\rangle, \tilde{V}), (\langle \nabla f, \tilde{V}\rangle, \tilde{V})) < 0$$

because V is timelike. Therefore, using the Cauchy-Schwartz innequality:

$$\begin{aligned} -(\langle \nabla f, \tilde{V}\rangle)^2 + |\tilde{V}|^2 &< 0 \\ \Rightarrow \qquad |V|^2 &< |\nabla f|^2 |\tilde{V}|^2 \\ \Rightarrow \qquad 1 &< |\nabla f|^2 \end{aligned}$$

On the other hand, from the definition of the derivative:

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - f(0) - \langle \nabla f, x\rangle}{|x|} &= 0 \\ \overset{f(0)=0}{\Rightarrow} \qquad \lim_{x\to 0} \frac{f(x) - \langle \nabla f, x\rangle}{|x|} &= 0 \end{aligned}$$

Since the limit exists, we may evaluate it through the direction $x = t\nabla f, t > 0$. In this case:

$$\begin{aligned} \lim_{t\to 0+} \frac{f(t\nabla f) - \langle \nabla f, t\nabla f\rangle}{t|\nabla f|} &= 0 \\ \Rightarrow \qquad \lim_{t\to 0^+} \frac{f(t\nabla f) - t|\nabla f|^2}{t|\nabla f|} &= 0 \\ \Rightarrow \qquad \lim_{t\to 0^+} \frac{f(t\nabla f)}{t|\nabla f|} &= |\nabla f|^2 > 1 \end{aligned}$$

Therefore, $\exists t > 0$ such that $f(t\nabla f) > t|\nabla f| \Rightarrow 0 > -f(t\nabla f)^2 + |t\nabla f|^2$. So the point $(f(t\nabla f), t\nabla f) \in \exp_p^{-1}(H^+(S))$ is timelike, which violates the achronality of $H^+(S)$. Contradiction. Summing up, $H^+(S)$ cannot be differentiable at p. □

A side note to be taken from the proof above is that there is no tangent timelike vector to $H^+(S)$, justifying the claim we have done before that $H^+(S)$, even when it is a smooth manifold, is not a Lorentzian manifold with the metric induced by g.

With this proposition we cleared up one direction of the equivalence, that $H^+(S)$ being differentiable at a point p implies there is one single lightlike direction tangent to the horizon at p. The other, that the existence of a single lightlike tangent direction at a point implies differentiability at p, has been first proved for $p \in H^+(S)_N$, in (CHRUŜCIEL; GALLOWAY, 1998), and then extended for all p of multiplicity one in (BEEM; KRÓLAK, 1998).

Before presenting the general result, we bring the proof given in (CHRUŜCIEL; GALLOWAY, 1998) because it is not based on the multiplicity of the point, but on the fact that it is a point in the interior of a generator inside $H^+(S)$. This result is interesting because it gives a first idea for the general proof and as a consequence, when added to proposition 3.3, that a point $p \in H^+(S)$ of multiplicity higher than one is not in $H^+(S)_N$, thus all generators leave $H^+(S)$ at p.

Proposition 3.4. *If $p \in H^+(S)_N$, $H^+(S)$ is differentiable at p.*

Proof. This proves 3.2 $III \Rightarrow I$.

Let $p \in H^+(S)_N$, U be a normal neighborhood in M around p for which there are $r, \delta > 0$ such that $\exp_p^{-1}(U) = (-\delta, \delta) \times B_r^3(0)$, using the notation from proposition 1.11, and e_0 be future-directed timelike. Furthermore, from the definition of $H^+(S)_N$, there is a future-directed lightlike curve $\gamma : (-\epsilon, \epsilon) \to U \cap H^+(S)$ such that $\gamma(0) = p$. Also, let's call $q_+ := \gamma(\epsilon/2)$ and $q_- := \gamma(-\epsilon/2)$.

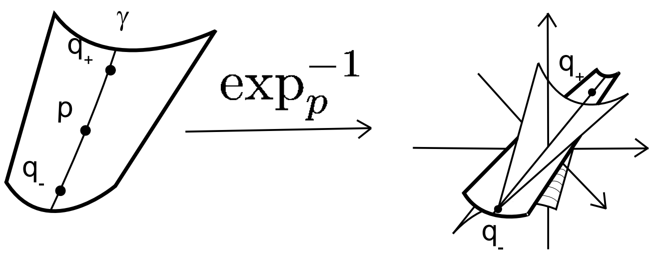


Figure 18 – The graphics of f_+ and f_- when seen as smooth submanifolds of T_pM “squeeze” the graphic of f in-between, guaranteeing its smoothness.

From proposition 1.11, there is $f : B_r^3(0) \to \mathbb{R}$ such that $\exp_p^{-1}(H^+(S))$ is the graphic of f. Call $\tilde{q}_+ := \exp_p^{-1}(q_+)$ and $\tilde{q}_- := \exp_p^{-1}(q_-)$. Now, since γ is future-directed lightlike and $H^+(S)$ is achronal, $0 \in S_+ := J^-(\tilde{q}_+)\backslash[I^-(\tilde{q}_+) \cup \{\tilde{q}_+\}]$ while at the same time $0 \in S_- := J^+(\tilde{q}_-)\backslash[I^+(\tilde{q}_-) \cup \{\tilde{q}_-\}]$. Both S_+ and S_- are smooth manifolds in $\exp_p^{-1}(U)$.

At the same time, since e_0 is future-directed timelike, if we take a point $(t, x) \in \exp_p^{-1}(U)$ we have that:

$$\begin{cases} t \leqslant f(x) \iff (t,x) \in \exp_p^{-1}(\mathcal{F}(S)) \\ t > f(x) \iff (t,x) \in \exp_p^{-1}([\mathcal{F}(S)]^c) \end{cases}$$

It follows that if $(t, x) \in S_+$, $t \leqslant f(x)$ and if $(t, x) \in S_-$, $t \geqslant f(x)$. From this fact and from the smoothness of S_+ and S_-, there is $r' > 0$ and functions $f_+, f_- : B^3_{r'}(0) \to \mathbb{R}$ such that $S_+ \cap B^3_{r'}(0) \times (-\delta, \delta)$ and $S_- \cap B^3_{r'}(0) \times (-\delta, \delta)$ are the graphics of f_+ and f_-, respectively. From the beginning of the paragraph we can see that, for any $x \in B^3_{r'}(0)$:

$$f_+(x) \leqslant f(x) \leqslant f_-(x) \quad [1]$$

But $0 \in [S_+ \cap S_- \cap \exp_p^{-1}(H^+(S))]$, hence, $f_+(0) = f(0) = f_-(0)$. Together with [1] and the fact that both f_+ and f_- are differentiable at 0, this yields that f is differentiable at 0 and so $H^+(S)$ is differentiable at p. □

Again, the multiplicity of p never appeared in this proof. Therefore if there is more than one generator crossing p, $H^+(S)$ is not differentiable at p, hence by this last proposition, p is not in the interior of any of the generators in $H^+(S)$, so they all leave $H^+(S)$ at p.

Also, it should be stressed that $p \in H^+(S)_N$ allowed us to find q_+ in $H^+(S)$ and therefore guaranteed that f_+ was below f everywhere. In contrast, in the case $p \in H^+(S)_+$, $q_+ \in [\mathcal{F}(S)]^c$, and the construction does not work. The proof of the general case in (CHRUŠCIEL; GALLOWAY, 1998) is much more intricate, owing much of its length to the construction of its many elements. We attempt here to make it clearer, more mathematically detailed and to avoid the use of coordinates as much as possible. In order to make the construction a little more organized we resource to a list of items, but perhaps it remains somewhat clumsy.

Theorem 3.5. *If $p \in H^+(S)$ and the multiplicity of p is one, $H^+(S)$ is differentiable at p.*

Proof. This proves 3.2 $IV \Rightarrow I$.

To the proof, we will need:

- X an unitary future-directed timelike vector field in M;
- $p \in H^+(S)$ a point of multiplicity one;
- N_p a convex normal neighborhood around p;

- h a Riemannian metric on M;
- $a, \delta > 0$ such that $\exp_p\left(a.B^h_\delta(0)\right) \subset N_p$ and, if $q \in \exp_p\left(B^h_\delta(0)\right)$ and $h_q(v,v) < \delta$, the geodesic through q with tangent vector v is well-defined in the interval $(-a, a)$, with $B^h_\delta(0)$ the ball with respect to the metric h (the existence of such a and δ is guaranteed by theorem 1 in section 2.3 of (PERKO, 1991));
- $\gamma : (-a, a) \to N_p$ a future-directed lightlike geodesic such that $\gamma(0) = p$ and γ is a piece of the generator of $H^+(S)$ through p. From corollary 1.17, $\gamma(t) \in H^+(S), \forall t < 0$;
- $r := \gamma(-a/2)$ and $\bar{p} := \exp_r^{-1}(p)$;
- TM the tangent bundle of M and $\Pi : TM \to M$ the canonical projection;
- $\tilde{M} := \{0_p \in T_pM | p \in M\} \subset TM$ the set of vectors corresponding to the null vector field on M, which is closed in TM;
- $E \subset TM$ the set of vectors $v \in T_qM$ such that $\exp_q v$ is well-defined for $q \in M$. As proved in section 9.3 from (HICKS, 1965), E is open in TM;
- $\begin{array}{rccc} \hat{\exp} : & E \cap \Pi^{-1} & \to & N_p \\ & v \in T_qM & \mapsto & \exp_q v \end{array}$. As proved in the same section from (HICKS, 1965), $\hat{\exp}$ is smooth;
- $\begin{array}{rccc} \hat{G} : & TM & \to & \mathbb{R} \\ & v \in T_qM & \mapsto & g_q(v,v) \end{array}$ which is smooth;
- $\begin{array}{rccc} \hat{G}_X : & TM & \to & \mathbb{R} \\ & v \in T_qM & \mapsto & g_q(X_q,v) \end{array}$, also smooth;
- $N(\bar{p}) \subset TM$ an open set such that $\bar{p} \in N(\bar{p}) \subset (E \cap \Pi^{-1}(N_p)) \cap \bar{M}^c$, which is possible because $\bar{p} \notin \bar{M}$;
- $L(\bar{p}) := \hat{G}^{-1}(\{0\}) \cap \hat{G}_X^{-1}(-\infty, 0) \cap N(\bar{p})$. Then, since the generator from r to p is future-directed, $\bar{p}$ is future-directed lightlike, hence $\bar{p} \in L(\bar{p})$.

Before continuing, we must establish that $L(\bar{p})$ is a manifold. Given $v \in \tilde{M}^c$, $v \in T_qM$:

$$\begin{aligned} \hat{G}(v + t.X_q) &= g_q(v,v) + 2tg_q(v,X) + t^2 g_q(X,X) \\ \Rightarrow \quad \frac{d}{dt}\hat{G}(v + t.X_q) &= 2g_q(v,X) + 2tg_q(X,X) \\ \Rightarrow \quad \frac{d}{dt}\hat{G}(v + t.X_q)|_{t=0} &= 2g_q(v,X) \end{aligned}$$

Hence, since v is lightlike, $D_v\hat{G}$ is non-singular for all $v \in L$. It follows that, when $\hat{G}$ is regarded as a function restricted to $N(\bar{p}) \cap \hat{G}_X^{-1}(-\infty, 0)$, $\hat{G}^{-1}(\{0\})$ is a submanifold of $N(\bar{p}) \cap \hat{G}_X^{-1}(-\infty, 0)$ of codimension 1, i.e., $L(\bar{p})$ is a submanifold of $N(\bar{p})$ of dimension 7.

Now we may continue our list of elements needed for the proof:

- $X_0 := X_p, X_1 := \dot{\gamma}(0)$ and X_2 and X_3 an g-orthonormal basis for $X_0^{\perp} \cap X_1^{\perp}$, hence $\{X_0, X_1, X_2, X_3\}$ is a basis for T_pM;
- $\epsilon, \eta > 0$ such that $\tilde{N}_p := (-\epsilon, \epsilon) \times B_\eta^3(0) \subset a.B_\delta^h(0)$, $(-\epsilon, \epsilon)$ in the direction of X_0 and $B_\eta^3(0)$ an Euclidean ball of radius η and center 0 in the hyperplane generated by $\{X_1, X_2, X_3\}$ as in proposition 1.11;
- $f : B_\eta^3(0) \to \mathbb{R}$ a Lipschitz function such that $\exp_p^{-1}(H^+(S)) \cap \tilde{N}_p$ is the graphic of f. [1]

The proof reduces to prove that f is differentiable at 0, which we will do by contradiction. Assume f is not differentiable at 0. In particular, $D_0 f \neq 0$, hence, from the definition of the derivative, there is a sequence $\{q_k\}_{k\in\mathbb{N}} \subset B_\eta^3(0)$ with $q_k \xrightarrow{k\to\infty} 0$ and for which $\exists C > 0$ such that $\forall k, \exists n(k) > k$ such that:

$$\frac{|f(q_n)|}{|q_n|} \geqslant C \quad [1]$$

- $\tilde{q}_n := \exp_p(f(q_n), q_n) \in H^+(S)$.

Since $q_n \xrightarrow{n\to\infty} 0$ and f is continuous, $f(q_n) \xrightarrow{n\to\infty} f(0)$, thus $(f(q_n), q_n) \xrightarrow{n\to\infty} 0$ and $\tilde{q}_n \xrightarrow{n\to\infty} p$. Then, there is $n_0 \in \mathbb{N}$ such that for $n > n_0$, $\tilde{q}_n \in \exp_p(B_\delta^h(0))$.

- $\gamma_n : \left[-a + \dfrac{a}{n}, a - \dfrac{a}{n}\right] \to M$, for $n > n_0$, is a parametrization by h-arc length of a generator through $\tilde{q}_n$ with $\gamma_n(0) = \tilde{q}_n$.

From the definition of a and of δ, $\gamma_n \not\subset \exp_p\left(B_{\frac{a}{2}\delta}^h(0)\right)$, for the geodesic can be extended to time $a - \xi$ for any $\xi > 0$. Now, p is an accumulation point for γ_n and lemma 1.13, including condition II, provides there is $\lambda : [-a, a] \to M$ a continuous future-directed causal curve such that, $\gamma(0) = p$ and there is a subsequence $\{\gamma_{n_j}\} \subset \{\gamma_n\}$ such that $\gamma_{n_j} \to \lambda$ uniformly on each compact subset of M.

But $\gamma_n\left(\left[-a + \dfrac{a}{n}, 0\right]\right) \subset H^+(S)$ for each n, hence, since $H^+(S)$ is closed, $\lambda([-a, 0]) \subset H^+(S)$. As a consequence, as $H^+(S)$ is achronal, λ is lightlike, thus a generator of $H^+(S)$ through p. But the multiplicity of p is one, thus $\lambda|_{(-a,0]} = \gamma$ and,

[1] Observe that in the other examples in this work f has been a function over $X_0^{\perp}$, but X_0 is transverse both to $X_0^{\perp}$ and $B_\eta^3(0)$, so we can move from one domain to the other.

since $\overline{\exp_p(B^h_{a\delta}(0))}$ is compact, the convergence is uniform and there is $j_0 > 0$ such that $j > j_0 \Rightarrow \gamma_{n_j}\left(\left[-a + \dfrac{a}{n_j}, 0\right]\right) \subset \exp_p\left(B^h_{a\delta}(0)\right)$.

To finish our proof, we still need more auxiliary objects.

- $\pi_0 : T_pM \to T_pM$ is the projection onto the subspace generated by X_0 with respect to the $\{X_0, X_1, X_2, X_3\}$ basis;
- $F : L(\bar{p}) \to \mathbb{R}$ given by $F(v) = \pi_0(\exp_p^{-1}(\exp_q(v)))$, with $v \in T_qM$. We may say F is smooth because $L(\bar{p})$ is a submanifold of E;
- $u_j := \gamma_{n_j}\left(-a + \dfrac{a}{n_j}\right)$, for $j > j_0$;
- $F_{u_j} := F|_{L(\bar{p})\cap\Pi^{-1}(u_j)}$ *and* $F_r := F|_{L(\bar{p})\cap\Pi^{-1}(r)}$.

If $v \in T_rM \cap L(\bar{p})$, $\exp_r(v) \in J^+(r)$, hence, in the same way as f_- in the proof of proposition 3.4, for $r \in H^+(S)$

$$F_r(v) \geqslant f\left(\exp_p^{-1}(\exp_r(v)) - F_r(v)\right). \; [2]$$

- $S_{u_j} := \exp_p^{-1}(\exp_{u_j}(L(\bar{p}) \cap \Pi^{-1}(u_j)))$ and $S_r := \exp_p^{-1}(\exp_r(L(\bar{p}) \cap \Pi^{-1}(r)))$. Each S_{u_j} and S_r are lightlike manifolds, for they are images of Minkowski lightcones by $\exp_p^{-1} \circ \exp_{u_j}$ and $\exp_p^{-1} \circ \exp_r$, respectively.

Now, X_1 is tangent to the image S_r, which is a lightlike manifold since it is an image of a Minkowski lightcone by $\exp_p^{-1} \circ \exp_r$. Hence, X_1 is normal to the hyperplane tangent to S_r (see proposition 1.19). At 0, it is the plane generated by $\{X_1, X_2, X_3\}$. This has two implications.

First, there is $\eta_2 > 0$ such that $\eta > \eta_2$ and $\tilde{F}_r : B^3_{\eta_2}(0) \to \mathbb{R}$ is such that $S_r \cap (-\epsilon, \epsilon) \times B^3_{\eta_2}(0)$ is the graphic of $\tilde{F}_r$. From the definition of F

$$F_r(\exp_r^{-1}(\exp_p(\tilde{F}_r(x), x))) = \tilde{F}_r(x), \forall x \in B^3_{\eta_2}(0) \; [3]$$

Let j_1 be such that $j_1 > j_0$ and $j > j_1 \Rightarrow q_{n_j} \in B^3_{\eta_2}(0)$.

Second, $F_r(\exp_r^{-1}(\gamma(t)))$ is well-defined, for $\gamma(t)$ is a future-directed lightlike curve based on r, and it is constant and equal to 0, since $\exp_p^{-1}(\gamma(t))$ is the X_1-axis of our chosen basis. Then

$$\begin{aligned} \frac{d}{dt}F_r(\exp_r^{-1}(\gamma(t))) &\equiv 0 \\ \overset{[3]}{\Rightarrow} \quad \frac{d}{dt}\tilde{F}_r(\exp_p^{-1}(\gamma(t))) &\equiv 0 \quad [4] \end{aligned}$$

For $\exp_p^{-1}(\gamma(t)) \in S_r, \forall t$.

On the other hand, $D_0 \exp_p(X_1)$ is the tangent along a geodesic both through p and r. Hence, by the Gauss Lemma (see section 1.4.3) both $\exp_p$ and $\exp_r$ are isometries along γ. As a consequence, X_1 is normal also to the plane tangent to $\tilde{F}_r^{-1}(0)$ at 0, thus being tangent to $\operatorname{grad} \tilde{F}_r(0)$. Along with [4] this gives:

$$\operatorname{grad} \tilde{F}_r(\bar{p}) = 0$$

In addition, from [3], $F_r(\exp_r^{-1}(\exp_p(\tilde{F}_r(q_{n_j}), q_{n_j}))) = \tilde{F}_r(q_{n_j})$, hence $\tilde{F}_r(q_{n_j}) \geqslant f(q_{n_j})$. Thus, for $j > j_1$:

$$0 = \lim_{j\to\infty} \frac{|\tilde{F}_r(q_{n_j}) - \tilde{F}_r(0) - D_0\tilde{F}_r(q_{n_j})|}{|q_{n_j}|} = \lim_{j\to\infty} \frac{|\tilde{F}_r(q_{n_j})|}{|q_{n_j}|} \geqslant \lim_{j\to\infty} \frac{f(q_{n_j})}{|q_{n_j}|}$$

So, for [1] to be possible, $\exists j_2 > j_1$ such that $j > j_2$ implies $f(q_{n_j}) < 0$. [1] becomes

$$\frac{-f(q_{n_j})}{|q_{n_j}|} \geqslant C \Rightarrow f(q_{n_j}) \leqslant -C|q_{n_j}| \ [1']$$

In the same fashion, as X_1 is the lightlike tangent to S_r along γ, $\dot{\gamma}_{n_j}$ is the lightlike tangent to S_{u_j}, for any j. Hence, for each j, there is $\xi_j > 0$ and $\tilde{F}_{u_j} : B^3_{\xi_j}(q_{n_j}) \to \mathbb{R}$ such that $S_{u_j} \cap [(-\epsilon, \epsilon) \times B^3_{\xi_j}(q_{n_j})]$ is the graphic of $\tilde{F}_{u_j}$ (check footnote 1 and remember X_0 is not in $T_{q_{n_j}} S_{u_j}$ because S_{u_j} is lightlike). As before:

$$F_{u_j}(\exp_{u_j}^{-1}(\exp_p(\tilde{F}_{u_j}(x), x))) = \tilde{F}_{u_j}(x), \forall x \in B^3_{\xi_j}(q_{n_j}) \ [5]$$

Note that the equalities [3] and [5] suggest we can somehow unify the $\tilde{F}_{(.)}$ smoothly, but we do not have an obvious regular domain V in some hypersurface of T_pM where to define a smooth function that coincides with the $\tilde{F}_r$ and the $\tilde{F}_{u_j}$'s appropriately. Yet.

- $\begin{array}{rccl} A: & N(p) \times (-\epsilon, \epsilon) \times B^3_{\eta_2}(0) & \to & TM \\ & (z, y, x) & \mapsto & \exp_z^{-1}(\exp_p(y, x)) \end{array}$ is a smooth function such that $A(r, 0, 0) = \bar{p}$.

Now, $\hat{G} \circ A$ is a smooth real function such that $(\hat{G} \circ A)(r, 0, 0) = 0$. Also, since X_0 is timelike, it follows from proposition 1.27 that $A(r, y, 0)$ is timelike in T_rM for any $y > 0$ and spacelike for $y < 0$. Hence, $\partial_y(\hat{G} \circ A)(r, 0, 0) \neq 0$. It follows, then, from the implicit function theorem, that there is $U \subset N(p)$ an open set such that $r \in U$, $\eta_3 > 0$, $\eta_2 > \eta_3$, and $\tilde{F} : U \times B^3_{\eta_3}(0) \to \mathbb{R}^3$ such that $(\hat{G} \circ A)(z, \tilde{F}(z, x), x) = 0, \forall z \in U, x \in B^3_{\eta_3}(0)$. In other words, $A(z, \tilde{F}(z, x), x) \in L(\bar{p})$.

Now, since $u_j \to r$ and $q_{n_j} \to 0$, there is $j_3 > j_2$ such that if $j > j_3$, $u_j \in U$ and $q_{n_j} \in B^3_{\eta_3}(0)$. Also, since $A(z, \tilde{F}(z, x), x) \in L(\bar{p})$, we can define $F \circ A$ in $U \times B^3_{\eta_3}(0)$ and, for $j > j_3$, $\tilde{F}(u_j, .) = \tilde{F}_{u_j}(.)$ in $B^3_{\eta_3}(0) \cap B^3_{\xi_j}(q_{n_j})$ and $\tilde{F}(r, .) = \tilde{F}_r(.)$ in $B^3_{\eta_3}(0)$.

Now we can finally prove the theorem.

Since $\tilde{F}$ is smooth:

$$\partial_x \tilde{F}(u_j, q_{n_j}) = D_{q_{n_j}} \tilde{F}_{u_j} \to \partial_x \tilde{F}(r, 0) = D_0 \tilde{F}_r = 0$$

So, there is $j_4 > j_3$ such that if $j > j_4$:

$$|D_{q_{n_j]}} \tilde{F}_{u_j}(\Delta v)| \leqslant \frac{C}{3}|\Delta v| \quad [6]$$

Additionally, if we fix $z \in U$, and take $\eta_4 > 0$, $\eta_3 > \eta_4$, we have, from the Taylor's remainder theorem applied to $\tilde{F}(z, .)$ that given $x, x_0 \in B^3_{\eta_4}(0)$:

$$\tilde{F}(z, x) = \tilde{F}(z, x_0) + \partial_x \tilde{F}(z, x_0)(x - x_0) + R(z, x)(x - x_0, x - x_0) \quad [7]$$

with $R(x)$ a quadratic form satisfying

$$|R_{lm}(z, x)| \leqslant \frac{1}{l!m!} \max \left\{ \left| \frac{\partial^2 \tilde{F}(z, y)}{\partial x_h \partial x_i} \right| | h, i = 1, 2, 3, y \in \overline{B^3_{\eta_4}(0)} \right\} \quad [8]$$

Now, choose $\tilde{U}$ a precompact open set in $N(\bar{p})$ such that $r \in \overline{\tilde{U}} \subset U$. Hence, since $\tilde{U} \times B^3_{\eta_4}(0)$ is precompact and $\tilde{F}$ is smooth, there is $M > 0$ such that for all $h, i \in \{1, 2, 3\}$, $(z, y) \in \tilde{U} \times B^3_{\eta_4}(0)$, $\left| \frac{\partial^2 \tilde{F}(z, y)}{\partial x_h \partial x_i} \right| < M$. Along with [8], that guarantees there is $\tilde{M} > 0$ such that,$\forall (z, x) \in \tilde{U} \times B^3_{\eta_4}(0)$:

$$R(z, x)(x - x_0, x - x_0) \leqslant \tilde{M}|x - x_0|^2 \quad [9]$$

Finally, choose $\eta_5 > 0$ such that $\frac{C}{3\tilde{M}}, \eta_4 > \eta_5$, $j_5 > j_4$ such that, if $j > j_5$, $q_{n_j} \in B^3_{\eta_5}(0)$ and $u_j \in \tilde{U}$. Then if we take $z = u_j$, $x = 0$ and $x_0 = q_{n_j}$ and replace in equation [7] we get:

$$\tilde{F}_{u_j}(0) = \tilde{F}_{u_j}(q_{n_j}) + D_{q_{n_j}} \tilde{F}_{u_j}(-q_{n_j}) + R(u_j, 0)(-q_{n_j}, -q_{n_j})$$

But $\tilde{F}_{u_j}(q_{n_j}) = f(q_{n_j})$ from the definition of F and from equation [5], since q_{n_j} is in the generator through u_j. Using this fact and inequalities [$1'$], [6] and [9] we have:

$$\tilde{F}_{u_j}(0) \leqslant -C|q_{n_j}| + \frac{C}{3}|q_{n_j}| + \tilde{M}.\frac{C}{3\tilde{M}}|q_{n_j}| < 0$$

Hence, S_{u_j} crosses the X_0 axis strictly below 0. In other words, there is $v_j \in S_{u_j}$ such that $v_j \ll 0$. It follows from proposition 1.27 that $u_j \prec \exp_p(v_j) \ll p \Rightarrow u_j \ll p$. But $u_j, p \in H^+(S)$, which is achronal. Contradiction. It follows that [1] is false, hence $H^+(S)$ is indeed differentiable at p. $\square$

If we go back to section 3.1 and look at the proposed vector field V, we may use the sequence γ_n we constructed along the previous proof to check the regularity of V. Specifically, we prove that $V : H^+(S) \to TM$ is continuous:

If all points in $H^+(S)$ have multiplicity one and $\{\tilde{q}_n\}_{n\in\mathbb{N}} \subset H^+(S)$ is a sequence such that $\tilde{q}_n \overset{n\to\infty}{\longrightarrow} p$. Let $\{\tilde{q}_{n_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{\tilde{q}_n\}_{n\in\mathbb{N}}$. If for each n_j we define γ_{n_j} to be the future-directed generator through $\tilde{q}_{n_j}$ parametrized by h-arc length such that $\gamma_{n_j}(0) = \tilde{q}_{n_j}$, there is a subsequence $\{\gamma_{j_k}\}_{k\in\mathbb{N}} \subset \{\gamma_{n_j}\}$ that converges uniformly to a causal curve γ through p. Since $\gamma \subset H^+(S)$, γ is the unique generator through p and $\dot{\gamma}_{j_k}(0) = V(\tilde{q}_{j_k}) \overset{k\to\infty}{\longrightarrow} \dot{\gamma}(0) = V(p)$. As that is true for every subsequence of $\{\tilde{q}_n\}_{n\in\mathbb{N}}$, $V(\tilde{q}_n) \overset{n\to\infty}{\longrightarrow} V(p)$ and V is continuous.

Although our naive conditions for the dynamical system in the first paragraph guarantee the continuity of our vector field, we cannot go one step further and try to prove that V is C^1, because that is not the case in general. To show that we bring the example presented in (BEEM; KRÓLAK, 1998) for a horizon that is C^1 but not C^2.

Example 3.6. *Let M be $U \times \mathbb{R}$, with U the subset of $\mathbb{R}^2$ such that $x > -1$, $y > -1$ and for $x \in [-1, 0]$, $y > -\sqrt{1-x^2}$. Let g be the usual plane metric in $\mathbb{R}^3$ with signature $(+,+,-)$ restricted to $U \times \mathbb{R}$. Let S be the hypersurface $U \times \{0\}$. Now, $H^+(S)$ is the graphic of a function over U and, for $x < 0$ or $y < 0$, the points $(x, y, t) \in H^+(S)$ have multiplicity one hence $H^+(S)$ is differentiable at those points.*

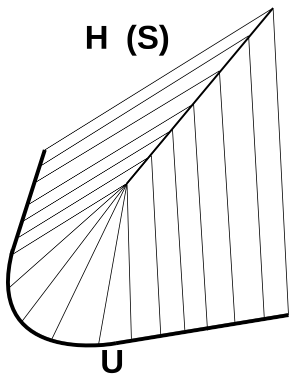


Figure 19 – Plot of the horizon of the surface $U \times \{0\}$. The generators leaving the lines $x = -1$ meet those leaving the line $y = -1$ at the points $z = x + 1 = y + 1$, $x, y > 0$, at points of multiplicity 2. While point $(0, 0, 1)$ has multiplicity $+\infty$. The horizon is not differentiable at those points.

If we try to write the function $f : U \to \mathbb{R}$ of which $H^+(S)$ is the graphic we will notice that in the region R given by $x < 0$ or $y < 0$ we have:

$$f(x,y) = \begin{cases} y+1, \ if\ x \geqslant 0 \ and\ y < 0 \\ x+1, \ if\ y \geqslant 0 \ and\ x < 0 \\ 1-\sqrt{x^2+y^2}, \ if\ x < 0 \ and\ y < 0 \end{cases}$$

Hence, we have a tangent vector field to $H^+(S)$, unitary in the Euclidean metric, which gives the direction of the generators at each point of the horizon above region R:

$$V(x,y) = \begin{cases} (0, 1/\sqrt{2}, 1/\sqrt{2}), \ if\ x \geqslant 0\ and\ y < 0 \\ (1/\sqrt{2}, 0, 1/\sqrt{2}), \ if\ y \geqslant 0\ and\ x < 0 \\ \left(-\frac{x}{\sqrt{2(x^2+y^2)}}, -\frac{y}{\sqrt{2(x^2+y^2)}}, \frac{1}{\sqrt{2}}\right), \ if\ x < 0\ and\ y < 0 \end{cases}$$

Now, if we fix $y \in (-1,0)$, the first coordinate function

$$V^1(x,y) = \begin{cases} -\frac{x}{\sqrt{2(x^2+y^2)}}, \ for\ x < 0 \\ 0, \ for\ x \geqslant 0 \end{cases}$$

is continuous but not C^1, which reflects the fact that the horizon itself is not C^2.

From the discussion throughout this chapter we see that if we expect our vector field to agree with the conditions for the stability theorems mentioned in chapter 2, specifically to be C^1, we must impose it by hand, although the continuity of the vector field is guaranteed by the uniqueness of the definition of V done in section 3.1.

Theorem 3.2 is of broader importance in the construction of a bridge between the generators of the horizon and its differentiability, so we prove the two remaining equivalences, always adapting the proofs from (BEEM; KRÓLAK, 1998).

Proposition 3.7. *If $H^+(S)$ is differentiable in an open subset W, $H^+(S)$ is C^1 in W.*

Proof. This proves 3.2 $I \Rightarrow II$.

From propositions 3.3 and 3.5 we have that the hypothesis is equivalent to say that every point $p \in W$ is of multiplicity one and hence, from the discussion on section 3.1, we may fix a Riemannian metric h on M and define a tangent vector field $V : W \to TH^+(S)$ which is unitary with respect to h, future-directed and tangent to the generator through each point $p \in M$. In the discussion that follows proposition 3.5 we have seen that V is continuous.

Now, for each $p \in W$, take N_p a normal neighborhood of M around p with respect to the orthonormal basis $\{X_0, X_1, X_2, X_3\}$ of T_pM, with X_0 future-directed timelike. Call the coordinate function $\exp_p^{-1}(q) := (x^0(q), ..., x^3(q))$ and $\{\partial_{x^0}(q), ..., \partial_{x^3}(q)\}$ the coordinate basis at each point of N_p. Inside at least an open subset $U \subset N_p$ we can guarantee that ∂_{x^0} is future-directed timelike and that there is a function $f : \tilde{U} \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\exp_p^{-1}(H^+(S))$ is the graphic of f in $\tilde{U} \times \mathbb{R}$, as seen in proposition 1.11.

Going back to V, we know V is tangent to a generator of $H^+(S)$ at each point of W and, since $H^+(S)$ is differentiable, that there is a well-defined tangent plane $T_qH^+(S)$

at each $q \in W$, hence, $V(q) \in T_q H^+(S)$. But $V(q)$ is lightlike and $T_q H^+(S)$ is a lightlike vector space. It follows that, as shown in proposition 1.19, $T_q H^+(S) = V(q)^{\perp}$.

But, since $H^+(S)$ is differentiable, f is differentiable and the vectors

$$\partial_{x^i} f := (f_{x^i}, 0, ..., 1, ..., 0)$$

1 being the coordinate x^i of $\partial_{x^i} f$, $i \in \{1, 2, 3\}$, are tangent to the graphic of f. Thus, $\partial_i f := (\exp_p)_* \partial_{x^i} f$ is normal to the vector field V for each i. Fixing i we can write this in coordinates on the tangent spaces of $H^+(S)$:

$$\begin{aligned} g(V, \partial_i f) &= 0 \\ \Rightarrow \quad g_{kj} V^k \partial_i^j f &= 0 \\ \Rightarrow \quad (g_{k1} V^k) f_{x^i} + g_{ki} V^k &= 0 \end{aligned}$$

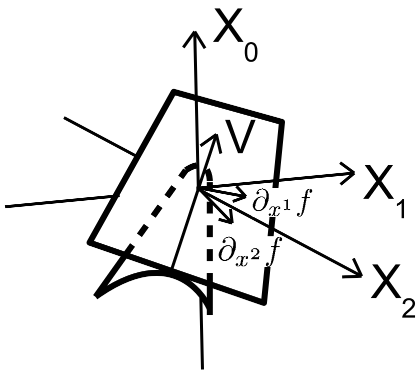


Figure 20 – The partial derivative vectors ∂_{x_i}, $i \in \{1, 2, 3\}$, are on the tangent plane of the graphic of f at 0, which is the plane normal to V.

But $g_{k1} V^k = g(\partial_{x^0}, V)$, which is everywhere non-zero, since V and ∂_{x^0} are future-directed in U. It follows from this, from the continuity of the metric and from the continuity of V that the partial derivatives of f are all continuous in U. Since f is differentiable in U, f is C^1 in U. Hence, $H^+(S)$ is C^1 in W. □

And finally, after this very geometrical argument, we use a result from the theory of ODE's to prove the last equivalence of theorem 3.2.

Proposition 3.8. *If $H^+(S)$ is C^1 in an open subset W, there are no endpoints of generators of the horizon inside W.*

Proof. This proves 3.2 $II \Rightarrow III$.

Since $H^+(S)$ is C^1, we may use h, the auxiliary Riemannian metric on M, to define an h-unitary vector field V tangent to $H^+(S)$ at each point $p \in W$. Then, as proved after proposition 3.5, V is continuous in W. Therefore, given $p \in W$, we can choose

(ψ_p, U_p) a coordinate chart for $H^+(S)$ around p and use Peano's Existence Theorem for the differential equation:

$$\begin{cases} \dot{\gamma}(t) = ((\psi_p)_* V)(\gamma(t)) \\ \gamma(0) = \psi_p(p) \end{cases}$$

to guarantee there is a solution $\gamma : (-\epsilon, \epsilon) \to \psi_p(U_p)$ for some $\epsilon > 0$.

Since $\psi_p^{-1} \circ \gamma$ is tangent to V at each point, it is a generator through p and $\gamma(\epsilon/2) \in H^+(S)$. Hence, since there is only one generator through p and given proposition 1.16 holds, p is not an endpoint in $H^+(S)$. □

This last proof collaterally highlights the importance of the hypothesis that W is open in Peano's Theorem, since our example 3.6 showed that the integral curves of V may leave the horizon even when it is C^1. Also, this result shows that, if p is an endpoint of multiplicity one, it is in the boundary of the set of points where $H^+(S)$ fails to be differentiable. However, endpoints of multiplicity greater than one may be found far from other endpoints of multiplicity one, such as seen in the same example 3.6, where all endpoints have multiplicity at least two.

4 From Dynamical Systems to Horizons

The core of (CHRUŠCIEL; ISENBERG, 1994) is the explicit construction of a spacetime with a horizon diffeomorphic to a prescribed compact 3-manifold Σ with the orbits of a prescribed vector field X as generators. It is important to note that at this point we are restricting ourselves to compact horizons, and this restriction agrees with our characterization result for structurally stable dynamical systems. On the other hand, the choice of the dimension for the manifold is convenient for the writing of the proof, since explicit calculations are being done, but is not necessary for the construction to work. The only limitation is that the dimension of the horizon is one less than the dimension of the spacetime generated.

In the following we adapt the construction of the metric from (CHRUŠCIEL; ISENBERG, 1994) by breaking it into more pieces, allowing us to use a slightly more general fashion of the metric in other examples ahead in the text. The idea is to construct a spacetime akin to Taub-NUT space: a cylinder with basis Σ which is globally hyperbolic under some horizontal section but with the time direction tilting as the height rises until it is parallel to the dividing section, where the global hyperbolicity breaks, making the section an horizon.

The cornerstone of the construction is lemma 3.2 at (CHRUŠCIEL; ISENBERG, 1994):

Lemma 4.1. *Let Σ be a compact 3-manifold and X a nowhere vanishing vector field over Σ. Take $\mu > 0$ and assume you have a spacetime (M, g) with $M = \Sigma \times (-\mu, \mu)$. Assume also there is Z a vector field over M such that $Z|_{\Sigma\times\{0\}} = X$. If the following hold:*

a. $g(Z, Z)|_{\Sigma\times\{0\}} = 0$,

b. If t is a parameter for the interval $(-\mu, \mu)$ in a parametrization of the product $\Sigma \times (-\mu, \mu)$, $t < 0 \Rightarrow dt(T) > 0, \forall T$ future-directed timelike.

Then we have:

1. $(\tilde{M}, \tilde{g}) := (\Sigma \times (-\mu, 0), g|_{\Sigma\times(-\mu,0)})$ is globally hyperbolic;

2. $\mathcal{H} := \Sigma \times \{0\}$ is a future Cauchy horizon for $(\tilde{M}, \tilde{g}) \subset (M, g)$;

3. X is tangent to the null generators of $\mathcal{H}$.

We then present a skeleton for a metric satisfying the hypothesis of the lemma, which may be specified for constructing examples of spacetimes that satisfy some desired condition.

Proposition 4.2. *If Σ, X, M, Z and t are as in lemma 4.1, assume there are β 1-form in M, ν a symmetric 2-form in M and $\chi : M^4 \to \mathbb{R}$ such that:*

1. $\beta(Z) > 0$ *and* $\beta(\partial_t) = 0$;
2. $\nu(Z, .) = \nu(\partial_t, .) = 0$;
3. $\nu(Y, Y) > 0, \forall Y \in T\Sigma$ *s.t.* $\{Y, Z\}$ *is L.I.;*
4. $t < 0 \Rightarrow \chi(p, t) > -2\dfrac{dt(Z)}{\beta(Z)}$;
5. $\chi(p, 0) \equiv 0$.

Then the metric

$$g = \chi\beta \otimes \beta + dt \otimes \beta + \beta \otimes dt + \nu$$

satisfies the hypothesis of lemma 4.1.

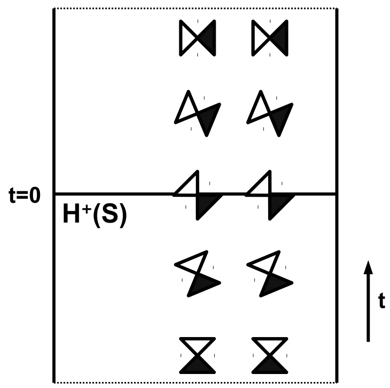


Figure 21 – The value of $\chi(p, t)$ determines the rotation of the light cones at $t \equiv C$. If it satisfies condition 4 we are in a region of global hyperbolicity with regard to the t-sections, if it satisfies the reverse inequality we are at a region with closed timelike curves of constant coordinate t. The boundary condition $\chi(p, 0) = 0 = dt(Z)|_{t=0}$ gives the horizon, with the light cones tangent to the hypersurface $t = 0$.

Proof. We only have to perform the calculations:

First, we show that g is indeed a Lorentzian metric. If $(p, t) \in M$, we can choose the basis $\{Z(p, t), \partial_t(p, t), W, Y\}$, with $W, Y \in Tp\Sigma \cap \ker(\beta(p, t))$, for $T_{(p,t)}M$. In that basis, $g_{(p,t)}$ assumes the matrix

$$[g_{(p,t)}] = \begin{bmatrix} \chi(t)\beta^2(Z) + 2\beta(Z)dt(Z) & \beta(Z) & 0 & 0 \\ \beta(Z) & 0 & 0 & 0 \\ 0 & 0 & \nu(W, W) & \nu(W, Y) \\ 0 & 0 & \nu(Y, W) & \nu(Y, Y) \end{bmatrix}$$

which is a Lorentzian metric, since $\nu|_{span<W,Y>}$ is a positive definite inner product and $\beta(Z) \neq 0$.

1. If $t = 0, \chi(p,t) = 0$ and:

$$\begin{aligned} g(Z,Z)|_{\Sigma\times\{0\}} &= 2dt(Z)\beta(Z) + \nu(Z,Z) \\ &= 2dt(X)\beta(X) \\ &= 0 \end{aligned}$$

2. We invert the matrix $[g_{(p,t)}]$. To simplify the writing, we call $\nu|_{span<W,Y>} := \tilde{\nu}$:

$$[g^{-1}_{(p,t)}] = \begin{bmatrix} 0 & \dfrac{1}{\beta(Z)} & 0 & 0 \\ \dfrac{1}{\beta(Z)} & -\chi(t) - 2\dfrac{dt(Z)}{\beta(Z)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\det\tilde{\nu}}\nu(Y,Y) & -\dfrac{1}{\det\tilde{\nu}}\nu(W,Y) \\ 0 & 0 & -\dfrac{1}{\det\tilde{\nu}}\nu(Y,W) & \dfrac{1}{\det\tilde{\nu}}\nu(W,W) \end{bmatrix}$$

Now, since we can find a local chart around any (p,t) with $\{Z(p,t), \partial_t(p,t), W, Y\}$ as coordinate vectors at (p,t):

$$g(\nabla t, \nabla t) = g^{-1}(dt, dt) = -\left(\chi(t) + 2\frac{dt(Z)}{\beta(Z)}\right) < 0$$

Therefore we can choose $-\nabla t$ as a global future timelike field in $\tilde{M}$. Thus, by the choice of time direction, $\forall T$ future-directed timelike vector field in $\tilde{M}$:

$$0 > g(-\nabla t, T) = -dt(T)$$

□

Note that, although the statement of last proposition is cumbersome, for the case $\Sigma = N \times \gamma$, with γ a curve tangent to X and N a Riemannian manifold, β and ν might be chosen to be the component forms of a product metric in $M = N \times \gamma \times (-\mu, \mu)$.

Finally, we show how a metric like the one in the paper (CHRUŜCIEL; ISENBERG, 1994) may be constructed inside the framework we defined in our proposition 4.2. Note that the metric constructed here differs from the one in the paper with respect to the definition of ν.

Proposition 4.3. *If Σ is a compact 3-manifold and X a nowhere vanishing vector field over Σ there is a spacetime (M, g) containing a Cauchy horizon $\mathcal{H}$ diffeomorphic to Σ such that the generators of $\mathcal{H}$ are the orbits of the flow generated by X.*

Proof. Choose $\mu > 0$, possibly $\mu = \infty$, and define $M := \Sigma \times (-\mu, \mu)$. Let Z be the Lie parallel field with respect to ∂_t with initial value X (t the parameter of the interval $(-\mu, \mu)$ as before). Now we have Σ, X, M and Z as in lemma 4.1 and all we need to do is build the metric of proposition 4.2:

1. Let h be any Riemannian metric on M and define

$$\beta(\cdot) := h(\partial_t, \partial_t)h(Z, \cdot) - h(Z, \partial_t)h(\partial_t, \cdot)$$

Since Z and ∂_t are nowhere parallel, it follows from the Cauchy-Schwartz inequality that β satisfies condition 1 of proposition 4.2.

2. Let h be as in the previous item and define

$$\begin{cases} \tilde{h} := h - \dfrac{h(\partial_t, \cdot) \otimes h(\partial_t, \cdot)}{h(\partial_t, \partial_t)} \\ \nu := \tilde{h} - \dfrac{\tilde{h}(Z, \cdot) \otimes \tilde{h}(Z, \cdot)}{\tilde{h}(Z, Z)} \end{cases}$$

It is easy to see that $\tilde{h}(\partial_t, \cdot) \equiv 0$. Also, it is shown with the Cauchy-Schwartz inequality that $\tilde{h}(V, V) > 0, \forall V$ such that $\{V, \partial_t\}$ is L.I. In particular, $\tilde{h}$ is a Riemannian metric when restricted to $T\Sigma$. Therefore one can redo the same reasoning to see that ν satisfies properties 2 and 3 from proposition 4.2.

3. Since Z is Lie-parallel with respect to ∂_t, $dt(Z) \equiv 0$. Therefore, if you pick any $\phi(t)$ such that

$$\begin{cases} \phi(t) > 0, t < 0 \\ \phi(0) = 0 \end{cases}$$

and define $\chi(p, t) := \phi(t)$, χ satisfies conditions 4 and 5 of proposition 4.2.

Setting all that, we have everything demanded by proposition 4.2 to take $g := \chi\beta \otimes \beta + dt \otimes \beta + \beta \otimes dt + \nu$ that satisfies the conditions of lemma 4.1 and, thus, the restrictions we prescribed for (M, g). □

Note that to guarantee $\beta(\cdot)$ is smooth, from its definition, we must assure Z also is, which demands both X and Σ to be smooth. This construction does not allow, in general, to build smooth spacetimes which have horizons that are not.

On the other hand, we can use the framework developed in this chapter to address the question of stability of horizons, which is in the roots of our work. As suggested by chapter 2, our main hope was to be able to tackle the question of the stability of the behavior of the generators of horizons through the study of their dynamics. That point is discussed at (HAWKING, 1992) and motivated the discussion on (CHRUŜCIEL; ISENBERG, 1994), which was our starting point.

But there is an underlying untouched question that should be set in order to give meaning to this program: **are horizons stable in the topology of metrics?** The reason is simply that if a perturbation on the metric destroys the horizon, the discussion of the stability of the generators becomes meaningless.

As we shall present here, the general straightforward answer to the question in bold above is: **no**. In the following we bring an example of a spacetime with a Cauchy horizon that disappears along a curve of metrics but for one point. The example comes from (CHRUŜCIEL; ISENBERG, 1997) and uses the constructions developed in this chapter.

Example 4.4. *Let (N, h) be a compact 2-manifold with Riemannian metric h, and equip $\mathbb{R} \times \mathbb{S}^1$ with the Lorentzian metric $\tilde{g} = t^2 d\theta \otimes d\theta + dt \otimes d\theta + d\theta \otimes dt$, with θ a global parameter for $\mathbb{S}^1$ and t a global parameter for $\mathbb{R}$.*

If we define (M, g) by $M := \mathbb{R} \times \mathbb{S}^1 \times N$ with g the product metric of $(\mathbb{R} \times \mathbb{S}^1, \tilde{g})$ and (N, h) we may call:

$$\begin{cases} \Sigma := \mathbb{S}^1 \times N \\ Z := \partial_\theta \\ X := \partial_\theta|_{t=0} \\ \beta := d\theta \\ \nu := g - t^2 d\theta \otimes d\theta + dt \otimes d\theta + d\theta \otimes dt \\ \chi(t, \theta, p) := t^2 \end{cases}$$

and we see that g is in the form described in proposition 4.2.

On the other hand, if we define the metric

$$g_\epsilon := (t^2 + \epsilon)\beta \otimes \beta + dt \otimes \beta + \beta \otimes dt + \nu$$

we see that, if $\epsilon > 0$, $\chi_\epsilon := t^2 + \epsilon$ satisfies condition 4 of proposition 4.2 for all $t \in \mathbb{R}$, thus (M, g_ϵ) is globally hyperbolic for all $\epsilon > 0$.

There was a major gap in last example's conclusion: the topology of the space of metrics that justifies regarding $\epsilon\beta \otimes \beta$ as a perturbation was not mentioned. That is the way presented in (CHRUŜCIEL; ISENBERG, 1997), where the example appears in section 2 and the topology (which we will present ahead) deep into section 3. The unstated claim is that whatever "reasonable" topology is chosen, given an open set U containing a metric g and a symmetric bilinear form ν, there should be a small enough $\epsilon > 0$ such that $\epsilon\nu \in U$. That is a fine heuristics to determine what topology to use for the metrics in a Lorentzian manifold, and it is a simple and effective way to check if a property breaks

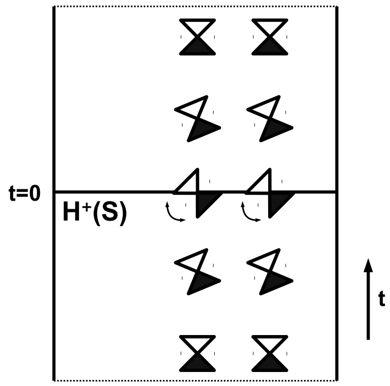


Figure 22 – The behavior of the spacetime in example 4.4 is very similar to the one depicted in figure 21, but the fact that $\chi(p,t)$ goes back to being positive at points of positive t guarantees that the horizon is the thin boundary between two regions of global hyperbolicity. Hence, the addition of any $\epsilon > 0$ to χ tweaks the light cones a little and destroys the horizon.

under small perturbations, but it is not useful to prove that some property is actually stable with respect to a topology.

In the work (CHRUŠCIEL; ISENBERG, 1997) the stability of a class of horizons satisfying certain topological and geometric conditions is proved in its theorem 2. In order to do so it specifies (*en passant*) a topology for the set of Lorentzian metrics on a manifold M, which we will call $Lor(M)$ for simplicity. But for some differences in notation and writing, it is defined in the following way:

Definition 4.5. *Let (M, g) be a spacetime, with g of class C^k, such that $H \subset M$ is a C^{k+1} Cauchy horizon for some partial Cauchy surface in M. Then the C^k-topology for the C^k metrics on M is generated by the sets:*

$$\mathcal{O}_k(h, \epsilon, U) := \{\hat{g} | ||g - h||_{C^k(\overline{U})} < \epsilon\}.$$

In this definition, h is a Lorentzian metric on M, $\epsilon > 0$, U a precompact open set in H, $||f||_{C^k(\overline{U})} = \sum_{i=0}^{k} \sup_{p \in \overline{U}} |f^{(i)}(x)|$ and $|f(x)|$ is a norm of $f(x)$ in 4×4-matrix space.

This is close to the definition of Whitney's strong topology on M, with the difference that it takes into account differences in the metrics only near H. The definition of this topology relies on choosing local charts on M. It would be interesting to see if the calculations on stability of horizons performed in (CHRUŠCIEL; ISENBERG, 1997) could get neater through the use of jets (see for example section 2.4 of (HIRSCH, 1976)).

The topology presented in definition 4.5 is compatible with the initial idea of perturbing metrics by the addition of a bilinear form of the kind $\epsilon\nu$. In fact, if we dropped the condition that $U \subset H$ and if M were compact, $g \in \mathcal{O}_k(h, \epsilon, M)$ implies that if $\nu := g - h$, $g = h + \nu$, with ν a symmetric bilinear form of norm less than ϵ. So the topology would be exactly the same as the one defined by ϵ perturbations.

That topology is, by no means, the only one defined for $Lor(M)$. For example, article (BEEM, 1995) presents some notion of stability for horizons and some results on those. The topology may be defined with less elements and relies more on the fact that $Lor(M)$ is a set of Lorentzian metrics:

Definition 4.6. *Given M a 4-manifold, let's define a relation $<$ in $Lor(M)$ by:*

$$g < h \iff g(v,v) \leqslant 0 \textit{ implies } h(v,v) < 0, \forall v \neq 0.$$

So a topology in $Lor(M)$ is generated by the sets

$$W(g,h,K) := \{\hat{g} \in Lor(M) | g < \hat{g} < h \textit{ on } K\}$$

for $g < h$ on K.

Although the symbol "$<$" suggests otherwise, $<$ induces only a partial order on $Lor(M)$, so the topology defined above is not exactly an order topology, although some calculations may be done in the same way. The procedure of "adding symmetric bilinear forms of small norm" is compatible with this topology, at least if the bilinear form added has compact support. It would be interesting to check the compatibility of the topologies defined in definitions 4.5 and 4.6 and to know which is more friendly to be related to the topology of the dynamical systems on horizons in M. That will not be done in this work, though.

5 Conclusion

The various considerations in chapter 3 should rise awareness to the fact that the correspondence between horizons and dynamical systems is not one-to-one. In principle, horizons might be non-differentiable manifolds, and in this case defining a vector field tangent to its generators would be an ill-defined procedure given theorem 3.2. Even when horizons are differentiable, the machinery presented in chapter 2 might be unsuited to extract information about the horizon, as showed by example 3.6 where the generators were C^0 but not C^1, as necessary for the Pallis-Smale Stability Conjecture to have effect. Even though, the construction presented in chapter 4 has been useful in providing counterexamples to claims regarding spacetimes in general, as showed in (CHRUŜCIEL; ISENBERG, 1994).

This work has approached the question of horizons and dynamical systems as a topological, generic question. The interest of General Relativity, though, is focused on spacetimes that obey certain geometric restrictions, such as Einstein's equations or the energy conditions. Imposing these conditions to the spacetime presented in chapter 4, for example, could bring to light some properties of stable generators of horizons and hence, since stability is usually assumed as a physically relevant property, of physically reasonable horizons. As a more ambitious plan, combining stability conditions to geometric restrictions on horizons in general we could get a better understanding of **which** horizons are effectively reasonable and how they behave.

6 Conclusão

As várias considerações do capítulo 3 deveriam despertar atenção ao fato de que a correspondência entre horizontes e sistemas dinâmicos não é bijetiva. A princípio, horizontes podem ser variedades não-diferenciáveis e, nesse caso, a definição de um campo de vetores tangente aos seus geradores pode ser um procedimento mal definido, como mostra o teorema 3.2. Mesmo se o horizonte for diferenciável, as ferramentas apresentadas no capítulo 2 podem ser inadequadas para extrair informações sobre o horizonte, como visto no exemplo 3.6, no qual os geradores são C^0, mas não C^1, o que é necessário para a validade da Conjectura de Estabilidade de Pallis-Smale. Mesmo assim, a construção apresentada no capítulo 4 se mostrou útil para a produção de contra-exemplos para afirmações sobre espaços-tempos em geral, como mostrado em (CHRUÈSCIEL; ISENBERG, 1994).

Este trabalho abordou a questão dos horizontes e sistemas dinâmicos de uma forma topológica, genérica. No entanto, o interesse da Relatividade Geral é focado em espaços-tempos que obedecem certas restrições geométricas, como as equações de Einstein ou as condições de energia. Impor essas condições ao espaço-tempo construído no capítulo 4, por exemplo, poderia lançar luz sobre algumas propriedades de geradores estáveis de horizontes e assim, como a estabilidade é geralmente considerada uma propriedade fisicamente relevante, de horizontes fisicamente razoáveis. Como um plano mais ambicioso, combinar condições de estabilidade com restrições geométricas nos horizontes poderia trazer um entendimento melhor sobre **quais** horizontes são realmente razoáveis e como eles se comportam.

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