Métodos para Equações do Transporte com Dados Aleatórios

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Resumo

Modelos matemáticos para processos do mundo real freqüentemente têm a forma de sistemas de equações diferenciais parciais. Estes modelos usualmente envolvem parâmetros como, por exemplo, os coeficientes no operador diferencial, e as condições iniciais e de fronteira. Tipicamente, assume-se que os parâmetros são conhecidos, ou seja, os modelos são considerados determinísticos. Entretanto, em situações mais reais esta hipótese freqüentemente não se verifica dado que a maioria dos parâmetros do modelo possui uma característica aleatória ou estocástica. Modelos avançados costumam levar em consideração esta natureza estocástica dos parâmetros. Em vista disso, certos componentes do sistema são modelados como variáveis aleatórias ou funções aleatórias. Equações diferenciais com parâmetros aleatórios são chamadas equações diferenciais aleatórias (ou estocásticas). Novas metodologias matemáticas têm sido desenvolvidas para lidar com equações diferenciais aleatórias, entretanto, este problema continua sendo objeto de estudo de muitos pesquisadores. Assim sendo, é importante a busca por novas formas (numéricas ou analíticas) de tratar equações diferenciais aleatórias. Durante a realização do curso de doutorado, vislumbrando a possibilidade de aplicações futuras em problemas de fluxo de fluidos em meios porosos (dispersão de poluentes e fluxos bifásicos, por exemplo), desenvolvemos trabalhos relacionados à equação do transporte linear unidimensional aleatória e ao problema de Burgers-Riemann unidimensional aleatório. Nesta tese, apresentamos uma nova metodologia, baseada nas idéias de Godunov, para tratar a equação do transporte linear unidimensional aleatória e desenvolvemos um eficiente método numérico para os momentos estatísticos da equação de Burgers-Riemann unidimensional aleatória. Para finalizar, apresentamos também novos resultados para o caso multidimensional: mostramos que algumas metodologias propostas para aproximar a média estatística da solução da equação do transporte linear multidimensional aleatória podem ser válidas para todos os momentos estatísticos da solução.

Abstract

Mathematical models for real-world processes often take the form of systems of partial differential equations. Such models usually involve certain parameters, for example, the coefficients in the differential operator, and the initial and boundary conditions. Usually, all the model parameters are assumed to be known exactly. However, in realistic situations many of the parameters may have a random or stochastic character. More advanced models must take this stochastic nature into account. In this case, the components of the system are then modeled as random variables or random fields. Differential equations with random parameters are called random (or stochastic) differential equations. New mathematical methods have been developed to deal with this kind of problem, however, solving this problem is still the goal of several researchers. Thus, it is important to look for new approaches (numerical or analytical) to deal with random differential equations. Throughout the realization of the doctorate and looking toward future applications in porous media flow (pollution dispersal and two phase flows, for instance) we developed works related to the one-dimensional random linear transport equation and to the one-dimensional random Burgers-Riemann problem. In this thesis, based on Godunov's ideas, we present a new methodology to deal with the one-dimensional random linear transport equation, and develop an efficient numerical scheme for the statistical moments of the solution of the one-dimensional random Burgers-Riemann problem. Finally, we also present new results for the multidimensional case: we have shown that some approaches to approximate the mean of the solution of the multidimensional random linear transport equation may be valid for all statistical moments of the solution.

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Capítulo 1

Introdução

1.1 Motivação e objetivos

Modelos matemáticos para problemas físicos freqüentemente têm a forma de sistemas de equações diferenciais parciais. Estes modelos usualmente envolvem parâmetros como, por exemplo, os coeficientes no operador diferencial, e as condições iniciais e de fronteira. Tipicamente, assume-se que os parâmetros são conhecidos, ou seja, os modelos são considerados determinísticos. Entretanto, em situações mais reais esta hipótese freqüentemente não se verifica dado que a maioria dos parâmetros do modelo possui uma característica aleatória ou estocástica. Modelos mais avançados costumam levar em consideração esta natureza estocástica dos parâmetros. Em vista disso, certos componentes do sistema são modelados como variáveis aleatórias ou funções aleatórias. Equações diferenciais com parâmetros aleatórios são chamadas equações diferenciais aleatórias (ou estocásticas). Novas metodologias matemáticas têm sido desenvolvidas para lidar com equações diferenciais aleatórias (veja [13, 17, 18, 22, 28, 30, 44], por exemplo); entretanto, este problema continua sendo objeto de estudo de muitos pesquisadores.

Dentre as várias metodologias para tratar equações diferenciais aleatórias podemos citar os *métodos para equações dos momentos* (veja [28, 44], por exemplo). Nestes métodos o objetivo é obter equações diferenciais determinísticas que governem os momentos estatísticos da solução do problema aleatório. A mais relevante destas equações é a equação diferencial para a média (esperança matemática) da solução, chamada *equação efetiva*. Estas equações diferenciais determinísticas são então resolvidas numericamente ou analiticamente. Convém ressaltar que tal estratégia não é simples de ser aplicada e os métodos oriundos desta metodologia possuem inúmeras restrições de validade, conseqüências de várias aproximações que são necessárias durante o processo (veja [44], por exemplo).

Na maioria dos casos as soluções numéricas de equações diferenciais parciais com co-

eficientes aleatórios são calculadas (aproximadas) usando o conhecido *método de Monte Carlo* (veja [10, 35], por exemplo). Neste caso, os parâmetros aleatórios do modelo são amostrados repetidamente reduzindo o problema à solução de uma equação diferencial determinística para cada amostra. As propriedades estocásticas da solução são subseqüentemente determinadas pela análise estatística do conjunto de soluções obtidas. De um modo geral, este método demanda geradores de números aleatórios e um número excessivo de simulações numéricas, ou seja, o custo computacional é alto.

Assim sendo, é importante a busca por novas técnicas (numéricas ou analíticas) de tratar equações diferenciais aleatórias. Durante a realização do curso de doutorado e vislumbrando a possibilidade de futuras aplicações em problemas de fluxo de fluidos em meios porosos (dispersão de poluentes e fluxos bifásicos, por exemplo), desenvolvemos trabalhos relacionados aos seguintes problemas:

- a equação do transporte linear unidimensional aleatória:

$$\begin{aligned} &\frac{\partial}{\partial t}Q(x,t) + A(t,x)\frac{\partial}{\partial x}Q(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &Q(x,0) = Q_0(x), \end{aligned} \tag{1.1}$$

  onde a velocidade $A(t,x)$ e a condição inicial são funções aleatórias. Esta equação é freqüente, por exemplo, em problemas de dispersão de poluentes no qual a variabilidade da permeabilidade do meio poroso tem como conseqüência um campo aleatório de velocidades, utilizado para calcular a concentração do poluente.

- o problema de Burgers-Riemann unidimensional aleatório:

$$\begin{aligned} &\frac{\partial}{\partial t}Q(x,t) + \frac{1}{2}\frac{\partial}{\partial x}Q^2(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &Q(x,0) = \begin{cases} Q_L, & \text{if} \quad x < 0, \\ Q_R, & \text{if} \quad x > 0, \end{cases} \end{aligned} \tag{1.2}$$

  onde Q_L and Q_R são variáveis aleatórias. Aqui a aleatoriedade aparece somente na condição inicial. A versão determinística de (1.2) foi introduzida por Burgers [1] como um modelo simplificado para capturar características básicas em dinâmica dos gases. Mas, em vez de modelar um processo físico, a Equação de Burgers invíscida tem sido usada no desenvolvimento de métodos numéricos e teóricos para equações hiperbólicas determinísticas. A investigação dos momentos estatísticos da solução deste problema foi nosso primeiro passo na direção do tratamento de leis de conservação não-lineares com parâmetros aleatórios.

1.2 Organização da tese

O texto desta tese está organizado de modo a agrupar (cronologicamente) os principais artigos, publicados e/ou submetidos para publicação, que foram resultados da pesquisa realizada. A seguir apresentamos uma breve relato sobre o conteúdo de cada capítulo.

1.2.1 Capítulo 2

A note on the Riemann problem for the random transport equation
(Ref. [4]; publicado no journal: *Computational & Applied Mathematics*).
Apresentamos uma expressão explícita para a solução do problema (1.1), com $A(t,x) = A$ uma variável aleatória, e com condição inicial dada por

$$Q(x,0) = \begin{cases} Q_0^+, & x > 0, \\ Q_0^-, & x < 0, \end{cases}$$

onde os estados iniciais, Q_0^- e Q_0^+, são variáveis aleatórias. Este problema é conhecido como *problema de Riemann aleatório*. Sua solução é fundamental no desenvolvimento de esquemas numéricos com condição inicial mais geral, via método de Godunov [14, 27] ou método de Glimm [12] (*random choice method*). Mostramos que esta solução (função aleatória) é de similaridade e, admitindo independência estatística entre a velocidade, A, e os estados iniciais, Q_0^- e Q_0^+, obtemos uma expressão para os momentos estatísticos.

1.2.2 Capítulo 3

A finite volume method for the mean of the solution of the random linear transport equation (Ref. [8]; publicado no *Journal of Applied Mathematics and Computation*).
Utilizando as idéias do trabalho [4] e o método de Godunov [14], construímos um esquema numérico explícito para a média da solução da Equação (1.1), onde $A(t,x) = A$ é uma variável aleatória e a condição inicial, $Q(x,0) = Q_0(x)$, é uma função aleatória. Sob algumas hipóteses na discretização obtemos condições de estabilidade do método e mostramos sua consistência com uma equação do tipo advecção-difusão determinística. Vários exemplos computacionais mostram uma boa concordância dos resultados quando comparados com o método de Monte Carlo.

1.2.3 Capítulo 4

A numerical scheme for the variance of the solution of the random linear transport equation (Ref. [5]; publicado no *Journal of Applied Mathematics and Computation*).

Avançando no conhecimento dos momentos estatísticos da solução de (1.1), com $A(t,x) = A$ uma variável aleatória e $Q_0(x)$ uma função aleatória, propomos um esquema numérico explícito para a variância da solução. Obtemos as condições de estabilidade do método proposto e mostramos sua consistência com um sistema determinístico (para a média e a variância) de equações do tipo advecção-difusão não-homogêneo desacoplado. Testes computacionais são apresentados para avaliar a proposta.

1.2.4 Capítulo 5

Statistical moments of the random linear transport equation
(Ref. [6]; submetido para o *Journal of Computational Physics*).
Neste trabalho generalizamos as idéias de [5, 8] e apresentamos um esquema numérico para os momentos da solução da equação do transporte linear unidimensional aleatória (1.1), com $A(t,x) = A$ uma variável aleatória e $Q_0(x)$ uma função aleatória. O esquema é baseado na solução de problemas de Riemann e no método de Godunov. Mostramos que o esquema é consistente e estável com uma equação do tipo advecção-difusão. Além disso, no caso em que a velocidade é normalmente distribuída, obtemos um sistema de equações diferenciais parciais para os momentos e momentos centrais da solução. Testes computacionais são apresentados para avaliar a proposta.

1.2.5 Capítulo 6

Statistical moments of the solution of the random Burgers-Riemann problem
(Ref. [7]; submetido para o *Journal of Mathematics and Computers in Simulation*).
Neste trabalho apresentamos uma expressão para a solução da Equação de Burgers aleatória (1.2). A solução aleatória permite expressões integrais para os momentos estatísticos da solução. Usando idéias de integração numérica, propomos um algoritmo eficiente para calcular os momentos estatísticos da solução. Testes computacionais são apresentados para validar a proposta.

1.2.6 Capítulo 7

On the evaluation of moments for solute transport by random velocity fields
(Ref. [9]; submetido para o *Journal of Applied Numerical Mathematics*).
Apresentamos um útil resultado para a equação do transporte linear aleatória multidimensional. Basicamente, provamos que algumas metodologias baseadas em *averaging approach* para aproximar a média estatística da solução da equação do transporte linear aleatória (Equação 1.1, multidimensional) podem ser válidas para todos os momentos

estatísticos da solução. Com este resultado podemos obter mais informações estatísticas sobre a solução aleatória, como ilustrado em dois exemplos particulares.

Capítulo 2

A note on the Riemann problem for the random transport equation

Abstract

We present an explicit expression to the solution of the random Riemann problem for the one-dimensional random linear transport equation. We show that the random solution is a similarity solution and the statistical moments have very simple expressions. Furthermore, we verify that the mean, variance, and 3rd central moment agree quite well with the Monte Carlo method. We point out that our approach could be useful in designing numerical methods for more general random transport problems.

Keyword: random linear transport equation, Riemann problem, statistical moments.

2.1 Introduction

Conservation laws are differential equations arising from physical principles of the conservation of mass, energy or momentum. The simplest of these equations is the one-dimensional advective equation and its solution plays a role in more complex problems such as the numerical solution of nonlinear conservation laws [6]. This linear initial value problem can, for instance, model the concentration, or density, of a chemical substance transported by a one dimensional fluid that flows with a known velocity. The deterministic problem is to find $u(x,t)$ such that

$$\begin{cases} u_t + a(x)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

It is well known that the solution to (2.1) is the initial condition transported along the characteristic curves. The characteristic system associated to (2.1) is defined by ordinary

differential equations:

$$\begin{cases} \dfrac{dx}{dt} = a(x), \quad x(0) = x_0, \\ \dfrac{d[u(x(t),t)]}{dt} = 0, \quad u(x,0) = u_0(x), \end{cases} \tag{2.2}$$

where the last equation is along the characteristic curve, $x(t)$, given by the first equation. If a is constant, the characteristics are straight lines and the analytic solution is $u(x,t) = u_0(x - at)$.

The complexity of natural phenomena compels us to study partial differential equations with random data. For example, (2.1) may model the flux of a two phase equal viscosity miscible fluid in a porous media. The total velocity is obtained from Darcy's law and it depends on the geology of the porous media. Thus, the external velocity is defined by a given statistics. Also, the prediction of the initial state of the process is obtained from data acquired with a few number of exploratory wells using geological methods.

Our aim in this paper is to study the random Riemann problem:

$$\begin{cases} U_t + AU_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ U(x,0) = U_0(x) = \begin{cases} U_0^+, & x > 0, \\ U_0^-, & x < 0, \end{cases} \end{cases} \tag{2.3}$$

where A, U_0^- and U_0^+ are random variables.

Several numerical methods which were developed to solve the deterministic problem (2.1) use solutions of Riemann problems. For instance, the Random Choice Method, developed by Glimm [2], and the Godunov's method [4, 6]. These methods suggest that the random Riemann solutions can be used for designing numerical methods to random transport equations, where the velocity and the initial condition are random fields. Our preliminary results in this direction, i.e., using Godunov's method with random Riemann solutions in the averaging step, are promising.

Besides the well-developed theoretical methods such as Ito integrals, Martingales and Wiener measure [5, 7, 9, 10] to deal with stochastic differential equations, two types of methods are normally used in the construction of solutions for random partial differential equations. The first is based on the Monte Carlo method which, in general, demands massive numerical simulations (see [8], for example), and the second is based on effective equations (see [3], for example), deterministic differential equations whose solutions are the statistical means of (2.3).

It is well known that, for each realization $A(\omega)$ and $U_0(x,\omega)$, of A and $U_0(x)$, respectively, one has a deterministic problem that can be solved analytically using the characteristics' method. Therefore, if the probability of the realizations is known then the random solution $U(x,t,\omega)$, and its probability, can be found analytically.

On the other hand, if we have precise information about the velocity we may consider a mixed deterministic-random version for (2.2):

$$\begin{cases} \dfrac{dx}{dt} = a, \quad x(0) = x_0, \\ \dfrac{d[U(x(t),t)]}{dt} = 0, \quad U(x,0) = U_0(x). \end{cases} \tag{2.4}$$

This mixed formulation gives the characteristic straight lines $x(t) = x_0 + at$ and a random ordinary differential equation along these straight lines. The formulation (2.4) is convenient to our future arguments because, for each realization $U_0(x,\omega)$ of $U_0(x)$, the random function $(x,t) \mapsto U(x,t,\omega) = U_0(x - at, \omega)$ solves (2.4). This means that for precise values of the velocity the random initial conditions are "transported" along the straight lines.

In this paper, we use (2.4) to find the random Riemann solution to (2.3). The procedure and the theoretical consequences are presented in Section 2.2. In Section 2.3 we assess our results by comparing them with the Monte Carlo method.

2.2 The Riemann problem

In this section we study the random Riemann initial value problem:

$$\begin{cases} \dfrac{dX}{dt} = A, \quad X(0) = x_0, \\ \dfrac{d[U(X,t)]}{dt} = 0, \qquad U(x,0) = \begin{cases} U_0^+, & x > 0, \\ U_0^-, & x < 0, \end{cases} \end{cases} \tag{2.5}$$

where A, U_0^- and U_0^+ are random variables. We assume the statistical independence of A and both U_0^- and U_0^+, and that their cumulative probability functions, F_A and $F_{U^-U^+}$, are known.

In our approach we focus on partial realizations in (2.5), i.e., we consider only $A(\omega)$, letting the data U_0^- and U_0^+ out of the realizations. This kind of decoupling of the system (2.5) allows us to use the solution of (2.4). To simplify, let us consider A continuously varying in some interval $[a_m, a_M]$, $a_m < a_M$.

We recall that each realization $A(\omega)$ yields the random function $(x,t) \mapsto U_0(x - A(\omega)t)$, i.e., the initial condition at $x_0 = x - A(\omega)t$. Also, as illustrated in Figure 2.1, for a fixed $(\bar{x}, \bar{t}\,)$ we have $\bar{x} - a_M\bar{t} \le x_0 \le \bar{x} - a_m\bar{t}$. Hence, the solution at $(\bar{x}, \bar{t}\,)$ depends upon the initial data in the interval $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$. As shown in Figure 2.1, this interval is determined by two characteristics $x - a_M t =$ constant and $x - a_m t =$ constant, both

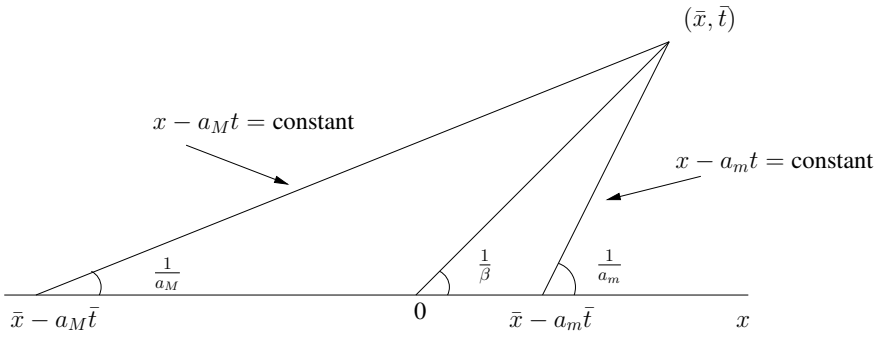

Figure 2.1: Interval of dependence

passing through $(\bar{x}, \bar{t}\,)$. From now on the interval $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$ will be referred to as the *interval of dependence* of the point $(\bar{x}, \bar{t}\,)$.

To separate the contributions of the left state, U_0^-, and right state, U_0^+, to the solution at $(\bar{x}, \bar{t}\,)$, we shall call $\beta = \bar{x}/\bar{t}$ and define the following disjoint subsets of $[a_m, a_M]$:

$$M^- = \{a;\ x_a = \bar{x} - a\bar{t} < 0\} \quad \text{and} \quad M^+ = \{a;\ x_a = \bar{x} - a\bar{t} > 0\}\,.$$

Comparing the slopes of the characteristics (see Figure 2.1), we can rewrite these sets as

$$M^- = \left\{a;\ \frac{1}{a_M} \le \frac{1}{a} < \frac{1}{\beta}\right\} = \{a;\ \beta < a \le a_M\}$$

and

$$M^+ = \left\{a;\ \frac{1}{\beta} < \frac{1}{a} \le \frac{1}{a_m}\right\} = \{a;\ a_m \le a < \beta\}\,.$$

Thus, the probability of occurrence of the sets M^+ and M^- can be calculated using the cumulative probability function of the velocity:

$$P(M^+) = F_A(\beta) = \theta \quad \text{and} \quad P(M^-) = 1 - F_A(\beta) = 1 - \theta. \tag{2.6}$$

The solution to (2.5) is given by the following result:

Proposition 2.1. *Let $(\bar{x}, \bar{t}\,)$, $\bar{t} > 0$, be an arbitrary point and $\beta = \bar{x}/\bar{t}$. The solution to (2.5) at $(\bar{x}, \bar{t}\,)$ is the random variable*

$$U(\bar{x}, \bar{t}\,) = (1 - X)U_0^- + XU_0^+ = U_0^- + X\left(U_0^+ - U_0^-\right), \tag{2.7}$$

where X is the Bernoulli random variable with $P(X = 0) = 1 - F_A(\beta)$ and $P(X = 1) = F_A(\beta)$.

Proof. To prove this proposition we use the characteristics $x - a_m t = 0$ and $x - a_M t = 0$ to divide the semi-plane $t \geq 0$ into three regions, $R_1 = \{(x,t);\ x < a_m t\}$, $R_2 = \{(x,t);\ a_m t \leq x \leq a_M t\}$, and $R_3 = \{(x,t);\ x > a_M t\}$, and we demonstrate (2.7) for each one of this regions.

If $(\bar{x}, \bar{t}\,) \in R_2$, we may divide the interval of dependence into two sub-intervals: $I^- = [\,\bar{x} - a_M\bar{t},\ 0\,)$ and $I^+ = [\,0,\ \bar{x} - a_m\bar{t}\,]$. In a realization such that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^-$, only the left state will contribute to the solution. On the other hand, we also conclude that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^-$ if and only if $A(\omega) \in M^-$, and therefore the probability of occurrence of I^- is equal to the probability of occurrence of M^-. Thus, from (2.6) it follows that $P(I^-) = P(M^-) = 1 - F_A(\beta)$. Otherwise, in a realization such that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^+$, the contribution will be due only to the right state and we use the same arguments above to conclude that $P(I^+) = P(M^+) = F_A(\beta)$. Finally, taking in account the probability of occurrence of U_0^- and U_0^+, the solution is obtained "weighting" their respective probabilities, i.e., $U(\bar{x}, \bar{t}\,) = (1-X)U_0^- + XU_0^+$, where X is the Bernoulli random variable with $P(X = 1) = F_A(\beta)$ and $P(X = 0) = 1 - F_A(\beta)$.

If $(\bar{x}, \bar{t}\,) \in R_1$ then $\bar{x} - a_m\bar{t} < 0$ and all the points of the interval of dependence, $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$, are negatives. Therefore, only the left state contributes to the solution, i.e., $U(\bar{x}, \bar{t}\,) = U_0^-$ with probability one. In this case the solution is (2.7) with $F_A(\beta) = 0$. On the other hand, if $(\bar{x}, \bar{t}\,) \in R_3$ only the right state contributes to the solution and we have $U(\bar{x}, \bar{t}\,) = U_0^+$ with probability one, i.e., (2.7) with $F_A(\beta) = 1$.

□

Corollary 2.1. *The solution of (2.5) is constant along the rays x/t = constant, i.e., the random solution is a similarity function.*

Proof. This result follows directly from (2.7) since if x/t = constant then $F_A\,(x/t)$ = constant.

□

Proposition 2.2. *If (x,t) is fixed, $n \in \mathbb{N}$, $n \geq 1$, and we assume the statistical independence of A and both U_0^- and U_0^+, then the nth moment of the random solution (2.7) is given by:*

$$\langle U^n(x,t)\rangle = \left\langle (U_0^-)^n \right\rangle + F_A\left(\frac{x}{t}\right)\left\{\left\langle (U_0^+)^n \right\rangle - \left\langle (U_0^-)^n \right\rangle\right\}. \tag{2.8}$$

Proof. From Proposition 2.1,

$$U(x,t) = U_0^- + X\left(U_0^+ - U_0^-\right) = (1-X)U_0^- + XU_0^+,$$

where $X = X(x,t)$ is the Bernoulli random variable:

$$X = \begin{cases} 1, & P\left(X=1\right) = F_A\left(\frac{x}{t}\right) = \theta \\ 0, & P\left(X=0\right) = 1 - F_A\left(\frac{x}{t}\right) = 1-\theta. \end{cases}$$

It is easy to see that $\langle X^j \rangle = F_A\left(\frac{x}{t}\right) = \theta$, for all $j = 1, 2, 3, ...$
To prove (2.8) we first need the following results:

- For all $n \geq 1$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0, \tag{2.9}$$

  where $\binom{n}{j}$ is the binomial coefficient.

- For $n \geq 1$ and $1 \leq j \leq n-1$,

$$\begin{aligned} &\left\langle (1-X)^{n-j}\, X^j \right\rangle = \\ &= \left\langle X^j \sum_{m=0}^{n-j} (-1)^m \binom{n-j}{m} X^m \right\rangle = \left\langle \sum_{m=0}^{n-j} (-1)^m \binom{n-j}{m} X^{m+j} \right\rangle = \\ &= \sum_{m=0}^{n-j} (-1)^m \binom{n-j}{m} \underbrace{\left\langle X^{m+j} \right\rangle}_{\theta} = \theta \underbrace{\sum_{m=0}^{n-j} (-1)^m \binom{n-j}{m}}_{\text{zero by (2.9)}} = 0. \end{aligned} \tag{2.10}$$

- For $n \geq 1$,

$$\begin{aligned} &\left\langle (1-X)^n \right\rangle = \left\langle \sum_{j=0}^{n} (-1)^j \binom{n}{j} X^j \right\rangle = \\ &= 1 + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \underbrace{\left\langle X^j \right\rangle}_{\theta} = 1 + \theta \underbrace{\sum_{j=1}^{n} (-1)^j \binom{n}{j}}_{-1 \; by \; (2.9)} = 1 - \theta. \end{aligned} \tag{2.11}$$

Now, assuming the independence of A and both U_0^- and U_0^+, we have:

$$\begin{aligned}\langle U^n(x,t)\rangle &= \left\langle \left[(1-X)U_0^- + XU_0^+\right]^n \right\rangle = \\ &= \left\langle \sum_{j=0}^{n} \binom{n}{j} (1-X)^{n-j} X^j \left(U_0^-\right)^{n-j} \left(U_0^+\right)^j \right\rangle = \\ &= \underbrace{\langle (1-X)^n \rangle}_{1-\theta \text{ by (2.11)}} \left\langle \left(U_0^-\right)^n \right\rangle + \underbrace{\langle X^n \rangle}_{\theta} \left\langle \left(U_0^+\right)^n \right\rangle + \\ &+ \sum_{j=1}^{n-1} \binom{n}{j} \underbrace{\left\langle (1-X)^{n-j} X^j \right\rangle}_{\text{zero by (2.10)}} \left\langle \left(U_0^-\right)^{n-j} \left(U_0^+\right)^j \right\rangle = \\ &= (1-\theta) \left\langle \left(U_0^-\right)^n \right\rangle + \theta \left\langle \left(U_0^+\right)^n \right\rangle .\end{aligned}$$

□

Corollary 2.2. *For a fixed (x,t), $\theta = F_A(x/t)$, and considering the independence of A and both U_0^- and U_0^+, the mean of the solution (2.5) is*

$$\langle U(x,t)\rangle = (1-\theta)\langle U_0^-\rangle + \theta\langle U_0^+\rangle = \langle U_0^-\rangle + \theta\left[\langle U_0^+\rangle - \langle U_0^-\rangle\right]. \tag{2.12}$$

Proof. The expression (2.12) follows from (2.8) with $n = 1$.

□

Proposition 2.3. *Let (x_1,t_1) and (x_2,t_2) be fixed. Define $\beta_j = x_j/t_j$, $\theta_j = F_A(\beta_j)$, and $H(\beta_j) = U(x_j,t_j)$ as in (2.7) ($j = 1,2$). Also, consider the statistical independence of A and both U_0^- and U_0^+. If $\beta_1 \neq \beta_2$ then the covariance between $H(\beta_1)$ and $H(\beta_2)$ is*

$$\begin{aligned}Cov[H(\beta_1), H(\beta_2)] = (1-\theta_1)(1-\theta_2)Var[U_0^-] + \theta_1\theta_2 Var[U_0^+] + \\ + \{\theta_1(1-\theta_2) + \theta_2(1-\theta_1)\} Cov[U_0^-, U_0^+].\end{aligned} \tag{2.13}$$

On the other hand, if $\beta_1 = \beta_2$ then the variance is given by

$$\begin{aligned}Var[H(\beta_1)] = Var[U(x_1,t_1)] = Var[U_0^-] + \theta_1\left\{Var[U_0^+] - Var[U_0^-]\right\} + \\ + \theta_1(1-\theta_1)[\langle U_0^+\rangle - \langle U_0^-\rangle]^2.\end{aligned} \tag{2.14}$$

Proof. At first, from (2.7) and (2.12) we can observe that

$$\begin{aligned}\langle H(\beta_1)\rangle\langle H(\beta_2)\rangle &= \left\{\langle U_0^-\rangle + \theta_1[\langle U_0^+\rangle - \langle U_0^-\rangle]\right\}\left\{\langle U_0^-\rangle + \theta_2[\langle U_0^+\rangle - \langle U_0^-\rangle]\right\} = \\ &= \langle U_0^-\rangle^2 + (\theta_1+\theta_2)\langle U_0^-\rangle[\langle U_0^+\rangle - \langle U_0^-\rangle] + \theta_1\theta_2[\langle U_0^+\rangle - \langle U_0^-\rangle]^2,\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\langle H(\beta_1)H(\beta_2)\rangle &= \left\langle \left\{U_0^- + X(\beta_1)[U_0^+ - U_0^-]\right\}\left\{U_0^- + X(\beta_2)[U_0^+ - U_0^-]\right\}\right\rangle = \\ &= \langle (U_0^-)^2\rangle + \left\langle \left\{X(\beta_1) + X(\beta_2)\right\}\left\{U_0^-[U_0^+ - U_0^-]\right\}\right\rangle + \\ &+ \langle X(\beta_1)X(\beta_2)\rangle\langle [U_0^+ - U_0^-]^2\rangle,\end{aligned}$$

where $X(\beta_j)$ (j=1,2) is the Bernoulli random variable defined in (2.7).
Since

$$\begin{aligned}Var[U_0^+ - U_0^-] &= Var[U_0^-] + Var[U_0^+] - 2\mathrm{Cov}[U_0^-, U_0^+] = \\ &= \langle [U_0^+ - U_0^-]^2\rangle - [\langle U_0^+\rangle - \langle U_0^-\rangle]^2,\end{aligned}$$

we have

$$\begin{aligned}\langle H(\beta_1)H(\beta_2)\rangle &= \langle (U_0^-)^2\rangle + (\theta_1+\theta_2)\langle U_0^-[U_0^+ - U_0^-]\rangle + \langle X(\beta_1)X(\beta_2)\rangle[\langle U_0^+\rangle - \langle U_0^-\rangle]^2 + \\ &+ \langle X(\beta_1)X(\beta_2)\rangle\left\{Var[U_0^-] + Var[U_0^+] - 2\mathrm{Cov}[U_0^-, U_0^+]\right\}. \qquad (2.16)\end{aligned}$$

From (2.15) and (2.16) it follows that

$$\begin{aligned}\mathrm{Cov}[H(\beta_1), H(\beta_2)] &= \langle H(\beta_1)H(\beta_2)\rangle - \langle H(\beta_1)\rangle\langle H(\beta_2)\rangle = \\ &= Var[U_0^-] + (\theta_1+\theta_2)\left\{\langle U_0^-[U_0^+ - U_0^-]\rangle - \langle U_0^-\rangle[\langle U_0^+\rangle - \langle U_0^-\rangle]\right\} + \\ &+ \langle X(\beta_1)X(\beta_2)\rangle\left\{Var[U_0^-] + Var[U_0^+] - 2\mathrm{Cov}[U_0^-, U_0^+]\right\} + \\ &+ \left\{\langle X(\beta_1)X(\beta_2)\rangle - \theta_1\theta_2\right\}[\langle U_0^+\rangle - \langle U_0^-\rangle]^2 = \\ &= Var[U_0^-] + (\theta_1+\theta_2)\left\{\mathrm{Cov}[U_0^-, U_0^+] - Var[U_0^-]\right\} + \\ &+ \langle X(\beta_1)X(\beta_2)\rangle\left\{Var[U_0^-] + Var[U_0^+] - 2\mathrm{Cov}[U_0^-, U_0^+]\right\} + \\ &+ \left\{\langle X(\beta_1)X(\beta_2)\rangle - \theta_1\theta_2\right\}[\langle U_0^+\rangle - \langle U_0^-\rangle]^2,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mathrm{Cov}[H(\beta_1), H(\beta_2)] &= \left\{1 - \theta_1 - \theta_2 + \langle X(\beta_1)X(\beta_2)\rangle\right\}Var[U_0^-] + \langle X(\beta_1)X(\beta_2)\rangle Var[U_0^+] + \\ &+ \left\{\theta_1 + \theta_2 - 2\langle X(\beta_1)X(\beta_2)\rangle\right\}\mathrm{Cov}[U_0^-, U_0^+] + \\ &+ \left\{\langle X(\beta_1)X(\beta_2)\rangle - \theta_1\theta_2\right\}[\langle U_0^+\rangle - \langle U_0^-\rangle]^2. \qquad (2.17)\end{aligned}$$

Now we can observe that

$$\langle X(\beta_1)X(\beta_2)\rangle = \begin{cases}\theta_1\theta_2 & \text{if } \beta_1 \neq \beta_2, \\ \theta_1 & \text{if } \beta_1 = \beta_2.\end{cases} \qquad (2.18)$$

Therefore, if $\beta_1 \neq \beta_2$ we have from (2.17) and (2.18):

$$\begin{aligned}\mathrm{Cov}[H(\beta_1), H(\beta_2)] &= \left\{1 - \theta_1 - \theta_2 + \theta_1\theta_2\rangle\right\}Var[U_0^-] + \theta_1\theta_2 Var[U_0^+] + \\ &+ \left\{\theta_1 + \theta_2 - 2\theta_1\theta_2\right\}\mathrm{Cov}[U_0^-, U_0^+] = \\ &= (1-\theta_1)(1-\theta_2)Var[U_0^-] + \theta_1\theta_2 Var[U_0^+] + \\ &+ \left\{\theta_1(1-\theta_2) + \theta_2(1-\theta_1)\right\}\mathrm{Cov}[U_0^-, U_0^+].\end{aligned}$$

Otherwise, if $\beta_1 = \beta_2$ it follows that

$$\begin{aligned} Var[H(\beta_1)] &= [1-\theta_1]Var[U_0^-] + \theta_1 Var[U_0^+] + \theta_1(1-\theta_1)[\langle U_0^+\rangle - \langle U_0^-\rangle]^2 \\ &= Var[U_0^-] + \theta_1\left\{Var[U_0^+] - Var[U_0^-]\right\} + \theta_1(1-\theta_1)[\langle U_0^+\rangle - \langle U_0^-\rangle]^2. \end{aligned}$$

□

As an illustration we plot in Figure 2.2 the mean of the solution at $t = T$, $\langle U(x,T)\rangle$, using (2.12). We can observe a diffusive behavior in the interval $[a_m T,\ a_M T]$, called by some authors the mixing zone. In this mixing zone, $\langle U(x,T)\rangle$ is the mean of the left state added to the product of the cumulative probability function of the velocity and the jump between the means of right and left states. This illustration also shows that the shape of the cumulative probability function of A controls the mixing zone: only symmetric density functions will produce antisymmetrical mixing zones. Our computational tests will make clear this remark (see Figures 2.3-2.4 (symmetric) and Figures 2.5-2.6 (nonsymmetric)).

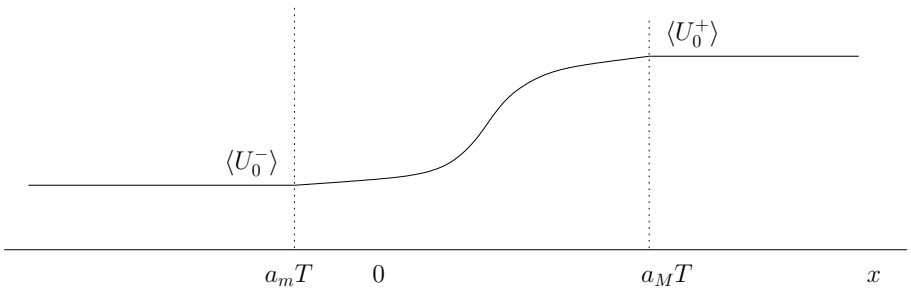

Figure 2.2: $\langle U(x,T)\rangle$, fixed T

The length of this mixing zone is studied by some authors (see [1, 3, 11, 12], for example) using the effective equation methodology. For instance, the effective equation for the linear transport with random velocity, $\nu(x)$, is

$$\frac{\partial\langle c\rangle}{\partial t} + \langle\nu(x)\rangle\,\frac{\partial\langle c\rangle}{\partial x} - D(t)\,\frac{\partial^2\langle c\rangle}{\partial x^2} = 0,$$

with the dissipation coefficient given by

$$D(t) = \int_0^t \langle\delta\nu(x-st)\delta\nu(x)\rangle ds.$$

If the random velocity is constant then

$$D(t) = \int_0^t \langle\delta\nu^2\rangle ds = \sigma^2 t,$$

where σ is the standard deviation of ν.

We shall compare a particular solution of the effective equation methodology with our expression for the mean, (2.12). If we take the initial condition

$$\langle c(x,0)\rangle = U_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases}$$

for both the effective equation and problem (2.5), we can show that the analytical expressions for the mean are:

(i) using the effective equation:

$$\langle c(x,t)\rangle = \frac{1}{2}\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x-\langle\nu\rangle t}{l(t)}} e^{-\omega^2}\, d\omega\right\},$$

where $l(t) = 2\left[\int_0^t D(\omega)\, d\omega\right]^{\frac{1}{2}}$ is the mixing length;

(ii) using (2.12) with a normally distributed random velocity, $A \sim N(\langle\nu\rangle, \sigma)$:

$$\langle U(x,t)\rangle = \frac{1}{2}\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x-\langle\nu\rangle t}{\sqrt{2}\sigma t}} e^{-\omega^2}\, d\omega\right\}.$$

Comparing these expressions, they will be equal only if the mixing length satisfies $l(t) = \sqrt{2}\sigma t$ or, equivalently, if the diffusion coefficient of the effective equation is $D(t) = \sigma^2 t$, i.e., the same dissipation coefficient for the constant velocity case.

2.3 Monte Carlo simulations

To assess our results we compare the expressions for the mean, variance and 3rd central moment with the Monte Carlo method. We use suites of realizations of A, U_0^- and U_0^+ considering: the independence of A and both U_0^- and U_0^+; U_0^- and U_0^+ have a bivariate normal distribution with $\langle U_0^-\rangle = 1$, $\langle U_0^+\rangle = 0$, $Var\left[U_0^-\right] = 0.16$, $Var\left[U_0^+\right] = 0.25$ and $Cov\left(U_0^-, U_0^+\right) = 0.12$. We plot the results in $T = 0.4$ and $T = 0.8$. In order to investigate the influence of the velocity randomness we use two models: (i) A is normally distributed, $A \sim N(1, 0.6)$, in Figures 2.3 and 2.4; (ii) A is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.15)$, in Figures 2.5 and 2.6. All Monte Carlo simulations were performed with 1 500 realizations and recalling that the solution to (2.5), at (x,t), for a single realization $\left(A(\omega),\ U_0^-(\omega),\ U_0^+(\omega)\right)$ of $\left(A,\ U_0^-,\ U_0^+\right)$, is

$$U(x,t) = U_0(x - A(\omega)t) = \begin{cases} U_0^-(\omega), & x - A(\omega)t < 0, \\ U_0^+(\omega), & x - A(\omega)t > 0. \end{cases}$$

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

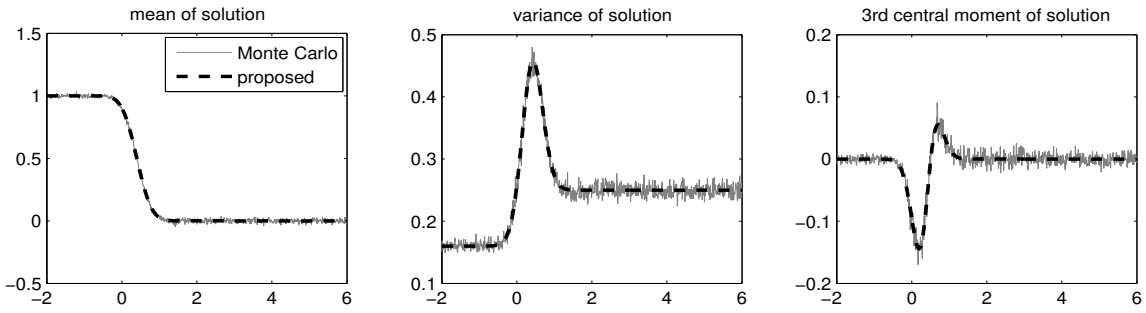

Figure 2.3: A is normal, $A \sim N(1, 0.6)$, and $T = 0.4$.

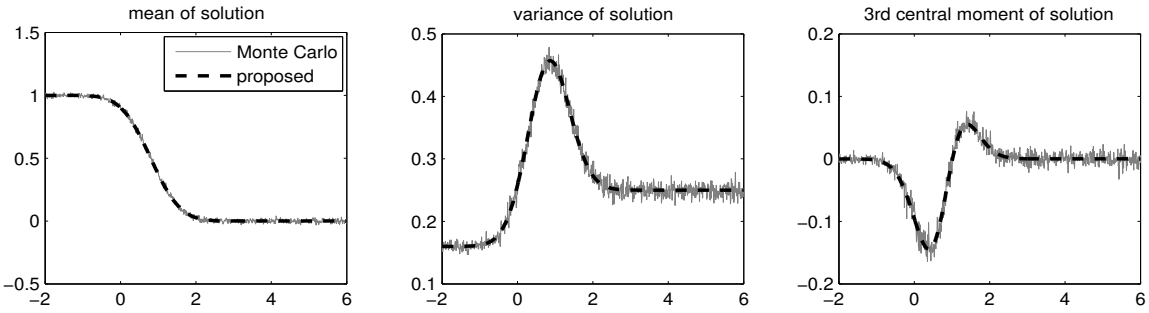

Figure 2.4: A is normal, $A \sim N(1, 0.6)$, and $T = 0.8$.

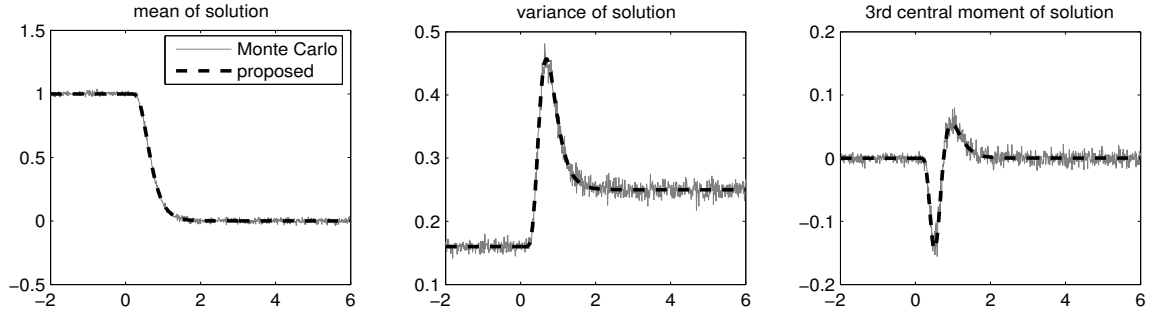

Figure 2.5: A is lognormal, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.15)$, and $T = 0.4$.

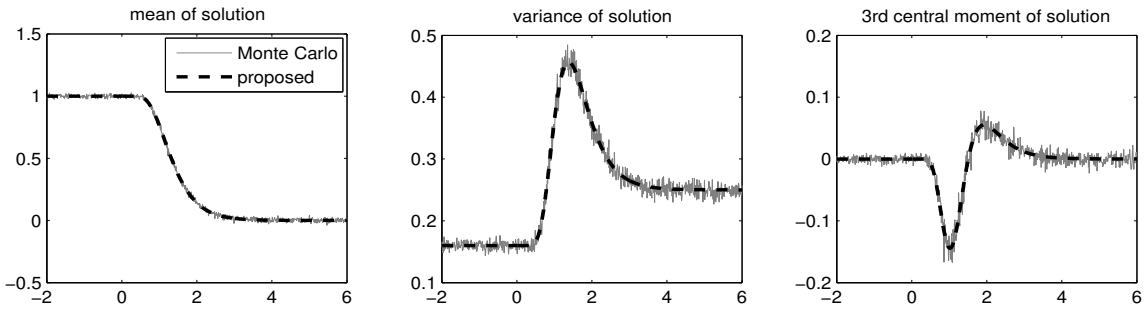

Figure 2.6: A is lognormal, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.15)$, and $T = 0.8$.

2.4 Concluding remarks

In this article we present an expression to the solution of the random Riemann problem for the linear transport equation with random velocity. As far as we know, this approach does not appear in the literature and we believe that it can be useful in the development of numerical procedures for more general random partial differential equations. Expression (2.7) shows us that if the statistics of the velocity is known then the local behavior of the solution is independent of the physical mechanisms governing the process. The procedure also shows agreement with the effective equation methodology when the velocity is a normal random variable; however, it seems to us that the random expression to the solution yields more information about the process.

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Capítulo 3

A finite volume method for the mean of the solution of the random linear transport equation

Abstract

We present a numerical scheme, based on Godunov's method (REA algorithm), for the statistical mean of the solution of the one-dimensional random linear transport equation, with homogeneous random velocity and random initial condition. Numerical examples are considered to validate our method.

Keyword: random linear transport equation, finite volume schemes, Riemann problem, statistical mean, Godunov's method (REA algorithm).

3.1 Introduction

Conservation laws are differential equations arising from physical principles of the conservation of mass, energy or momentum. The simplest of these equations is the one-dimensional advective equation and its solution plays a role in more complex problems such as the numerical solution for nonlinear conservation law. This linear initial value problem can, for instance, model the concentration, or density, of a chemical substance transported by a one-dimensional fluid that flows with a known velocity. In the deterministic case, we want to find $q(x,t)$ such that:

$$\begin{cases} q_t + a(x)q_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ q(x,0) = q_0(x). \end{cases} \tag{3.1}$$

It is well known that the solution to (3.1) is the initial condition transported along the characteristic curves.

The complexity of natural phenomena compels us to study partial differential equations with random data. For example, (3.1) may model the flux of a two phase equal viscosity miscible fluid in a porous media. The total velocity is obtained from Darcy's law and it depends on the geology of the porous media. Thus, the external velocity is defined by given statistics. Also, the prediction of the initial state of the process is obtained from data acquired from a small number of exploratory wells using geological methods.

In this work, we are concerned with the numerical solution of the random version of the problem (3.1), i.e., the stochastic transport equation,

$$Q_t(x,t) + AQ_x(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{3.2}$$

with a homogeneous random transport velocity, A, and stochastic initial condition, $Q(x,0) = Q_0(x)$.

A mathematical basis for the solution of stochastic, or random, partial differential equations has not been complete yet. Besides the well-developed theoretical methods such as Ito integrals, Martingales and Wiener measure [7, 8], two types of methods are normally used in the construction of solutions for random partial differential equations. The first method is based on the Monte Carlo method, which in general demands massive numerical simulations using high resolution methods (see [6]), and the second is based on the effective equations (see [2]), the deterministic differential equations whose solutions are the statistical means of (3.2).

The solution of (3.2) is a random function. For a particular case when the initial condition is given by

$$Q(x,0) = \begin{cases} Q_0^+, & x > 0, \\ Q_0^-, & x < 0, \end{cases} \tag{3.3}$$

with Q_0^- and Q_0^+ random variables, we have shown in [1] that the solution of the Riemann problem (3.2)-(3.3) is

$$Q(x,t) = Q_0^- + X\left(Q_0^+ - Q_0^-\right), \tag{3.4}$$

where X is the Bernoulli random variable with $P(X=0) = 1 - F_A(x/t)$ and $P(X=1) = F_A(x/t)$; here $F_A(x)$ is the cumulative probability function of the random variable A.

Also, according to [1], considering the independence of A and both Q_0^- and Q_0^+, the statistical mean of the solution of the Riemann problem (3.2)-(3.3) for a fixed (x,t) is

$$\langle Q(x,t)\rangle = \langle Q_0^-\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right]. \tag{3.5}$$

Besides the formal verification of the explicit expression (3.4), in [1] we compare (3.5) with the mean given by an effective equation to (3.2) and also show that the Monte Carlo

method agree quite well with (3.5). We can observe that (3.5) gives the mean, $\langle Q(x,t)\rangle$, without considering either the effective equation or the Monte Carlo method.

In this paper, we use these results to design a numerical scheme to approximate the statistical mean for (3.2) with a more general initial condition. The method is based on the Riemann problems solution, Godunov's ideas, and finite volume methods – widely used in high-resolution methods for deterministic conservation laws (see [5], Ch. 4).

The outline of this paper is as follows. In Section 3.2 we deduce the explicit numerical scheme using the ideas of Godunov's reconstruct-evolve-average algorithm. The analysis of stability and convergence of the method is presented in Section 3.3. Finally, in Section 3.4 we present and compare some numerical examples.

3.2 The numerical scheme

In this section, we present the numerical method for the mean of the solution of (3.2). Initially, we discretize both space and time assuming uniform mesh spacing with Δx and Δt, respectively. We denote the spatial and time grid points by $x_j = j\Delta x$ and $t_n = n\Delta t$, respectively. In a context of finite volume methods, denoting the jth grid cell by $\mathrm{C}_j = (x_{j-1/2},\ x_{j+1/2})$, where $x_{j\pm1/2} = x_j \pm \Delta x/2$, the value denoted by Q_j^n approximates the average value of the random function $Q(x,t_n)$ over the jth grid cell:

$$Q_j^n \approx \frac{1}{\Delta x}\int_{\mathrm{C}_j} Q(x,t_n)dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} Q(x,t_n)dx. \tag{3.6}$$

We follow the basic ideas of REA algorithm, **R**econstruct-**E**volve-**A**verage, a finite volume algorithm originally proposed by [3] as a method for solving the nonlinear Euler equations of gas dynamics.

Assuming that the cell averages at time t_n, Q_j^n, are known, we summarize the REA algorithm (see [5], Ch. 4) in three steps:

Step 1. Reconstruct a piecewise polynomial function $\tilde{Q}(x,t_n)$, defined for all x, from the cell averages Q_j^n. In our case we use the piecewise constant function with Q_j^n in the jth grid cell, i.e., $\tilde{Q}(x,t_n) = Q_j^n$, for $x \in \mathrm{C}_j$.

Step 2. Evolve the equation exactly, or approximately, with this initial data to obtain $\tilde{Q}(x,t_{n+1})$ a time Δt later. In our case we can evolve exactly using (3.4).

Step 3. Average $\tilde{Q}(x,t_{n+1})$ over each grid cell to obtain the new cell averages, i.e.,

$$Q_j^{n+1} = \frac{1}{\Delta x}\int_{\mathrm{C}_j} \tilde{Q}(x,t_{n+1})dx.$$

The piecewise constant function, step 1, defines a set of Riemann problems in each $x = x_{j-1/2}$: differential equation (3.2) with the initial condition

$$Q(x, t_n) = \begin{cases} Q_{j-1}^n, & x < x_{j-1/2}, \\ Q_j^n, & x > x_{j-1/2}. \end{cases} \tag{3.7}$$

Therefore, we may use (3.4) to solve each Riemann problem:

$$Q(x, t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right], \tag{3.8}$$

where, for x fixed, $X(x)$ is the Bernoulli random variable:

$$X(x) = \begin{cases} 1, & P\left(X(x) = 1\right) = F_A(x), \\ 0, & P\left(X(x) = 0\right) = 1 - F_A(x). \end{cases} \tag{3.9}$$

As in the deterministic case the solution at $t_{n+1/2}$, $\tilde{Q}(x, t_{n+1/2})$, can be constructed by piecing together the Riemann solutions, provided that the half time step $\Delta t/2$ is short enough such that adjacent Riemann problems have not started to interact yet. This requires that Δx and Δt must be chosen satisfying:

$$Q(x_{j-1},\ t_{n+1/2}) \approx Q_{j-1}^n \quad \text{and} \quad Q(x_j,\ t_{n+1/2}) \approx Q_j^n,$$

where the symbol " $\approx$ " means "*sufficiently near to*".

Substituting the above conditions into (3.8) we must have $X\left(-\Delta x/\Delta t\right) = 0$ and $X\left(\Delta x/\Delta t\right) = 1$ both with probability sufficiently near to 1. From (3.9) this means the following conditions:

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{3.10}$$

Remark 3.1. *We may regard (3.10) as a kind of CFL condition for the method: the interval* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *must contain the effective support of the density probability function of A. The word effective support means that outside* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *the probability of A is sufficiently near to zero, i.e., it can be disregarded. The existence of the effective support is ensured by Chebyshev's inequality: for any* $k > 0$, $P\{|A - \langle A\rangle| \geq k\sigma_A\} \leq 1/k^2$, *where* σ_A *is the standard variation of A.*

Under hypothesis (3.10) we may finish the step 2 taking

$$\tilde{Q}(x, t_{n+1/2}) = \sum_j Q(x, t_{n+1/2})\ \mathbf{1}_{[x_{j-1},\, x_j]}, \tag{3.11}$$

where $\mathbf{1}_{[x_{j-1},\, x_j]}$ is the characteristic function of $[x_{j-1},\ x_j]$.

In step 3 of REA algorithm we use (3.11) to calculate $Q_{j-1/2}^{n+1/2}$:

$$\begin{aligned} Q_{j-1/2}^{n+1/2} &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \tilde{Q}(x,t_{n+1/2})dx \\ &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \left\{ Q_{j-1}^n + X\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right]\right\} dx \\ &= Q_{j-1}^n + \frac{1}{\Delta x}\left\{\int_{x_{j-1}}^{x_j} X\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right) dx\right\}\left[Q_j^n - Q_{j-1}^n\right] \\ &= Q_{j-1}^n + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)\, dx\right\}\left[Q_j^n - Q_{j-1}^n\right]. \end{aligned} \tag{3.12}$$

The cell averages, $Q_{j-1/2}^{n+1/2}$, define new Riemann problems at x_j. We repeat the procedure above to obtain the solution in C_j at t_{n+1}:

$$\begin{aligned} Q_j^{n+1} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \left\{ Q_{j-1/2}^{n+1/2} + X\left(\frac{x-x_j}{\Delta t/2}\right)\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]\right\} dx = \\ &= Q_{j-1/2}^{n+1/2} + \frac{1}{\Delta x}\left\{\int_{x_{j-1/2}}^{x_{j+1/2}} X\left(\frac{x-x_j}{\Delta t/2}\right) dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right] = \\ &= Q_{j-1/2}^{n+1/2} + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]. \end{aligned} \tag{3.13}$$

Lemma 3.1. *Let* $Y = \int_{-\xi}^{\xi} X(x)dx$ *be a random variable with* $\xi > 0$ *and* $X(x)$ *the random field defined in (3.9). Then* $P\{Y = \langle Y\rangle\} = 1$.

Proof. Since $\langle Y\rangle = \left\langle \int_{-\xi}^{\xi} X(x)dx \right\rangle = \int_{-\xi}^{\xi} \langle X(x)\rangle dx = \int_{-\xi}^{\xi} F_A(x)dx$, we have

$$\begin{aligned} \langle Y^2\rangle &= \left\langle \left[\int_{-\xi}^{\xi} X(x)dx\right]^2 \right\rangle = \left\langle \int_{-\xi}^{\xi}\int_{-\xi}^{\xi} X(x_1)X(x_2)dx_1dx_2 \right\rangle \\ &= \int_{-\xi}^{\xi}\int_{-\xi}^{\xi} \langle X(x_1)X(x_2)\rangle dx_1dx_2 = \int_{-\xi}^{\xi}\int_{-\xi}^{\xi} \langle X(x_1)\rangle\langle X(x_2)\rangle dx_1dx_2 \\ &= \int_{-\xi}^{\xi}\int_{-\xi}^{\xi} F_A(x_1)F_A(x_2)dx_1dx_2 = \left[\int_{-\xi}^{\xi} F_A(x)dx\right]^2 = \langle Y\rangle^2. \end{aligned}$$

Therefore, $Var(Y) = \langle Y^2\rangle - \langle Y\rangle^2 = 0$ and thus $P\{Y = \langle Y\rangle\} = 1$.

□

From this result we can conclude that

$$\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx = \left\langle \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx \right\rangle = \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} \langle X(x)\rangle dx = \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx,$$

and thus rewrite (3.12)–(3.13) as

$$Q_{j-1/2}^{n+1/2} = Q_{j-1}^n + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)\ dx\right\}\left[Q_j^n - Q_{j-1}^n\right] \tag{3.14}$$

and

$$Q_j^{n+1} = Q_{j-1/2}^{n+1/2} + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]. \tag{3.15}$$

Lemma 3.2. *Let A be a random variable and $[-\xi, \xi]$ an effective support of the density probability function, f_A, of A. Then*

$$\int_{-\xi}^{\xi} F_A(x)\,dx \ \approx\ \xi - \langle A\rangle. \tag{3.16}$$

Proof. Using the hypothesis and integration by parts in the definition of the statistical mean of A we have:

$$\langle A\rangle = \int_{-\infty}^{\infty} x\ f_A(x)\,dx \approx \int_{-\xi}^{\xi} x\ f_A(x)\,dx = x\ F_A(x)|_{-\xi}^{\xi} - \int_{-\xi}^{\xi} F_A(x)\,dx.$$

Since $F_A(-\xi) \approx 0$ and $F_A(\xi) \approx 1$ we obtain the result.

□

Using (3.16) as an approximation of the integral in (3.14) and (3.15), and denoting $\lambda = (\Delta t\langle A\rangle)/\Delta x$, we define the two-step numerical scheme:

$$Q_{j-1/2}^{n+1/2} = \frac{1}{2}\left[Q_{j-1}^n + Q_j^n\right] - \frac{\lambda}{2}\left[Q_j^n - Q_{j-1}^n\right]$$

and

$$Q_j^{n+1} = \frac{1}{2}\left[Q_{j-1/2}^{n+1/2} + Q_{j+1/2}^{n+1/2}\right] - \frac{\lambda}{2}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right].$$

Joining these expressions we can summarize the two-step scheme above in the explicit method:

$$Q_j^{n+1} = Q_j^n - \frac{\lambda}{2}\left[Q_{j+1}^n - Q_{j-1}^n\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right]. \tag{3.17}$$

Taking the statistical mean in (3.17) we obtain the explicit scheme for the mean of the solution to (3.2):

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\lambda}{2}\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right], \tag{3.18}$$

where $\lambda = (\Delta t\langle A\rangle)/\Delta x$.

Remark 3.2. *The numerical method (3.18) is conservative, in the sense that it can be rewritten as*

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right],$$

where $F_{j-1/2}^n = (1/2)\langle A\rangle[\langle Q_{j-1}^n\rangle + \langle Q_j^n\rangle] - (1/4)\langle A\rangle\,(1\lambda+\lambda)\,[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle]$ *is an approximation to the average flux at* $x = x_{j-1/2}$.

3.3 Numerical analysis of the scheme

In this section, we analyze the convergence of the method (3.18) and show its stability and consistency with an advective-diffusive equation.

Proposition 3.1. *For* $(\Delta x^2/\Delta t) = \nu$ *fixed the numerical scheme (3.18) is an* $\mathcal{O}(\Delta x^2)$ *approximation for* $u(x,t)$*, solution of the deterministic differential equation*

$$u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx}. \tag{3.19}$$

Proof. Let $u(x,t)$ be a smooth function such that $u(x_j,t_n) = \langle Q_j^n\rangle$. Thus, by (3.18) we have

$$\begin{aligned} u(x,t+\Delta t) = u(x,t) - \frac{\Delta t}{2\Delta x}\langle A\rangle\left[u(x+\Delta x,t) - u(x-\Delta x,t)\right] + \\ + \frac{1}{4}\left[1 + \left(\frac{\Delta t}{\Delta x}\langle A\rangle\right)^2\right]\left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\right], \end{aligned}$$

and using Taylor's expansion it follows

$$\begin{aligned} &\left\{u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + ...\right\} + \langle A\rangle\left\{u_x + \frac{\Delta x^2}{6}u_{xxx} + ...\right\} = \\ &= \frac{1}{4}\left(\frac{\Delta x^2}{\Delta t} + \Delta t\langle A\rangle^2\right)\left\{u_{xx} + \frac{\Delta x^2}{2}u_{xxxx} + ...\right\}. \end{aligned}$$

Since $(\Delta x^2/\Delta t) = \nu$ is fixed, we have $\Delta t = (\Delta x^2/\nu) = \mathcal{O}(\Delta x^2)$. Thus, grouping the terms of the same order we arrive at the expression:

$$u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx} + \mathcal{O}(\Delta x^2).$$

□

Proposition 3.2. *The numerical method (3.18) is stable under the conditions (3.10) and*

$$\frac{\Delta t}{\Delta x}|\langle A\rangle| \leq 1. \tag{3.20}$$

Proof. Using the von Neumann analysis (see [9]) it follows that the amplification factor associated to (3.18) is

$$\begin{aligned} g(\theta) &= 1 - \frac{\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{1}{4}(1+\lambda^2)\left(e^{i\theta} - 2 + e^{-i\theta}\right) \\ &= 1 + \frac{1}{2}(1+\lambda^2)(\cos\theta - 1) - i\ \lambda \sin\theta \\ &= 1 - (1+\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right) - i\ 2\lambda \sin\left(\frac{\theta}{2}\right)\ \cos\left(\frac{\theta}{2}\right), \end{aligned}$$

for $\theta \in [-\pi, \pi]$.
The magnitude of the amplification factor $g(\theta)$ is given by,

$$\begin{aligned} |g(\theta)|^2 &= \left\{1 - (1+\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right)\right\}^2 + 4\lambda^2 \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\ &= 1 - \left[2(1+\lambda^2) - 4\lambda^2\right]\ \sin^2\left(\frac{\theta}{2}\right) + \left[(1+\lambda^2)^2 - 4\lambda^2\right]\ \sin^4\left(\frac{\theta}{2}\right) \\ &= 1 - 2(1-\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right) + (1-\lambda^2)^2\ \sin^4\left(\frac{\theta}{2}\right) \\ &= \left[1 - (1-\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right)\right]^2, \quad \theta \in [-\pi,\ \pi]. \end{aligned}$$

Therefore, if $|\lambda| \leq 1$ we have $|g(\theta)| \leq 1$, for all $\theta \in [-\pi, \pi]$.

□

Remark 3.3. *We can show that the conditions in (3.10) are sufficient for (3.20). In fact, using Lemma 3.2:*

$$0 \leq \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx \approx \frac{\Delta x}{\Delta t} - \langle A \rangle \leq \frac{2\Delta x}{\Delta t}.$$

Thus $-\Delta x/\Delta t \leq \langle A \rangle \leq \Delta x/\Delta t$ *or* $|\langle A \rangle| \leq \Delta x/\Delta t$, *i.e.,* $\Delta t\,|\langle A \rangle|\,/\Delta x \leq 1$. *With this remark we conclude that the conditions (3.10) ensure the stability of the proposed scheme.*

Remark 3.4. *Under the stability conditions (3.10) and the consistency (Proposition 3.1) we have the convergence of the means calculated by (3.18) to the solution of equation (3.19).*

Proposition 3.3. *Under the conditions (3.10), the numerical scheme (3.18) is total variation diminishing (TVD), i.e.,* $TV(Q^{n+1}) \leq TV(Q^n)$.

Proof. We observe that (3.18) can be rewritten as

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \underbrace{\frac{(1+\lambda)^2}{4}}_{\alpha}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right] + \underbrace{\frac{(1-\lambda)^2}{4}}_{\beta}\left[\langle Q_j^{n+1}\rangle - \langle Q_j^n\rangle\right].$$

According to Harten's theorem [4] the sufficient conditions to ensure the TVD property of a method are: $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \leq 1$. From Remark 3.3 we have $|\lambda| \leq 1$. Thus, these three conditions are satisfied under hypothesis (3.10).

□

3.4 Numerical examples

To assess our method for the mean of the linear advective equation with random data we present two numerical examples. In Example 3.1 we solve a Riemann problem with random velocity and deterministic initial condition; in this case the exact solution, $\langle Q(x,t)\rangle$, is known. In Example 3.2 we apply our method in a problem with random velocity and initial condition being a correlated random field. In both examples we use A normally, lognormally, and uniformly distributed, respectively, to compare the effects of the velocity distribution.

Example 3.1.

Let us consider the PDE (3.2) with the deterministic initial condition

$$Q(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

In Figures 3.1 – 3.3 we compare the approximations of the mean calculated using (3.18) with the exact values given by (3.5): $\langle Q(x,t)\rangle = 1 - F_A(x/t)$. We plot the results at $T = 0.1$ and $T = 0.3$ (figures (a) and (b), respectively). To observe the influence of the velocity variation we use three models: [i] A is normally distributed, $A \sim N(1.0, 0.8)$, in Figure 3.1; [ii] A is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.25)$, in Figure 3.2; [iii] A is uniformly distributed in $[0.75, 1.25]$, in Figure 3.3. The values of Δt and Δx are presented in the captions of the figures. The figures in this example, especially Figure 3.3, also help us in the verification of the "high-resolution" of the proposed method in the sense that the numerical dispersion of the method does not give a false appearance to the mixing zone derived from the variability of the velocity.

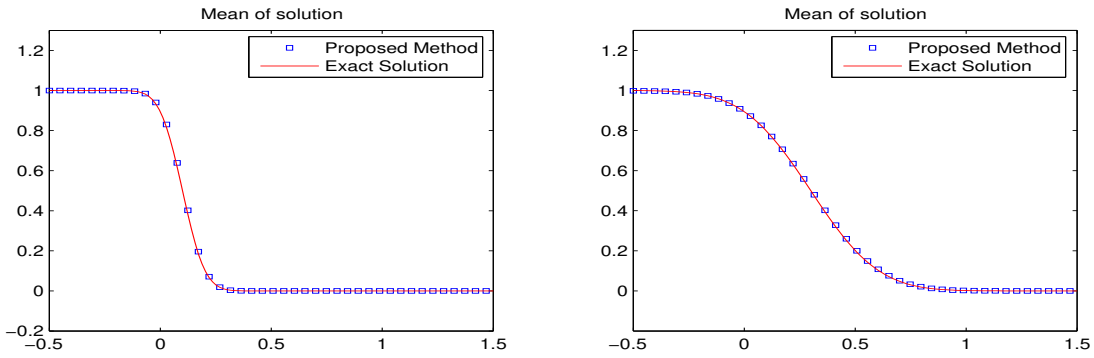

Figure 3.1: $\Delta x = 0.016$, $\Delta t = 0.002$ (a), and $\Delta t = 0.00065$ (b).

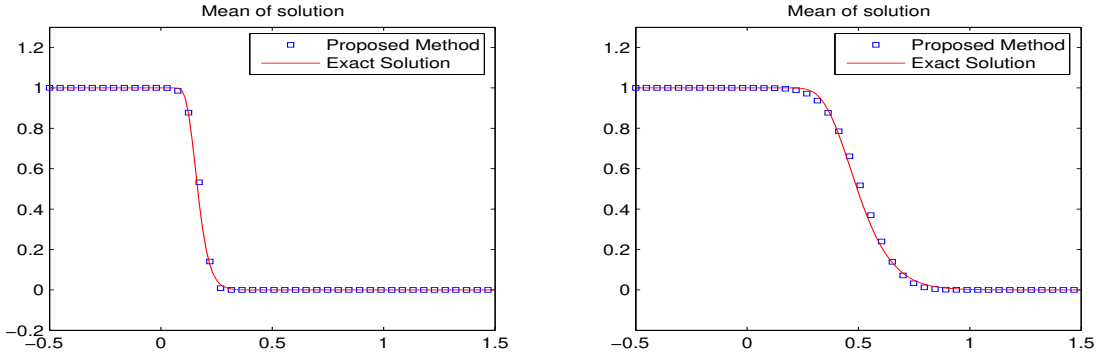

Figure 3.2: $\Delta x = 0.016$, $\Delta t = 0.005$ (a), and $\Delta t = 0.0022$ (b).

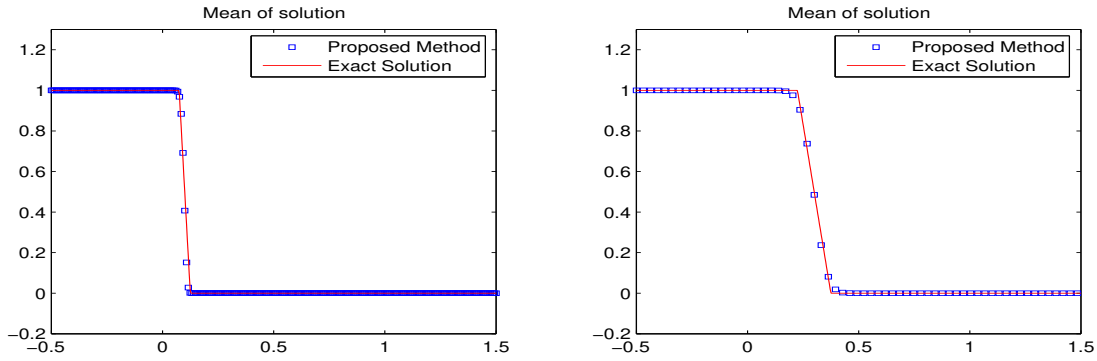

Figure 3.3: $\Delta x = 0.004$, $\Delta t = 0.003$ (a), and $\Delta t = 0.001$ (b).

Example 3.2.

Here we consider the PDE (3.2) with a normal random initial condition, $Q_0(x)$. The mean of $Q_0(x)$ is

$$\langle Q_0(x)\rangle = \begin{cases} 1, & x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, & \text{otherwise}, \end{cases} \tag{3.21}$$

and the covariance is defined by $\mathrm{Cov}(x,\tilde{x}) = \sigma^2 \exp\left(-\beta|x-\tilde{x}|\right)$, where $\mathrm{Var}[Q_0(x)] = \sigma^2$ is constant and $\beta > 0$ governs the decay rate of the spatial correlation. In the tests we use $\beta = 40$ and $\sigma^2 = 0.2$. The numerical results are compared with the Monte Carlo method using suites of realizations of A and $Q_0(x)$, with A and $Q_0(x)$ statistically independents. The solution of (3.2)–(3.21) for a single realization $A(\omega)$ and $Q_0(x,\omega)$, of A and $Q_0(x)$, respectively, is given by $Q(x,t,\omega) = Q_0(x - A(\omega)t, \omega)$. The realizations of the correlated random field $Q_0(x)$ are generated using the *matriz decomposition method*, a direct method for generating correlated random fields (see [10], Ch. 3, for example). We use the Monte Carlo method with 1500 realizations, and plot the results at $T = 0.1$ and $T = 0.3$ (figures (a) and (b), respectively). Again, we use three models of velocity: [i] A is normally distributed, $A \sim N(1.0, 0.8)$, in Figure 3.4; [ii] A is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.25)$, in Figure 3.5; [iii] A is uniformly distributed in $[0.75, 1.25]$, in Figure 3.6. The values of Δt and Δx are the same used in Example 3.1. In fact, the known solution of the Riemann problem allows to choose good values for Δt and Δx. Once these values were calibrated, they are used in the general initial condition problem with success, as show the results presented here. The numerical tests have shown that a good choice for ν in (3.19) is $\nu = 2\mathrm{Var}[A]T$.

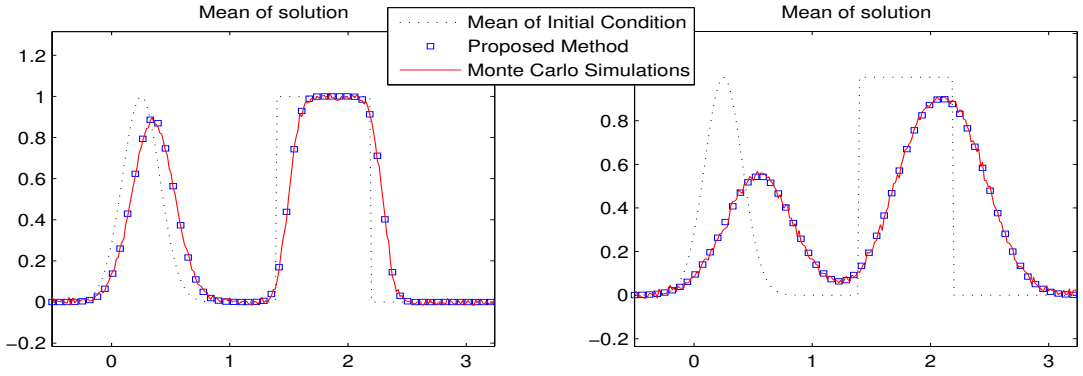

Figure 3.4: $\Delta x = 0.016$, $\Delta t = 0.002$ (a), and $\Delta t = 0.00065$ (b).

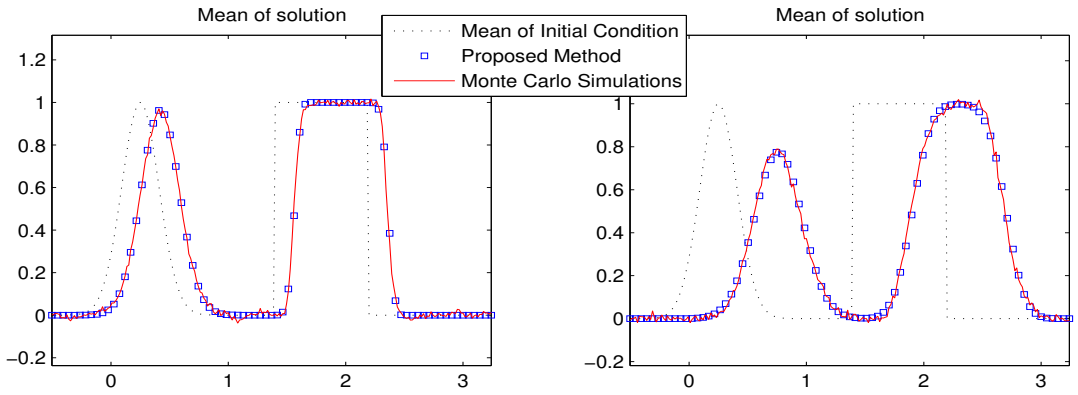

Figure 3.5: $\Delta x = 0.016$, $\Delta t = 0.005$ (a), and $\Delta t = 0.0022$ (b).

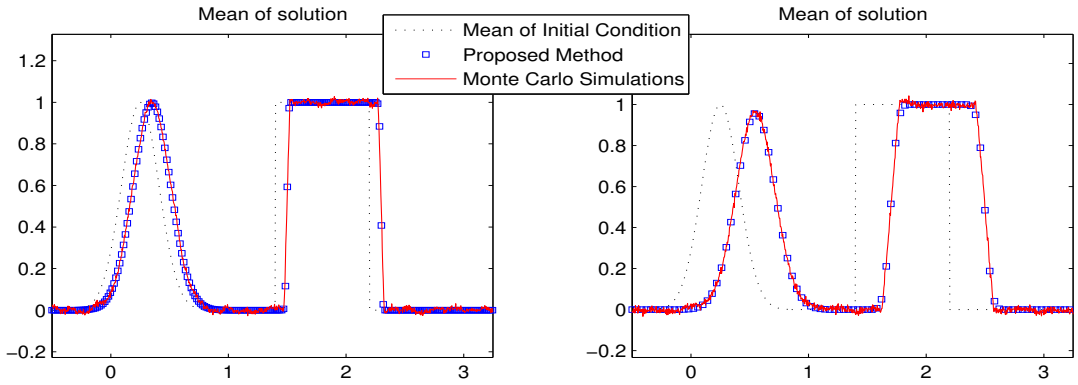

Figure 3.6: $\Delta x = 0.004$, $\Delta t = 0.003$ (a), and $\Delta t = 0.001$ (b).

3.5 Concluding remarks

In this article, we present a numerical scheme for the statistical mean of the solution of the random linear transport equation. The random data are the velocity (random variable) and the initial condition (random function). To design the method we use the basic ideas of Godunov's method (REA algorithm) with a known expression for the random Riemann problem solution. We obtain the stability condition of the method and show its consistency with a deterministic advective-diffusive equation, which means convergence of the method. The examples show good agreement of the results with the Monte Carlo method.

As far as we know, this methodology has not been studied yet. The advantages of the algorithm are: it does not require an effective equation and it does not demand the great number of realizations necessary in the Monte Carlo method. We believe that this methodology can also be applied to solve more general problems, and also to obtain information about other statistical moments of the solution.

Acknowledgments

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Capítulo 4

A numerical scheme for the variance of the solution of the random linear transport equation

Abstract

We present a numerical scheme, based on Godunov's method (REA algorithm), for the variance of the solution of the one-dimensional random linear transport equation with homogeneous random velocity and stochastic initial condition. We obtain the stability conditions of the method and we also show its consistency with a deterministic nonhomogeneous advective-diffusive equation, which means convergence. Numerical results are considered to validate our scheme.

Keyword: random linear transport equation, finite volume schemes, Godunov's method.

4.1 Introduction

In this work, we are concerned with the variance of the solution of the random transport equation,

$$\begin{cases} Q_t(x,t) + AQ_x(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), \end{cases} \tag{4.1}$$

with a homogeneous random transport velocity, A, and stochastic initial condition, $Q_0(x)$. The solution, $Q(x,t)$, is a random function. For the particular case, Riemann problem (4.1) with

$$Q(x,0) = \begin{cases} Q_0^- & \text{if } x < 0, \\ Q_0^+ & \text{if } x > 0, \end{cases} \tag{4.2}$$

where Q_0^- and Q_0^+ are random variables, we presented in [1] the expression for its solution:

$$Q_R(x,t) = Q_0^- + X\left(Q_0^+ - Q_0^-\right), \tag{4.3}$$

where X is the Bernoulli random variable with $P(X=0) = 1 - F_A(x/t)$ and $P(X=1) = F_A(x/t)$; here $F_A(x)$ is the cumulative probability function of the random variable A.

Also, according to [1], and considering the statistical independence between A and both Q_0^-, Q_0^+, the mean and variance of the solution are given by

$$\langle Q_R(x,t)\rangle = \langle Q_0^-\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right] \tag{4.4}$$

and

$$\begin{aligned} Var[Q_R(x,t)] =& Var[Q_0^-] + F_A\left(\frac{x}{t}\right)\left[Var[Q_0^+] - Var[Q_0^-]\right] + \\ & F_A\left(\frac{x}{t}\right)\left[1 - F_A\left(\frac{x}{t}\right)\right]\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right]^2, \end{aligned} \tag{4.5}$$

respectively.

In our point of view, the special attraction of (4.3), (4.4), and (4.5) is their utilization in discretizations of stochastic equations, like (4.1). In [2] we present an explicit method to calculate the mean of $Q(x,t)$, the solution of (4.1) with $Q(x,0) = Q_0(x)$ a random function. In that report, we show that Godunov's method provides a numerical scheme for the statistical mean which is, under certain assumptions on the discretization, stable and consistent with an advective-diffusive equation. Therefore, besides the scheme itself, the numerical approach also gives an effective equation compatible with one published in the literature.

The aim of this paper is to improve the knowledge of the random solution of (4.1). Basically, we present a numerical method to calculate the variance of $Q(x,t)$, which is the quantity most commonly used to specify the dispersion of the distribution around its mean.

The outline of this paper is as follows. In Section 4.2 we deduce the explicit numerical scheme using Godunov's ideas. Consistency, stability and convergence are analyzed in Section 4.3. Finally, in Section 4.4 we present some numerical examples.

4.2 The numerical scheme

In this section, we present the numerical scheme for the variance of the solution of (4.1). We denote the spatial and time grid points by $x_j = j\Delta x$ and $t_n = n\Delta t$, respectively, and the jth grid cell is $\mathcal{C}_j = (x_{j-1/2},\ x_{j+1/2})$, $x_{j\pm1/2} = x_j \pm \Delta x/2$. Let Q_j^n be an approximation

of the cell average of $Q(x,t_n)$:

$$Q_j^n \simeq \frac{1}{\Delta x}\int_{\mathcal{C}_j} Q(x,t_n)dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} Q(x,t_n)dx. \tag{4.6}$$

Assuming that the cell averages at time t_n, Q_j^n, are known, we summarize the REA, **R**econstruct-**E**volve-**A**verage, algorithm [4, 5] in three steps:

Step 1. Reconstruct a piecewise polynomial function, $\tilde{Q}(x,t_n)$, from the cell averages Q_j^n. In our case we use the piecewise constant function with Q_j^n in the jth cell, i.e., $\tilde{Q}(x,t_n) = Q_j^n$ for all $x \in \mathcal{C}_j$.

Step 2. Evolve the equation exactly, or approximately, with this initial data to obtain $\tilde{Q}(x,t_{n+1})$ a time Δt later.

Step 3. Average $\tilde{Q}(x,t_{n+1})$ over each grid cell to obtain the new cell averages, i.e.,

$$Q_j^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_j} \tilde{Q}(x,t_{n+1})dx.$$

At a time t_n, the piecewise constant function, step 1, defines a set of Riemann problems in each $x = x_{j-1/2}$: the differential equation (4.1) with the initial condition

$$Q(x,t_n) = \begin{cases} Q_{j-1}^n & \text{if} \quad x < x_{j-1/2} \\ Q_j^n & \text{if} \quad x > x_{j-1/2}. \end{cases} \tag{4.7}$$

We may use (4.3) to find a local solution to each Riemann problem at a time $\Delta t/2$ later:

$$Q(x,t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right], \tag{4.8}$$

where, for x sufficiently close to $x_{j-1/2}$, $X(x)$ is the Bernoulli random variable:

$$X(x) = \begin{cases} 1, & P\left(X(x)=1\right) = F_A(x) \\ 0, & P\left(X(x)=0\right) = 1 - F_A(x). \end{cases} \tag{4.9}$$

Also, according to (4.4) and (4.5), and denoting $\Theta_{j-1/2}(x) = F_A\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)$, we have:

$$\langle Q(x,t_{n+1/2})\rangle = \langle Q_{j-1}^n\rangle + \Theta_{j-1/2}(x)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right] \tag{4.10}$$

and

$$\begin{aligned} Var[Q(x,t_{n+1/2})] =& Var[Q_{j-1}^n] + \Theta_{j-1/2}(x)\left[Var[Q_j^n] - Var[Q_{j-1}^n]\right] + \\ & \Theta_{j-1/2}(x)\left(1 - \Theta_{j-1/2}(x)\right)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \end{aligned} \tag{4.11}$$

Therefore, the variance of the solution at $t_{n+1/2}$, $Var[\tilde{Q}(x, t_{n+1/2})]$, can be constructed by piecing together the local values of the variance, (4.11), provided that the half time step $\Delta t/2$ is short enough such that adjacent Riemann problems do not interact between themselves. This requires that Δx and Δt must be chosen satisfying:

$$Var[Q(x_{j-1}, t_{n+1/2})] \approx Var[Q^n_{j-1}] \quad \text{and} \quad Var[Q(x_j, t_{n+1/2})] \approx Var[Q^n_j],$$

where the symbol " $\approx$ " means "*sufficiently near to*". Substituting these conditions in (4.11), the following conditions must be satisfied:

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{4.12}$$

Remark 4.1. *We may regard (4.12) as a kind of* **CFL** *condition for the method. The interval* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *must contain the "effective support" of the density probability function of A. This means that outside* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *the probability of A is sufficiently near to zero, i.e., it can be disregarded. The existence of an effective support is ensured by Chebyshev's inequality:* $P\{|A - \langle A\rangle| \geq k\sigma_A\} \leq 1/k^2$*, for all* $k > 0$*, where* σ_A *is the standard variation of A. Therefore, if we take* $1/k^2$ *sufficiently close to zero, to escape from the interaction between solutions of Riemann problems we must take* $(|\langle A\rangle| + k\sigma_A)\,\Delta t/\Delta x \leq 1$*.*

Under the hypothesis (4.12), the expression (4.11) defines $Var[\tilde{Q}(x, t_{n+1/2})]$, $x \in [x_{j-1}, x_j]$; its cell average will be denoted by

$$V^{n+1/2}_{j-1/2} = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} Var[\tilde{Q}(x, t_{n+1/2})]\, dx.$$

Therefore, using (4.11) we have the cell average of the variance in $[x_{j-1}, x_j]$ at $t = t_{n+1/2}$:

$$\begin{aligned} V^{n+1/2}_{j-1/2} =& \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \left\{V^n_{j-1} + \Theta_{j-1/2}(x)\left[V^n_j - V^n_{j-1}\right]\right\} dx + \\ & \underbrace{\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \Theta_{j-1/2}(x)\left(1 - \Theta_{j-1/2}(x)\right) dx}_{\Gamma} \left[\langle Q^n_j\rangle - \langle Q^n_{j-1}\rangle\right]^2. \end{aligned}$$

Preliminary computational tests have shown that Γ reduces excessively the contribution of $\left[\langle Q^n_j\rangle - \langle Q^n_{j-1}\rangle\right]^2$ to $V^{n+1/2}_{j-1/2}$. The following approximation provides better results:

$$\Gamma = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \Theta_{j-1/2}(x)\left[1 - \Theta_{j-1/2}(x)\right] dx \;\simeq\; \Theta_{j-1/2}(\zeta)\left[1 - \Theta_{j-1/2}(\zeta)\right],$$

where $\zeta \in [x_{j-1},\ x_j]$ is such that

$$\Theta_{j-1/2}(\zeta)\left[1-\Theta_{j-1/2}(\zeta)\right] = \max_{x\in[x_{j-1},\,x_j]} \Theta_{j-1/2}(x)\left[1-\Theta_{j-1/2}(x)\right].$$

It is straightforward to show that ζ must satisfy $\Theta_{j-1/2}(\zeta) = 1/2$.

Thus, $\Theta_{j-1/2}(\zeta)\left[1-\Theta_{j-1/2}(\zeta)\right] = 1/4$ and, changing variables in the other integral, we obtain

$$V_{j-1/2}^{n+1/2} = V_{j-1}^n + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)\,dx\right\}\left[V_j^n - V_{j-1}^n\right] + \frac{1}{4}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \quad (4.13)$$

Lemma 4.1. *Let A be a random variable and $[-\xi,\xi]$ an effective support of the density probability function, f_A, of A, i.e., $F_A(-\xi)\approx 0$ and $F_A(\xi)\approx 1$. Then*

$$\int_{-\xi}^{\xi} F_A\,(x)\,dx \ \approx\ \xi - \langle A\rangle. \quad (4.14)$$

Proof. See [2].

□

Substituting (4.14) in (4.13), and denoting $\lambda = \Delta t\langle A\rangle/\Delta x$, we have:

$$V_{j-1/2}^{n+1/2} = \frac{1}{2}\left[V_j^n + V_{j-1}^n\right] - \frac{\lambda}{2}\left[V_j^n - V_{j-1}^n\right] + \frac{1}{4}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \quad (4.15)$$

Now we can repeat the same procedure to obtain approximations of the solution in $[x_{j-1/2}, x_{j+1/2}]$ at t_{n+1}:

$$V_j^{n+1} = \frac{1}{2}\left[V_{j+1/2}^{n+1/2} + V_{j-1/2}^{n+1/2}\right] - \frac{\lambda}{2}\left[V_{j+1/2}^{n+1/2} - V_{j-1/2}^{n+1/2}\right] + \frac{1}{4}\left[\langle Q_{j+1/2}^{n+1/2}\rangle - \langle Q_{j-1/2}^{n-1/2}\rangle\right]^2. \quad (4.16)$$

The ideas of the Godunov method were also used in [2] to design a scheme for approximations of the statistical means of (4.1):

$$\langle Q_{j-1/2}^{n+1/2}\rangle = \frac{1}{2}\left[\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right] - \frac{\lambda}{2}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right] \quad (4.17)$$

and

$$\langle Q_j^{n+1}\rangle = \frac{1}{2}\left[\langle Q_{j+1/2}^{n+1/2}\rangle + \langle Q_{j-1/2}^{n+1/2}\rangle\right] - \frac{\lambda}{2}\left[\langle Q_{j+1/2}^{n+1/2}\rangle - \langle Q_{j-1/2}^{n+1/2}\rangle\right], \quad (4.18)$$

or, joining these expressions,

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\lambda}{2}\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right]. \quad (4.19)$$

Using (4.15) and (4.17) in (4.16), we can summarize the two-step scheme for the variance in the explicit form:

$$\begin{aligned} V_j^{n+1} =& V_j^n - \frac{\lambda}{2}\left[V_{j+1}^n - V_{j-1}^n\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[V_{j+1}^n - 2V_j^n + V_{j-1}^n\right] + \\ & \frac{1}{8}(1-\lambda)\left[\langle Q_{j+1}^n\rangle - \langle Q_j^n\rangle\right]^2 + \frac{1}{8}(1+\lambda)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2 + \\ & \frac{1}{16}\left\{\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] - \lambda\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right]\right\}^2, \end{aligned} \tag{4.20}$$

where $\lambda = \Delta t\langle A\rangle/\Delta x$.

4.3 Numerical analysis of the scheme

In this section, we analyze some numerical aspects of the method (4.19)–(4.20), for the mean and variance of the solution of (4.1). We obtain the stability conditions of the scheme and we also show its consistency with a deterministic nonhomogeneous advective-diffusive system.

Proposition 4.1. *For $\Delta x^2/\Delta t = \nu$ fixed, the numerical scheme (4.19)-(4.20) is an $\mathcal{O}(\Delta x^2)$ approximation for $u(x,t)$ and $v(x,t)$, solutions of the deterministic system of partial differential equations:*

$$\begin{cases} u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx} \\ v_t + \langle A\rangle v_x = \frac{\nu}{4}v_{xx} + \frac{\nu}{2}u_x^2. \end{cases} \tag{4.21}$$

Proof. Let $v(x,t)$ and $u(x,t)$ be smooth functions such that $v(x_j,t_n) = V_j^n$ and $u(x_j,t_n) = \langle Q_j^n\rangle$. We have shown in [2] that if $\Delta x^2/\Delta t = \nu$ is fixed then the numerical scheme (4.19) gives an $\mathcal{O}(\Delta x^2)$ approximation for $u(x,t)$, solution of the differential equation $u_t + \langle A\rangle u_x = (\nu/4)u_{xx}$.
Also, using Taylor's expansion in (4.20) we have

$$\begin{aligned} & \left[v_t + \frac{\Delta t}{2}v_{tt} + \mathcal{O}(\Delta t^2)\right] + \langle A\rangle\left[v_x + \mathcal{O}(\Delta x^2)\right] = \frac{\nu}{4}\left(1+\lambda^2\right)\left[v_{xx} + \mathcal{O}(\Delta x^2)\right] + \\ & \frac{\nu}{8}(1-\lambda)\left[u_x + \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right]^2 + \frac{\nu}{8}(1+\lambda)\left[u_x - \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right]^2 + \\ & \frac{\nu}{4}\left\{\left[u_x + \mathcal{O}(\Delta x^2)\right] - \frac{\lambda}{2}\left[u_{xx} + \mathcal{O}(\Delta x^2)\right]\right\}^2. \end{aligned}$$

Since $\Delta x^2/\Delta t = \nu$ is fixed, we have $\lambda = \Delta t\langle A\rangle/\Delta x = \Delta x\langle A\rangle/\nu = \mathcal{O}(\Delta x)$ and $\Delta t = \mathcal{O}(\Delta x^2)$.

Thus, grouping the terms of the same order, we obtain

$$v_t + \langle A \rangle v_x = \frac{\nu}{4} v_{xx} + \frac{\nu}{2} u_x^2 + \mathcal{O}(\Delta x^2).$$

□

Remark 4.2. *Although the consistency result is for any ν, computational tests have shown that a good choice for $\nu = \Delta x^2/\Delta t$, in (4.21), to calculate $\langle Q(x,T) \rangle$ and $Var(Q(x,T))$, is $\nu = 2\,Var[A]T$.*

Remark 4.3. *The modified equations in (4.21) constitute a decoupled deterministic non-homogeneous advective-diffusive system whose transport terms are the mean of the velocity, and the diffusive terms are the same. The source term in the second equation involves the spatial derivative of the mean, given by the first equation.*

Remark 4.4. *In [2] we have shown that the stability condition of (4.19) is (4.12). On the other hand, since the terms corresponding to the mean can be considered source terms, the method for the variance, (4.20), has the same stability conditions, i.e., (4.12). As a linear problem, we have convergence.*

4.4 Numerical examples

To assess our method for the variance we present two numerical examples. In Example 4.1 we solve a Riemann problem in which the exact values of $\langle Q(x,t) \rangle$ and $\mathrm{Var}[Q(x,t)]$ are known. In Example 4.2 we apply the method in a problem with random velocity and a correlated random field as the initial condition. In both examples we use A normally and lognormally distributed. The value of Δx is presented in the caption of the figures. The value of Δt was chosen based on Remark 4.2, i.e., we used $\nu = \Delta x^2/\Delta t = 2\mathrm{Var}[A]T$. All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

Example 4.1.

Let us consider the problem (4.1)–(4.2) where the random variables U_0^- and U_0^+ have a bivariate normal distribution defined by: $\langle U_0^- \rangle = 1$ (mean of U_L); $\langle U_0^+ \rangle = 0$ (mean of U_R); $Var[U_0^-] = 0.16$ (variance of U_0^-); $Var[U_0^+] = 0.25$ (variance of U_0^+); and $\rho = 0$ (correlation coefficient between U_0^- and U_0^+). In Figures 4.1 – 4.4 we compare the approximations of the mean and variance calculated using (4.19) and (4.20), respectively, with the exact values given by (4.4) and (4.5). We plot the results at $T = 0.3$ and $T = 0.5$. To observe the influence of the velocity variation we use two models: [i] A is normally distributed, $A \sim N(1.0, 0.6)$, in Figures 4.1 and 4.2; [ii] A is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.25)$, in Figures 4.3 and 4.4.

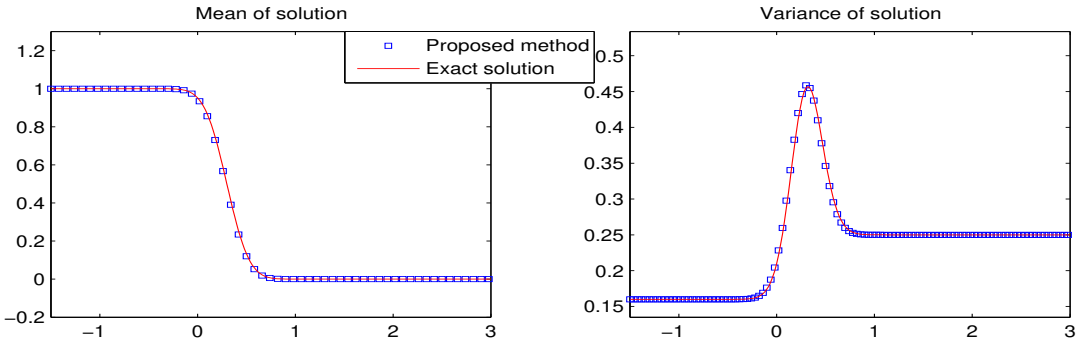

Figure 4.1: $\Delta x = 0.02$ and $T = 0.3$.

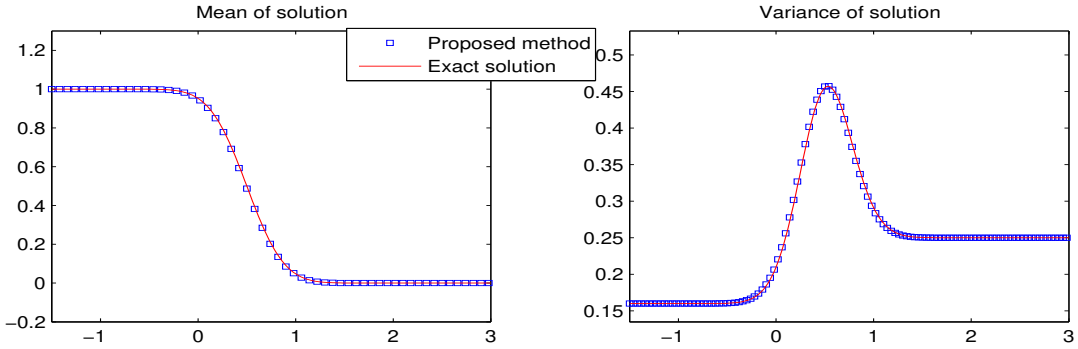

Figure 4.2: $\Delta x = 0.02$ and $T = 0.5$.

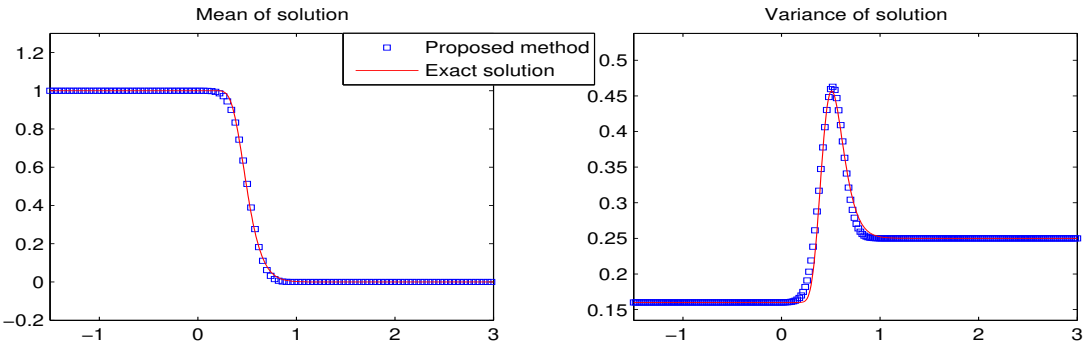

Figure 4.3: $\Delta x = 0.01$ and $T = 0.3$.

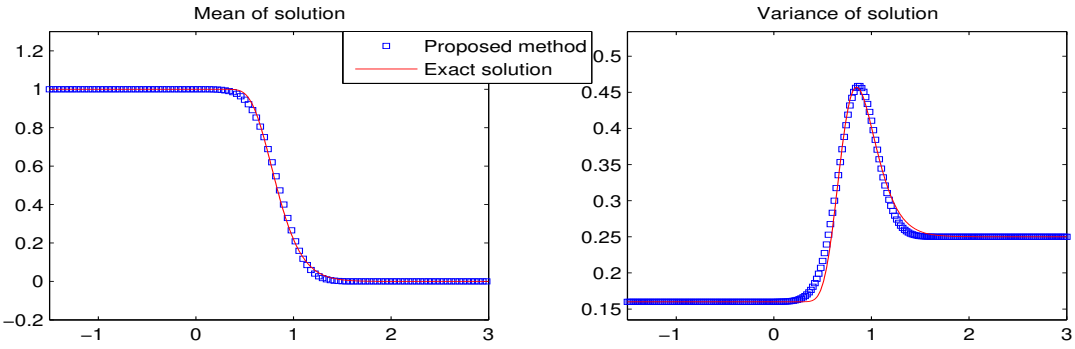

Figure 4.4: $\Delta x = 0.01$ and $T = 0.5$.

Example 4.2.

In this example we consider the random partial differential equation (4.1) with initial condition being the normal random field with mean

$$\langle Q_0(x)\rangle = \begin{cases} 1, & x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, & \text{otherwise}, \end{cases} \tag{4.22}$$

and covariance $\mathrm{Cov}(x, \tilde{x}) = \sigma^2 \exp(-\beta|x - \tilde{x}|)$, where $\mathrm{Var}[Q_0(x)] = \sigma^2$ is constant. The parameter $\beta > 0$ governs the decay rate of the spatial correlation. In the tests we use $\beta = 40$ and $\sigma^2 = 0.12$. The numerical results are compared with the Monte Carlo method using suites of realizations of A and $Q_0(x)$, with A and $Q_0(x)$ statistically independent. The analytical solution of each realization $A(\omega)$ and $Q_0(x, \omega)$ is given by $Q(x, t, \omega) = Q_0(x - A(\omega)t, \omega)$. The 2000 realizations of the correlated random field $Q_0(x)$ are generated using the *matriz decomposition method*, a direct method for generating correlated random fields (see [9], Ch. 3, for example). As in the previous example, we use two models of velocity: [i] A is normally distributed, $A \sim N(-0.5, 0.6)$, in Figures 4.5 and 4.6; [ii] A is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.15, 0.25)$, in Figures 4.7 and 4.8.

4.5 Concluding remarks

In this paper, we extend the ideas presented in our previous work [2] to obtain more information about the statistical properties of the solution to the one-dimensional random linear transport equation. We show that the ideas of the Godunov method can also be used to design a numerical scheme to calculate the variance of the solution: (4.19) and (4.20). We also present the stability conditions and the consistency of the numerical scheme with the decoupled system of advective-diffusive equations (4.21). Computational results are

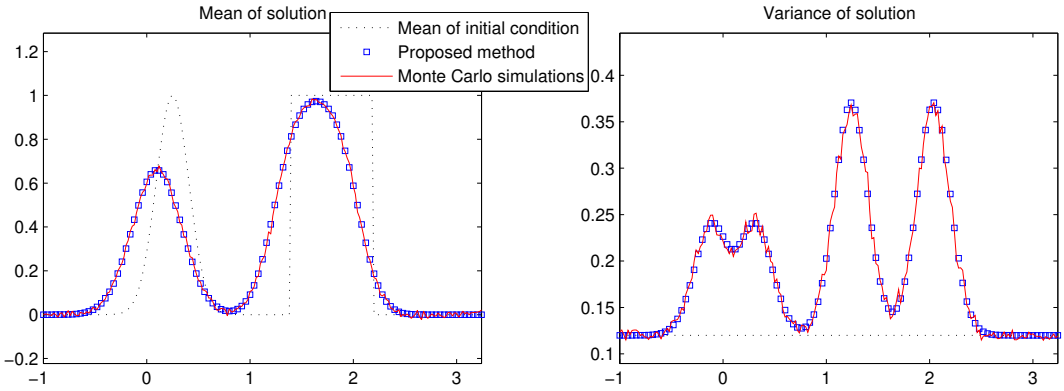

Figure 4.5: $\Delta x = 0.02$ and $T = 0.3$.

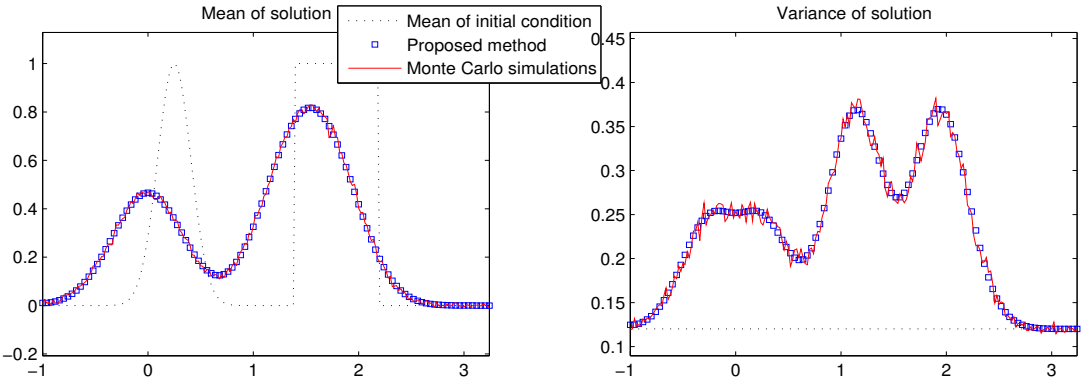

Figure 4.6: $\Delta x = 0.02$ and $T = 0.5$.

compared with the exact solution, in the Riemann problem, and with the Monte Carlo method in a more general situation. As far as we know, this kind of methodology has not been used to deal with differential equations with uncertainties in the parameters.

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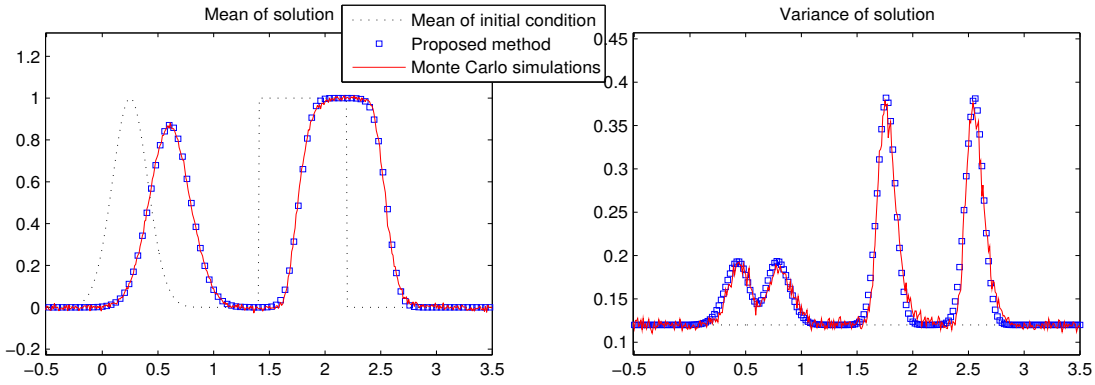

Figure 4.7: $\Delta x = 0.01$ and $T = 0.3$.

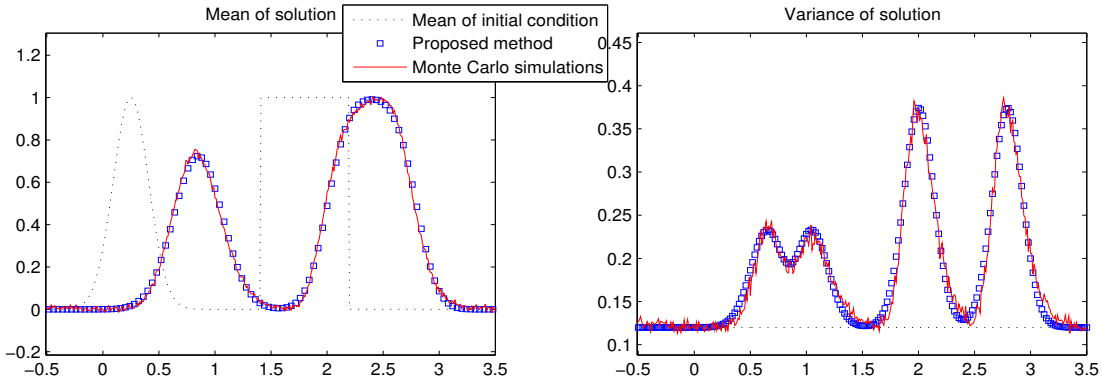

Figure 4.8: $\Delta x = 0.01$ and $T = 0.5$.

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Capítulo 5

Statistical moments of the random linear transport equation

Abstract

This paper deals with a numerical scheme to approximate the mth moment of the solution of the one-dimensional random linear transport equation. The initial condition is assumed to be a random function and the transport velocity is a random variable. The scheme is based on local Riemann problem solutions and Godunov's method. We show that the scheme is consistent and stable with an advective-diffusive equation. Furthermore, in the case where the velocity is normally distributed we obtain partial differential equations for the moments and the central moments. Numerical examples are added to illustrate our approach.

Keyword: random linear transport equation, Riemann problem, statistical moments, Godunov's method, numerical methods for random partial differential equations.

5.1 Introduction

Partial differential equations have been important models during the last centuries, mainly because they have the fundamental support of differential calculus, numerical methods, and computers. However, the formulation of a physical process as a partial differential equation demands experiments to measure the data, for example, the diffusion coefficient, permeability of a porous media, initial conditions, boundary conditions and so on. This means that the interpretation of the data as random variables is more realistic in some practical situations. Differential equations with random parameters are called Random Differential Equations; new mathematical methods have been developed to deal with this kind of problems (see [6, 8, 12, 15], for example).

We are interested in the solution of random linear transport equations given by

$$\begin{cases} Q_t(x,t) + AQ_x(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), \end{cases} \tag{5.1}$$

where A is a random variable and $Q_0(x)$ is a random function.

The solution for the random Riemann problem to (5.1),

$$Q_0(x) = \begin{cases} Q_L, & \text{if} \quad x < 0, \\ Q_R, & \text{if} \quad x > 0, \end{cases} \tag{5.2}$$

with Q_L and Q_R being random variables is given by [1]:

$$Q(x,t) = Q_L + X\left(\frac{x}{t}\right)(Q_R - Q_L). \tag{5.3}$$

In (5.3) X is the Bernoulli random variable with $P\{X(\xi) = 1\} = F_A(\xi)$, the cumulative probability function of the random variable A. Furthermore, in case of independence between A and both Q_L and Q_R, the mth moment of $Q(x,t)$, $\langle Q^m(x,t)\rangle$, $m \in \mathbb{N}$, $m \geq 1$, is given by

$$\langle Q^m(x,t)\rangle = \langle Q_L^m\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_R^m\rangle - \langle Q_L^m\rangle\right]. \tag{5.4}$$

The closed solution (5.3) and Godunov's ideas [7, 9, 10] are used in [4] and [2] to design numerical methods to compute the mean and the variance of the solution to (5.1). The methods are explicit and neither demand generation of random numbers (as does the Monte Carlo method [5, 11, 14, 17]), nor require differential equations governing the statistical moments (as in the effective equations methodology [6, 17]). Moreover, the schemes are stable and consistent with an advective-diffusive equation which agrees with the effective equation to the expectation presented in the literature (see [6], for example). In [3] we use the idea of collecting deterministic realizations through their probability functions to solve the nonlinear random Riemann-Burgers equation.

In this paper, we deal with the general moments of the solution to (5.1). The outline of this paper is as follows. In Section 5.2 we use (5.3) and (5.4) to design a numerical method to the mth statistical moment of the solution to the general problem (5.1). We present the CFL condition under which the local solutions do not interact between themselves. In Section 5.3 we show the stability of the numerical scheme and its consistency with an advective-diffusive equation. We show that the diffusion coefficient is related with the probability density function of the velocity by Eq. (5.18), which has a simple solution in the normal velocity case. In Section 5.4 we present a decoupled system of partial differential equations to be satisfied by the central moments of the random solution. All the partial differential equations in this paper are linear. In fact, denoting by $\mathbb{L}(u) = u_t + \langle A\rangle u_x - \nu u_{xx}$, the equations are of the form: $\mathbb{L}(u) = 0$, for the moments, and $\mathbb{L}(u) = f$, for the central moments. Computational experiments and comparisons with the Monte Carlo method are presented in Section 5.5.

5.2 The numerical scheme

In this section, we present the numerical method for the mth statistical moment of the solution to (5.1). The method is based on the juxtaposition of Riemann problems whose solutions are given by (5.3). We discretize both space and time assuming a uniform mesh spacing: $x_j = j\Delta x$, $x_{j\pm1/2} = x_j \pm (\Delta x/2)$, $t_n = n\Delta t$, $t_{n\pm1/2} = t_n \pm (\Delta t/2)$, for $\Delta x, \Delta t > 0$. In Figure 5.1 we present a schematic diagram of the algorithm. Let us assume that the random variables Q_j^n and the mth moments $\langle Q_j^{m,n} \rangle = \langle Q^m(x_j, t_n) \rangle$ are known at $t = t_n$.

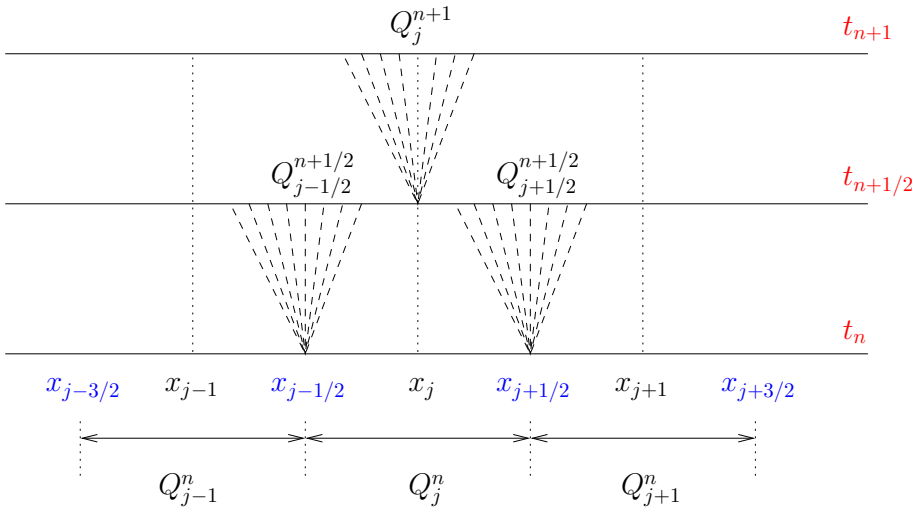

Figure 5.1: Schematic diagram of the algorithm.

In the following we use the ideas of REA, **R**econstruct-**E**volve-**A**verage, algorithm [7, 10] to approximate $\langle Q_j^{m,n+1} \rangle = \langle Q^m(x_j, t_{n+1}) \rangle$.

Step 1. We reconstruct the piecewise random constant function $\tilde{Q}(x, t_n)$ from Q_j^n, i.e, $\tilde{Q}(x, t_n) = Q_j^n$ for $x \in [x_{j-1/2}, x_{j+1/2}]$.

The piecewise constant random function $\tilde{Q}(x, t_n)$ defines a set of local random Riemann problems, each one centered at $x = x_{j-1/2}$,

$$\begin{aligned} &Q_t(x,t) + AQ_x(x,t) = 0, \quad t > t_n, \quad x \in \mathbb{R}, \\ &Q(x,t_n) = \begin{cases} Q_{j-1}^n, & \text{if} \quad x < x_{j-1/2}, \\ Q_j^n, & \text{if} \quad x > x_{j-1/2}. \end{cases} \end{aligned} \tag{5.5}$$

Step 2. From (5.3) and (5.4), the local solutions of (5.5) and the respective statistical moments are given by

$$G_{j-1/2}(x, t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right] \tag{5.6}$$

and

$$\langle G^m_{j-1/2}(x,t_{n+1/2})\rangle = \langle Q^{m,n}_{j-1}\rangle + F_A\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[\langle Q^{m,n}_{j}\rangle - \langle Q^{m,n}_{j-1}\rangle\right]. \tag{5.7}$$

The global solution at $t = t_{n+1/2}$, $\tilde{Q}(x,t_{n+1/2})$, can be constructed by piecing together the local random Riemann solutions (5.6), provided that $\Delta t/2$ is sufficiently small such that adjacent local random Riemann solutions do not interact. Therefore, taking into account the similarity property of the random Riemann solutions, Δx and Δt must be chosen such that:

$$G_{j-1/2}(x,t_{n+1/2})\big|_{x=x_{j-1}} \approx Q^n_{j-1}, \quad G_{j-1/2}(x,t_{n+1/2})\big|_{x=x_j} \approx Q^n_j,$$

where the symbol " $\approx$ " means "*sufficiently near to*". By substituting these conditions in (5.6) we must have

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{5.8}$$

Remark 5.1. *We may regard (5.8) as the* **CFL** *condition for the method: the interval* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *must contain an effective support of the density probability function of* A*. This means that the probability of* A *outside of the interval* $[-\Delta x/\Delta t, \Delta x/\Delta t]$ *is sufficiently near to zero, and then may be disregarded. The existence of an effective support is ensured by Chebyshev's inequality:* $P\{|A - \langle A\rangle| \geq k\sigma_A\} \leq 1/k^2$*, for all* $k > 0$*, where* σ_A *is the standard variation of* A*. Therefore, if we take* $1/k^2$ *sufficiently close to zero, to escape from the interaction between solutions of Riemann problems we must take* $(|\langle A\rangle| + k\sigma_A)\,\Delta t/\Delta x \leq 1$*.*

Under condition (5.8), we conclude Step 2 by taking

$$\tilde{Q}(x,t_{n+1/2}) = \sum_{j-1/2} G_{j-1/2}(x,t_{n+1/2})\ \mathbf{1}_{[x_{j-1},x_j]}$$

where $\mathbf{1}_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. From (5.7) it follows that

$$\langle \tilde{Q}^m(x,t_{n+1/2})\rangle = \sum_{j-1/2} \langle G^m_{j-1/2}(x,t_{n+1/2})\rangle\ \mathbf{1}_{[x_{j-1},x_j]}. \tag{5.9}$$

In a similar way, using the values at $t = t_{n+1/2}$, we obtain

$$\langle \hat{Q}^m(x,t_{n+1})\rangle = \sum_{j} \langle G^m_{j}(x,t_{n+1})\rangle\ \mathbf{1}_{[x_{j-1/2},x_{j+1/2}]}. \tag{5.10}$$

Step 3. We use (5.10) to approximate $\langle Q_j^{m,n+1}\rangle$ as the average value of $\langle \hat{Q}^m(x,t_{n+1})\rangle$ over the interval $[x_{j-1/2}, x_{j+1/2}]$:

$$\begin{aligned}\langle Q_j^{m,n+1}\rangle &\simeq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \langle \hat{Q}^m(x,t_{n+1})\rangle dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \langle G_j^m(x,t_{n+1})\rangle dx \\ &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \left\{ \langle Q_{j-1/2}^{m,n+1/2}\rangle + F_A\left(\frac{x-x_j}{\Delta t/2}\right)\left[\langle Q_{j+1/2}^{m,n+1/2}\rangle - \langle Q_{j-1/2}^{m,n+1/2}\rangle\right]\right\} dx \\ &= \langle Q_{j-1/2}^{m,n+1/2}\rangle + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[\langle Q_{j+1/2}^{m,n+1/2}\rangle - \langle Q_{j-1/2}^{m,n+1/2}\rangle\right]. \end{aligned} \tag{5.11}$$

Likewise, we use (5.9) to approximate $\langle Q_{j-1/2}^{m,n+1/2}\rangle$:

$$\begin{aligned}\langle Q_{j-1/2}^{m,n+1/2}\rangle &\simeq \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \langle \tilde{Q}^m(x,t_{n+1/2})\rangle dx = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \langle G_{j-1/2}^m(x,t_{n+1/2})\rangle dx \\ &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \left\{ \langle Q_{j-1}^{m,n}\rangle + F_A\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[\langle Q_j^{m,n}\rangle - \langle Q_{j-1}^{m,n}\rangle\right]\right\} dx \\ &= \langle Q_{j-1}^{m,n}\rangle + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[\langle Q_j^{m,n}\rangle - \langle Q_{j-1}^{m,n}\rangle\right]. \end{aligned} \tag{5.12}$$

The following result is proved in [4]:

Lemma 5.1. *Let Y be a random variable and $[-\xi,\xi]$ an effective support of the density probability function, f_Y, of Y, i.e., $F_Y(-\xi)\approx 0$ and $F_Y(\xi)\approx 1$. Then*

$$\int_{-\xi}^{\xi} F_Y(x)dx \;\approx\; \xi - \langle Y\rangle. \tag{5.13}$$

Inserting (5.13) in (5.11) and (5.12), and denoting $\lambda = \Delta t\langle A\rangle/\Delta x$, gives

$$\langle Q_j^{m,n+1}\rangle = \frac{1}{2}\left[\langle Q_{j-1/2}^{m,n+1/2}\rangle + \langle Q_{j+1/2}^{m,n+1/2}\rangle\right] - \frac{\lambda}{2}\left[\langle Q_{j+1/2}^{m,n+1/2}\rangle - \langle Q_{j-1/2}^{m,n+1/2}\rangle\right] \tag{5.14}$$

and

$$\langle Q_{j-1/2}^{m,n+1/2}\rangle = \frac{1}{2}\left[\langle Q_{j-1}^{m,n}\rangle + \langle Q_j^{m,n}\rangle\right] - \frac{\lambda}{2}\left[\langle Q_j^{m,n}\rangle - \langle Q_{j-1}^{m,n}\rangle\right]. \tag{5.15}$$

Grouping these expressions we summarize the two-step scheme (5.14)–(5.15) in the one-step explicit method:

$$\begin{aligned}\langle Q_j^{m,n+1}\rangle \;=\; & \langle Q_j^{m,n}\rangle - \frac{\lambda}{2}\left[\langle Q_{j+1}^{m,n}\rangle - \langle Q_{j-1}^{m,n}\rangle\right] + \\ & + \frac{1}{4}\left(1+\lambda^2\right)\left[\langle Q_{j+1}^{m,n}\rangle - 2\langle Q_j^{m,n}\rangle + \langle Q_{j-1}^{m,n}\rangle\right]. \end{aligned} \tag{5.16}$$

Remark 5.2. *The numerical scheme (5.16) is conservative, i.e., it can be rewritten as*

$$\langle Q_j^{m,n+1}\rangle = \langle Q_j^{m,n}\rangle - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^{m,n} - F_{j-1/2}^{m,n}\right],$$

where $F_{j-1/2}^{m,n} = (1/2)\langle A\rangle[\langle Q_{j-1}^{m,n}\rangle + \langle Q_j^{m,n}\rangle] - (1/4)\langle A\rangle\,(1/\lambda + \lambda)\,[\langle Q_j^{m,n}\rangle - \langle Q_{j-1}^{m,n}\rangle]$ *is an approximation to the average flux at* $x = x_{j-1/2}$.

5.3 Numerical analysis of the scheme

The scheme (5.16) is a generalization of a previously studied scheme to the mean ($m = 1$) of the solution to (5.1). Therefore, we can use the same arguments used in [4] to show

- *consistency*: if $\nu = \Delta x^2/(4\Delta t)$ is fixed then the numerical scheme (5.16) yields an $\mathcal{O}(\Delta x^2)$ approximation for the solution of the partial differential equation

$$u_t + \langle A\rangle u_x = \nu u_{xx}; \tag{5.17}$$

- *stability*: the numerical method (5.16) is stable under the CFL condition (5.8).

As a linear problem, the convergence of (5.16) to the differential equation (5.17) is a consequence of the Lax Equivalence Theorem, no matter what $\nu = \Delta x^2/(4\Delta t)$ is. The following proposition gives an additional information about the diffusion associated with the random velocity, A.

Proposition 5.1. *The diffusion coefficient in (5.17) must satisfy*

$$-f_A'\left(\frac{x}{t}\right)\nu(x,t) = f_A\left(\frac{x}{t}\right)(x - \langle A\rangle t), \tag{5.18}$$

where $f_A(\xi) = d[F_A(\xi)]/d\xi$ *is the density probability function of* A.

Proof. As a general differential equation, (5.17) must be satisfied by a particular solution. The random Riemann problem (5.1)–(5.2) is a particular case of (5.1) with known moments given by (5.4):

$$\langle Q^m(x,t)\rangle = \langle Q_L^m\rangle + F_A\left(\frac{x}{t}\right)[\langle Q_R^m\rangle - \langle Q_L^m\rangle].$$

Direct derivations and substitution of this solution in (5.17) gives (5.18), a necessary condition to $\nu(x,t)$.

□

5.3.1 The Normal case

Let $A \sim N(\langle A\rangle, \sigma_A)$. Using the normal probability density function in (5.18) we obtain $\nu = \sigma_A^2 t$. In this case, the differential equation (5.17) turns to be

$$u_t + \langle A\rangle u_x = (\sigma_A^2 t)u_{xx}, \quad t > 0, \tag{5.19}$$

which agrees with the effective equation for the statistical mean presented by some authors (see [6], for example). We emphasize that our convergence results show that the differential equation which describes the evolution of all the moments is the same. Using (5.18) we may also show that if $\nu(x,t)$ depends only on t then A is normally distributed.

Now let $t = t_f$ be fixed, and select Δt and Δx such that

$$\frac{\Delta x^2}{4\Delta t} = \nu = \frac{1}{2}(\sigma_A^2 t_f). \tag{5.20}$$

The convergence results show that our method converges to the solution of the differential equation

$$u_t + \langle A\rangle u_x = \frac{1}{2}(\sigma_A^2 t_f)u_{xx}. \tag{5.21}$$

The solutions of (5.19) and (5.21), $u_1(x,t)$ and $u_2(x,t)$, respectively, both with $u(x,0) = g(x)$, are equal at $t = t_f$. Indeed, according to [13] we have

$$u_1(x,t_f) = \frac{1}{\sqrt{\pi}\xi_1(t_f)}\int_{-\infty}^{+\infty} \exp\left[-\left(\frac{x - \langle A\rangle t_f - \omega}{\xi_1(t_f)}\right)^2\right] g(\omega)\, d\omega, \tag{5.22}$$

where

$$\xi_1(t_f) = 2\left[\int_0^{t_f} (\sigma_A^2 s)\, ds\right]^{1/2} = \sqrt{2}\sigma_A t_f.$$

On the other hand, the solution to (5.21) is also given by (5.22) with

$$\xi_2(t_f) = 2\left[\int_0^{t_f} [(\sigma_A^2 t_f)/2]\, ds\right]^{1/2}$$

instead of $\xi_1(t_f)$. Since $\xi_1(t_f) = \xi_2(t_f)$ then $u_1(x,t_f) = u_2(x,t_f)$.

The condition (5.20) can be rewritten as $\Delta x/\Delta t = 2\sigma_A^2 t_f/\Delta x$. Thus, the CFL condition (5.8) may be satisfied for Δx sufficiently small.

5.4 The system of partial differential equations for the central moments

Central moments of a given random function $Q(x,t)$ are deterministic functions defined by $\mu_m = \langle (Q - \langle Q\rangle)^m\rangle$, $m \in \mathbb{N}$, $m \geq 2$. The most used central moment is the variance,

$m = 2$, which was introduced by K. F. Gauss (1777-1855) as a measure of dispersion of the distribution of $Q(x,t)$. But high order central moments are also useful information concerning random variables [12, 15].

In the following we show that the central moment $\mu_m(x,t)$, if sufficiently smooth, satisfies an advective-diffusive equation with the source term defined by the expectation and the central moments $\mu_{m-1}(x,t)$ and $\mu_{m-2}(x,t)$. Here, we may extend the definition of central moments for $m \geq 0$ since $\mu_0 = 1$ and $\mu_1 = 0$.

We may use algebraic manipulations to show that

(i) If $k \leq m-2$ then

$$\binom{m}{k+2}(k+1)(k+2) = \binom{m}{k}(m-k)(m-k-1). \tag{5.23}$$

(ii) If $k \leq m-1$ then

$$\binom{m}{k+1}(k+1) = \binom{m}{k}(m-k). \tag{5.24}$$

(iii)

$$\mu_m = \langle Q^m \rangle - \sum_{k=2}^{m-1} \binom{m}{k} \mu_k \langle Q \rangle^{m-k} \; - \; \langle Q \rangle^m. \tag{5.25}$$

Proposition 5.2. *Let Z(x,t) be a random function whose statistical moments satisfy (5.17), i.e., the advective-diffusive equations:*

$$\langle Z^m \rangle_t + \langle A \rangle \langle Z^m \rangle_x = \nu \langle Z^m \rangle_{xx}, \tag{5.26}$$

$m \in \mathbb{N}$, $m \geq 1$. *Then the central moments,* $\mu_m(x,t) = \langle [Z - \langle Z \rangle]^m \rangle$, $m \in \mathbb{N}$, $m \geq 2$, *satisfy the advective-diffusive equations with source term:*

$$\mu_{m,t} + \langle A \rangle \mu_{m,x} - \nu \mu_{m,xx} = 2m\nu\mu_{m-1,x} \, \langle Z \rangle_x + m(m-1)\nu\mu_{m-2} \langle Z \rangle_x^2, \tag{5.27}$$

where $\mu_0 = 1$ *and* $\mu_1 = 0$.

Proof. The proof is based on the induction principle. Since $\mu_2(x,t) = \langle Z^2(x,t) \rangle - \langle Z(x,t) \rangle^2$, $\mu_1(x,t) = 0$ and $\mu_0(x,t) = 1$, direct substitution and derivations show (5.27) for $k = 2$. As the induction hypothesis we assume that (5.27) is true for $k = 3 : (m-1)$, and our task is to prove that (5.27) is true for $k = m$.

From (5.25) we have $\mu_m(x,t) = \langle Z^m \rangle - \sum_{k=2}^{m-1} \binom{m}{k} \mu_k \langle Z \rangle^{m-k} \; - \; \langle Z \rangle^m$. By differentiating this expression with respect to t and x, grouping conveniently the terms, and using (5.26)

we arrive at

$$\begin{aligned}
&\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = \\
&\quad - \sum_{k=2}^{m-1} \binom{m}{k} \langle Z\rangle^{m-k} \left\{\mu_{k,t} + \langle A\rangle\mu_{k,x} - \nu\mu_{k,xx}\right\} + \\
&\quad + 2\,\nu \sum_{k=2}^{m-1} \binom{m}{k} (m-k)\, \mu_{k,x}\, \langle Z\rangle^{m-k-1}\, \langle Z\rangle_x + \\
&\quad + \nu \sum_{k=2}^{m-2} \binom{m}{k} (m-k)\,(m-k-1)\, \mu_k\, \langle Z\rangle^{m-k-2}\, (\langle Z\rangle_x)^2 + \\
&\quad + \nu\, m\,(m-1)\, \langle Z\rangle^{m-2} \langle Z\rangle_x^2. \qquad (5.28)
\end{aligned}$$

Using the induction hypothesis in the first sum in (5.28), and separating the last term of the second and third sums, we obtain

$$\begin{aligned}
&\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = \\
&\quad - \sum_{k=2}^{m-1} \binom{m}{k} \langle Z\rangle^{m-k} \left\{2k\nu\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\nu\mu_{k-2}(\langle Z\rangle_x)^2\right\} + \\
&\quad + 2\,\nu \sum_{k=2}^{m-2} \binom{m}{k} (m-k)\, \mu_{k,x}\, \langle Z\rangle^{m-k-1}\, \langle Z\rangle_x + \\
&\quad + \nu \sum_{k=2}^{m-3} \binom{m}{k} (m-k)\,(m-k-1)\, \mu_k\, \langle Z\rangle^{m-k-2}\, \langle Z\rangle_x^2 + \\
&\quad + 2\,m\,\nu\,\mu_{m-1,x}\,\langle Z\rangle_x + m\,(m-1)\,\nu\,\mu_{m-2}\,\langle Z\rangle_x^2 + \\
&\quad + \underbrace{\nu\, m\,(m-1)\,\langle Z\rangle^{m-2}\langle Z\rangle_x^2}_{\text{equal the first sum with k=2.}},
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = 2m\nu\mu_{m-1,x}\langle Z\rangle_x + m(m-1)\nu\mu_{m-2}\langle Z\rangle_x^2 - \\
&\quad - \nu \sum_{k=3}^{m-1} \binom{m}{k} \langle Z\rangle^{m-k} \left\{2k\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\mu_{k-2}\langle Z\rangle_x^2\right\} + \\
&\quad + \nu \sum_{k=2}^{m-2} \binom{m}{k} (m-k)\, 2\, \mu_{k,x}\, \langle Z\rangle^{m-k-1}\, \langle Z\rangle_x + \\
&\quad + \nu \sum_{k=2}^{m-3} \binom{m}{k} (m-k)\,(m-k-1)\, \mu_k\, \langle Z\rangle^{m-k-2}\, \langle Z\rangle_x^2. \qquad (5.29)
\end{aligned}$$

To show that the three sums on the right side of (5.29) are zero, we open the first one of them:

$$\sum_{k=3}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\left\{2\,k\,\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\mu_{k-2}\langle Z\rangle_x^2\right\} \underbrace{=}_{\mu_1=0}$$

$$= \sum_{k=3}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\,2k\mu_{k-1,x}\langle Z\rangle_x + \sum_{k=4}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\,k(k-1)\mu_{k-2}\langle Z\rangle_x^2 =$$

$$= 2\sum_{k=2}^{m-2}\binom{m}{k+1}(k+1)\langle Z\rangle^{m-k-1}\mu_{k,x}\langle Z\rangle_x +$$

$$+ \sum_{k=2}^{m-3}\binom{m}{k+2}(k+1)(k+2)\langle Z\rangle^{m-k-2}\mu_k\langle Z\rangle_x^2 \underbrace{=}_{\text{using (5.23) and (5.24)}}$$

$$= 2\sum_{k=2}^{m-2}\binom{m}{k}(m-k)\mu_{k,x}\langle Z\rangle^{m-k-1}\langle Z\rangle_x +$$

$$+ \sum_{k=2}^{m-3}\binom{m}{k}(m-k)(m-k-1)\mu_k\langle Z\rangle^{m-k-2}\langle Z\rangle_x^2.$$

Therefore, from (5.29) we arrive at (5.27).

□

Remark 5.3. *In Section 5.3 we have shown that the numerical method (5.16), for the moments, is stable and consistent with (5.17). Since we have used the same method (5.16) to compute the central moments, we conclude that the method for the central moments is stable and consistent with (5.27), equation (5.17) with a source term.*

5.5 Computational tests

In this section, we present some examples to assess our approach. In Examples 5.1 and 5.2 the initial condition allows exact statistical moments of the solution. We use Riemann initial conditions defined by bivariate normal distributions; in this case the solutions for the moments are given by (5.4). In order to investigate the influence of the randomness we use two models: in Example 5.1 the velocity, A, is normally distributed, and in Example 5.2 the velocity is lognormally distributed. In both cases we compare the exact solutions, given by (5.4), with the solutions yielded by the numerical scheme (5.16) for some statistical moments. In Example 5.3 we apply our method in the problem (5.1) where the initial condition is a normal random function and the transport velocity is a normal random variable. The numerical experiments presented in this section were done

in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

Example 5.1.

Let us consider the random Riemann problem (5.1)–(5.2) where the random velocity is normally distributed, $A \sim N(1.0, 0.8)$, and the random variables Q_L and Q_R have a bivariate normal distribution defined by: $\langle Q_L \rangle = 1.0$ (mean of Q_L); $\langle Q_R \rangle = 0.0$ (mean of Q_R); $\sigma_L = 0.4$ (standard deviation of Q_L); $\sigma_R = 0.5$ (standard deviation of Q_R); and $\rho = 0.4$ (correlation coefficient between Q_L and Q_R). In Figure 5.2 we compare the exact values for the mean, variance, 3rd central moment, and 4th central moment with the computations using (5.16) at $t_f = 0.4$, and Δt and Δx satisfying (5.20).

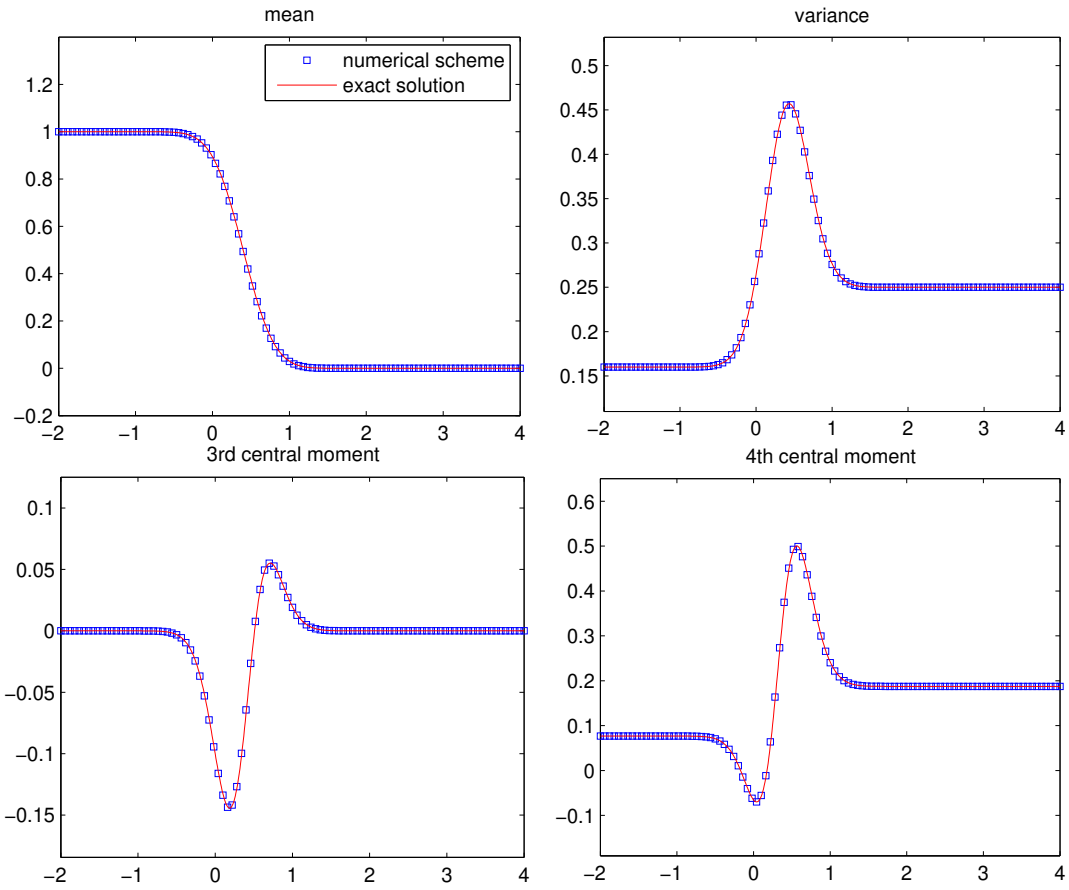

Figure 5.2: $A \sim N(1.0, 0.8)$, $\Delta x = 0.01$, $\Delta t = 0.000195$, and $t_f = 0.4$.

Example 5.2.

To check the influence of the velocity distribution we consider the random Riemann problem (5.1)–(5.2) in which the random velocity is lognormally distributed, $A = exp(\xi)$,

$\xi \sim N(0.5, 0.35)$. The initial condition (Q_L, Q_R) has a bivariate normal distribution defined by: $\langle Q_L \rangle = 1.0$; $\langle Q_R \rangle = 0.15$; $\sigma_L = 0.36$; $\sigma_R = 0.25$; and $\rho = 0.4$. Taking the lognormal distribution, $A = exp(\xi)$, $\xi \sim N(\mu_\xi, \sigma_\xi)$, in (5.18) we obtain

$$\nu(x,t) = \frac{\sigma_\xi^2 \left(\frac{x}{t}\right)\left(\frac{x}{t} - \langle A \rangle t\right)}{(\sigma_\xi^2 - \mu_\xi) + \ln\left(\frac{x}{t}\right)}. \tag{5.30}$$

This mean that it is not possible to find constants Δx and Δt such that $(\Delta x^2)/(4\Delta t) = \nu$, the consistency condition. Moreover, the diffusion coefficient given by (5.30) may be physically inappropriate since it can assume negative values. If we use (5.20) as in the previous example the results loose quality as shown in Figure 5.3.

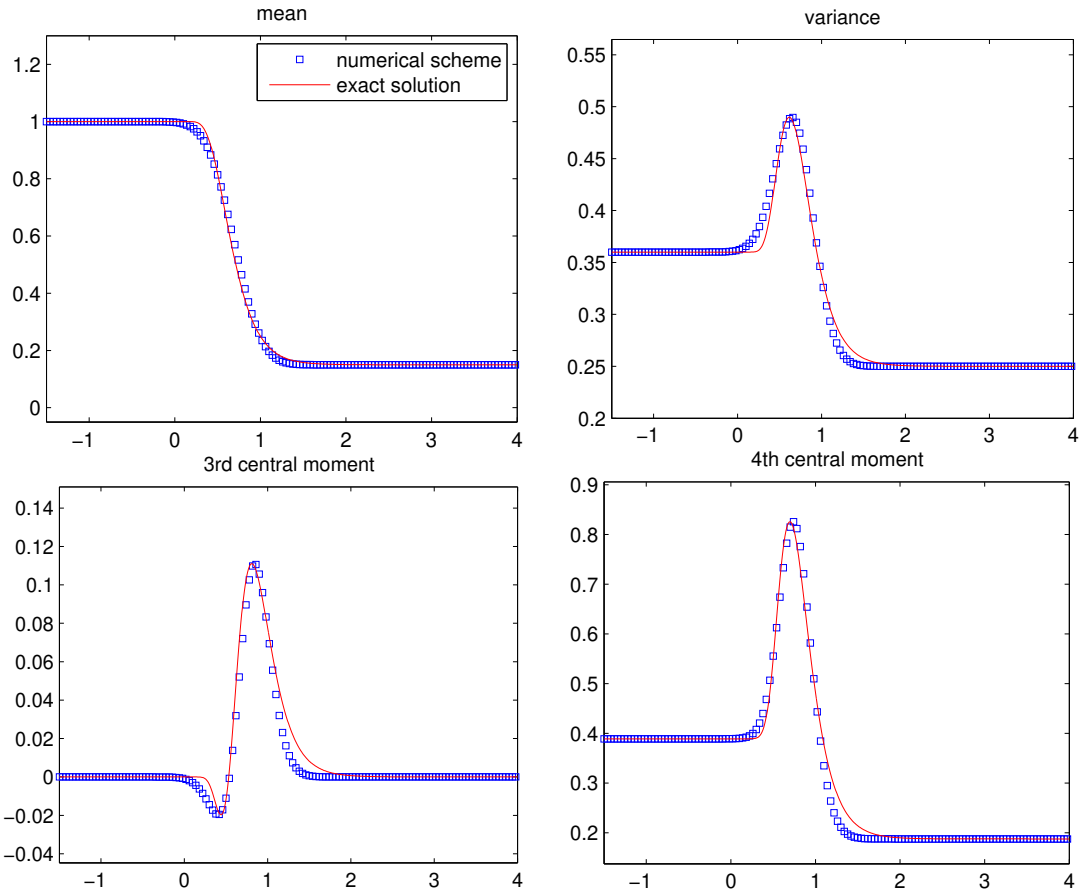

Figure 5.3: $A = exp(\xi)$, $\xi \sim N(0.5, 0.35)$, $\Delta x = 0.01$, $\Delta t = 0.000312$, and $t_f = 0.4$.

Example 5.3.

In this example we test our method for the random partial differential equation (5.1) in which A is normal, $A \sim N(-0.5, 0.6)$, and $Q_0(x)$ is a normal random function with

mean

$$\langle Q_0(x)\rangle = \begin{cases} 1, & x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, & \text{otherwise}, \end{cases} \tag{5.31}$$

and covariance $\mathrm{Cov}(x,\tilde{x}) = \sigma^2 \exp\left(-\beta|x-\tilde{x}|\right)$, where $\mathrm{Var}[Q_0(x)] = \sigma^2$ is constant and $\beta > 0$ governs the decay rate of the spatial correlation. We use $\beta = 0.3$ and $\sigma^2 = 0.16$. The numerical results are compared with Monte Carlo simulations using suites of realizations of A and $Q_0(x)$, where A and $Q_0(x)$ are statistically independents. Observe that each realization $A(\omega)$ and $Q_0(x,\omega)$ implies analytical solution given by $Q(x,t,\omega) = Q_0(x - A(\omega)t, \omega)$. To generate the realizations required by Monte Carlo simulations we use random numbers generator of MATLAB. Comparisons with the Monte Carlo method, with 30 000 realizations, are plotted in Figure 5.4.

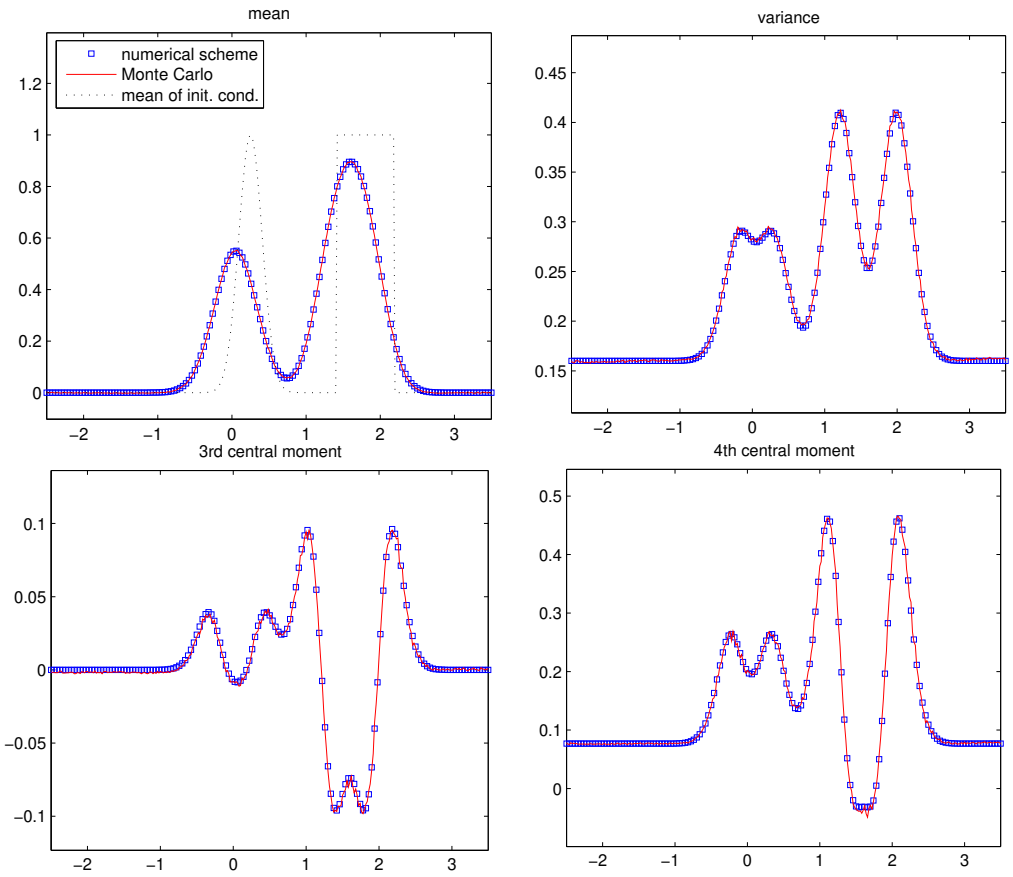

Figure 5.4: $A \sim N(-0.5, 0.6)$, $\Delta x = 0.02$, $\Delta t = 0.000138$, and $t_f = 0.4$.

5.6 Conclusions

In this paper, we have used the Godunov ideas to obtain a numerical scheme for the statistical moments of the solution of the one-dimensional random linear transport equation. We consider the velocity as a random variable and the initial condition as a random function. We have used an explicit solution of the random Riemann problem to evolve in the REA algorithm. Moreover, we have shown that the scheme is stable and consistent with an advective-diffusive equation. A particular Riemann problem solution is used to find the diffusion coefficient of the differential equations for the statistical moments. Also, we have obtained the differential equations for the central moments of the solution. Computational tests have illustrated our theoretical results.

Acknowledgments

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Capítulo 6

Statistical moments of the solution of the random Burgers-Riemann problem

Abstract

We solve Burgers' equation with random Riemann initial conditions. The closed solution allows simple expressions for its statistical moments. Using these ideas we design an efficient algorithm to calculate the statistical moments of the solution. Our methodology is an alternative to the Monte Carlo method. The present approach does not demand a random numbers generator as does the Monte Carlo method. Computational tests are added to validate our approach.

Keyword: random Burgers' equation, Monte Carlo method, Riemann problem, statistical moments, numerical methods for random partial differential equations.

6.1 Introduction

When the data of a differential equation, the coefficients or the initial conditions, are random variables its solution is a random function; this kind of mathematical problem has been called a random differential equation. A great number of practical processes under current investigations falls on the stochastic modeling; we may quote the models in control, communications, economic systems, chemical kinetics, biosciences, statistical mechanics and spatial areas and so on. The methodology to understand and solve differential equations with uncertainties has stimulated studies under several points of view. Since the solution is a random function, one particular solution corresponding to a spe-

cific realization is not of concern: it is important to know the statistical properties of the solution such as its mean, variance, or other statistical moments.

Some methods for random differential equations are categorized as moment equations methods. In these methods the purpose is to obtain differential equations governing the statistical moments. The most important of these equations is the differential equation for the expectation (mean), which is called for some authors as effective equation. As far as we know, no effective equation is known for the nonlinear problem discussed in this paper.

The Monte Carlo method is an alternative in solving random differential equations. Partial differential equations and the Monte Carlo method have been related for more than a century, since the works developed by Lord Rayleigh (1899), Courant *et al* (1928), and Kolmogorov (1931). For instance, Courant *et al* showed that a particular finite difference equation for the two dimensional Dirichlet boundary value problem and a two dimensional random walk produce the same results. In modern terms the Monte Carlo method originated from Los Alamos and the atomic bomb project. Now it is being used in many scientific fields [6, 20]. The basic idea is to solve a large number of deterministic differential equations choosing particular values for the random variables according to their assumed probabilistic distribution. The statistical information of the random solution is estimated using these realizations. The Monte Carlo method can be used in either linear or nonlinear random differential equations. However, the exceptionally large volume of calculations, and the difficulty for generating random numbers limit the significance of this method.

In a different direction we have been studying numerical methods for the random transport equation. In the linear case our ideas were inspired by Godunov's method [9, 15] for the deterministic transport equation. In [3] we present an explicit expression for the random solution to one-dimensional random advective equations where the constant velocity and the Riemann initial condition are random variables. This closed solution yields simple expressions for its statistical moments, and computational experiments show good agreement between our expressions and the Monte Carlo method for the first three moments. The closed solution for random Riemann problems and Godunov's ideas are used in [5] and [4] to design numerical methods to calculate the mean and variance of the solution to transport equations with more general initial condition (random fields). Our methods are explicit and do not demand differential equations governing the statistical moments, the effective equations. Furthermore, our scheme is consistent and stable with the diffusive effective equation presented in the literature [8]. Computational experiments have shown good agreements with the Monte Carlo method.

In this paper, we generalize our previous ideas to solve the random Riemann problem

for Burgers' equation

$$\begin{aligned} &\frac{\partial}{\partial t}U(x,t)+\frac{1}{2}\frac{\partial}{\partial x}U^2(x,t)=0, \quad t>0, \quad x\in\mathbb{R}, \\ &U(x,0)=\begin{cases} U_L, & \text{if} \quad x<0, \\ U_R, & \text{if} \quad x>0, \end{cases} \end{aligned} \tag{6.1}$$

where U_L and U_R are random variables. Here the randomness appears only because of the initial condition. The deterministic version of (6.1) was introduced by Burgers [1] as the simplest model that captures some key features of gas dynamics, the nonlinear hyperbolic term. But, rather than modeling a physical process, the inviscid Burgers equation has been widely used for developing both theoretical and numerical methods in the literature of deterministic hyperbolic equations.

Taking into account that several numerical methods to deal with deterministic conservation laws use solutions of Riemann problems (*Random Choice Method* developed by Glimm [7], and *Godunov's method* [9, 15], for example), we believe that the results of the current paper may be useful in developing numerical methods for more general random conservation laws. Moreover, since the mathematical theory of methods to random partial differential equations are difficult and not complete yet (see [13, 16, 19], for example), numerical methods can be a good alternative to deal with random differential equations.

Kim (2006) presents a scheme to calculate the statistical moments of the random Burgers'equation in [11]. Nevertheless, the author considers the simple case where the random initial condition is an explicit function of the spatial variable, and of the normal random variable with zero mean and unit variance. The author uses Wiener chaos expansion to separate random and deterministic effects, and utilizes the Lax-Wendroff method to discretize the deterministic system of partial differential equations that governs the propagation of randomness.

In this paper, we use two basic ideas to construct the solution, and its moments, to (6.1): (i) the realizations of the probabilistic problem are nonlinear transport equations whose analytical solutions are known (shock and rarefaction waves); and (ii) the random solution and its statistical moments, as functions of the initial condition and its joint density function, are found using geometrical partitions of the phase plane (U_L, U_R). Integrations on these sets are the shock and rarefaction averaging process.

The outline of this paper is as follows. In Section 2 we deduce an explicit solution to problem (6.1). We also show the similarity of the solution as well as present an expression for its statistical moments. Based on bidimensional midpoint quadrature formula, in Section 3 we suggest an efficient algorithm to approximate the statistical moments. Finally, we present some computational tests and conclusions.

6.2 The random solution

In this section, we construct the solution to (6.1), the one-dimensional Burgers' equation with random Riemann initial condition. We assume that the random initial states, U_L and U_R, and their joint probability density function, $f_{U_L U_R}$, are known. For a single realization, $U_L(\omega)$ and $U_R(\omega)$, of U_L and U_R, respectively, we have the deterministic Burgers-Riemann problem:

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t,\omega)+\frac{1}{2}\frac{\partial}{\partial x}u^2(x,t,\omega)=0, \quad t>0, \quad x\in\mathbb{R}, \\ &u(x,0,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad x<0, \\ U_R(\omega), & \text{if} \quad x>0. \end{cases} \end{aligned} \tag{6.2}$$

Physically correct solutions to (6.2), i.e., entropy solutions, are the rarefaction or shock waves (see [14, 15], for example):

(a) If $U_L(\omega) < U_R(\omega)$ then the solution is the rarefaction wave emanating from $(x,t)=(0,0)$

$$u(x,t,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad \frac{x}{t}<U_L(\omega), \\ \frac{x}{t}, & \text{if} \quad U_L(\omega)\le\frac{x}{t}\le U_R(\omega), \\ U_R(\omega), & \text{if} \quad \frac{x}{t}>U_R(\omega). \end{cases} \tag{6.3}$$

(b) If $U_L(\omega) > U_R(\omega)$ then the solution is the shock wave

$$u(x,t,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad \frac{x}{t}<s(\omega), \\ U_R(\omega), & \text{if} \quad \frac{x}{t}>s(\omega), \end{cases} \tag{6.4}$$

with the shock velocity, $s(\omega) = (1/2)\,[U_L(\omega)+U_R(\omega)]$, given by the Rankine-Hugoniot jump condition.

Thus, holding (x,t) fixed and considering the rarefaction and shock solutions together, we can join (6.3)-(6.4) to express $u(x,t,\omega)$ as

$$u(x,t,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad \beta<U_L(\omega) \quad \text{and} \quad U_L(\omega)<U_R(\omega), \\ \beta, & \text{if} \quad U_L(\omega)\le\beta\le U_R(\omega) \quad \text{and} \quad U_L(\omega)<U_R(\omega), \\ U_R(\omega), & \text{if} \quad \beta>U_R(\omega) \quad \text{and} \quad U_L(\omega)<U_R(\omega), \\ U_L(\omega), & \text{if} \quad \beta<\frac{1}{2}[U_L(\omega)+U_R(\omega)] \quad \text{and} \quad U_L(\omega)>U_R(\omega), \\ U_R(\omega), & \text{if} \quad \beta>\frac{1}{2}[U_L(\omega)+U_R(\omega)] \quad \text{and} \quad U_L(\omega)>U_R(\omega), \end{cases} \tag{6.5}$$

where $\beta = x/t$.

To simplify (6.5) we define the following mutually exclusive subsets of the phase plane

(U_L, U_R):

$$\begin{array}{lll}
\mathcal{R}_r^- & = \{(U_L,U_R) \text{ such that } U_L < U_R \text{ and } \beta < U_L\}; \\
\mathcal{R}_r^0 & = \{(U_L,U_R) \text{ such that } U_L < U_R \text{ and } U_L \leq \beta \leq U_R\}; \\
\mathcal{R}_r^+ & = \{(U_L,U_R) \text{ such that } U_L < U_R \text{ and } \beta > U_R\}; \\
\mathcal{R}_s^- & = \left\{(U_L,U_R) \text{ such that } U_L > U_R \text{ and } \beta < \tfrac{1}{2}[U_L+U_R]\right\}; \\
\mathcal{R}_s^+ & = \left\{(U_L,U_R) \text{ such that } U_L > U_R \text{ and } \beta > \tfrac{1}{2}[U_L+U_R]\right\}.
\end{array} \tag{6.6}$$

In this way, for a fixed $\beta = x/t$, we can rewrite the solution (6.5) as follows:

$$u(x,t,\omega) = \begin{cases} U_L(\omega), & \text{if } (U_L(\omega),U_R(\omega)) \in \mathcal{R}_r^- \bigcup \mathcal{R}_s^- = \mathcal{R}^-(\beta), \\ \beta, & \text{if } (U_L(\omega),U_R(\omega)) \in \mathcal{R}_r^0 = \mathcal{R}^0(\beta), \\ U_R(\omega), & \text{if } (U_L(\omega),U_R(\omega)) \in \mathcal{R}_r^+ \bigcup \mathcal{R}_s^+ = \mathcal{R}^+(\beta). \end{cases} \tag{6.7}$$

In Figure 6.1 we illustrate the phase plane as $\mathcal{R}^-(\beta) \bigcup \mathcal{R}^0(\beta) \bigcup \mathcal{R}^+(\beta)$; as we can see this partition of the phase plane depends exclusively of $\beta = x/t$.

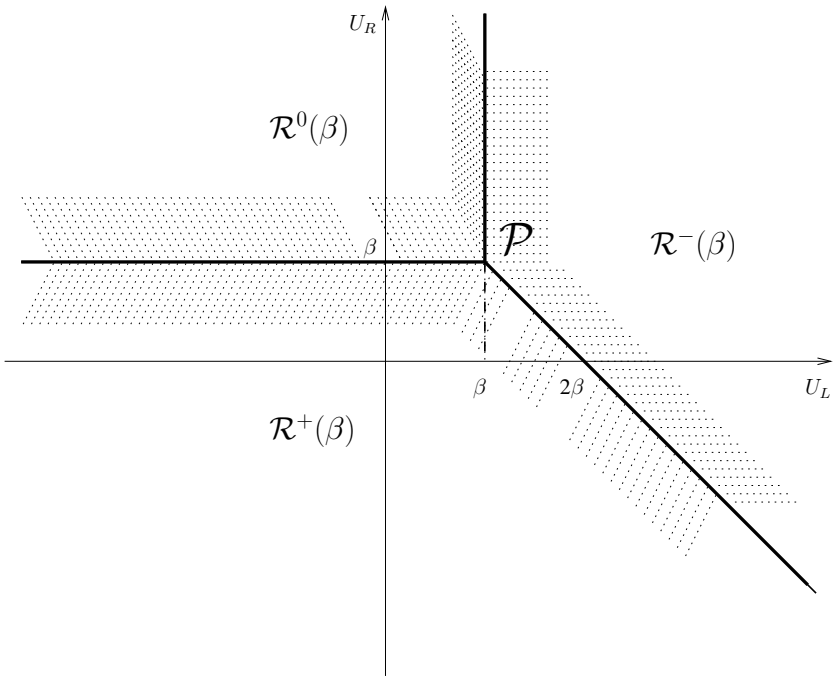


Figure 6.1: Integration regions.

Let $\mathcal{X}_A$ be the characteristic function of A, a set in (U_L, U_R) plane:

$$\mathcal{X}_A = \begin{cases} 1, & \text{if } (U_L,U_R) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Using $\mathcal{X}_A$ in (6.7), the arguments so far summarized prove the proposition:

Proposition 6.1. *The solution to the random Burgers-Riemann problem (6.1), in a fixed (x,t), is the random function*

$$U(x,t) = U_L \mathcal{X}_{\mathcal{R}^-} + \beta \mathcal{X}_{\mathcal{R}^0} + U_R \mathcal{X}_{\mathcal{R}^+}, \tag{6.8}$$

where $\beta = x/t$, and $\mathcal{X}_{\mathcal{R}^-}$, $\mathcal{X}_{\mathcal{R}^0}$, and $\mathcal{X}_{\mathcal{R}^+}$ are the characteristic functions of the mutually exclusive sets defined in (6.7).

Remark 6.1. *Expression (6.8) is the same for all (x,t) such that $x/t = \beta$. Therefore, $U(x,t)$ is a similarity function.*

In the following corollary, the expression (6.8) and the joint probability density function of U_L and U_R are used to calculate the statistical properties of the random solution.

Corollary 6.1. *The mth statistical moment of $U(x,t)$, for a fixed (x,t), $\beta = x/t$, is*

$$\begin{aligned} \langle U^m(x,t)\rangle = \int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L,u_R) du_L du_R + \\ +\beta^m \int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L,u_R) du_L du_R + \int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L,u_R) du_L du_R. \end{aligned} \tag{6.9}$$

Proof. From (6.8), since $\mathcal{R}^-(\beta)$, $\mathcal{R}^0(\beta)$ and $\mathcal{R}^+(\beta)$ are mutually exclusive sets, we have

$$\begin{aligned} \langle U^m(x,t)\rangle &= \int\int_{\mathbb{R}\times\mathbb{R}} \left[u_L \mathcal{X}_{\mathcal{R}^-} + \beta \mathcal{X}_{\mathcal{R}^0} + u_R \mathcal{X}_{\mathcal{R}^+}\right]^m f_{U_L U_R}(u_L,u_R) du_L du_R = \\ &= \int\int_{\mathbb{R}\times\mathbb{R}} \left[u_L^m \mathcal{X}_{\mathcal{R}^-} + \beta^m \mathcal{X}_{\mathcal{R}^0} + u_R^m \mathcal{X}_{\mathcal{R}^+}\right] f_{U_L U_R}(u_L,u_R) du_L du_R = \\ &= \int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L,u_R) du_L du_R + \beta^m \int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L,u_R) du_L du_R + \\ &+ \int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L,u_R) du_L du_R. \end{aligned}$$

□

Effective values of the moments (6.9) require the calculations of three double integrals for each value of β. In some particular cases we can calculate these integrals exactly. For instance, if U_L and U_R are independent random variables and uniformly distributed in the interval $[-a,a]$, some calculations show that the mean of the solution to (6.1) is given by

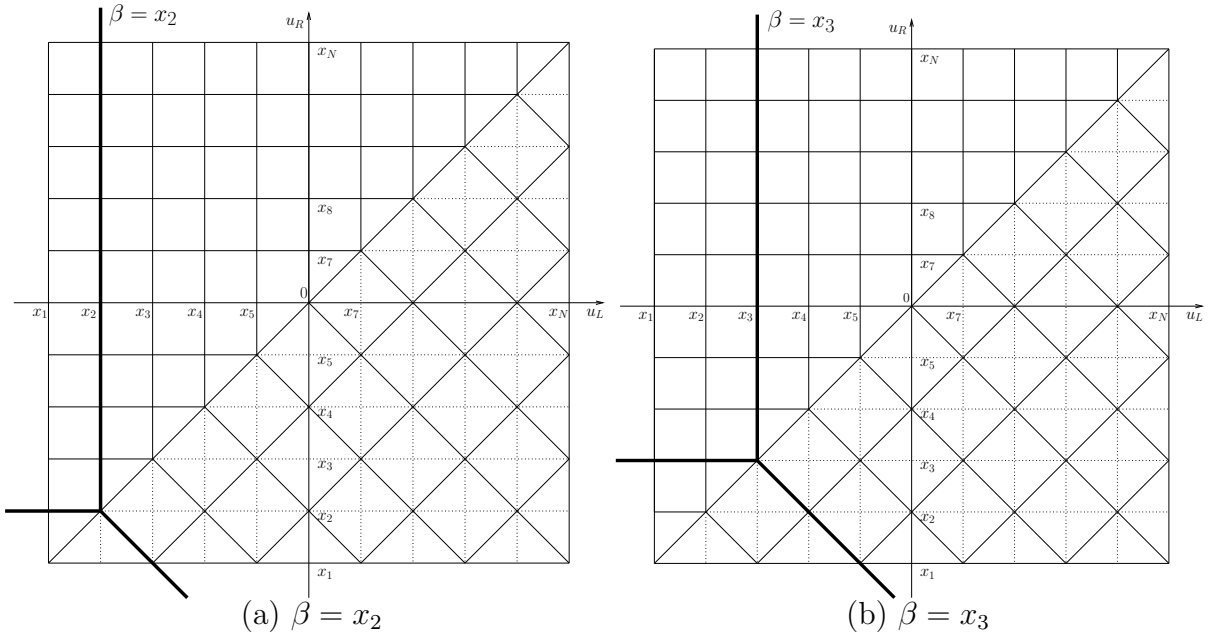
$$\langle U(x,t)\rangle = \begin{cases} -\frac{\beta}{4a^2}\left[\text{sign}(\beta)\beta - a\right]^2, & \text{if} \quad -a \le \beta \le a, \\ 0, & \text{otherwise}, \end{cases} \tag{6.10}$$

where $\beta = x/t$. We will use this solution in Example 6.1 as a test problem to assess the performances between the Monte Carlo method and our algorithm. However, in general we must use numerical integration to calculate (6.9). In the following section we describe a useful way to do that.

6.3 The algorithm

To design an efficient algorithm to calculate the statistical moments using (6.9), we take a square in the phase plane (U_L, U_R), $\Gamma_M = [-M, M] \times [-M, M]$, which contains the effective support of $f_{U_L U_R}$; this means that outside Γ_M the values of the density probability function, $f_{U_L U_R}$, are sufficiently near to zero.

As shown in Figure 6.1, the point $\mathcal{P} = (\beta, \beta)$, $\beta = x/t$, is critical to define $\mathcal{R}^-$, $\mathcal{R}^0$ and $\mathcal{R}^+$ regions. This point moves in northeast (southwest) direction as β increases (decreases). Without loss of generality, we will take $t = 1$ and use the similarity property (Remark 6.1) to obtain the solution for $t > 0$. Therefore, we may identify $x_j = \beta_j$ and take the same discretization grid for β, U_L, and U_R, as illustrated in Figure 6.2.



(a) $\beta = x_2$ (b) $\beta = x_3$

Figure 6.2: Discretization scheme of the Γ_M square.

Notation:

- $\{x_j = -M + jh;\ (j = 1 : N)\}$ is the β-mesh with $\Delta\beta = h > 0$; $x_1 = -M$; $x_N = M$; $x_{j+1/2} = x_j + h/2$ $(j = 1 : N - 1)$; N is an odd number;
- $I_l \approx \int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L, u_R) du_L du_R$;
- $I_0 \approx \int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L, u_R) du_L du_R$;
- $I_r \approx \int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L, u_R) du_L du_R$.

We initiate the calculations taking $\beta = \beta_1 = x_1$. In this case $\Gamma_M \subset \mathcal{R}^-(\beta_1)$ and, consequently, $I_l = \langle U_L^m \rangle$, $I_0 = 0$ and $I_r = 0$; these values are used to initiate the algorithm. To save memory, the temporary calculations to update I_l, I_0 and I_r in β_j-step, $x_j = x_{j-1} + h$ $(j = 2, 3, \ldots, N)$, are done in the "S_j^h-strip": $S_j^h = \Gamma_M \bigcap \{\mathcal{R}^-(\beta_j) \setminus \mathcal{R}^-(\beta_{j-1})\}$. This strip is a collection of squares (and half squares) with edges h and $\sqrt{2}h$ (see Figure 6.2). The integration is performed using the *bidimensional midpoint quadrature formula* (see [10]). To clarify the ideas, in Table 6.1 we summarized the first step of our algorithm.

Step 1
$\beta \leftarrow x_2$;
$I_l \leftarrow I_l - h^2 x_{1+1/2}^m \left\{ \sum_{i=1}^{N-2} f\left(x_{1+1/2}, x_{N-i+1/2}\right) + \frac{1}{2} f\left(x_{1+1/2}, x_{1+1/2}\right) \right\}$ $-h^2 x_2^m f\left(x_2, x_1\right)$;
$I_0 \leftarrow I_0 + h^2 \sum_{i=1}^{N-2} f\left(x_{1+1/2}, x_{N-i+1/2}\right)$;
$I_r \leftarrow I_r + \frac{1}{2} h^2 x_{1+1/2}^m f\left(x_{1+1/2}, x_{1+1/2}\right) + h^2 x_1^m f\left(x_4, x_1\right)$;
$\langle U(x_2, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r$.

Table 6.1: Illustration of the first step of Algorithm 1

Repeating this idea in the next β_j-steps, we formulate Algorithm 1.

Remark 6.2. *Observe that the Γ_M-discretization scheme has the recursive advantage: the solution at β_j can be calculated just by updating the solution at β_{j-1}.*

Remark 6.3. *The main advantage of Algorithm* 1 *is that it does not require a random numbers generator (massive simulation of data with a known probability distribution) as does the Monte Carlo method. Furthermore, as we will see in Examples 6.1 and 6.4, its convergence is faster than the Monte Carlo method.*

Algorithm 1

N is an odd number;

$I_l = \langle U_L^m \rangle$; $I_0 = 0$; $I_r = 0$;

for $k = 1 : \frac{N-1}{2}$ **do**

$\beta \leftarrow x_{k+1}$;

$$I_l \leftarrow I_l - h^2 x_{k+1/2}^m \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$$
$$-2h^2 \sum_{i=k+1}^{2k-1} x_i^m f\left(x_i, x_{2k-i+1}\right) - h^2 x_{2k}^m f\left(x_{2k}, x_1\right);$$

$$I_0 \leftarrow I_0 + h^2 \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) - \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) \right\};$$

$$I_r \leftarrow I_r + h^2 x_{k+1/2}^m \left\{ \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$$
$$+2h^2 \sum_{i=k+1}^{2k-1} x_{2k-i+1}^m f\left(x_i, x_{2k-i+1}\right) + h^2 x_1^m f\left(x_{2k}, x_1\right);$$

$\langle U(x_{k+1}, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r$;

end for

for $k = \frac{N+1}{2} : (N-1)$ **do**

$\beta \leftarrow x_{k+1}$;

$$I_l \leftarrow I_l - h^2 x_{k+1/2}^m \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$$
$$-2h^2 \sum_{i=k+1}^{N-1} x_i^m f\left(x_i, x_{2k-i+1}\right) - h^2 x_N^m f\left(x_N, x_{2k-N+1}\right);$$

$$I_0 \leftarrow I_0 + h^2 \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) - \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) \right\};$$

$$I_r \leftarrow I_r + h^2 x_{k+1/2}^m \left\{ \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$$
$$+2h^2 \sum_{i=k+1}^{N-1} x_{2k-i+1}^m f\left(x_i, x_{2k-i+1}\right) + h^2 x_{2k-N+1}^m f\left(x_N, x_{2k-N+1}\right);$$

$\langle U(x_{k+1}, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r$;

end for

6.4 Computational tests

In this section, we present some examples to assess and illustrate our approach. In Example 6.1 we take an initial condition that allows exact calculations for the mean. In the following examples the initial condition has a bivariate normal distribution. In these examples the mean, variance, 3rd central moment, and 4th central moment of the solution are obtained by Algorithm 1 and compared with the Monte Carlo method. To generate the realizations $(U_L(\omega), U_R(\omega))$, required by the Monte Carlo method, we use random numbers generators of MATLAB. The analytical solution for each realization, $(U_L(\omega), U_R(\omega))$, is given by (6.3) or (6.4). We compare the performances of the methods. We also plot the solution of the deterministic problem where the initial condition is the statistical mean of the random data. Some authors ([17], for example) use the name "*naive*" for this solution. The numerical experiments presented in this section were done in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

Example 6.1.

We use (6.10) to calculate exact values of the mean of the solution to (6.1) with the Riemann initial condition:

$$U(x,0) = \begin{cases} U_L, & \text{if} \quad x < 0, \\ U_R, & \text{if} \quad x > 0, \end{cases}$$

where U_L and U_R are independent random variables uniformly distributed in the interval $[-1, 1]$. The mean, at $t = 0.4$ and $t = 0.8$, is plotted in Figure 6.3. Absolute errors of approximations given by the Monte Carlo method and Algorithm 1 are compared in Table 6.2, using $\langle U(x,1)\rangle$, $x \in [-1,1]$. The CPU times are also presented in this table.

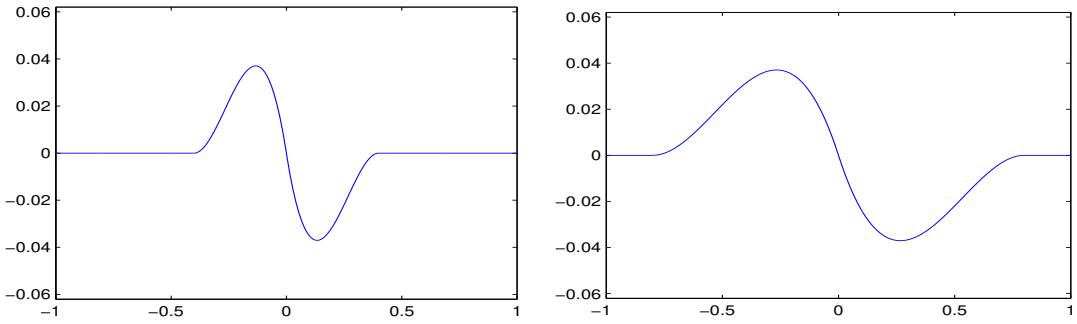


Figure 6.3: Mean at $t = 0.4$ (left) and $t = 0.8$ (right).

Monte Carlo method			Algorithm 1		
realizations (N_r)	absolute error ($\|\| \, . \, \|\|_\infty$)	CPU time (sec)	number of partitions (N)	absolute error ($\|\| \, . \, \|\|_\infty$)	CPU time (sec)
1 000	0.0268	0.071	201	2.49×10^{-5}	0.084
5 000	0.0124	0.358			
10 000	0.0091	0.718			
30 000	0.0048	2.154			
50 000	0.0037	3.599			
100 000	0.0027	7.223			

Table 6.2: Absolute errors and CPU times; $h = 0.01$.

Example 6.2.

Let us consider the problem (6.1) with U_L and U_R having bivariate normal distribution defined by: $\langle U_L \rangle = 0.1$ (mean of U_L); $\langle U_R \rangle = 0.9$ (mean of U_R); $\sigma_L = 0.3$ (standard deviation of U_L); $\sigma_R = 0.2$ (standard deviation of U_R); and $\rho = 0.42$ (correlation coefficient between U_L and U_R). Since the probability density function with these data has the effective support in the semiplane $U_L < U_R$, the rarefaction wave solutions dominate. Figure 6.4 illustrates the mean (compared with the naive solution), variance, 3rd central moment, and 4th central moment calculated at $t = 1$ for $x \in [-3, 3]$. As we can see, the randomness of the initial conditions smoothen the edges of the naive solution, as in the random linear transport equations.

Example 6.3.

To illustrate a shock-dominant case, we changed the data of U_L and U_R used in the previews example: $\langle U_L \rangle = 0.9$; $\langle U_R \rangle = 0.1$; $\sigma_L = 0.3$; $\sigma_R = 0.2$; and $\rho = 0.42$. With these data, the bivariate normal probability density function has the effective support in the semiplane $U_L > U_R$. In Figure 6.5 we plot the mean, variance, 3rd central moment, and 4th central moment calculated at $t = 1$ for $x \in [-3, 3]$. Again, the randomness of the initial conditions smoothen the edges of the naive solution.

Example 6.4.

In this example, we also consider a bivariate normal distribution with data that mix rarefaction and shock waves in the realizations: $\langle U_L \rangle = 0.2$; $\langle U_R \rangle = 0.4$; $\sigma_L = 0.2$; $\sigma_R = 0.5$; and $\rho = 0.42$. In Figure 6.6 we present approximations to the mean, variance, 3rd central moment, and 4th central moment computed using the Monte Carlo method and Algorithm 1. We also include the naive solution. Since $\langle U_L \rangle < \langle U_R \rangle$ the naive solution is a rarefaction wave. This example emphasizes the difference between the mean of the solution and the solution computed using means of the data. Here, the effect of the randomness is more than to smoothen edges: as shown in Figure 6.6 the mean of

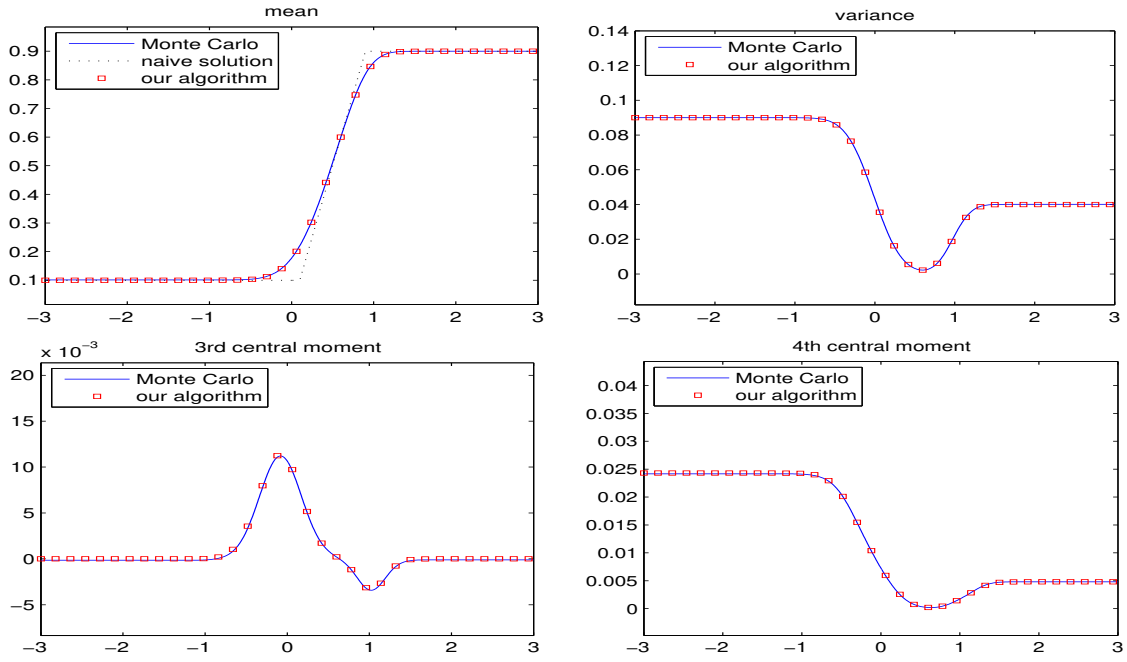


Figure 6.4: Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

the solution is a humped function. In Table 6.3 we compare the performances between the Monte Carlo method and Algorithm 1 in calculating $\langle U(x,1)\rangle$, $x \in [-3,3]$, taking into account the error estimates of each method and the CPU time. For instance, in the approximations plotted in Figure 6.6 the Monte Carlo method has taken 8.675 sec while Algorithm 1 has taken 0.991sec.

Monte Carlo method			Algorithm 1		
realizations (N_r)	estimate of error $\mathcal{O}(1/\sqrt{N_r})$	CPU time (sec)	number of partitions (N)	estimate of error $\mathcal{O}(1/N^2)$	CPU time (sec)
1 000	0.0316	0.185	601	0.00018	0.991
5 000	0.0141	0.877			
10 000	0.0100	1.744			
30 000	0.0063	5.210			
50 000	0.0044	8.675			
100 000	0.0031	17.294			

Table 6.3: Absolute errors and CPU times; $h = 0.01$ (600 subintervals).

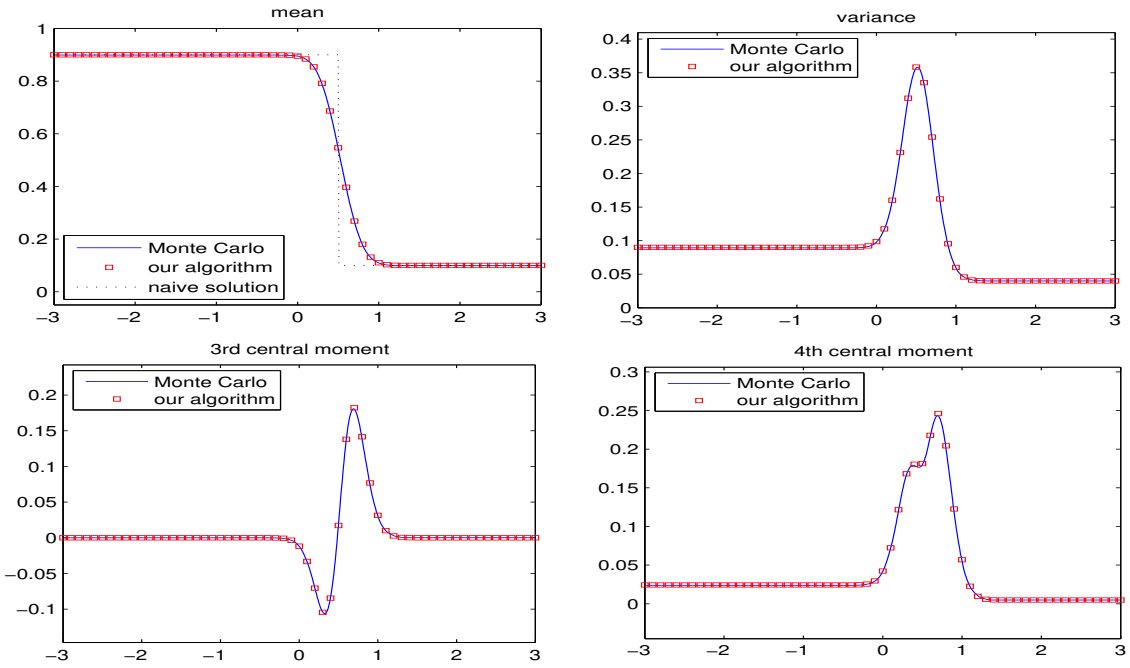


Figure 6.5: Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

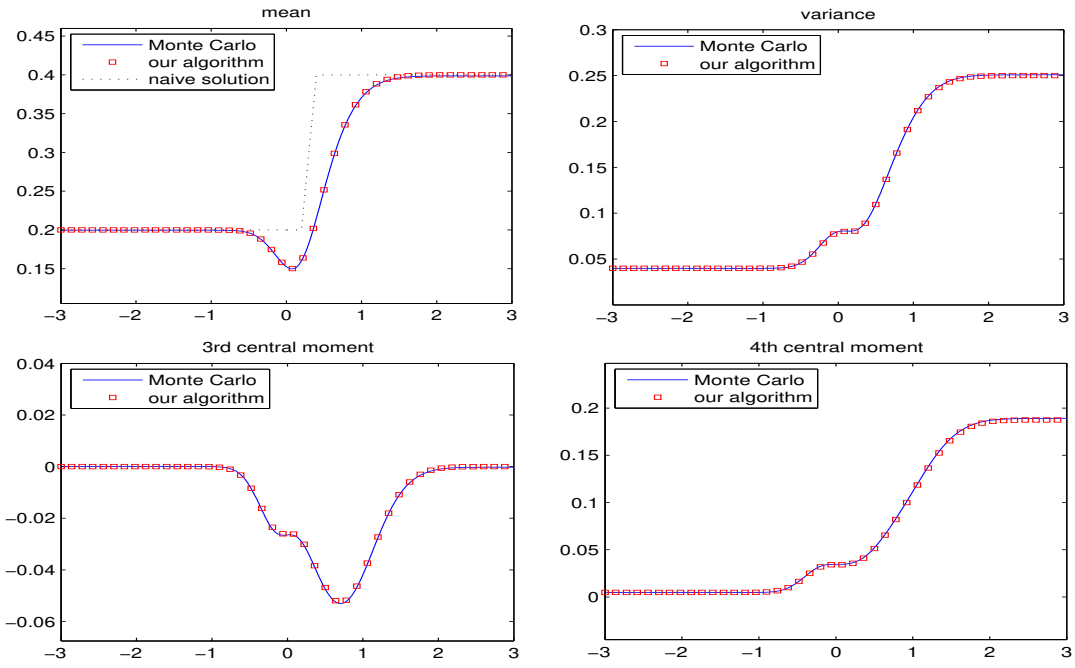


Figure 6.6: Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

6.5 Concluding remarks

We have used the basic solutions to nonlinear conservation laws, the shock and rarefaction waves, to construct the random solution for Burgers' equation with random Riemann initial condition. These basic solutions are grouped to deduce simple expressions to calculate the statistical properties of the random solution by integrations in three mutually exclusive cones in the phase plane (Figure 6.1). We also design an algorithm to calculate the integrals, in case of difficult analytic expressions of the joint density distribution of the initial condition. Our approach outperformed the Monte Carlo method in terms of accuracy and computational cost. We believe that this approach can be also used to solve more general problems.

Acknowledgments

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Capítulo 7

On the evaluation of moments for solute transport by random velocity fields

Abstract

In this note, we consider the random linear transport equation. We indicate that standard averaging approaches to obtain an equation for the evolution of the statistical mean of the solution may also be valid for all the statistical moments of the solution. With this result we can obtain more statistical information about the random solution, as illustrated in two particular examples.

Keyword: random linear transport equation, random velocity field, averaging approach, statistical moments, Gaussian process, Telegraph process.

7.1 Introduction

In this note, we consider the transport of a passive scalar by an incompressible random velocity field as described by the equation

$$U_t(\mathbf{x},t) + \nabla.[\mathbf{V}(\mathbf{x},t)U(\mathbf{x},t)] = 0, \quad U(\mathbf{x},t_0) = g(\mathbf{x}), \tag{7.1}$$

where U is the density of a passively advected agent (concentration of a chemical species, temperature, etc.), $\mathbf{V}$ is a random velocity field, and $g(\mathbf{x})$ is the deterministic initial distribution of the scalar. The subscript t in $U_t(\mathbf{x},t)$ denotes the partial derivative with respect to this variable. Taking into account the incompressibility of $\mathbf{V}$, i.e., $\nabla \cdot \mathbf{V} = 0$, we rewrite equation (7.1) as

$$U_t(\mathbf{x},t) + V_i(\mathbf{x},t)U_{x_i}(\mathbf{x},t) = 0, \quad U(\mathbf{x},t_0) = g(\mathbf{x}), \tag{7.2}$$

where repeated indices indicate summation.

Standard approaches (see, e.g., [2, 6, 7, 9, 10, 12]) to derive an equation for the mean of U use the Reynolds decomposition

$$V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$$

(angle brackets denote ensemble averaging) in (7.2) to obtain the following non-closed averaged equation

$$\langle U\rangle_t + \langle V_i(\mathbf{x},t)\rangle\langle U\rangle_{x_i} + \langle V_i'(\mathbf{x},t)U_{x_i}\rangle = 0. \tag{7.3}$$

The basic difficulty with such approaches lies in the necessity to approximate (model) the unknown correlation moment between the random velocity fluctuations and $U(\mathbf{x},t)$, the term $\langle V_i'(\mathbf{x},t)U_{x_i}(\mathbf{x},t)\rangle$ in (7.3). Moreover, the knowledge of the mean, $\langle U(\mathbf{x},t)\rangle$, is not enough to provide a detailed understanding of the random transport process. One must at least examine higher moments of $U(\mathbf{x},t)$. With that in mind, it is our purpose to show that for the linear transport equation (7.1), some approaches used to approximate $\langle V_i'(\mathbf{x},t)U_{x_i}(\mathbf{x},t)\rangle$ in (7.3) may be also used to approximate all the moments of the solution. In Section 7.2 we present this result, and in Sections 7.3–7.4 we illustrate the approach with two examples.

7.2 Main result

Proposition 7.1. *Let $V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$ in (7.2). Then*

$$\langle U^m(\mathbf{x},t)\rangle_t + \langle V_i(\mathbf{x},t)\rangle\langle U^m(\mathbf{x},t)\rangle_{x_i} + \langle V_i'(\mathbf{x},t)U^m_{x_i}(\mathbf{x},t)\rangle = 0, \tag{7.4}$$

where $\langle U^m(\mathbf{x},t)\rangle$, $m\in\mathbb{Z}$, $m\geq 1$, is the mth moment of the solution to (7.2).

Proof. Notice that $U^m(\mathbf{x},t)$, $m\in\mathbb{Z}$, $m\geq 1$, satisfies an equation like (7.2). Indeed, differentiating $U^m(\mathbf{x},t)$ with respect to t and x, and using (7.2) we obtain

$$(U^m)_t + V_i(\mathbf{x},t)(U^m)_{x_i} = m\,U^{(m-1)}[\phi U_t(\mathbf{x},t) + V_i(\mathbf{x},t)U_{x_i}(\mathbf{x},t)] = 0.$$

Averaging this expression and using the Reynolds decomposition of the velocity, $V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$, yields (7.4).

□

Remark 7.1. *In [1] we have shown that in one-dimensional transport problems with a constant random velocity, V, if the partial differential equation for the moments is an advection-diffusion equation with diffusion coefficient ν, then ν must satisfy the equation $-f_V'(x/t)\nu = f_V(x/t)(x-\langle V\rangle t)$, where $f_V(\xi)$ is the probability density function of V. For such problems equation (7.3) is closed with $-\langle V'U_x\rangle = \nu\langle U\rangle_{xx}$.*

7.3 First application: Gaussian processes

Consider the following one-dimensional version of problem (7.1):

$$U_t(x,t) + V(t)U_x(x,t) = 0, \quad U(x,0) = \mathbf{H}(-x), \tag{7.5}$$

where $\mathbf{H}(x)$ is the Heaviside function, and the random velocity, $V(t)$, is Gaussian with $\langle V(t)\rangle = V$ constant and an exponentially decaying covariance function, $\mathrm{Cov}_V(t,\tau) = \sigma_V^2 \exp(-|t-\tau|/\beta)$. The covariance function is parameterized by the variance, $\mathrm{Var}[V(t)] = \sigma_V^2$ (which is assumed to be constant), and by the correlation length, $\beta > 0$, which governs the decay rate of the time correlation.

According to [6, 9, 10], the correlation moment between the random flow-velocity and the random concentration U can be written in the form

$$\langle V'(t)U_x(x,t)\rangle = -\left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right) \langle U(x,t)\rangle_{xx}. \tag{7.6}$$

Thus, the mean concentration is exactly governed by

$$\begin{aligned} &\langle U(x,t)\rangle_t + V\langle U(x,t)\rangle_x = \left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right) \langle U(x,t)\rangle_{xx}, \\ &\langle U(x,0)\rangle = \mathbf{H}(-x). \end{aligned} \tag{7.7}$$

In view of Proposition 7.1 we can use (7.6) to calculate all the moments, i.e., the mth moment satisfies the following equation:

$$\begin{aligned} &\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right) \langle U^m(x,t)\rangle_{xx}, \\ &\langle U^m(x,0)\rangle = [\mathbf{H}(-x)]^m = \mathbf{H}(-x). \end{aligned} \tag{7.8}$$

The solution to (7.8) is

$$\langle U^m(x,t)\rangle = \frac{1}{2}\mathrm{erfc}\left(\frac{x-Vt}{\xi(t)}\right), \tag{7.9}$$

where $\mathrm{erfc}(x)$ is the complementary error function and

$$\xi(t) = 2\left[\int_0^t \int_0^\eta \mathrm{Cov}_V(\eta,\tau) d\tau d\eta\right]^{1/2}.$$

We now compare the moments (7.9) with those yielded by the Monte Carlo method. To generate the realizations $V(t,\omega)$ required by the Monte Carlo method, we use the subroutine [*mvnrnd.m*] of MATLAB. The analytical solution for each realization is

$$U(x,t,\omega) = U\left(x - \int_0^t V(s,\omega)\, ds, 0\right) = \mathbf{H}\left(\int_0^t V(s,\omega)\, ds - x\right).$$

In our numerical experiments the integration of $V(t,\omega)$ is performed using the Simpson's quadrature rule (see [3], for example). Figures 7.1 and 7.2 illustrate the mean, variance, and third central moment of the solution to (7.5) computed using the averaging approach and the Monte Carlo method (with 50 000 realizations). The plots correspond to the following data: $\langle V(t)\rangle = V = -0.2$; $\sigma_V^2 = 0.4$; $t = 0.6$; $\Delta t = 0.001$; and $\Delta x = 0.0005$. In Figure 7.1 we use $\beta = 0.1$ and in Figure 7.2 we use $\beta = 1.0$, i.e, a more correlated field. All the numerical experiments were done in double precision with some MATLAB codes on a 1.73Ghz Intel Core Duo 2 with 2Gb of memory.

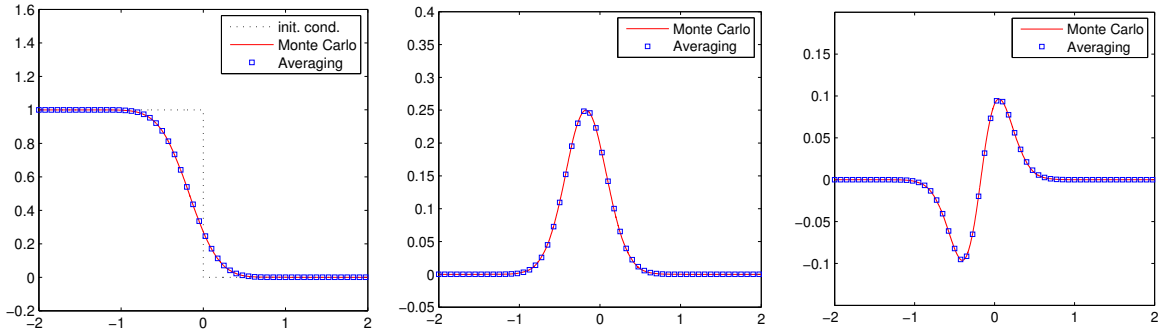

Figure 7.1: Mean (left), variance (middle), and third central moment (right) of the solution to (7.5); $\beta = 0.1$.

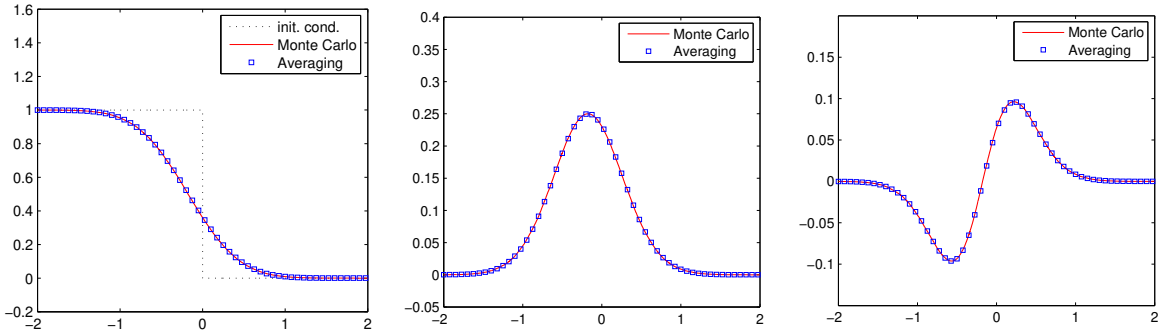

Figure 7.2: Mean (left), variance (middle), and third central moment (right) of the solution to (7.5); $\beta = 1.0$.

7.3.1 The probability density function

For this particular example, we have shown that each moment of the solution to (7.5), $\langle U^m(x,t)\rangle$ (even in the case of a more general random initial condition, $G(x)$) satisfies

the following advection-diffusion equation:

$$\begin{aligned} &\psi_t(x,t) + V\psi_x(x,t) = \beta(t)\,\psi_{xx}(x,t), \\ &\psi(x,0) = \langle G(x)^m \rangle, \end{aligned} \tag{7.10}$$

where $\beta(t) = \displaystyle\int_0^t \mathrm{Cov}_V(t,\tau)\; d\tau$. As a consequence, the probability density function for the random solution $U(x,t)$, $f_U(u;x,t)$, also satisfies an initial value problem for the advection-diffusion equation (7.10), i.e.,

$$\begin{aligned} &(f_U)_t + V(f_U)_x = \beta(t)\,(f_U)_{xx}, \\ &f_U(u;x,0) = f_G(u;x). \end{aligned} \tag{7.11}$$

Indeed, the Fourier transform of $f_U(u;x,t)$, under the assumption that the probability density function is uniquely determined by its moments (see, e.g., [5] for conditions for uniqueness in the problems of moments), is

$$\widehat{f_U}(\omega;x,t) = \sum_{j=0}^{\infty} \frac{(i\omega)^j}{j!} \langle U^m(x,t) \rangle, \tag{7.12}$$

where $\phi\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \beta(t)\,\langle U^m(x,t)\rangle_{xx}$. Taking the derivative with respect to t and x in (7.12), we arrive at

$$(\widehat{f_U})_t + V(\widehat{f_U})_x = \beta(t)\,(\widehat{f_U})_{xx}. \tag{7.13}$$

Since the variable ω does not appear in the derivatives, we can go back to the variable u and find (7.11). The respective initial condition follows from the probability density function of $G(x)$. This result for the density probability of $U(x,t)$, $f_U(u;x,t)$ agrees with that presented in [11] on page 247 using a different methodology.

7.4 Second application: Telegraph processes

In this section, we consider the one-dimensional transport with the random telegraph process (see [4, 8], for example) as a model for the velocity, $V(t)$. According to [10], this is a convenient model of a function that has finite jumps in random times. The random telegraph process is a stochastic process $V(t)$ defined by

$$V(t) = V + \xi(-1)^{N(t)}, \tag{7.14}$$

where the state space of $V(t)$ is $\{V-\alpha_0, V+\alpha_0\}$, the times at which the process changes the values $(V-\alpha_0)$ and $(V+\alpha_0)$ are distributed according to a *Poisson process* $N(t)$

with intensity rate λ, and ξ is a random variable independent of $N(t)$ and such that $P\{\xi = \alpha_0\} = 1/2 = P\{\xi = -\alpha_0\}$. This process is stationary (see [4], for more details) with mean $\langle V(t)\rangle = V$ and covariance $\mathrm{Cov}_V(t,\tau) = \alpha_0^2 \exp(-2\lambda|t-\tau|)$.

According to [6, 10], the correlation moment between $V'(t)$ and $U(x,t)$ is exactly given by

$$\langle V'(t)U(x,t)\rangle = -\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U(x - V(t-\tau),\tau)\rangle \, d\tau. \tag{7.15}$$

Using (7.15) in (7.3) we obtain the differential equation for the mean concentration,

$$\langle U(x,t)\rangle_t + V\langle U(x,t)\rangle_x = \frac{\partial}{\partial x}\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U(x - V(t-\tau),\tau)\rangle \, d\tau. \tag{7.16}$$

Proposition 7.1 asserts that Equation (7.16) is the same for all statistical moments, i.e., the mth moment satisfies the equation

$$\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \frac{\partial}{\partial x}\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U^m(x - V)(t-\tau),\tau)\rangle \, d\tau.$$

The analysis of the exact solution to (7.16) is presented in [10].

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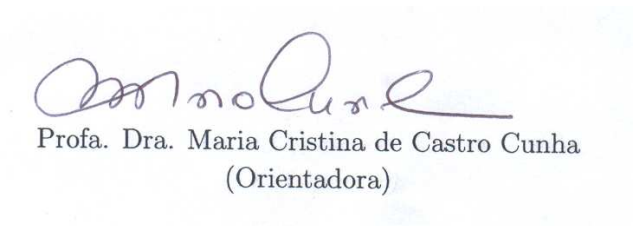
Métodos para Equações do Transporte com Dados Aleatórios

Este exemplar corresponde à redação final da Tese devidamente corrigida e defendida por Fabio Antonio Dorini e aprovada pela Banca Examinadora.

Campinas, 17 de dezembro de 2007.



Profa. Dra. Maria Cristina de Castro Cunha
(Orientadora)

Tese apresentada ao Programa de Pós Graduação do Instituto de Matemática, Estatística e Computação Científica, UNICAMP, como requisito parcial para a obtenção do título de Doutor em Matemática Aplicada.

Capítulo 8

Conclusões e trabalhos futuros

8.1 Conclusões

A apresentação desta tese na forma cronológica em que nosso trabalho foi sendo desenvolvido e submetido a revistas especializadas reflete como fomos nos aprofundando numa area bastante fértil.

O desafio inicial de conhecer métodos para lidar com incertezas em parâmetros de equações diferenciais como modelos matemáticos foi motivado pela simulação de fluxo em meios porosos. Historicamente estes modelos foram incorporando os conceitos de permeabilidade, permeabilidade relativa, porosidade, pressão capilar, além dos mais tradicionais na mecânica dos fluidos, como a viscosidade. Experimentos em laboratório são usados na avaliação destas propriedades e como tal estão sujeitos a erros de medição e dificuldade de obtenção de amostras compatíveis com os reservatórios em analise. Assim, deve-se considerar a aleatoriedade dos parâmetros que alimentam as várias equações diferenciais que são usadas como modelo de fluxos em meios porosos.

Os trabalhos apresentados nesta tese representam nossa contribuição na metodologia aplicável as equações diferenciais com dados aleatórios.

8.2 Trabalhos futuros

Nesta seção resumimos três problemas que estão em estudo e nos quais usamos algumas idéias desenvolvidas nesta tese.

8.2.1 Problema 1.

As equações diferenciais parciais que foram objeto de estudo nesta tese são casos particulares da lei de conservação mais geral:

$$\begin{aligned} &\frac{\partial}{\partial t}Q(x,t)+\frac{\partial}{\partial x}f(Q(x,t))=0, \quad t>0, \quad x\in\mathbb{R}, \\ &Q(x,0)=Q_0(x). \end{aligned} \tag{8.1}$$

onde Q é a quantidade conservada e $f(Q)$ é a função fluxo. Aplicações desta equação aparecem em problemas de recuperação de petróleo, dispersão de poluentes, fluxo de gases, fluxo de tráfego, dentre outros. Visando a dar continuidade aos nossos estudos, e baseados nas idéias apresentadas em [6, 17], vamos investir na busca das propriedades estatísticas da solução da Equação de Buckley-Leverett unidimensional aleatória, que modela fluxos bifásicos imiscíveis e incompressíveis em um meio poroso,

$$\begin{aligned} &\frac{\partial}{\partial t}S(x,t)+\frac{\partial}{\partial x}f(S(x,t))=0, \quad t>0, \quad x\geq 0, \\ &S(0,t)=S^-, \quad t>0, \\ &S(x,0)=S^+, \quad x>0, \end{aligned} \tag{8.2}$$

onde $S(x,t)$ representa a saturação de água no meio poroso e a função fluxo, determinada usando a *Lei de Darcy* e a incompressibilidade das duas fases dos fluidos, é dada por:

$$f(S)=\frac{S^\beta}{S^\beta+\alpha(1-S)^\beta}. \tag{8.3}$$

De acordo com a literatura especializada [2, 17, 18], as principais fontes de incerteza (aleatoriedade) em (8.2)–(8.3) aparecem em α (razão entre as viscosidades dos fluidos), β, e nos estados S^- e S^+. O expoente β é oriundo da hipótese que a permeabilidade relativa é governada por uma lei de potência; isto é, assume-se que a permeabilidade (que é função da saturação S) é proporcional a S^β, onde o expoente β é obtido a partir de resultados experimentais via ajuste de curvas.

8.2.2 Problema 2.

A concentração de um soluto não-reagente $S(\mathbf{x},t)$ na região $\|\mathbf{x}\|=\|(x_1,x_2,...,x_N)\|<\infty$, $t\geq t_0$ é descrita pelas equações

$$\phi S_t(\mathbf{x},t)+\nabla.[\mathbf{V}(\mathbf{x},t)S(\mathbf{x},t)]=0, \quad S(\mathbf{x},t_0)=g(\mathbf{x}), \tag{8.4}$$

onde ϕ é a porosidade e $V(\mathbf{x},t)$ é a velocidade aleatória de Darcy. Assumimos que $g(\mathbf{x})$ é a concentração inicial (não aleatória) e ϕ é uma constante (não aleatória). Levando em

consideração a condição de incompressibilidade do fluido em (8.4), isto é, $\text{div}[\mathbf{V}(\mathbf{x},t)] = 0$, temos

$$\phi S_t(\mathbf{x},t) + V_i(\mathbf{x},t)S_{x_i}(\mathbf{x},t) = 0, \quad S(\mathbf{x},t_0) = g(\mathbf{x}). \tag{8.5}$$

Outra simplificação em (8.4) seria considerar a velocidade aleatória uma função apenas do tempo – essa hipótese certamente simplifica o problema, porém o mesmo continua suficientemente interessante e complicado. Assim, nosso problema de interesse é o seguinte:

$$\phi S_t(\mathbf{x},t) + V_i(t)S_{x_i}(\mathbf{x},t) = 0, \quad S(\mathbf{x},t_0) = g(\mathbf{x}). \tag{8.6}$$

É fato que para cada realização $\mathbf{V}(t,\omega)$ da velocidade aleatória $\mathbf{V}(t)$, a solução, $S(\mathbf{x},t,\omega)$, é constante ao longo das curvas características

$$\frac{\partial}{\partial t}\mathbf{X}(t) = \mathbf{V}(t,\omega), \quad \mathbf{X}(0) = \mathbf{X}_0, \tag{8.7}$$

ou, equivalentemente, $S(\mathbf{X}(t),t,\omega) = g(\mathbf{X}_0)$, onde $\mathbf{X}(t) = \mathbf{X}_0 + \displaystyle\int_0^t \mathbf{V}(\tau,\omega)d\tau$.

Deste modo, a solução para (8.6) pode ser expressa como

$$S(\mathbf{x},t) = g\left(\mathbf{x} - \int_0^t \mathbf{V}(\tau)d\tau\right). \tag{8.8}$$

Agora, denotando $\mathbf{A}(t)$ (vetor aleatório N-dimensional) como

$$\mathbf{A}(t) = \int_0^t \mathbf{V}(\tau)d\tau, \tag{8.9}$$

segue que $S(\mathbf{x},t) = g\left(\mathbf{x} - \mathbf{A}(t)\right)$.

Os resultados até o momento apresentados provam o seguinte resultado:

Proposition 8.1. *O m-ésimo momento estatístico, $m \in \mathbb{Z}$, $m \geq 1$, da solução para (8.6) é dado por*

$$\langle S^m(\mathbf{x},t)\rangle = \int_{\mathbb{R}^N} g(\mathbf{x}-\mathbf{a})^m f_{\mathbf{A}(t)}(\mathbf{a})\, d\mathbf{a}, \tag{8.10}$$

onde $f_{\mathbf{A}(t)}(\mathbf{a})$ é a função de densidade de probabilidade conjunta do vetor aleatório $\mathbf{A}(t)$.

Remark 8.1. *De acordo com (8.10), se $g(\mathbf{x}) = \delta(\mathbf{x})$ é uma função Delta então*

$$\langle S^m(\mathbf{x},t)\rangle = \int_{\mathbb{R}^N} \delta(\mathbf{x}-\mathbf{a})^m f_{\mathbf{A}(t)}(\mathbf{a})\, d\mathbf{a} = f_{\mathbf{A}(t)}(\mathbf{x}). \tag{8.11}$$

Em vista destes resultados, seria interessante entender as propriedades estatísticas do vetor aleatório $\mathbf{A}(t)$ como função das propriedades estatísticas de $\mathbf{V}(t)$. Por exemplo, a partir de [19], página 162, segue que:

Proposition 8.2. *Se* $\mathbf{V}(t)$ *é um processo aleatório Gaussiano então* $\mathbf{A}(t)$ *é também um processo Gaussiano.*

Neste caso, as propriedades estatísticas (média e covariância) de $\mathbf{A}(t)$ são facilmente obtidas via integração das propriedades estatísticas do processo Gaussiano $\mathbf{V}(t)$. Tendo em vista o exposto, nosso interesse futuro é entender o processo aleatório $\mathbf{A}(t)$ a partir das informações estatísticas do processo $\mathbf{V}(t)$.

8.2.3 Problema 3.

Outro problema de interesse é investigar a possibilidade de aplicação da metodologia apresentada no Capítulo 4 a fim propor esquemas numéricos para

- a equação de advecção unidimensional aleatória:

$$\begin{aligned} &\frac{\partial}{\partial t}Q(x,t) + A(x)\frac{\partial}{\partial x}Q(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &Q(x,0) = Q_0(x), \end{aligned} \tag{8.12}$$

  onde a velocidade de transporte $A(x)$ e a condição inicial são funções aleatórias.

- a equação de advecção bidimensional aleatória:

$$\begin{cases} Q_t(x,y,t) + A(x,y)Q_x(x,y,t) + B(x,y)Q_y(x,y,t) = 0, \quad t > 0, \quad x,y \in \mathbb{R}, \\ Q(x,y,0) = Q_0(x,y), \end{cases} \tag{8.13}$$

  onde as velocidades de transporte, $A(x,y)$ e $B(x,y)$, e a condição inicial, $Q_0(x,y)$, são campos aleatórios. Métodos do tipo direções alternadas (ADI) [27, 40], que têm sido usados com sucesso para tratar equações diferenciais parciais multidimensionais determinísticas, seriam utilizados para dividir o problema bidimensional em dois problemas unidimensionais.

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