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Counting numerical semigroups by genus and even gaps and some generalizations. Patterns on numerical semigroups

Contagem de semigrupos numéricos pelo gênero e lacunas pares e generalizações. *Patterns* em semigrupos numéricos

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Supervisor: Fernando Eduardo Torres Orihuela

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“The task is not so much
to see what no one yet has seen,
but to think what no body yet has thought
about that which everyone sees."
(Arthur Schopenhauer)

“A tarefa não é tanto
ver aquilo que ninguém viu,
mas pensar o que ninguém ainda pensou
sobre aquilo que todo mundo vê."
(Arthur Schopenhauer)

Resumo

Neste trabalho, apresentamos uma abordagem para o problema de contagem de semigrupos numéricos pelo gênero, usando o fato de que cada semigrupo numérico de gênero g possui uma quantidade de lacunas pares γ e o número n_g dos semigrupos de gênero g pode ser calculado como a soma dos números $N_\gamma(g)$, que denota a quantidade de semigrupos numéricos de gênero g e γ lacunas pares. Um dos principais resultados do trabalho é o fato de $N_\gamma(g)$ é constante para γ fixado e $g \geqslant 3\gamma$. De forma natural estudamos o comportamento da sequência $N_\gamma(3\gamma)$.

Motivados pela similaridade entre as sequências de Fibonacci e (n_g), estudamos o comportamento assintótico de sequências envolvendo os números n_g. Usando as ideias do Capítulo 2 deste trabalho, estudamos uma generalização natural de semigrupo γ-hiperelíptico. Ao final do trabalho, introduzimos o conceito de *patterns* e tentamos entender como eles podem ser aplicados a problemas envolvendo semigrupos numéricos.

Palavras-chave: semigrupos numéricos. semigrupos γ-hiperelípticos. gênero. *patterns*.

Abstract

In this work, we present an approach to the problem of counting numerical semigroups by genus, using the fact that each numerical semigroup with genus g has a number of even gaps γ and the number n_g, that denotes the number of numerical semigroups of genus g, can be computed as a sum of the numbers $N_\gamma(g)$, which denotes the number of numerical semigroups with genus g and γ even gaps. One of the results of this work is the fact that $N_\gamma(g)$ is constant for a fixed γ and $g \geqslant 3\gamma$. Naturally, we study the behaviour of the sequence $N_\gamma(3\gamma)$.

Motivated by similarity between the Fibonacci and (n_g) sequences, we study the asymptotic behaviour of sequences involving the numbers n_g. By using some ideas of Chapter 2 of this work, we study a natural generalization of γ-hyperelliptic semigroup. At the end, we introduce the concept of patterns and we try to understand how they can be applied to the problems involving numerical semigroups.

Keywords: numerical semigroups. γ-hyperelliptic semigroups. genus. patterns

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H_g	hyperelliptic semigroup with genus g
$\lfloor x \rfloor$	integer part of the real number x
n_g	number of numerical semigroups with genus g
$N_\gamma(g)$	number of numerical semigroups with genus g and γ even gaps
S	numerical semigroup
S_{g+1}	ordinary semigroup with genus g
$\mathbb{N}_0$	set of non-negative integers
$\mathcal{S}_g$	set of numerical semigroups with genus g
$\mathcal{S}_\gamma(g)$	set of numerical semigroups with genus g and γ even gaps

Contents

Introduction

A numerical semigroup S is a cofinite submonoid of $\mathbb{N}_0$. The elements of $G(S) := \mathbb{N}_0 \backslash S$ are the *gaps* of S and the cardinality of $G(S)$ is the *genus* of S, denoted by $g(S)$. The Frobenius number of S, denoted by $F(S)$, is the largest element of $G(S)$, while the multiplicity of S, denoted by $m(S)$, is the first positive integer in S.

Given a non-negative integer g, what is the number n_g of numerical semigroups with genus g? One can prove that n_g is always finite (see Corollary 1.1), but it seems that determining its exactly value is a difficult task. For instance, $n_0 = 1$, since $\mathbb{N}_0$ is the only numerical semigroup with genus 0, $n_1 = 1$, since $\mathbb{N}_0 \backslash \{1\}$ is the only numerical semigroup with genus 1 and $n_2 = 2$, since $\mathbb{N}_0 \backslash \{1, 2\}$ and $\mathbb{N}_0 \backslash \{1, 3\}$ are the numerical semigroups with genus 2.

In 2007, Bras-Amorós and de Mier (7) used representation of a numerical semigroup by Dyck paths to prove that $n_g \leqslant C_g = \dfrac{1}{g+1}\dbinom{2g}{g}$, for all g, where C_g is the Catalan number of order g.

A natural approach to computing n_g is the so called *semigroup tree*; see (4), (6). In 2008, Bras-Amorós (3) calculated the 50 first elements of that sequence being some of them $1, 1, 2, 4, 7, 12, 23, 39, \ldots$. By using these computations, she conjectured the following statements:

(I) $\displaystyle\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi := \frac{1+\sqrt{5}}{2}$ (golden ratio);

(II) $\displaystyle\lim_{g\to\infty} \frac{n_g + n_{g+1}}{n_{g+2}} = 1.$

(III) $n_g + n_{g+1} \leqslant n_{g+2}$, for all g positive integer;

Notice that (I) above implies (II), since $\dfrac{n_{g+1}+n_g}{n_{g+2}} = \dfrac{n_{g+1}}{n_{g+2}} + \dfrac{n_g}{n_{g+1}} \cdot \dfrac{n_{g+1}}{n_{g+2}}$ and $\dfrac{1}{\varphi} + \dfrac{1}{\varphi^2} = 1$ (φ is a root of $x^2 - x - 1 = 0$).

In 2009, Bras-Amorós (4) used some techniques in multisets and the semigroup tree to prove that $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$, for all $g \geqslant 3$, where (F_n) is the Fibonacci sequence. Here, the lower and upper bounds are interesting, since they are well known.

In 2010, Elizalde (11) used generating functions and improved the former bounds. He proved that there are a_g and c_g coefficients of explicit generating functions such that $a_g \leqslant n_g \leqslant c_g$, for all $g \geqslant 1$.

Also in 2010, Zhao (30) worked on this problem and realized that most of numerical semigroups with a fixed genus satisfy the following property: the Frobenius number of the numerical semigroup is less than three times its multiplicity. He defined t_g as the number of numerical semigroups S with genus g such that $F(S) < 3 \cdot m(S)$ and he conjectured that $\lim_{g\to\infty} \frac{t_g}{n_g} = 1$. In that paper, he also proved the interesting fact that the number of numerical semigroups S with genus g such that $F(S) < 2 \cdot m(S)$ is exactly F_{g+1}, the Fibonacci number of order $g + 1$.

Finally, in 2013, Zhai (29) proved that Bras-Amorós was right about the asymptotic behaviour of (n_g) sequence, confirming Zhao's conjecture. In fact, he proved the following:

(A) If t_g is the number of numerical semigroups S with genus g that satisfy $F(S) < 3m(S)$, then $\sup_{g\in\mathbb{N}} \frac{t_g}{\varphi^g} < \infty$. In particular, $\lim_{g\to\infty} \frac{t_g}{\varphi^g} = \mu$ is a real number.

(B) $\lim_{g\to\infty} \frac{t_g}{n_g} = 1$. In particular, $\lim_{g\to\infty} \frac{n_g}{\varphi^g} = \mu$.

More precisely,

Zhai's Theorem. *Let (n_g) be the sequence of number of numerical semigroups with genus g. Then*

$$\lim_{g\to\infty} \frac{n_g}{\varphi^g} = \mu,$$

where μ is a constant greater then 3.78.

Since $\frac{n_{g+1}}{n_g} = \frac{n_{g+1}}{\varphi^{g+1}} \cdot \frac{\varphi^g}{n_g} \cdot \varphi$, one can conclude that $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi$ and $\lim_{g\to\infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1$, as we pointed out before. Although, checking if $n_g + n_{g+1} \leqslant n_{g+2}$, for all g remains as an open problem and it seems to be a very hard problem according to its asymptotic behaviour. In fact, even the weaker version of this problem remains open. We present it as

Question 1.

Is it true that $n_g < n_{g+1}$ *for* $g > 0$?

We do observe that (I) above implies the existance of a positive constant C_0 such that $n_g < n_{g+1}$ holds for $g \geqslant C_0$; however the true value of C_0 seems to be out of reach. Even if we knew C_0, it would still probably be difficult to answer Question 1. We can justify this statement as follows: up to 2015, $n_{67} = 377\ 866\ 907\ 506\ 273$ is the greatest n_g known so far; see (12).

In 2012, some authors worked with other invariant of numerical semigroups to count them by genus. For instance, Blanco and Rosales (2) considered the following partition of $\mathcal{S}_g$: $\bigcup_{F=g}^{2g-1} \mathcal{S}(F,g)$, where $\mathcal{S}(F,g)$ denotes the set of numerical semigroups with genus g and Frobenius number F.

Kaplan (14) counted numerical semigroups by genus and multiplicity. It is possible to rewrite

$$n_g = \sum_{m=0}^{g+1} N(m,g),$$

where $N(m,g)$ denotes the number of numerical semigroups with multiplicity m and genus g. He proved that if $2g < 3m$, then $N(m,g) = N(m-1,g-1) + N(m-1,g-2)$ by using some combinatorial methods; see also (15).

Bras-Amorós (5) counted numerical semigroups by genus and ordinarization number (see Definition 1.1). It is possible to rewrite

$$n_g = \sum_{r=0}^{\lfloor \frac{g}{2} \rfloor} n_{g,r},$$

where $n_{g,r}$ denotes the number of numerical semigroups with ordinarization number r and genus g. She proved that if $r > \max\left\{\frac{g}{3}+1, \left\lfloor \frac{g+1}{2} \right\rfloor - 14\right\}$, then $n_{g,r} \leqslant n_{g+1,r}$. She also conjectured that if $r > \frac{g}{3}$, then $n_{g,r} \leqslant n_{g+1,r}$.

In 1997, Torres (27) was interested in some aspects of algebraic curves. He worked with Weierstrass semigroups at ramified points of double covering of curves of large genus. The link between this work and computing n_g is the so called γ-hyperelliptic semigroups which simply are numerical semigroups having γ even gaps. In fact, we compute n_g as follows. Let $N_\gamma(g)$ be the number of γ-hyperelliptic semigroups with genus g. Then (see Lemma 2.1)

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g).$$

In **Chapter 1**, we present some well-known results about numerical semigroups, which are fundamental for this work.

In **Chapter 2**, we present γ-hyperelliptic semigroups so we count numerical semigroups by genus and even gaps. It gives an approach to deal with Question 1. In particular, we obtain some interesting results about the sequence $N_\gamma(g)$ such as:

Theorem 2.1. *Let $g \geqslant 3\gamma$. Then $N_\gamma(g) = N_\gamma(3\gamma)$.*

Theorem 2.2. *Let $g < 3\gamma$. Then $N_\gamma(g) < N_\gamma(3\gamma)$.*

In **Chapter 3**, motivated by Theorem 2.1 we study the sequence $(N_\gamma(3\gamma))$. Surprisely enough, we prove that it coincides with the sequence (f_γ) introduced by Bras-Amorós in (5) in connection with ordinarization transform (see Definition 1.1) of a numerical semigroup. We have

Theorem 3.1. *Let γ be a non-negative integer. Then $f_\gamma = N_\gamma(3\gamma)$.*

Next we compute some particular values of $N_\gamma(3\gamma)$ in order to understand the asymptotic behaviour of the sequence. We prove

Theorem 3.2. *Let $\epsilon > 0$. Then*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0 \quad \textit{and} \quad \lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

This result shows that there is a function A such that $2^\gamma < A(\gamma) \leqslant (2\varphi)^\gamma$ and $f_\gamma \sim A(\gamma)$, for large enough γ. By making some calculations, it seems that $A(\gamma) = n_{2\gamma}$. More precisely, we conjecture that there is a positive constant C such that $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{n_{2\gamma}} = C$. We prove the following

Theorem 3.3. *If there is a positive constant C such that $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{n_{2\gamma}} = C$, then $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{f_{\gamma-1}} = \varphi^2$.*

By using some other techniques, we also prove the following conditional result

Theorem 3.4. *If $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{f_{\gamma-1}} = \varphi^2$, then $\lim_{\gamma\to\infty} \dfrac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i} = \varphi$.*

In **Chapter 4**, we present some other Fibonacci-like behaviour of the sequence (n_g). The main technique for those results are based on the proof of Theorem 3.4; in fact, the following limits hold true (their inspiration comes from some asymptotic properties of Fibonacci numbers - see Corollary 4.1):

Theorem 4.1. *Let (n_g) be the sequence of number of numerical semigroups with genus g. Then*

(1) $\lim_{g\to\infty} \dfrac{n_{g+1}}{n_g} = \varphi$ *and* $\lim_{g\to\infty} \dfrac{n_{g-1}+n_g}{n_{g+1}} = 1$.

(2) $\lim_{g\to\infty} \dfrac{n_g^2}{n_{g+1}n_{g-1}} = 1$.

(3) $\lim_{g\to\infty} \dfrac{\sum_{i=0}^{g} n_i}{n_{g+1}} = \varphi$. *and* $\lim_{g\to\infty} \dfrac{\sum_{i=0}^{g-1} n_i}{n_{g+1}} = 1$.

(4) $\lim_{g\to\infty} \dfrac{\sum_{i=0}^{g} n_i^2}{n_g^2} = \varphi$ *and* $\lim_{g\to\infty} \dfrac{\sum_{i=0}^{g} n_i^2}{n_{g+1}^2} = \dfrac{1}{\varphi}$.

(5) $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g}} = \varphi$ *and* $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g+1}} = 1.$

We also prove an asymptotic property of the sequence (n_g) which is different from Fibonacci sequence. We recall that $\frac{F_g^2 + F_{g+1}^2}{F_{2g+1}} = 1$, for all g. We prove the following

Theorem 4.2. *Let (n_g) be the sequence of number of numerical semigroups with genus g. Then*

$$\lim_{g\to\infty} \frac{n_g^2 + n_{g+1}^2}{n_{2g+1}} = \sqrt{5}\mu > 8.$$

In **Chapter 5**, we generalize γ-hyperelliptic semigroups by considering (M, γ)-semigroups and proving some analogous results. We also realize that it also generalizes numerical semigroups containing a fixed integer. In particular, we prove

Theorem 5.1. *Let S be a (M, γ)-hyperelliptic semigroup with genus g. Then*

$$2g \geqslant (M+1)\gamma.$$

We also prove that, in general, this bound is sharp.

Finally, in **Chapter 6**, we present *patterns on numerical semigroups*. In particular, we study linear patterns with length two; we prove some results in order to try to characterize the ones which are admitted by a fixed numerical semigroup. Summarizing:

Theorem 6.1. *Let S be a numerical semigroup and let $p(X,Y) = aX - bY$ be a pattern which is not admitted by S. Then either*

- *$a = b = h$, with $h \in G(S)$.*
- *$\frac{c(S)}{m(S)} > a - b \geqslant 1$, with a and $b \in G(S) \backslash PF(S)$.*

1 Basic results on numerical semigroups

A numerical semigroup S is a cofinite submonoid of $\mathbb{N}_0$, i.e., S is subset of $\mathbb{N}_0$ such that $0 \in S$, S is closed under addition and the complement of S in $\mathbb{N}_0$ is finite. For a numerical semigroup S, there are some invariants associated to it. We present them below as following:

- $G(S) := \mathbb{N}_0 \backslash S$ is the set of *gaps* of S,
- $g(S) := \#G(S)$ is the *genus* of S,
- $m(S) := \min\{s \in S : s \neq 0\}$ is the *multiplicity* of S,
- $c(S) := \min\{s \in S : s + n \in S, \forall n \in \mathbb{N}_0\}$ is the *conductor* of S,
- $F(S) := c(S) - 1$ is the *Frobenius number* of S.

We also consider the set of gaps of a numerical semigroup with genus g as $\{\ell_1 < \cdots < \ell_g\}$. To complete our first definitions: $\mathcal{S}_g := \{S : g(S) = g\}$ and $n_g := \#\mathcal{S}_g$ are the set of numerical semigroups with genus g and the number of numerical semigroups with genus g, respectively.

Throughout this chapter, we will recall some properties of numerical semigroups. A good reference for this topic is (21) and for sake of completeness, we state proofs of the results. We start with a well-known result:

Proposition 1.1. *Let S be a numerical semigroup with genus g. Then $2g + \mathbb{N}_0 \subseteq S$.*

Proof. If $g = 0$, then $S = \mathbb{N}_0$ and the lemma follows. If $g \geqslant 1$, then there are, at least, g non-gaps in $[1, 2g]$ (otherwise, S would have more than g gaps). Let $\rho_1 < \cdots < \rho_g$ be those non-gaps and suppose that there is some gap $\ell \geqslant 2g$. Then all the numbers $\ell - \rho_i$ are gaps (otherwise, $S \ni (\ell - \rho_i) + \rho_i = \ell \notin S$). Thus, S would have, at least, $g + 1$ gaps, which is a contradiction. Hence, all gaps of S are lower than $2g$ and the result follows. □

Next result proves that each number n_g is finite.

Corollary 1.1. *If $g \geqslant 1$, then*

$$n_g \leqslant \binom{2g-2}{g-1}.$$

Proof. Let S be a numerical semigroup with genus $g \geqslant 1$. We know that 1 is a gap of S and, by Proposition 1.1, all gaps of S are less than $2g$. Hence there are exactly $2g-2$ possibilities for other gaps of S so we choose $g-1$ as gaps and the result follows. □

We recall that the integer part of a real number x, denoted by $\lfloor x \rfloor$, is the greatest integer which is lower than or equal to x. From this definition, if x is a real number, then

$$x - 1 < \lfloor x \rfloor \leqslant x.$$

Another consequence of Proposition 1.1 is about the generators of a numerical semigroup.

Corollary 1.2. *Every numerical semigroup is finitely generated.*

Proof. Let S be a numerical semigroup with genus g and write $S \cap [0, 2g-1] = \{\alpha_1, \dots, \alpha_g\}$. We claim that $\Gamma := \{\alpha_1, \dots, \alpha_g, 2g, 2g+1, \dots, 4g-1\}$ generates S. In fact, let $s \in S$ with $s \geqslant 4g$. Using Euclides algorithm, we rewrite $s = (k-1) \cdot 2g + \ell$, where $k = \left\lfloor \frac{s}{2g} \right\rfloor \geqslant 2$ and $2g \leqslant \ell \leqslant 4g-1$. Thus, s is a linear combination of two elements of Γ and we conclude that the finite set Γ generates S. □

If S is generated by $\{a_1, \dots, a_n\}$, then we write $S = \langle a_1, \dots, a_n \rangle$. On the other hand, given positive integers $a_1, \dots, a_n$, the set $\langle a_1, \dots, a_n \rangle$ is a numerical semigroup if and only if $\gcd(a_1, \dots, a_n) = 1$ to ensure condition on the complement of S in $\mathbb{N}_0$. If there is no subset of $\{a_1, \dots, a_n\}$ that generates S, we say that it is the *minimal set* of generators. One can prove that it is unique. The number of elements of the minimal system of generators of a numerical semigroup S is called *embedding dimension* of S and it is denoted by $e(S)$. Notice that the procedure we did in last corollary implies that a numerical semigroup with genus g satisfies $e \leqslant 3g$.

There are some relations between the genus of a numerical semugroup and other invariants of it. Now, we do a brief survey:

Proposition 1.2. *Let S be a numerical semigroup with genus g and conductor c. Then $g+1 \leqslant c \leqslant 2g$.*

Proof. Proposition 1.1 ensures the upper bound, since $c \leqslant 2g$. For computing the lower bound, suppose that there is a numerical semigroup with conductor c and genus g satisfying $c \leqslant g$. Then the number of gaps of it would be, at most, $g-1$ and this is a contradiction. Hence $c \geqslant g+1$. □

Remark 1.1. *In general, those bounds are sharp. In fact, $S_{g+1} = \{0, g+1, \rightarrow\}$ has conductor $g+1$ and genus g and $H_g = \{0, 2, 4, \dots, 2g, \rightarrow\}$ has conductor $2g$ and genus g. The numerical semigroups S_{g+1} and H_g are called ordinary semigroup with genus g and hyperelliptic semigroup with genus g, respectively.*

Remark 1.2. *Numerical semigroups with genus g and concuctor c that satisfy $c = 2g$ are called symmetric semigroups. The ones that satisfy $c = 2g - 1$ are called pseudo-symmetric semigroups. Notice that all hyperelliptic semigroups are symmetric semigroups, but the converse is false. For instance, $\{0, 3, 4, 6, \rightarrow\}$ has genus 3 and conductor 6, thus it is symmetric.*

Observe that the hyperelliptic semigroup with genus g can be written as $\langle 2, 2g+1 \rangle$ and $\{0, 3, 4, 6, \rightarrow\}$ can be written as $\langle 3, 4 \rangle$; both of them have two generators. A well known result by Sylvester (24) is about genus and conductor of two-generated semigroups: let a and b be coprime integers. If $S = \langle a, b \rangle$, then $c(S) = (a-1)(b-1)$ and $g(S) = (a-1)(b-1)/2$. This proves that two-generated numerical semigroups are always symmetric. On the other hand, $\langle 4, 6, 7 \rangle$ is not a two-generated numerical semigroup, but it is symmetric ($g = 5$ and $c = 10$).

There is a construction by Kunz and Waldi (16) that shows us a way to find all symmetric semigroups containing an integer $p \geqslant 3$. In fact, one can prove that, looking at numerical semigroups containing an integer p as integer points of a polyhedral at $\mathbb{R}^p$ (under a bijective map), symmetric semigroups are associated with points of some faces of this polyhedral.

Next result relates the multiplicity and the genus of a numerical semigroup.

Proposition 1.3. *Let S be a numerical semigroup with genus g and multiplicity m. Then $m \leqslant g + 1$.*

Proof. If a numerical semigroup has multiplicity m and genus g satisfying $m \geqslant g + 2$, then the number of gaps would be, at least, $g + 1$ and this is a contradiction. Hence $m \leqslant g + 1$. □

Remark 1.3. *Ordinary semigroups attain maximum multiplicity.*

The *Apéry set* of an element k of a numerical semigroup S is

$$Ap(S, k) := \{s \in S : s - k \notin S\}.$$

A consequence of this definition is that $\#Ap(S, k) = k$. In fact, for each $i \in [0, k-1]$, let $s_i := \min\{\ell \in S : \ell \equiv i \pmod{k}\}$. Then, $Ap(S, k) = \{s_0 = 0, s_1, \ldots, s_{k-1}\}$.

It is natural considering the Apéry set of the multiplicity of a numerical semigroup. This set has m elements and $\{m\} \cup Ap(S, m) \backslash \{0\}$ generates S, since each element $s \in S$ can be written as $s = k_1 \cdot m + i = k_2 \cdot m + s_i$, for some i. Thus, $e \leqslant m$. Using Proposition 1.3, we conclude that a numerical semigroup with genus g and embedding dimension e satisfies $e \leqslant g+1$. In fact this bound is sharp, since the ordinary semigroup of genus g has embedding dimension $g+1$; it can be written as $\langle g+1, g+2, \ldots, 2g+1\rangle$.

On the other hand, 1 is a lower bound for embedding dimension of a numerical semigroup. The unique numerical semigroup with embedding dimension 1 is $\mathbb{N}_0$ and, for each $k \geqslant 2$, there are infinitely many numerical semigroups with embedding dimension k (if $a_1, \ldots, a_k$ are coprime, then $\langle a_1, \ldots, a_k\rangle$ is a numerical semigroup). Looking for a better lower bound for embedding dimension of a numerical semigroup S depending on its invariants seems to be an interesting problem and, in fact, it is an old one. In 1978, Wilf (28) conjectured that $F(S)+1 \leqslant e(S)n(S)$, for every numerical semigroup S, where $n(S) = \#S \cap [0, F(S)]$. It can be written as $e(S) \geqslant \dfrac{F(S)+1}{F(S)+1-g(S)}$, since $n(S) = F(S)+1-g(S)$. Several authors investigated this problem, giving partial answers, but it remains as an open problem, up to now.

Bras-Amorós (5) introduced the notion of ordinarization transform of a numerical semigroup. Let S be a non-ordinary semigroup with Frobenius number $F(S)$ and multiplicity $m(S)$. The ordinarization transform of S is the numerical semigroup $S' = \{F(S)\} \cup S \backslash \{m(S)\}$ which has also genus g. For example, if $S = \mathbb{N}_0 \backslash \{1,2,3,6,7\}$, then $S' = \mathbb{N}_0 \backslash \{1,2,3,4,6\}$, since $F(S) = 7$ and $m(S) = 4$. Clearly, if we make this operation several times, we obtain an ordinary semigroup. Hence, for a fixed S, we can consider the least number $r(S)$ of such operations we should do to obtain an ordinary semigroup. This number is called ordinarization number of S. Naturally, the ordinarization number of an ordinary semigroup is 0. More precisely:

Definition 1.1. *Let $\mathcal{S}_g$ be the set of all numerical semigroups with genus g and let S_{g+1} be the ordinary semigroup with genus g. Consider the map:*

$$\begin{array}{rcccl} \mathbf{T} & : & \mathcal{S}_g \backslash \{S_{g+1}\} & \to & \mathcal{S}_g \\ & & S & \mapsto & (S \cup \{F(S)\}) \backslash \{m(S)\}, \end{array}$$

The image $\mathbf{T}(S)$ is the ordinarization tranform of S and the integer $r(S) := \min\{r \in \mathbb{N} : \mathbf{T}^r(S) = S_{g+1}\}$ is the ordinarization number of S. Naturally, $r(S_{g+1}) := 0$.

In that paper, she proved that

Proposition 1.4. *(5, Lemma 2) Let S be a numerical semigroup with genus g and ordinarization number r. Then $r \leqslant \left\lfloor \dfrac{g}{2} \right\rfloor$.*

Proof. We recall that an ordinary semigroup has multiplicity $g+1$. Writing $S \cap [1, g] = \{s_1 < \cdots < s_k\}$, we conclude that k is exactly the number of ordinarization of S, since all numbers less than g must be changed with all gaps greater than g for some ordinarization step. Hence, $r = k$. Notice that

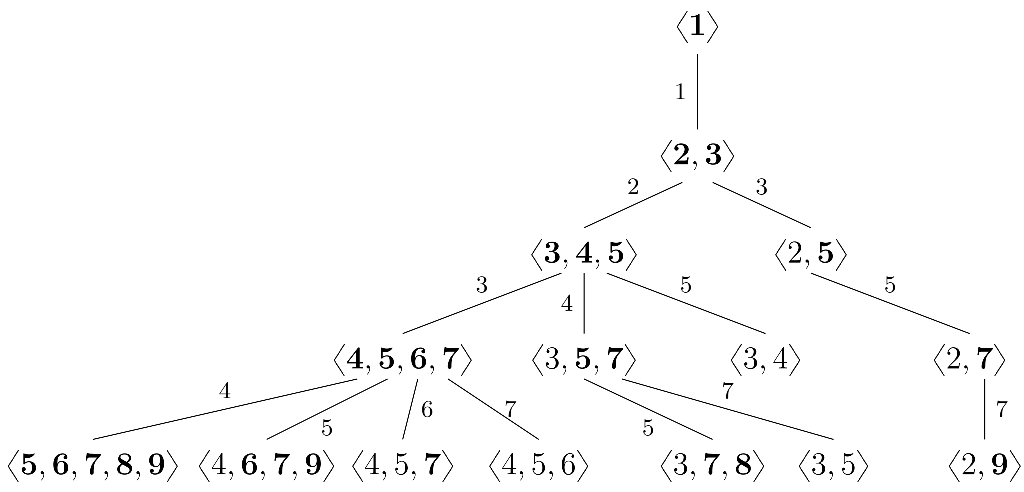
$$s_1, \ldots, s_r, s_1 + s_r, \ldots, s_r + s_r$$

are $2r$ different non-gaps of S greater than 0 and less than $2g$. Since $S \cap [1, 2g] = g$, we conclude that $2r \leqslant g$. Thus, $r \leqslant \left\lfloor \frac{g}{2} \right\rfloor$ (because r is an integer). $\square$

Remark 1.4. *(5, Lemma 4) This bound is sharp, since the hyperelliptic semigroup with genus g has ordinarization number $r = \left\lfloor \frac{g}{2} \right\rfloor$.*

The so called *semigroup tree* is an useful combinatorial object in this theory. First evidences for it are (20) and (22). In 2009, Bras-Amorós (4) introduced formally the semigroup tree. Each numerical semigroup is represented once and the construction is inductive. This representation is interesting, since it splits numerical semigroups by genus in each depth.

Here, we give an idea of its construction, following (4), where the set of minimal generators and Frobenius number of a numerical semigroup play an important role. The root of this tree is $\mathbb{N}_0$, which is the unique numerical semigroup of genus 0 and can be written as $\langle 1 \rangle$. The unique generator of this numerical semigroup is 1 and it is greater than Frobenius number -1 (it is a convention). For next depth, we take out 1 of numerical semigroup, creating the semigroup $\langle 2, 3 \rangle$. The Frobenius number of this numerical semigroup is 1 and all minimal generators are greater than 1. By taking out 2, we obtain the numerical semigroup $\langle 3, 4, 5 \rangle$ and by taking out 3, we obtain the numerical semigroup $\langle 2, 5 \rangle$. We can do this procedure as many times as we want. One can contruct all numerical semigroups with genus up to 4.



2 Counting numerical semigroups by genus and even gaps

In this chapter, we deal with Question 1 by taking into consideration the effect of even gaps on the structure of numerical semigroups (cf (27)). Thus we use γ-hyperelliptic semigroups, which simply are numerical semigroups with γ even gaps. This terminology comes from theory of double covering (cf (25)). Some examples are $\langle 2, 2g+1 \rangle$, which are the only 0-hyperelliptic semigroups (also named hyperelliptic) and $\mathbb{N}_0 \backslash \{1, 2\}$, which is 1-hyperelliptic. Notice that a γ-hyperelliptic semigroup with genus g satisfies $\gamma \leqslant g$. For g and γ non-negative integers, we set

$$\mathcal{S}_\gamma(g) := \{S \in \mathcal{S}_g : \gamma(S) = \gamma\} \quad \text{and} \quad N_\gamma(g) := \#\mathcal{S}_\gamma(g).$$

Each numerical semigroup S with genus g has a number of even gaps that lies in $[0, g] \cap \mathbb{Z}$. Then we can write

$$\mathcal{S}_g := \bigcup_{\gamma=0}^{g} \mathcal{S}_\gamma(g) \quad \text{and} \quad n_g = \sum_{\gamma=0}^{g} N_\gamma(g). \tag{2.1}$$

By using this approach, finding out precise values or even lower and upper bound for $N_\gamma(g)$ are of interest. In this chapter we prove the following:

Theorem 2.1. *Let $g \geqslant 3\gamma$. Then $N_\gamma(g) = N_\gamma(3\gamma)$.*

This result states that, for a fixed γ, the sequence $(N_\gamma(g))_g$ is constant for $g \geqslant 3\gamma$ and it is equal to $N_\gamma(3\gamma)$. Another result we prove in this chapter is:

Theorem 2.2. *Let $g < 3\gamma$. Then $N_\gamma(g) < N_\gamma(3\gamma)$.*

2.1 On γ-hyperelliptic semigroups

In this section, we recall some important results about γ-hyperelliptic semigroups. Some of them can be found in (27), where there is a geometric interest; see also (18) and (19). Here, we are interested only in arithmetic properties of numerical semigroups.

Lemma 2.1. *(27, Lemma 2.1) Let S be a γ-hyperelliptic semigroup with genus g. Then*

$$2g \geqslant 3\gamma.$$

Proof. If $g \geqslant 2\gamma$, then $2g \geqslant 4\gamma \geqslant 3\gamma$ and the lemma follows. Assume $g \leqslant 2\gamma - 1$. The numerical semigroup S has γ even gaps and $g - \gamma$ odd gaps in $[1, 2g - 1]$. Hence, S has $g - \gamma$ even non-gaps and γ odd non-gaps in $[1, 2g]$. Let $q_1 < \cdots < q_\gamma$ be the even gaps of S and let $r_\gamma < \cdots < r_1$ be the odd non-gaps in $S \cap [1, 2g]$. Notice that $q_\gamma - r_i \notin S$, for all $i \in \{1, \ldots, \gamma\}$ (otherwise, $S \ni r_i + (q_\gamma - r_i) = q_\gamma \notin S$). For each i such that $q_\gamma - r_i \geqslant 1$, all numbers $q_\gamma - r_i$ are odd gaps.

Claim. $q_\gamma \leqslant 4g - 4\gamma$.

Suppose $q_\gamma \geqslant 4g - 4\gamma + 2$. Since $r_i \leqslant 2g - 2i + 1$, then for all $i \geqslant 2\gamma - g$ $(\geqslant 1)$,

$$\begin{aligned} q_\gamma - r_i &\geqslant (4g - 4\gamma + 2) - (2g - 2i + 1) \\ &\geqslant 2g - 4\gamma + 1 + 2i \\ &\geqslant 2g - 4\gamma + 1 + 4\gamma - 2g = 1. \end{aligned}$$

We proved that all numbers $q_\gamma - r_{2\gamma - g} < \cdots < q_\gamma - r_\gamma$ are different gaps of S, so there are, at least, $\gamma - (2\gamma - g) + 1 = g - \gamma + 1$ different odd gaps in $S \cap [1, 2g]$, which is a contradiction. Hence, $q_\gamma \leqslant 4g - 4\gamma$.

There are $2g - 2\gamma$ even numbers in $[2, 4g - 4\gamma]$, which γ of them are gaps. Hence, $2g - 2\gamma \geqslant \gamma$ and we conclude that $2g \geqslant 3\gamma$. □

Remark 2.1. *If γ is even, then $\mathbb{N}_0 \backslash (\{2, 4, \ldots, 2\gamma\} \cup \{1, 3, \ldots, \gamma - 1\})$ is a numerical semigroup with genus $g = \gamma + \frac{\gamma}{2}$, i.e, $2g = 3\gamma$. This proves that the bound found in Lemma 2.1 is sharp. All those numerical semigroups are of maximal embedding dimension and they can be described as $\langle \gamma + 1, \gamma + 3, \ldots, 3\gamma + 1 \rangle$ (see example 2.1 for uniqueness).*

Using Lemma 2.1, we can rewrite (2.1) as

$$\mathcal{S}_g := \bigcup_{\gamma=0}^{\lfloor \frac{2g}{3} \rfloor} \mathcal{S}_\gamma(g) \quad \text{and} \quad n_g = \sum_{\gamma=0}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g).$$

Definition 2.1. *The* one half *of a numerical semigroup S is $S/2 := \{s \in \mathbb{N}_0 : 2s \in S\}$.*

Proposition 2.1. *Let S be a γ-hyperelliptic semigroup. Then $4\gamma + 2\mathbb{N}_0 \subseteq S$.*

Proof. We prove that $S/2$ is a numerical semigroup with genus γ:

- $S/2 \subseteq \mathbb{N}_0$ by definition;
- $0 \in S/2$: it follows from the fact that $0 = 2 \cdot 0$;
- $a, b \in S/2 \Rightarrow a + b \in S/2$: by definition $2a$ and $2b \in S$. Since S is a numerical semigroup, then $2a + 2b = 2(a + b) \in S$ and it proves that $a + b \in S/2$.

- $\mathbb{N}_0\backslash(S/2)$ has γ elements: notice that $\mathbb{N}_0\backslash(S/2) = \{\ell \in \mathbb{N}_0 : 2\ell \notin S\}$. Those elements are the even gaps of S and we know that there are exactly γ of them.

By Lemma 1.1, $2\gamma + \mathbb{N}_0 \subseteq S/2$. Then $4\gamma + 2\mathbb{N}_0 \subseteq 2(S/2) \subset S$ and the result follows. □

From now on, we assume g and γ non-negative integers such that $2g \geqslant 3\gamma$. We have a natural parametrization of the family $\mathcal{S}_\gamma(g)$ onto $\mathcal{S}_\gamma$ given by

$$\begin{array}{rcccl} \mathbf{x}_g & : & \mathcal{S}_\gamma(g) & \rightarrow & \mathcal{S}_\gamma \\ & & S & \mapsto & S/2 \end{array}$$

which is a surjective map. In fact, given $T \in \mathcal{S}_\gamma$, we have $\mathbf{x}_g(S) = T$, for

$$S := 2T \cup \{2g - 2\gamma + i : i \in \mathbb{N}\} \in \mathcal{S}_\gamma(g),$$

being $2T := \{2t : t \in T\}$. If $g = 3\gamma$, we denote $\mathbf{x} := \mathbf{x}_{3\gamma}$.

Remark 2.2. *Indeed, any $S \in \mathcal{S}_\gamma(g)$ can be uniquely written as*

$$S = 2(S/2) \cup \{o_\gamma < \ldots < o_1\} \cup S_{2g}$$

where $o_\gamma, \ldots, o_1$ are certain odd numbers in $[1, 2g-1]$.

Remark 2.3. *For g and γ non-negative integers, we can write*

$$N_\gamma(g) = \sum_{T \in \mathcal{S}_\gamma} \#\mathbf{x}_g^{-1}(T).$$

Let $S \in \mathcal{S}_\gamma(g)$. Remark 2.2 tell us that there is natural way to obtain the even gaps (and non-gaps) of S from a numerical semigroup $T \in \mathcal{S}_\gamma$: if $\{2q_1, \ldots, 2q_\gamma\}$ is the set of even gaps of S, then it can be obtained by duplicating the gaps from $\mathbb{N}_0\backslash\{q_1, \ldots, q_\gamma\} \in \mathcal{S}_\gamma$. Moreover, T is uniquely determined ($T = S/2$).

Reciprocally, given $T \in \mathcal{S}_\gamma$ we can obtain a γ-hyperelliptic semigroup S of some genus g by duplicating the gaps of T and making a suitable choice on the odd numbers that are gaps or non-gaps of S. Notice that not all choices for odd integers return numerical semigroups (some sets are not closed under addition) and in general S is not uniquely determined (in general, the fiber $\mathbf{x}_g^{-1}(T)$ has more than one element).

Now we look for conditions that a γ-hyperelliptic semigroup with some genus $g \geqslant 3\gamma/2$ has to attain. In fact, let $T \in \mathcal{S}_\gamma$ and consider all sets

$$S = 2T \cup \{o_\gamma < \ldots < o_1\} \cup S_{2g},$$

where the numbers o_i are odd integers in $[1, 2g-1]$.

From definition, $S \subseteq \mathbb{N}_0$, $0 \in S$ and $\mathbb{N}_0 \backslash S$ has γ even elements. There are g odd numbers in $[1, 2g-1]$, such that γ of them lies in S; thus $\mathbb{N}_0 \backslash S$ has $g-\gamma$ odd elements and $\#(\mathbb{N}_0 \backslash S) = g$. Hence, we conclude that $S \in \mathcal{S}_\gamma(g)$ if, and only if, S is closed under addition. Moreover, the even part of S is closed under addition, since T is.

We look for necessary conditions on S so it is a numerical semigroup. Let $T \in \mathcal{S}_\gamma$. Writing $T = \mathbb{N}_0 \backslash \{q_1, \ldots, q_\gamma\}$ and $T \cap [0, 2\gamma] = \{t_0 = 0, t_1, \ldots, t_\gamma = 2\gamma\}$, we have that $2t_0 = 0 < 2t_1 < \cdots < 2t_{\gamma-1} < 2t_\gamma = 4\gamma$ are even non-gaps of S and $2q_1 < \cdots < 2q_\gamma$ are the even gaps of S.

Here we write the first odd number in S, o_γ, as $O = 2k+1$. By closed under addition condition, we conclude that $O + 2t_i$ must be a non-gap of S, for all i, and $(O + 4\gamma) + \mathbb{N}_0 \subseteq S$. Notice that, at first, there are no necessary conditions on each number $O + 2q_i$. We illustrate it below:

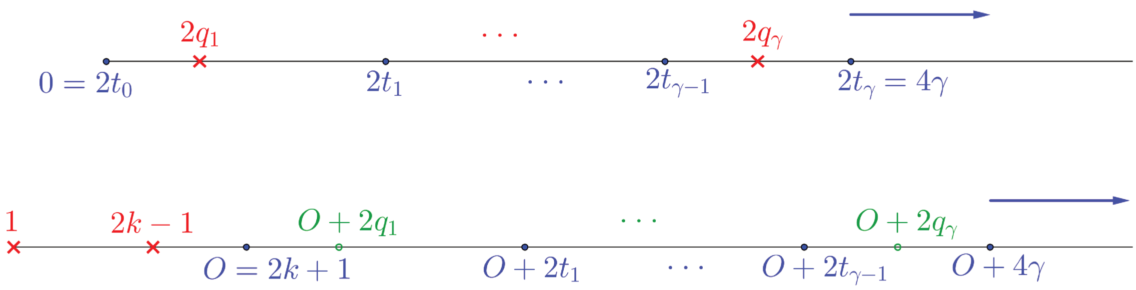


Figure 1 – Configuration of a γ-hyperelliptic semigroup

By computing these conditions, we can obtain upper and lower bounds for the first odd integer in a γ-hyperelliptic semigroup S with genus g only depending on γ and g.

Lemma 2.2. *Let S be a γ-hyperelliptic semigroup with genus g and let O be the first odd number in S. Then*

$$2g - 4\gamma + 1 \leqslant O \leqslant 2g - 2\gamma + 1.$$

Proof. Write $O = 2k+1$ and let $2q_1, \ldots, 2q_\gamma$ be the even gaps of S. Then the k numbers $1, 3, \ldots, 2k-1$ are odd gaps of S and the only possibilities for other gaps of S are $O + 2q_1, \ldots, O + 2q_\gamma$. Thus $\gamma + k \leqslant g \leqslant 2\gamma + k$. Hence $g - 2\gamma \leqslant k \leqslant g - \gamma$ and we conclude that $2g - 4\gamma + 1 \leqslant O = 2k+1 \leqslant 2g - 2\gamma + 1$. □

Proposition 2.2. *Let S be a γ-hyperelliptic semigroup with genus g such that $g \leqslant 2\gamma - 1$ and let O be the first odd number in S. Then $O \geqslant 4\gamma - 2g + 1$.*

Proof. Consider $o_\gamma < \cdots < o_{4\gamma-2g}$ as the first odd integers of S. Then the numbers $o_\gamma + o_\gamma < \cdots < o_\gamma + o_{4\gamma-2g}$ are $2g - 3\gamma + 1$ even elements of S. By the Claim in the proof of Lemma 2.1, we have that the largest even gap of S is at most $4g - 4\gamma$ and there are $2g - 3\gamma$ even numbers in $[2, 4g - 4\gamma] \cap S$. Hence, $o_\gamma + o_{4\gamma-2g}$ is an even non-gap of S greater than or equal to $4g - 4\gamma + 2$. Also, for all $i \in \{1, \ldots, 4\gamma - 2g\}$, $o_i \leqslant 2g - (2i - 1)$. Hence, $o_\gamma \geqslant (4g - 4\gamma + 2) - (2g - 2(4\gamma - 2g) + 1) = 4\gamma - 2g + 1$. □

Remark 2.4. *We can rewrite the bounds of Lemma 2.2 as* $|2g - 4\gamma| + 1 \leqslant O \leqslant 2g - 2\gamma + 1$.

Example 2.1. *We study γ-hyperelliptic semigroups S with genus g that attain extremal cases for the inequality $2g \geqslant 3\gamma$. In particular, we prove that $N_\gamma(3\gamma/2) = 1$, if γ is even and $N_\gamma((3\gamma + 1)/2) = (\gamma + 3)/2$ if γ is odd (see Tables 1, 2 and 3 for more details).*

Case γ even. We look for $2g = 3\gamma$. If $\gamma = 0$, then $g = 0$ and the numerical semigroup is $\mathbb{N}_0$. By Remark 2.4, we conclude that $\gamma + 1 \leqslant O \leqslant \gamma + 1$, i.e., $O = \gamma + 1$. In the interval $[\gamma + 1, 2g - 1] = [\gamma + 1, 3\gamma - 1]$, there are γ odd numbers. Hence, we conclude that $(o_\gamma, \ldots, o_1) = (\gamma + 1, \ldots, 3\gamma - 1)$. By closed under addition property of numerical semigroups, we conclude that $\{2\gamma + 2i : i \in \mathbb{N}\} \subset S$ and the set of even gaps of S is $\{2, \ldots, 2\gamma\}$. Hence there is exactly one γ-hyperelliptic semigroup with genus $3\gamma/2$.

Case γ odd. We look for $2g = 3\gamma + 1$. By Remark 2.4, we conclude that $\gamma \leqslant O \leqslant \gamma + 2$, i.e., $O = \gamma$ or $O = \gamma + 2$. By Claim in proof of Lemma 2.1, the largest even gap is at most $4g - 4\gamma = 2\gamma + 2$. Thus, $\{2\gamma + 2i + 2 : i \in \mathbb{N}\} \subset S$ and the γ even gaps of S are elements of $\{2, \ldots, 2\gamma + 2\}$, i.e., there is exactly one even non-gap in that set.
Case $O = \gamma + 2$. The interval $[\gamma + 2, 2g - 1] = [\gamma + 2, 3\gamma]$ contains exactly γ odd numbers. Thus, $(o_\gamma, \ldots, o_1) = (\gamma + 2, \ldots, 3\gamma)$. The possibilities for that even non-gap are $\gamma + 3, \gamma + 5, \ldots, 2\gamma + 2$ (if we choose some $x \leqslant \gamma + 1$, then $2x \leqslant 2\gamma + 2$ would lies in S and S would not be γ-hyperelliptic). Hence, there are exactly $(\gamma + 1)/2$ γ-hyperelliptic semigroup with genus $(3\gamma + 1)/2$ and $O = \gamma + 2$.
Case $O = \gamma$. In this case, we conlude that $2\gamma \in S$. Hence, the set of even gaps is $\{2, \ldots, 2\gamma - 2, 2\gamma + 2\}$. If $\gamma + 2$ is an odd non-gap, then $\gamma + (\gamma + 2) = 2\gamma + 2$ is a non-gap and this is a contradiction. Hence, $(o_\gamma, \ldots, o_1) = (\gamma, \gamma + 4, \ldots, 3\gamma)$ and there is exactly one γ-hyperelliptic semigroup with genus $(3\gamma + 1)/2$ and $O = \gamma$.

2.2 Proofs of Theorems 2.1 and 2.2

We introduce a definition we use in this section.

Definition 2.2. *Let $t \in \mathbb{Z}$. The t-translation of a numerical semigroup S is the map $\Phi_t : S \to \mathbb{Z}$ defined by*

$$s \mapsto \begin{cases} s & \text{if } s \equiv 0 \pmod{2}, \\ s - 2t & \text{otherwise}. \end{cases}$$

For $t = g - 3\gamma$, we denote Φ_t by Φ.

Proposition 2.3. *If $S \in \mathcal{S}_\gamma(g)$, then $\Phi(S) \in \mathcal{S}_\gamma(3\gamma)$.*

Proof. Let $t = g - 3\gamma$ and O be the first odd integer in S. From the definition, $\Phi(S)$ has γ even gaps. By making t-translation, we observe that the first odd number in $\Phi(S)$ is $O - 2t$ and number of odd gaps in $\Phi(S)$ is $(g - \gamma) - t = g - \gamma - g + 3\gamma = 2\gamma$. Hence, the genus of $\Phi(S)$ is 3γ. Finally, we prove that $\Phi(S)$ is closed under addition. Let a and $b \in \Phi(S)$. If a and b even numbers, then $a + b \in \Phi(S)$. If a is even and $b = c - 2t$ is odd, with $c \in S$, then $a + b = a + c - 2t = d - 2t$, with $d \in S$, since S is closed addition. Hence, $a + b \in \Phi(S)$. If $a = c - 2t$ and $b = d - 2t$ are odd, then $a + b = (c + d) - 4t \geqslant 2(2g - 4\gamma + 1) - 4(g - 3\gamma) = 4\gamma + 2$, since, by Lemma 2.2, $c, d \geqslant O \geqslant 2g - 4\gamma + 1$. Hence, $a + b \in \Phi(S)$ and we conclude that $\Phi(S) \in S_\gamma(3\gamma)$. ☐

Definition 2.2, with $t = g - 3\gamma$, induces the map

$$\begin{array}{rcl} \tilde{\Phi} \;:\; \mathcal{S}_\gamma(g) & \longrightarrow & \mathcal{S}_\gamma(3\gamma) \\ S & \longmapsto & \Phi(S). \end{array}$$

It is clear that for different γ-hyperelliptic semigroups with same genus g, they have, at least, one different gap from each other. Hence, the images of those numerical semigroups under $\tilde{\Phi}$ are also different. Hence, $\tilde{\Phi}$ is an injective map and we conclude that $\#S_\gamma(g) = N_\gamma(g) \leqslant N_\gamma(3\gamma) = \#S_\gamma(3\gamma)$, for all g.

In general, Definition 2.2 induces a map $\tilde{\Phi}_t : \mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma(g - t)$ for any $t \in \mathbb{Z}$ such that $2(g - t) \geqslant 3\gamma$. This map can be seen as follows in next figure, where $S \in \mathcal{S}_\gamma(g)$.

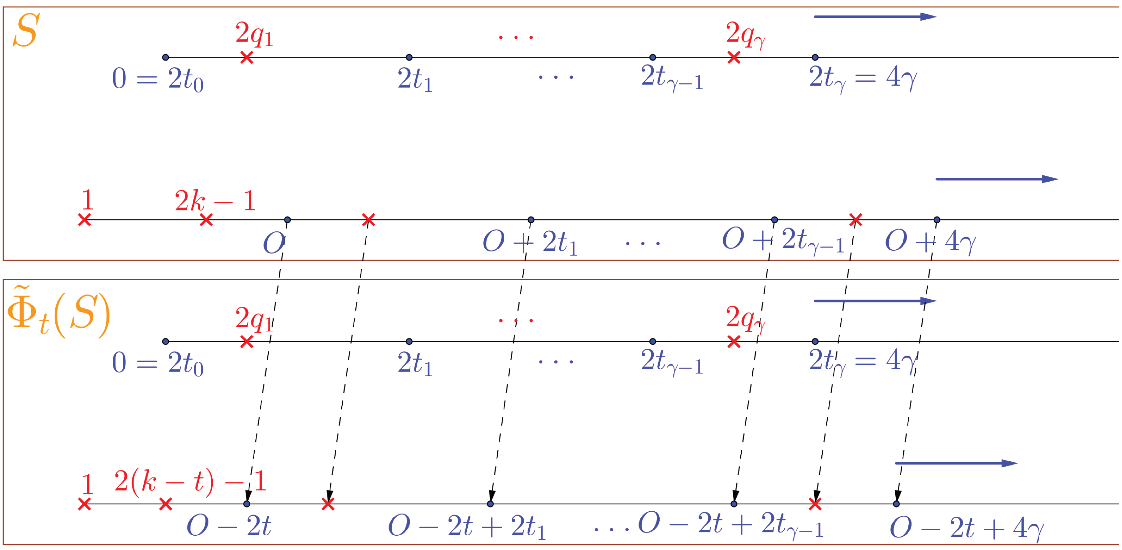


Figure 2 – Map $\tilde{\Phi}_t$, with $t > 0$

Before proving Theorem 2.1, we explain why we have to consider $g \geqslant 3\gamma$. Observe the Figure 1: in that case, Lemma 2.2 implies that $O \geqslant 2\gamma + 1 > q_\gamma \geqslant q_i$, for all i and there is no situations like $q_i = O + 2q_j$ or $q_i = O + 2t_j$. If $q_i = O + 2q_j$, then a green point $O + 2q_j$ on Figure 2 is automatically a red point, changing the global configuration. If $q_i = O + 2t_j$, then a blue point $O + 2t_j$ on Figure 2 is also a red point q_i, which leads to a contradiction. Also, all sum of odd non-gaps lies in the numerical semigroup, since, for O_1 and O_2 odd non-gaps, $O_1 + O_2 \geqslant 2O \geqslant 4\gamma + 2$.

Theorem 2.1. *Let $g \geqslant 3\gamma$. Then $N_\gamma(g) = N_\gamma(3\gamma)$.*

Proof. We only have to prove that map $\tilde{\Phi}$ defined above is surjective. For a given $T \in S_\gamma(3\gamma)$, natural candidate for pre-image of it under $\tilde{\Phi}$ is S, which is obtained from T by a $(-t)$-translation ($t := g - 3\gamma \geqslant 0$). Notice that:

- $g(S) = g(S(3\gamma)) + t = 3\gamma + (g - 3\gamma) = g$;
- $\gamma(S) = \gamma$, since $\gamma(S(3\gamma)) = \gamma$;
- S is closed under addition, since T is closed under addition and the first odd integer in S is greater than or equal to the first odd integer in T.

Thus, $S \in \mathcal{S}_\gamma(3\gamma)$.

□

For a given g, we can rewrite n_g and n_{g+1} as

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g)$$

and

$$n_{g+1} = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g+1) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2(g+1)}{3} \rfloor} N_\gamma(g+1).$$

Theorem 2.1 states that $N_\gamma(g) = N_\gamma(g+1)$, for $\gamma \leqslant \frac{g}{3}$. Then we have the following equivalence to Question 1.

Corollary 2.1. *$n_g \leqslant n_{g+1}$ if, and only if,* $\displaystyle\sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g) \leqslant \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2(g+1)}{3} \rfloor} N_\gamma(g+1)$.

By using GAP (13) and the package NumericalSgps (10), we computed, with help of Maria Bras-Amorós and Pedro A. García-Sánchez, a few values of $N_\gamma(g)$. We thank them for their valuable contribution.

g \ γ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	n_g
0	**1**																	1
1	1																	1
2	1	1																2
3	1	**2**	1															4
4	1	2	4															7
5	1	2	6	3														12
6	1	2	**7**	12	1													23
7	1	2	7	19	10													39
8	1	2	7	21	32	4												67
9	1	2	7	**23**	51	33	1											118
10	1	2	7	23	62	91	18											204
11	1	2	7	23	65	142	98	5										343
12	1	2	7	23	**68**	174	257	59	1									592
13	1	2	7	23	68	192	412	271	25									1001
14	1	2	7	23	68	197	514	678	197	6								1693
15	1	2	7	23	68	**200**	570	1100	793	92	1							2857
16	1	2	7	23	68	200	602	1409	1855	606	33							4806
17	1	2	7	23	68	200	609	1595	2999	2191	343	7						8045
18	1	2	7	23	68	200	**615**	1693	3890	4993	1836	138	1					13467
19	1	2	7	23	68	200	615	1744	4472	8126	6033	1130	43					22464
20	1	2	7	23	68	200	615	1756	4797	10723	13317	5335	544	8				37396
21	1	2	7	23	68	200	615	**1764**	4959	12528	21764	16447	3624	191	1			62194
22	1	2	7	23	68	200	615	1764	5034	13616	29209	35392	15365	1897	53			103246
23	1	2	7	23	68	200	615	1764	5053	14191	34628	57925	44575	11098	804	9		170963
24	1	2	7	23	68	200	615	1764	**5060**	14469	38096	78602	93919	43262	6485	254	1	282828

Table 1 – A few values for $N_\gamma(g)$ (part 1)

g \ γ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
25	1	2	7	23	68	200	615	1764	5060	14589	40098	94469	154077	119669	33525	3013	64
26	1	2	7	23	68	200	615	1764	5060	14611	41086	105074	211576	247756	120881	20945	1153
27	1	2	7	23	68	200	615	1764	5060	**14626**	41541	111426	257734	407238	320649	98104	10873
28	1	2	7	23	68	200	615	1764	5060	14626	41725	114889	290192	565331	652952	?	?
29	1	2	7	23	68	200	615	1764	5060	14626	41765	116546	310511	697502	1073955	853254	?
30	1	2	7	23	68	200	615	1764	5060	14626	**41785**	117238	322103	794314	1504305	1714253	?
31	1	2	7	23	68	200	615	1764	5060	14626	41785	117497	328098	858020	1877733	2822586	2264348
32	1	2	7	23	68	200	615	1764	5060	14626	41785	117555	330854	895949	2162304	3988248	4493022
33	1	2	7	23	68	200	615	1764	5060	14626	41785	**117573**	331977	916624	2356790	5034455	7403032
34	1	2	7	23	68	200	615	1764	5060	14626	41785	117573	332373	926905	2477817	5861266	10542852
35	1	2	7	23	68	200	615	1764	5060	14626	41785	117573	332439	931437	2546526	6448122	13450037

Table 2 – A few values for $N_\gamma(g)$ (part 2)

g \ γ	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	n_g
25																		467224
26	10																	770832
27	335	1																1270267
28	?	?																2091030
29	?	?	11															3437839
30	?	?	?	1														5646773
31	?	?	?	?														9266788
32	?	?	?	?	12													15195070
33	5987936	?	?	?	?	1												24896206
34	11753434	?	?	?	?	?												40761087
35	19377030	15796993	?	?	?	?	13											66687201

Table 3 – A few values for $N_\gamma(g)$ (part 3)

Using this table, we conjecture a stronger condition in relation to any of the equivalences in last Corollary.

Conjecture 2.1. *Let γ be a non-negative integer. Then*

$$N_\gamma(g) \leqslant N_\gamma(g+1), \forall g.$$

Notice that Theorem 2.1 ensures that, when γ is fixed, then Conjecture 2.1 is true for all $g \geqslant 3\gamma$. Now we prove Theorem 2.2.

Theorem 2.2. *Let $g < 3\gamma$. Then $N_\gamma(g) < N_\gamma(3\gamma)$.*

Proof. If $\gamma = 0$ or 1, then first two columns on Table 1 ensure the condition. Suppose $\gamma \geqslant 2$. We prove that $\tilde{\Phi}$ is not surjective. Let

$$T = \mathbb{N}_0 \backslash (\{2, 6, \ldots, 4\gamma - 2\} \cup \{1, 3, \ldots, 2\gamma - 1, 2\gamma + 3, 2\gamma + 7, \ldots, 6\gamma - 1\}) \in S_\gamma(3\gamma),$$

which has first odd non-gap $O = 2\gamma + 1$ and $4\gamma - 2$ as a gap that is greater than or equal to 6. Suppose that there is $S \in \mathcal{S}_\gamma(g)$ such that $\tilde{\Phi}(S) = T$ and let $t := g - 3\gamma$. If $g < 2\gamma$, then 1 or 3 is the first odd number in S. If $g \geqslant 2\gamma$, then $O + 2t = 2g - 4\gamma + 1$ is the first odd number in S. Then

$$\begin{cases} S \ni 2 \cdot (2g - 4\gamma + 1) = 4(g - 2\gamma) + 2 \notin S, & \text{if } 2\gamma \leqslant g < 3\gamma \\ S \ni 1 + 1 = 2 \notin S \text{ or } S \ni 3 + 3 = 6 \notin S, & \text{if } 3\gamma/2 \leqslant g < 2\gamma \end{cases}$$

and this is a contradiction. Hence, there is no $S \in \mathcal{S}_\gamma(g)$ such that $\tilde{\Phi}(S) = T$. □

To end up this section, we prove a result about symmetric semigroups. At first, it will not contribute for solving Conjecture 2.1, but it is an interesting result in itself.

Proposition 2.4. *Let $g \geqslant 3\gamma$. Then there are, at least, n_γ γ-hyperelliptic semigroups with genus g which are symmetric.*

Proof. Let $T \in \mathcal{S}_\gamma$ and write $T = \mathbb{N}_0 \backslash \{q_1, \ldots, q_\gamma\}$. We prove that if

$$S := 2T \cup \{2g - 1 - 2t : t \in \mathbb{Z} \backslash T\},$$

then $S \in \mathcal{S}_\gamma(g)$ and it is symmetric.

- S has γ even gaps, since $2T = 2\mathbb{N}_0 \backslash \{2q_1, \ldots, 2q_\gamma\}$.

- $\#(\mathbb{N}_0 \backslash S) = g$:
  We showed that S has γ even gaps. We have to prove that S has $g - \gamma$ odd gaps. Since $\mathbb{Z}\backslash T = \{\ldots, -2, -1, q_1, \ldots, q_\gamma\}$, then odd gaps of S is $\{2g-1-2q_\gamma, \ldots, 2g-1-2q_1, 2g+1, \rightarrow\}$. Notice that $2g-1-2q_\gamma \geqslant 2\gamma+1 \geqslant 1$, since $g \geqslant 3\gamma$ and $q_\gamma \leqslant 2\gamma - 1$ (here, it would be enought $g \geqslant 2\gamma$). Then the set of odd gaps of S is $([1, 2g-1] \cap (2\mathbb{Z}+1))\backslash\{2g-1-2q_\gamma, \ldots, 2g-1-2q_1\}$, which has $g-\gamma$ elements.

- S is a numerical semigroup:
  We already proved that $\mathbb{N}_0\backslash S$ is a finite set. Since $0 \in T$, then $0 = 2 \cdot 0 \in 2T \subset S$. We show that S is closed under addition. Let $2t_1$ and $2t_2$, with t_1 and $t_2 \in T$, be even elements of S. Then $2t_1 + 2t_2 = 2(t_1 + t_2) \in S$, since T is closed under addition. Let $2t_1$ and $2g-1-2t_2$, with $t_1 \in T$ and $t_2 \in \mathbb{Z}\backslash T$, be an even element of S and an odd element of S, respectively. Then $a := 2t_1 + (2g-1-2t_2) = 2g-1-2(t_2-t_1)$. Notice that $t_2 - t_1 \in \mathbb{Z}\backslash T$. Otherwise, $T \ni (t_2 - t_1) + t_1 = t_2 \notin T$. Hence $a \in S$. Finally, let $2g-1-2t_1$ and $2g-1-2t_2$, with t_1 and $t_2 \in \mathbb{Z}\backslash T$, be odd elements of S. Then $b := (2g-1-2t_1) + (2g-1-2t_2) = 2(2g-(t_1+t_2)-1)$. Since $t_1, t_2 \leqslant 2\gamma - 1$ and $g \geqslant 3\gamma$, then $2g - (t_1+t_2) - 1 \geqslant 2\gamma + 1$. Hence, $b \geqslant 4\gamma + 2$ and $b \in S$.

- S is symmetric:
  We proved that the odd part of S is
  $$\{2g-1-2q_\gamma < \ldots < 2g-1-2q_1 < 2g+1, \rightarrow\}.$$
  Since $q_1 > 0$, then $2g-1 \notin S$. Hence, Frobenius number of S is $2g-1$ and we conclude that S is symmetric.

There are exactly n_γ possibilities for T under this construction, hence the result follows. □

Remark 2.5. *In this construction, we chose* $o_i = 2g - 1 - 2q_i$.

Corollary 2.2. *Let* g *and* γ *be non-negative integers such that* $g \geqslant 3\gamma$. *Then* $N_\gamma(g) \geqslant n_\gamma$.

Proof. By Remark 2.3 and Proposition 2.4, we have

$$N_\gamma(g) = \sum_{T \in \mathcal{S}_\gamma} \#\mathbf{x}_g^{-1}(T) \geqslant \sum_{T \in \mathcal{S}_\gamma} 1 = n_\gamma.$$

□

3 On the sequence $(N_\gamma(3\gamma))$

Bras-Amorós (5) introduced the ordinarization transform of a numerical semigroup. By making some calculations, she obtained a sequence (f_γ). After some nice discussions with Maria Bras-Amorós, we prove the following:

Theorem 3.1. *Let γ be a non-negative integer. Then $f_\gamma = N_\gamma(3\gamma)$.*

At first, these sequences do not seem to have a closely relation. We also prove an asymptotic result for the sequence f_γ:

Theorem 3.2. *Let $\epsilon > 0$. Then*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0 \quad \text{and} \quad \lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

Finally, we prove two conditional results based on the following conjecture we did on the sequence (f_γ).

Conjecture 3.1. *There is a positive constant C such that $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{n_{2\gamma}} = C$.*

Theorem 3.3. *If Conjecture 3.1 holds, then $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{f_{\gamma-1}} = \varphi^2$.*

Theorem 3.4. *If $\lim_{\gamma\to\infty} \dfrac{f_\gamma}{f_{\gamma-1}} = \varphi^2$, then $\lim_{\gamma\to\infty} \dfrac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i} = \varphi$.*

3.1 A relation with Bras-Amorós' sequence

Bras-Amorós (5) proved that if g and r are integers such that $\dfrac{g+2}{3} \leqslant r \leqslant \left\lfloor \dfrac{g}{2} \right\rfloor$, then the number of numerical semigroups with genus g and ordinarization number r depends only on $\gamma := \left\lfloor \dfrac{g}{2} \right\rfloor - r$. She constructed a sequence (f_γ) that returns this number of numerical semigroups. In this section, we prove that her sequence (f_γ) and our sequence $(N_\gamma(3\gamma))$ are the same. Before that, we recall some definitions.

Definition 3.1. *(5, p. 2514) Let S be a numerical semigroup.*

(1) *A finite set $B \subseteq \mathbb{N}_0$ is called S-*closed *if for $b \in B$ and $s \in S$ we have either $b + s \in B$ or $b + s > \max(B)$.*

(2) *We let $C(S, i)$ denote the collection of S-closed sets B such that $0 \in B$ and $\#B = i$.*

Observe that if $B \in C(S,i)$, then we can write $B = \{0 = b_0 < \cdots < b_{i-1} = \max B\}$.

Proposition 3.1. *Let* $S \in \mathcal{S}_\gamma$. *If* $B \in C(S,\gamma+1)$, *then* $\max B \leqslant 2\gamma$.

Proof. Suppose that $\max B > 2\gamma$. Since $0 \in B$, then, for all $s \in S$, we have $0 + s = s \in B$ or $0 + s = s > 2\gamma$. Hence, $S \cap [0, 2\gamma] \subseteq B \backslash \{\max B\}$. Notice that $\#(B\backslash\{\max B\}) = \gamma$ and $\#(S \cap [0, 2\gamma]) = \gamma + 1$, since $g(S) = \gamma$ and this is a contradiction. □

If γ in a non-negative integer, then f_γ is defined as

$$f_\gamma := \sum_{S \in \mathcal{S}_\gamma} \#C(S, \gamma + 1).$$

Now we are ready to conclude that the sequences coincide.

Theorem 3.1. *Let* γ *be a non-negative integer. Then* $f_\gamma = N_\gamma(3\gamma)$.

Proof. For a fixed γ, let $\mathcal{S}_\gamma = \bigcup_{i=1}^{n_\gamma} T_i$, where each numerical semigroup with genus γ is represented by exactly one T_i. We can write

$$f_\gamma = \#C(T_1, \gamma + 1) + \cdots + \#C(T_{n_\gamma}, \gamma + 1),$$

and, by Remark 2.3,

$$N_\gamma(3\gamma) = \#\mathbf{x}^{-1}(T_1) + \cdots + \#\mathbf{x}^{-1}(T_{n_\gamma}).$$

We prove that $\#C(T_i, \gamma + 1) = \#\mathbf{x}^{-1}(T_i)$, for each $i \in \{1, \ldots, n_\gamma\}$ and the result follows.

Following Bras-Amorós construction, in (5, Theorem 13), define, for each $i \in \{1, \ldots, n_\gamma\}$,

$$\begin{array}{rccl} \varphi_i : & C(T_i, \gamma+1) & \longrightarrow & \mathbf{x}^{-1}(T_i) \\ & B = \{0 = b_0 < \cdots < b_\gamma\} & \longmapsto & 2T_i \cup \{2b - 2b_\gamma + 6\gamma + 1 : b \in B\} \cup (6\gamma + 3 + \mathbb{N}_0). \end{array}$$

Now we prove that φ_i is a bijection.

- φ_i is injective:
  Clearly, for $B_1 \neq B_2$, we have $\varphi_i(B_1) \neq \varphi_i(B_2)$.
- φ_i is well defined:
  - ⋆ $\varphi_i(B)$ has γ even gaps, since T_i has genus γ and all even gaps of $\varphi_i(B)$ lies in $2T_i$.

- ⋆ $\varphi_i(B)$ has 2γ odd gaps:
  First of all, notice that, from Proposition 3.1

$$2b - 2b_\gamma + 6\gamma + 1 \geqslant 2 \cdot 0 - 2 \cdot 2\gamma + 6\gamma + 1 = 2\gamma + 1.$$

  Hence, condition on the first odd integer in $\varphi_i(B)$ is verified (this proves that for $g < 3\gamma$ the construction is not good). There are $3\gamma + 1$ odd numbers in $[1, 6\gamma + 1]$, which $\gamma + 1$ belongs to B. Hence, S has 2γ odd gaps.

- ⋆ $\varphi_i(B)$ is closed under addition:
  We already know that even part of $\varphi_i(B)$ is closed under addition, since T_i is a numerical semigroup. Let $2t$ be an even number in $\varphi_i(B)$ ($t \in T_i$) and $2b - 2b_\gamma + 6\gamma + 1$ and $2b' - 2b_\gamma + 6\gamma + 1$ be odd numbers in S ($b, b' \in B$). Then
  - $a := 2t + (2b - 2b_\gamma + 6\gamma + 1) = 2(t + b) - 2b_\gamma + 6\gamma + 1$. Since $t + b \in B$ or $t + b \geqslant b_\gamma$, we conclude that $a \in \{2b - 2b_\gamma + 6\gamma + 1 : b \in B\} \subset \varphi_i(B)$ or $a \geqslant 6\gamma + 1$ (then $a \in \varphi_i(B)$).
  - $c := (2b - 2b_\gamma + 6\gamma + 1) + (2b' - 2b_\gamma + 6\gamma + 1) = 2(b + b') - 4b_\gamma + 12\gamma + 2 \geqslant 2 \cdot 0 - 8\gamma + 12\gamma + 2 = 4\gamma + 2$. Since T_i has genus γ, then, by Proposition 2.1, $c \in \varphi_i(B)$.

- φ_i is surjective:
  Let $S \in \mathbf{x}^{-1}(T)$. Since the genus of S is 3γ, we conclude that $(6\gamma + 1) + \mathbb{N}_0 \subseteq S$. There are 3γ odd numbers in $[1, 6\gamma - 1]$, which γ are non-gaps and 2γ are gaps. Hence, we can write $S = 2T_i \cup \{O_0 < \ldots < O_\gamma = 6\gamma + 1\} \cup (6\gamma + 3 + \mathbb{N}_0)$. We prove that $S = \varphi_i(B)$, where

$$B = \left\{ \frac{O_i - O_0}{2} : i \in \{0, \ldots, \gamma\} \right\}.$$

  Notice that $\varphi_i(B)$ is equal to

$$\begin{aligned} & 2T_i \cup \left\{ 2 \cdot \frac{O_i - O_0}{2} - 2 \cdot \frac{O_\gamma - O_0}{2} + 6\gamma + 1 : i \in \{0, \ldots, \gamma\} \right\} \cup (6\gamma + 3 + \mathbb{N}_0) \\ = \; & 2T_i \cup \{(O_i - O_0) - (6\gamma + 1 - O_0) + 6\gamma + 1 : i \in \{0, \ldots, \gamma\}\} \cup (6\gamma + 3 + \mathbb{N}_0) \\ = \; & 2T_i \cup \{O_i : i \in \{0, \ldots, \gamma\}\} \cup (6\gamma + 3 + \mathbb{N}_0) = S. \end{aligned}$$

  To finish, we prove that $B \in C(T_i, \gamma + 1)$. In fact, $(O_0 - O_0)/2 = 0 \in B$ and $\#B = \#\{0, \ldots, \gamma\} = \gamma + 1$. Also, B is a T_i-closed set: we prove that $b_k + t \in B$ or $b_k + t > b_\gamma$, where $b_k = (O_k - O_0)/2 \in B$ and $t \in T_i$.

$$\frac{O_k - O_0}{2} + t \in B \Leftrightarrow \exists j \text{ such that } \frac{O_k - O_0}{2} + t = \frac{O_j - O_0}{2} \Leftrightarrow O_k + 2t = O_j$$

  and

$$\frac{O_k - O_0}{2} + t > \frac{O_\gamma - O_0}{2} \Leftrightarrow O_k + 2t > O_\gamma = 6\gamma + 1.$$

Since S is closed under addition, we conclude that one of situations above occur. Hence, B is a T_i-closed set.

□

3.2 Bounds for f_γ

In Section 2.2, we proved that $N_\gamma(g)$ is constant for $g \geqslant 3\gamma$ and it is equal to $N_\gamma(3\gamma)$. In last section, we proved that $f_\gamma = N_\gamma(3\gamma)$, where (f_γ) is the sequence introduced by Bras-Amorós (5). In this section, we are interested in finding bounds for the sequence f_γ (in sense of $N_\gamma(3\gamma)$ definition), specially to estimate its asymptotic behaviour.

We recall that $\mathbf{x} : \mathcal{S}_\gamma(3\gamma) \to \mathcal{S}_\gamma, S \mapsto S/2$, is a surjective map and computing $\#\mathbf{x}^{-1}(T)$, for each T, is enough to calculate f_γ, since

$$f_\gamma = \sum_{T \in \mathcal{S}_\gamma} \#\mathbf{x}^{-1}(T). \tag{3.1}$$

Let $S \in \mathcal{S}_\gamma(3\gamma)$. There are $T \in \mathcal{S}_\gamma$ and $o_\gamma < \cdots < o_1$ odd integers in $[2\gamma + 1, 6\gamma - 1]$ such that

$$S = 2T \cup \{o_\gamma < \cdots < o_1\} \cup S_{6\gamma}\,. \tag{3.2}$$

Question 3.1. *Which sets given by (3.2), with $T \in \mathcal{S}_\gamma$ and $o_\gamma < \cdots < o_1$ odd integers in $[2\gamma + 1, 6\gamma - 1]$, are in fact numerical semigroups?*

Proposition 3.2. *A necessary and sufficient condition on S given in (3.2) so it lies in $\mathcal{S}_\gamma(3\gamma)$ is checking the properties $2t + o_i = o_j$ or $2t + o_i \geqslant 6\gamma + 1$, for all $t \in T$ and $i, j \in \{1, \ldots, \gamma\}$.*

Proof. By construction, $S \subseteq \mathbb{N}_0$, $0 \in S$, $\#(\mathbb{N}_0 \backslash S) = 3\gamma$, $\#(2\mathbb{N}_0 \backslash 2T) = \gamma$, $2T + 2T \subseteq 2T$ and $o_i + o_j \geqslant 2o_\gamma \geqslant 4\gamma + 2$ (thus, $o_i + o_j \in 2T$). Hence, a necessary and sufficient condition so $S \in \mathcal{S}_\gamma(3\gamma)$ is that $2T + \{o_j : j \in \{1, \ldots, \gamma\}\} \subseteq S$ and it is equivalent to $2t + o_i = o_j$ or $2t + o_i \geqslant 6\gamma + 1$, for all $t \in T$ and $i, j \in \{1, \ldots, \gamma\}$. □

Remark 3.1. *In our opinion, Proposition 3.2 is the main connection between the sequences $(N_\gamma(3\gamma))$ and (f_γ).*

If $S \in \mathcal{S}_\gamma(3\gamma)$ given in (3.2), then, by Lemma 2.2, $2\gamma + 1 \leqslant o_\gamma \leqslant 4\gamma + 1$. Hence, we can write $o_\gamma = 2\gamma + 2i + 1$, for some $i \in \{0, \ldots, \gamma\}$. It is natural to define, for each i,

$$\mathbf{x}^{-1}(T^i) := \{S \in \mathbf{x}^{-1}(T) : o_\gamma(S) = 2\gamma + 2i + 1\},$$

and we can rewrite equation (3.1) as

$$f_\gamma = \sum_{T \in \mathcal{S}_\gamma} \sum_{i=0}^{\gamma} \#\mathbf{x}^{-1}(T^i). \tag{3.3}$$

Proposition 3.3. *Let $T \in \mathcal{S}_\gamma$. Then $\#\mathbf{x}^{-1}(T^i) \geqslant 1$, for all $i \in \{0, \ldots, \gamma\}$.*

Proof. For $T \in \mathcal{S}_\gamma$ and $o_\gamma = 2\gamma + 2i + 1$, rewrite the set S as

$$S = 2T \cup \{o_\gamma + 2t : t \in T \text{ and } t \leqslant 2\gamma - i - 1\} \cup X \cup S_{6\gamma}\,.$$

If X has the largest odd integers of the set $[2\gamma+1, 6\gamma-1] \backslash \{o_\gamma + 2t : t \in T \text{ and } t \leqslant 2\gamma - i - 1\}$ such that $\#X + \#\{o_\gamma + 2t : t \in T \text{ and } t \leqslant 2\gamma - i - 1\} = \gamma$, then $S \in \mathbf{x}^{-1}(T^i)$ and the the result follows. □

Corollary 3.1. *Let γ be a non-negative integer. Then $f_\gamma \geqslant (\gamma + 1) \cdot n_\gamma$.*

Proof. By (3.3),

$$f_\gamma \geqslant \sum_{T \in \mathcal{S}_\gamma} \sum_{i=0}^{\gamma} 1 = (\gamma + 1) \cdot n_\gamma\,.$$

□

Remark 3.2. *We have $\#\mathbf{x}^{-1}(T^0) = 1 = \#\mathbf{x}^{-1}(T^\gamma)$, for all $T \in \mathcal{S}_\gamma$.*

1) Let $i = 0$. One can easily see that

$$\{o_\gamma < \cdots < o_1\} = \{2\gamma + 1 + t : t \in T \text{ and } t \leqslant 2\gamma - 1\},$$

since $\#T \cap [0, 2\gamma - 1] = \gamma$.

2) Let $i = \gamma$. One can easily see that

$$\{o_\gamma < \cdots < o_1\} = \{4\gamma + 1, 4\gamma + 3, \ldots, 6\gamma - 1\}.$$

In particular, $f_0 = \sum_{T \in \mathcal{S}_0} 1 = n_0 = 1$ and $f_1 = \sum_{T \in \mathcal{S}_1} 2 = 2n_1 = 2$.

Now we point out other approach to Question 3.1. The interval $[2\gamma + 1, 6\gamma - 1]$ has 2γ odd integers and we can write

$$[2\gamma + 1, 6\gamma - 1] \cap (2\mathbb{Z} + 1) = \{o_\gamma < \cdots < o_1\} \cup \{\omega_1 < \cdots < \omega_\gamma\}.$$

We keep the notation $o_\gamma = 2\gamma + 2i + 1$ and now we assume $1 \leqslant i \leqslant \gamma - 1$. By construction, we conclude that

$$(\omega_1, \ldots, \omega_i) = (2\gamma + 1, \ldots, 2\gamma + 2i - 1).$$

Hence, for each $i \in [1, \gamma - 1] \cap \mathbb{Z}$, we have to look for good choices for odd integers $\{\omega_{i+1}, \cdots, \omega_\gamma\}$ in the following context: given $T \in \mathcal{S}_\gamma$, we have to check which choices for $\{\omega_{i+1} < \ldots < \omega_\gamma\}$ (from the set $\{o_\gamma + 2q : q \in G(T)$ and $q \leqslant 2\gamma - i - 1\}$, by Proposition 3.2) such that

$$[2\gamma+1, 6\gamma-1] \cap (2\mathbb{Z}+1) \backslash \{2\gamma+1 < \cdots < 2\gamma+2i-1 < \omega_{i+1} < \cdots < \omega_\gamma\} = \{o_\gamma < \cdots < o_1\}$$

gives a positive answer to Question 3.1.

At first, it gives an upper bound to $\#\mathbf{x}^{-1}(T^i)$, for all i and T.

Proposition 3.4. *Let $T \in \mathcal{S}_\gamma$. Then $\#\mathbf{x}^{-1}(T^i) \leqslant \binom{\gamma}{i}$, for all $i \in \{0, \ldots, \gamma\}$.*

Proof. If $1 \leqslant i \leqslant \gamma - 1$, then $\#\{o_\gamma + 2q : q \in G(T)$ and $q \leqslant 2\gamma - i - 1\} \leqslant \#\{o_\gamma + 2q : q \in G(T)\} = \gamma$. Thus, there are, at most, $\binom{\gamma}{\gamma - i}$ possibilities for choosing $\omega_j' s$. The symmetry of binomial coefficient and Remark 3.2 complete the proof. ☐

Corollary 3.2. *Let γ be a non-negative integer. Then $f_\gamma \leqslant 2^\gamma \cdot n_\gamma$.*

Proof. By (3.3),

$$f_\gamma \leqslant \sum_{T \in \mathcal{S}_\gamma} \sum_{i=0}^{\gamma} \binom{\gamma}{i} = 2^\gamma \cdot n_\gamma \,.$$

☐

Putting together Corollaries 3.1 and 3.2, we obtain

Proposition 3.5. *Let γ be a non-negative integer. Then*

$$n_\gamma \cdot (\gamma + 1) \leqslant f_\gamma \leqslant n_\gamma \cdot 2^\gamma.$$

Corollary 3.3. *Let φ be the golden ratio. Then*

1) $\lim_{\gamma \to \infty} \dfrac{f_\gamma}{\varphi^\gamma} = \infty$;

2) for $\epsilon > 0$, $\lim_{\gamma \to \infty} \dfrac{f_\gamma}{(2\varphi + \epsilon)^\gamma} = 0$.

Proof. We recall that (29, Theorem 1), $\lim_{\gamma \to \infty} \dfrac{n_\gamma}{\varphi^\gamma}$ is a real number μ.

1) By Proposition 3.5,

$$\frac{f_\gamma}{\varphi^\gamma} \geqslant \frac{n_\gamma}{\varphi^\gamma} \cdot (\gamma + 1).$$

The right-hand of this inequality goes to ∞ as γ goes to ∞, so result follows.

2) By Proposition 3.5, for each $\epsilon > 0$,

$$\frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} \leqslant \frac{n_\gamma}{\varphi^\gamma}\cdot\frac{\varphi^\gamma\cdot 2^\gamma}{(2\varphi+\epsilon)^\gamma} = \frac{n_\gamma}{\varphi^\gamma}\cdot\left(\frac{2\varphi}{2\varphi+\epsilon}\right)^\gamma.$$

The right-hand of this inequality goes to 0 as γ goes to ∞, so result follows.

□

Remark 3.3. *We can make a similar procedure to compute $N_\gamma(g)$, for any g such that $2g \geqslant 3\gamma$. In fact, consider $\mathbf{x}_g : \mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma, S \mapsto S/2$. We know that calculating $\#\mathbf{x}^{-1}(T)$, for each $T \in \mathcal{S}_g$, is enough to calculate $N_\gamma(g)$, since*

$$N_\gamma(g) = \sum_{T\in\mathcal{S}_\gamma} \#\mathbf{x}_g^{-1}(T).$$

Given $S \in \mathcal{S}_\gamma(g)$, there are $T \in \mathcal{S}_\gamma$ and $o_\gamma < \cdots < o_1$ odd integers in $[2g-4\gamma+1, 2g-2\gamma-1]$ such that

$$S = 2T \cup \{o_\gamma < \ldots < o_1\} \cup S_{2g}.$$

Notice that if $g \geqslant 3\gamma$, then $2g-4\gamma+1 \geqslant 2\gamma+1$. Then all possible o_i's are greater than or equal to $2\gamma+1$. Hence, all elements of the set $\{o_i : i \in \{1,\ldots,\gamma\}\}$ satisfy $o_i+o_j \geqslant 4\gamma+2$, hence $o_i+o_j \in 2T$. It gives us a way to see what is t-translation, in Definition 2.2. If $g < 3\gamma$, we can have situation such as $o_i \leqslant 2\gamma-1$ and $o_i+o_j \notin 2T$. It gives us a way to see that not all t-translations work well in this case.

Remark 3.4. *Forgetting for a moment our results from Section 2.1, we observe that, with this construction, Proposition 3.5 can be extended for all $g \geqslant 3\gamma$. In particular, inequality $N_\gamma(g) \leqslant n_\gamma\cdot 2^\gamma$ still holds for all $g \geqslant 3\gamma$. Notice that upper bound does not depend on g. It implies that for a fixed γ, $N_\gamma(g)$ is an integer and a bounded sequence. By using Bolzano-Weierstrass theorem, we conclude that there is an integer sequence $\{g_n\}$ such that $N_\gamma(g_i) = N_\gamma(g_j)$. It could be a previous evidence for Theorem 2.1.*

In general, we have a self-inclusion $\{\omega_{i+1} < \cdots < \omega_\gamma\} \subset \{o_\gamma+2q : q \in G(T) \text{ and } q \leqslant 2\gamma-i-1\}$. In these cases, choosing the numbers $\omega_{i+1} < \cdots < \omega_\gamma$ implies, automatically, in the existence of $q \in G(T)$ and $q \leqslant 2\gamma-i-1$ such that $o_j = o_\gamma+2q$. Depending on the index i and the numerical semigroup T, it is possible that exist more than one q in this situation.

Thus, we are again concerned with closed under addition condition. We have to check, among the elements of the form $o_\gamma+2q$, with $q \in G(T)$ and $q \leqslant 2\gamma-i-1$, for which of them there are $t \in T$ and $\tilde{q} \in G(T)$ and $\tilde{q} \leqslant 2\gamma-i-1$ such that $o_\gamma+2q+2t = o_\gamma+2\tilde{q}$? Last equality is equivalent to $t = \tilde{q}-q$ and implies the following

Proposition 3.6. *Let $T \in \mathcal{S}_\gamma$ and $q \in G(T)$, with $q \leqslant 2\gamma - i - 1$. If there are $t \in T$ and $\tilde{q} \in G(T)$, with $\tilde{q} \leqslant 2\gamma - i - 1$, such that $\tilde{q} - q = t$, then*

$$o_\gamma + 2\tilde{q} \in \{\omega_{i+1}, \ldots, \omega_\gamma\} \text{ implies that } o_\gamma + 2q \in \{\omega_{i+1}, \ldots, \omega_\gamma\}.$$

Hence, it is important to compute all positive differences of elements in $G(T)$. We introduce a graph related to T. The vertex of this graph are the elements of $G(T)$. There is an edge connecting two vertex if, an only if, the difference between the numbers represented by its vertex lies in T.

Example 3.1. *Consider $T = \mathbb{N}_0 \backslash \{1, 2, 3, 6\} \in \mathcal{S}_4$. The graph related to T is the following*

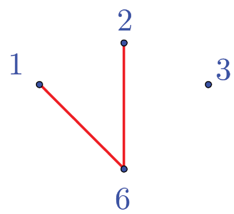


Figure 3 – Graph related to $T = \mathbb{N}_0 \backslash \{1, 2, 3, 6\}$

We want to construct all numerical semigroups of the form

$$S = 2T \cup \{o_4 < o_3 < o_2 < o_1\} \cup S_{24}.$$

Observe that $2\gamma + 2i + 1 = 9 + 2i$, $2\gamma - i - 1 = 7 - i$ and

$(i = 1)$: $\{11 + 2q : q \in G(T) \text{ and } q \leqslant 6\} = \{13, 15, 17, 23\}$,

$(i = 2)$: $\{13 + 2q : q \in G(T) \text{ and } q \leqslant 5\} = \{15, 17, 19\}$,

$(i = 3)$: $\{15 + 2q : q \in G(T) \text{ and } q \leqslant 4\} = \{17, 19, 21\}$.

In $(i = 1)$, we have to choose $\{\omega_2 < \omega_3 < \omega_4\}$ from $\{13, 15, 17, 23\}$ with the following restriction: if $23 \in \{\omega_2 < \omega_3 < \omega_4\}$, then 13 and $15 \in \{\omega_2 < \omega_3 < \omega_4\}$. Hence, if $\omega_4 = 23$, then $o_2 = 13$ and $o_3 = 15$. If $\omega_4 \neq 23$, then $o_4 = 17$, $o_3 = 15$ and $o_2 = 13$. Thus, $\#\mathbf{x}^{-1}(T^1) = 2$.

In $(i = 2)$, we have to choose $\{\omega_3 < \omega_4\}$ among $\{15, 17, 19\}$ with no restriction. Thus, $\#\mathbf{x}^{-1}(T^2) = \binom{3}{2} = 3$.

In $(i = 3)$, we have to choose $\{\omega_4\}$ among $\{17, 19, 21\}$ with no restriction. Thus, $\#\mathbf{x}^{-1}(T^3) = \binom{3}{1} = 3$.

Hence, we conclude that $\#\mathbf{x}^{-1}(T) = \sum_{i=0}^{4} \#\mathbf{x}^{-1}(T^i) = 1 + 2 + 3 + 3 + 1 = 10$.

Following same idea in Example 3.1, we compute f_2.

Example 3.2. *($\gamma = 2$ and $g = 6$)*
Numerical semigroups with genus 2 are $\langle 3,4,5\rangle = \mathbb{N}_0\backslash\{1,2\}$ *and* $\langle 2,5\rangle = \mathbb{N}_0\backslash\{1,3\}$. *Observe that* $2\gamma + 2i + 1 = 5 + 2i$, $2\gamma - i - 1 = 3 - i$.

The graph related to $A_1 := \mathbb{N}_0\backslash\{1,2\}$ *is the following*

Figure 4 – Graph related to $A_1 = \mathbb{N}_0\backslash\{1,2\}$

Now, we compute $\#\mathbf{x}^{-1}(A_1^1)$. *Observe that* $\{7 + 2q : q \in G(T)$ *and* $q \leqslant 2\} = \{9, 11\}$. *Hence, we have to choose* $\{\omega_2\}$ *among* $\{9, 11\}$ *with no restriction. Thus,* $\#\mathbf{x}^{-1}(A_1^1) = \binom{2}{1} = 2$ *and we obtain* $\#\mathbf{x}^{-1}(A_1) = \sum_{i=0}^{2} \#\mathbf{x}^{-1}(A_1^i) = 1 + 2 + 1 = 4$.

The graph related to $A_2 := \mathbb{N}_0\backslash\{1,3\}$ *is the following*

Figure 5 – Graph related to $A_2 = \mathbb{N}_0\backslash\{1,3\}$

Now, we compute $\#\mathbf{x}^{-1}(A_2^1)$. *Observe that* $\{7+2q : q \in G(T)$ *and* $q \leqslant 2\} = \{9\}$. *Hence, we have to choose* $\{\omega_2\}$ *among* $\{9\}$ *with no restriction. Thus,* $\#\mathbf{x}^{-1}(A_2^1) = \binom{1}{1} = 1$ *and we obtain* $\#\mathbf{x}^{-1}(A_2) = \sum_{i=0}^{2} \#\mathbf{x}^{-1}(A_2^i) = 1 + 1 + 1 = 3$.

Thus, we conclude that $f_2 = \#\mathbf{x}^{-1}(A_1) + \#\mathbf{x}^{-1}(A_2) = 4 + 3 = 7$.

With a similar process for other values of γ we obtain the following table:

γ	$n_\gamma \cdot (\gamma+1)$	f_γ	$n_\gamma \cdot 2^\gamma$
0	1	1	1
1	2	2	2
2	6	7	8
3	16	23	32
4	35	68	112
5	72	200	384
6	161	615	1472
7	312	1764	4992
8	603	5060	17152

Table 4 – First bounds for f_γ

First values of γ show us that the bounds obtained in Proposition 3.5 are far from f_γ, so it is of interest getting better bounds.

For $\gamma \geqslant 2$, we define, for each $k \in \{0, \ldots, \gamma-1\}$,

$$T_k = \mathbb{N}_0 \backslash \{1, \ldots, \gamma-1, \gamma+k\} \in \mathcal{S}_\gamma.$$

For a fixed k, we are interested in finding out which sets of the form

$$S = 2T_k \cup \{o_\gamma < \cdots < o_1\} \cup S_{6\gamma}$$

with $o_\gamma < \ldots < o_1$ odd integers in $[2\gamma+1, 6\gamma-1]$ are numerical semigroups, i.e., which of them returns a positive answers to Question 3.1, with $T = T_k$.

As usually, we write $o_\gamma = 2\gamma + 2i + 1$, for some $i \in \{0, \ldots, \gamma\}$. With same procedure we did, we know that it is equivalent to find out, for a fixed T_k, which choices for $\{\omega_{i+1} < \cdots < \omega_\gamma\}$ from the set $\{o_\gamma + 2q : q \in G(T_k) \text{ and } q \leqslant 2\gamma - i - 1\}$ such that $[2\gamma+1, 6\gamma-1] \cap (2\mathbb{Z}+1) \backslash \{2\gamma+1 < \cdots < 2\gamma+2i-1 < \omega_{i+1} < \cdots < \omega_\gamma\} = \{o_\gamma < \cdots < o_1\}$ gives a positive answer to Question 3.1.

For all $i \in \{0, \ldots, \gamma\}$, one has

$$\{o_\gamma+2, \ldots, o_\gamma+2\gamma-2, o_\gamma+2\gamma+2k\} \supseteq \{o_\gamma+2q : q \in G(T_k) \text{ and } q \leqslant 2\gamma-i-1\}$$

and

$$\{o_\gamma+2q : q \in G(T_k) \text{ and } q \leqslant 2\gamma-i-1\} \supseteq \{o_\gamma+2, \ldots, o_\gamma+2\gamma-2\}.$$

Observe that

1. If $\gamma + k \leqslant 2\gamma - i - 1$, i.e., $i \leqslant \gamma - k - 1$, then

$$\{o_\gamma+2q : q \in G(T_k) \text{ and } q \leqslant 2\gamma-i-1\} = \underbrace{\{o_\gamma+2, \ldots, o_\gamma+2\gamma-2, o_\gamma+2\gamma+2k\}}_{\gamma \text{ elements}}.$$

2. Otherwise, i.e., $i > \gamma - k - 1$,

$$\{o_\gamma + 2q : q \in G(T_k) \text{ and } q \leqslant 2\gamma - i - 1\} = \underbrace{\{o_\gamma + 2, \ldots, o_\gamma + 2\gamma - 2\}}_{\gamma-1 \text{ elements}}.$$

The graph associated to T_k is the following:

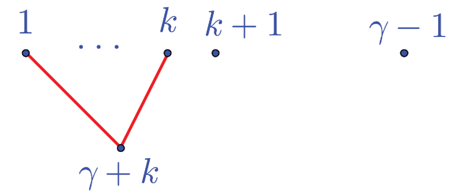


Figure 6 – Graph related to $T_k = \mathbb{N}_0 \backslash \{1, \ldots, \gamma - 1, \gamma + k\}$

By using this graph, we conclude that if $i \leqslant \gamma - k - 1$, then

$$\#\mathbf{x}^{-1}(T_k^i) = \binom{1}{0} \cdot \binom{\gamma - 1}{\gamma - i} + \binom{1}{1} \cdot \binom{\gamma - 1 - k}{\gamma - i - k - 1}.$$

Here, $\binom{1}{0}$ means that $\gamma + k \notin \{\omega_{i+1}, \ldots, \omega_\gamma\}$ and $\binom{1}{1}$ means that $\gamma + k \in \{\omega_{i+1}, \ldots, \omega_\gamma\}$. The graph tells us that there is no restriction in the first case and it tells us that $\{1, \ldots, k, \gamma + k\} \subset \{\omega_{i+1}, \ldots, \omega_\gamma\}$ in the second case.

If $i > \gamma - k - 1$, then

$$\#\mathbf{x}^{-1}(T_k^i) = \binom{\gamma - 1}{\gamma - i}.$$

Thus, we have

Proposition 3.7. *For all* $k \in \{0, \ldots, \gamma - 1\}$,

$$\#\mathbf{x}^{-1}(T_k) = \sum_{i=0}^{\gamma} \#\mathbf{x}^{-1}(T_k^i) = 2^{\gamma-1} \cdot \left(1 + \frac{1}{2^k}\right).$$

Proof. By last computations, we have

$$\begin{aligned}
\#\mathbf{x}^{-1}(T_k) &= \sum_{i=0}^{\gamma} \#\mathbf{x}^{-1}(T_k^i) \\
&= 2 + \sum_{i=1}^{\gamma-k-1} \left[\binom{\gamma-1}{\gamma-i} + \binom{\gamma-1-k}{\gamma-i-k-1} \right] + \sum_{i=\gamma-k}^{\gamma-1} \binom{\gamma-1}{\gamma-i} \\
&= 2 + \sum_{i=1}^{\gamma-1} \binom{\gamma-1}{\gamma-i} + \sum_{i=1}^{\gamma-k-1} \binom{\gamma-1-k}{\gamma-i-k-1} \\
&= 2 + \sum_{i=1}^{\gamma-1} \binom{\gamma-1}{i-1} + \sum_{i=1}^{\gamma-k-1} \binom{\gamma-1-k}{i} \\
&= 2 + (2^{\gamma-1} - 1) + (2^{\gamma-1-k} - 1) \\
&= 2^{\gamma-1} \cdot \left(1 + \frac{1}{2^k}\right).
\end{aligned}$$

□

Corollary 3.4. *Let* $M_\gamma := 2^\gamma \left(1 + \frac{\gamma}{2}\right) - 1$. *Then*

$$M_\gamma + (n_\gamma - \gamma)(\gamma + 1) \leqslant f_\gamma \leqslant M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma$$

Proof. It follows from the equality $M_\gamma = \sum_{k=0}^{\gamma-1} \#\mathbf{x}^{-1}(T_k)$ and Proposition 3.5. □

With those bounds, we construct the following table:

γ	$M_\gamma + (n_\gamma - \gamma) \cdot (\gamma + 1)$	f_γ	$M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma$
0	1	1	1
1	2	2	2
2	7	7	7
3	23	23	27
4	62	68	95
5	153	200	266
6	374	615	1343
7	831	1764	4671
8	1810	5060	16383

Table 5 – Better bounds for f_γ

For $\gamma \geqslant 3$, we can make a similar procedure with the numerical semigroups

$$U_k = \mathbb{N}_0 \backslash \{1, \ldots, \gamma - 2, \gamma, \gamma + k\} \in \mathcal{S}_\gamma,$$

for $k \in \{1, \ldots, \gamma - 1\} \backslash \{\gamma - 2\}$.

It is possible to show that, in this case,

$$\sum_{k=1}^{\gamma-3} \#\mathbf{x}^{-1}(U_k) + \#\mathbf{x}^{-1}(U_{\gamma-1}) = 2^\gamma \left(\frac{3\gamma}{8} - \frac{5}{16}\right) - 3.$$

However, asymptotic behaviour of this function is still of type $\gamma \cdot 2^\gamma$ and we cannot get significantly better bounds.

Nevertheless, we can use Corollary 3.4 to prove the following:

Corollary 3.5.

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

Proof. From last Corollary, we have $f_\gamma \geqslant M_\gamma + (n_\gamma - \gamma)\cdot(\gamma+1)$. Then,

$$\begin{aligned} \frac{f_\gamma}{2^\gamma} &\geqslant \left(\frac{\gamma}{2}+1\right) - \frac{1}{2^\gamma} + (\gamma+1)\cdot\frac{n_\gamma}{2^\gamma} - \frac{\gamma^2+\gamma}{2^\gamma} \\ &\geqslant \left(\frac{\gamma}{2}+1\right) - \frac{1}{2^\gamma} - \frac{\gamma^2+\gamma}{2^\gamma}. \end{aligned}$$

Since the right-hand of this inequality goes to ∞ as γ goes to ∞, we conclude that

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

□

Corollaries 3.3 and 3.5 can be put together and we have the following result about the asymptotic behaviour of f_γ:

Theorem 3.2. *Let* $\epsilon > 0$. *Then*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0 \quad \textit{and} \quad \lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

This result ensures that if f_γ grows exponentially, then it grows like β^γ, for some $\beta \in (2, 2\varphi]$. We construct a table with a few values of f_γ and $f_\gamma/f_{\gamma-1}$:

γ	f_γ	$f_\gamma/f_{\gamma-1}$
0	1	
1	2	2.00
2	7	3.50
3	23	3.29
4	68	2.96
5	200	2.94
6	615	3.08
7	1764	2.87
8	5060	2.87
9	14626	2.89
10	41785	2.86
11	117573	2.81
12	332475	2.83
13	933891	2.81
14	2609832	2.79

Table 6 – A few values of f_γ and $f_\gamma/f_{\gamma-1}$

By using those values, we suspect that $\lim_{\gamma\to\infty} \frac{f_\gamma}{f_{\gamma-1}} = \varphi^2 \approx 2.618$ and this is a previous evidence that $\beta = \varphi^2$. Next table shows us that first values of f_γ are closely to $n_{2\gamma}$. We recall that from Zhai's Theorem, the sequence $(n_{2\gamma})$ grows like $\varphi^{2\gamma}$.

γ	f_γ	$n_{2\gamma}$	$f_\gamma/n_{2\gamma}$
0	1	1	1.00
1	2	2	1.00
2	7	7	1.00
3	23	23	1.00
4	68	67	1.01
5	200	204	0.98
6	615	592	1.04
7	1764	1693	1.04
8	5060	4806	1.05
9	14626	13467	1.09
10	41785	37396	1.12
11	117573	103246	1.14
12	332475	282828	1.18
13	933891	770832	1.21
14	2609832	2091030	1.25

Table 7 – A few values of $f_\gamma, n_{2\gamma}$ and $f_\gamma/n_{2\gamma}$

By using Table 7, we conjecture that the sequence f_γ grows like $n_{2\gamma}$. More precisely:

Conjecture 3.1. *There is a positive constant C such that*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{n_{2\gamma}} = C.$$

Proposition 3.8. *Conjecture 3.1 is equivalent to the following: there is a positive constant $\tilde{C}$ such that*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{\varphi^{2\gamma}} = \tilde{C}.$$

Proof. On one hand, we write $\frac{f_\gamma}{\varphi^{2\gamma}} = \frac{f_\gamma}{n_{2\gamma}} \cdot \frac{n_{2\gamma}}{\varphi^{2\gamma}}$. On the other hand, $\frac{f_\gamma}{n_{2\gamma}} = \frac{f_\gamma}{\varphi^{2\gamma}} \cdot \frac{\varphi^{2\gamma}}{n_{2\gamma}}$. Now, we use Zhai's Theorem and the proof follows. □

3.3 Further results

In Section 3.2, we conjectured an item about the asymptotic behaviour of f_γ. In this section, we prove some conditional results based on that conjecture. Those suspecting get stronger when we look at the following table:

γ	f_γ	$n_{2\gamma}$	$f_\gamma/f_{\gamma-1}$	$f_\gamma/n_{2\gamma}$	$f_{\gamma+1}/\sum_{i=0}^{\gamma} f_i$
0	1	1		1.00	2.00
1	2	2	2.00	1.00	2.33
2	7	7	3.50	1.00	2.30
3	23	23	3.29	1.00	2.06
4	68	67	2.96	1.01	1.98
5	200	204	2.94	0.98	2.04
6	615	592	3.08	1.04	1.93
7	1764	1693	2.87	1.04	1.89
8	5060	4806	2.87	1.05	1.89
9	14626	13467	2.89	1.09	1.87
10	41785	37396	2.86	1.12	1.83
11	117573	103246	2.81	1.14	1.83
12	332475	282828	2.83	1.18	1.82
13	933891	770832	2.81	1.21	1.80
14	2609832	2091030	2.79	1.25	

Table 8 – A few values of $f_\gamma, n_{2\gamma}, f_\gamma/f_{\gamma-1}, f_\gamma/n_{2\gamma}$ and $f_{\gamma+1}/\sum_{i=0}^{\gamma} f_i$

Theorem 3.3. *If Conjecture 3.1 holds, then* $\lim_{\gamma\to\infty} \frac{f_\gamma}{f_{\gamma-1}} = \varphi^2$.

Proof. We can rewrite $\frac{f_\gamma}{f_{\gamma-1}} = \frac{f_\gamma}{n_{2\gamma}} \cdot \frac{n_{2\gamma}}{n_{2\gamma-1}} \cdot \frac{n_{2\gamma-1}}{n_{2\gamma-2}} \cdot \frac{n_{2(\gamma-1)}}{f_{\gamma-1}}$. Then

$$\lim_{\gamma\to\infty}\frac{f_\gamma}{f_{\gamma-1}} = \lim_{\gamma\to\infty}\frac{f_\gamma}{n_{2\gamma}}\cdot\lim_{\gamma\to\infty}\frac{n_{2\gamma}}{n_{2\gamma-1}}\cdot\lim_{\gamma\to\infty}\frac{n_{2\gamma-1}}{n_{2\gamma-2}}\cdot\lim_{\gamma\to\infty}\frac{n_{2(\gamma-1)}}{f_{\gamma-1}},$$

since all limits involved exist. Hence,

$$\lim_{\gamma\to\infty}\frac{f_\gamma}{f_{\gamma-1}} = C\cdot\varphi\cdot\varphi\cdot\frac{1}{C} = \varphi^2.$$

□

Theorem 3.4. *If* $\lim_{\gamma\to\infty}\frac{f_\gamma}{f_{\gamma-1}} = \varphi^2$*, then* $\lim_{\gamma\to\infty}\frac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i} = \varphi$*.*

Proof. First of all, notice that $\frac{1}{\varphi^2}\approx 0.382$. Since $\lim_{\gamma\to\infty}\frac{f_\gamma}{f_{\gamma+1}} = \frac{1}{\varphi^2}$, let, for each $\epsilon\in\left(0,\frac{1}{3}\right)$,

$$\gamma_0(\epsilon) := \min\left\{i\in\mathbb{N} : \frac{1}{\varphi^2}-\epsilon < \frac{f_j}{f_{j+1}} < \frac{1}{\varphi^2}+\epsilon, \forall j\geqslant i\right\}.$$

For $\gamma > \gamma_0(\epsilon) = \gamma_0$,

$$\begin{aligned}
\frac{f_0+\cdots+f_{\gamma_0-1}+f_{\gamma_0}+\cdots+f_\gamma}{f_{\gamma+1}} &= \frac{f_0+\cdots+f_{\gamma_0-1}}{f_{\gamma+1}}+\frac{f_{\gamma_0}}{f_{\gamma+1}}+\frac{f_{\gamma_0+1}}{f_{\gamma+1}}+\cdots+\frac{f_\gamma}{f_{\gamma+1}}\\
&= \frac{f_0+\cdots+f_{\gamma_0-1}}{f_{\gamma+1}}+\frac{f_{\gamma_0}}{f_{\gamma_0+1}}\cdot\frac{f_{\gamma_0+1}}{f_{\gamma_0+2}}\cdot\ldots\cdot\frac{f_\gamma}{f_{\gamma+1}}+\\
&\quad+\frac{f_{\gamma_0+1}}{f_{\gamma_0+2}}\cdot\frac{f_{\gamma_0+2}}{f_{\gamma_0+3}}\cdot\ldots\cdot\frac{f_\gamma}{f_{\gamma+1}}+\cdots+\frac{f_\gamma}{f_{\gamma+1}}\\
&< \frac{f_0+\cdots+f_{\gamma_0-1}}{f_{\gamma+1}}+\left(\frac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0+1}+\\
&\quad+\left(\frac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0}+\cdots+\left(\frac{1}{\varphi^2}+\epsilon\right)
\end{aligned}$$

For each $\epsilon\in\left(0,\frac{1}{3}\right)$, $f_0+\cdots+f_{\gamma_0(\epsilon)-1}$ is a finite number. Also, by hypothesis, (f_γ) is an increasing sequence, for large enough γ. If the sequence is bounded, then it converges to a number. Thus $\lim_{\gamma\to\infty}\frac{f_\gamma}{f_{\gamma-1}} = 1$, which is a contradiction. Hence, the sequence is not bounded and we conclude that $\lim_{\gamma\to\infty} f_\gamma = \infty$ and

$$\lim_{\gamma\to\infty}\frac{f_0+\cdots+f_{\gamma_0(\epsilon)-1}}{f_{\gamma+1}} = 0.$$

Notice that

$$\lim_{\gamma\to\infty}\left[\left(\frac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0+1}+\left(\frac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0}+\cdots+\left(\frac{1}{\varphi^2}+\epsilon\right)\right]$$

is the geometric series with radius $\dfrac{1}{\varphi^2}+\epsilon$ that lies in $(0.38, 0.72)$. Thus, this value is

$$\frac{\frac{1}{\varphi^2}+\epsilon}{1-\left(\frac{1}{\varphi^2}+\epsilon\right)} = \frac{1+\epsilon\varphi^2}{\varphi^2-(1+\epsilon\varphi^2)}.$$

Hence, if $A(\epsilon,\gamma) := \dfrac{f_0+\cdots+f_{\gamma_0-1}}{f_{\gamma+1}} + \left(\dfrac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0+1} + \left(\dfrac{1}{\varphi^2}+\epsilon\right)^{\gamma-\gamma_0} + \cdots + \left(\dfrac{1}{\varphi^2}+\epsilon\right)$, then, for each $\epsilon \in \left(0, \dfrac{1}{3}\right)$,

$$\frac{f_0+\cdots+f_{\gamma_0(\epsilon)-1}+f_{\gamma_0(\epsilon)}+\cdots+f_\gamma}{f_{\gamma+1}} < A(\epsilon,\gamma) \xrightarrow{\gamma\to\infty} \frac{1+\epsilon\varphi^2}{\varphi^2-(1+\epsilon\varphi^2)}.$$

By making similar process, we conclude that

$$\begin{aligned}\frac{f_0+\cdots+f_{\gamma_0(\epsilon)-1}+f_{\gamma_0(\epsilon)}+\cdots+f_\gamma}{f_{\gamma+1}} &> \frac{f_0+\cdots+f_{\gamma_0(\epsilon)-1}}{f_{\gamma+1}} + \left(\frac{1}{\varphi^2}-\epsilon\right)^{\gamma-\gamma_0+1} + \\ &\quad + \left(\frac{1}{\varphi^2}-\epsilon\right)^{\gamma-\gamma_0} + \cdots + \left(\frac{1}{\varphi^2}-\epsilon\right) \\ &\xrightarrow{\gamma\to\infty} \frac{1-\epsilon\varphi^2}{\varphi^2-(1-\epsilon\varphi^2)}.\end{aligned}$$

Observe that, in this case, the geometric series radius is $\dfrac{1}{\varphi^2}-\epsilon$ that lies in $(0.04, 0.39)$.

Hence, for all $\epsilon \in \left(0, \dfrac{1}{3}\right)$

$$\frac{1-\epsilon\varphi^2}{\varphi^2-(1-\epsilon\varphi^2)} \leqslant \lim_{\gamma\to\infty} \frac{f_0+\cdots+f_\gamma}{f_{\gamma+1}} \leqslant \frac{1+\epsilon\varphi^2}{\varphi^2-(1+\epsilon\varphi^2)}.$$

By making $\epsilon \to 0$, we conclude that

$$\lim_{\gamma\to\infty} \frac{\sum_{i=0}^{\gamma} f_i}{f_{\gamma+1}} = \frac{1}{\varphi^2-1} = \frac{1}{\varphi},$$

thus

$$\lim_{\gamma\to\infty} \frac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i} = \varphi,$$

completing the proof. □

4 Fibonacci like and not like behaviour of (n_g)

Let (F_n) be the Fibonacci sequence, i.e., $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for all $n \in \mathbb{N}_0$. The first few numbers in this sequence are $(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots)$.

We recall that Bras-Amorós (3) conjectured two items about asymptotic behaviour of (n_g) (being them similar to Fibonacci sequence) and Zhai (29) proved that she was right.

In this chapter, we prove some other asymptotic properties of (n_g) sequence that are the same as Fibonacci sequence. We also prove an asymptotic property of (n_g) that is different from Fibonacci sequence.

4.1 Fibonacci like behaviour of (n_g)

In this section, we prove analogous asymptotic properties of sequence (n_g) that are similar to Fibonacci sequence.

Proprieties. *Let (F_n) be the Fibonacci sequence. Then*

(1) $F_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$ *(Binet's formula).*

(2) $F_n^2 = F_{n+1}F_{n-1} + (-1)^n$.

(3) $F_0 + F_1 + \cdots + F_{n-1} = F_{n+1} - 1$.

(4) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

(5) $F_0 + F_2 + \cdots + F_{2n} = F_{2n+1}$.

Corollary 4.1. *Let (F_n) be the Fibonacci sequence. Then*

(1) $\lim_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \varphi$ *and* $\lim_{n\to\infty} \dfrac{F_{n-1} + F_n}{F_{n+1}} = 1.$

(2) $\lim_{n\to\infty} \dfrac{F_n^2}{F_{n+1}F_{n-1}} = 1.$

(3) $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n} F_i}{F_{n+1}} = \varphi$ *and* $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n-1} F_i}{F_{n+1}} = 1.$

(4) $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n} F_i^2}{F_n^2} = \varphi$ *and* $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n} F_i^2}{F_{n+1}^2} = \dfrac{1}{\varphi}.$

(5) $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n} F_{2i}}{F_{2n}} = \varphi$ *and* $\lim_{n\to\infty} \dfrac{\sum_{i=0}^{n} F_{2i}}{F_{2n+1}} = 1.$

Theorem 4.1. *Let* (n_g) *be the sequence of number of numerical semigroups with genus* g*. Then*

(1) $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi$ *and* $\lim_{g\to\infty} \frac{n_{g-1}+n_g}{n_{g+1}} = 1.$

(2) $\lim_{g\to\infty} \frac{n_g^2}{n_{g+1}n_{g-1}} = 1.$

(3) $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_i}{n_{g+1}} = \varphi.$ *and* $\lim_{g\to\infty} \frac{\sum_{i=0}^{g-1} n_i}{n_{g+1}} = 1.$

(4) $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_i^2}{n_g^2} = \varphi$ *and* $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_i^2}{n_{g+1}^2} = \frac{1}{\varphi}.$

(5) $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g}} = \varphi$ *and* $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g+1}} = 1.$

Item (1) is a consequence of Zhai's Theorem, noticed by him. Item (2) is a consequence of (1). For items (3), (4) and (5), we use same ideas we used in proof of Theorem 3.4.

Proof. (of item (2)) We only have to observe that

$$\frac{n_g^2}{n_{g+1}n_{g-1}} = \underbrace{\frac{n_g}{n_{g+1}}}_{\to\frac{1}{\varphi}} \cdot \underbrace{\frac{n_g}{n_{g-1}}}_{\to\varphi} \xrightarrow{g\to\infty} 1.$$

□

Proof. (of item (3)) We enumerate the steps of the proof:

I. $\lim_{g\to\infty} \frac{n_g}{n_{g+1}} = \frac{1}{\varphi} \approx 0.62.$

II. For each $\epsilon \in \left(0, \frac{1}{3}\right)$ define $g_0(\epsilon) := \min\left\{i \in \mathbb{N} : \frac{1}{\varphi} - \epsilon < \frac{n_j}{n_{j+1}} < \frac{1}{\varphi} + \epsilon, \forall j \geqslant i\right\}.$

III. For $g \gg 0$, (n_g) is increasing. Since $\lim_{g\to\infty} \frac{n_g}{n_{g+1}} \neq 1$, then $\lim_{g\to\infty} n_g = \infty$.

IV. For $g > g_0(\epsilon) = g_0$,

$$\begin{aligned}\frac{n_0+\cdots+n_{g_0-1}+n_{g_0}+\cdots+n_g}{n_{g+1}} &< \frac{n_0+\cdots+n_{g_0-1}}{n_{g+1}} + \left(\frac{1}{\varphi}+\epsilon\right)^{g-g_0+1} + \\ &\quad + \left(\frac{1}{\varphi}+\epsilon\right)^{g-g_0} + \cdots + \left(\frac{1}{\varphi}+\epsilon\right) \\ &\xrightarrow{g\to\infty} \frac{1+\epsilon\varphi}{\varphi-(1+\epsilon\varphi^2)}.\end{aligned}$$

and

$$\begin{aligned}\frac{n_0+\cdots+n_{g_0-1}+n_{g_0}+\cdots+n_g}{n_{g+1}} &> \frac{n_0+\cdots+n_{g_0-1}}{n_{g+1}}+\left(\frac{1}{\varphi}-\epsilon\right)^{g-g_0+1}+\\ &\quad+\left(\frac{1}{\varphi}-\epsilon\right)^{g-g_0}+\cdots+\left(\frac{1}{\varphi}-\epsilon\right)\\ &\xrightarrow{g\to\infty} \frac{1-\epsilon\varphi}{\varphi-(1-\epsilon\varphi^2)}.\end{aligned}$$

V. Hence, for all $\epsilon\in\left(0,\frac{1}{3}\right)$,

$$\frac{1-\epsilon\varphi}{\varphi-(1-\epsilon\varphi)} \leqslant \lim_{g\to\infty}\frac{n_0+\cdots+n_g}{n_{g+1}} \leqslant \frac{1+\epsilon\varphi}{\varphi-(1+\epsilon\varphi)}.$$

VI. By making $\epsilon\to 0$, we conclude that $\lim_{g\to\infty}\frac{\sum_{i=0}^{g} n_i}{n_{g+1}}=\frac{1}{\varphi-1}=\varphi$.

To the other part, notice that

$$\underbrace{\frac{n_0+\cdots+n_g}{n_{g+1}}}_{\to\varphi}=\frac{n_0+\cdots+n_{g-1}}{n_{g+1}}+\underbrace{\frac{n_g}{n_{g+1}}}_{\to\frac{1}{\varphi}}.$$

Hence

$$\lim_{g\to\infty}\frac{\sum_{i=0}^{g-1} n_i}{n_{g+1}}=\varphi-\frac{1}{\varphi}=1,$$

compliting the proof. □

Proof. (of item (4)) We enumerate the steps of the proof:

I. $\lim_{g\to\infty}\frac{n_g}{n_{g+1}}=\frac{1}{\varphi}\approx 0.62.$

II. For each $\epsilon\in\left(0,\frac{1}{3}\right)$ define $g_0(\epsilon):=\min\left\{i\in\mathbb{N}:\frac{1}{\varphi}-\epsilon<\frac{n_j}{n_{j+1}}<\frac{1}{\varphi}+\epsilon,\forall j\geqslant i\right\}$.

III. For $g\gg 0$, (n_g) is increasing. Since $\lim_{g\to\infty}\frac{n_g}{n_{g+1}}\neq 1$, then $\lim_{g\to\infty}n_g=\infty$.

IV. For $g>g_0(\epsilon)=g_0$,

$$\begin{aligned}\frac{n_0^2+\cdots+n_{g_0-1}^2+n_{g_0}^2+\cdots+n_g^2}{n_g^2} &< \frac{n_0^2+\cdots+n_{g_0-1}^2}{n_g^2}+\left[\left(\frac{1}{\varphi}+\epsilon\right)^2\right]^{g-g_0+1}+\\ &\quad+\left[\left(\frac{1}{\varphi}+\epsilon\right)^2\right]^{g-g_0}+\cdots+\left(\frac{1}{\varphi}+\epsilon\right)^2+1\\ &\xrightarrow{g\to\infty} \frac{\varphi^2}{\varphi^2-(1+2\varphi\epsilon+\varphi^2\epsilon^2)}.\end{aligned}$$

and

$$\frac{n_0^2+\cdots+n_{g_0-1}^2+n_{g_0}^2+\cdots+n_g^2}{n_g^2} > \frac{n_0^2+\cdots+n_{g_0-1}^2}{n_g^2}+\left[\left(\frac{1}{\varphi}-\epsilon\right)^2\right]^{g-g_0+1}+$$
$$+\left[\left(\frac{1}{\varphi}-\epsilon\right)^2\right]^{g-g_0}+\cdots+\left(\frac{1}{\varphi}-\epsilon\right)^2+1$$
$$\xrightarrow{g\to\infty} \frac{\varphi^2}{\varphi^2-(1-2\varphi\epsilon+\varphi^2\epsilon^2)}.$$

V. Hence, for all $\epsilon \in \left(0,\frac{1}{3}\right)$,

$$\frac{\varphi^2}{\varphi^2-(1-2\varphi\epsilon+\varphi^2\epsilon^2)} \leqslant \lim_{g\to\infty}\frac{n_0^2+\cdots+n_g^2}{n_g^2} \leqslant \frac{\varphi^2}{\varphi^2-(1+2\varphi\epsilon+\varphi^2\epsilon^2)}.$$

VI. By making $\epsilon \to 0$, we conclude that $\lim_{g\to\infty}\frac{\sum_{i=0}^g n_i^2}{n_g^2}=\frac{\varphi^2}{\varphi^2-1}=\frac{\varphi^2}{\varphi}=\varphi.$

Since $\frac{\sum_{i=0}^g n_i^2}{n_{g+1}^2}=\frac{\sum_{i=0}^g n_i^2}{n_g^2}\cdot\frac{n_g^2}{n_{g+1}^2}$, then $\lim_{g\to\infty}\frac{\sum_{i=0}^g n_i^2}{n_{g+1}^2}=\varphi\cdot\frac{1}{\varphi^2}=\frac{1}{\varphi}.$ □

Proof. (of item (5)) We enumerate the steps of the proof:

I. $\lim_{g\to\infty}\frac{n_g}{n_{g+2}}=\frac{1}{\varphi^2}\approx 0.38.$

II. For each $\epsilon \in \left(0,\frac{1}{3}\right)$ define $g_0(\epsilon) := \min\left\{i \in 2\mathbb{N} : \frac{1}{\varphi^2}-\epsilon < \frac{n_j}{n_{j+2}} < \frac{1}{\varphi^2}+\epsilon, \forall j \geqslant i\right\}.$

III. For $g \gg 0$, (n_g) is increasing. Since $\lim_{g\to\infty}\frac{n_g}{n_{g+1}} \neq 1$, then $\lim_{g\to\infty} n_g=\infty$.

IV. For $g > g_0(\epsilon) = g_0$,

$$\frac{n_0+n_2\cdots+n_{g_0-2}+n_{g_0}+\cdots+n_{2g}}{n_{2g}} < \frac{n_0+\cdots+n_{g_0-2}}{n_{2g}}+\left(\frac{1}{\varphi^2}+\epsilon\right)^{g-\frac{g_0}{2}}+$$
$$+\left(\frac{1}{\varphi^2}+\epsilon\right)^{g-\frac{g_0}{2}-1}+\cdots+\left(\frac{1}{\varphi^2}+\epsilon\right)+1$$
$$\xrightarrow{g\to\infty} \frac{\varphi^2}{\varphi^2-(1+\varphi^2\epsilon)}.$$

and

$$\frac{n_0+n_2\cdots+n_{g_0-2}+n_{g_0}+\cdots+n_{2g}}{n_{2g}} > \frac{n_0+\cdots+n_{g_0-2}}{n_{2g}}+\left(\frac{1}{\varphi^2}-\epsilon\right)^{g-\frac{g_0}{2}}+$$
$$+\left(\frac{1}{\varphi^2}-\epsilon\right)^{g-\frac{g_0}{2}-1}+\cdots+\left(\frac{1}{\varphi^2}-\epsilon\right)+1$$
$$\xrightarrow{g\to\infty} \frac{\varphi^2}{\varphi^2-(1-\varphi^2\epsilon)}.$$

V. Hence, for all $\epsilon \in \left(0, \frac{1}{3}\right)$,

$$\frac{\varphi^2}{\varphi^2 - (1 - \varphi^2\epsilon)} \leqslant \lim_{g\to\infty} \frac{n_0 + \cdots + n_{2g}}{n_{2g}} \leqslant \frac{\varphi^2}{\varphi^2 - (1 + \varphi^2\epsilon)}.$$

VI. By making $\epsilon \to 0$, we conclude that $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g}} = \frac{\varphi^2}{\varphi^2 - 1} = \frac{\varphi^2}{\varphi} = \varphi$.

Since $\frac{\sum_{i=0}^{g} n_{2i}}{n_{2g+1}} = \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g}} \cdot \frac{n_{2g}}{n_{2g+1}}$, then $\lim_{g\to\infty} \frac{\sum_{i=0}^{g} n_{2i}}{n_{2g+1}} = \varphi \cdot \frac{1}{\varphi} = 1$. □

4.2 Fibonacci not like behaviour of (n_g)

Zhai's Theorem shows us that

$$\lim_{g\to\infty} \frac{n_g}{\varphi^g} = \mu,$$

where μ is a constant greater than 3.78. By Binet's formula, we have

$$\lim_{n\to\infty} \frac{F_n}{\varphi^n} = \frac{1}{\sqrt{5}} \approx 0.45.$$

It shows us that (n_g) and (F_n) are not (asymptotically) the same. In Theorem 4.1, we obtained some asymptotic properties of sequence (n_g) that are exactly the same as the Fibonacci sequence. Now we prove an asymptotic property that (n_g) satisfy that is different from Fibonacci sequence. We know that $F_n^2 + F_{n+1}^2 = F_{2n+1}$, for all n. Thus, $\lim_{n\to\infty} \frac{F_n^2 + F_{n+1}^2}{F_{2n+1}} = 1$. We prove

Theorem 4.2. *Let (n_g) be the sequence of number of numerical semigroups with genus g. Then*

$$\lim_{g\to\infty} \frac{n_g^2 + n_{g+1}^2}{n_{2g+1}} = \sqrt{5}\mu > 8.$$

Proof. We can rewrite

$$\begin{aligned} \frac{n_g^2 + n_{g+1}^2}{n_{2g+1}} &= \frac{n_g^2}{\varphi^{2g}} \cdot \frac{\varphi^{2g+1}}{n_{2g+1}} \cdot \frac{1}{\varphi} + \frac{n_{g+1}^2}{\varphi^{2(g+1)}} \cdot \frac{\varphi^{2g+1}}{n_{2g+1}} \cdot \varphi \\ &= \underbrace{\left(\frac{n_g}{\varphi^g}\right)^2}_{\to\mu^2} \cdot \underbrace{\frac{\varphi^{2g+1}}{n_{2g+1}}}_{\to\frac{1}{\mu}} \cdot \frac{1}{\varphi} + \underbrace{\left(\frac{n_{g+1}}{\varphi^{g+1}}\right)^2}_{\to\mu^2} \cdot \underbrace{\frac{\varphi^{2g+1}}{n_{2g+1}}}_{\to\frac{1}{\mu}} \cdot \varphi \\ &\xrightarrow{g\to\infty} \mu\left(\frac{1}{\varphi} + \varphi\right) = \mu\sqrt{5} > 8, \end{aligned}$$

completing the proof. □

5 A generalization of γ-hyperelliptic semigroups

In this chapter, we generalize the idea of γ-hyperelliptic semigroups (cf. (26)) and prove some basic results which naturally generalizes results from Chapter 2. Torres (26) studied some arithmetic properties of generalized γ-hyperelliptic semigroups, when he was dealing with M-sheeted covering of curves (in particular, it generalizes the double covering of curves).

Let $S \in \mathcal{S}_g$ with set of gaps $G(S)$. For a positive integer M, we define

$$\gamma_M(S) = \#\{k \in G(S) : k \equiv 0 \pmod{M}\}.$$

If $\gamma_M(S) = \gamma$, we say that S is a (M, γ)-hyperelliptic semigroup. Notice that a $(1, \gamma)$-hyperelliptic semigroup is a numerical semigroup with genus γ and a $(2, \gamma)$-hyperelliptic semigroup is a γ-hyperelliptic semigroup.

Let γ, g and M non-negative integers. We define $\mathcal{S}_{(M,\gamma)}(g) := \{S \in \mathcal{S}_g : \gamma_M(S) = \gamma\}$ and $N_{(M,\gamma)}(g) := \#\mathcal{S}_{(M,\gamma)}(g)$. Throughout this chapter M is an integer greater than or equal to 2 and we write $(\equiv_M i)$ to denote the set of integers congruent to i modulo M and $(\not\equiv_M i)$ for its complement in $\mathbb{Z}$.

Proposition 5.1. *Let S be a (M, γ)-hyperelliptic semigroup with genus g. Then*

$$\gamma \leqslant \left\lfloor \frac{2g}{M} \right\rfloor.$$

Proof. The gaps of S lies in $[1, 2g]$, but there are $\left\lfloor \frac{2g}{M} \right\rfloor$ multiples of M in that interval; the result follows. □

In order to generalize last proposition, we follow the ideas of Lemma 2.1 to get a sharper bound relating γ, g and M for (M, γ)-hyperelliptic semigroups with genus g.

Theorem 5.1. *Let S be a (M, γ)-hyperelliptic semigroup with genus g. Then*

$$2g \geqslant (M + 1)\gamma.$$

Proof. If $2g \geqslant 2M\gamma$, then $2g \geqslant (M + 1)\gamma$, since $2M \geqslant M + 1$, for all $M \geqslant 1$. Suppose $2g < 2M\gamma$. In $[1, 2g]$, there are $\left\lfloor \frac{2g}{M} \right\rfloor$ numbers $(\equiv_M 0)$, which γ are gaps and $\left\lfloor \frac{2g}{M} \right\rfloor - \gamma$ are non-gaps, and $2g - \left\lfloor \frac{2g}{M} \right\rfloor$ numbers ($\not\equiv_M 0$), which $g - \gamma$ are gaps and $g - \left\lfloor \frac{2g}{M} \right\rfloor + \gamma$ are non-gaps. Let $q_1 < \cdots < q_\gamma$ be the gaps ($\equiv_M 0$) of S and $r_{g - \lfloor \frac{2g}{M} \rfloor + \gamma} < \cdots < r_1$ be the non gaps ($\not\equiv_M 0$) in $[1, 2g]$. Notice that $q_\gamma - r_i \notin S$, for all i (otherwise, $S \ni r_i + (q_\gamma - r_i) = q_\gamma \notin S$). For i such that $q_\gamma - r_i \geqslant 0$, all numbers $q_\gamma - r_i$ are gaps ($\not\equiv_M 0$). Notice that

$$\begin{cases} r_{(M-1)\ell+1} \leqslant 2g - (M\ell + 1) = 2g - [(M-1)\ell + 1] - \ell \\ r_{(M-1)\ell+2} \leqslant 2g - (M\ell + 2) = 2g - [(M-1)\ell + 2] - \ell \\ \qquad \vdots \\ r_{(M-1)\ell+M-1} \leqslant 2g - (M\ell + M - 1) = 2g - [(M-1)\ell + M - 1] - \ell \end{cases}$$

Writing $i = (M-1)\ell + k$, with $k \in \{1, \ldots, M-1\}$, we conclude that

$$r_i \leqslant 2g - i - \left\lfloor \frac{i}{M-1} \right\rfloor + \epsilon$$

where $\epsilon = 0$, if $k \neq M-1$ and $\epsilon = 1$, if $k = M-1$.

Claim. $q_\gamma \leqslant 2(g - \gamma)\frac{M}{M-1}$.

We follow the idea of the proof of the Claim in Lemma 2.1: we suppose $q_\gamma \geqslant 2(g-\gamma)\frac{M}{M-1} + 1$ and we construct $g - \gamma + 1$ gaps ($\not\equiv_M 0$), which leads to a contradiction.

We know that for $i \in \left\{2\gamma - \left\lfloor \frac{2g}{M} \right\rfloor, \ldots, g - \left\lfloor \frac{2g}{M} \right\rfloor + \gamma\right\}$, $q_\gamma - r_i$ is a gap ($\not\equiv_M 0$). For all $i \geqslant 2\gamma - \left\lfloor \frac{2g}{M} \right\rfloor$, we obtain

$$\begin{aligned} q_\gamma - r_i &\geqslant \left[2(g-\gamma)\frac{M}{M-1} + 1\right] - \left[2g - i - \left\lfloor \frac{i}{M-1} \right\rfloor + \epsilon\right] \\ &= 2(g-\gamma)\frac{M}{M-1} + 1 - 2g + i + \left\lfloor \frac{i}{M-1} \right\rfloor - \epsilon \\ &\geqslant \frac{2M}{M-1}g - \frac{2M}{M-1}\gamma + 1 - 2g + \left(2\gamma - \left\lfloor \frac{2g}{M} \right\rfloor\right) + \left\lfloor \frac{\left(2\gamma - \lfloor \frac{2g}{M} \rfloor\right)}{M-1} \right\rfloor - \epsilon \\ &> 2g\left(\frac{M}{M-1} - 1\right) - 2\gamma\left(\frac{M}{M-1} - 1\right) + 1 - \frac{2g}{M} + \frac{2\gamma - \lfloor \frac{2g}{M} \rfloor}{M-1} - 1 - \epsilon \\ &\geqslant 2g\left(\frac{1}{M-1}\right) - 2\gamma\left(\frac{1}{M-1}\right) - \frac{2g}{M} + \frac{2\gamma}{M-1} - \frac{\lfloor \frac{2g}{M} \rfloor}{M-1} - \epsilon \\ &\geqslant 2g\left(\frac{1}{M-1} - \frac{1}{M}\right) - \frac{2g}{M(M-1)} - \epsilon \\ &\geqslant 2g\left(\frac{1}{M(M-1)} - \frac{1}{M(M-1)}\right) - 1 \\ &= -1. \end{aligned}$$

Thus, $q_\gamma - r_i \geqslant 0$, for all $i \geqslant 2\gamma - \left\lfloor \frac{2g}{M} \right\rfloor$. Then, there are, at least, $g-\gamma+1$ different gaps ($\not\equiv_M 0$). In fact, we proved that all numbers $q_\gamma - r_{2\gamma-\lfloor \frac{2g}{M} \rfloor} < \cdots < q_\gamma - r_{g-\lfloor \frac{2g}{M} \rfloor+\gamma}$ are different gaps ($\not\equiv_M 0$) and this is a contradiction. Thus, $q_\gamma \leqslant 2(g-\gamma)\dfrac{M}{M-1}$.

There are $\left\lfloor \dfrac{2(g-\gamma)\frac{M}{M-1}}{M} \right\rfloor = \left\lfloor \dfrac{2(g-\gamma)}{M-1} \right\rfloor$ numbers ($\equiv_M 0$) in $\left[M, 2(g-\gamma)\dfrac{M}{M-1}\right]$, where γ are gaps. Thus, $\dfrac{2(g-\gamma)}{M-1} \geqslant \left\lfloor \dfrac{2(g-\gamma)}{M-1} \right\rfloor \geqslant \gamma$ and we conclude that $2g-2\gamma \geqslant (M-1)\gamma$. Hence, $2g \geqslant (M+1)\gamma$. □

Proposition 5.2. *Theorem 5.1 is sharp.*

Proof. We construct a (M,γ)-hyperelliptic semigroup with genus g that satisfies the equality $2g = (M+1)\gamma$. We consider two cases: γ even and γ odd.

- If γ is even:

In this case, a numerical semigroup satisfying $2g = (M+1)\gamma$ is

$$S := \mathbb{N}_0 \backslash \left(\{M, 2M, \ldots, M\gamma\} \cup \bigcup_{i=1}^{M-1} \left\{ i, M+i, \ldots, M\left(\frac{\gamma}{2}-1\right)+i \right\} \right).$$

Notice that $0 \in S$ and S has γ gaps ($\equiv_M 0$). For each $i \in \{1, \ldots, M-1\}$, there are $\gamma/2$ gaps ($\equiv_M i$). Hence, $g(S) = \gamma + (M-1)\gamma/2 = (M+1)\gamma/2$.

For each $i \not\equiv_M 0$, the first non-gap ($\equiv_M i$) is $M\gamma/2 + i$ and all numbers $M\gamma/2 + ki$, with $k \in \mathbb{N}$ are non-gaps. The first non-gap ($\equiv_M 0$) is $M(\gamma+1)$. Hence, for all $a, b \in S$, it follows that $a + b > M\gamma/2 + M\gamma/2 = M\gamma$ and $a + b \in S$.

- If γ is odd:

In this case, M must be odd (otherwise, equality does not occur). Let

$$S' := \mathbb{N}_0 \backslash \left(\{M, 2M, \ldots, M\gamma\} \cup \bigcup_{i=1}^{M-1} \left\{ i, M+i, \ldots, M\left(\frac{\gamma-1}{2}-1\right)+i \right\} \right).$$

With a similar procedure to the other case, we obtain $g(S') = \gamma + (M-1)(\gamma-1)/2 = (M+1)(\gamma-1)/2 + 1$. We want to construct S turning some non-gaps of S' into gaps of S such that $g(S) = (M+1)\gamma/2$. Hence, we have open $(M-1)/2$ non-gaps ($\not\equiv_M 0$) carefully. In order to obtain a closed under addition set, we can open first $(M-1)/2$ numbers of form $M(\gamma-1)/2 + i$. Hence,

$$S = S' \backslash \left\{ \frac{M(\gamma-1)}{2} + 1, \ldots, \frac{M(\gamma-1)}{2} + \frac{M-1}{2} \right\}.$$

Now we show that S is, in fact, closed under addition. Let $a, b \in S$ with $a \equiv_M i$ and $b \equiv_M j$.

- ⋆ If $i = j = 0$, then $a + b \in S$, since $a + b \equiv_M 0$ and $a + b \geqslant M(\gamma + 1)$.
- ⋆ If $i = 0$ and $j \neq 0$, then $a + b \in S$, since $a + b \equiv_M j$ and $a + b \geqslant b$.
- ⋆ If $0 < i, j \leqslant (M-1)/2$, then $a = M(\gamma+1)/2 + i$ and $b = M(\gamma+1)/2 + j$. Thus, $a + b = M(\gamma + 1) + (i + j) \in S$.
- ⋆ If $0 < i \leqslant (M-1)/2 < j \leqslant M-1$, then $a = M(\gamma+1)/2+i$ and $b = M(\gamma-1)/2+j$. Thus, $a + b = M\gamma + (i + j) \in S$.
- ⋆ If $i, j > (M-1)/2$, then $a = M(\gamma-1)/2 + i$ and $b = M(\gamma-1)/2 + j$. Thus, $a + b = M(\gamma - 1) + (i + j) > M(\gamma - 1) + (M - 1) = M\gamma - 1$. Since $i + j \not\equiv_M 0$, $a + b > M\gamma$ and we conclude that $a + b \in S$.

□

Remark 5.1. *Theorem 5.1 improves the Proposition 5.1. For example, if $M = 3$, Proposition 5.1 gives $g \geqslant \dfrac{3\gamma}{2}$ and Theorem 5.1 gives $2g \geqslant 4\gamma$, i.e., $g \geqslant 2\gamma$.*

Definition 5.1. *The quocient of a numerical semigroup S by a positive integer M is $S/M := \{s \in \mathbb{N}_0 : Ms \in S\}$.*

Let γ be a non-negative integer. There is a natural parametrization of the family $\mathcal{S}_{(M,\gamma)}(g)$ onto $\mathcal{S}_\gamma$ given by

$$\begin{array}{rcccl} \mathbf{x}_{M,g} & : & \mathcal{S}_{(M,\gamma)}(g) & \to & \mathcal{S}_\gamma \\ & & S & \mapsto & S/M \end{array}$$

which is a surjective map.

Remark 5.2. *Indeed, any $S \in \mathcal{S}_{(M,\gamma)}(g)$ can be uniquely written as*

$$S = M(S/M) \cup \{o_{g+\gamma-\lfloor \frac{2g}{M} \rfloor} < \cdots < o_1\} \cup S_{2g}$$

where $o_{g+\gamma-\lfloor \frac{2g}{M} \rfloor}, \ldots, o_1$ are certain numbers ($\not\equiv_0 M$) in $[1, 2g-1]$.

Remark 5.3. *For g and γ non-negative integers, we write*

$$N_{(M,\gamma)}(g) = \sum_{T \in \mathcal{S}_\gamma} \#\mathbf{x}_{M,g}^{-1}(T).$$

Let $S \in \mathcal{S}_{(M,\gamma)}(g)$. Remark 5.2 tell us that there is natural way to obtain the gaps and non-gaps ($\equiv_M 0$) of S from a numerical semigroup $T \in \mathcal{S}_\gamma$: if $\{Mq_1, \ldots, Mq_\gamma\}$ is the set of gaps ($\equiv_M 0$) of S, then it can be obtained by multiplying by M the gaps from $\mathbb{N}_0 \backslash \{q_1, \ldots, q_\gamma\} \in \mathcal{S}_\gamma$. Moreover, T is uniquely determined ($T = S/M$).

Reciprocally, given $T \in \mathcal{S}_\gamma$ we can obtain a (M,γ)-hyperelliptic semigroup S of some genus g by multiplying by M the gaps of T and making a suitable choice on the numbers ($\not\equiv_M 0$) that are gaps or non-gaps of S. Notice that not all choices for those integers return numerical semigroups (some sets are not closed under addition) and in general S is not uniquely determined (in general, the fiber $\mathbf{x}_{M,g}^{-1}(T)$ has more than one element).

Now we look for conditions that a (M,γ)-hyperelliptic semigroup with some genus $g \geqslant (M+1)\gamma/2$ has to attain. In fact, let $T \in \mathcal{S}_\gamma$ and consider all sets

$$S = MT \cup \{o_{g+\gamma-\lfloor \frac{2g}{M} \rfloor} < \ldots < o_1\} \cup S_{2g},$$

where the numbers o_i are numbers ($\not\equiv_M 0$) less than $2g$.

From definition, $S \subseteq \mathbb{N}_0$, $0 \in S$ and $\mathbb{N}_0 \backslash S$ has γ elements ($\not\equiv_M 0$). There are $2g - \left\lfloor \dfrac{2g}{M} \right\rfloor$ numbers ($\not\equiv_M 0$) in $[1, 2g-1]$, such that $g + \gamma - \left\lfloor \dfrac{2g}{M} \right\rfloor$ of them lies in S. Thus $\mathbb{N}_0 \backslash S$ has $g - \gamma$ elements ($\not\equiv_M 0$) and $\#(\mathbb{N}_0 \backslash S) = g$. Hence we conclude that $S \in \mathcal{S}_{(M,\gamma)}(g)$ if, and only if, S is closed under addition. Moreover, the part ($\equiv_M 0$) of S is closed under addition, since T is.

Let $T \in \mathcal{S}_\gamma$, where $T = \mathbb{N}_0 \backslash \{q_1, \ldots, q_\gamma\}$ and $T \cap [0, 2\gamma] = \{t_0 = 0, t_1, \ldots, t_\gamma = 2\gamma\}$, we have that $Mt_0 = 0 < Mt_1 < \cdots < Mt_{\gamma-1} < Mt_\gamma = 2M\gamma$ are non-gaps ($\equiv_M 0$) of S and $Mq_1 < \ldots < Mq_\gamma$ are the gaps ($\not\equiv_M 0$) of S.

Let $O_i := \min\{s \in S : s \equiv i \pmod M\} = Mk_i + i$, for $i \in \{1, \ldots M-1\}$. By closed under addition condition, we conclude that $O_j + 2t_i$ must be a non-gap of S, for all i and j and $(O_j + 2M\gamma) + \mathbb{N}_0 \subseteq S$, for all j. Notice that, at first, there are no necessary conditions on each number $O_j + Mq_i$. We illustrate it below:

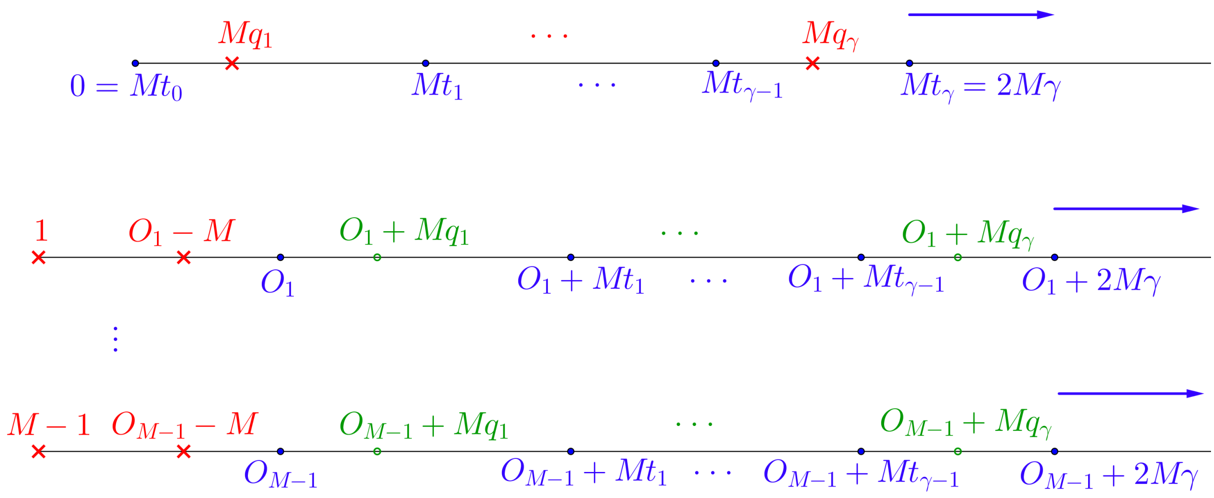


Figure 7 – Configuration of a (M,γ)-hyperelliptic semigroup

We obtain upper and lower bounds for the sum of the numbers O_i's in a (M,γ)-hyperelliptic semigroup S with genus g only depending on γ, g and M.

Proposition 5.3. *Let S be a (M,γ)-hyperelliptic semigroup with genus g and $O_i := \min\{s \in S : s \equiv i \pmod{M}\}$, for $i \in \{1,\dots,M-1\}$. Then*

$$Mg - M^2\gamma + \binom{M}{2} \leqslant \sum_{i=1}^{M-1} O_i \leqslant Mg - M\gamma + \binom{M}{2}.$$

Proof. For each i, let $O_i = Mk_i + i$ and $Mq_1, \dots, Mq_\gamma$ be the gaps $(\equiv_M 0)$ of S. Then the k_i numbers $i, \dots, Mk_i - (M-i)$ are gaps $(\equiv_M i)$ of S and the only possibilities for other gaps $(\equiv_M i)$ of S are $O_i + Mq_1, \dots, O_i + Mq_\gamma$. Thus $\gamma + \sum_i k_i \leqslant g \leqslant \gamma + \sum_i k_i + (M-1)\gamma$ and $g - M\gamma \leqslant \sum_i k_i \leqslant g - \gamma$. Multiplying last inequality by M and adding $\sum_i i$, we obtain

$$Mg - M^2\gamma + \sum_{i=1}^{M-1} i \leqslant \sum_{i=1}^{M-1} (Mk_i + i) \leqslant Mg - M\gamma + \sum_{i=1}^{M-1} i, \text{ i.e.,}$$

$$Mg - M^2\gamma + \binom{M}{2} \leqslant \sum_{i=1}^{M-1} O_i \leqslant Mg - M\gamma + \binom{M}{2}.$$

□

In particular, last result improves an immediate consequence of (26, Lemma 2.1). He proves that for in an integer O in a (M,γ)-hyperelliptic semigroup with genus g such that $\gcd(O,M) = 1$, the following holds

$$O \geqslant \frac{2g - 2M\gamma}{M-1} + 1.$$

This implies that

$$\sum_{i=1}^{M-1} O_i \geqslant 2g - 2M\gamma + (M-1).$$

Remark 5.4. *Notice that our $(M,0)$-hyperelliptic semigroups are those considered by Kunz-Waldi (16, p.1). In fact, $N_{(M,0)}(g)$ is the number of numerical semigroups with genus g containing the integer M.*

We recall that a *quasipolynomial of degree d* is a function $f : \mathbb{N} \to \mathbb{C}$ of the form $f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \ldots + c_0(n)$ such that each $c_i(n)$ is a periodic functions, with integer periods and $c_d \neq 0$. This means c_i depends only on the congruence class modulo an integer.

Using Kunz and Waldi (16) results, we conclude that $N_{(M,0)}(g)$ agrees with a quasipolynomial of degree $M-2$. For instance,

$$N_{(2,0)}(g) = 1, \quad N_{(3,0)}(g) = \left\lfloor \frac{g}{3} \right\rfloor + 1 \quad \text{and} \quad N_{(4,0)}(g) = \left\lfloor \frac{g^2}{12} + \frac{g}{2} \right\rfloor + 1.$$

We compute some values of $N_{(3,\gamma)}(g)$ and $N_{(4,\gamma)}(g)$ with GAP (13) and the package NumericalSgps (10). There are two tables with those computations in the end of this chapter.

We would not be a surprise if the following holds: let γ be a non-negative integer. Then, for all $M \geqslant 2$, $N_{(M,\gamma)}(g) \leqslant N_{(M,\gamma)}(g+1)$, for all g.

Remark 5.5. *If there is some M such that*

$$N_{(M,\gamma)}(g) \leqslant N_{(M,\gamma)}(g+1), \text{ for all } g,$$

then $n_g \leqslant n_{g+1}$, for all g, since

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{2g}{M+1} \rfloor} N_{(M,\gamma)}(g).$$

$g \setminus \gamma$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	n_g
0	1														1
1	1														1
2	1	1													2
3	2	2													4
4	2	4	1												7
5	2	4	6												12
6	3	6	12	2											23
7	3	8	19	9											39
8	3	8	26	28	2										67
9	4	10	33	57	14										118
10	4	12	41	94	51	2									204
11	4	12	49	127	132	19									343
12	5	14	57	170	255	88	3								592
13	5	16	66	208	414	263	29								1001
14	5	16	74	250	591	618	136	3							1693
15	6	18	82	300	787	1153	478	33							2857
16	6	20	91	345	998	1858	1282	203	3						4806
17	6	20	99	388	1199	2688	2800	801	44						8045
18	7	22	107	440	1423	3622	5153	2406	283	4					13467
19	7	24	116	485	1671	4632	8345	5885	1244	55					22464
20	7	24	124	529	1898	5694	12266	12285	4174	391	4				37396
21	8	26	132	581	2142	6787	16814	22260	11502	1878	64				62194
22	8	28	141	626	2398	7962	21818	36073	26731	6951	506	4			103246
23	8	28	149	670	2630	9176	27162	53527	53950	20882	2707	74			170963
24	9	30	157	722	2876	10406	32819	74283	96496	53441	10926	658	5		282828
25	9	32	166	767	3133	11665	38677	97756	156275	118854	36000	3796	94		467224
26	9	32	174	811	3365	12936	44744	123272	233470	234058	100478	16633	845	5	770832

Table 9 – A few values for $N_{(3,\gamma)}(g)$

$g \backslash \gamma$	0	1	2	3	4	5	6	7	8	9	10	n_g
0	1											1
1	1											1
2	2											2
3	3	1										4
4	4	3										7
5	5	5	2									12
6	7	10	6									23
7	8	16	15									39
8	10	22	33	2								67
9	12	31	58	17								118
10	14	42	96	49	3							204
11	16	52	141	118	16							343
12	19	65	205	247	56							592
13	21	80	285	441	171	3						1001
14	24	94	378	735	425	37						1693
15	27	111	487	1122	939	167	4					2857
16	30	130	618	1627	1838	532	31					4806
17	33	148	757	2258	3239	1470	470					8045
18	37	169	913	3036	5331	3437	540	4				13467
19	40	192	1089	3971	8215	7141	1752	64				22464
20	44	214	1275	5065	12043	13515	4838	397	5			37396
21	48	239	1476	6325	16971	23554	11916	1614	51			62194
22	52	266	1699	7735	23097	38547	26180	5383	287			103246
23	56	292	1929	9301	30480	59635	52515	15452	1298	5		170963
24	61	321	2177	11028	39281	88113	97443	39302	5004	98		282828
25	65	352	2444	12909	49572	125218	168717	90543	16632	766	6	467224

Table 10 – A few values for $N_{(4,\gamma)}(g)$

6 Patterns on numerical semigroups

In this chapter, we discuss another topic related to numerical semigroups. We forget, for a moment, the problem of counting numerical semigroups by genus and we now study the concept of *Patterns*. It was introduced by Bras-Amorós and García-Sánchez (8) as a generalization of Arf semigroups (numerical semigroups S that satisfies the following: for all $x \geqslant y \geqslant z$, with x, y and $z \in S$, $x + y - z \in S$); see (1) and (17). A linear pattern is a linear homogeneous polynomial with non-zero and integer coefficients. In 2013, Bras-Amorós, García-Sánchez and Vico-Oton (9) introduced non-homogeneous patterns and in 2016, Stokes (23) worked with patterns of ideal of numerical semigroups.

6.1 Some definitions and previous results on patterns

In this section, we present definitions and important results. Some of them can be found in (8) and (23).

Following (8), a *pattern* p of length n is a homogeneous linear polynomial $p = \sum_{i=1}^{n} a_i X_i$, where $a_i \in \mathbb{Z} \backslash \{0\}$ and the set $\{p(s_1, \ldots, s_n) : s_1 \geqslant \cdots \geqslant s_n, s_i \in S\}$ is denoted by $p(S)$. A numerical semigroup S *admits a pattern* p if $p(S) \subseteq S$ and a pattern p is said *admissible* if there is a numerical semigroup S that admits p.

Lemma 6.1. *(8, Theorem 12) Let $p = \sum_{i=1}^{n} a_i X_i$ be a pattern. The following are equivalent:*

1) p is admissible.

2) $\sum_{i=1}^{n'} a_i \geqslant 0$, for all $n' \leqslant n$.

3) $p(\mathbb{N}_0) \subseteq \mathbb{N}_0$.

Proof. 1) $\Rightarrow$ 2) : Suppose that there is n' such that $\sum_{i=1}^{n'} a_i < 0$. If S is a numerical semigroup and s is a non-zero element of S, then $p(\underbrace{s, \ldots, s}_{n'}, 0, \ldots, 0) = s \cdot \sum_{i=1}^{n'} a_i < 0$, thus not in S. Hence, p is not admissible.
2) $\Rightarrow$ 3) : Let $z_1 \geqslant \cdots \geqslant z_n$ be non-negative integers. Then $p(z_1, \ldots, z_n) \geqslant z_n \cdot \sum_{i=1}^{n} a_i \geqslant 0$, thus a non-negative integer.
3) $\Rightarrow$ 1) : It follows from the fact that $\mathbb{N}_0$ admits p. $\square$

A pattern p is *strongly admissible* if p and p' are admissible, where

$$p' = \begin{cases} p - x_1, & \text{if } a_1 > 1 \\ p(0, x_1, \dots, x_{n-1}), & \text{otherwise.} \end{cases}$$

Lemma 6.2. *(23, p. 186) Let $p = \sum_{i=1}^{n} a_i X_i$ be a pattern. Then p is a strongly admissible pattern if, and only if, $\sum_{i=1}^{n'} a_i \geqslant 1$, for all $n' \leqslant n$.*

Proof. By Proposition 6.1, we have $\sum_{i=1}^{n'} a_i \geqslant 0, \forall n' \leqslant n$.

- Let $a_1 = 1$:
  Since $p' = \sum_{i=2}^{n} a_i X_{i-1}$ is admissible, then $\sum_{i=2}^{n'} a_i \geqslant 0, \forall n' \leqslant n$. Thus, $\sum_{i=1}^{n'} a_i \geqslant 1, \forall n' \leqslant n$.
- Let $a_1 > 1$:
  Since $p' = (a_1 - 1)X_1 + \sum_{i=2}^{n} a_i X_i$ is admissible, then $-1 + \sum_{i=1}^{n'} a_i \geqslant 0, \forall n' \leqslant n$. Thus, $\sum_{i=1}^{n'} a_i \geqslant 1, \forall n' \leqslant n$.

Now assume that $\sum_{i=1}^{n'} a_i \geqslant 1 > 0$, for all $n' \leqslant n$. By Lemma 6.1, p is admissible. Next we show that p' is also admissible.

- Let $a_1 = 1$:
  In this case, $p' = \sum_{i=2}^{n} a_i X_{i-1}$. Then $\sum_{i=2}^{n'} a_i \geqslant 1 - a_1 = 0$, for all n' such that $2 \leqslant n' \leqslant n$. Thus, p' is admissible.
- Let $a_1 > 1$:
  In this case, $p' = (a_1 - 1)X_1 + \sum_{i=2}^{n} a_i X_i$. Then $-1 + \sum_{i=1}^{n'} a_i \geqslant 0$, for all $n' \leqslant n$. Thus, p' is admissible.

□

Let S be a numerical semigroup and let p be a pattern (from now on, a pattern means an admissible pattern). Now, we look for conditions on the set $p(S)$ so it is a numerical semigroup. Notice that $p(S) \subseteq \mathbb{N}_0$, $0 = p(0, \dots, 0) \in p(S)$ and $p(S)$ is closed under addition. In fact, if $s_1 \geqslant \cdots \geqslant s_n$ and $r_1 \geqslant \cdots \geqslant r_n$, with s_i and $r_j \in S$, then $s_1 + r_1 \geqslant \cdots \geqslant s_n + r_n$ are all elements in S; since p is linear, $p(s_1, \dots, s_n) + p(r_1, \dots, r_n) =$

$p(s_1 + r_1, \ldots, s_n + r_n) \in S$. Thus, $p(S)$ is a numerical semigroup if, and only if, $\mathbb{N}_0 \backslash p(S)$ is finite.

A pattern $p = \sum_{i=1}^{n} a_i X_i$ is *premonic* if there is $n' \leqslant n$ such that $\sum_{i=1}^{n'} a_i = 1$ and it is *monic* if $a_1 = 1$. In particular, all monic patterns are also premonic.

Lemma 6.3. *(8, Proposition 22) Let p be a premonic pattern and S be a numerical semigroup. Then $p(S)$ is a numerical semigroup.*

Proof. We show that $S \subseteq p(S)$ and then $\mathbb{N}_0 \backslash p(S) \subseteq \mathbb{N}_0 \backslash S$, being the last one a finite set, since S is a numerical semigroup. Let $s \in S$ and observe that there is $n' \leqslant n$ such that $\sum_{i=1}^{n'} a_i = 1$. Then $s = s \cdot 1 + 0 = \sum_{i=1}^{n'} a_i \cdot s + \sum_{i=n'+1}^{n} a_i \cdot 0 = p(\underbrace{s, \ldots, s}_{n'}, 0 \ldots, 0)$. □

Corollary 6.1. *Let p be a premonic pattern admitted by a numerical semigroup S. Then $p(S) = S$.*

Proof. It follows from the inclusion $p(S) \subseteq S$. □

A pattern $p = \sum_{i=1}^{n} a_i X_i$ is *primitive* if $\gcd(a_1, \ldots, a_n) = 1$. By gcd properties, a premonic pattern is also a primitive pattern. Next result improves Proposition 6.3.

Lemma 6.4. *(23, Corollary 4) Let $p = \sum_{i=1}^{n} a_i X_i$ be a pattern and S be a numerical semigroup. The set $p(S)$ is a numerical semigroup if, and only if, p is primitive.*

Proof. We only have to prove that $\mathbb{N}_0 \backslash p(S)$ is finite if, and only if, p is primitive.

Suppose p not primitive, i.e., $\gcd(a_1, \ldots, a_n) = d > 1$. Then $p(S) \subseteq d\mathbb{N}_0$ and $\mathbb{N}_0 \backslash p(S)$ is an infinite set. On the other hand, suppose p primitive.

Claim. *$p(S) \nsubseteq d\mathbb{Z}$, for all $d > 1$.*

Let d be an integer greater than 1, $c = c(S)$ be the conductor of S and $s \in S \cap d\mathbb{Z}$, with $s \geqslant c$. By hypothesis, $s + d > s + 1 > s$ are elements of S, with $d \mid s, d \mid s + d$ and $s + 1 \equiv 1 \pmod{d}$. Also, there is a_k such that $d \nmid a_k$. Then,

$$p(S) \ni p(\underbrace{s + d, \ldots, s + d}_{k-1}, s + 1, \underbrace{s, \ldots, s}_{n-k}) \equiv a_k(s + 1) \equiv a_k \not\equiv 0 \pmod{d}$$

and the Claim follows.

Now, since $p(S)$ is closed under addition and $p(S) \nsubseteq d\mathbb{Z}$, for all $d > 1$, it follows that $p(S)$ has finite complement in S. □

Next, we obtain a condition involving the multiplicities of the numerical semigroups S and $p(S)$.

Proposition 6.1. *Let S be a numerical semigroup of multiplicity $m(S) = m$ and $p = \sum_{i=1}^{n} a_i X_i$ be a strongly admissible primitive pattern. Then $m \mid m(p(S))$.*

Proof. First of all, notice that $p(S)$ is a numerical semigroup, since p is primitive. We show that there is $j \in \{1, \dots, n\}$ such that $m(p(S)) = m \cdot \sum_{i=1}^{j} a_i$. Let $k \in \{1, \dots, n\}$ and $s_1, \dots, s_k \in S \backslash \{0\}$ such that $s_1 \geqslant \cdots \geqslant s_k$. Then

$$
\begin{aligned}
p(s_1, \dots, s_k, 0, \dots, 0) &= a_1 s_1 + a_2 s_2 + \cdots + a_k s_k \\
&\geqslant (a_1 + a_2) s_2 + a_3 s_3 + \cdots + a_k s_k \\
&\geqslant (a_1 + a_2 + a_3) s_3 + \cdots + a_k s_k \\
&\vdots \\
&\geqslant (a_1 + \cdots + a_k) s_k \\
&\geqslant m \cdot \sum_{i=1}^{k} a_i \\
&= p(\underbrace{m, \dots, m}_{k}, 0, \dots, 0).
\end{aligned}
$$

Let $j \in \{1, \dots, n\}$ such that $\sum_{i=1}^{j} a_i \leqslant \sum_{i=1}^{k} a_i$, for all k. Thus, $p(\underbrace{m, \dots, m}_{j}, 0, \dots, 0) \leqslant p(\underbrace{m, \dots, m}_{k}, 0, \dots, 0)$ and we conclude that $m(p(S)) = p(\underbrace{m, \dots, m}_{j}, 0, \dots, 0) = m \cdot \sum_{i=1}^{j} a_i$. □

Let $\mathcal{P}_n^d(S)$ be the set of patterns of length at most n and degree at most d admitted by S. We write $\mathcal{P}_n(S)$ to denote the set of linear patterns of length at most n admitted by a numerical semigroup S.

Lemma 6.5. *([23], Lemma 31) The set $\mathcal{P}_n^d(S)$ with the usual sum is a monoid, for all numerical semigroups S.*

Proof. Let S be a numerical semigroup. It is clear that 0 is admitted by S. Let p and $q \in \mathcal{P}_n^d(S)$. Since $(p+q)(s_1, \dots, s_n) = p(s_1, \dots, s_n) + q(s_1, \dots, s_n) \in S, \forall s_1 \geqslant \cdots \geqslant s_n$, with $s_i \in S$, then $p + q$ is also admitted by S. □

6.2 Linear patterns of length two

In this section, we present results we obtain after some nice discussions with Klara Stokes.

Let S be a numerical semigroup. We observe that $\mathcal{P}_n(S) \subseteq \mathcal{P}_{n+1}(S)$, $\mathcal{P}_0(S) = \{0\}$ and $\mathcal{P}_1(S) = \{aX : a \geqslant 0\}$ (otherwise, if $0 \neq s \in S \subseteq \mathbb{N}_0$, then $p(s) = as \leqslant -s < 0$). We can identify $\mathcal{P}_1(S)$ with $\mathbb{N}_0$.

Proposition 6.2. *Let S be a numerical semigroup. Then $\mathcal{P}_n(S) \subseteq \mathcal{P}_n(\mathbb{N}_0)$.*

Proof. If $p \in \mathcal{P}_n(S)$, then p is admissible. By Proposition 6.1, $\mathbb{N}_0$ admits p. □

We have $\mathcal{P}_n(\mathbb{N}_0) = \left\{ \sum_{i=1}^{n} a_i X_i : \sum_{i=1}^{n'} a_i \geqslant 0, \forall n' \leqslant n \right\}$ and we can identify $\mathcal{P}_n(\mathbb{N}_0)$ with integer points $(a_1, \dots, a_n)$ at $\mathbb{R}^n$ such that $\sum_{i=1}^{n'} a_i \geqslant 0$, for all $n' \leqslant n$. From now on, we identify $\mathcal{P}_n(\mathbb{N}_0)$ with $\Omega_n := \left\{ (a_1, \dots, a_n) \in \mathbb{Z}^n : \sum_{i=1}^{n'} a_i \geqslant 0, \forall n' \leqslant n \right\}$ and $\mathcal{P}_n(S)$ with $\Omega_n(S) := \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_1 X_1 + \dots + a_n X_n \in \mathcal{P}_n(S)\}$. Notice that $\mathbb{N}_0^n \subseteq \Omega_n(S) \subseteq \Omega_n$, for all numerical semigroups S.

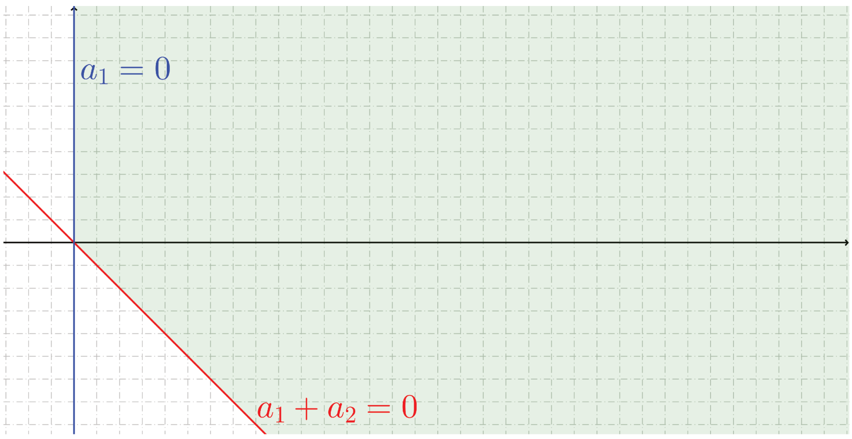


Figure 8 – Ω_2 consists of integer points at green's region

Let S be a numerical semigroup. Computing $\Omega_2(S)$ is equivalent to check which patterns $p(X, Y) = aX + bY$ are admitted by S. We consider the case $a \neq 0$ and $b \neq 0$ (otherwise, we obtain patterns of length 0 or 1). We have to check which points of $\Omega_2^-(S) := \{(a, b) \in \mathbb{Z}^2 : a > 0, a + b \geqslant 0 \text{ and } b < 0\}$ also lies in $\Omega_2(S)$. Since the coefficient of Y has to be negative, we consider patterns of the form $p(X, Y) = aX - bY$, with $a, b > 0$ and $a - b \geqslant 0$.

Proposition 6.3. *Let S be a numerical semigroup and $p(X, Y) = aX - bY$ such that $a, b \geqslant 1$ and $a - b = 0$. Then p is admitted by S if, and only if, $a \in S$.*

Proof. Rewrite $p(X,Y) = a(X-Y) + (a-b)Y$ and let s_1 and $s_2 \in S$, with $s_1 \geqslant s_2$. Then $p(s_1,s_2) = a(s_1-s_2) + (a-b)s_2 = a(s_1-s_2)$.
($\Rightarrow$) Since s_1 and s_2 are arbitrary numbers in S, we can choose them such that $s_1 - s_2 = 1$ (for example, s_2 is the conductor of S and $s_1 = s_2 + 1$). In this case, $p(s_1,s_2) = a$. Since p is admitted by S, we conclude that $a \in S$.
($\Leftarrow$) Since $a \in S$ and $s_1 \geqslant s_2$, then $p(s_1,s_2) = a(s_1-s_2) \in S$. Then, p is admitted by S. □

Next result characterizes all patterns of length two admitted by hyperelliptic or ordinary semigroups.

Proposition 6.4. *If S is hyperelliptic or ordinary, then the only patterns of length two not admitted by S are of the form*

$$p(X,Y) = hX - hY, \text{ with } h \in G(S).$$

Proof. Let $p(X,Y) = aX - bY$ be an admissible pattern (so $a \geqslant 1, b > 0$ and $a - b \geqslant 0$). For $s \in S$, $p(s,s) = (a-b)s \in S$. Let s_1 and $s_2 \in S$, with $s_1 > s_2$. Write $p(s_1,s_2) = a(s_1-s_2) + (a-b)s_2$.

- If $a, b \geqslant 1$ and $a - b = 0$, then we can apply Proposition 6.3. We conclude that p is not admitted by S if, and only if, $p(X,Y) = hX - hY$, with $h \in G(S)$.
- If $a, b \geqslant 1$ and $a - b \geqslant 1$, then $p(s_1,s_2) = a(s_1-s_2) + (a-b)s_2 \geqslant s_1$. If S is ordinary, then $s_1 \geqslant c(S)$, for all $s_1 \in S$ (notice that $s_1 > s_2$ implies $s_1 \neq 0$), hence $p(s_1,s_2) \in S$ and p is admitted by S. If S is hyperelliptic, then there are two possibilities for s_1: $s_1 \geqslant c(S)$, which is similar to the previous case, or s_1 even, which implies that s_2 is also even and we conclude that $p(s_1,s_2)$ is also even. Thus $p(s_1,s_2) \in S$. Hence p is always admitted by S.

□

By using similar proof, we generalize the last result.

Corollary 6.2. *Let A be a positive integer and $S := A\mathbb{N}_0 \cup \{c(S), \rightarrow\}$. Then the only patterns of length two not admitted by S are of the form*

$$p(X,Y) = hX - hY, \text{ with } h \in G(S).$$

Remark 6.1. *The numerical semigroup $S = \langle 3,4 \rangle$ does not admit $p(X,Y) = 2X - Y$, since $p(4,3) = 5 \notin S$. This is an example of a numerical semigroup that does not admit a pattern of length two different from $p(X,Y) = hX - hY$, with $h \in G(S)$.*

Proposition 6.5. *Let S be a numerical semigroup and $p(X,Y) = aX - bY$. If $a \in S$ or $b \in S$, then p is admitted by S.*

Proof. If $a \in S$, write $p(X,Y) = a(X-Y) + (a-b)Y$. Let $s_1 \geqslant s_2$, with s_1 and $s_2 \in S$. Then

$$p(s_1,s_2) = \underbrace{a}_{\in S}\underbrace{(s_1 - s_2)}_{\geqslant 0} + \underbrace{(a-b)}_{\geqslant 0}\underbrace{s_2}_{\in S} \in S.$$

If $b \in S$, write $p(X,Y) = (a-b)X + b(X-Y)$. Let $s_1 \geqslant s_2$, with s_1 and $s_2 \in S$. Then

$$p(s_1,s_2) = \underbrace{(a-b)}_{\geqslant 0}\underbrace{s_1}_{\in S} + \underbrace{b}_{\in S}\underbrace{(s_1 - s_2)}_{\geqslant 0} \in S.$$

□

We have already characterized all patterns $p(X,Y) = aX - bY$ admitted by all numerical semigroup with $a = b$. From now on, we assume $a - b \geqslant 1$, in other words, that p is strongly admissible.

We recall that the pseudo-Frobenius set of a numerical semigroup S is

$$PF(S) = \{x \in \mathbb{Z} : x \notin S \text{ and } x + s \in S, \forall s \in S\backslash\{0\}\}.$$

In particular, $F(S) \in PF(S)$.

Lemma 6.6. *Let $z \in PF(S)$ and k be a non-negative integer. Then $kz \in PF(S)$ or $kz \in S$.*

Proof. If $k = 0$, then $kz + s = s \in S$, for all $s \in S$. If $k \geqslant 1$ and $kz \in S$, then we are done. Now, suppose $k \geqslant 1$ and $kz \notin S$. We proceed by induction on k. Let $k = 1$. Since $z \in PF(S)$, then for all $s \in S\backslash\{0\}$, $z + s \in S$. For any positive integer k and all $s \in S$, we write $kz + s = (k-1)z + s + z$. By induction hypothesis, $(k-1)z + s \in S$ and the proof follows. □

Proposition 6.6. *Let S be a numerical semigroup and $p(X,Y) = aX - bY$, with $a - b \geqslant 1$. If $a \in PF(S)$ or $b \in PF(S)$, then p is admitted by S.*

Proof. If $a \in PF(S)$, write $p(X,Y) = a(X-Y) + (a-b)Y$. Let $s_1 \geqslant s_2$, with s_1 and $s_2 \in S$. Then

$$p(s_1,s_2) = \underbrace{\underbrace{a}_{\in PF(S)}\underbrace{(s_1 - s_2)}_{\geqslant 0}}_{\in S \text{ or } \in PF(S), \text{ by Lemma 6.6}} + \underbrace{\underbrace{(a-b)}_{\geqslant 1}\underbrace{s_2}_{\in S}}_{\in S} \in S.$$

If $b \in PF(S)$, write $p(X,Y) = (a-b)X + b(X-Y)$. Similarly, for $s_1 \geqslant s_2$, with s_1 and $s_2 \in S$ $p(s_1,s_2) = (a-b)s_1 + b(s_1 - s_2) \in S$. □

Proposition 6.7. *Let $p(X,Y) = aX - bY$ be a pattern and S be a numerical semigroup, with conductor $c(S)$ and pseudo-Frobenius set $PF(S)$. If $a(s_1 - s_2) \in S \cup PF(S)$, for all $s_1 > s_2 \in S$ with $s_2 < \dfrac{c(S)}{a-b}$, then p is admitted by S.*

Proof. Rewrite $p(X,Y) = a(X-Y)+(a-b)Y$. If $s_2 \geqslant \dfrac{c(S)}{a-b}$, then $(a-b)s_2 \geqslant c(S)$. Hence, for all $s_1 \geqslant s_2$, $p(s_1,s_2) = a(s_1 - s_2) + (a-b)s_2 \in S$. If $s_2 < \dfrac{c(S)}{a-b}$, then $(a-b)s_2 \in S$, because $a - b \geqslant 0$ and $s_2 \in S$. Since $a(s_1 - s_2) \in S \cup PF(S)$, then $p(s_1, s_2)$ is the sum of a pseudo-Frobenius (or an element of S) and an element of S, hence, by Lemma 6.6, it lies in S. □

Proposition 6.8. *Let $p(X,Y) = aX - bY$ be a pattern and let S be a numerical semigroup with conductor $c(S)$ and multiplicity $m(S)$. If $a - b \geqslant \dfrac{c(S)}{m(S)}$, then p is admitted by S.*

Proof. Let $s_1 \geqslant s_2$, with s_1 and $s_2 \in S$. If $s_2 = 0$, then $p(s_1,s_2) = as_1 \in S$. Otherwise, rewrite $p(X,Y) = a(X-Y)+(a-b)Y$. Then $p(s_1,s_2) \geqslant a(s_1 - s_2) + (a-b)m(S) \geqslant c(S)$ and the proof follows. □

We can summarize the results obtained in this section as

Theorem 6.1. *Let S be a numerical semigroup and let $p(X,Y) = aX - bY$ be a pattern which is not admitted by S. Then either*

- *$a = b = h$, with $h \in G(S)$.*
- *$\dfrac{c(S)}{m(S)} > a - b \geqslant 1$, with a and $b \in G(S) \backslash PF(S)$.*

Remark 6.2. *If S is a numerical semigroup, then all patterns $p(X,Y) = hX - hY$, with $h \in G(S)$, are not admitted by S.*

Theorem 6.1 provides an algorithm to decide which patterns p of length two are admitted by a fixed numerical semigroup S.

Example 6.1. *Let $S = \langle 3, 4\rangle$. Then $G(S) = \{1,2,5\}, PF(S) = \{5\}, c(S) = 6, m(S) = 3$ and $c(S)/m(S) = 2$. All possible patterns that do not admit S are*

$$\begin{array}{lll} X - Y & & \\ 2X - Y & 2X - 2Y & \\ \star & \star & 5X - 5Y \end{array}$$

Remark 6.2 ensures that $X - Y, 2X - 2Y$ and $5X - 5Y$ are not admitted by S. Also, $2X - Y$ is not admitted, since $2 \cdot 4 - 3 = 5 \notin S$.

With this algorithm, we can compute lower and upper bounds for the quantity of linear patters of length two which are not admitted by a fixed numerical semigroup S. We recall that the type of a numerical semigroup is the number of elements in $PF(S)$ and it is denoted by $t(S)$.

Proposition 6.9. *Let S be a numerical semigroup and define $N_p(S)$ as the number of patterns of length two not admitted by S. Then*

$$g(S) \leqslant N_p(S) \leqslant g(S) + \binom{g(S) - t(S)}{2}.$$

Proof. We justify the lower bound by using Proposition 6.3. If $p(X, Y) = aX - bY$ is a pattern not admitted by S, then $a = b = h \in G(S)$ or $a - b \geqslant 1$ and $a, b \in G(S) \backslash PF(S)$. There are exactly $g(S) + \binom{g(S) - t(S)}{2}$ of such elements. □

Remark 6.3. *Hyperelliptic semigroups and ordinary semigroups reach the lower bound and $\langle 3, 4\rangle$ reaches the upper bound.*

6.3 An interplay between patterns, the counting problem and (M, γ)-hyperelliptic semigroups

As we mentioned before, Zhao and Zhai considered numerical semigroups S with genus g such that $F(S) < 3 \cdot m(S)$, i.e., $c(S)/m(S) \leqslant 3$. We recall that the number of those numerical semigroups is denoted by t_g. Zhai proved that $\lim_{g \to \infty} \frac{t_g}{n_g} = 1$. Hence, for $g \gg 0$, most of numerical semigroups S satisfy $c(S)/m(S) \leqslant 3$. It is clear that hyperelliptic semigroups S with genus g satisfy $c(S)/m(S) = g$. Thus, if $g \geqslant 4$, then $t_g < n_g$. For hyperelliptic semigroups, Proposition 6.4 ensures that the patterns not admitted by them are $p(X, Y) = hX - hY$, with $h \in G(S)$.

Proposition 6.8 shows that for numerical semigroups S such that $c(S)/m(S) \leqslant 3$ the only possibilities for patterns of length two not admitted by S are $p(X, Y) = aX - bY$, where $a - b \in \{1, 2\}$ and $a, b \in G(S) \backslash PF(S)$.

In this way, patterns $p(X, Y) = aX - bY$, with $a - b \geqslant 3$, not admitted by a numerical semigroup can be rare. In fact, the only possibilities for that is if the numerical semigroup does not satisfy $c(S)/m(S) \leqslant 3$. However, next proposition comes to show that we can obtain pattern $p(X, Y) = aX - bY$ not admitted by a numerical semigroup, with $a - b$ arbitrarily large. It also generalizes Example 6.1.

Proposition 6.10. *Let $S = \langle k + 1, k + 2\rangle$, with $k \geqslant 1$. Then $p(X, Y) = kX - Y$ is not admitted by S.*

Proof. Notice that $m(S) = k + 1, c(S) = k(k + 1)$ and $c(S)/m(S) = k$. We show that $p(k + 2, k + 1)$ is the Frobenius number of S. In fact, $p(k + 2, k + 1) = k(k + 2) - (k + 1) = k^2 + k - 1$. On the other hand, $F(S) = k(k + 1) - 1 = k^2 + k - 1$ and we are done. □

Remark 6.4. *Proposition 6.8 shows that there is no pattern $p(X, Y) = aX - bY$, with $a - b \geqslant k$ not admitted by the numerical semigroup presented Proposition 6.10. Thus, the pattern presented there is extremal.*

To end up this chapter we propose an approach to compute (M, γ)-hyperelliptic semigroups by using patterns, as follows.

Let $p(X_1, \ldots, X_n) = \sum a_i X_i$ be a linear admissible pattern and S be a numerical semigroup with genus γ. By Lemma 6.4, $p(S)$ is a numerical semigroup if, and only if, $\gcd(a_1, \ldots, a_n) = 1$. In those cases, it is clear that all numbers of the form $a_1 \cdot s \in p(S)$, for $s \in S$. Hence, we can think $p(S)$ as a $(a_1, \tilde{\gamma})$-hyperelliptic semigroup, where $\tilde{\gamma} \leqslant \gamma$. For instance, if $a_1 = 2$, then we are dealing with γ-hyperelliptic semigroups.

Let S be a numerical semigroup and $p(X, Y) = 2X - bY$ be an admissible primitive pattern, i.e., b is odd and $b \leqslant 1$. One can prove that if $b \neq 1$, then p is always admitted by S, i.e., $p(S) \subseteq S$. Hence, when we start with a numerical semigroup S and we apply the pattern p, we obtain a numerical semigroup $p(S)$ such that $g(p(S)) \geqslant g(S)$. For the cases that $b = 1$, the pattern $2X - Y$ is premonic and it is also equivalent to the Arf pattern. Hence, if S is an Arf semigroup, then $p(S) = S$. By this reason, the interest seems to be working with non-Arf semigroups A. In those cases, $p(A) \nsubseteq A$. For instance, if $A = \langle 3, 4 \rangle = \mathbb{N}_0 \backslash \{1, 2, 5\}$, then $p(A) = \langle 3, 4, 5 \rangle = \mathbb{N}_0 \backslash \{1, 2\}$.

7 Main results and future work

We left some open questions in this work. Here we summarize our contribution and what remains open. We also give some ideas that can be useful to solve them.

Chapter 2

We proved that the sequence $N_\gamma(g)$ satisfy the following:

- If $g \geqslant 3\gamma$, then $N_\gamma(g) = N_\gamma(3\gamma)$.
- If $g < 3\gamma$, then $N_\gamma(g) < N_\gamma(3\gamma)$.

At first, we cannot compare the numbers $N_\gamma(g_1)$ and $N_\gamma(g_2)$, if $\frac{3\gamma}{2} \leqslant g_1 < g_2 < 3\gamma$. It seems that, under this hypothesis, $N_\gamma(g_1) < N_\gamma(g_2)$. We tried some similar techniques to prove it, but we could not solve the problem completely. It remains as an open problem. It is important to recall that if one can prove that, then it implies that the sequence (n_g) is increasing, giving a positive answer to Question 1.

An interesting fact arises when we do observe Tables 1, 2 and 3 is the following: when g is fixed, the sequence $(N_\gamma(g))_\gamma$ seems to be increasing up to $\left\lfloor \frac{g}{2} \right\rfloor$ and decreasing then. A future work is studying it.

Chapter 3

We studied the sequence $(N_\gamma(3\gamma))$ and we proved that it coincides with the sequence $(f\gamma)$ considered by Bras-Amorós. We also proved some asymptotic properties of this sequence (φ is the golden ratio and ϵ is any positive number):

- $\lim\limits_{\gamma\to\infty} \dfrac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0.$
- $\lim\limits_{\gamma\to\infty} \dfrac{f_\gamma}{2^\gamma} = \infty.$

It means that the sequence f_γ grows more quickly than 2^γ and more slowly than or equal to $(2\varphi)^\gamma$. Some computational experiments suggest that f_γ grows like $\varphi^{2\gamma}$. Proving it remains as an open problem.

We also proved that if f_γ grows like $\varphi^{2\gamma}$, then the following asymptotic properties of f_γ holds true:

- $\lim\limits_{\gamma\to\infty} \dfrac{f_\gamma}{f_{\gamma-1}} = \varphi^2$.

- $\lim\limits_{\gamma\to\infty} \dfrac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i} = \varphi$.

Chapter 4

This chapter was constructed just after we realized that the ideas of the proof of Theorem 3.4 could be applied on some Fibonacci like asymptotic properties of the sequence (n_g). We also realized that there is an asymptotic property of it that do not coincides with the Fibonacci sequence. As a future work, we can compare some other properties between those sequences.

Chapter 5

We studied a natural generalization of γ-hyperelliptic semigroups, the so called (M,γ)-hyperelliptic semigroups. We proved that all those semigroups with a fixed genus g satisfy

- $2g \geqslant (M+1)\gamma$ and this bound is sharp.

We also prove that if O_i is the first element congruent to i modulo M in a (M,γ)-hyperelliptic semigroup with genus g, then

- $Mg - M^2\gamma + \binom{M}{2} \leqslant \sum_{i=1}^{M-1} O_i \leqslant Mg - M\gamma + \binom{M}{2}$.

We also realized that this generalization can be applied to deal with the problem of deciding if the sequence (n_g) is increasing by considering the numbers $N_{(M,\gamma)}(g)$. Some computational experiments we did with $M = 3$ and $M = 4$ suggest that, when γ is fixed, the sequence $(N_{(M,\gamma)}(g))_g$ is increasing. If this sequence is in fact increasing for some integer M, then it implies that the sequence (n_g) is also increasing.

As a future work, we can approach this problem by trying to find a bijection with Kunz and Waldi's approach to deal with numerical semigroups that contain a fixed integer M.

Chapter 6

In this chapter, we deal with patterns on numerical semigroups, specially the ones that can be written as $p(X,Y) = aX - bY$, with non-negative integers $a \geqslant b$. We proved that if p is not admitted by a fixed numerical semigroup S, then either

- $a = b = h$, with $h \in G(S)$;
- $\frac{c(S)}{m(S)} > a - b \geqslant 1$, with a and $b \in G(S) \backslash PF(S)$.

We also prove that there are numerical semigroups that attains last upper bound. More precisely,

- $\langle k+1, k+2 \rangle$, with $k \geqslant 1$ does not admit $p(X,Y) = kX - Y$.

As a future work, we can consider admissible primitive patterns $p(X,Y) = MX - bY$ (and in general, $p = MX_1 + \sum a_i X_i$ with length greater than 2) to construct numerical semigroups. In fact, $p(S)$ is a $(M, \tilde{\gamma})$-hyperelliptic semigroup, with $\tilde{\gamma} \leqslant g(S)$. Is it possible to apply those ideas on the problem of the sequence (n_g)?

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