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APPROXIMATION PROPERTY, REFLEXIVITY AND COMPLEMENTED SUBSPACES ON HOMOGENEOUS POLYNOMIALS

PROPRIEDADE DE APROXIMAÇÃO, REFLEXIVIDADE E SUBESPAÇOS COMPLEMENTADOS EM POLINÔMIOS HOMOGÊNEOS

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Orientador: Sahibzada Waleed Noor

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Dedicated to the memory of Jorge Mujica (1946-2017).

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RESUMO

A propriedade da aproximação foi introduzida por Grothendieck (GROTHENDIECK, 1955). Enflo (ENFLO, 1973) deu o primeiro exemplo de um espaço de Banach sem a propriedade da aproximação. O contraexemplo de Enflo é um espaço de Banach construído artificialmente. O primeiro espaço de Banach sem a propriedade da aproximação definido naturalmente foi dado por Szankowski (SZANKOWSKI, 1981), que provou que o espaço $\mathcal{L}(\ell_2; \ell_2)$ de todos os operadores lineares e contínuos em ℓ_2 não tem a propriedade da aproximação. Recentemente Dineen e Mujica (DINEEN; MUJICA, 2015) provaram que se $1 < p \leqslant q < \infty$, então $\mathcal{L}(\ell_p; \ell_q)$ não tem a propriedade da aproximação. Eles também provaram que se $1 < p < \infty$, então o espaço $\mathcal{P}(^n\ell_p)$ de todos os polinômios n-homogêneos contínuos em ℓ_p não tem a propriedade da aproximação para cada $n \geqslant p$. Primeiramente, neste trabalho usamos os métodos de Dineen e Mujica (DINEEN; MUJICA, 2015) e Godefroy e Saphar (GODEFROY; SAPHAR, 1989) para apresentar alguns exemplos naturais de espaços de Banach de operadores lineares e polinômios homogêneos sem a propriedade da aproximação.

Emmanuele (EMMANUELE, 1992) e John (JOHN, 1992) mostraram que se c_0 está imerso no espaço $\mathcal{L}_K(E; F)$ de todos os operadores compactos de E em F, então $\mathcal{L}_K(E; F)$ não é complementado no espaço $\mathcal{L}(E; F)$ de todos os operadores lineares e contínuos de E em F para cada E e F espaços de Banach de dimensão infinita. Seja $\mathcal{P}_K(^nE; F)$ (resp. $\mathcal{P}_w(^nE; F)$) o subespaço de todos os polinômios n- homogêneos contínuos $P \in \mathcal{P}(^nE; F)$ que são compactos (resp. fracamente contínuos em conjuntos limitados). Neste trabalho mostramos que se $\mathcal{P}_K(^nE; F)$ contém uma cópia isomorfa de c_0, então $\mathcal{P}_K(^nE; F)$ não é complementado em $\mathcal{P}(^nE; F)$. Da mesma maneira, nós mostramos que se $\mathcal{P}_w(^nE; F)$ contém uma cópia isomorfa de c_0, então $\mathcal{P}_w(^nE; F)$ não é complementado em $\mathcal{P}(^nE; F)$.

Finalmente, nesta tese nós provamos que se E e F são espaços de Banach reflexivos e G é um subespaço linear fechado de $\mathcal{L}_K(E; F)$, então G somente pode ser reflexivo ou não isomorfo a um espaço dual. Esse resultado generaliza (FEDER, 1975, Theorem 2) e dá a solução para o problema proposto por Feder (FEDER et al., 1980, Problem 1).

Palavras-chave: Espaço de Banach, operador linear, operador compacto, polinômio homogêneo, propriedade da aproximação, subespaço complementado.

ABSTRACT

The approximation property was introduced by Grothendieck (GROTHENDIECK, 1955). Enflo (ENFLO, 1973) gave the first example of a Banach space without the approximation property. Enflo's counterexample is an artificially constructed Banach space. The first naturally defined Banach space without the approximation property was given by Szankowski (SZANKOWSKI, 1981), who proved that the space $\mathcal{L}(\ell_2; \ell_2)$ of continuous linear operators on ℓ_2 does not have the approximation property. Recently Dineen and Mujica (DINEEN; MUJICA, 2015) proved that if $1 < p \leqslant q < \infty$, then $\mathcal{L}(\ell_p; \ell_q)$ does not have the approximation property. They also proved that if $1 < p < \infty$, then the space $\mathcal{P}(^n\ell_p)$ of continuous n-homogeneous polynomials on ℓ_p does not have the approximation property for every $n \geqslant p$. Firstly, in this work by using the methods of Dineen and Mujica (DINEEN; MUJICA, 2015) and Godefroy and Saphar (GODEFROY; SAPHAR, 1989), we present many naturally examples of Banach spaces of linear operators and homogeneous polynomials which do not have the approximation property.

Emmanuele (EMMANUELE, 1992) and John (JOHN, 1992) showed that if c_0 embeds on the space $\mathcal{L}_K(E; F)$ of all compact operators from E into F, then $\mathcal{L}_K(E; F)$ is not complemented on the space $\mathcal{L}(E; F)$ of all continuous linear operators from E into F for every E and F infinite dimensional Banach spaces. Let $\mathcal{P}_K(^nE; F)$ (resp. $\mathcal{P}_w(^nE; F)$) denote the subspace of all continuous n- homogeneous polynomials $P \in \mathcal{P}(^nE; F)$ which are compact (resp. weakly continuous on bounded sets). In this work we show that if $\mathcal{P}_K(^nE; F)$ contains an isomorphic copy of c_0, then $\mathcal{P}_K(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$. Likewise, we show that if $\mathcal{P}_w(^nE; F)$ contains an isomorphic copy of c_0, then $\mathcal{P}_w(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$.

Finally, in this thesis we prove that if E and F are reflexive Banach spaces and G is a closed linear subspace of $\mathcal{L}_K(E; F)$ then G is either reflexive or non-isomorphic to a dual space. This result generalizes (FEDER, 1975, Theorem 2) and gives the solution to a problem posed by Feder (FEDER et al., 1980, Problem 1).

Keywords: Banach space, linear operator, compact operator, homogeneous polynomial, approximation property, complemented subspace.

List of symbols

$\mathbb{N}$	The natural numbers.
$\mathbb{R}$	The real numbers.
$\mathbb{C}$	The complex numbers.
$\mathbb{K}$	The scalar field $\mathbb{R}$ or $\mathbb{C}$.
E, F	Banach spaces.
B_E	The closed unit ball in E.
$B(a;r)$	$\{x \in E : \| x - a \| < r\}$.
E'	The topological dual of E.
$\mathcal{L}(E;F)$	The space of all bounded linear operators from E into F.
$\mathcal{F}(E;F)$	The space of finite rank operators from E into F.
$\mathcal{L}_K(E;F)$	The subspace of all $T \in \mathcal{L}(E;F)$ which are compact.
$\mathcal{L}_{wK}(E;F)$	The subspace of all $T \in \mathcal{L}(E;F)$ which are weakly compact.
J_E	The canonical embedding from E into E''.
$E \hookrightarrow F$	The space E is isomorphic to a subspace of F.
$d(w,p)$	The Lorentz sequence space.
ℓ_∞	The collection of bounded sequences of scalars $x = (x_n)$, with the norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} \lvert x_n \rvert$.
c_0	The sequences of scalars that converges to zero endowed with the norm $\| \cdot \|_\infty$.
T'	The adjoint operator of T.
$\mathcal{L}_a(^nE;F)$	The vector space of all n linear mappings $A : \underbrace{E \times E \times \ldots \times E}_{n} \to F$.
$\mathcal{L}(^nE;F)$	The subspace of all $A \in \mathcal{L}_a(^nE;F)$ which are continuous.
$\mathcal{P}_a(^nE;F)$	The vector space of all n-homogeneous polynomials from E into F.
$\mathcal{P}(^nE;F)$	The subspace of all $P \in \mathcal{P}_a(^nE;F)$ which are continuous.

$\mathcal{P}_f(^nE;F)$	The subspace of $\mathcal{P}(^nE;F)$ generated by all polynomials of the form $P(x) = (\phi(x))^n b$, with $\phi \in E'$ and $b \in F$.
$\mathcal{P}_A(^nE;F)$	$\overline{\mathcal{P}_f(^nE;F)}^{\\|\cdot\\|}$.
$\mathcal{P}_w(^nE;F)$	The subspace of all $P \in \mathcal{P}(^nE;F)$ which are weakly continuous on bounded sets.
$\mathcal{P}_K(^nE;F)$	The subspace of all $P \in \mathcal{P}(^nE;F)$ which are compact.
$\mathcal{P}_{wK}(^nE;F)$	The subspace of all $P \in \mathcal{P}(^nE;F)$ which are weakly compact.
$\otimes_n E$	The n- fold tensor product of E.
$\otimes_{n,s} E$	The n- fold symmetric tensor product of E.
$\hat{\otimes}_{n,s,\pi} E$	The n- fold symmetric projective tensor product of E.
τ_c	The compact-open topology.
$M^\perp$	The annihilator of M in E', that is, the collection of all continuous linear functionals on the Banach space E which vanish on the subset M of E.
$\sigma(E;E')$	Weak topology on E.
$\sigma(E';E)$	Weak-star topology on E'.
AP	Approximation property.
BAP	Bounded approximation property.
CAP	Compact approximation property.
$w.u.C.$	Weakly unconditionally Cauchy.

Contents

Introduction

Recall that a Banach space E has the *approximation property* if the identity operator on E can be approximated by finite rank operators uniformly on compact sets of E. The *approximation property* was introduced by A. Grothendieck (GROTHENDIECK, 1955) in 1955, but the origins of the notion trace back to the Lwów School of Mathematics of 1930s. A result which goes back to the beginnings of functional analysis asserts that the compact operators on a Hilbert space are exactly those operators which are limits in norm of operators of finite rank. One part of this assertion is trivially true for every pair of Banach spaces E and F. More precisely, if $\mathcal{L}(E;F)$ denotes the Banach space of all bounded linear operators from E into F with the sup norm, then each $T \in \mathcal{L}(E;F)$ for which $\lim\limits_{n\to\infty} \|T_n - T\| = 0$ for suitable $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$, with $\dim T_n(E) < \infty$, is a compact operator. It was realized a long ago that the converse assertion is also true for many examples of spaces E and F besides Hilbert spaces, for example, if F is a Banach space with a Schauder basis. The question whether the converse assertion is true for arbitrary Banach spaces E and F (which was called for obvious reasons the approximation problem) remained open for a long time.

The Problem 153 of the *Scottish Book* (MAULDIN,) was enunciated as follows:

Given a continuous function $f = f(s,t)$ defined on $[0,1] \times [0,1]$ and a number $\epsilon > 0$; do there exist numbers $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n$, such that

$$\left| f(s,t) - \sum_{k=1}^{n} a_k f(s,b_k) f(c_k,t) \right| < \epsilon$$

for all $s, t \in [0,1]$?

This problem was posed by Mazur in 1936 and according to Pelczyński (PIETSCH, 2007, p.287) Mazur knew that the positive answer to Problem 153 would imply the positive answer to the approximation problem.

A related open question, the basis problem, asked whether every separable Banach space has a Schauder basis. A negative solution to the approximation problem also gives a negative solution to the basis problem, because every Banach space with a Schauder basis satisfies the approximation property.

The approximation problem was solved (in the negative) by Per Enflo in 1972. Enflo constructed a separable reflexive Banach space without the approximation property and consequently gave a negative solution to the basis problem. The result of Enflo was improved by Davie (DAVIE, 1973),(DAVIE, 1975), Figiel (FIGIEL, 1974) and Szankowski (SZANKOWSKI, 1978) as follows:

For every $p \in [1, \infty]$, $p \neq 2$, there exists a subspace E_p of the space ℓ_p which does not have the approximation property. Moreover, $E_\infty \subset c_0$.

Enflo's counterexample is an artificially constructed Banach space. The first natural example of a Banach space without the approximation property was given by Szankowski (SZANKOWSKI, 1981) in 1981, who proved that the space $\mathcal{L}(\ell_2; \ell_2)$ of continuous linear operators on ℓ_2 does not have the approximation property. The situation for $\mathcal{L}(\ell_1; \ell_1)$ remains open. In 1989 Godefroy and Saphar (GODEFROY; SAPHAR, 1989) proved that, if $\mathcal{L}_K(\ell_2; \ell_2)$ denotes the subspace of all compact members of $\mathcal{L}(\ell_2; \ell_2)$, then the quotient $\mathcal{L}(\ell_2; \ell_2)/\mathcal{L}_K(\ell_2; \ell_2)$ does not have the approximation property. Recently Dineen and Mujica (DINEEN; MUJICA, 2015) proved that if $1 < p \leqslant q < \infty$, then $\mathcal{L}(\ell_p; \ell_q)$ does not have the approximation property. They also proved that if $1 < p < \infty$, then the space $\mathcal{P}(^n\ell_p)$ of continuous n-homogeneous polynomials on ℓ_p does not have the approximation property, for every $n \geqslant p$.

In this work we will give new natural examples of Banach spaces of linear operators and homogeneous polynomials which do not have the approximation property. We use the methods of Dineen and Mujica (DINEEN; MUJICA, 2015) and Godefroy and Saphar (GODEFROY; SAPHAR, 1989) to achieve our aim. Among other results, the main results that we prove are the following:

Theorem 0.0.1. *If $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then $\mathcal{L}(E; F)$ does not have the approximation property.*

This result improves a previous result of Dineen and Mujica (DINEEN; MUJICA, 2015). We also show that:

Theorem 0.0.2. *If $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then the quotient $\mathcal{L}(E; F)/\mathcal{L}_K(E; F)$ does not have the approximation property.*

This result improves a previous result of Godefroy and Saphar (GODEFROY; SAPHAR, 1989).

We also present examples of Banach spaces of linear operators defined on Pelczynski's universal space U_1, Orlicz sequence spaces ℓ_{M_p} and Lorentz sequence spaces $d(w, p)$ which do not have the approximation property.

Finally, we present examples of Banach spaces of homogeneous polynomials without the approximation property, such as the following:

Theorem 0.0.3. *If $1 < p < \infty$ and E is a closed infinite dimensional subspace of ℓ_p, then $\mathcal{P}(^nE)$ does not have the approximation property, for every $n \geqslant p$.*

This result improves another result of Dineen and Mujica (DINEEN; MUJICA, 2015).

Theorem 0.0.4. *If $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then $\mathcal{P}(^nE;F)$ does not have the approximation property, for every $n \geqslant 1$.*

We also show that if $n < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then the quotient $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ does not have the approximation property.

Now, let us consider the long standing conjecture:

The space $\mathcal{L}_K(E;F)$ of compact linear operators is either equal to the space $\mathcal{L}(E;F)$ or uncomplemented in $\mathcal{L}(E;F)$.

Several authors have treated this problem and gave an affirmative answer in certain cases (see for instance Kalton (KALTON, 1974), Emmanuelle (EMMANUELE, 1992), John (JOHN, 1992), Bator and Lewis (GHENCIU, 2005a) and Ghenciu (GHENCIU, 2005b), among others).

Kalton (KALTON, 1974) studied the structure of the space $\mathcal{L}_K(E;F)$ and he showed in particular the following result:

Theorem. *Let E be a Banach space with an unconditional finite dimensional expansion of the identity $(A_n)_{n\in\mathbb{N}}$. If F is any infinite dimensional Banach space, the following are equivalent.*

1. $\mathcal{L}_K(E;F) = \mathcal{L}(E;F)$.
2. $\mathcal{L}_K(E;F)$ *contains no copy of* c_0.
3. $\mathcal{L}(E;F)$ *contains no copy of* ℓ_∞.
4. $\mathcal{L}_K(E;F)$ *is complemented in* $\mathcal{L}(E;F)$.
5. *For* $T \in \mathcal{L}(E;F)$, *the series* $\sum_{n=1}^{\infty} T \circ A_n$ *converges in norm.*

Emmanuele (EMMANUELE, 1992) and John (JOHN, 1992) showed independently that if c_0 embeds in $\mathcal{L}_K(E;F)$ then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$, for every infinite dimensional Banach spaces E and F . John (JOHN, 1992) gave a sufficient condition for $\mathcal{L}_K(E;F)$ to contain a copy of c_0, more precisely he proved that if there is a noncompact operator $T : E \to F$ which factors through a Banach space G having an unconditional basis, then $\mathcal{L}_K(E;F)$ contains a copy of c_0. John also proved that if

E and F are infinite dimensional Banach spaces such that each non compact operator $T \in \mathcal{L}(E;F)$ factors through a Banach space G with an unconditional basis, then the following conditions are equivalent:

1. $\mathcal{L}_K(E;F) = \mathcal{L}(E;F)$.
2. $\mathcal{L}(E;F)$ contains no copy of ℓ_∞.
3. $\mathcal{L}_K(E;F)$ contains no copy of c_0.
4. $\mathcal{L}_K(E;F)$ is complemented in $\mathcal{L}(E;F)$.

Let $\mathcal{P}_w(^nE;F)$ be the space of all n-homogeneous polynomials from E into F which are weakly continuous on bounded sets. When $n = 1$ we have $\mathcal{P}_w(^nE;F) = \mathcal{L}_K(E;F)$. González and Gutiérrez (GONZÁLEZ; GUTIÉRREZ, 2000) obtained a polynomial version of the aforementioned result of Kalton. They proved the following result:

Theorem. *Suppose that E has an unconditional finite dimensional expansion of the identity and let $n \in \mathbb{N}$ $(n > 1)$. Then the following assertions are equivalent:*

1. *$\mathcal{P}(^nE;F) = \mathcal{P}_w(^nE;F)$.*
2. *$\mathcal{P}_w(^nE;F)$ contains no copy of c_0.*
3. *$\mathcal{P}(^nE;F)$ contains no copy of ℓ_∞.*
4. *$\mathcal{P}_w(^nE;F)$ is complemented in $\mathcal{P}(^nE;F)$.*

Ghenciu (GHENCIU, 2005b) obtained the following result:

Let E and F be Banach spaces, and let G be a Banach space with an unconditional basis (g_n) and coordinate functionals (g'_n).

(a) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a seminormalized basic sequence in F and $(S'(g'_n))$ is not relatively compact in E', then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.

(b) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a seminormalized basic sequence in F and $(S'(g'_n))$ is not relatively weakly compact in E', then $\mathcal{L}_{wK}(E;F)$ is not complemented in $\mathcal{L}(E;F)$.

This result generalizes results of several authors (see for instance (EMMANUELE, 1991; GHENCIU, 2005a; FEDER, 1982)). In this thesis we obtain polynomial versions of the preceding results.

The most important results obtained are the following:

Theorem 0.0.5. *Let E and F be Banach spaces, and let G be a Banach space with an unconditional basis (g_n) and coordinate functionals (g'_n). If there exist operators $R \in \mathcal{L}(G; F)$ and $S \in \mathcal{L}(E; G)$ such that $(R(g_n))$ is a seminormalized basic sequence in F and $(S'(g'_n))$ is not relatively compact in E', then $\mathcal{P}_w(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$ for every $n \in \mathbb{N}$.*

The next proposition is a polynomial version of (EMMANUELE, 1992, Theorem 2) and (JOHN, 1992, Theorem 1).

Proposition 0.0.6. *Let E be an infinite dimensional Banach space and $n > 1$. If $\mathcal{P}_w(^nE; F)$ contains a copy of c_0, then $\mathcal{P}_w(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$.*

The following theorem is a polynomial version of (JOHN, 1992, Theorem 2).

Theorem 0.0.7. *Let E and F be Banach spaces and $P \in \mathcal{P}(^nE; F)$ such that $P \notin \mathcal{P}_w(^nE; F)$. Suppose that P admits a factorization $P = Q \circ T$ through a Banach space G with an unconditional finite dimensional expansion of the identity, where $T \in \mathcal{L}(E; G)$ and $Q \in \mathcal{P}(^nG; F)$. Then $\mathcal{P}_w(^nE; F)$ contains a copy of c_0 and thus $\mathcal{P}_w(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$.*

We say that E is a *dual space* if there exists a Banach space X such that E is isometrically isomorphic to X'. In 1957 R. Schatten (SCHATTEN, 1957) proved that, if H is an infinite dimensional Hilbert space, then the space $\mathcal{L}_K(H; H)$ is not a dual space. In this moment it was natural to ask if it was possible to generalize the result of Schatten. We know that, in special cases, $\mathcal{L}_K(E; F)$ is reflexive (for example, if $E = \ell_p$, $F = \ell_q$, $1 < q < p < \infty$, see. J. R. Holub (HOLUB, 1971)). In 1975 Feder and Saphar (FEDER, 1975) proved that, if E and F are reflexive Banach spaces and G is a closed linear subspace of $\mathcal{L}_K(E; F)$ which contains the space $\mathcal{F}(E, F)$ of all finite rank linear operators from E into F, then G is either reflexive or is not a dual space. Later, in 1980 Feder (FEDER et al., 1980) showed that if E and F are reflexive Banach spaces such that F or E' is a subspace of a Banach space with an unconditional basis, then the space $\mathcal{L}_K(E; F)$ is either reflexive or non-isomorphic to a dual space. But the following question posed in (FEDER et al., 1980) remains open:

Question. *Let E and F be reflexive Banach spaces. Is $\mathcal{L}_K(E; F)$ either reflexive or non-isomorphic to a dual space?*

In this work, we obtain a positive answer for the previous question. In fact, we prove the following more general result:

Theorem 0.0.8. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E; F)$. Then G is either reflexive or non-isomorphic to a dual space.*

We also prove that if E and F are reflexive Banach spaces, then the space $\mathcal{P}_w(^nE; F)$ of all n-homogeneous polynomials from E into F which are weakly continuous on bounded sets is either reflexive or non-isomorphic to a dual space. As other consequences of this result we also obtain two conditions, one that ensures that $\mathcal{P}_w(^nE; F)$ is non-isomorphic to a dual space and other such that $\mathcal{P}_w(^nE; E)$ is non-isomorphic to a dual space. More specifically, we prove the following corollaries:

Corollary 0.0.9. *Let E and F be reflexive Banach spaces such that E has the CAP. If $\mathcal{P}_w(^nE; F) \neq \mathcal{P}(^nE; F)$, then $\mathcal{P}_w(^mE; F)$ is not isomorphic to a dual space for every $m \geqslant n$.*

Corollary 0.0.10. *Let E be a reflexive infinite dimensional Banach space with the CAP. Then $\mathcal{P}_w(^nE; E)$ is non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Finally, we prove that if E and F are reflexive Banach spaces, then the space $\mathcal{P}_A(^nE; F)$ is either reflexive or non-isomorphic to a dual space. Hence, we obtain a generalization of a result due to Boyd and Ryan (BOYD; RYAN, 2001, Theorem 21).

This work is organized as follows:

In Chapter 1 we introduce basic definitions and properties of Banach space theory that we will use in next sections.

In Chapter 2 we prove Theorems 0.0.1, 0.0.2,0.0.3 and 0.0.4, among others.

In Chapter 3 we mainly show Proposition 0.0.6 and Theorems 0.0.5 and 0.0.7. We also obtain a generalization of González and Gutiérrez (GONZÁLEZ; GUTIÉRREZ, 2000, Theorem 7).

Finally, in Chapter 4 we present the proof of Theorem 0.0.8 and Corollaries 0.0.9 and 0.0.10.

1 Preliminaries

In this chapter we introduce basic concepts and essential notation to the development of the next chapters. In Section 1 we present basic definitions and properties concerning Banach spaces. In Section 2 we study multilinear mappings which will be used later to define homogeneous polynomials. In Section 3 we define the symmetric projective tensor product for stating a linearization theorem due to Ryan. In Section 4 we analyse the main results obtained for Banach spaces of linear operators without the approximation property that will be generalized in the next chapter. Finally, in the last section we enunciate some theorems about reflexivity and copies of c_0 in spaces of compact operators that will be extended to homogeneous polynomials.

1.1 Basic concepts in Banach spaces

Definition 1.1.1. *Let E and F be Banach spaces. The space of all* bounded linear operators *from E into F is denoted by $\mathcal{L}(E;F)$. If $F = \mathbb{K}$, $\mathcal{L}(E;F)$ is the topological dual of E that is denoted by E'.*

The following spaces are special classes of bounded linear operators.

Definition 1.1.2. *Let $\mathcal{F}(E;F)$ denote the subspace of $\mathcal{L}(E;F)$ generated by all bounded linear operators of the form $T(x) = \phi(x)b$, with $\phi \in E'$ and $b \in F$. The normed space $\mathcal{F}(E;F)$ is called the space of* finite rank operators*.*

Definition 1.1.3. *Let $\mathcal{L}_K(E;F)$ denote the subspace of all $T \in \mathcal{L}(E;F)$ which are compact, that is which map bounded sets onto relatively compact sets.*

Definition 1.1.4. *Let $\mathcal{L}_{wK}(E;F)$ denote the subspace of all $T \in \mathcal{L}(E;F)$ which are weakly compact, that is which map bounded sets onto relatively weakly compact sets.*

Given $f \in E'$ and $x \in E$ we shall often write $\left\langle f, x \right\rangle$ instead of $f(x)$.

Definition 1.1.5. *Let $J_E : E \to E''$ denote the* canonical injection *from E into E'' defined as follows: given $x \in E$, the map $f \to \left\langle f, x \right\rangle$ is a continuous linear functional on E'; thus it is an element of E'', which we denote by $J_E(x)$. We have*

$$\left\langle J_E(x), f \right\rangle = \left\langle f, x \right\rangle$$

for all $x \in E$ and $f \in E'$. The Banach space E is said to be reflexive *if $J_E(E) = E''$.*

Definition 1.1.6. *An isomorphism between Banach spaces E and F is an linear operator $T : E \to F$ bijective, such that T and its inverse T^{-1} are continuous. An isomorphism T is an isometric isomorphism if $\|T(x)\| = \|x\|$ for each $x \in E$. We say that E is isomorphic or isometrically isomorphic to F, if there is an isomorphism or an isometric isomorphism between E and F, respectively.*

Definition 1.1.7. *We say that E is a* dual space *if there exists a Banach space X, such that E is isometrically isomorphic to X'.*

Definition 1.1.8. *The series $\sum_{n=1}^{\infty} x_n$ of elements of E is* unconditionally convergent *if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges (in norm) over all permutations π of $\mathbb{N}$.*

Definition 1.1.9. *The series $\sum_{n=1}^{\infty} x_n$ of elements of E is* weakly unconditionally Cauchy *(w.u.C. in short) if $\sum_{n=1}^{\infty} |x'(x_n)| < \infty$ for all $x' \in E'$ or, equivalently if*

$$\sup\left\{\left\|\sum_{n\in F} x_n\right\|; F \subset \mathbb{N}, F finite\right\} < \infty.$$

Definition 1.1.10. *A sequence $(e_n)_{n\in\mathbb{N}}$ in E is said to be a* Schauder basis *if each $x \in E$ has a unique series representation of the form $x = \sum_{n=1}^{\infty} e'_n(x)e_n$, where $e'_n(x) \in \mathbb{K}$ for every $n \in \mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ which is a Schauder basis of its closed linear span is called a* basic sequence.

Remark 1.1.11. *If E has a Schauder basis $(e_n)_{n\in\mathbb{N}}$ then the coordinate functionals $e'_n : x \in E \to e'_n(x) \in \mathbb{K}$ and the mappings $T_n : x \in E \to \sum_{i=1}^{n} e'_i(x)e_i \in E$ are linear. If E_n denotes the subspace generated by $e_1, e_2, \ldots, e_n$ then T_n is a projection from E onto E_n.*

Example 1.1.12. *The unit vectors $e_n = (0, 0, \ldots, \underbrace{1}_{n}, 0, \ldots)$ form a Schauder basis for c_0 and ℓ_p with $1 \leqslant p < \infty$. An example of a basis in the space c of convergent sequences of scalars, is given by $x_1 = (1, 1, \ldots)$ and $x_n = e_{n-1}$ for $n > 1$. The expansion of $x = (a_1, a_2, \ldots) \in c$ with respect to this basis is*

$$x = (\lim_{n\to\infty} a_n)x_1 + (a_1 - \lim_{n\to\infty} a_n)x_2 + (a_2 - \lim_{n\to\infty} a_n)x_3 + \ldots$$

Definition 1.1.13. *Let $(e_n)_{n\in\mathbb{N}}$ be a basic sequence in a Banach space E. A sequence of non-zero vectors $(u_j)_{j\in\mathbb{N}}$ in E of the form $u_j = \sum_{n=p_j+1}^{p_{j+1}} a_n e_n$, with $(a_n)_{n\in\mathbb{N}}$ scalars and $p_1 < p_2 < \ldots$ an increasing sequence of integers, is called a* block basis sequence *of $(e_n)_{n\in\mathbb{N}}$.*

Definition 1.1.14. *A Schauder basis $(e_n)_{n\in\mathbb{N}} \subset E$ is* shrinking *if the coordinate functionals $(e'_n)_{n\in\mathbb{N}}$ form a basis of the dual space E'.*

Definition 1.1.15. *A sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is a* semi-normalized basic sequence *if $(x_n)_{n\in\mathbb{N}}$ is a Schauder basis for the closed subspace $M = \overline{[x_n : n \in \mathbb{N}]}$, and moreover there are constant a and b such that $0 < a < \|x_n\| < b$ for all $n \in \mathbb{N}$.*

Definition 1.1.16. *A Schauder basis $(x_n)_{n\in\mathbb{N}} \subset E$ is* unconditional *if whenever the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges, it converges unconditionally. A Schauder basis $(x_n)_{n\in\mathbb{N}}$ is unconditional if and only if, there is $C \geqslant 0$ such that*

$$\left\| \sum_{k=1}^{n} \epsilon_k \alpha_k x_k \right\| \leqslant C \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|$$

for all $n \in \mathbb{N}$, all scalar coefficients α_k and all signs $\epsilon_k = \pm 1$.

Definition 1.1.17. *A Banach space E contains a copy of F if there exists a closed subspace Z of E, such that F is isomorphic to Z. We write $F \hookrightarrow E$ when E contains a copy of F.*

Definition 1.1.18. *Two bases, $(e_n)_{n\in\mathbb{N}}$ of E and $(f_n)_{n\in\mathbb{N}}$ of F, are called* equivalent *if there is an isomorphism T from E onto F for which $T(e_n) = f_n$ for all $n \in \mathbb{N}$.*

The following spaces will be used in the next chapter.

Theorem 1.1.19. *((LINDENSTRAUSS; TZAFRIRI, , Theorem 2.d.10)) There exists a separable Banach space U_1 having an unconditional basis $(e_n)_{n\in\mathbb{N}}$ such that every unconditional basic sequence (in an arbitrary separable Banach space) is equivalent to a subsequence of $(e_n)_{n\in\mathbb{N}}$.*

Definition 1.1.20. *(see (LINDENSTRAUSS; TZAFRIRI, , p. 175)) Let $1 \leqslant p < \infty$ and let $w = \{w_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of positive numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. Let*

$$d(w,p) = \left\{ x = (\xi_n)_{n=1}^{\infty} \subset \mathbb{K} : \|x\| = \sup_{\pi} \left(\sum_{n=1}^{\infty} |\xi_{\pi(n)}|^p w_n \right)^{1/p} < \infty \right\},$$

where π ranges over all permutations of $\mathbb{N}$. Then $d(w,p)$ is a Banach space, called a Lorentz sequence space.

Proposition 1.1.21. *((LINDENSTRAUSS; TZAFRIRI, , Theorem 4.e.3)) Let $(e_n)_{n\in\mathbb{N}}$ be the unit vector basis of a Lorentz sequence space $d(w,p)$ with $p \geqslant 1$. Then every normalized block basis sequence $u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i e_i$, $n = 1, 2, \ldots$ such that $\lim_{i\to\infty} a_i = 0$ contains, for every*

$\epsilon > 0$, a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ which is $1+\epsilon$ equivalent to the unit vector basis of ℓ_p and so that $[u_{n_j}]_{j\in\mathbb{N}}$ is complemented in $d(w,p)$. Consequently, every infinite dimensional subspace of $d(w,p)$ contains complemented subspaces which are nearly isometric to ℓ_p.

The following result due to Bessaga and Pelczyński characterizes w.u.C. series.

Theorem 1.1.22. *((BESSAGA; PEŁCZYŃSKI, 1958, Lemma 2)) or ((DIESTEL, 2012, Theorem 6)) The following statements regarding a formal series $\sum_{n=1}^{\infty} x_n$ in a Banach space are equivalent:*

1. *$\sum_{n=1}^{\infty} x_n$ is w.u.C.*
2. *There is a $C > 0$ such that for any $(t_n)_{n\in\mathbb{N}} \in \ell_\infty$*
$$\sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} t_k x_k \right\| \leqslant C \sup_{n\in\mathbb{N}} |t_n|.$$
3. *For any $(t_n)_{n\in\mathbb{N}} \in c_0$, $\sum_{n=1}^{\infty} t_n x_n$ converges.*

Theorem 1.1.23. *((DIESTEL, 2012, Theorem 8)) Let E be a Banach space. Then, in order that each series $\sum_{n=1}^{\infty} x_n$ in E with $\sum_{n=1}^{\infty} |x'(x_n)| < \infty$ for each $x' \in E'$ be unconditionally convergent, it is both necessary and sufficient that E contains no copy of c_0.*

Definition 1.1.24. *Let E be a Banach space. We say that E has an* unconditional finite dimensional expansion of the identity *if there is a sequence of bounded linear operators $A_n : E \to E$ of finite rank, such that for $x \in E$*

$$\sum_{n=1}^{\infty} A_n(x) = x$$

unconditionally.

Remark 1.1.25. *In particular, each Banach space with an* unconditional Schauder basis *has an unconditional finite dimensional expansion of the identity.*

Definition 1.1.26. *A function μ from a field Σ of subsets of a set Ω to a Banach space E is called a* finitely additive measure*, or simply a* vector measure*, if whenever E_1 and E_2 are disjoint members of Σ then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.*

Definition 1.1.27. *Let Σ be a field of subsets of a set Ω, and let $\mu : \Sigma \to E$ be a vector measure. μ is said to be* strongly additive *if the series $\sum_{n=1}^{\infty} \mu(A_n)$ converges in norm for each sequence (A_n) of pairwise disjoint members of Σ.*

Theorem 1.1.28. *(DIESTEL-FAIRES)((DIESTEL; UHL, 1977, Theorem 2)) Let Σ be a field of subsets of the set Ω and $G : \Sigma \to E$ be a bounded vector measure. If G is not strongly additive, then E contains a copy of c_0. If in addition Σ is a σ- field, then the above statement remains true if the space c_0 is replaced by the space ℓ_∞.*

Definition 1.1.29. *Let $T : E \to F$ be a bounded linear operator. Then the adjoint operator $T' : F' \to E'$ of T is defined by*

$$\left\langle T'(g), x \right\rangle = \left\langle g, T(x) \right\rangle$$

for all $g \in F'$ and $x \in E$.

Definition 1.1.30. *Let E be a Banach space and let F be a closed subspace of E. We say that F is a complemented subspace of E if there exists a projection $\pi : E \to E$ such that $\pi(E) = F$.*

Definition 1.1.31. *If $M \subset E$ is a linear subspace we set*

$$M^\perp = \{f \in E'; f(x) = 0, \forall x \in M\}.$$

We say that $M^\perp$ is the annihilator or the space orthogonal to M.

Definition 1.1.32. *Let E and F be Banach spaces, and $U \subset E$ be an open subset of E. A function $f : E \to F$ is called* Fréchet differentiable *at $x \in U$ if there exists $T \in \mathcal{L}(E; F)$ such that*

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A(h)\|}{\|h\|} = 0.$$

Definition 1.1.33. *A subset of a Banach space is called* conditionally weakly compact *if every sequence in it has a weakly Cauchy subsequence.*

Definition 1.1.34. *A sequence (x_n) in a Banach space E is called* weakly Cauchy *if for every $\varphi \in E'$ the sequence $(\varphi(x_n))$ is Cauchy in the scalar field.*

Definition 1.1.35. *A Banach space E is* weakly sequentially complete *if every weakly Cauchy sequence is weakly convergent in E.*

Example 1.1.36. *If E is reflexive, then B_E is weakly compact. It follows from Smulian Theorem that every bounded sequence in E admits a weakly convergent subsequence. In particular, every reflexive Banach space is weakly sequentially complete.*

1.2 Homogeneous polynomials

This section is devoted to the study of homogeneous polynomials in Banach spaces, this concept is the basis for this work. Firstly, we introduce the concept of multilinear mappings.

Definition 1.2.1. *For each $n \in \mathbb{N}$ we shall denote by $\mathcal{L}_a({}^nE;F)$ the vector space of all n- linear mappings $A : \underbrace{E \times E \times \ldots \times E}_{n} \to F$, whereas we shall denote by $\mathcal{L}({}^nE;F)$ the subspace of all continuous members of $\mathcal{L}_a({}^nE;F)$. For each $A \in \mathcal{L}_a({}^nE;F)$ we define*

$$\|A\| = \sup\{\|A(x_1, x_2, \ldots, x_n)\| : x_j \in E, \max_j \|x_j\| \leqslant 1\}.$$

When $n = 1$, we shall write $\mathcal{L}_a({}^1E;F) = \mathcal{L}_a(E;F)$ and $\mathcal{L}({}^1E;F) = \mathcal{L}(E;F)$. When $F = \mathbb{K}$, we shall write $\mathcal{L}_a({}^nE;\mathbb{K}) = \mathcal{L}_a({}^nE)$ and $\mathcal{L}({}^nE;\mathbb{K}) = \mathcal{L}({}^nE)$.

Example 1.2.2. *Given $\varphi_1, \varphi_2, \ldots, \varphi_n \in E'$, then the n- linear mapping*

$$A(x_1, x_2, \ldots, x_n) = \varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_n(x_n)$$

for all $x_1, x_2, \ldots, x_n \in E$, belongs to $\mathcal{L}({}^nE;\mathbb{K})$.

Proposition 1.2.3. *((MUJICA, 1985, Proposition 1.2)) For each $A \in \mathcal{L}_a({}^nE;F)$ the following conditions are equivalent:*

1. *A is continuous.*

2. *A is continuous at the origin.*

3. *$\|A\| < \infty$.*

Proposition 1.2.4. *((MUJICA, 1985, Proposition 1.3)) $\mathcal{L}({}^nE;F)$ is a Banach space under the norm $A \to \|A\|$.*

Definition 1.2.5. *$\mathcal{L}^s({}^nE;F)$ denotes the subspace of all $A \in \mathcal{L}({}^nE;F)$ which are symmetric, that is $A(x_1, x_2, \ldots, x_n) = A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for each permutation σ of $\{1, 2, \ldots, n\}$. When $F = \mathbb{K}$, we write $\mathcal{L}^s({}^nE)$ instead of $\mathcal{L}^s({}^nE;\mathbb{K})$. S_n denotes the group of all permutations of $\{1, 2, \ldots, n\}$. For each $A \in \mathcal{L}_a({}^nE;F)$ and $x \in E$, we define $Ax^n = A(\underbrace{x, x, \ldots, x}_{n}) = \hat{A}(x)$.*

Definition 1.2.6. *A mapping $P : E \to F$ is said to be an n-* homogeneous polynomial *if there exists $A \in \mathcal{L}_a({}^nE;F)$ such that $P(x) = Ax^n = \widehat{A}(x)$ for every $x \in E$. We shall denote by $\mathcal{P}_a({}^nE;F)$ the vector space of all n- homogeneous polynomials from E into F. We shall represent by $\mathcal{P}({}^nE;F)$ the subspace of all continuous members of $\mathcal{P}_a({}^nE;F)$. For each $P \in \mathcal{P}({}^nE;F)$ we define*

$$\|P\| = \sup\{\|P(x)\| : x \in E, \|x\| \leqslant 1\}.$$

When $F = \mathbb{K}$, we shall write $\mathcal{P}_a({}^nE;\mathbb{K}) = \mathcal{P}_a({}^nE)$ and $\mathcal{P}({}^nE;\mathbb{K}) = \mathcal{P}({}^nE)$.

Example 1.2.7. *Let* $P((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} x_n^m$*, for* $(x_n)_{n\in\mathbb{N}} \in \ell_2$ *and* $m \geqslant 2$*. Then* $P \in \mathcal{P}(^m\ell_2)$*.*

Proposition 1.2.8. *((MUJICA, 1985, Corollary 2.3))*

1. *A polynomial* $P \in \mathcal{P}_a(^nE;F)$ *is continuous if and only if* $\|P\| < \infty$*.*
2. $\mathcal{P}(^nE;F)$ *is a Banach space under the norm* $P \to \|P\|$*.*
3. *The mapping* $A \to \hat{A}$ *induces an isomorphism between* $\mathcal{L}^s(^nE;F)$ *and* $\mathcal{P}(^nE;F)$*.*

The following definitions give us special classes of homogeneous polynomials.

Definition 1.2.9. *Let* $\mathcal{P}_f(^nE;F)$ *denote the subspace of* $\mathcal{P}(^nE;F)$ *generated by all polynomials of the form* $P(x) = (\phi(x))^n b$*, with* $\phi \in E'$ *and* $b \in F$*. We denote by* $\mathcal{P}_A(^nE;F)$ *the closure of* $\mathcal{P}_f(^nE;F)$ *with respect to the norm topology.*

Definition 1.2.10. *Let* $\mathcal{P}_w(^nE;F)$ *denote the subspace of* $\mathcal{P}(^nE;F)$ *formed by all* P *which are* weakly continuous on bounded sets*, that is the restriction* $P|_B : B \to F$ *is continuous for each bounded set* $B \subset E$*, when* B *and* F *are endowed with the weak topology and the norm topology, respectively.*

Definition 1.2.11. *The subspace* $\mathcal{P}_K(^nE;F)$ *of compact polynomials of* $\mathcal{P}(^nE;F)$ *is formed by all polynomials that send bounded sets onto relatively compact sets.*

Definition 1.2.12. *The subspace* $\mathcal{P}_{wK}(^nE;F)$ *of weakly compact polynomials of* $\mathcal{P}(^nE;F)$ *is formed by all polynomials that send bounded sets onto relatively weakly compact sets.*

We always have the inclusions

$$P_f(^nE;F) \subset P_A(^nE;F) \subset P_w(^nE;F) \subset \mathcal{P}_K(^nE;F) \subset \mathcal{P}_{wK}(^nE;F) \subset \mathcal{P}(^nE;F).$$

We refer to (DINEEN, 2012) or (MUJICA, 1985) for background information on the theory of polynomials on Banach spaces.

Remark 1.2.13. *((DINEEN; MUJICA, 2015, Remark 3.3)) If* $1 < p < \infty$ *and* $n < p$*, then* $\mathcal{P}(^n\ell_p)$ *is a reflexive Banach space with a Schauder basis.*

Proposition 1.2.14. *((ARON; SCHOTTENLOHER, 1976, Proposition 5.3))or ((BLASCO, 1997, Proposition 5)) Let* $m, n \in \mathbb{N}$*,* $m \leqslant n$*. Then* $\mathcal{P}(^mE;F)$ *is isomorphic to a complemented subspace of* $\mathcal{P}(^nE;F)$*.*

The following lemma is a special case of (GONZÁLEZ; GUTIÉRREZ, 1995, Corollary 5). This lemma will be very important to prove the main result proposed in the last chapter.

Lemma 1.2.15. *((BU; JI; WONG, 2015, Lemma 4.1)) Let E and F be reflexive Banach spaces. Let $P_m, P \in \mathcal{P}_w(^nE, F)$ for each $m \in \mathbb{N}$. Then $\lim_{m\to\infty} P_m = P$ weakly in $\mathcal{P}_w(^nE, F)$ if and only if $\lim_{m\to\infty} y'(P_m(x)) = y'(P(x))$ for every $x \in E$ and every $y' \in F'$.*

Theorem 1.2.16. *((GONZÁLEZ; GUTIÉRREZ, 2000, Theorem 3)) The space $\mathcal{P}_w(^nE; F)$ contains a copy of ℓ_∞ if and only if either F contains a copy of ℓ_∞ or E contains a complemented copy of ℓ_1.*

Theorem 1.2.17. *((GONZÁLEZ; GUTIÉRREZ, 2000, Lemma 6)) Suppose E has an unconditional finite dimensional expansion of the identity and let $P \in \mathcal{P}(^nE; F)$. Then there is a w.u.C. series $\sum_{i=1}^{\infty} P_i$ in $\mathcal{P}_w(^nE; F)$ such that, for all $x \in E$, $P(x) = \sum_{i=1}^{\infty} P_i(x)$ unconditionally.*

1.3 Linearization theorem for n-homogeneous polynomials

An important tool in this work is a linearization theorem due to Ryan (RYAN, 1980).

The n- *fold tensor product*, $\otimes_n E$, of the vector space E can be constructed as a space of linear functionals on $\mathcal{L}_a(^nE)$, in the following way: for $x_1, x_2, \ldots, x_n \in E$, we denote by $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ the funcional given by evaluation at the point $(x_1, x_2, \ldots, x_n)$. In other words,

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n)(A) = \left\langle A, x_1 \otimes x_2 \otimes \ldots \otimes x_n \right\rangle = A(x_1, x_2, \ldots, x_n)$$

for each n- linear form $A \in \mathcal{L}_a(^nE)$. The n- *fold tensor product* $\otimes_n E$ is the subspace of the algebraic dual of $\mathcal{L}_a(^nE)$ spanned by these elements. Thus, a typical tensor in $\otimes_n E$ has the form

$$u = \sum_{j=1}^{m} \lambda_j x_j^1 \otimes x_j^2 \otimes \ldots \otimes x_j^n,$$

where $\lambda_j \in \mathbb{K}$ and $x_j^i \in E$ for $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant n$.

Proposition 1.3.1. *((RYAN, 2002, Proposition 2.1)) If $\sum_{j=1}^{m} \lambda_j x_j^1 \otimes x_j^2 \otimes \ldots \otimes x_j^n$ is a representation of u, then*

$$\pi(u) = \inf\left\{ \sum_{j=1}^{m} |\lambda_j| \|x_j^1\| \|x_j^2\| \ldots \|x_j^n\| : u = \sum_{j=1}^{m} \lambda_j x_j^1 \otimes x_j^2 \otimes \ldots \otimes x_j^n \right\}$$

is a norm on $\otimes_n E$. Moreover, $\pi(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \|x_1\| \|x_2\| \ldots \|x_n\|$ for every $x_1, x_2, \ldots, x_n \in E$.

Definition 1.3.2. *For $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in \otimes_n E$, let $x_1 \otimes_s x_2 \otimes_s \ldots \otimes_s x_n$ denote its symmetrization, that is,*

$$x_1 \otimes_s x_2 \otimes_s \ldots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(n)},$$

let $\otimes_{n,s} E$ denote the n- fold symmetric tensor product *of E, that is, the linear span of $\{x_1 \otimes_s x_2 \otimes_s \ldots \otimes_s x_n : x_1, x_2, \ldots, x_n \in E\}$ in $\otimes_n E$. Let $\hat{\otimes}_{n,s,\pi} E$ denote the n-* fold symmetric projective tensor product *of E, that is, the completion of $\otimes_{n,s} E$, under the* symmetric projective tensor norm π *defined previously.*

The next important theorem is due to Blasco.

Theorem 1.3.3. *((BLASCO, 1997, Theorem 3)) The space $\otimes_{n,s,\pi} E$ is a complemented subspace of $\otimes_{n+1,s,\pi} E$, for each positive integer n.*

Definition 1.3.4. *Let $\mathcal{C}(E; F)$ denote the vector space of all continuous mappings from E into F. When $F = \mathbb{K}$ we shall write $\mathcal{C}(E; \mathbb{K}) = C(E)$. The* compact-open topology *or* topology of compact convergence *is the locally convex topology τ_c on $\mathcal{C}(E; F)$ which is generated by the seminorms of the form $f \to \sup_{x \in K} \|f(x)\|$, where K varies among all the compact subsets of E.*

We will use the following version of Ryan's linearization theorem, which appeared in (MUJICA, 1991).

Theorem 1.3.5. *For each Banach space E and each $n \in \mathbb{N}$ let*

$$Q(^n E) = (\mathcal{P}(^n E), \tau_c)',$$

with the norm induced by $\mathcal{P}(^n E)$, and let

$$\delta_n : x \in E \to \delta_x \in Q(^n E)$$

denote the evaluation mapping, that is, $\delta_x(P) = P(x)$ for all $x \in E$ and $P \in \mathcal{P}(^n E)$. Then $Q(^n E)$ is a Banach space and $\delta_n \in \mathcal{P}(^n E; Q(^n E))$. The pair $(Q(^n E), \delta_n)$ has the following universal property: for each Banach space F and each $P \in \mathcal{P}(^n E; F)$, there is a unique operator $T_p \in \mathcal{L}(Q(^n E); F)$ such that $T_p \circ \delta_n = P$. The mapping

$$P \in \mathcal{P}(^n E; F) \to T_p \in \mathcal{L}(Q(^n E); F)$$

is an isometric isomorphism. Moreover $P \in \mathcal{P}_K(^n E; F)$ if and only if $T_p \in \mathcal{L}_K(Q(^n E); F)$, and $P \in \mathcal{P}_{wK}(^n E; F)$ if and only if $T_p \in \mathcal{L}_{wK}(Q(^n E); F)$. Furthermore $Q(^n E)$ is isometrically isomorphic to $\hat{\otimes}_{n,s,\pi} E$.

Definition 1.3.6. *Consider a linear ordering $\prec$ of $\mathbb{N}^2$ such that $(1,1) \prec (2,1) \prec (2,2) \prec (1,2) \prec (3,1) \prec (3,2) \prec (3,3) \prec (2,3) \prec (1,3) \prec (4,1) \prec \ldots$. Clearly $(\mathbb{N}^2, \prec)$, as a linearly ordered set, is isomorphic to the usual integers $(\mathbb{N}, <)$.*

The next theorem will be used in the next chapter.

Theorem 1.3.7. *((FABIAN et al., 2011, Corollary 16.69)) Let E be a Banach space with a shrinking Schauder basis $(e_n)_{n\in\mathbb{N}}$, and F be a Banach space with a Schauder basis $(f_n)_{n\in\mathbb{N}}$. Then $\{e'_n \otimes f_j\}(n,j) \in (\mathbb{N}^2, \prec)$ is a Schauder basis of $\mathcal{L}_K(E;F)$. Moreover, if both $(e'_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are shrinking, then $\{e'_n \otimes f_j\}(n,j) \in (\mathbb{N}^2, \prec)$ is a shrinking basis of $\mathcal{L}_K(E;F)$.*

Finally, we introduce the concept of holomorphic function.

Definition 1.3.8. *Let U be an open subset of E. A mapping $f : U \to F$ is said to be* holomorphic *or* analytic *if for each $a \in U$ there exists a ball $B(a;r) \subset U$ and a sequence of polynomials $P_n \in \mathcal{P}(^nE;F)$ such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x-a)$$

uniformly for $x \in B(a;r)$. We shall denote by $\mathcal{H}(U;F)$ the vector space of all holomorphic mappings from U into F. When $F = \mathbb{K}$ then we shall write $\mathcal{H}(U;\mathbb{K}) = \mathcal{H}(U)$.

1.4 The approximation property

In this section we enunciate some theorems about the approximation property that will be used in the next chapter.

Definition 1.4.1. *Let E be a Banach space. E is said to have the* approximation property *(AP in short) if given $K \subset E$ compact and $\epsilon > 0$, there exists $T \in \mathcal{F}(E;E)$ such that $\|Tx - x\| < \epsilon$ for every $x \in K$.*

Definition 1.4.2. *A Banach space E is said to have the* bounded approximation property *(BAP in short) if there exists $\lambda \geqslant 1$ so that for every compact $K \subset E$ and for every $\epsilon > 0$, there exists $T \in \mathcal{F}(E;E)$ such that $\|T\| \leqslant \lambda$ and $\|Tx - x\| < \epsilon$ for every $x \in K$.*

Example 1.4.3. *((MUJICA, 1985, Theorem 27.4)) Every Banach space with Schauder basis has the BAP.*

Proposition 1.4.4. *((MUJICA, 1985, Proposition 27.2)) For a Banach space E the following conditions are equivalent:*

1. *E has the approximation property.*
2. *Each $T \in \mathcal{L}(E;E)$ can be uniformly approximated on compact sets by operators of finite rank.*

3. *For each Banach space F, each $T \in \mathcal{L}(E; F)$ can be uniformly approximated on compact sets by operators of finite rank.*

4. *For each Banach space F, each $T \in \mathcal{L}(F; E)$ can be uniformly approximated on compact sets by operators of finite rank.*

Definition 1.4.5. *A Banach space E is said to have the* compact approximation property *(CAP in short) if given $K \subset E$ compact and $\epsilon > 0$, there exists $T \in \mathcal{L}_K(E; E)$ such that $\|Tx - x\| < \epsilon$ for every $x \in K$.*

The following results give us examples of Banach spaces without the approximation property. These examples will be used to prove some results in the next chapter.

Proposition 1.4.6. *((DINEEN; MUJICA, 2015, Proposition 2.1)) If $1 < p, q < \infty$, then $\mathcal{L}(L_p[0,1]; L_q[0,1])$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular $\mathcal{L}(L_p[0,1]; L_q[0,1])$ does not have the approximation property.*

Proposition 1.4.7. *((DINEEN; MUJICA, 2015, Proposition 2.2)) If $1 < p \leqslant q < \infty$, then $\mathcal{L}(\ell_p; \ell_q)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular $\mathcal{L}(\ell_p; \ell_q)$ does not have the approximation property.*

Proposition 1.4.8. *((DINEEN; MUJICA, 2015, Proposition 2.4))*

1. *If E and F contain complemented subspaces isomorphic to ℓ_2, then $\mathcal{L}(E; F)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular, $\mathcal{L}(E; F)$ does not have the approximation property.*

2. *If E contains a complemented subspace isomorphic to ℓ_2, then $\mathcal{L}(E; E')$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular, $\mathcal{L}(E; E')$ does not have the approximation property.*

Theorem 1.4.9. *((DINEEN; MUJICA, 2015, Theorem 3.2)) If $1 < p < \infty$ and $n \geqslant p$, then $\mathcal{P}(^n\ell_p)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular, $\mathcal{P}(^n\ell_p)$ does not have the approximation property.*

Corollary 1.4.10. *((BU; JI; WONG, 2015, Corollary 4.4)) Assume that both E and F are reflexive.*

1. *If $\mathcal{P}_w(^nE; F) = \mathcal{P}(^nE; F)$, then $\mathcal{P}_w(^nE; F)$ is reflexive.*

2. *If E has the CAP, then $\mathcal{P}_w(^nE; F)$ is reflexive if and only if $\mathcal{P}_w(^nE; F) = \mathcal{P}(^nE; F)$.*

Theorem 1.4.11. *((PEŁCZYŃSKI, 1960, Lemma 2) or (LINDENSTRAUSS; TZAFRIRI, , Proposition 2.a.2)) Let E be an infinite dimensional subspace of ℓ_p, $1 \leqslant p < \infty$. Then E contains a subspace F which has a complement in ℓ_p and is isomorphic to ℓ_p.*

Theorem 1.4.12. *((BANACH, 1987, p.206, 12.5)) Let $2 < p < \infty$ and let E be a subspace of $L_p[0;1]$ not isomorphic to any Hilbert space. Then E contains a complemented subspace isomorphic to ℓ_p.*

Theorem 1.4.13. *((GODEFROY; SAPHAR, 1989, Theorem 2.4)) Let E be a Banach space and M be a closed subspace of E such that $M^\perp$ is complemented in E'. If M has the BAP, then E/M has the AP implies that E has the AP.*

Theorem 1.4.14. *((GODEFROY; SAPHAR, 1989, Corollary 2.8)) Let H be an infinite dimensional Hilbert space, and $\mathcal{L}_K(H;H)$ be the space of compact operators on H. Then the quotient algebra $\mathcal{L}(H;H)/\mathcal{L}_K(H;H)$ does not have the A.P.*

Lemma 1.4.15. *((JOHNSON, 1979, Lemma 1)) Let E and F be Banach spaces and suppose F has the BAP. Then there is a projection P on $\mathcal{L}(E;F)'$ such that $\|P\| \leqslant \lambda$, the range of P is isomorphic to $\mathcal{L}_K(E;F)'$ (isometric if $\lambda = 1$) and the kernel of P is $\mathcal{L}_K(E;F)^\perp$.*

1.5 Complemented subspaces and reflexivity in the space of bounded linear operators

In this section we enunciate some important results about copies of c_0 and reflexivity in spaces of compact operators. The following three theorems will be generalized in the last chapter.

Theorem 1.5.1. *((FEDER, 1975, Theorem 2)) Let E and F be reflexive Banach spaces and G a closed linear subspace of $\mathcal{L}_K(E;F)$ which contains the space $\mathcal{F}(E,F)$ of all finite rank linear operators from E into F. Then G is either reflexive or is not a dual space.*

Theorem 1.5.2. *((FEDER et al., 1980, Theorem 5)) Let E and F be reflexive Banach spaces such that F or E' is a subspace of a Banach space with an unconditional basis. Then $\mathcal{L}_K(E;F)$ is either reflexive or non-isomorphic to a dual space.*

Definition 1.5.3. *If E is a Banach space then $P \in \mathcal{P}_\alpha(^nE)$ is an* integral polynomial *or a* polynomial of integral type *if there exists a regular Borel measure μ of finite variation on $(\overline{B_{E'}}, \sigma(E';E))$ such that*

$$P(x) = \int_{\overline{B_{E'}}} \phi(x)^n d\mu(\phi) \tag{1.1}$$

for all $x \in E$. We write $\mathcal{P}_I(^nE)$ for the space of all n- homogeneous integral polynomials on E. We define the integral norm of an integral polynomial P, $\|P\|_I$, as the infimum of $\|\mu\|$ taken over all regular Borel measures which satisfy (1.1). With the integral norm $\mathcal{P}_I(^nE)$ becomes a Banach space.

Theorem 1.5.4. *((BOYD; RYAN, 2001, Theorem 21)) Let E be a reflexive Banach space with one of the following conditions holding:*

1. *$B_{E'}$ has a Fréchet differentiable norm.*
2. *E is separable and $\mathcal{P}_I(^nE')$ is weakly sequentially complete.*

Then $\mathcal{P}_A(^nE)$ is either reflexive or not isometric to a dual space.

The following corollaries will be generalized in Chapter 3.

Corollary 1.5.5. *((GHENCIU, 2005b, Corollary 2)) If $c_0 \hookrightarrow F$ and E' contains a w' null sequence $(x'_n)_{n\in\mathbb{N}}$ which is not w null, then $\mathcal{L}_{wK}(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Corollary 1.5.6. *((GHENCIU, 2005b, Corollary 3)) Assume that E contains a complemented copy of c_0 and $c_0 \hookrightarrow F$. Then $\mathcal{L}_{wK}(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Corollary 1.5.7. *((GHENCIU, 2005b, Corollary 4)) If $c_0 \hookrightarrow F$ and E is an infinite dimensional Banach space, then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Corollary 1.5.8. *((GHENCIU, 2005b, Corollary 5)) Assume that $\mathcal{L}(E;\ell_1) \neq \mathcal{L}_K(E;\ell_1)$ and that F contains a copy of ℓ_1. Then $\mathcal{L}_{wK}(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Corollary 1.5.9. *((GHENCIU, 2005b, Corollary 6)) If E contains a complemented copy of ℓ_1 and F is infinite dimensional, then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Theorem 1.5.10. *((EMMANUELE, 1992, Theorem 2)),((JOHN, 1992, Theorem 1)) or ((GHENCIU; LEWIS, 2011, Corollary 11)) Assume that $c_0 \hookrightarrow \mathcal{L}_K(E;F)$. Then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

Lemma 1.5.11. *((GONZÁLEZ; GUTIÉRREZ, 2000, Lemma 5)) Suppose E contains a complemented copy of ℓ_1. Then $\mathcal{P}_w(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for all F and $n > 1$.*

Theorem 1.5.12. *((JOHN, 1992, Theorem 2)) Let E, F be arbitrary Banach spaces and $T : E \to F$ a non-compact operator. Suppose that T admits a factorization $T = AB$ through a Banach space Z with an unconditional basis (countable or uncountable). Then the space $\mathcal{L}_K(E;F)$ contains an isomorphic copy of c_0 and thus $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.*

The following results will be generalized in chapter 3.

Theorem 1.5.13. *((KALTON, 1974, Theorem 6)) Let E be a Banach space with an unconditional finite dimensional expansion of the identity $(A_n)_{n\in\mathbb{N}}$. If F is any infinite dimensional Banach space the following are equivalent.*

1. $\mathcal{L}_K(E;F) = \mathcal{L}(E;F)$.
2. $\mathcal{L}_K(E;F)$ *contains no copy of* c_0.
3. $\mathcal{L}(E;F)$ *contains no copy of* ℓ_∞.
4. $\mathcal{L}_K(E;F)$ *is complemented in* $\mathcal{L}(E;F)$.

Theorem 1.5.14. *((GONZÁLEZ; GUTIÉRREZ, 2000, Theorem 7)) Suppose E has an unconditional finite dimensional expansion of the identity and let $n > 1$. Then the following conditions are equivalent:*

1. $\mathcal{P}_w(^nE;F) = \mathcal{P}(^nE;F)$.
2. $\mathcal{P}_w(^nE;F)$ *contains no copy of* c_0.
3. $\mathcal{P}(^nE;F)$ *contains no copy of* ℓ_∞.
4. $\mathcal{P}_w(^nE;F)$ *is complemented in* $\mathcal{P}(^nE;F)$.

Theorem 1.5.15. *((JOHN, 1992, Remark 3e))) Suppose that E and F are infinite dimensional Banach spaces, such that each non compact operator $T \in \mathcal{L}(E;F)$ factors through a Banach space Z with an unconditional basis, then the following conditions are equivalent:*

1. $\mathcal{L}_K(E;F)$ *contains a copy of* c_0.
2. $\mathcal{L}(E;F)$ *contains a copy of* c_0.
3. $\mathcal{L}(E;F)$ *contains a copy of* ℓ_∞.
4. $\mathcal{L}_K(E;F) \neq \mathcal{L}(E;F)$.
5. $\mathcal{L}_K(E;F)$ *is not complemented in* $\mathcal{L}(E;F)$.

2 Banach spaces of linear operators and homogeneous polynomials without the approximation property

In this Chapter, by using the methods of Dineen and Mujica (DINEEN; MUJICA, 2015) and Godefroy and Saphar (GODEFROY; SAPHAR, 1989), we present many examples of Banach spaces of linear operators and homogeneous polynomials which do not have the approximation property.

In Section 2.1 we present some examples of Banach spaces of linear operators without the approximation property. Among other results, we show that if $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then $\mathcal{L}(E;F)$ does not have the approximation property. This improves a result of Dineen and Mujica (DINEEN; MUJICA, 2015). We also show that if $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then the quotient $\mathcal{L}(E;F)/\mathcal{L}_K(E;F)$ does not have the approximation property. This improves a result of Godefroy and Saphar (GODEFROY; SAPHAR, 1989).

In Section 2.2 we present more examples of Banach spaces of linear operators without the approximation property. Our examples are Banach spaces of linear operators on Pelczynski's universal space U_1, on Orlicz sequence spaces ℓ_{M_p}, and on Lorentz sequence spaces $d(w,p)$.

In Section 2.3 we present examples of Banach spaces of homogeneous polynomials without the approximation property. Among other results we show that if $1 < p < \infty$ and E is a closed infinite dimensional subspace of ℓ_p, then $\mathcal{P}(^nE)$ does not have the approximation property for every $n \geqslant p$. This improves another result of Dineen and Mujica (DINEEN; MUJICA, 2015). We also show that if $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then $\mathcal{P}(^nE;F)$ does not have the approximation property for every $n \geqslant 1$. We also prove that if $n < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively, then the quotient $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ does not have the approximation property.

2.1 Banach spaces of linear operators without the approximation property

The following well-known proposition will be repeatedly used throughout this chapter.

Proposition 2.1.1. *Let E and F be Banach spaces. E is isomorphic to a complemented subspace of F if and only if there are $A \in \mathcal{L}(E;F)$ and $B \in \mathcal{L}(F;E)$ such that $B \circ A = I$.*

Proof. ($\Rightarrow$) Let $\varphi : E \to M$ be an isomorphism, where M is a complemented subspace of F. Denote by $j : M \to F$ the inclusion map and $\pi : F \to M$ the projection. Consider $A = j \circ \varphi$ and $B = \varphi^{-1} \circ \pi$, then $B \circ A(e) = \varphi^{-1} \circ \pi \circ j \circ \varphi(e) = \varphi^{-1} \circ j \circ \varphi(e) = e$ for every $e \in E$. Therefore $B \circ A = I$.

($\Leftarrow$) Let $M = A(E)$. Since $B \circ A = I$, then A is an injective operator, thus M is isomorphic to E. We take $\pi = A \circ B : F \to M$, then π is a projection. Hence E is isomorphic to a complemented subspace of F. □

Proposition 2.1.2. *Let E, F, M, and N be Banach spaces. If E and F contain complemented subspaces isomorphic to M and N, respectively, then $\mathcal{L}(E;F)$ contains a complemented subspace isomorphic to $\mathcal{L}(M;N)$.*

Proof. By hypothesis there are $A_1 \in \mathcal{L}(M;E)$, $B_1 \in \mathcal{L}(E;M)$, $A_2 \in \mathcal{L}(N;F)$, $B_2 \in \mathcal{L}(F;N)$ such that $B_1 \circ A_1 = I$ and $B_2 \circ A_2 = I$. Consider the operators

$$C : S \in \mathcal{L}(M;N) \to A_2 \circ S \circ B_1 \in \mathcal{L}(E;F)$$

and

$$D : T \in \mathcal{L}(E;F) \to B_2 \circ T \circ A_1 \in \mathcal{L}(M;N).$$

Then $D \circ C = I$ and the desired conclusion follows. □

The next theorem improves Proposition 1.4.8.

Theorem 2.1.3. *Let $1 < p \leqslant q < \infty$. If E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively, then $\mathcal{L}(E;F)$ does not have the approximation property.*

Proof. By Proposition 2.1.2 $\mathcal{L}(E;F)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_p;\ell_q)$. Then the conclusion follows from Proposition 1.4.7. □

The next result improves Proposition 1.4.7.

Theorem 2.1.4. *Let $1 < p \leqslant q < \infty$, and let E and F be closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively. Then $\mathcal{L}(E;F)$ does not have the approximation property.*

Proof. By Theorem 1.4.11 E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively. Then the desired conclusion follows from Theorem 2.1.3. $\square$

The next result complements Proposition 1.4.6.

Theorem 2.1.5. *Let $2 < p \leqslant q < \infty$, and let E and F be closed infinite dimensional subspaces of $L_p[0,1]$ and $L_q[0,1]$, respectively, with F not isomorphic to ℓ_2. Then $\mathcal{L}(E;F)$ does not have the approximation property.*

Proof.

(i) If E is not isomorphic to ℓ_2, then it follows from Theorem 1.4.12 that E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively. Then the desired conclusion follows from Theorem 2.1.3.

(ii) If E is isomorphic to ℓ_2, then the same argument shows that $\mathcal{L}(E;F)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_q)$, and the desired conclusion follows as before.

$\square$

The next result improves Theorem 1.4.14.

Theorem 2.1.6. *If $1 < p \leqslant q < \infty$, then $\mathcal{L}(\ell_p;\ell_q)/\mathcal{L}_K(\ell_p;\ell_q)$ does not have the approximation property.*

Proof. By Lemma 1.4.15 $\mathcal{L}_K(\ell_p;\ell_q)^\perp$ is a complemented subspace of $\mathcal{L}(\ell_p;\ell_q)'$. By Theorem 1.3.7, $\mathcal{L}_K(\ell_p;\ell_q)$ has a Schauder basis. If we assume that $\mathcal{L}(\ell_p;\ell_q)/\mathcal{L}_K(\ell_p;\ell_q)$ has the approximation property, then Theorem 1.4.13 would imply that $\mathcal{L}(\ell_p;\ell_q)$ has the approximation property, thus contradicting Proposition 1.4.7. $\square$

Remark 2.1.7. *If $1 < q < p < \infty$, then $\mathcal{L}(\ell_p;\ell_q) = \mathcal{L}_K(\ell_p;\ell_q)$ by a result of Pitt (PITT, 1936). Hence the restriction $p \leqslant q$ in the preceding theorem cannot be deleted.*

Proposition 2.1.8. *If E and F contain complemented subspaces isomorphic to M and N, respectively, then $\mathcal{L}(E;F)/\mathcal{L}_K(E;F)$ contains a complemented subspace isomorphic to $\mathcal{L}(M;N)/\mathcal{L}_K(M;N)$.*

Proof. By hypothesis there are $A_1 \in \mathcal{L}(M;E)$, $B_1 \in \mathcal{L}(E;M)$, $A_2 \in \mathcal{L}(N;F)$, $B_2 \in \mathcal{L}(F;N)$ such that $B_1 \circ A_1 = I$ and $B_2 \circ A_2 = I$. Let $C : \mathcal{L}(M;N) \to \mathcal{L}(E;F)$ and $D : \mathcal{L}(E;F) \to \mathcal{L}(M;N)$ be the operators from the proof of Proposition 2.1.2. Since $C(\mathcal{L}_K(M;N)) \subset \mathcal{L}_K(E;F)$ and $D(\mathcal{L}_K(E;F)) \subset \mathcal{L}_K(M;N)$, the operators

$$\tilde{C} : [S] \in \mathcal{L}(M;N)/\mathcal{L}_K(M;N) \to [A_2 \circ S \circ B_1] \in \mathcal{L}(E;F)/\mathcal{L}_K(E;F)$$

and

$$\tilde{D} : [T] \in \mathcal{L}(E;F)/\mathcal{L}_K(E;F) \to [B_2 \circ T \circ A_1] \in \mathcal{L}(M;N)/\mathcal{L}_K(M;N)$$

are well defined, and $\tilde{D} \circ \tilde{C} = I$, thus completing the proof. □

Theorem 2.1.9. *Let $1 < p \leqslant q < \infty$. If E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively, then $\mathcal{L}(E;F)/\mathcal{L}_K(E;F)$ does not have the approximation property.*

Proof. By Proposition 2.1.8 $\mathcal{L}(E;F)/\mathcal{L}_K(E;F)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_p;\ell_q)/\mathcal{L}_K(\ell_p;\ell_q)$. Then the desired conclusion follows from Theorem 2.1.6. □

By combining Theorem 2.1.9 and Theorem 1.4.11 we obtain the following theorem.

Theorem 2.1.10. *Let $1 < p \leqslant q < \infty$, and let E and F be closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively. Then $\mathcal{L}(E;F)/\mathcal{L}_K(E;F)$ does not have the approximation property.*

2.2 Concrete examples of Banach spaces of linear operators without the approximation property

Applying Theorem 2.1.3 we obtain particular examples of Banach spaces without the approximation property.

Example 2.2.1. *Let U_1 denote the universal space of Pelczynski (see Theorem 1.1.19). U_1 is a Banach space with an unconditional basis with the property that every Banach space with an unconditional basis is isomorphic to a complemented subspace of U_1. Since every ℓ_p $(1 \leqslant p < \infty)$ has an unconditional basis, it follows that every ℓ_p $(1 \leqslant p < \infty)$ is isomorphic to a complemented subspace of U_1. By Theorem 2.1.3 none of the spaces $\mathcal{L}(U_1;U_1)$, $\mathcal{L}(U_1;\ell_q)$ $(1 < q < \infty)$ or $\mathcal{L}(\ell_p;U_1)$ $(1 < p < \infty)$ have the approximation property.*

Definition 2.2.2. *(see (LINDENSTRAUSS; TZAFRIRI, , p. 137))*

An Orlicz function M is a continuous convex nondecreasing function $M : [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. Let

$$\ell_M = \left\{ x = (\xi_n)_{n=1}^{\infty} \subset \mathbb{K} : \sum_{n=1}^{\infty} M(|\xi_n|/\rho) < \infty \quad for\ some \quad \rho > 0 \right\}.$$

Then ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0; \sum_{n=1}^{\infty} M(|\xi_n|/\rho) \leqslant 1 \right\}.$$

ℓ_M is called an Orlicz sequence space.

Example 2.2.3. *Consider the Orlicz function $M_p(t) = t^p(1 + |\log t|)$ if $t > 0$ and $M(0) = 0$, for $1 < p < \infty$. Then the Orlicz sequence space ℓ_{M_p} contains complemented subspaces isomorphic to ℓ_p (see (LINDENSTRAUSS; TZAFRIRI, , p. 157)). If $1 < p \leqslant q < \infty$, then by Theorem 2.1.3, $\mathcal{L}(\ell_{M_p}; \ell_{M_q})$ does not have the approximation property.*

Example 2.2.4. *It follows from Proposition 1.1.21 that every closed infinite dimensional subspace of the Lorentz sequence space $d(w, p)$ contains a complemented subspace isomorphic to ℓ_p. By Theorem 2.1.3, if $1 < p \leqslant q < \infty$, and E and F are closed infinite dimensional subspaces of $d(w, p)$ and $d(w, q)$, respectively, then $\mathcal{L}(E; F)$ does not have the approximation property.*

2.3 Spaces of homogeneous polynomials without the approximation property

An important tool in this section is a linearization theorem due to Ryan (Theorem 1.3.5).

Proposition 2.3.1. *If E and F contain complemented subspaces isomorphic to M and N, respectively, then $\mathcal{P}(^nE; F)$ contains a complemented subspace isomorphic to $\mathcal{P}(^nM; N)$.*

Proof. By hypothesis there are $A_1 \in \mathcal{L}(M; E)$, $B_1 \in \mathcal{L}(E; M)$, $A_2 \in \mathcal{L}(N; F)$, $B_2 \in \mathcal{L}(F; N)$ such that $B_1 \circ A_1 = I$ and $B_2 \circ A_2 = I$. Consider the operators

$$C : P \in \mathcal{P}(^nM; N) \to A_2 \circ P \circ B_1 \in \mathcal{P}(^nE; F)$$

and

$$D : Q \in \mathcal{P}(^nE; F) \to B_2 \circ Q \circ A_1 \in \mathcal{P}(^nM; N).$$

Then $D \circ C = I$ and the desired conclusion follows. □

Corollary 2.3.2. *If E contains a complemented subspace isomorphic to M, then $\mathcal{P}(^nE)$ contains a complemented subspace isomorphic to $\mathcal{P}(^nM)$.*

Proof. Take $F = N = \mathbb{K}$ in Proposition 2.3.1. □

The next result improves Theorem 1.4.9.

Theorem 2.3.3. *Let $1 < p < \infty$. If E contains a complemented subspace isomorphic to ℓ_p, then $\mathcal{P}(^nE)$ does not have the approximation property for every $n \geqslant p$.*

Proof. By Corollary 2.3.2 $\mathcal{P}(^nE)$ contains a complemented subspace isomorphic to $\mathcal{P}(^n\ell_p)$. Then the conclusion follows from Theorem 1.4.9. □

Theorem 2.3.3 can be used to produce many additional counterexamples. For instance, by combining Theorem 2.3.3 and Theorem 1.4.11 we obtain the following result.

Theorem 2.3.4. *Let $1 < p < \infty$ and let E be a closed infinite dimensional subspace of ℓ_p. Then $\mathcal{P}(^nE)$ does not have the approximation property for every $n \geqslant p$.*

In a similar way we may obtain scalar-valued polynomial versions of Theorem 2.1.5 and Examples 2.2.1, 2.2.3 and 2.2.4.

Theorem 2.3.5. *Let $1 < p \leqslant q < \infty$. If E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively, then $\mathcal{P}(^nE; F)$ does not have the approximation property for every $n \geqslant 1$.*

Proof. By Proposition 1.2.14 $\mathcal{L}(E; F)$ is isomorphic to a complemented subspace of $\mathcal{P}(^nE; F)$. Then the desired conclusion follows from Theorem 2.1.3. □

Theorem 2.3.5 can be used to produce many additional counterexamples. For instance, by combining Theorem 2.3.5 and Theorem 1.4.11 we obtain the following result.

Theorem 2.3.6. *Let $1 < p \leqslant q < \infty$ and let E and F be closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively. Then $\mathcal{P}(^nE; F)$ does not have the approximation property for every $n \geqslant 1$.*

In a similar way we may obtain vector-valued polynomial versions of Theorem 2.1.5 and Examples 2.2.1, 2.2.3 and 2.2.4. We leave the details to the reader.

Theorem 2.3.7. *If $n < p \leqslant q < \infty$. Then $\mathcal{P}(^n\ell_p; \ell_q)/\mathcal{P}_K(^n\ell_p; \ell_q)$ does not have the approximation property.*

Proof. By Theorem 1.3.5 we can write

$$\mathcal{P}(^n\ell_p; \ell_q) = \mathcal{L}(Q(^n\ell_p); \ell_q)$$

and

$$\mathcal{P}_K(^n\ell_p; \ell_q) = \mathcal{L}_K(Q(^n\ell_p); \ell_q).$$

We apply Theorem 1.4.13. By Lemma 1.4.15 $\mathcal{L}_K(Q(^n\ell_p); \ell_q)^\perp$ is a complemented subspace of $\mathcal{L}(Q(^n\ell_p); \ell_q)'$. By Remark 1.2.13 $\mathcal{P}(^n\ell_p)$ is a reflexive Banach space with a Schauder basis. Hence $Q(^n\ell_p)$ is also a reflexive Banach space with a Schauder basis. Then by Theorem 1.3.7 $\mathcal{L}_K(Q(^n\ell_p); \ell_q)$ has a Schauder basis. If we assume that $\mathcal{L}(Q(^n\ell_p); \ell_q)/\mathcal{L}_K(Q(^n\ell_p); \ell_q)$ has the approximation property, then Theorem 1.4.13 would imply that $\mathcal{L}(Q(^n\ell_p); \ell_q)$ has the approximation property. But this contradicts Theorem 2.1.3, since $\ell_p = Q(^1\ell_p)$ is a complemented subspace of $Q(^n\ell_p)$, by Theorem 1.3.3. This completes the proof. □

Proposition 2.3.8. *If E and F contain complemented subspaces isomorphic to M and N, respectively, then $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ contains a complemented subspace isomorphic to $\mathcal{P}(^nM;N)/\mathcal{P}_K(^nM;N)$.*

Proof. By hypothesis there are $A_1 \in \mathcal{L}(M;E)$, $B_1 \in \mathcal{L}(E;M)$, $A_2 \in \mathcal{L}(N;F)$, $B_2 \in \mathcal{L}(F;N)$ such that $B_1 \circ A_1 = I$ and $B_2 \circ A_2 = I$. Let

$$C : \mathcal{P}(^nM;N) \to \mathcal{P}(^nE;F)$$

and

$$D : \mathcal{P}(^nE;F) \to \mathcal{P}(^nM;N)$$

be the operators from the proof of Proposition 2.3.1. Since $C(\mathcal{P}_K(^nM;N)) \subset \mathcal{P}_K(^nE;F)$ and $D(\mathcal{P}_K(^nE;F)) \subset \mathcal{P}_K(^nM;N)$, the operators

$$\tilde{C} : [P] \in \mathcal{P}(^nM;N)/\mathcal{P}_K(^nM;N) \to [A_2 \circ P \circ B_1] \in \mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$$

and

$$\tilde{D} : [Q] \in \mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F) \to [B_2 \circ Q \circ A_1] \in \mathcal{P}(^nM;N)/\mathcal{P}_K(^nM;N)$$

are well-defined and $\tilde{D} \circ \tilde{C} = I$, thus completing the proof. □

Theorem 2.3.9. *Let $n < p \leqslant q < \infty$. If E and F contain complemented subspaces isomorphic to ℓ_p and ℓ_q, respectively, then $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ does not have the approximation property.*

Proof. By Proposition 2.3.8 $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ contains a complemented subspace isomorphic to $\mathcal{P}(^n\ell_p;\ell_q)/\mathcal{P}_K(^n\ell_p;\ell_q)$. Thus the desired conclusion follows from Theorem 2.3.7. □

Theorem 2.3.9 can be used to produce many additional counterexamples. For instance by combining Theorem 2.3.9 and Theorem 1.4.11 we obtain the following theorem.

Theorem 2.3.10. *Let $n < p \leqslant q < \infty$, and let E and F be closed infinite dimensional subspaces of ℓ_p and ℓ_q, respectively. Then $\mathcal{P}(^nE;F)/\mathcal{P}_K(^nE;F)$ does not have the approximation property.*

The interest in the study of the approximation property in spaces of homogeneous polynomials begun in 1976 with a paper of Aron and Schottenloher (ARON; SCHOTTENLOHER, 1976). They began the study of the approximation property on the space $\mathcal{H}(E)$ of all holomorphic functions on E under various topologies. Among many other results they proved that $(\mathcal{H}(E), \tau_w)$ has the approximation property if and only if $\mathcal{P}(^n E)$ has the approximation property for every $n \in \mathbb{N}$. Here τ_w denotes the compact-ported topology introduced by Nachbin. They also proved that $\mathcal{P}(^n \ell_1)$ has the approximation property for every $n \in \mathbb{N}$. Ryan (RYAN, 1980) proved that $\mathcal{P}(^n c_0)$ has a Schauder basis, and in particular has the approximation property, for every $n \in \mathbb{N}$. Tsirelson (TSIREL'SON, 1974) constructed a reflexive Banach space X, with an unconditional Schauder basis, which contains no subspace isomorphic to any ℓ_p. By using a result of Alencar, Aron and Dineen (ALENCAR; ARON; DINEEN, 1984), Alencar (ALENCAR, 1985) proved that $\mathcal{P}(^n X)$ has a Schauder basis, and in particular has the approximation property, for every $n \in \mathbb{N}$. In a series of papers Dineen and Mujica (DINEEN; MUJICA, 2004) (DINEEN; MUJICA, 2010) (DINEEN; MUJICA, 2012) have extended some of the results of Aron and Schottenloher (ARON; SCHOTTENLOHER, 1976) to spaces of holomorphic functions defined on arbitrary open sets.

3 Complemented subspaces of homogeneous polynomials

3.1 The main results

The proofs of our main results rests mainly on the following theorem of Ghenciu (GHENCIU, 2005b), which generalizes results of several authors (EMMANUELE, 1991),(GHENCIU, 2005a), (FEDER, 1982).

Theorem 3.1.1. *((GHENCIU, 2005b, Theorem 1)) Let E and F be Banach spaces, and let G be a Banach space with an unconditional basis (g_n) and coordinate functionals (g'_n).*

(a) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a semi-normalized basic sequence in F and $(S'(g'_n))$ is not relatively compact in E', then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$.

(b) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a semi-normalized basic sequence in F and $(S'(g'_n))$ is not relatively weakly compact in E', then $\mathcal{L}_{wK}(E;F)$ is not complemented in $\mathcal{L}(E;F)$.

Emmanuele (EMMANUELE, 1992) and John (JOHN, 1992) independently proved that if $\mathcal{L}_K(E;F)$ contains a copy of c_0, then $\mathcal{L}_K(E;F)$ is not complemented in $\mathcal{L}(E;F)$ (see Theorem 1.5.10). They also proved that if there exists a noncompact operator $T \in \mathcal{L}(E;F)$ which factors through a Banach space with an unconditional basis, then $\mathcal{L}_K(E;F)$ contains a copy of c_0. Clearly Theorem 3.1.1 (a) follows from these results.

The following results are polynomial versions of Theorem 3.1.1.

Theorem 3.1.2. *Let E and F be Banach spaces, and let G be a Banach space with an unconditional basis (g_n) and coordinate functionals (g'_n).*

(a) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a semi-normalized basic sequence in F and $(S'(g'_n))$ is not relatively compact in E', then $\mathcal{P}_K(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.

(b) If there exist operators $R \in \mathcal{L}(G;F)$ and $S \in \mathcal{L}(E;G)$ such that $(R(g_n))$ is a semi-normalized basic sequence in F and $(S'(g'_n))$ is not relatively weakly compact in E', then $\mathcal{P}_{wK}(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.

Proof. (a) The case $n = 1$ follows from Theorem 3.1.1 (a). If $n \in \mathbb{N}$, then by Theorem 1.3.5 there exists an isomorphism

$$P \in \mathcal{P}(^nE; F) \to T_P \in \mathcal{L}(\hat{\bigotimes}_{n,s,\pi} E; F).$$

Furthermore $P \in \mathcal{P}_K(^nE; F)$ if and only if $T_P \in \mathcal{L}_K(\hat{\bigotimes}_{n,s,\pi} E; F)$. Suppose that $\mathcal{P}_K(^nE; F)$ is complemented in $\mathcal{P}(^nE; F)$. Then $\mathcal{L}_K(\hat{\bigotimes}_{n,s,\pi} E; F)$ is complemented in $\mathcal{L}(\hat{\bigotimes}_{n,s,\pi} E; F)$. Let $\pi : \mathcal{L}(\hat{\bigotimes}_{n,s,\pi} E; F) \to \mathcal{L}_K(\hat{\bigotimes}_{n,s,\pi} E; F)$ be a projection. By Theorem 1.3.3 E is isomorphic to a complemented subspace of $\hat{\bigotimes}_{n,s,\pi} E$. Hence there exist operators $A \in \mathcal{L}(E; \hat{\bigotimes}_{n,s,\pi} E)$ and $B \in \mathcal{L}(\hat{\bigotimes}_{n,s,\pi} E; E)$ such that $B \circ A = I$. Consider the operator

$$\rho : T \in \mathcal{L}(E; F) \to \pi(T \circ B) \circ A \in \mathcal{L}_K(E; F).$$

If $T \in \mathcal{L}_K(E; F)$, then $T \circ B \in \mathcal{L}_K(\hat{\bigotimes}_{n,s,\pi} E; F)$ and therefore $\pi(T \circ B) \circ A = T \circ B \circ A = T$. Thus $\rho : \mathcal{L}(E; F) \to \mathcal{L}_K(E; F)$ is a projection, contradicting the case $n = 1$.

(b) The proof of (b) is almost identical to the proof of (a), but using that $P \in \mathcal{P}_{wK}(^nE; F)$ if and only if $T_P \in \mathcal{L}_{wK}(\hat{\bigotimes}_{n,s,\pi} E; F)$, Theorem 1.3.5. □

The method of the proof of Theorem 3.1.2 does not work to prove the next theorem, since it is not true in general that $P \in \mathcal{P}_w(^nE; F)$ if and only if $T_P \in \mathcal{L}_w(\hat{\bigotimes}_{n,s,\pi} E; F)$ (for example, the polynomial $P(x) = \sum_{n=1}^{\infty} x_n^2$ for all $x = (x_n) \in \ell_2$, is such that $P \in \mathcal{P}_K(^2\ell_2) - \mathcal{P}_w(^2\ell_2)$ and $T_P \in \mathcal{L}_K(\hat{\bigotimes}_{2,s,\pi} \ell_2) = \mathcal{L}_w(\hat{\bigotimes}_{2,s,\pi} \ell_2)$, see ((ARON; HERVÉS; VALDIVIA, 1983, p. 192)). Thus we have to proceed differently.

Theorem 3.1.3. *Let E and F be Banach spaces, and let G be a Banach space with an unconditional basis (g_n) and coordinate functionals (g_n'). If there exist operators $R \in \mathcal{L}(G; F)$ and $S \in \mathcal{L}(E; G)$ such that $(R(g_n))$ is a seminormalized basic sequence in F and $(S'(g_n'))$ is not relatively compact in E', then $\mathcal{P}_w(^nE; F)$ is not complemented in $\mathcal{P}(^nE; F)$ for every $n \in \mathbb{N}$.*

Proof. It follows from results of Aron and Prolla (ARON; PROLLA, 1980) and Aron, Hervés and Valdivia (ARON; HERVÉS; VALDIVIA, 1983) that $\mathcal{P}_w(^nE; F) \subset \mathcal{P}_K(^nE; F)$ for every $n \in \mathbb{N}$, and the fact that $\mathcal{P}_w(^1E; F) = \mathcal{P}_K(^1E; F)$ follows from (ARON; PROLLA, 1980, Proposition 2.5). Thus the case $n = 1$ follows from Theorem 3.1.1 (a). To prove the theorem by induction on n it suffices to prove that if $\mathcal{P}_w(^{n+1}E; F)$ is complemented in $\mathcal{P}(^{n+1}E; F)$, then $\mathcal{P}_w(^nE; F)$ is complemented in $\mathcal{P}(^nE; F)$. Aron and Schottenloher (ARON; SCHOTTENLOHER, 1976, Proposition 5.3) proved that $\mathcal{P}(^nE; F)$ is isomorphic to a complemented subspace of $\mathcal{P}(^{n+1}E; F)$ when F is the scalar field, but their proof works equally well when F is an arbitrary Banach space (see (BLASCO, 1997, Proposition 5)). Thus there exist operators $A \in \mathcal{L}(\mathcal{P}(^nE; F); \mathcal{P}(^{n+1}E; F))$ and $B \in \mathcal{L}(\mathcal{P}(^{n+1}E; F); \mathcal{P}(^nE; F))$ such that $B \circ A = I$. The operator A is of the form

$$A(P)(x) = \varphi_0(x)P(x)$$

for every $P \in \mathcal{P}(^nE;F)$ and $x \in E$, where $\varphi_0 \in E'$ verifies that $\|\varphi_0\| = 1 = \varphi_0(x_0)$, where $x_0 \in E$ and $\|x_0\| = 1$. It is clear that if $P \in \mathcal{P}_w(^nE;F)$, then $A(P) \in \mathcal{P}_w(^{n+1}E;F)$. On the other hand, the operator B is of the form $B = A^{-1} \circ D$, where $D : \mathcal{P}(^{n+1}E;F) \to \mathcal{P}(^{n+1}E;F)$ is defined by $D(P)(x) = P(x) - P(x - \varphi_0(x)x_0)$ for every $P \in \mathcal{P}(^{n+1}E;F)$ and $x \in E$. Is not difficult to prove that $B(\mathcal{P}_w(^{n+1}E;F)) \subset \mathcal{P}_w(^nE;F)$, (see (CA;, 2012, p. 597)). Let us assume that $\mathcal{P}_w(^{n+1}E;F)$ is complemented in $\mathcal{P}(^{n+1}E;F)$, and let $\pi : \mathcal{P}(^{n+1}E;F) \to \mathcal{P}_w(^{n+1}E;F)$ be a projection. Consider the operator

$$\rho = B \circ \pi \circ A : \mathcal{P}(^nE;F) \to \mathcal{P}_w(^nE;F).$$

If $P \in \mathcal{P}_w(^nE;F)$, then $A(P) \in \mathcal{P}_w(^{n+1}E;F)$, and therefore

$$\rho(P) = B \circ \pi \circ A(P) = B \circ A(P) = P.$$

Thus $\rho : \mathcal{P}(^nE;F) \to \mathcal{P}_w(^nE;F)$ is a projection, and therefore $\mathcal{P}_w(^nE;F)$ is complemented in $\mathcal{P}(^nE;F)$. This completes the proof.

□

Ghenciu (GHENCIU, 2005b) derived as corollaries of Theorem 3.1.1 results of several authors (EMMANUELE, 1991), (GHENCIU, 2005a), (FEDER, 1982), (KALTON, 1974) and (JOHN, 1992). We now apply Theorems 3.1.2 and 3.1.3 to obtain polynomial versions of those corollaries.

Corollary 3.1.4. *If F contains a copy of c_0 and E' contains a weak-star null sequence which is not weakly null, then $\mathcal{P}_{wK}(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.*

Corollary 3.1.5. *If F contains a copy of c_0 and E contains a complemented copy of c_0, then $\mathcal{P}_{wK}(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.*

Corollary 3.1.6. *If F contains a copy of ℓ_1 and $\mathcal{L}(E;\ell_1) \neq \mathcal{L}_K(E;\ell_1)$, then $\mathcal{P}_{wK}(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.*

When $n = 1$, Corollaries 3.1.4, 3.1.5 and 3.1.6 correspond to Corollaries 1.5.5, 1.5.6 and 1.5.8, respectively. Ghenciu derived those corollaries by observing that E and F satisfy the hypothesis of Theorem 3.1.1 (b). Since the hypothesis of Theorem 3.1.1 (b) coincides with the hypothesis of Theorem 3.1.2 (b), we see that Corollaries 3.1.4, 3.1.5 and 3.1.6 follow from Theorem 3.1.2 (b).

Corollary 3.1.7. *If F contains a copy of c_0 and E is infinite dimensional, then:*

(a) $\mathcal{P}_K(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.

(b) $\mathcal{P}_w(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$ for every $n \in \mathbb{N}$.

Corollary 3.1.8. *If E contains a complemented copy of ℓ_1 and F is infinite dimensional, then:*

(a) $\mathcal{P}_K({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$ for every $n \in \mathbb{N}$.

(b) $\mathcal{P}_w({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$ for every $n \in \mathbb{N}$.

When $n = 1$ Corollaries 3.1.7 and 3.1.8 correspond to Corollaries 1.5.7 and 1.5.9, respectively. Ghenciu derived those corollaries by observing that E and F satisfy the hypothesis of Theorem 3.1.1 (a). Since the hypothesis of Theorem 3.1.1 (a) coincide with the hypothesis of Theorems 3.1.2 (a) and 3.1.3, we see that Corollaries 3.1.7 and 3.1.8 follow from Theorems 3.1.2 (a) and 3.1.3.

Finally we present the following corollary.

Corollary 3.1.9. *If E contains a copy of ℓ_1 and F contains a copy of ℓ_p, with $2 \leqslant p < \infty$, then:*

(a) $\mathcal{P}_K({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$ for every $n \in \mathbb{N}$.

(b) $\mathcal{P}_w({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$ for every $n \in \mathbb{N}$.

Proof. We follow an argument of Emmanuele (EMMANUELE, 1992, p. 334). By a result of Pelczynski (PELCZYNSKI, 1968), if E contains a copy of ℓ_1, then E has a quotient isomorphic to ℓ_2 (see also the proof of (ARON; DIESTEL; RAJAPPA, 1985)). Let $S : E \to \ell_2$ be the quotient mapping, and let $R : \ell_2 \hookrightarrow \ell_p \subset F$ be the natural inclusion. Since $S' : \ell_2 \to E'$ is an embedding, the hypothesis of Theorems 3.1.2 (a) and 3.1.3 are clearly satisfied. □

Proposition 3.1.10. *Let E and F be infinite dimensional Banach spaces. If $\mathcal{P}_K({}^nE;F)$ contains a copy of c_0, then $\mathcal{P}_K({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$.*

Proof. By Theorem 1.3.5 we have that $P \in \mathcal{P}_K({}^nE;F)$ if and only if $T_P \in \mathcal{L}_K(\hat{\otimes}_{n,s,\pi}E;F)$. Thus the result follows from Theorem 1.5.10. □

The next proposition is a polynomial version of Theorem 1.5.10. The proof is based on ideas of (GHENCIU; LEWIS, 2011, Corollary 11).

Proposition 3.1.11. *Let E be an infinite dimensional Banach space and $n > 1$. If $\mathcal{P}_w({}^nE;F)$ contains a copy of c_0, then $\mathcal{P}_w({}^nE;F)$ is not complemented in $\mathcal{P}({}^nE;F)$.*

Proof. By Corollary 3.1.7 and Lemma 1.5.11 we may suppose without loss of generality that F contains no copy of c_0 and E contains no complemented copy of ℓ_1. By Theorem

1.2.16, $\mathcal{P}_w(^nE;F)$ contains no copy of ℓ_∞. Let (P_i) be a copy of the unit vector basis (e_i) of c_0 in $\mathcal{P}_w(^nE;F)$. Then

$$\sup\left\{\left\|\sum_{i\in F} e_i\right\|; F\subset\mathbb{N}, F finite\right\}=1.$$

By a result of Bessaga and Pelczynski (BESSAGA; PEŁCZYŃSKI, 1958) (see also Theorem 1.1.22) the series $\sum_{i=1}^{\infty} e_i$ is weakly unconditionally Cauchy in c_0. This implies that the series $\sum_{i=1}^{\infty} P_i$ is weakly unconditionally Cauchy in $\mathcal{P}_w(^nE;F)$. For every $\varphi\in F'$ and $x\in E$ we consider the continuous linear functional

$$\psi : P\in\mathcal{P}_w(^nE;F)\to\varphi(P(x))\in\mathbb{K}.$$

Since the series $\sum_{i=1}^{\infty} P_i$ is weakly unconditionally Cauchy in $\mathcal{P}_w(^nE;F)$,

$$\sum_{i=1}^{\infty}|\psi(P_i)|=\sum_{i=1}^{\infty}|\varphi(P_i(x))|<\infty$$

for every $\varphi\in F'$ and $x\in E$. This shows that $\sum_{i=1}^{\infty} P_i(x)$ is weakly unconditionally Cauchy in F for each $x\in E$. Finally, since F contains no copy of c_0, an application of Theorem 1.1.23 shows that $\sum_{i=1}^{\infty} P_i(x)$ converges unconditionally in F for each $x\in E$. Let $\mu:\wp(\mathbb{N})\to\mathcal{P}(^nE;F)$ be the finitely additive vector measure defined by $\mu(A)(x)=\sum_{i\in A} P_i(x)$ for each $x\in E$ and $A\subset\mathbb{N}$. Suppose there is a projection $\pi:\mathcal{P}(^nE;F)\to\mathcal{P}_w(^nE;F)$. Then $\pi(P_i)=P_i$ for each $i\in\mathbb{N}$. If the sequence $(\|P_i\|)$ does not converge to zero, then there is $\epsilon>0$ and a subsequence (i_k) of $\mathbb{N}$, such that $\|P_{i_k}\|>\epsilon$ for each $k\in\mathbb{N}$. But this implies that the measure $\pi\circ\mu:\wp(\mathbb{N})\to\mathcal{P}_w(^nE;F)$ is not strongly additive. Then the Diestel-Faires Theorem 1.1.28 would imply that $\mathcal{P}_w(^nE;F)$ contains a copy of ℓ_∞. Therefore $\|P_i\|\to 0$, but this is absurd too, because (P_i) is a copy of (e_i). This complete the proof. □

The following theorem is a polynomial version of Theorem 1.5.12.

Theorem 3.1.12. *Let E and F be Banach spaces and $P\in\mathcal{P}(^nE;F)$ such that $P\notin\mathcal{P}_w(^nE;F)$. Suppose that P admits a factorization $P=Q\circ T$ through a Banach space G with an unconditional finite dimensional expansion of the identity, where $T\in\mathcal{L}(E;G)$ and $Q\in\mathcal{P}(^nG;F)$. Then $\mathcal{P}_w(^nE;F)$ contains a copy of c_0 and thus $\mathcal{P}_w(^nE;F)$ is not complemented in $\mathcal{P}(^nE;F)$.*

Proof. The case $n=1$ follows from Theorem 1.5.12.

Case $n > 1$: Since G has an unconditional finite dimensional expansion of the identity, by Theorem 1.2.17 there is a sequence $(Q_i) \subset \mathcal{P}_w({}^nG; F)$ so that $Q(z) = \sum_{i=1}^{\infty} Q_i(z)$ unconditionally for each $z \in G$. Hence $P(x) = \sum_{i=1}^{\infty} Q_i(T(x))$ unconditionally for each $x \in E$. Since $Q_i \in \mathcal{P}_w({}^nG; F)$ for every $i \in \mathbb{N}$, it follows that $Q_i \circ T \in \mathcal{P}_w({}^nE; F)$ for every $i \in \mathbb{N}$. By the uniform boundedness principle, we have

$$\sup\left\{\left\| \sum_{i \in F} Q_i \circ T \right\|; F \subset \mathbb{N}, F finite\right\} < \infty.$$

Again by Theorem 1.1.22 the series $\sum_{i=1}^{\infty} Q_i \circ T$ is weakly unconditionally Cauchy in $\mathcal{P}_w({}^nE; F)$. Since $P \notin \mathcal{P}_w({}^nE; F)$, an application of Theorem 1.1.23 shows that $\mathcal{P}_w({}^nE; F)$ contains a copy of c_0, and therefore by Proposition 3.1.11 $\mathcal{P}_w({}^nE; F)$ is not complemented in $\mathcal{P}({}^nE; F)$. $\square$

Corollary 3.1.13. *Let E and F be Banach spaces, with E infinite dimensional, and let $n > 1$. If each $P \in \mathcal{P}({}^nE; F)$ such that $P \notin \mathcal{P}_w({}^nE; F)$ admits a factorization $P = Q \circ T$, where $T \in \mathcal{L}(E; G)$, $Q \in \mathcal{P}({}^nG; F)$ and G is a Banach space with an unconditional finite dimensional expansion of the identity, then the following conditions are equivalent:*

(1) $\mathcal{P}_w({}^nE; F)$ contains a copy of c_0.

(1′) $\mathcal{P}_K({}^nE; F)$ contains a copy of c_0.

(2) $\mathcal{P}_w({}^nE; F)$ is not complemented in $\mathcal{P}({}^nE; F)$.

(2′) $\mathcal{P}_K({}^nE; F)$ is not complemented in $\mathcal{P}({}^nE; F)$.

(3) $\mathcal{P}_w({}^nE; F) \neq \mathcal{P}({}^nE; F)$.

(3′) $\mathcal{P}_K({}^nE; F) \neq \mathcal{P}({}^nE; F)$.

(4) $\mathcal{P}({}^nE; F)$ contains a copy of c_0.

(5) $\mathcal{P}({}^nE; F)$ contains a copy of ℓ_∞.

Proof. (1) $\Rightarrow$ (2) by Proposition 3.1.11.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) by Theorem 3.1.12.

(1) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (3) suppose (4) holds and (3) does not hold. Then $\mathcal{P}_w({}^nE; F) = \mathcal{P}({}^nE; F) \supset c_0$. Thus (1) holds, and therefore (3) holds, a contradiction.

(5) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5) Since (4) $\Rightarrow$ (1′) $\mathcal{P}_K(^nE;F)$ contains a copy of c_0. By Theorem 1.3.5 $\mathcal{P}(^nE;F)$ and $\mathcal{P}_K(^nE;F)$ are isometrically isomorphic to $\mathcal{L}(\widehat{\otimes}_{n,s,\pi}E;F)$ and $\mathcal{L}_K(\widehat{\otimes}_{n,s,\pi}E;F)$, respectively. Thus $\mathcal{L}_K(\widehat{\otimes}_{n,s,\pi}E;F)$ contains a copy of c_0. Since E is infinite dimensional, $\widehat{\otimes}_{n,s,\pi}E$ is also infinite dimensional.Then by combining the proofs of Theorem 1.5.13 $(iii) \Rightarrow (ii)$ and Theorem 1.5.15 (2) $\Rightarrow$ (3) we can conclude that $\mathcal{L}(\widehat{\otimes}_{n,s,\pi}E;F)$ contains a copy of ℓ_∞ and the result follows.

Thus (1), (2), (3), (4) and (5) are equivalent.

(1) $\Rightarrow$ (1′) is obvious.

(1′) $\Rightarrow$ (2′) by Proposition 3.1.10.

(2′) $\Rightarrow$ (3′) is obvious.

(3′) $\Rightarrow$ (3) is obvious.

Since (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (1′), the proof of the corollary is complete. □

Remark 3.1.14. *In particular if E has an unconditional finite dimensional expansion of the identity we obtain Theorem 1.5.14. The assumptions of this corollary apply also if F is a complemented subspace of a space with an unconditional basis.*

4 On the reflexivity of $\mathcal{P}_w(^nE;F)$

An important result of Feder (FEDER et al., 1980) states that if E and F are reflexive Banach spaces such that F or E' is a subspace of a Banach space with an unconditional basis, then the space $\mathcal{L}_K(E;F)$ of all compact linear operators from E to F is either reflexive or non-isomorphic to a dual space. In (FEDER, 1975), Feder and Saphar proved that if E and F are reflexive Banach spaces and G is a closed linear subspace of $\mathcal{L}_K(E;F)$ which contains the space $\mathcal{F}(E,F)$ of all finite rank linear operators from E to F, then G is either reflexive or is not a dual space. But the following question posed in (FEDER et al., 1980) remains open:

Question. *Let E and F be reflexive Banach spaces. Is $\mathcal{L}_K(E;F)$ either reflexive or non-isomorphic to a dual space?.*

In this Chapter, we obtain a positive answer for the previous question. In fact, we prove the following more general result:

Theorem. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E;F)$. Then G is either reflexive or non-isomorphic to a dual space.*

We also prove that if E and F are reflexive Banach spaces, then the space $\mathcal{P}_w(^nE;F)$ of all n-homogeneous polynomials from E into F which are weakly continuous on bounded sets is either reflexive or non-isomorphic to a dual space. As other consequences of this result we also obtain two conditions, one that ensures that $\mathcal{P}_w(^nE;F)$ is non-isomorphic to a dual space and other such that $\mathcal{P}_w(^nE;E)$ is non-isomorphic to a dual space (see Corollaries 4.1.8 and 4.1.9). Finally, we prove that if E and F are reflexive Banach spaces, then the space $\mathcal{P}_A(^nE;F)$ is either reflexive or non-isomorphic to a dual space. Hence, we obtain a generalization of Boyd and Ryan Theorem 1.5.4.

4.1 The main result

To prove the main result, we need the following proposition, which is a special case of (Bu, 2013, Theorem 2.5).

Proposition 4.1.1. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E;F)$. Then G is reflexive if and only if it is weakly sequentially complete.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) By (Bu, 2013, Lemma 2.4) B_G is conditionally weakly compact and hence, relatively weakly compact if G is weakly sequentially complete. Therefore G is reflexive. $\square$

The following lemma is a special case of (KALTON, 1974, Corollary 3).

Lemma 4.1.2. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E;F)$. Let $T_m, T \in G$ for each $m \in \mathbb{N}$. Then $\lim_{m\to\infty} T_m = T$ weakly in G if and only if $\lim_{m\to\infty} y'(T_m(x)) = y'(T(x))$ for every $x \in E$ and every $y' \in F'$.*

Proof. ($\Rightarrow$) For every $y' \in F'$ and $x \in E$, consider the linear functional

$$\psi_{y',x} : T \in G \to y'(T(x)) \in \mathbb{K}.$$

Since $\psi_{y',x} \in G'$ and $\lim_{m\to\infty} T_m = T$ weakly in G, then it follows that $\lim_{m\to\infty} y'(T_m(x)) = y'(T(x))$ for every $x \in E$ and every $y' \in F'$.

($\Leftarrow$) Let $\varphi \in G'$. By the Hahn-Banach Theorem there is $\tilde{\varphi} \in \mathcal{L}_K(E;F)'$ such that $\tilde{\varphi}|_{G'} = \varphi$. On the other hand, by (KALTON, 1974, Corollary 3) $\lim_{m\to\infty} T_m = T$ weakly in $\mathcal{L}_K(E;F)$, that is, $\lim_{m\to\infty} \varphi(T_m) = \lim_{m\to\infty} \tilde{\varphi}(T_m) = \tilde{\varphi}(T) = \varphi(T)$. This complete the proof. $\square$

Theorem 4.1.3. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E;F)$. Then G is either reflexive or non-isomorphic to a dual space.*

Proof. Suppose that G is isomorphic to the conjugate of a Banach space X. Let $\varphi : X' \to G$ be an isomorphism. By using Proposition 4.1.1 we only need to prove that G is weakly sequentially complete. Let (T_m) be a weakly Cauchy sequence in G. For every $y' \in F'$ and $x \in E$, consider the linear functional

$$\psi_{y',x} : T \in G \to y'(T(x)) \in \mathbb{K}.$$

Since $\psi_{y',x} \in G'$ we have that

$$\lim_{m\to\infty} \psi_{y',x}(T_m) = \lim_{m\to\infty} y'(T_m(x))$$

exists for every $y' \in F'$ and $x \in E$. Let $(J_G(T_m))$ in G''. By the Banach-Alaoglu-Bourbaki Theorem, there exists a subnet $(J_G(T_\alpha))$ of $(J_G(T_m))$ and $\theta \in G''$ such that $\lim J_G(T_\alpha) = \theta$ in the $\sigma(G'', G')$-topology. In particular, $\lim \left\langle J_G(T_\alpha), \psi_{y',x} \right\rangle = \lim y'(T_\alpha(x)) = \theta(\psi_{y',x})$ for every $y' \in F'$ and $x \in E$. Since $\lim_{m\to\infty} y'(T_m(x))$ exists and $(y'(T_\alpha(x)))$ is a subnet of $(y'(T_m(x)))$ for every $y' \in F'$ and $x \in E$, it follows that

$$\lim_{m\to\infty} y'(T_m(x)) = \lim y'(T_\alpha(x)) = \theta(\psi_{y',x})$$

for every $y' \in F'$ and $x \in E$. Now, we want to prove that

$$\pi : \phi \in G'' \to J_G \circ \varphi \circ J'_X \circ (\varphi'')^{-1}(\phi) \in J_G(G)$$

is a projection. Note that

$$\left\langle J'_X \circ (\varphi'')^{-1}(J_G(T)), z\right\rangle = \left\langle (\varphi'')^{-1}(J_G(T)), J_X(z)\right\rangle = \left\langle (\varphi^{-1})''(J_G(T)), J_X(z)\right\rangle$$
$$= \left\langle J_G(T), (\varphi^{-1})'(J_X(z))\right\rangle = \left\langle (\varphi^{-1})'(J_X(z)), T\right\rangle = \left\langle J_X(z), \varphi^{-1}(T)\right\rangle = \left\langle \varphi^{-1}(T), z\right\rangle$$

for each $T \in G$ and $z \in X$. This implies that

$$J'_X \circ (\varphi'')^{-1}(J_G(T)) = \varphi^{-1}(T)$$

and then $\pi \circ J_G = J_G$. Thus π is a projection and so

$$G'' = J_G(G) \oplus \ker(\pi).$$

Let $T \in G$ and $\eta \in \ker(\pi)$ be such that $\theta = J_G(T) + \eta$. Since $\eta \in \ker(\pi)$ and $J_G \circ \varphi$ is injective, we have $J'_X \circ (\varphi'')^{-1}(\eta) = 0$. On the other hand,

$$\eta(\psi_{y',x}) = \left\langle (\varphi'')^{-1}(\eta), \varphi'(\psi_{y',x})\right\rangle = \left\langle J''_X(\varphi'(\psi_{y',x}), (\varphi'')^{-1}(\eta)\right\rangle = \left\langle \varphi'(\psi_{y',x}), J'_X \circ (\varphi'')^{-1}(\eta)\right\rangle = 0.$$

Hence

$$\lim_{m\to\infty} y'(T_m(x)) = \theta(\psi_{y',x}) = \left\langle J_G(T), \psi_{y',x}\right\rangle + \left\langle \eta, \psi_{y',x}\right\rangle = y'(T(x)),$$

for every $y' \in F'$ and $x \in E$. By Lemma 4.1.2 it follows that $\lim_{m\to\infty} T_m = T$ weakly in G. This completes the proof. □

The previous result gives an affirmative answer of (FEDER et al., 1980, Problem 1) and consequently is a generalization of (FEDER, 1975, Theorem 2) and (FEDER et al., 1980, Theorem 5).

The proof of the next theorem is similar to the proof of Theorem 4.1.3, but using (BU; JI; WONG, 2015, Lemma 4.1) and (BU; JI; WONG, 2015, Theorem 4.2) instead of Lemma 4.1.2 and Proposition 4.1.1, respectively.

Theorem 4.1.4. *Let E and F be reflexive Banach spaces, then $\mathcal{P}_w(^n E; F)$ is either reflexive or non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Remark 4.1.5. *Note that Theorem 4.1.4 does not work for $\mathcal{P}_K(^n E; F)$ instead of $\mathcal{P}_w(^n E; F)$. In fact, $\mathcal{P}_K(^2\ell_2) = \mathcal{P}(^2\ell_2) = \mathcal{L}(\ell_2; \ell_2) = (\hat{\otimes}_{2,s,\pi}\ell_2)'$ is a dual space that is not reflexive.*

Corollary 4.1.6. *Let E and F be reflexive Banach spaces. If $\mathcal{P}(^n E; F)$ is isomorphic to $\mathcal{P}_w(^n E; F)$, then $\mathcal{P}(^n E; F)$ is reflexive.*

Proof. Since $\mathcal{P}(^n E; F)$ is a dual space, then the conclusion follows from Theorem 4.1.4. □

The next proposition is a particular case of (ALENCAR; ARON; DINEEN, 1984, Proposition 5.3).

Proposition 4.1.7. *Let E and F be Banach spaces. Then $\mathcal{P}_w({}^nE;F)$ is isomorphic to a closed subspace of $\mathcal{P}_w({}^mE;F)$ for every $m \geqslant n$.*

Proof. To prove the proposition by induction on n it suffices to prove that $\mathcal{P}_w({}^nE;F)$ is isomorphic to a closed subspace of $\mathcal{P}({}^{n+1}E;F)$. Choose $\varphi \in E'$ such that $\varphi \neq 0$. Define $\rho : \mathcal{P}_w({}^nE;F) \to \mathcal{P}_w({}^{n+1}E;F)$ by $\rho(Q)(x) = \varphi(x)Q(x)$ for all $x \in E$. It is clear that ρ is an injective linear operator. Therefore $\mathcal{P}_w({}^nE;F)$ is isomorphic to $\rho(\mathcal{P}_w({}^nE;F)) \subset \mathcal{P}_w({}^{n+1}E;F)$. This completes the proof. □

Corollary 4.1.8. *Let E and F be reflexive Banach spaces such that E has the CAP. If $\mathcal{P}_w({}^nE;F) \neq \mathcal{P}({}^nE;F)$, then $\mathcal{P}_w({}^mE;F)$ is not isomorphic to a dual space for every $m \geqslant n$.*

Proof. By Theorem 4.1.4 we only need to prove that $\mathcal{P}_w({}^mE;F)$ is not reflexive for every $m \geqslant n$. By Proposition 4.1.7 we have that $\mathcal{P}_w({}^nE;F)$ is isomorphic to a closed subspace of $\mathcal{P}_w({}^mE;F)$ for every $m \geqslant n$. If we assume that $\mathcal{P}_w({}^mE;F)$ is reflexive for some $m \geqslant n$, then $\mathcal{P}_w({}^nE;F)$ is also reflexive. Since E has the CAP, then by (BU; JI; WONG, 2015, Corollary 4.4) we have that $\mathcal{P}_w({}^nE;F) = \mathcal{P}({}^nE;F)$, but this contradicts the hypothesis. □

Corollary 4.1.9. *Let E be a reflexive infinite dimensional Banach space with the CAP. Then $\mathcal{P}_w({}^nE;E)$ is non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Proof. By the Riesz Theorem $\mathcal{L}_K(E;E) \neq \mathcal{L}(E;E)$. Now the result follows from Corollary 4.1.8. □

Definition 4.1.10. *For every $P \in \mathcal{P}({}^nE;F)$, consider $A_P \in \mathcal{L}({}^nE;F)$ such that $P(x) = \hat{A}_P(x)$ for each $x \in E$. We define $\hat{d}^{n-1}P : E \to \mathcal{P}({}^{n-1}E;F)$, see (DINEEN, 2012, p.13), by*

$$\hat{d}^{n-1}P(x)(y) = A_P(x, y, \ldots, y),$$

for every x, y in E.

Proposition 4.1.11. *Let E and F be Banach spaces. If $P \in \mathcal{P}_A({}^nE;F)$ then $\hat{d}^{n-1}P(x) \in \mathcal{P}_A({}^{n-1}E;F)$ for every $x \in E$.*

Proof. Firstly, we want to prove that if $P \in \mathcal{P}_f({}^nE;F)$ then $\hat{d}^{n-1}P(x) \in \mathcal{P}_f({}^{n-1}E;F)$ for every $x \in E$. For each $P \in \mathcal{P}_f({}^nE;F)$ there exists $\phi_j \in E'$ and $b_j \in F$ with $1 \leqslant j \leqslant m$, such that $P(x) = \sum_{j=1}^{m} \phi_j^n(x)b_j$ for each $x \in E$. Thus $\hat{d}^{n-1}P(x)(y) = \sum_{j=1}^{m} \phi_j^{n-1}(y)\phi_j(x)b_j$ for each $y \in E$. Therefore $\hat{d}^{n-1}P(x) \in \mathcal{P}_f({}^{n-1}E;F)$ for every $x \in E$. Now, if $P \in \mathcal{P}_A({}^nE;F)$

then there exists a sequence (P_k) in $\mathcal{P}_f(^nE;F)$, such that $\lim_{k\to\infty}\|P_k - P\| = 0$. If $\|x\| = 0$ then $\hat{d}^{n-1}P(x)$ is a null polynomial, for this reason we only need taking $\|x\| > 0$. Now,

$$\lim_{k\to\infty}\|\hat{d}^{n-1}P_k(x) - \hat{d}^{n-1}P(x)\| = \|x\|\lim_{k\to\infty}\sup_{\|y\|=1}\left\|A_{P_k}\left(\frac{x}{\|x\|}, y, \ldots, y\right) - A_P\left(\frac{x}{\|x\|}, y, \ldots, y\right)\right\|$$
$$\leqslant \|x\|\lim_{k\to\infty}\|A_{P_k} - A_P\| \leqslant \|x\|\frac{n^n}{n!}\lim_{k\to\infty}\|P_k - P\| = 0.$$

Hence $\hat{d}^{n-1}P(x) \in \mathcal{P}_A(^{n-1}E;F)$ for every $x \in E$. □

The proof of the next theorem is based on ideas of (BU; JI; WONG, 2015, Theorem 4.2).

Lemma 4.1.12. *Let E and F be reflexive Banach spaces. Then $\mathcal{P}_A(^nE;F)$ is reflexive if and only if it is weakly sequentially complete.*

Proof. $(\Rightarrow)$ Immediate.

$(\Leftarrow)$ It follows from Proposition 4.1.1 that the theorem holds for $n = 1$. Using induction, we assume that the theorem holds for $n-1$ and we will show that the theorem holds for n, where $n \geqslant 2$.

To do this, we suppose that $\mathcal{P}_A(^nE;F)$ is weakly sequentially complete. We want to show that $\mathcal{P}_A(^nE;F)$ is reflexive. It follows from Proposition 4.1.7 that $\mathcal{P}_A(^{n-1}E;F)$ is isomorphic to a closed subspace of $\mathcal{P}_A(^nE;F)$. Thus $\mathcal{P}_A(^{n-1}E;F)$ is also weakly sequentially complete. By the induction hypothesis, $\mathcal{P}_A(^{n-1}E;F)$ is reflexive. To show that $\mathcal{P}_A(^nE;F)$ is reflexive, we only need to show that every bounded sequence in $\mathcal{P}_A(^nE;F)$ has a weakly Cauchy subsequence. Take any bounded sequence (P_k) in $\mathcal{P}_A(^nE;F)$. By (DINEEN, 2012, p.88, Proposition 2.6) $\hat{d}^{n-1}P_k \in \mathcal{L}_K(E;\mathcal{P}(^{n-1}E;F))$. Since $P_k \in \mathcal{P}_A(^nE;F)$, it follows from Proposition 4.1.11 that $\hat{d}^{n-1}P_k(x) \in \mathcal{P}_A(^{n-1}E;F)$ for every $x \in E$, and hence, $\hat{d}^{n-1}P_k \in \mathcal{L}_K(E;\mathcal{P}_A(^{n-1}E;F))$. Note that E and $\mathcal{P}_A(^{n-1}E;F)$ are reflexive and note that $(\hat{d}^{n-1}P_k)$ is a bounded sequence in $\mathcal{L}_K(E;\mathcal{P}_A(^{n-1}E;F))$, It follows from (Bu, 2013, Lemma 2.4) that $(\hat{d}^{n-1}P_k)$ has a weakly Cauchy subsequence, without loss of generality, say $(\hat{d}^{n-1}P_k)$.

For every $x \in E$ and every $y' \in F'$, define a linear functional $\phi_{x,y'}$ in $\mathcal{P}_A(^{n-1}E;F)$ by $\phi_{x,y'}(P) = \left\langle P(x), y'\right\rangle$ for every $P \in \mathcal{P}_A(^{n-1}E;F)$. Then $\phi_{x,y'} \in \mathcal{P}_A(^{n-1}E;F)'$. Since $(\hat{d}^{n-1}P_k)$ is a weakly Cauchy sequence in $\mathcal{L}_K\big(E;\mathcal{P}_A(^{n-1}E;F)\big)$, it follows that $\left(\left\langle\hat{d}^{n-1}P_k(x), \phi_{x,y'}\right\rangle\right)$ is a Cauchy sequence. Note that $\left\langle\hat{d}^{n-1}P_k(x), \phi_{x,y'}\right\rangle = \left\langle P_k(x), y'\right\rangle$. Thus $\left(\left\langle P_k(x), y'\right\rangle\right)$ is a Cauchy sequence. By (BU; JI; WONG, 2015, Lemma 4.1), (P_k) is a weakly Cauchy sequence in $\mathcal{P}_A(^nE;F)$. This complete the proof. □

The proof of the following theorem is almost identical to the proof of Theorem 4.1.3, but using (BU; JI; WONG, 2015, Lemma 4.1) and Lemma 4.1.12 instead of Lemma 4.1.2 and Proposition 4.1.1, respectively.

Theorem 4.1.13. *Let E and F be reflexive Banach spaces. Then $\mathcal{P}_A(^nE; F)$ is either reflexive or non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

The next result is a generalization of Theorem 1.5.4.

Corollary 4.1.14. *Let E be a reflexive Banach space. Then $\mathcal{P}_A(^nE)$ is either reflexive or non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Proof. Take $F = \mathbb{K}$ in Theorem 4.1.13. $\square$

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