UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

HERACLIO LEDGAR LÓPEZ LÁZARO

Asymptotic Analysis to non-Autonomous Systems of Incompressible non-Newtonian Fluids

Análise Assintótica para Sistemas não-Autônomos de Fluidos Incompressíveis não-Newtonianos

Campinas

2020

Heraclio Ledgar López Lázaro

Asymptotic Analysis to non-Autonomous Systems of Incompressible non-Newtonian Fluids

Análise Assintótica para Sistemas não-Autônomos de Fluidos Incompressíveis não-Newtonianos

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Supervisor: Gabriela Del Valle Planas

Co-supervisor: Pedro Marín Rubio

Este exemplar corresponde à versão final da Tese defendida pelo aluno Heraclio Ledgar López Lázaro e orientada pela Profa. Dra. Gabriela Del Valle Planas.

Campinas

2020

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

L881a López Lázaro, Heraclio Ledgar, 1991-
Asymptotic analysis to non-autonomous systems of incompressible non-newtonian fluids / Heraclio Ledgar López Lázaro. – Campinas, SP : [s.n.], 2020.

Orientador: Gabriela Del Valle Planas.
Coorientador: Pedro Marín Rubio.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Comportamento assintótico de soluções. 2. Fluidos não-newtonianos. 3. Soluções fracas (Matemática). 4. Modelo de Ladyzhenskaya. I. Planas, Gabriela Del Valle, 1972-. II. Marín Rubio, Pedro, 1976-. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações para Biblioteca Digital

Título em outro idioma: Análise assintótica para sistemas não-autônomos de fluidos incompressíveis não-newtonianos
Palavras-chave em inglês:
Asymptotic behavior of solutions
Non-newtonian fluids
Weak solutions (Mathematics)
Ladyzhenskaya model
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Gabriela Del Valle Planas [Orientador]
Anne Caroline Bronzi
Marcelo Martins dos Santos
Paulo Mendes de Carvalho Neto
Ma To Fu
Data de defesa: 27-03-2020
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0003-0457-117X
- Currículo Lattes do autor: http://lattes.cnpq.br/1643805445053989

Tese de Doutorado defendida em 27 de março de 2020 e aprovada

pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). GABRIELA DEL VALLE PLANAS

Prof(a). Dr(a). ANNE CAROLINE BRONZI

Prof(a). Dr(a). MARCELO MARTINS DOS SANTOS

Prof(a). Dr(a). PAULO MENDES DE CARVALHO NETO

Prof(a). Dr(a). MA TO FU

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Este trabajo está dedicado a la mujer que me crió y amó
incondicionalmente, quien me definió como um
hombre de bien y me enseñó que en los mínimos detalles
está la felicidad. Todo esto es por ti, mi madre-abuelita querida.
Señora María del Rosario Seminario Rubio.
Te amo hasta el cielo.

Acknowledgements

Antes de tudo, graças a Deus pela força e segurança que ele me deu em tudo este caminho, *Porque dele e por ele, e para ele, são todas as coisas; glória, pois, a ele eternamente. Amém. Romanos 11:36*

Agradeço à minha avó, María del Rosario, embora ela não esteja mais conosco, ela é a principal autora de todas as minhas realizações, *Infinitas gracias mi viejita hermosa*. Aos meus pais, Sra. Blanca Lázaro e Sr. Alfonso López, que com amor e, acima de tudo, confiança deixaram tomar minhas próprias decisões e me apoiaram até o fim, *Infinitas gracias papitos lindos*. À minha companheira de vida, luta, aventura, minha namorada Jackeline del Carmen, que faz parte deste trabalho, porque o viu nascer há 11 anos e tem estado em todas as etapas até hoje, *Gracias mi amor*. Para meus irmãos Lenard, Lenz e Levi, pelo apoio incondicional de estar sempre lá para, mim *Gracias mis cholos vagos*.

Agradeço à Professora Dra. Gabriela Del Valle Planas, pela confiança, orientação, comprometimento e dedicação proporcionada no processo de doutorado, que fez que seja uma etapa muito agradável, abrindo inúmeraveis novas oportunidades neste maravilhoso mundo da pesquisa. Agradeço ao Professor Dr. Pedro Marin Rubio, pela amizade, oportunidade de trabalhar com ele e por ter aberto as portas da Universidad de Sevilla, tornando esta uma estância confortável.

Um agradecimento especial aos meus grandes amigos, Lizeth Santa Cruz e John Matos, que fazem parte da minha família e da minha vida.

Um agradecimento especial ao meu bom amigo e irmão da vida, Carlos Troncos Sanchez, pela ajuda e conselhos na redação deste trabalho.

No final, mas não por último, para agradecer a confiança e o laço inabalável, meu bom amigo de aventura, bom irmão dos conselhos e compadre, porque a vida queria dessa maneira, Sr. Martin Salinas Albarran. Obrigado por tudo.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001 e do Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, processo nº 2018/140649-2.

Agradeço ao SANTANDER e IMUS pelo apoio financeiro para a realização do período co-tutela/sandwich na Universidad de Sevilla.

"Não vos amoldeis às estruturas deste mundo,
mas transformai-vos pela renovação da mente,
a fim de distinguir qual é a vontade de Deus:
o que é bom, o que Lhe é agradável, o que é perfeito.
(Bíblia Sagrada, Romanos 12, 2)

Resumo

Neste trabalho investigamos o comportamento assintótico de modelos matemáticos que descrevem o escoamento de fluidos incompressíveis não-Newtonianos, com e sem termos de retardo. Supomos que o fluido ocupa um domínio limitado com fronteira regular em duas ou três dimensões. Nestes modelos o tensor de estresse, associado à viscosidade do fluido, é caracterizado como sendo uma função que satisfaz condições de p-coercividade e $(p-1)$-crescimento.

Para o modelo sem termo de retardo, além de resultados de existência, unicidade e regularidade de solução, mostramos existência de atratores do tipo *pullback* sobre universos temperados nos espaços de Banach $(L^2)^n$ e $(W_0^{1,p})^n$ com divergente nulo, denotados por H e V_p, respectivamente. Em H, o comportamento assintótico *pullback* é analisado usando um método de energia para $p \geqslant 1 + 2n/(n+2)$. Neste caso, as soluções fracas definem um processo multívoco que é semi-contínuo superiormente e fechado. Para o estudo de atratores do tipo *pullback* em V_p, além de maior regularidade na força externa, será necessário que o tensor de estresse seja um potencial e $p \geqslant 5/2$ se $n = 3$ ou $p > 2$ se $n = 2$. Assim, o processo definido sobre V_p torna-se um processo unívoco, devido à unicidade da solução fraca. Finalmente, mostramos um resultado de comparação dos atratores em H e V_p.

Para o modelo com termos de retardo, primeiramente mostramos a existência de soluções fracas para $p \geqslant 1 + 2n/(n+2)$. A partir das soluções fracas é definido um processo multívoco e provada a existência de atratores do tipo *pullback*, em universos temperados, definidos sobre dois espaços de Banach diferentes. A existência dos atratores dependerá de p e de um parâmetro relacionado aos coeficientes de viscosidade do tensor de estresse e aos parâmetros associados aos termos de retardo.

Palavras-chave: fluidos incompressíveis não-Newtonianos; atratores *pullback*; processo multívoco; retardos; modelo de Ladyzhenskaya.

Abstract

In this work, we investigate the asymptotic behavior of mathematical models that describe the flow of non-Newtonian incompressible fluids, with and without delay terms. We assume that the fluid occupies a smooth bounded domain in two or three dimensions. In these models, the stress tensor, associated with fluid viscosity, is characterized as a function that satisfies p-coercivity and $(p-1)$-growth conditions.

For the model without delay term, in addition to results of existence, uniqueness and regularity of the solution, we show the existence of pullback attractors on tempered universes in the Banach spaces $(L^2)^n$ and $(W_0^{1,p})^n$ with divergence-free, denoted by H and V_p, respectively. In H, the pullback asymptotic behavior is analyzed by an energy method for $p \geqslant 1 + 2n/(n+2)$. In this case, weak solutions define a multi-valued process that is upper semi-continuous with closed values. For the study of pullback attractors in V_p, besides the higher regularity on the external force, it will be necessary the stress tensor to be a potential and $p \geqslant 5/2$ if $n = 3$ or $p > 2$ if $n = 2$. Thus, the process defined on V_p becomes a single-valued process, due to the uniqueness of the weak solution. Finally, we show a comparison result between the attractors in H and V_p.

For the model with delay terms, we first show the existence of weak solutions for $p \geqslant 1 + 2n/(n+2)$. From the weak solutions, a multi-valued process is defined and the existence of pullback attractors, in tempered universes, defined over two different Banach spaces is proven. The existence of the attractors will depend on p and a parameter related to the viscosity coefficients of the stress tensor and the parameters associated with the delay terms.

Keywords: incompressible non-Newtonian fluids; pullback attractors; multi-valued process; delays; Ladyzhenskaya model

Contents

Introduction **12**

1 NOTATIONS AND IMPORTANT RESULTS **20**
1.1 Notations **20**
1.1.1 Basic Notations 20
1.1.2 Functional Spaces 21
1.1.3 Notation of Functional Analysis 23
1.2 Results of Analysis and Topology **24**
1.3 Results of Functional Analysis **24**
1.3.1 Compactness, Inequalities and Embedding Results 24
1.3.2 Interpolation Results 27
1.3.3 Basis Consisting of Eigenfunctions 27
1.4 Ordinary Differential Equations **29**
1.4.1 Ordinary Differential Equations **(ODE)** 29
1.4.2 Delay Ordinary Differential Equations **(DODE)** 30
1.5 Pullback Attractors **31**
1.5.1 Pullback Attractor for Closed Process 31
1.5.2 Pullback Attractor for Multi-valued Process 33
1.5.3 Comparison of attractors 35
1.6 Compressible and Incompressible Fluids **36**

2 A CLASS OF INCOMPRESSIBLE NON-NEWTONIAN FLUIDS . . **44**
2.1 Statement of the Problem **44**
2.2 Existence of Weak Solutions to (LM) **49**
2.3 Uniqueness of Weak Solutions to (LM) **57**
2.4 A Regularity Result **59**
2.5 Existence of Pullback Attractors **62**
2.5.1 Pullback attractor in H 62
2.5.2 Pullback Attractor in V_p 72

3 A CLASS OF INCOMPRESSIBLE NON-NEWTONIAN FLUID WITH DELAY **81**
3.1 Statement of the Problem **81**
3.2 Existence of Weak Solutions to (LMD) **83**
3.3 Uniqueness of Weak Solution to (LMD) **86**
3.4 Existence of Pullback Attractors in C_H and M^2_H **89**

3.5 Continuity of the Processes $U(\cdot,\cdot)$ and $S(\cdot,\cdot)$ 101

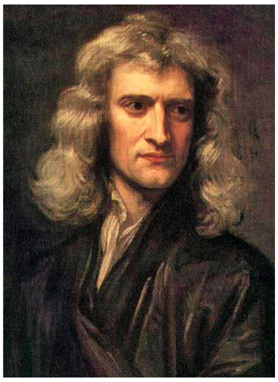
4 FINAL COMMENTS AND FUTURE PROPOSALS 106

BIBLIOGRAPHY 109

Introduction

Mathematical models that describe the behavior of fluid flows as the air, water, oil, blood, and many other magnitudes have been subject of interest in the last two hundred years to the chemistry, physics, biology, economics, among others areas. Let us do a little and brief tour through the history of dynamical systems and its importance. Let us start in the 16th century with big minds like Issac Newton, who contributed to the mathematical modeling through the formalization of classical mechanics, and Johannes Kepler, who contributed with works in celestial mechanics. They started new mathematical concepts, extracted from physics, such as the dynamical systems that are still studied today.

Isaac Newton (1642-1727)



Johannes Kepler (1571-1630)

French mathematician Henri Poincaré is considered one of the creators of the modern theory of the dynamical systems. His work "Les méthodes nouvelles de mécanique céleste", published in 1892 [52], allows us to understand the stability, periodicity, and asymptotic behavior of solutions of a differential equation without the need to explicitly know them.

In 1927, the American mathematician George Birkhoff, internationally recognized by the so-called Ergodic Theorem, published the work Dynamical Systems [5], considered the first book in the area of dynamical systems.

Following this timeline, let us think about the next question "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" This question can be interpreted like is it possible that two relatively close objects, as they evolve over time, could differ greatly? The answer to this question can be found in chaos theory.

The chaos theory is a branch of nonlinear dynamical systems theory which is very sensitive to variations in initial conditions. One of the pioneers in the chaos theory was Edward Lorenz, American meteorologist, mathematician, and philosopher. He contributed

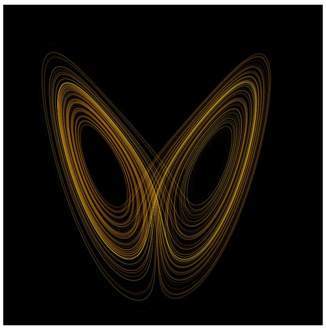
Henri Poincaré (1854-1912)

George Birkhoff (1884-1944)

with a mathematical model that describes the behavior of convection in the atmosphere. With this model, he realized that at minimal alterations in the initial data, they resulted in widely divergent solutions, whose phenomenon is known as the “butterfly effect”. Another interesting observation is that solutions oscillate irregularly without repeating themselves in a bounded region of the phase space; in a modern language, this is known as an attractor.

Edward Lorenz (1917-2008)

Lorenz Attractor

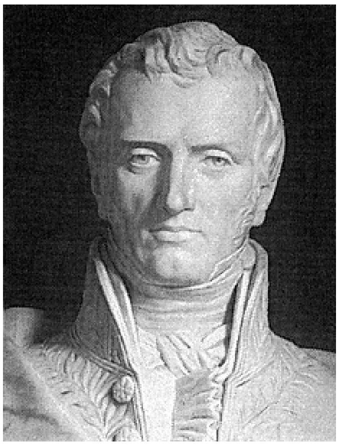


The study of the asymptotic behavior of solutions to differential equations is fundamentally important to understand the behavior of the solutions for long periods of time. And for this study, the concept of attractor has been widely used in recent years.

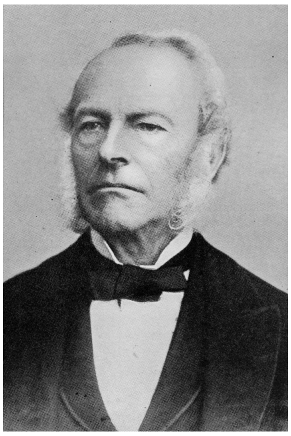
An attractor is a subset of the phase-space toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if they are slightly disturbed. For autonomous dynamical systems, the concept of a global attractor is widely employed, see, for instance, [34, 49, 54, 56]. For non-autonomous dynamical systems, the study of pullback attractors has received a lot of attention, since it can be seen as a generalization of the global attractor, see, among others, [17, 35]. This is not the only approach, uniform attractors, skew-product flows and attractors are other valid options.

An example where the existence and uniqueness of the solution, and their asymptotic behavior which are studied in the autonomous and non-autonomous cases, is the well-known Navier-Stokes system, named after Claude-Louis Navier and George Gabriel Stokes; [23, 42, 45, 54]. This system is obtained by applying the principles of

Claude L. H. Navier (1785-1836)



George G. Stokes (1819-1903)



conservation and thermodynamics to a volume of fluid. This system is useful to describe the velocity and pressure of incompressible Newtonian fluids and it is given by

$$\frac{\partial \mathbf{u}}{\partial t} - \nu_0 \Delta \mathbf{u} = -\sum_{i=1}^{n} u_i \frac{\partial \mathbf{u}}{\partial x_i} - \nabla p + \mathbf{f} \quad \text{in } \Omega \times (0, \infty), \tag{1}$$

$$\mathrm{div}_x \mathbf{u} = 0 \quad \text{in } \Omega \times (0, \infty), \tag{2}$$

where $\mathbf{u} = (u_1, \cdots, u_n)$ is the fluid velocity, p is the pressure, $\mathbf{f}$ an external force, $\nu_0 > 0$ is the viscosity and $\Omega \subset \mathbb{R}^n$ is a domain.

Solutions of the Navier–Stokes equations often include turbulence, which remains one of the greatest unsolved problems in Physics, despite its immense importance in other science as Engineering. For the Navier–Stokes existence and smoothness problem, mathematicians have not proved yet that given smooth initial conditions, for the three-dimensional Navier-Stokes system, the smooth solutions always exist; and if they existed, they would have bounded energy. This is called the Navier–Stokes existence and smoothness problem. In May 2000, the Clay Mathematics Institute made this problem one of its seven Millennium Prize problems in mathematics. The institute offers a US$ 1,000,000 prize to the first person who provides a solution for a specific statement of the problem [16].

On the other hand, regarding the asymptotic behavior of the solutions of the Navier-Stokes system in the autonomous case, i.e., when the external force $\mathbf{f}$ is independent of t, on a smooth bounded domain $\Omega \subset \mathbb{R}^2$, the existence of global attractor can be proved "easily", since the solutions are regular, [42, 54, 56]. But, the three-dimensional case is more involved because the uniqueness of the solution is still unknown. Therefore, other concepts are introduced to be able to investigate the asymptotic behavior of solutions, [17, 19, 23, 24, 57]. In the non-autonomous case, i.e., when the external force $\mathbf{f}(t)$ depends on t, the asymptotic behavior of the solutions was analyzed by several approaches cited above, in particular, within the framework of pullback attractors. For example, in the two-dimensional case, the pullback dynamics was analyzed by using the energy method on the spaces H and V, which consist, roughly speaking, of all functions in $L^2(\Omega)^2$ and $W_0^{1,2}(\Omega)^2$ with divergence-free, respectively; see [27, 28, 31]. Other related results about the Navier-Stokes system can be seen in [10, 11, 48, 55].

There are also other types of fluids in nature that in mathematical physics are called non-Newtonian fluids [23, 39, 42, 45]; for example, blood, butter, lava, etc.; unlike the Newtonian fluids, their viscosity varies as the temperature and shear stress are applied. In 1969, Russian mathematician Olga Aleksandrovna Ladyzhenskaya proposed some mathematical models, known as Ladyzhenskaya models [21, 37, 38, 39, 41], for incompressible viscous flows or incompressible non-Newtonian fluids.

Olga Ladyzhenskaya (1922-2004)

They are described by

$$\frac{\partial u_j}{\partial t} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\big(\mathbb{S}^i_{jk}(\mathbf{e}(\mathbf{u}))\big) = -\sum_{i=1}^{n} u_i \frac{\partial u_j}{\partial x_i} - \frac{\partial p}{\partial x_j} + f_j \quad \text{in } \Omega \times (0,\infty), \tag{3}$$

$$\mathrm{div}_x \mathbf{u} = 0 \quad \text{in } \Omega \times (0,\infty), \tag{4}$$

$i = 1, 2$ and $j = 1, 2$ or $j = 1, 2, 3$, and $\mathbb{S}(\mathbf{e}(\mathbf{u}))$ is defined by

$$\mathbb{S}^1(\mathbf{e}(\mathbf{u})) = \nu_1 \mathbf{e}(\mathbf{u}) + \nu_2 |\mathbf{e}(\mathbf{u})|^{p-2} \mathbf{e}(\mathbf{u}),$$

or its special case

$$\mathbb{S}^2(\mathbf{e}(\mathbf{u})) = \big(\nu_1 + \nu_2 \|\mathbf{e}(\mathbf{u})\|^2_{L^2(\Omega)^n}\big)\mathbf{e}(\mathbf{u}),$$

where $\nu_1, \nu_2 > 0$ are the viscosity coefficients, $\mathbb{S}^i(\mathbf{e}(\mathbf{u}))$, $i = 1$ or 2, are the stress tensors associated to this system and $\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u}^T + \nabla \mathbf{u})$ is the symmetric gradient. Many models, also known as variants of the Navier-Stokes system or modified Navier-Stokes equations are characterized by the stress tensor that we will denote by $\mathbb{S}$. When $\mathbb{S}(\mathbf{e}(\mathbf{u})) = 2\nu \mathbf{e}(\mathbf{u})$ with $\nu > 0$, then the equations are reduced to the Navier-Stokes equations (1).

For non-Newtonian fluids, in general, the stress tensor is a nonlinear convex function (in some cases the stress tensor is a potential, see definition 1.48) and satisfies the following conditions: there exist constants $C_1, C_2 > 0$ and parameters $p > 1$ and $q \in [p-1, p)$ such that, for all $\mathbf{M} \in \mathbb{R}^{n^2}_{sym}$,

$$\mathbb{S}(\mathbf{M}) : \mathbf{M} \geqslant C_1 |\mathbf{M}|^p \qquad (p \text{ - } \textbf{coercivity} \text{ condition}),$$

$$|\mathbb{S}(\mathbf{M})| \leqslant C_2 (1 + |\mathbf{M}|)^q \qquad (q \text{ - } \textbf{growth} \text{ condition}).$$

For non-Newtonian fluids, in typical cases, the stress tensor has the form:

$$\mathbb{S}(\mathbf{e}(\mathbf{u})) = 2(\nu + \nu_0 |\mathbf{e}(\mathbf{u})|^{p-2})\mathbf{e}(\mathbf{u}) \quad \text{or} \quad \mathbb{S}(\mathbf{e}(\mathbf{u})) = 2(\nu + \nu_0 |\mathbf{e}(\mathbf{u})|^2)^{\frac{p-2}{2}} \mathbf{e}(\mathbf{u}), \quad p > 1,$$

with $\nu, \nu_0 > 0$.

Therefore, the motion of incompressible non-Newtonian fluids on a domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, can be characterized by the velocity field $\mathbf{u} = (u_1, \cdots, u_n)$ and the pressure π, and governed by the system of $n + 1$ equations given by

$$\frac{\partial \mathbf{u}}{\partial t} - \text{div}_x \mathbb{S}(\mathbf{e}(\mathbf{u})) = -\text{div}_x(\mathbf{u} \otimes \mathbf{u}) - \nabla_x \pi + \mathbf{f}(t) \quad \text{in} \quad \Omega_{\tau,T}, \tag{5}$$

$$\text{div}_x \mathbf{u} = 0 \quad \text{in} \quad \Omega_{\tau,T}, \tag{6}$$

subject to initial and boundary conditions. Here, $\Omega \subset \mathbb{R}^n, n = 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$, $\tau < T$, where we have denoted by $\Omega_{\tau,T} := (\tau, T) \times \Omega$ and $\mathbf{u} \otimes \mathbf{u} = (u_i u_j)_{i,j}$.

Next, we recall some results about existence, uniqueness and asymptotic behavior of the solutions associated with the system (5)-(6) in the autonomous and non-autonomous cases. Let us start with the autonomous case, i.e., when the external force $\mathbf{f}$ is independent of time t. J. Nečas et al. [45, chapter 5] studied the uniqueness and existence of measure-valued, weak, and strong solutions subject to space-periodic boundary conditions and Dirichlet conditions. J. Málek and J. Nečas [44] proved the existence of a global attractor with finite fractal dimension for three-dimensional flow of incompressible fluids. J. Málek and D. Pražák [46] introduced a new criterion for the finiteness of the fractal dimension of the attractor through the method of short trajectories for the cases: two-dimensional if $p \geqslant 2$ and three-dimensional if $p \geqslant 11/5$, subject to space-periodic boundary conditions. E. Feireisl and D. Pražák [23] analyzed on the asymptotic behavior on $L^2(\Omega)^n$ for incompressible non-Newtonian fluids when $p \geqslant 2$ and for $1 < p < 2$ only in the two-dimensional case. D. Pražák and J. Žabenský [53] showed that there exists an exponential attractor for a perturbed three-dimensional Ladyzhenskaya model. Finally, for the non-autonomous case, i.e., the external force $\mathbf{f} = \mathbf{f}(t)$ dependents of time t, very recently, Yang et al. [60] showed a result that establishes the existence of finite-dimensional pullback attractors in a general setting involving tempered universes for one of the models proposed by Ladyzhenskaya. Other results can be seen in [63, 64, 65].

As we can see, there exist many results on the theory of the existence of solutions and global attractors for the autonomous case. However, the pullback dynamics for incompressible non-Newtonian fluids is much less explored. Therefore, the first aim of this thesis is to investigate the existence of pullback attractors for this kind of problems.

We observe that there are some difficulties in analyzing the pullback dynamics of non-Newtonian fluids in comparison to Newtonian fluids. Even though if it were possible to consider the three-dimensional case with the uniqueness of solutions in the case that p is large enough; there would still be some additional obstacles in proving the asymptotic compactness of the process. This happens mainly because there is no higher regularity, for instance $W^{2,2}$ - regularity, to the solution. In this way, we have to explore the p-integrability of the solutions as well as the regularity of the time partial derivative to achieve our objective.

The second aim of this thesis is to study the well-posedness and the existence of pullback attractors on tempered universes, for the system (5)-(6) with a delay term:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}_x \mathbb{S}(\mathbf{e}(\mathbf{u})) + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) = -\nabla_x \pi + \mathbf{f}(t) + \mathbf{g}(t, \mathbf{u}_t) \quad \text{in } \Omega_{\tau,T}, \tag{7}$$

$$\mathrm{div}_x \mathbf{u} = 0 \quad \text{in } \Omega_{\tau,T}, \tag{8}$$

$$\mathbf{u} = 0 \quad \text{on } (\tau, T) \times \partial\Omega, \tag{9}$$

$$\mathbf{u}(\tau, x) = \mathbf{u}^\tau(x) \quad x \in \Omega, \tag{10}$$

$$\mathbf{u}(\tau + t, x) = \phi(t, x) \quad \text{in } \Omega_h, \tag{11}$$

where $\mathbf{u}^\tau$ is the velocity of fluid in the initial time $t = \tau$, $g(\cdot, \mathbf{u}_\cdot)$ is the delay term, with $\mathbf{u}_t(s) = \mathbf{u}(t+s)$ for all $s \in (-h, 0)$, and ϕ is the initial condition defined on $\Omega_h := (-h, 0) \times \Omega$.

Delay differential equations, also known as functional differential equations were originated in problems in geometry and number theory, see [3, 58]; and it has been studied for at least 200 years. The subject gained much attention after 1940 due to the consideration of meaningful models to physical systems and control. It is probably true that many scientists of that time were well aware of the fact that hereditary effects occur in physical systems, but this effect was often ignored because of the lack of a well-established theory.

This kind of equations appears in various applications, such as viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flow, neural networks, combustion, the interaction of species, microbiology, learning models, epidemiology, physiology, as well as many others, see Kolmanovskii and Myshkis [36].

Let us mention some historical facts about systems with delays. For instance, at the IV-International Congress of Mathematicians, based in Rome in 1908, the French mathematician Charles Émile Picard [1, 51] made the following statements about the importance of considering hereditary effects in physical systems:

Les équations différentielles de la mécanique classique sont telles qu'il en résulte que le mouvement est déterminé par la simple connaissance des positions et des vitesses, c'est-à-dire par l'état à un instant donné et à l'instant infiniment voison.

Les états antérieurs n'y intervenant pas, l'hérédité y est un vain mot. L'application de ces équations oú le passé ne se distingue pas de l'avenir, oú les mouvements sont de nature réversible, sont donc inapplicables aux êtres vivants.

Nous pouvons réver d'équations fonctionnelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre un temps passé trés éloigné et le temps actuel, qui apporteront la part de l'hérédité.

The Italian mathematician and physicist Vito Volterra studied integro-differential equations that model viscoelasticity [33], and in 1930, he wrote a book on the impact of hereditary effects on models for the interaction of species [58].

Nowadays, there is a vast literature regarding systems of equations with delays, for example, see [3, 43]. Concerning with delay systems for incompressible Newtonian fluids, we must mention T. Caraballo and J. Real [13, 14, 15]. In these papers, the authors showed the existence of weak solutions to 2D Navier–Stokes equations, when their external force contains some hereditary characteristics and besides that, they proved the existence of pullback attractors. J. García-Luengo, P. Marín-Rubio and J. Real [29, 30] analyzed some new regularity results of pullback attractors for 2D Navier-Stokes equations with delays. In the case of incompressible non-Newtonian fluids with delays involving fourth-order operators, C. Zhao et al. [62, 66] investigated the existence of solutions and pullback attractors and L. Liu, T. Caraballo and X. Fu [43] analyzed their exponential stability.

Having mentioned some historical facts about the development of physical and mathematical concepts for a better understanding of the importance about the study of attractors for differential equations with and without delay, we present the organization of this thesis and describe the main results.

In Chapter 1, we give the preliminaries, notations and results that we will use in the development of all this work, as interpolation theorems in Banach space, compact embeddings, existence and uniqueness of solutions for ordinary differential equations with delay and without delay **(DODE)**, **(ODE)**. We recall the theory of the existence of pullback attractors for upper-semicontinuous multi-valued process and closed process. Finally, we give a brief introduction about the theory of incompressible and compressible non-Newtonian fluids.

In Chapter 2, concerning the existence and uniqueness of weak solutions, we start with the formulation and justification of the problem with Cauchy-Dirichlet conditions (5)-(6) and the stress tensor satisfying p-coercivity and $(p-1)$-growth conditions. Then, we recall and prove the existence of weak solutions, when $p \geqslant 1 + 2n/(n+2)$ (this proof is similar to the one described in [42, Theorem 5.1]). Moreover, we show the uniqueness of weak solution, in the cases: $p \geqslant 2$ if $n = 2$, and if the weak solutions belong to $L^{\frac{2p}{2p-3}}(\tau, T; V_p)$ for $n = 3$ (this result can also be found in [23, Theorem 7.6]). Finally, we give a regularity result for weak solutions that can be approximated by sequences of regular solutions in the following cases: $p \geqslant 12/5$ when $n = 3$, and $p > 2$ when $n = 2$ (this result can be found in [23, Theorem 7.32]). In section 2.5, we show the existence of pullback attractors on tempered universes for the non-autonomous incompressible non-Newtonian fluids on the Hilbert space H for $p \geqslant 1 + 2n/(n+2)$. To this end, we perform an analysis on a multi-valued process defined from weak solutions, since the uniqueness of weak solutions is not guaranteed. We conclude this chapter showing the existence of pullback attractors on V_p. In order to do that, we built tempered universes on V_p from tempered universes defined on H, for $p \geqslant 5/2$ if $n = 3$ and for $p > 2$ if $n = 2$. Here, it is necessary to ask for more regularity on the external force and initial conditions. Moreover, the stress tensor should be a potential (see definition 1.48) to make the estimates that allow us to show that the process defined on V_p is asymptotically compact.

In Chapter 3, we first establish the formulation of the problem with Cauchy-Dirichlet condition for incompressible non-Newtonian fluids with delay, given by the system (7)-(11). After that we show the existence and uniqueness of weak solutions, when the initial conditions belong to H and to $L^2_H = L^2(-h, 0; H)$, with $h > 0$. In section 3.4, we show the existence of pullback attractors for the upper-semicontinuous multi-valued process in tempered universes defined on the phase spaces $C_H := C([-h, 0]; H)$ and $M^2_H := H \times L^2(-h, 0; H)$ when $p \geqslant 1 + 2n/(n+2)$. Finally, we prove that the process is a continuous process on the Banach spaces C_H and M^2_H when $p \geqslant (n+2)/2$.

In Chapter 4, we show a summary of all results established in this research, and finally, we conclude with some observations and future proposals.

1 Notations and Important Results

In this section, we introduce notations that are used in functional analysis, partial differential equations and the theory of dynamical systems. Moreover, we state some important results that we will use in the development of all the work, as interpolation theorems in Banach spaces, compact embeddings, existence and uniqueness of solutions for ordinary differential equations with delay and without delay. We recall the theory of the existence of pullback attractors for upper-semicontinuous multi-valued process and closed process. Finally, we give an introduction to the theory of incompressible and compressible non-Newtonian fluids. All results can be found in the books [2, 7, 22, 23, 25, 42, 50, 45].

1.1 Notations

1.1.1 Basic Notations

- We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of the natural and real numbers, respectively.
- $\big(\mathbb{R}^n, |\cdot|\big)$ is the n-dimensional Euclidean spaces, for $n = 1, 2, 3, ...$, and their elements will be denoted by $\mathbf{a} = (a_1, ..., a_n)$.
- We denote by $\mathbb{R}^2_d = \{(s,t) \in \mathbb{R}^2 : s \leqslant t\}$.
- Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the scalar product of vectors $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ is denoted by
$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_i b_i.$$
- In general, we are going to consider Ω an open bounded subset of $\mathbb{R}^n$.
- $\partial\Omega$ will denote the boundary of the set Ω.
- Given $T, \tau \in \mathbb{R}$, with $T > \tau$, we denote by $\Omega_{\tau,T} = (\tau, T) \times \Omega$ and by $\Sigma_{\tau,T} = (\tau, T) \times \partial\Omega$.
- The set $\mathbb{R}^{n^2}$ will denote the set of square matrices $n \times n$, and their elements will be denoted by $\mathbf{A} = (a_{i,j})_{i,j=1}^n$. And $\mathbb{R}^{n^2}_{sym}$ will denote the set of symmetric square matrices.
- The transpose of a square matrix $\mathbf{A} = (a_{i,j})_{i,j=1}^n$ is $\mathbf{A}^T = (a_{j,i})_{i,j=1}^n$.
- The scalar product of tensors $\mathbf{A} = (a_{i,j})_{i,j=1}^n$, $\mathbf{B} = (b_{i,j})_{i,j=1}^n$ reads
$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{n} a_{i,j} b_{j,i} = a_{i,j} b_{j,i}.$$

- The symbol $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of vectors $\mathbf{a}$, $\mathbf{b}$, specifically,

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{a} \otimes \mathbf{b})_{i,j} = a_i b_j.$$

- The product of a matrix $\mathbf{A}$ with a vector $\mathbf{b}$ is a vector $\mathbf{Ab}$ whose components are

$$(\mathbf{Ab})_i = \sum_{j=1}^{n} A_{i,j} b_j \quad for\ i = 1, \cdots, n.$$

- The product of a matrix $\mathbf{A} = (a_{i,j})_{i,j=1}^{n}$ and a matrix $\mathbf{B} = (b_{i,j})_{i,j=1}^{n}$ is a matrix $\mathbf{AB}$ with components

$$(\mathbf{AB})_{i,j} = \sum_{r=1}^{n} a_{i,r} b_{r,j}.$$

1.1.2 Functional Spaces

- $C(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is continuous function}\}$.
- $C^k(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is } k\text{-times continuously differentiable}\}$.
- $C^k(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{R} : f \text{ is } k\text{-times continuously differentiable}\}$.
- $C^\infty(\Omega) = \cap_{k=1}^{\infty} C^k(\Omega)$ and $C^\infty(\overline{\Omega}) = \cap_{k=1}^{\infty} C^k(\overline{\Omega})$.
- $C_c^k(\Omega) = \{f \in C^k(\Omega) : \text{supp}(f) \text{ is compact}\}$, $\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$.
- $\mathcal{D}(\Omega)$ is the space $C_c^\infty(\Omega)$, with the following sense of convergence: $f_n \to f$ in $\mathcal{D}(\Omega)$, if there exists a compact $K \subset \Omega$ such that $\{f_n\} \subset C_c^\infty(K)$ and $D^\alpha f_n \to D^\alpha f$ uniformly for any α, where α denoted the multi-index. We say that $\mathcal{D}(\Omega)$ is the space of test functions.
- $\mathcal{D}'(\Omega)$ will denote the topological dual of $\mathcal{D}(\Omega)$.
- For $p \in [1, +\infty)$, $L^p(\Omega)$ is the set of all the p-integrable measurable functions, with norm defined by

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

- For $p = +\infty$, $L^\infty(\Omega)$ is the set of all the essentially bounded measurable functions, with norm defined by

$$\|f\|_\infty = \|f\|_{L^\infty(\Omega)} := ess\ sup_\Omega |f|.$$

- $L^p(\Omega)^n = \{\mathbf{f} = (f_1, \ldots, f_n) : f_i \in L^p(\Omega),\ \forall i = 1, ..., n\}$, with norm

$$\|\mathbf{f}\|_p = \left(\sum_{k=1}^{n} \|f_i\|_p^p \right)^{1/p}.$$

- Let $\tau \leqslant T$ be and $h > 0$. Given a space X and a function $\mathbf{u} : [\tau - h, T] \to X$, for each $t \in [\tau, T]$ we denote by $\mathbf{u}_t : [-h, 0] \to X$ such that $\mathbf{u}_t(s) = \mathbf{u}(t + s)$ for all $s \in [-h, 0]$.

Remark 1.1. *For all $p \in [1, +\infty]$, we know that $\big(L^p(\Omega), \|\cdot\|_p\big)$ is a Banach space, and for the particular case $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with inner product*

$$(f, g) = \int_\Omega f(x)g(x)dx.$$

We will denote the norm in $L^2(\Omega)$ by $|\cdot|_2$.

Remark 1.2. *For all $p \in [1, +\infty]$, we know that $\big(L^p(\Omega)^n, \|\cdot\|_p\big)$ is a Banach space, and for the particular case $p = 2$, the space $L^2(\Omega)^n$ is a Hilbert space with inner product*

$$(\mathbf{f}, \mathbf{g}) = \sum_{i=1}^{n} (f_i, g_i).$$

We will denote the norm in $L^2(\Omega)^n$ by $|\cdot|_2$.

- $W^{k,p}(\Omega) = \{f : D^\alpha f \in L^p(\Omega),\ \forall |\alpha| \leqslant k\}$ with $k \in \mathbb{N}$, $p \in [0, \infty]$, α is a multi-index and $D^\alpha f$ is the derivative of order α of f in the sense of the distributions. This space is provided with the norm

$$\|f\|_{k,p} = \begin{cases} \Big(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p^p \Big)^{1/p} & \text{if } p \in [1, \infty), \\ \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_\infty & \text{if } p = \infty. \end{cases}$$

- $W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{k,p}(\Omega)}$, with the norm of $W^{k,p}(\Omega)$

- $W^{k,p}(\Omega)^n = \{\mathbf{f} = (f_1, \ldots, f_n) : f_i \in W^{k,p}(\Omega)\}$, with norm

$$\|\mathbf{f}\|_{k,p} = \Big(\sum_{i=1}^{n} \|f_i\|_{k,p}^p \Big)^{1/p}$$

- $W_0^{k,p}(\Omega)^n = \{\mathbf{f} = (f_1, \ldots, f_n) : f_i \in W_0^{k,p}(\Omega)\}$, with the norm of $W^{k,p}(\Omega)^n$.

- Let X be a Banach space with norm $\|\cdot\|_X$, for $-\infty < \tau < T < +\infty$, we denote by $C([\tau, T]; X)$ the space of the continuous functions from $[\tau, T]$ into X, which is a Banach space with norm

$$\|\mathbf{f}\|_{C([\tau,T];X)} = \sup_{t \in [\tau,T]} \|\mathbf{f}(t)\|_X.$$

- For $1 \leqslant p < +\infty$ and $-\infty \leqslant \tau < T \leqslant +\infty$, then $L^p(\tau, T; X)$ is the Banach space of all measurable functions in Bochner sense $\mathbf{f} : (\tau, T) \to X$ for which the norm

$$\|\mathbf{f}\|_{L^p(\tau,T;X)} = \left(\int_\tau^T \|\mathbf{f}(t)\|_X^p dt \right)^{1/p}$$

is finite.

- For $p = \infty$, $L^\infty(\tau, T; X)$ is the Banach space of all measurable functions in Bochner sense $\mathbf{f} : (\tau, T) \to X$ for which the norm

$$\|\mathbf{f}\|_{L^\infty(\tau,T;X)} = ess \sup_{t \in (\tau,T)} \|\mathbf{f}(t)\|_X$$

is finite.

- Given the sets X and Y, we denote by $C(X, Y) = \{f : X \to Y : f \text{ is continuous}\}$.

- Given a Banach space X, $\eta > 0$ and $q \in [1, +\infty)$, we denote by $\mathcal{I}_X^{q,\eta} = \{\mathbf{f} \in L^q_{loc}(\mathbb{R}, X) : \int_{-\infty}^0 e^{\eta s} \|\mathbf{f}(s)\|_X^q ds < \infty\}$.

1.1.3 Notation of Functional Analysis

- Let X be a normed space, with norm $\| \cdot \|_X$. Let X^* be its topological dual.

  (a) We say that x_k converges to x in the weak topology of X, or that x_k converges weakly to x in X, if $f(x_k) \to f(x)$, for any $f \in X^*$. And we denote by

$$x_k \rightharpoonup x \quad \text{in} \quad X.$$

  (b) We say that f_k converges to f in the weak-$*$ topology of X^*, or that f_k converges weakly-$*$ to f in X^*, if $f_k(x) \to f(x)$, for any $x \in X$. And we denote by

$$f_k \overset{*}{\rightharpoonup} f \quad \text{in } X^*.$$

- Let X and Y be Banach spaces such that $X \subset Y$. We say that X is continuously embedded in Y and denote it by

$$X \hookrightarrow Y,$$

if there exists a constant $C > 0$ such that, $\|x\|_Y \leqslant C\|x\|_X$, for all $x \in X$.

- Let X and Y be Banach spaces such that $X \subset Y$. We say that, X is compactly embedded in Y, and denote it by

$$X \hookrightarrow\hookrightarrow Y,$$

if $X \hookrightarrow Y$ and each bounded sequence in X is relatively compact in Y.

1.2 Results of Analysis and Topology

Theorem 1.3. ***(Ascoli Theorem)*** *Let X be a space and let (Y, d) be a metric space. Consider $C(X, Y)$ with the topology of compact convergence; let $\mathcal{F}$ be subset of $C(X, Y)$.*

(a) *If $\mathcal{F}$ is equicontinuous under d and the set*

$$\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$$

has compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$.

(b) *The converse holds if X is locally compact Hausdorff.*

Proof. See [50, Theorem 47.1, pg. 290]. □

Lemma 1.4. ***(Green Theorem)*** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded with $\partial\Omega \in C^{0,1}$ or $\Omega = \mathbb{R}^n$ and let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward normal vector. Then for $u \in W^{1,1}(\Omega)$ we have*

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)dx = \int_{\partial\Omega} u\nu_i dS \quad i = 1, \ldots, n$$

where the values of u on $\partial\Omega$ are understood in the sense of traces.

Proof. See [45, Lemma 2.20, pg. 29]. □

1.3 Results of Functional Analysis

1.3.1 Compactness, Inequalities and Embedding Results

Theorem 1.5. ***(Alaoglu weak-∗ compactness)*** *Let X be a separable Banach space and let (f_n) be a bounded sequence in X^*. Then (f_n) has a weak-∗ convergent subsequence.*

Proof. See [45, Theorem 2.1 pg. 21]. □

Corollary 1.6. ***(Reflexive weak compactness)*** *Let X be a reflexive Banach space and let (x_n) be a bounded sequence in X. Then (x_n) has a subsequence that converges weakly in X.*

Proof. It follows from Theorem 1.5. □

Proposition 1.7. ***(Hölder Inequality)*** *For functions $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have that $fg \in L^1(\Omega)$ and*

$$\int_\Omega f(x)g(x)dx \leqslant \|f\|_p \|g\|_q.$$

When $p = q = 2$, Hölder inequality reduces to well known Schwarz inequality.

Theorem 1.8. ***(Korn Inequality)*** *Let $1 < p < \infty$. Then there exists a constant $c(p) = c(p)(\Omega)$ such that the inequality*

$$\|\mathbf{v}\|_{1,p} \leqslant c(p)\|\mathbf{e}(\mathbf{v})\|_p$$

is fulfilled for all $\mathbf{v} \in W_0^{1,p}(\Omega)^n$, where $\Omega \subset \mathbb{R}^n$ is open bounded domain with $\partial\Omega \in C^{0,1}$ and $\mathbf{e}(\mathbf{v}) = \dfrac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$.

Proof. See [45, Theorem 1.10 pg. 196]. □

In all this work we will denote $c_0 = c(2)$ if $p = 2$ and $\tilde{c}_0 = c(p)$ if $p \neq 2$, to the constants that appear in the Korn inequality.

Theorem 1.9. ***(General Sobolev Inequalities)*** *Consider $\Omega \subset \mathbb{R}^n$ an open bounded with $\partial\Omega \in C^{0,1}$ and let $0 \leqslant j < k$, $1 \leqslant p$, $q < \infty$. Put*

$$m_0 \equiv \frac{1}{p} - \frac{k-j}{n} \quad \textit{and} \quad m \equiv \frac{1}{m_0} \quad \textit{if } m_0 \neq 0.$$

- *Assume $m_0 > 0$. Then*

$$\begin{aligned} &W^{k,p}(\Omega) \hookrightarrow W^{j,m}(\Omega),\\ &W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,m_1}(\Omega), \quad m_1 < m,\\ &W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{j,m}(\mathbb{R}^n). \end{aligned}$$

- *Assume $m_0 < 0$. Then for $\alpha \in [0,1)$*

$$m_0 + \frac{\alpha}{n} = 0 \Rightarrow \begin{cases} W^{k,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega}),\\ W^{k,p}(\mathbb{R}^n) \hookrightarrow C^{j,\alpha}(\mathbb{R}^n), \end{cases}$$

$$m_0 + \frac{\alpha}{n} < 0 \Rightarrow W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{j,\alpha}(\overline{\Omega}).$$

- *Assume $m_0 = 0$. Then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,q}(\Omega) \quad q \in [1,\infty).$$

Proof. The proof can be seen in [45, Theorem 2.17 pg. 28] or [22]. □

Let H be a Hilbert space with a scalar product $(\cdot,\cdot)_H$ and let X be a Banach space such that

$$X \hookrightarrow H \simeq H^* \hookrightarrow X^*, \tag{1.1}$$

and

$$X \text{ is dense in } H. \tag{1.2}$$

Then, if $\mathbf{u} \in L^p(I;X)$, with I an interval, we denote by $\dfrac{d\mathbf{u}}{dt}$ the element of the space $L^q(I;X^*)$ (where $\dfrac{1}{p}+\dfrac{1}{q}=1$) such that

$$\int_I \left\langle \frac{d\mathbf{u}}{dt}, \mathbf{v} \right\rangle \phi(t)dt = -\int_I (\mathbf{u}(t),\mathbf{v})_H \phi'(t)dt,$$

for all $\mathbf{v} \in X$ and $\phi \in \mathcal{D}(I)$.

Theorem 1.10. *Let (1.1), (1.2) be satisfied and let $p \in (1,\infty)$. Then*

- $W \equiv \left\{ \mathbf{u} \in L^p(\tau,T;X) \mid \dfrac{d\mathbf{u}}{dt} \in L^q(\tau,T;X^*) \right\} \hookrightarrow C([\tau,T];H)$;
- *For all $\mathbf{u},\mathbf{v} \in W$ and all $s,t \in [\tau,T]$*

$$(\mathbf{u}(t),\mathbf{v}(t))_H - (\mathbf{u}(s),\mathbf{v}(s))_H = \int_s^t \left(\left\langle \frac{d\mathbf{u}(r)}{dt}, \mathbf{v}(r) \right\rangle + \left\langle \frac{d\mathbf{v}(r)}{dt}, \mathbf{u}(r) \right\rangle \right) dr.$$

In particular for $\mathbf{u}=\mathbf{v}$,

$$\frac{1}{2}\|\mathbf{u}(t)\|_H^2 - \frac{1}{2}\|\mathbf{u}(s)\|_H^2 = \int_s^t \left\langle \frac{d\mathbf{u}(r)}{dt}, \mathbf{u}(r) \right\rangle dr,$$

for all $s,t \in [\tau,T]$.

Proof. See [45, Lemma 2.45, pg. 35]. □

Theorem 1.11. ***(Aubin-Lions-Simon)*** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leqslant p,\ r \leqslant +\infty$. For $T > \tau$, we define*

$$E_{p,r} = \left\{ \mathbf{u} \in L^p(\tau,T;B_0) \mid \frac{d\mathbf{u}}{dt} \in L^r(\tau,T;B_2) \right\}$$

(i) *If $p < +\infty$, the embedding of $E_{p,r}$ into $L^p(\tau,T;B_1)$ is compact.*

(ii) *If $p = +\infty$ and if $r > 1$, the embedding of $E_{p,r}$ into $C([\tau,T];B_1)$ is compact.*

Proof. See [6, Theorem II.5.16, pg. 102]. □

Proposition 1.12. *Let X,Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Assume that $\mathbf{u}_n$ is bounded sequence in $L^\infty(t_0,T;X)$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $L^q(t_0,T;X)$ for some $q \in [1,\infty)$ and $\mathbf{u} \in C([t_0,T];Y)$. Then $\mathbf{u}(t) \in X$ and $\|\mathbf{u}(t)\|_X \leqslant \liminf_{n\to\infty} \|\mathbf{u}_n\|_{L^\infty(t_0,T;X)}$ for all $t \in [t_0,T]$.*

Proof. See [45, Lemma 11.2, pg. 288] or [28, Lemma 4.9]. □

Proposition 1.13. *Let X and Y be Banach spaces, with X reflexive and $X \hookrightarrow Y$. If $\mathbf{u} \in L^\infty(\tau,T;X) \cap C_w([\tau,T],Y)$, then $\mathbf{u} \in C_w([\tau,T],X)$ and $\mathbf{u}(t)$ belongs to X for all $t \in [\tau,T]$.*

Proof. See [2, Theorem 1.6, pg. 21]. □

1.3.2 Interpolation Results

Theorem 1.14. ***(Interpolation in L^p)*** *Assume $1 \leqslant p_2 \leqslant p \leqslant p_1$ and $u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$. Then*

$$\|u\|_p \leqslant \|u\|_{p_1}^{\alpha} \|u\|_{p_2}^{1-\alpha},$$

where $\dfrac{1}{p} = \dfrac{\alpha}{p_1} + \dfrac{1-\alpha}{p_2}$, $\alpha \in [0,1]$.

Proof. See [45, Corollary 2.10 pg. 26]. □

Theorem 1.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain, and $q \leqslant n \leqslant p < \infty$. then*

$$\|\mathbf{u}\|_p \leqslant \hat{c}\|\mathbf{u}\|_q^{1-\frac{q}{p}} \|\nabla \mathbf{u}\|_n^{\frac{q}{p}}, \tag{1.3}$$

for any $\mathbf{u} \in W_0^{1,n}(\Omega)^n$. Here the constant $\hat{c}$ is scaling invariant, but tends to ∞ as $p \to \infty$.

Proof. See [23, Theorem 9.1, pg. 259]. □

Remark 1.16. *A special case of the above result for two-dimensional domain is called Ladyzhenskaya inequality, for $\mathbf{u} \in W_0^{1,2}(\Omega)^2$*

$$\|\mathbf{u}\|_4 \leqslant \hat{c}|\mathbf{u}|_2^{\frac{1}{2}} |\nabla \mathbf{u}|_2^{\frac{1}{2}}. \tag{1.4}$$

Theorem 1.17. ***(Interpolation of Gagliardo-Nirenberg)*** *Let Ω be a bounded domain of $\mathbb{R}^n$ with $\partial\Omega$ of class C^m and let $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \leqslant r$, $q \leqslant \infty$. For any integer j, $0 \leqslant j < m$, and for any number α in the interval $[j/m, 1]$, such that*

$$\frac{1}{p} = \frac{j}{n} + \alpha\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-\alpha)\frac{1}{q}.$$

If $m - j - n/r$ is not a non-negative integer, then

$$\|D^j u\|_p \leqslant C\|u\|_{m,r}^{\alpha} \|u\|_q^{1-\alpha}, \tag{1.5}$$

where the constant C depends only on Ω, r, q, m, j and α.

If $1 < r < \infty$ and $m - j - n/r$ is a non-negative integer, then (1.5) holds for $\alpha \in [j/m, 1)$.

Proof. See [25, Theorem 10.1]. □

1.3.3 Basis Consisting of Eigenfunctions

The next results can be found in [45, Appendix **A.4** pg. 288].

For $s \geqslant 1$ and $p > 1$, let us define

$$\begin{aligned}
&\mathcal{V} := \{\varphi \in C_c^{\infty}(\Omega)^n : \mathrm{div}_x \varphi = 0\};\\
&H = \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega)^n - \text{norm};\\
&V_p = \text{closure of } \mathcal{V} \text{ in the } W^{1,p}(\Omega)^n - \text{norm};\\
&V^s = \text{closure of } \mathcal{V} \text{ in the } W^{s,2}(\Omega)^n - \text{norm}.
\end{aligned}$$

If $s = 1$ or $p = 2$, then V will denote the spaces V_2, V^1, respectively. Let us denote the dual space of V_p by V_p^* and $\langle \cdot, \cdot \rangle$ the duality between V_p and V_p^*, the scalar product in H is marked by $(\cdot, \cdot)$ while the scalar product in V^s is marked by $((\cdot, \cdot))_s$.

The spaces V_p and V^s can be characterized, as:

$$\begin{aligned} V_p &= \{\mathbf{u} \in W_0^{1,p}(\Omega)^n : \operatorname{div}_x \mathbf{u} = 0\}, \\ V^s &= \{\mathbf{u} \in W^{s,2}(\Omega)^n : \gamma(\mathbf{u}) = 0 \text{ at } \partial\Omega, \ \operatorname{div}_x \mathbf{u} = 0\}, \end{aligned}$$

where γ is the trace operator given by

$$\begin{aligned} \gamma : \quad W^{1,2}(\Omega) &\hookrightarrow H^{1/2}(\partial\Omega) \\ \mathbf{u} \quad &\mapsto \quad \mathbf{u}|_{\partial\Omega}. \end{aligned}$$

By $H^{-1/2}(\partial\Omega)^n$ we mean the dual space of $H^{1/2}(\partial\Omega)^n$. Defining

$$E(\Omega) \equiv \{\mathbf{u} \in L^2(\Omega)^n : \operatorname{div}_x \mathbf{u} \in L^2(\Omega)\},$$

it is possible to construct a trace operator

$$\widehat{\gamma} : E(\Omega) \hookrightarrow H^{-1/2}(\partial\Omega)^n,$$

such that $\widehat{\gamma}(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}$ for $\mathbf{u} \in C^1(\overline{\Omega})$, $\mathbf{n}$ being an outer normal unit vector of $\partial\Omega$. Then it holds that

$$H = \{\mathbf{u} \in L^2(\Omega)^n : \widehat{\gamma}(\mathbf{u}) = 0, \ \operatorname{div}_x \mathbf{u} = 0 \ \text{ in } \ \mathcal{D}'(\Omega)\}.$$

We are interested in the following spectral problem: Find $\mathbf{w}^r \in V^s$ and $\lambda_r \in \mathbb{R}$ satisfying

$$((\mathbf{w}^r, \mathbf{v}))_s = \lambda_r (\mathbf{w}^r, \mathbf{v}) \quad \mathbf{v} \in V^s. \tag{1.6}$$

Theorem 1.18. *There exist a countable set* $\{\lambda_k\}_{k=1}^{\infty}$ *and a corresponding family of eigenvectors* $\{\mathbf{w}^k\}_{k=1}^{\infty}$ *solving problem (1.6) such that*

- $(\mathbf{w}^i, \mathbf{w}^j) = \delta_{i,j} \quad \forall i, j \in \mathbb{N}$,
- $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ *and* $\lambda_k \to \infty$ *as* $k \to \infty$,
- $((\frac{\mathbf{w}^i}{\sqrt{\lambda_i}}, \frac{\mathbf{w}^j}{\sqrt{\lambda_j}}))_s = \delta_{i,j} \quad \forall i, j \in \mathbb{N}$,
- $\{\mathbf{w}\}_{k=1}^{\infty}$ *forms a basis of* V^s.

Moreover, defining $H^m \equiv \operatorname{span}\{\mathbf{w}^1, \ldots, \mathbf{w}^m\}$ *(a linear hull) and* $P^m : V^s \to H^m$ *by*

$$P^m(\mathbf{v}) = \sum_{i=1}^{m} (\mathbf{v}, \mathbf{w}^i)\mathbf{w}^i,$$

we obtain

$$\|P^m\|_{\mathcal{L}(V^s, V^s)} \leqslant 1, \quad \|P^m\|_{\mathcal{L}((V^s)^*, (V^s)^*)} \leqslant 1, \quad \|P^m\|_{\mathcal{L}(H,H)} \leqslant 1.$$

Proof. See [45, Appendix: **A.4** pg. 290]. □

1.4 Ordinary Differential Equations

1.4.1 Ordinary Differential Equations **(ODE)**

Given $t_0 \in \mathbb{R}$ and $\delta > 0$ let us denote by $I_\delta = (t_0 - \delta, t_0 + \delta)$. Let us consider for $\mathbf{c} : I_\delta \to \mathbb{R}^n$, the system of ordinary differential equations

$$\begin{cases} \dfrac{d\mathbf{c}}{dt}(t) = \mathbf{f}(t, \mathbf{c}(t)) \quad t \in I_\delta, \\ \mathbf{c}(t_0) = \mathbf{c}_0 \in \mathbb{R}^n. \end{cases} \tag{1.7}$$

Assume $\mathbf{f} : I_\delta \times \mathrm{K} \to \mathbb{R}^n$, where $\mathrm{K} = \{\mathbf{c} \in \mathbb{R}^n : |\mathbf{c} - \mathbf{c}_0| < M\}$ for some $M > 0$.

Definition 1.19. *A function $\mathbf{f} : I_\delta \times \mathrm{K} \to \mathbb{R}^n$ is said to satisfy the* ***Carathéodory Conditions*** *if*

- $\mathbf{c} \mapsto F_i(t, \mathbf{c})$ *is continuous for almost all $t \in I_\delta$,*
- $t \mapsto F_i(t, \mathbf{c})$ *is measurable for all $i = 1, \ldots, n$ and for all $\mathbf{c} \in \mathrm{K}$,*
- *There exists an integrable function $G : I_\delta \to \mathbb{R}$ such that*

$$|F_i(t, \mathbf{c})| \leqslant G(t) \quad \forall (t, \mathbf{c}) \in I_\delta \times \mathrm{K}, \quad \forall i = 1, \ldots, n.$$

Theorem 1.20. *Assume that $\mathbf{f}$ satisfies the Carathéodory conditions. Then there exist $\delta_0 \in (0, \delta)$ and a continuous function $\mathbf{c} : I_{\delta_0} \to \mathbb{R}^n$ such that*

- $\dfrac{d\mathbf{c}}{dt}$ *exists for almost all $t \in I_{\delta_0}$,*
- $\mathbf{c}$ *solves (1.20).*

Proof. See [45, Theorem 3.4 pg. 287]. □

Lemma 1.21. ***(Gronwall inequality)*** *Let $y : (\tau, T) \to \mathbb{R}$ and $g : (\tau, T) \to \mathbb{R}$ be non-negative functions, $g \in L^1(\tau, T)$. Suppose that inequality*

$$y(t) \leqslant C + \int_\tau^t g(s) y(s) ds,$$

holds for $t \in (\tau, T)$ with $C \in \mathbb{R}$. Then

$$y(t) \leqslant C \exp \int_\tau^t g(s) ds \quad t \in (\tau, T).$$

Proof. See [45, Lemma 3.5 pg. 288]. □

1.4.2 Delay Ordinary Differential Equations **(DODE)**

Delay differential equations arise from various applications, like biology, medicine, control theory, climate models, and many others. Their independent variables are the time t and one or more dimensional variable x , which usually represents the position in space but may also represent relative DNA content, size of cells, or their maturation level, or other values. The solutions (dependent variables) of delay partial differential equations may represent fluid velocity, temperature, voltage, concentrations or densities of various particles, for example cells, bacteria, chemicals, animals and so on.

Let us first establish suitable assumptions on the term in which the delay is present. Let X and Y be two separable Banach spaces and

$$g : [\tau, T] \times C([-h, 0]; X) \to Y$$

such that the following holds.

(I) For all $\xi \in C([-h, 0]; X)$, the mapping $t \in [\tau, T] \to g(t, \xi) \in Y$ is measurable.

(II) For each $t \in [\tau, T]$, $g(t, 0) = 0$.

(III) There exists $L_g > 0$ such that, for all $s \in [\tau, T]$ and for any $\xi, \eta \in C([-h, 0]; X)$,

$$\|g(s, \xi) - g(s, \eta)\|_Y \leqslant L_g \|\xi - \eta\|_{C([-h,0];X)}.$$

(IV) There exists $C_g > 0$ such that, for all $t \in [\tau, T]$ and for any $u, v \in C([\tau - h, T]; X)$,

$$\int_\tau^t \|g(s, u_s) - g(s, v_s)\|_Y^2 ds \leqslant C_g^2 \int_{\tau-h}^t \|u(s) - v(s)\|_X ds.$$

Remark 1.22. *Observe that conditions* ***(I)-(III)*** *above imply that, given* $u \in C([\tau - h, T]; X)$*, the function* $g_u : t \in [\tau, T] \to Y$ *defined by* $g_u(t) = g(t, u_t)$, $\forall t \in [\tau, T]$*, is measurable and, in fact, belongs to* $L^\infty(\tau, T; Y)$*. Then, thanks to* ***(IV)****, the mapping*

$$\mathcal{G} : u \in C([\tau - h, T]; X) \to g_u \in L^2(\tau, T; Y)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$*, which is uniformly continuous from* $L^2(\tau - h, T; X)$ *into* $L^2(\tau, T; Y)$*. From now on, we will write* $g(t, u_t) = \widetilde{\mathcal{G}}(t)$ *for each* $u \in L^2(\tau - h, T; X)$*, and thus for all* $t \in [\tau, T]$ *and any* $u, v \in L^2(\tau - h, T; X)$*, we get*

$$\int_\tau^t \|g(s, u_s) - g(s, v_s)\|_Y^2 ds \leqslant C_g^2 \int_{\tau-h}^t \|u(s) - v(s)\|_X^2 ds.$$

Theorem 1.23. *Let* $\mathbf{u}_0 \in \mathbb{R}^n$, $\phi \in L^2(-h, 0; \mathbb{R}^n)$, $k \in L^2(0, T; \mathbb{R}^n)$, $g : [0, T] \times C([0, T]; \mathbb{R}^n) \to \mathbb{R}^n$ *satisfying hypotheses* ***(I)-(IV)*** *with* $X = Y = \mathbb{R}^n$*, and* $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *a continuous function such that* $\mathbf{f}(t, 0) = 0$ *and for all* $m > 0$ *there exists* $L_m > 0$ *such that*

$$|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})| \leqslant L_m |\mathbf{u} - \mathbf{v}|, \quad \forall |\mathbf{u}| \leqslant m, \quad |\mathbf{v}| \leqslant m, \quad \forall t \in [0, T].$$

Then

(a) *For each $t_* \in [0,T]$ there exists at most one solution to problem*

$$\begin{cases} \text{to find } \mathbf{u} \in L^2(-h,t_*;\mathbb{R}^n) \cap C([0,t_*];\mathbb{R}^n) \text{ such that} \\ \mathbf{u}(t) = \phi(t), \quad t \in [-h,0] \\ \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{f}(s,\mathbf{u}(s))ds + \int_0^t g(s,\mathbf{u}_s)ds + \int_0^t k(s)ds \quad \forall t \in [0,t_*]; \end{cases} \tag{1.8}$$

(b) *there exists $t_* \in [0,T]$ such that there exists one (and only one) solution to problem* (1.8)*;*

(c) *suppose that there exists a constant $C > 0$ such that if $t_* \in [0,T]$ is such that there is a solution $\mathbf{u}$ of (1.8), then $\max_{t\in[0,t_*]} |\mathbf{u}(t)| \leqslant C$. Then, under this additional assumption, there exists a solution to problem* (1.8) *with $t_* = T$.*

Proof. See [13, Appendix **A**]. □

1.5 Pullback Attractors

In this section we are going to study the existence of pullback attractors for upper-semicontinuous multi-valued process and for closed process. At the end of this section we will give a comparison result for pullback attractors. All the results in the next subsections can be found in [17, 12, 28, 8, 32, 61, 18, 2].

Let (X, d_X) be a metric space and $\mathbb{R}^2_d = \{(t,s) \in \mathbb{R}^2 : t \geqslant s\}$. In what follows, we denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X.

We denote by $dist_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$dist_X(\mathcal{O}_1,\mathcal{O}_2) = \sup_{x\in\mathcal{O}_1} \inf_{y\in\mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\mathcal{O}_2 \subset X.$$

We consider a universe $\mathcal{D}$, that is a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ and a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 1.24. *A universe $\mathcal{D}$ is inclusion-closed if given $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D(t) \subset D'(t)$ for all $t \in \mathbb{R}$, it fulfils that $\widehat{D}' \in \mathcal{D}$.*

1.5.1 Pullback Attractor for Closed Process

Definition 1.25. *A process on X is a family of mappings $U(t,\tau) : X \to X$ for any pairs $(t,\tau) \in \mathbb{R}^2_d$, such that*

1. $U(\tau,\tau)x = x \ \forall \tau \in \mathbb{R} \ \forall x \in X$,

2. $U(t,\tau) = U(t,s)U(s,\tau) \ \forall \tau \leqslant s \leqslant t$.

As a convenient shorthand, we will refer to the process $U(\cdot,\cdot)$ rather than the process $U(t,\tau) : X \to X$ for any pairs $(t,\tau) \in \mathbb{R}^2_d$ in all that follows.

Definition 1.26. *A process $U(\cdot,\cdot)$ on X is said to be closed if for any $\tau \leqslant t$, and any sequence $\{x_k\} \subset X$ with $x_k \to x \in X$ and $U(t,\tau)x_k \to y \in X$, then $U(t,\tau)x = y$.*

Definition 1.27. *The family $\widehat{D}_0$ is said to be pullback $\mathcal{D}$-absorbing for a closed process $U(\cdot,\cdot)$ on X if for any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \leqslant t$ such that*

$$U(t,\tau)D(\tau) \subseteq D_0(t) \ \ for \ all \ \tau \leqslant \tau_0(t,\widehat{D}).$$

Definition 1.28. *Given a family of nonempty sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, a closed process $U(\cdot,\cdot)$ on X is pullback $\widehat{D}$-asymptotically compact if, for any $t \in \mathbb{R}$, any sequence $\{\tau_k\} \subset (-\infty,t]$ and $\{x_k\} \subset X$ such that $\tau_k \to -\infty$ and $x_k \in D(\tau_k)$ for all $k \in \mathbb{N}$, the sequence $\{U(t,\tau_k)x_k\}$ is relatively compact in X.*

Definition 1.29. *A closed process $U(\cdot,\cdot)$ on X is pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for all $\widehat{D} \in \mathcal{D}$.*

Definition 1.30. *A pullback $\mathcal{D}$-attractor for a closed process $U(\cdot,\cdot)$ on X is a family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ such that*

1. *for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X;*

2. *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \ \ \forall \widehat{D} \in \mathcal{D} \ \forall t \in \mathbb{R},$$

3. *$\mathcal{A}_{\mathcal{D}}$ is invariant, i.e.*

$$\mathcal{A}_{\mathcal{D}}(t) = U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) \ \ \forall (t,\tau) \in \mathbb{R}^2_d.$$

A pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}}$ is said to be minimal if it satisfies that if there exists another family of closed sets $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ such that it is pullback $\mathcal{D}$-attracting, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ for all $t \in \mathbb{R}$. The inclusion of minimality guarantees the uniqueness of the pullback attractor.

Denote by

$$\Lambda(\widehat{D}_0,t) := \bigcap_{\sigma \leqslant t} \overline{\bigcup_{\tau \leqslant \sigma} U(t,\tau)D_0(\tau)}^X \quad \forall t \in \mathbb{R} \tag{1.9}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Proposition 1.31. *If $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for a closed process $U(\cdot,\cdot)$, then*

$$\Lambda(\widehat{D},t) \subset \Lambda(\widehat{D}_0,t) \quad \text{for all } \widehat{D} \in \mathcal{D},\ t \in \mathbb{R}.$$

In addition, if $\widehat{D}_0 \in \mathcal{D}$, then

$$\Lambda(\widehat{D}_0,t) \subset \overline{D_0(t)} \quad \text{for all } t \in \mathbb{R}.$$

Proof. The proof can be seen in [28]. □

Proposition 1.32. *Assume that $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for a closed process $U(\cdot,\cdot)$ on X, which is pullback $\widehat{D}_0$-asymptotically compact. Then, the process $S(\cdot,\cdot)$ is also pullback $\mathcal{D}$-asymptotically compact.*

Proof. The proof can be seen in [28]. □

Theorem 1.33. *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 \subset \mathcal{P}(X)$ which pullback $\mathcal{D}$-absorbing, and assume that $U(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D},t)}^{X}, \quad t \in \mathbb{R},$$

is the minimal pullback attractor for the closed process $U(\cdot,\cdot)$ on X. Besides, if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0,t) \subset \overline{D_0(t)}^{X}$, for all $t \in \mathbb{R}$.

Proof. The proof can be seen in [28]. □

Remark 1.34. *If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies **1** - **3** of the Definition 1.30. A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $\widehat{D}_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed.*

1.5.2 Pullback Attractor for Multi-valued Process

Definition 1.35. *A multi-valued process (also called multi-valued non-autonomous dynamical system) $U(\cdot,\cdot)$ on X is a family of mappings $U(t,\tau) : X \to \mathcal{P}(X)$ for any pairs $(t,\tau) \in \mathbb{R}^2_d$, such that*

1. *$U(\tau,\tau)x = \{x\}$ $\forall \tau \in \mathbb{R}$ $\forall x \in X$,*
2. *$U(t,\tau)x \subset U(t,s)\big(U(s,\tau)x\big)$ $\forall \tau \leqslant s \leqslant t$ $\forall x \in X$, where $U(t,\tau)W := \bigcup_{y \in W} U(t,\tau)y$.*

Observe that if the relationship given in 2 is an equality instead of an inclusion, the multi-valued process $U(\cdot,\cdot)$ is called strict.

Definition 1.36. *A multi-valued process $U(\cdot,\cdot)$ on X is upper-semicontinuous if the mapping $U(t,\tau)$ is upper-semicontinuous from X into $\mathcal{P}(X)$ for all $(t,\tau)\in\mathbb{R}^2_d$, i.e. for any $x\in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau)x$ there exists a value $\epsilon>0$ such that $U(t,\tau)y\subset\mathcal{N}$ provided that $d_X(x,y)<\epsilon$.*

Definition 1.37. *The family $\widehat{D}_0$ is said to be pullback $\mathcal{D}$-absorbing for a multi-valued process $U(\cdot,\cdot)$ on X if for every $t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}$, there exists a $\tau_0(t,\widehat{D})\leqslant t$ such that*

$$U(t,\tau)D(\tau)\subseteq D_0(t)\ \ \forall\,\tau\leqslant\tau_0(t,\widehat{D}).$$

Definition 1.38. *Given a family of nonempty sets $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$, a multi-valued process $U(\cdot,\cdot)$ on X is pullback $\widehat{D}$-asymptotically compact if for any $t\in\mathbb{R}$ and any sequences $\{\tau_k\}\subset(-\infty,t]$ and $\{x_k\}\subset X$ such that $\tau_k\to-\infty$ and $x_k\in D(\tau_k)$, it fulfils that any sequence $\{y_k\}$ is relatively compact in X, where $y_k\in U(t,\tau_k)x_k$ for all k.*

Definition 1.39. *A multi-valued process $U(\cdot,\cdot)$ on X is pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D}\in\mathcal{D}$.*

Definition 1.40. *A pullback $\mathcal{D}$-attractor for a multi-valued process $U(\cdot,\cdot)$ on X is a family $\mathcal{A}_{\mathcal{D}}=\{\mathcal{A}_{\mathcal{D}}(t):t\in\mathbb{R}\}\subset\mathcal{P}(X)$ such that*

1. *for any $t\in\mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X;*
2. *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau\to-\infty} dist_X(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t))=0\ \ \forall\widehat{D}\in\mathcal{D}\ \forall t\in\mathbb{R},$$

   where $dist_X(\cdot,\cdot)$ denotes the Hausdorff semi-distance in X between two subsets of X;

3. *$\mathcal{A}_{\mathcal{D}}$ is negatively invariant, i.e.*

$$\mathcal{A}_{\mathcal{D}}(t)\subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)\ \ \forall(t,\tau)\in\mathbb{R}^2_d.$$

A pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}}$ is said to be minimal if it satisfies that if there exists another family of closed sets $\widehat{C}=\{C(t):t\in\mathbb{R}\}$ such that it is pullback $\mathcal{D}$-attracting, then $\mathcal{A}_{\mathcal{D}}(t)\subset C(t)$ for all $t\in\mathbb{R}$.

Remark 1.41. *Observe that pullback attractors are not unique in general (cf. [17]); however, the minimal pullback attractor is, therefore, in the sense of minimality, one recovers uniqueness of pullback attractor.*

We denote the omega-limit set of $\widehat{D}_0$ at time t by

$$\Lambda(\widehat{D}_0,t):=\bigcap_{\sigma\leqslant t}\overline{\bigcup_{\tau\leqslant\sigma}U(t,\tau)D_0(\tau)}^X,$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Theorem 1.42. *Assume that $U(\cdot,\cdot)$ is an upper-semicontinuous multi-valued process with closed values, $\widehat{D}_0 \in \mathcal{D}$ pullback $\mathcal{D}$-absorbing family and also suppose that $U(\cdot,\cdot)$ on X is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad \forall t \in \mathbb{R},$$

is the minimal pullback $\mathcal{D}$-attractor and $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

Proof. The proof can be seen in [8, 30]. □

Remark 1.43. *If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of $\mathcal{D}$ that satisfies the properties* $1-3$ *given above. In addition, if the multi-valued process $U(\cdot,\cdot)$ is strict, then $\mathcal{A}_{\mathcal{D}}$ is strictly invariant under the process $U(\cdot,\cdot)$, i.e.*

$$\mathcal{A}_{\mathcal{D}}(t) = U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) \ \ \forall (t,\tau) \in \mathbb{R}^2_d.$$

1.5.3 Comparison of attractors

We will denote by $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Corollary 1.44. *Under the assumptions of Theorem 1.42, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F^X$, then $\mathcal{A}_{\mathcal{D}_F^X} = \{\mathcal{A}_{\mathcal{D}_F^X}(t) : t \in \mathbb{R}\}$ where*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \overline{\bigcup_{B \text{ bounded}} \Lambda(B,t)}^X, \tag{1.10}$$

is the minimal pullback $\mathcal{D}_F^X$-attractor for the multi-valued process $U(\cdot,\cdot)$ and the following relationship holds

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \in \mathbb{R}.$$

Proof. The proof can be seen in [8, 30]. □

Remark 1.45. *It can be proved (see [47]) that, under the assumptions of the preceding Corollary, if for some $T \in \mathbb{R}$ the set $\bigcup_{t\leqslant T} D_0(t)$ is a bounded subset of X, then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \leqslant T.$$

Now, we establish an abstract result that allows to compare two attractors for a process under appropriate assumptions.

Theorem 1.46. *Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be two metric space such that $X_1 \subset X_2$ with continuous injection, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that we have a map $U(\cdot, \cdot)$ that acts as a process in both cases, i.e. $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$ is a process.*

For each $t \in \mathbb{R}$, let us denote

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \quad i = 1, 2,$$

where the subscript in i in the symbol of the omega-limit set Λ_i is used to denote the dependence of the respective topology.

Then,

$$\mathcal{A}_1(t) \subset \mathcal{A}_2(t) \quad \forall t \in \mathbb{R}.$$

Suppose moreover that the two following conditions are satisfied:

(i) *$\mathcal{A}_1(t)$ is a compact subset X_1 for all $t \in \mathbb{R}$,*

(ii) *for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and $t^*_{\widehat{D}_1} \leqslant t$ (both possibly depending on t and $\widehat{D}_2$), such that $U(\cdot, \cdot)$ is pullback $\widehat{D}_1$-asymptotically compact, for any $s \leqslant t^*_{\widehat{D}_1}$ there exists a $\tau_s \leqslant s$ such that*

$$U(s, \tau)D_2(\tau) \subset D_1(s) \quad \text{for all } \tau \leqslant \tau_s.$$

Then, under all the conditions above,

$$\mathcal{A}_1(t) = \mathcal{A}_2(t) \quad \forall t \in \mathbb{R}.$$

Proof. See [8, 30]. □

Remark 1.47. *It is important to observe that all the results established in this subsection are still valid for upper-semicontinuous multi-valued process being the proofs analogous, see [8, 9].*

1.6 Compressible and Incompressible Fluids

The aim of this section is to give a brief introduction to mathematical models of fluids in partial differential equations for a better physical understanding and to show the importance of the development of this work. The next physical concepts were extracted from the book [45, Chapter 1]. See also [23, 4, 20, 40, 39].

The following equations represent the governing principles in fluid mechanics: conservation of mass, momentum, and energy. These laws are presented in differential form, applicable at a point or to a fluid particle. All quantities are evaluated at $(t, x) \in [\tau, T) \times \Omega$, where $[\tau, T)$ is an interval of time and $\Omega \subset \mathbb{R}^n$ that can be physically interpreted as a domain occupied by the material at an instant of time $t \in [\tau, T)$.

- **Law of conservation of mass:** States that the rate at which mass enters the system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system.
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0, \tag{1.11}$$

- **Law of balance of momentum:** The local conservation of momentum is expressed by the Cauchy momentum equations
$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = \frac{\partial \mathbf{T}_{ij}}{\partial x_j} + \rho f_i, \tag{1.12}$$

- **Law of balance of energy:**
$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial(\rho e v_j)}{\partial x_j} = \frac{\partial(\mathbf{T}_{ij} v_k)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho r + \rho f_i v_j, \tag{1.13}$$

where ρ is fluid density, $\mathbf{v} = (v_1, \ldots, v_n)$ is the fluid velocity, $\mathbf{f} = (f_1, \ldots, f_n)$ is an external force, $\mathbf{T} = (\mathbf{T}_{ij})$ is the Cauchy stress tensor, $e = E + |\mathbf{v}|^2/2$ with E the specific internal energy of the material, $\mathbf{q} = (q_1, ..., q_n)$ is the spatial heat flux vector and r is the rate of external communication of heat to the body through radiation.

Next, we will present some of the most known and studied systems in fluid mechanics, which are deduced from the conservation laws.

I. Compressible non-Newtonian Fluids: Euler equations describe the movement of a non-viscous compressible fluid and the stress tensor is determined by the pressure, see [45], i.e.,
$$\mathbf{T} = -P(\rho, \theta)\mathbb{I},$$
where θ is the temperature and $\mathbb{I}$ is the identity matrix. If we take into consideration the viscous effects, the dependence of $\mathbf{T}$ on other quantities, say $\nabla \mathbf{v}$, $\nabla \theta$, should be assumed. In this case
$$\mathbf{T} = \hat{\mathbf{T}}(\rho, \theta, \nabla \theta, \nabla \mathbf{v}).$$
However, when the motion of the material is isothermal, i.e. the temperature $\theta = \theta_0 > 0$ is constant, the tensor function $\hat{\mathbf{T}}$ does not depend on θ and $\nabla \theta$. Thus,
$$\mathbf{T} = \bar{\mathbf{T}}(\rho, \nabla \mathbf{v}). \tag{1.14}$$
Consequently, the equations (1.11)-(1.12) are not coupled with (1.13) and can be considered separately. Thus, we can determine ρ and $\mathbf{v}$ from (1.11), (1.12) and then use (1.13) to calculate the remaining thermodynamical quantities.

It can be shows that (1.14) has the form

$$\mathbf{T} = -P(\mathbf{e})\mathbb{I} + \tilde{\mathbf{T}}(\rho, \mathbf{e}), \tag{1.15}$$

where $\mathbf{e} = \mathbf{e}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ is the symmetric part of the velocity gradient $\nabla \mathbf{v}$. Under the previous assumptions, system (1.11), (1.12) reads

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0, \\ \frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tilde{\mathbf{T}}_{ij}}{\partial x_j} + \rho f_i, \end{aligned} \tag{1.16}$$

for $i = 1, ..., n$.

II. Incompressible non-Newtonian Fluids: If a material is incompressible and homogeneous, $\rho(t,x) = \rho_0 > 0$ for all $(t,x) \in [\tau, T) \times \Omega$, and we consider an isothermal process, then

- from (1.14) it follows that
$$\mathbf{T} = -\pi\mathbb{I} + \tau^E,$$
where π is the so-called undetermined pressure and τ^E is the extra stress tensor;
- system (1.16) reduces to
$$\begin{aligned} \operatorname{div}_x \mathbf{v} = 0, \\ \rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau^E_{ij}}{\partial x_j} + \rho_0 f_i, \end{aligned} \tag{1.17}$$
for $i = 1, \ldots, n$.

For this type of fluids it is usual to assume that the extra stress τ^E is given by the sum of two symmetric tensors depending on $\mathbf{e} = \mathbf{e}(\mathbf{v})$. This means that, for all $(t,x) \in [\tau, T) \times \Omega$,

$$\tau^E = \mathbb{S}(\mathbf{e}(\mathbf{v})) + \sigma(\mathbf{e}(\mathbf{v})),$$

or equivalently

$$\tau^E = \mathbb{S}(\mathbf{e}) + \sigma(\mathbf{e}).$$

Usually, instead of $\mathbb{S}$, it is employed the notation τ. However, to avoid confusion with the initial time $t = \tau$, we choose to use $\mathbb{S}$.

In general, the p-coercivity and q-growth conditions are required on $\mathbb{S}$ and σ, i.e. there exist constants $C_1, C_2 > 0$ and parameters $p > 1$ and $q \in [p-1, p)$ such that for all $\mathbf{M} \in \mathbb{R}^{n^2}_{sym}$, it holds

$$\begin{cases} \mathbb{S}(\mathbf{M}) : \mathbf{M} \geqslant C_1 |\mathbf{M}|^p, \\ \sigma(\mathbf{M}) : \mathbf{M} \geqslant 0, \end{cases}$$

$$|\mathbb{S}(\mathbf{M}) + \sigma(\mathbf{M})| \leqslant C_2(1 + |\mathbf{M}|)^q.$$

Now, in order to develop a mathematical theory, the minimal conditions that we will require on the stress tensor $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ are:

$$\begin{aligned} \mathbb{S}(\mathbf{0}) &= \mathbf{0}, \\ \big(\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})\big) : (\mathbf{A} - \mathbf{B}) &\geqslant \nu_1\big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2}|\mathbf{A} - \mathbf{B}|^2, \\ |\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})| &\leqslant c_1\nu_1\big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2}|\mathbf{A} - \mathbf{B}|, \end{aligned} \tag{1.18}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n^2}_{sym}$. We conveniently denote by $\mu = \big(\frac{\nu_2}{\nu_1}\big)^{\frac{1}{p-2}}$ if $p \neq 2$ and $\mu = 1$ if $p = 2$, where ν_1, ν_2 are the viscosities (critical physical parameters of the problem).

Note that from (1.18) it follows the p-coercivity and $(p-1)$-growth conditions of $\mathbb{S}$, that is: there exist positive constants c_2, c_3 such that, for all $p \geqslant 2$ the stress tensor $\mathbb{S}$ satisfies:

$$\begin{aligned} \mathbb{S}(\mathbf{D}) : \mathbf{D} &\geqslant c_2(\nu_1|\mathbf{D}|^2 + \nu_2|\mathbf{D}|^p), \\ |\mathbb{S}(\mathbf{D})| &\leqslant c_3\nu_1(1 + \mu|\mathbf{D}|)^{p-1}, \end{aligned}$$

for all $\mathbf{D} \in \mathbb{R}^{n^2}_{sym}$.

Now, we are going to introduce some definitions and results related to the stress tensor $\mathbb{S}$. Then, we will present most known systems of equations for incompressible non-Newtonian fluids, such as the Ladyzhenskaya models and fluids of type power-law.

Definition 1.48. *A function $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ has a potential if there exists a function $\Phi : \mathbb{R}^{n^2} \to [0, +\infty)$ such that Φ is a C^2- function,*

$$\partial_{\mathbf{A}}\Phi(\mathbf{A}) = \mathbb{S}(\mathbf{A}), \tag{1.19}$$

$$\Phi(\mathbf{0}) = \partial_{\mathbf{A}}\Phi(\mathbf{0}) = \mathbf{0}, \tag{1.20}$$

$$\partial^2_{\mathbf{A}}\Phi(\mathbf{A}) : (\mathbf{B} \otimes \mathbf{B}) \geqslant C_1 \begin{cases} \nu_2|\mathbf{A}|^{p-2}|\mathbf{B}|^2, \\ \quad or \\ \nu_1(1 + \mu|\mathbf{A}|)^{p-2}|\mathbf{B}|^2, \end{cases} \tag{1.21}$$

$$|\partial^2_{\mathbf{A}}\Phi(\mathbf{A})| \leqslant C_2\nu_1(1 + \mu|\mathbf{A}|)^{p-2}, \tag{1.22}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n^2}_{sym}$.

We denote the components of the stress tensor $\mathbb{S}$ by $\tau_{ij} := \mathbb{S}_{ij}$, with this the expressions in (1.19)-(1.22) can be written as

$$
\begin{aligned}
\frac{\partial \Phi}{\partial a_{ij}}(\mathbf{A}) &= \tau_{ij}(\mathbf{A}), \\
\Phi(\mathbf{0}) = \frac{\partial \Phi}{\partial a_{ij}}(\mathbf{0}) &= 0, \\
\frac{\partial^2 \Phi(\mathbf{A})}{\partial a_{ij} \partial a_{lk}} b_{ij} b_{lk} &\geqslant C_1 \begin{cases} \nu_2 |\mathbf{A}|^{p-2} |\mathbf{B}|^2, \\ \quad \text{or} \\ \nu_1 (1 + \mu |\mathbf{A}|)^{p-2} |\mathbf{B}|^2, \end{cases} \\
\left| \frac{\partial^2 \Phi(\mathbf{A})}{\partial a_{ij} \partial a_{lk}} \right| &\leqslant C_2 \nu_1 (1 + \mu |\mathbf{A}|)^{p-2},
\end{aligned}
$$

for all $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}) \in \mathbb{R}^{n^2}_{sym}$.

Next Lemmas furnish some algebraic properties satisfied when $ has a potential.

Lemma 1.49. *Let $p > 1$ and $\$: \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$, $\Phi : \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy (1.19)-(1.22). Then there exist positive constants c_1, c_2, c_3, C_3, C_4 and C_5 such that for all $\mathbf{e} \in \mathbb{R}^{n^2}_{sym}$*

$$
\tau_{ij}(\mathbf{e}) e_{ij} \geqslant C_3 \begin{cases} \nu_2 |\mathbf{e}|^p, \\ \nu_1 |\mathbf{e}| (\mu^{p-1} |\mathbf{e}|^{p-1} - 1) \geqslant C_4 (|\mathbf{e}|^p - 1), \end{cases} \tag{1.23}
$$

$$
|\tau_{ij}(\mathbf{e})| \leqslant c_3 \nu_1 (1 + \mu |\mathbf{e}|)^{p-1}, \tag{1.24}
$$

$i, j = 1, \ldots, n$, and for all $\mathbf{e}, \hat{\mathbf{e}} \in \mathbb{R}^{n^2}_{sym}$

$$
(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\hat{\mathbf{e}}))(e_{ij} - \hat{e}_{ij}) \geqslant 0. \tag{1.25}
$$

Further, the inequality $(1.23)_2$ can be replaced by

$$
\tau_{ij}(\mathbf{e}) e_{ij} \geqslant C_3 \min\{|\mathbf{e}|^2, |\mathbf{e}|^p\}. \tag{1.26}
$$

Moreover, if $p \geqslant 2$ then

$$
\tau_{ij}(\mathbf{e}) e_{ij} \geqslant c_2 (\nu_1 |\mathbf{e}|^2 + \nu_2 |\mathbf{e}|^p), \tag{1.27}
$$

and there exists c_1 such that

$$
(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\hat{\mathbf{e}}))(e_{ij} - \hat{e}_{ij}) \geqslant c_1 \begin{cases} \nu_2 |\mathbf{e} - \hat{\mathbf{e}}|^p, \\ \nu_1 |\mathbf{e} - \hat{\mathbf{e}}|^2 + \nu_2 |\mathbf{e} - \hat{\mathbf{e}}|^p. \end{cases} \tag{1.28}
$$

If $p \in (1, 2)$ and $|\mathbf{e}|, |\hat{\mathbf{e}}| \leqslant R$ then there exists $C_5 = C_5(R)$ such that

$$
(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\hat{\mathbf{e}}))(e_{ij} - \hat{e}_{ij}) \geqslant C_5 |\mathbf{e} - \hat{\mathbf{e}}|^2. \tag{1.29}
$$

Proof. See [45, Lemma 1.19 pg. 198].

□

Lemma 1.50. *Let $p > 1$ and $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$, $\Phi : \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy (1.19)-(1.22). Then there exist positive constants c_5, c_6 and C_6 such that for all $\mathbf{e} \in \mathbb{R}^{n^2}_{sym}$*

$$c_6(1 + |\mathbf{e}|)^p \geqslant \Phi(\mathbf{e}) \geqslant C_6 \begin{cases} |\mathbf{e}|^p, \\ |\mathbf{e}|(|\mathbf{e}|^{p-1} - 1). \end{cases} \tag{1.30}$$

If $p \geqslant 2$ then

$$c_5\nu_1(1 + \mu|\mathbf{e}|)^{p-2}|\mathbf{e}|^2 \leqslant \Phi(\mathbf{e}). \tag{1.31}$$

Proof. See [45, Lemma 1.35 pg. 201]. □

Observe that, by Lemma 1.49, if the stress tensor $\mathbb{S}$ satisfies (1.19)-(1.22), then, also satisfies (1.18). In particular, it satisfies the p-coercivity and $(p-1)$-growth conditions.

Example 1.51. ***(Stokes' Law)*** *If the dependence of $\mathbb{S}$ on $\mathbf{e}$ is linear, i.e.*

$$\mathbb{S}(\mathbf{e}) = 2\nu\mathbf{e} \quad \nu > 0, \tag{1.32}$$

and $\sigma \equiv 0$, system (1.17) turns into the well-known Navier-Stokes system

$$\begin{gathered} \operatorname{div}_x \mathbf{v} = 0, \\ \rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \nu \Delta v_i + \rho_0 f_i, \end{gathered}$$

for $i = 1, 2, ..., n$. We observe that in this case

$$\mathbb{S}(\mathbf{M}) : \mathbf{M} = 2\nu|\mathbf{M}|^2.$$

Therefore, the p-coercivity condition is satisfied with $p = 2$.

Definition 1.52. *An incompressible fluid, the behavior of which is characterized by Stokes' law (1.32), is called* ***Newtonian fluid****. Fluids that cannot be described by (1.32) are usually called* ***non-Newtonian fluids****.*

Remark 1.53. *The main points on non-Newtonian fluid behavior are: the ability of the fluid to shear thin or shear thicken in shear flows; the presence of non-zero normal stress differences in shear flows; the ability of the fluid to exhibit stress relaxation and the ability of fluid to creep [26, pg. 129].*

Example 1.54. ***(Ladyzhenskaya models)*** *In 1969 O. Ladyzhenskaya proposed new equations for the description of the motion of viscous incompressible fluids [39]:*

$$\textbf{\textit{(LM1)}} \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - (\nu_0 + \nu_1 |\nabla \mathbf{v}|_2^2)\Delta \mathbf{v} + v_k \dfrac{\partial \mathbf{v}}{\partial x_k} = -\nabla p + \mathbf{f}(t, x), \\ \operatorname{div}_x \mathbf{v} = 0, \end{cases}$$

where $\nu_0, \nu_1 > 0$, or

$$\textbf{\textit{(LM2)}} \begin{cases} \dfrac{\partial v_i}{\partial t} - \dfrac{\partial}{\partial x_k}\mathbf{T}_{ik}(e_{jl}(\mathbf{v})) + v_k e_{ik}(\mathbf{v}) = -\dfrac{\partial q}{\partial x_i} + f_i, \\ \operatorname{div}_x \mathbf{v} = 0, \end{cases}$$

where, $q = \pi - \dfrac{1}{2}|\mathbf{v}|^2$ *and* $e_{ik}(\mathbf{v})$ *are the components of the symmetric gradient* $\mathbf{e}(\mathbf{v})$*, or its special case*

$$\textbf{\textit{(LM3)}} \begin{cases} \dfrac{\partial v_i}{\partial t} - \dfrac{\partial}{\partial x_k}[(\nu_2 + \nu_3|\mathbf{e}(\mathbf{v})|_2^2)e_{ik}(\mathbf{v})] + v_k e_{ik}(\mathbf{v}) = -\dfrac{\partial \pi}{\partial x_i} + f_i, \\ \operatorname{div}_x \mathbf{v} = 0, \end{cases}$$

where $\nu_2, \nu_3 > 0$.

For the system ***(LM2)****, the following hypotheses for the functions* $\mathbf{T}_{ik}(e_{jl}(\mathbf{v}))$ *are assumed:*

(i) $\mathbf{T}_{ik} = \mathbf{T}_{ki}$, $\mathbf{T}_{ik}(e_{jl}(\mathbf{v}))$ *are continuous functions on* $e_{jl}(\mathbf{v})$ *for all* $j, l = 1, \ldots, n$ *and*

$$|\mathbf{T}_{ik}(e_{jl}(\mathbf{v}))| \leqslant c\big(1 + |\mathbf{e}(\mathbf{v})|^{2\mu}\big)|\mathbf{e}(\mathbf{v})| \quad \textit{with} \;\; \mu \geqslant 1/4,$$

(ii)

$$|\mathbf{T}_{ik}(e_{jl}(\mathbf{v}))| \geqslant \nu_4|\mathbf{e}(\mathbf{v})|^2\big(1 + \varepsilon|\mathbf{e}(\mathbf{v})|^{2\mu}\big) \quad \textit{where} \;\; \nu_4, \varepsilon > 0,$$

(iii) *for any solenoidal vector functions* $\mathbf{u}, \mathbf{v} \in W^{1,2}(\Omega)^n \cap W^{1,2+2\mu}(\Omega)^n$ *with* $\mathbf{u} = \mathbf{v}$ *on* $\partial\Omega$*, the following inequality holds*

$$\int_\Omega [\mathbf{T}_{ik}(e_{jl}(\mathbf{u})) - \mathbf{T}_{ik}(e_{jl}(\mathbf{v}))](e_{ik}(\mathbf{u}) - e_{ik}(\mathbf{v})) \geqslant \nu_5 \int_\Omega \sum_{ik}(e_{ik}(\mathbf{u}) - e_{ik}(\mathbf{v}))^2,$$

where $\nu_5 > 0$.

Conditions ***(i)****-****(iii)*** *are satisfied, for instance, by*

$$\mathbf{T}_{ik}(e_{jl}(\mathbf{v})) = \beta(|\mathbf{e}(\mathbf{v})|_2^2)e_{ik}(\mathbf{v})$$

if the "viscosity coefficient" $\beta(s)$ *is a positive monotonically increasing function of* $s \geqslant 0$ *such that for large* s *the inequality*

$$c_1 s^\mu \leqslant \beta(s) \leqslant c_2 s^\mu \qquad \textit{with } c_1, c_2 > 0 \;\; \textit{and} \;\; \mu \geqslant 1/4.$$

Remark 1.55. *In a recent publication, the asymptotic pullback behavior of the solutions of system* ***(LM1)*** *was analyzed. Furthermore, the finite fractal dimension of the attractor is studied, see [60].*

Example 1.56. ***(Generalized Newtonian fluid and power-law fluid)*** *Let $\mathbb{S}$ be given by*

$$\mathbb{S}(\mathbf{e}) = 2\mu(|\mathbf{e}|^2)\mathbf{e} = 2\tilde{\mu}(\mathbf{e})\mathbf{e}, \tag{1.33}$$

and $\sigma \equiv 0$. The potential Φ is defined by

$$\Phi(\mathbf{e}) = \int_0^{|\mathbf{e}|^2} \mu(s)ds. \tag{1.34}$$

If, in particular we take

$$\mu(s) = \nu_0 s^{r/2}, \tag{1.35}$$

then

$$\Phi(\mathbf{e}) = \frac{2\nu_0}{r+2}|\mathbf{e}|^{r+2} \quad \text{and} \quad \mathbb{S}(\mathbf{e}) : \mathbf{e} = 2\nu_0|\mathbf{e}|^r\mathbf{e} : \mathbf{e} = 2\nu_0|\mathbf{e}|^{r+2}$$

and we see that the p-coercivity condition holds for $p = r + 2$.

Remark 1.57. *The fluids characterized by (1.33) are called* ***generalized Newtonian fluids*** *(even if they are non-Newtonian ones). The fluids described by (1.33) and (1.35) are called power-law fluids.*

Example 1.58. ***(Various variants of power-law fluids)*** *Let us consider*

$$\begin{array}{ll} (\mathbf{a}) & \mathbb{S}^1(\mathbf{e}) = 2\nu_0|\mathbf{e}|^r\mathbf{e}, \\ (\mathbf{b}) & \mathbb{S}^2(\mathbf{e}) = 2\nu_0(1+|\mathbf{e}|^r)\mathbf{e}, \\ (\mathbf{c}) & \mathbb{S}^3(\mathbf{e}) = 2\nu_0(1+|\mathbf{e}|^2)^{r/2}\mathbf{e}, \\ (\mathbf{d}) & \mathbb{S}^{3+i}(\mathbf{e}) = 2\nu_\infty\mathbf{e} + \mathbb{S}^i, \quad i = 1,2,3, \end{array} \tag{1.36}$$

where ν_0 and ν_∞ are positive constants. Using (1.33) and (1.34) it is easy to observe that the potential Φ^i corresponding to the tensor $\mathbb{S}^i$ defined in (1.36) for $r > -1$ and $i = 1, ..., 6$, are as follows:

$$\begin{array}{ll} (\mathbf{a}) & \Phi^1(\mathbf{e}) = \dfrac{2\nu_0}{r+2}|\mathbf{e}|^{r+2}, \\ (\mathbf{b}) & \Phi^2(\mathbf{e}) = \dfrac{2\nu_0}{r+2}(1+|\mathbf{e}|)^{r+2} - \dfrac{2\nu_0}{r+1}(1+|\mathbf{e}|)^{r+1} + \dfrac{2\nu_0}{(r+1)(r+2)}, \\ (\mathbf{c}) & \Phi^3(\mathbf{e}) = \dfrac{2\nu_0}{r+2}\big[(1+|\mathbf{e}|^2)^{\frac{r+2}{2}} - 1\big], \\ (\mathbf{d}) & \Phi^{3+i}(\mathbf{e}) = \nu_\infty|\mathbf{e}|^2 + \Phi^i(\mathbf{e}), \quad i = 1,2,3. \end{array}$$

Observe that the Ladyzhenskaya model ***(LM3)*** *is a type of power-law fluid, since the stress tensor associated to the system* ***(LM3)*** *is similar at the tensor given by $\mathbb{S}^4$ with $r = 2$.*

2 A Class of incompressible non-Newtonian Fluids

In this chapter we will discuss several topics starting with the formulation and statement of the problem, which consists of a model for incompressible non-Newtonian fluids. Moreover, with the Faedo-Galerkin method, we will prove existence of weak solutions. Regarding the uniqueness of weak solution, this result will be shown for weak solutions with a certain preset regularity. Also, we will show a regularity result, for which we will assume some additional hypotheses on the external force, the initial condition and the stress tensor. Finally, we will prove the existence of pullback attractors for an upper-semicontinuous multi-valued process defined on the Hilbert space H for tempered universes, and the existence of pullback attractors for a closed process defined on the Banach space V_p, in tempered universes built from the tempered universes of H.

2.1 Statement of the Problem

The physical properties of the fluids that we are interested in are encoded in the stress tensor $\mathbb{S}$, which is a function of $\mathbf{e}(\mathbf{u}) : \mathbb{R}^n \to \mathbb{R}^{n^2}_{sym}$, that is the symmetric part of the gradient of the fluid velocity $\mathbf{u}$, whose components are defined by

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \quad \text{with} \;\; i, j = 1, \dots, n.$$

Typical examples of stress tensors are

$$\mathbb{S}(\mathbf{e}(\mathbf{u})) = \nu_1 \mathbf{e}(\mathbf{u}) + \nu_2 |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u}), \tag{2.1}$$

or its special case

$$\mathbb{S}(\mathbf{e}(\mathbf{u})) = \big(\nu_1 + \nu_2 |\mathbf{e}(\mathbf{u})|_2^2\big)\mathbf{e}(\mathbf{u}). \tag{2.2}$$

As we saw in Section 1.6, Example 1.54, they were suggested by O. Ladyzhenskaya [41, 39, 40, 23]. They are known as Ladyzhenskaya Models or power-law fluids or rate-type fluids, and belong to a class so-called generalized Navier-Stokes fluids.

In this work, we are going to consider general assumptions on the behavior of stress tensor, conditions that will be given in (2.7) below, and, in particular, it satisfies the p-coercivity and the $(p-1)$-growth conditions. Let us move on to the formulation of the problem.

Let be $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, an open bounded domain with regular boundary $\partial\Omega$. For $s \geqslant 1$ and $p > 1$, let us recall the following spaces

$$\begin{aligned} \mathcal{V} &:= \{\varphi \in C_c^\infty(\Omega)^n : \mathrm{div}_x\varphi = 0\}; \\ H &= \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega)^n - \text{norm}; \\ V_p &= \text{closure of } \mathcal{V} \text{ in the } W^{1,p}(\Omega)^n - \text{norm}; \\ V^s &= \text{closure of } \mathcal{V} \text{ in the } W^{s,2}(\Omega)^n - \text{norm}. \end{aligned}$$

Given τ and T, with $\tau < T$, we consider the following system of partial differential equations with Dirichlet boundary condition for incompressible non-Newtonian fluids, that we will call problem **(LM)**:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}_x \mathbb{S}(\mathbf{e}(\mathbf{u})) = -\mathrm{div}_x(\mathbf{u}\otimes\mathbf{u}) - \nabla\pi + \mathbf{f}(t) \quad \text{in } \Omega_{\tau,T}, \tag{2.3}$$

$$\mathrm{div}_x\mathbf{u} = 0 \quad \text{in } \Omega_{\tau,T}, \tag{2.4}$$

$$\mathbf{u} = 0 \quad \text{on } (\tau,T)\times\partial\Omega, \tag{2.5}$$

$$\mathbf{u}(\tau,x) = \mathbf{u}_\tau(x) \quad x\in\Omega, \tag{2.6}$$

where $\Omega_{\tau,T} = (\tau,T)\times\Omega$, $\mathbf{u} = (u_1,\cdots,u_n)$ is the fluid velocity, π the pressure, $\mathbf{f} = (f_1,\cdots,f_n)$ an external force, $\mathbf{u}_\tau$ is the velocity of fluid at the initial time $t = \tau$ and $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ is the stress tensor satisfying:

$$\begin{aligned} \mathbb{S}(\mathbf{0}) &= \mathbf{0}, \\ \big(\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})\big) : (\mathbf{A}-\mathbf{B}) &\geqslant \nu_1\big(1+\mu(|\mathbf{A}|+|\mathbf{B}|)\big)^{p-2}|\mathbf{A}-\mathbf{B}|^2, \\ |\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})| &\leqslant c_1\nu_1\big(1+\mu(|\mathbf{A}|+|\mathbf{B}|)\big)^{p-2}|\mathbf{A}-\mathbf{B}|, \end{aligned} \tag{2.7}$$

for all $\mathbf{A},\mathbf{B} \in \mathbb{R}^{n^2}_{sym}$. We conveniently denote by $\mu = \big(\frac{\nu_2}{\nu_1}\big)^{\frac{1}{p-2}}$ if $p \neq 2$ and $\mu = 1$ if $p = 2$, where ν_1, ν_2 are the viscosities (critical physical parameters of the problem), see for example (2.1) or (2.2).

Note that from (2.7) it follows the p-coercivity and $(p-1)$-growth conditions of $\mathbb{S}$, that is: there exist positive constants c_2, c_3 such that, for all $p \geqslant 2$ the tensor stress $\mathbb{S}$ satisfies

$$\mathbb{S}(\mathbf{D}) : \mathbf{D} \geqslant c_2(\nu_1|\mathbf{D}|^2 + \nu_2|\mathbf{D}|^p), \tag{2.8}$$

and for all $p > 1$ the tensor stress $\mathbb{S}$ satisfies

$$|\mathbb{S}(\mathbf{D})| \leqslant c_3\nu_1(1+\mu|\mathbf{D}|)^{p-1}, \tag{2.9}$$

for all $\mathbf{D} \in \mathbb{R}^{n^2}_{sym}$.

Remark 2.1. *At some point in this work, we will have to work with the components of the function* $\mathbb{S}(\mathbf{D})$ *and we will conveniently denote by* $\tau_{ij}(\mathbf{D}) := \mathbb{S}_{ij}(\mathbf{D})$.

In order to obtain an operator formulation to system **(LM)**, we introduce some operators, related to the stress tensor $\mathbb{S}$ and the matrix $\mathbf{u} \otimes \mathbf{u} = (u_i u_j)$. See, for instance, [44].

Lemma 2.2. *Let $\pi : \Omega \to \mathbb{R}$ and $\phi, \mathbf{u} : \Omega \to \mathbb{R}^n$ be functions such that $\phi, \mathbf{u} \in \mathcal{V}$. Then*

$$(\nabla\pi, \phi) = -\int_\Omega \pi \frac{\partial \phi_i}{\partial x_i} = 0,$$

and

$$(\mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}), \phi) = \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \phi_i dx.$$

Proof. As $\mathbf{u}$ and ϕ are smooth functions with $\mathrm{div}_x \mathbf{u} = 0$, by the Green Theorem (Lemma 1.4), we have that

$$\begin{aligned}(\mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}), \phi) &= \int_\Omega \frac{\partial(u_i u_j)}{\partial x_j} \phi_i dx \\ &= \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \phi_i dx + \int_\Omega u_i \frac{\partial u_j}{\partial x_j} \phi_i dx \\ &= \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \phi_i dx,\end{aligned}$$

since $\mathrm{div}_x \mathbf{u} = 0$. On the other hand

$$(\nabla\pi, \phi) = \int_\Omega \frac{\partial \pi}{\partial x_i} \phi_i dx = -\int_\Omega \pi \frac{\partial \phi_i}{\partial x_i} = 0.$$

□

The proofs of the next Lemmas can be found in [44].

Lemma 2.3. *Let $\mathbf{u} \in L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ be arbitrary. Then*

$$\int_\tau^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i dx dt < \infty$$

for all $\mathbf{v} \in L^p(\tau, T; V_p)$ if $p \geqslant 1 + 2n/(n+2)$.

Proof. We will proof this lemma for the case $p < n$, since the case $p \geqslant n$ is easy. Thus, note that, due to the embedding $W^{1,p}(\Omega)^n \hookrightarrow L^{\frac{np}{n-p}}(\Omega)^n$ (Theorem 1.9) and the interpolation inequality (Theorem 1.14)

$$\|\mathbf{u}\|_\sigma \leqslant \|\mathbf{u}\|_2^{1-\alpha} \|\mathbf{u}\|_{np/(n-p)}^\alpha \quad \text{with} \quad \alpha = \frac{(\sigma - 2)np}{\sigma(np - 2(n-p))} \tag{2.10}$$

holds provided that $2 \leqslant \sigma \leqslant np/(n-p)$.

Next, we will to verify that if $\mathbf{u} \in L^p(\tau, T; V_p) \cap L^\infty(\tau, T; H)$ then

$$\mathbf{u} \in L^\sigma(\Omega_{\tau,T})^n = L^\sigma(\tau, T; L^\sigma(\Omega)^n), \tag{2.11}$$

with $\sigma = \dfrac{n+2}{n}p$. Indeed, again due to the embedding $W^{1,p}(\Omega)^n \hookrightarrow L^{\frac{np}{n-p}}(\Omega)^n$, i.e., there exists a constant $d > 0$ such that $\|\mathbf{u}\|_{np/(n-p)} \leqslant d\|u\|_{1,p}$. Then, from (2.10), we obtain

$$\int_\tau^T \|\mathbf{u}\|_\sigma^\sigma dt \leqslant \int_\tau^T |\mathbf{u}|_2^{(1-\alpha)\sigma} \|\mathbf{u}\|_{np/(n-p)}^{\alpha\sigma} dt \leqslant d\|\mathbf{u}\|_{L^\infty(\tau,T;H)}^{(1-\alpha)\sigma} \int_\tau^T \|\mathbf{u}\|_{1,p}^{\frac{p(\sigma-2)n}{(np-2(n-p))}} dt. \tag{2.12}$$

The right hand side of (2.12) is finite if $\dfrac{(\sigma-2)n}{np-2(n-p)} = 1$. Thus (2.11) is proved.

Now, integrating by parts and by using the Hölder inequality

$$\int_\tau^T \Big| \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i dx \Big| dt = \int_\tau^T \Big| \int_\Omega u_j \frac{\partial v_i}{\partial x_j} u_i dx \Big| dt \leqslant \|\nabla \mathbf{v}\|_{L^p(\Omega_{\tau,T})^n} \|\mathbf{u}\|_{L^{2q}(\Omega_{\tau,T})^n}^2,$$

where $q = \dfrac{p}{p-1}$.

Since $2q \leqslant \sigma$ ($\Leftrightarrow p \geqslant 1 + 2n/(n+2)$) and using (2.12), we have

$$\begin{aligned}
\int_\tau^T \int_\Omega |u_i(x,t)|^{2q} dxdt &\overset{Hölder}{\leqslant} C\Big(\int_\tau^T \int_\Omega |u_i(x,t)|^\sigma dxdt\Big)^{\frac{2q}{\sigma}} \\
&= C\Big(\int_\tau^T \|\mathbf{u}\|_\sigma^\sigma dt\Big)^{\frac{2q}{\sigma}} \\
&\leqslant \widetilde{C}^q \Big(\int_\tau^T \|\mathbf{u}\|_{1,p}^p dt\Big)^{\frac{2q}{\sigma}},
\end{aligned}$$

where $\widetilde{C} = C\big(d\|\mathbf{u}\|_{L^\infty(\tau,T;H)}^{(1-\alpha)\sigma}\big)^{\frac{2}{\sigma}}$. Therefore, we conclude that

$$\int_\tau^T | \int_\Omega (u_j \frac{\partial u_i}{\partial x_j} v_i)(x,t) dx| dt \leqslant \widetilde{C} \|\nabla \mathbf{v}\|_{L^p(\Omega_{\tau,T})^n} \|\mathbf{u}\|_{L^p(\tau,T;V_p)}^{\frac{2p}{\sigma}} < \infty. \tag{2.13}$$

□

Remark 2.4. *From the previous Lemma, we observe that for $\mathbf{u} \in L^\infty(\tau,T;H) \cap L^p(\tau,T;V_p)$, it holds $u_j \dfrac{\partial u_i}{\partial x_j} \in L^q(\Omega_{\tau,T})$ with $q = p/(p-1)$, if $p \geqslant 1 + 2n/(n+2)$.*

Definition 2.5. *Let us define the operator $B : L^\infty(\tau,T;H) \cap L^p(\tau,T;V_p) \to L^q(\tau,T;V_p^*)$*

$$\int_\tau^T \langle B(\mathbf{u})(t), \mathbf{v}(t) \rangle dt = \int_\tau^T \int_\Omega u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j} v_i(x,t) dxdt.$$

Lemma 2.6. *If $p \geqslant 1 + 2n/(n+2)$, then*

$$B : L^p(\tau,T;V_p) \cap L^\infty(\tau,T;H) \to L^q(\tau,T;V_p^*),$$

is a continuous operator.

Proof. From (2.13), we know that

$$\int_\tau^T \langle B(\mathbf{u})(t), \mathbf{v}(t)\rangle dt \leqslant \widetilde{C}\|\nabla \mathbf{v}\|_{L^p(\Omega_{\tau,T})}\|\mathbf{u}\|^{\frac{2p}{\sigma}}_{L^p(\tau,T;V_p)},$$

therefore, B is well defined.

Now, let $\{\mathbf{u}_m\}$ be a sequence belonging to $L^p(\tau,T;V_p)\cap L^\infty(\tau,T;H)$ such that

$$\begin{aligned} \mathbf{u}_m \to \mathbf{u} \quad &\text{in} \quad L^\infty(\tau,T;H),\\ \mathbf{u}_m \to \mathbf{u} \quad &\text{in} \quad L^p(\tau,T;V_p). \end{aligned} \tag{2.14}$$

Then we have that

$$\begin{aligned} \int_\tau^T \langle B(\mathbf{u}_m) - B(\mathbf{u}), \mathbf{v}\rangle dt = &\int_\tau^T\int_\Omega (u_j^m - u_j)(u_i^m - u_i)\frac{\partial v_i}{\partial x_j}dxdt + \int_\tau^T\int_\Omega (u_i u_j^m + u_j u_i^m)\frac{\partial v_i}{\partial x_j}dxdt\\ &- 2\int_\tau^T\int_\Omega u_j u_i \frac{\partial v_i}{\partial x_j}dxdt = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_\tau^T\int_\Omega (u_j^m - u_j)(u_i^m - u_i)\frac{\partial v_i}{\partial x_j}dxdt,\\ I_2 &= \int_\tau^T\int_\Omega (u_i u_j^m + u_j u_i^m)\frac{\partial v_i}{\partial x_j}dxdt - 2\int_\tau^T\int_\Omega u_j u_i\frac{\partial v_i}{\partial x_j}dxdt. \end{aligned}$$

By inequality (2.13), we get

$$|I_1| \leqslant \widehat{C}\|\nabla \mathbf{v}\|_{L^p(\Omega_{\tau,T})}\|\mathbf{u}_m - \mathbf{u}\|^{2(1-\alpha)}_{L^\infty(\tau,T;H)}\|\mathbf{u}_m - \mathbf{u}\|^{\frac{2p}{\sigma}}_{L^p(\tau,T;V_p)} \overset{(2.14)}{\to} 0.$$

Observe that from the convergence in (2.14), we have $|I_2| \to 0$. Therefore, we conclude that B is a continuous operator. □

Definition 2.7. *Let us define the operator* $\mathbb{T} : L^p(\tau,T;V_p) \to L^q(\tau,T;V_p^*)$ *by*

$$\int_\tau^T \langle \mathbb{T}(\mathbf{u})(t), \mathbf{v}(t)\rangle dt = \int_\tau^T\int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{v})dxdt.$$

Lemma 2.8. *Let* $\mathbb{S}$ *be a stress tensor satisfying (2.7). Then, for* $p > 1$ *we have that* $\mathbb{T} : L^p(\tau,T;V_p) \to L^q(\tau,T;V_p^*)$ *is a continuous operator.*

Proof.

$$\begin{aligned} \int_\tau^T |\int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{v})dx|dt &\overset{(2.9)}{\leqslant} c_3\nu_1\int_\tau^T\int_\Omega (1+\mu|\mathbf{e}(\mathbf{u})|)^{p-1}|\mathbf{e}(\mathbf{v})|dxdt\\ &\overset{Hölder}{\leqslant} c_3\nu_1\Big(\int_\tau^T \|1+\mu|\mathbf{e}(\mathbf{u})|\|_p^p dt\Big)^{\frac{p-1}{p}}\Big(\int_\tau^T \|\mathbf{e}(\mathbf{v})\|_p^p dt\Big)^{\frac{1}{p}}. \end{aligned}$$

Therefore, since $p > 1$ then, the operator $\mathbb{T}$ is well defined. Besides that, from the previous inequality and with the similar arguments as in Lemma 2.6, the operator $\mathbb{T} : L^p(\tau,T;V_p) \to L^q(\tau,T;V_p^*)$ is continuous. □

With the help of the previous Lemmas, we can now introduce the weak formulation to **(LM)**.

Definition 2.9. *Let $p > 1$, $\mathbf{u}_\tau \in H$ and $\mathbf{f} \in L^q(\tau, T; V_p^*)$, where $q = \dfrac{p}{p-1}$. Then a function $\mathbf{u}$ is a weak solution to system **(LM)** if*

$$\mathbf{u} \in L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p) \quad \textit{with} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^q(\tau, T; V_p^*), \tag{2.15}$$

and $\mathbf{u}$ satisfy

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \rangle + \langle \mathbb{T}(\mathbf{u}), \mathbf{v} \rangle + \langle B(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle, \tag{2.16}$$

for all $\mathbf{v} \in V_p$, and a.e. $\tau \leqslant t \leqslant T$, and

$$\mathbf{u}(\tau) = \mathbf{u}_\tau. \tag{2.17}$$

Remark 2.10. *By Theorem 1.10 a weak solution has a representative in the class*

$$\mathbf{u} \in C([\tau, T]; H), \tag{2.18}$$

whereby (2.17) makes sense. Besides, for any functions $\mathbf{u}$, $\mathbf{v}$, belonging to the class (2.15), it holds

$$(\mathbf{u}(t_2), \mathbf{v}(t_2)) - (\mathbf{u}(t_1), \mathbf{v}(t_1)) = \int_{t_1}^{t_2} \Big(\Big\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \Big\rangle + \Big\langle \frac{\partial \mathbf{v}}{\partial t}, \mathbf{u} \Big\rangle \Big) dt, \tag{2.19}$$

where on the right-hand side $\langle \cdot, \cdot \rangle$ stands for the duality between V_p^* and V_p.

2.2 Existence of Weak Solutions to (LM)

The proof of existence of weak solutions is standard and similar demonstrations can be found in [42, 45].

Theorem 2.11. ***(Existence)*** *Given τ, T with $\tau < T$, $\mathbf{u}_\tau \in H$ and $\mathbf{f} \in L^q(\tau, T; V_p^*)$. Then, if $p \geqslant 1 + \dfrac{2n}{n+2}$, there exists at least one weak solution of the problem **(LM)**.*

Proof. Let us consider

$$s > \frac{n}{2} + 1. \tag{2.20}$$

From Theorem 1.18 there exists a set $\{\mathbf{w}_r\}_{r=1}^\infty$ formed by the eigenfunctions to problem

$$((\mathbf{w}_r, \varphi))_s = \lambda_r (\mathbf{w}_r, \varphi) \quad \forall \varphi \in V^s, \tag{2.21}$$

which are orthonormal in H and orthogonal in V^s. We choose s satisfying (2.20) because of the following: if $\mathbf{v} \in W^{s,2}(\Omega)^n$ then $\nabla \mathbf{v} \in W^{s-1,2}(\Omega)^{n^2}$ and by Theorem 1.9

$$W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \text{ if } \frac{1}{2} - \frac{s-1}{n} < 0,$$

which is just (2.20). Consequently, for all $p > 1$ we have $\nabla \mathbf{v} \in L^p(\Omega)^{n^2}$ and $V^s \hookrightarrow V_p$.

- **Galerkin system and a priori estimates:**

Let us define $\mathbf{u}_m(t,x) = \sum_{r=1}^{m} \gamma_r^m(t)\mathbf{w}_r$, where the coefficients $\gamma_r^m(t)$ solve the so-called Galerkin system

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}_m(t),\mathbf{w}_j) + \langle \mathbb{T}(\mathbf{u}_m(t)),\mathbf{w}_j\rangle + \langle B(\mathbf{u}_m(t)),\mathbf{w}_j\rangle = \langle \mathbf{f}(t),\mathbf{w}_j\rangle \;\; 1 \leqslant j \leqslant m, \\ \mathbf{u}_m(\tau) = P^m\mathbf{u}_\tau. \end{cases} \tag{2.22}$$

Here, P^m is the orthogonal projector of H onto the linear hull of the first m eigenvectors $\mathbf{w}_j$, $j = 1, ..., m$, (see Theorem 1.18), therefore

$$P^m\mathbf{u}_\tau \to \mathbf{u}_\tau \;\; in \;\; L^2(\Omega)^n. \tag{2.23}$$

Observe that (2.22) is a system of ordinary differential equations in the unknown $\gamma^m(t) = (\gamma_1^m(t), \cdots, \gamma_m^m(t))$. The existence and uniqueness of solutions follow from Theorem 1.20, so there exists one solution defined in an interval $[\tau, t_m)$ with $\tau < t_m \leqslant T$. However, as can be deduced by the a priori estimates below, $t_m = T$.

We multiply the $j-$th equation of the Galerkin system (2.22) by $\gamma_j^m(t)$ and add the equations. The result can be written in the form

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}_m(t)|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u}_m)) : \mathbf{e}(\mathbf{u}_m)dx = \langle \mathbf{f}(t),\mathbf{u}_m\rangle, \tag{2.24}$$

since $\int_\Omega u_j^m \dfrac{\partial u_i^m}{\partial x_j} u_i^m dx = 0$.

Given $\epsilon > 0$, by (2.8), the Korn inequality and applying the the Young inequality in the right-hand side of (2.24), one has

$$\frac{d}{dt}|\mathbf{u}_m|_2^2 + \frac{2c_2\nu_1}{c_0^2}|\nabla\mathbf{u}_m|_2^2 + \big(\frac{2c_2\nu_2}{\tilde{c}_0^p} - \frac{2\epsilon^p}{p}\big)\|\nabla\mathbf{u}_m\|_p^p \leqslant \frac{2}{q\epsilon^q}\|\mathbf{f}(t)\|_*^q$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Integrating from τ to t, we deduce

$$\begin{aligned} &|\mathbf{u}_m(t)|_2^2 + \frac{2c_2\nu_1}{c_0^2}\int_\tau^t |\nabla\mathbf{u}_m(s)|_2^2 ds + \big(\frac{2c_2\nu_2}{\tilde{c}_0^p} - \frac{2\epsilon^p}{p}\big)\int_\tau^t \|\nabla\mathbf{u}_m(s)\|_p^p ds \\ &\leqslant |P^m\mathbf{u}_\tau|_2^2 + \frac{2}{q\epsilon^q}\int_\tau^t \|\mathbf{f}(s)\|_*^q ds. \end{aligned}$$

We know that $|P^m\mathbf{u}_\tau|_2 \leqslant |\mathbf{u}_\tau|_2$. So we choose $\epsilon > 0$ small enough and conclude that $t_m = T$, $\forall m \in \mathbb{N}$, and

$$\begin{aligned} &\{\mathbf{u}_m\}_{m=1}^\infty \;\text{ is bounded in } L^\infty(\tau,T;H), \\ &\{\mathbf{u}_m\}_{m=1}^\infty \;\text{ is bounded in } L^p(\tau,T;V_p). \end{aligned} \tag{2.25}$$

From (2.22) we have that

$$\frac{\partial \mathbf{u}_m}{\partial t} = -P^m \mathbb{T}(\mathbf{u}_m) - P^m B(\mathbf{u}_m) + P^m \mathbf{f}. \tag{2.26}$$

Therefore, from Lemmas 2.6 and 2.8, the sequence $\{\dfrac{\partial \mathbf{u}_m}{\partial t}\}$ is bounded in $L^q(\tau, T; V_p^*)$.

- **Limiting process I**

From (2.25), (2.26), the compactness theorems (Theorem 1.5 and Corollary 1.6) and the Aubin-Lions Theorem (Theorem 1.11), it follows that, up to subsequences,

$$\begin{aligned}
&\mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in} \;\; L^\infty(\tau, T; H), \\
&\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{in} \;\; L^p(\tau, T; V_p), \\
&\frac{\partial \mathbf{u}_m}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \quad \text{in} \;\; L^q(\tau, T; V_p^*), \\
&\mathbf{u}_m \to \mathbf{u} \quad \text{in} \;\; L^2(\tau, T; H) \;\; \text{and a.e. in } H \text{ and } \Omega_{\tau,T}, \\
&\mathbb{T}(\mathbf{u}_m) \rightharpoonup \mathcal{X} \quad \text{in} \;\; L^q(\tau, T; V_p^*).
\end{aligned} \tag{2.27}$$

Now, we are going to show that

$$\int_\tau^T \langle B(\mathbf{u}_m(s)), \mathbf{v}\rangle ds \overset{m\to\infty}{\longrightarrow} \int_\tau^T \langle B(\mathbf{u}(s)), \mathbf{v}\rangle ds \quad \forall \mathbf{v} \in V_p. \tag{2.28}$$

For each $\mathbf{w}_r$ with $r \in \mathbb{N}$, we have

$$\begin{aligned}
\int_\tau^T \int_\Omega (u_j^m u_i^m - u_j u_i) \frac{\partial w_i^r}{\partial x_j} dx dt &= \int_\tau^T \int_\Omega (u_j^m - u_j) u_i^m \frac{\partial w_i^r}{\partial x_j} dx dt \\
&\quad + \int_\tau^T \int_\Omega u_j (u_i^m - u_i) \frac{\partial w_i^r}{\partial x_j} dx dt \\
&= I_1 + I_2.
\end{aligned}$$

Then,

$$|I_1| \leqslant \|\nabla \mathbf{w}_r\|_\infty \int_\tau^T |\mathbf{u}_m - \mathbf{u}|_2 |\mathbf{u}^m|_2 dt \overset{(2.25)}{\leqslant} C \|\nabla \mathbf{w}_r\|_\infty \int_\tau^T |\mathbf{u}_m - \mathbf{u}|_2 dt$$

which tends to zero due to (2.27). On the other hand the weak convergence in (2.27), implies $|I_2| \to 0$ as $m \to \infty$. Since $\operatorname{span}\{\mathbf{w}_r\}$ is dense in V_p, given $\mathbf{v} \in V_p$ there exists a sequence $\{\mathbf{v}_k\}$ in $\operatorname{span}\{\mathbf{w}_r\}$ such that $\mathbf{v}_k \to \mathbf{v}$ in V_p. Then from estimate (2.13) given in Lemma 2.3, we obtain

$$\begin{aligned}
&\int_\tau^T \langle B(\mathbf{u}_m) - B(\mathbf{u}), \mathbf{v}\rangle dt \\
&= \int_\tau^T \langle B(\mathbf{u}_m), \mathbf{v} - \mathbf{v}_k\rangle dt + \int_\tau^T \langle B(\mathbf{u}_m) - B(\mathbf{u}), \mathbf{v}_k\rangle dt + \int_\tau^T \langle B(\mathbf{u}), \mathbf{v} - \mathbf{v}_k\rangle dt \\
&\leqslant \widetilde{C}_{\tau,T} \|\mathbf{v} - \mathbf{v}_k\|_{1,p} \big(\|\mathbf{u}_m\|_{L^p(\tau,T;V_p)}^{\frac{2p}{\sigma}} + \|\mathbf{u}\|_{L^p(\tau,T;V_p)}^{\frac{2p}{\sigma}} \big) + \int_\tau^T \langle B(\mathbf{u}_m) - B(\mathbf{u}), \mathbf{v}_k\rangle dt.
\end{aligned}$$

Thus (2.28) is true.

From (2.22), (2.27), and (2.28), we obtain

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in V_p \text{ and a.e. } t. \tag{2.29}$$

To conclude the proof of this theorem, we have to show that $\mathbf{u}(\tau) = \mathbf{u}_\tau$ and $\mathbb{T}(\mathbf{u}) = \mathcal{X}$.

- **Limiting process II**

Observe that

$$\frac{d}{dt}(\mathbf{u}_m(t), \mathbf{w}_r) + \langle \mathbb{T}(\mathbf{u}_m(t)), \mathbf{w}_r\rangle + \langle B(\mathbf{u}_m(t)), \mathbf{w}_r\rangle = \langle \mathbf{f}(t), \mathbf{w}_r\rangle,$$

is true, for all $\mathbf{w}_r$, $1 \leqslant r \leqslant m$.

Thus, integrating from τ to t, we obtain

$$(\mathbf{u}_m(t), \mathbf{w}_r) + \int_\tau^t \big(\langle \mathbb{T}(\mathbf{u}_m(s)), \mathbf{w}_r\rangle + \langle B(\mathbf{u}_m(s)), \mathbf{w}_r\rangle\big) ds = (\mathbf{u}_m(\tau), \mathbf{w}_r) + \int_\tau^t \langle \mathbf{f}(s), \mathbf{w}_r\rangle ds,$$

for all $\mathbf{w}_r$, $1 \leqslant r \leqslant m$.

From (2.23), (2.27) and passing to the limit in m, we have

$$(\mathbf{u}(t), \mathbf{w}_r) + \int_\tau^t \big(\langle \mathcal{X}, \mathbf{w}_r\rangle + \langle B(\mathbf{u}(s)), \mathbf{w}_r\rangle\big) ds = (\mathbf{u}_\tau, \mathbf{w}_r) + \int_\tau^t \langle \mathbf{f}(s), \mathbf{w}_r\rangle ds,$$

for all $\mathbf{w}_r$. Thus, by a density argument, we conclude that

$$(\mathbf{u}(t), \mathbf{v}) + \int_\tau^t \big(\langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}(s)), \mathbf{v}\rangle\big) ds = (\mathbf{u}_\tau, \mathbf{v}) + \int_\tau^t \langle \mathbf{f}(s), \mathbf{v}\rangle ds, \tag{2.30}$$

for all $\mathbf{v} \in V_p$.

Now, integrating (2.29) from τ to t and using (2.19), it follows that

$$(\mathbf{u}(t), \mathbf{v}) + \int_\tau^t \big(\langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}(s)), \mathbf{v}\rangle\big) ds = (\mathbf{u}(\tau), \mathbf{v}) + \int_\tau^t \langle \mathbf{f}(s), \mathbf{v}\rangle ds. \tag{2.31}$$

Therefore, comparing (2.30) and (2.31), we conclude that

$$\mathbf{u}(\tau) = \mathbf{u}_\tau.$$

The next argument is justified by the fact that the function $\mathbf{u}(t)$ cannot be chosen as a test function in the weak formulation, since, for the moment, it is only valid for all $\mathbf{v} \in V_p$. Therefore, from $\mathbf{u}$ we build a function $\mathbf{v}(t)$ (regularization of $\mathbf{u}$) that is defined in V_p for each $t \in [\tau, T]$.

We consider $s_\tau, s \in]\tau, T[$, $s_\tau < s$; and θ_m a continuous function, linear by parts on $[\tau, T]$, $\theta_m(t) = 1$ if $s_\tau + \frac{2}{m} < t < s - \frac{2}{m}$, $\theta_m(t) = 0$ if $t > s - \frac{1}{m}$ or $t < s_\tau + \frac{1}{m}$; and ρ_k a standard mollifier function in $\mathcal{D}(\mathbb{R})$, $\rho_k(t) = \rho_k(-t)$,

$$\int_{-\infty}^{+\infty} \rho_k(t)dt = 1, \ \mathrm{supp}(\rho_k) \subset \Big[-\frac{1}{k}, \frac{1}{k}\Big].$$

For $k > 2m$, we introduce

$$\mathbf{v} = \big((\theta_m \mathbf{u}) * \rho_k * \rho_k\big)\theta_m. \tag{2.32}$$

We denote by $\mathbf{u}' := \frac{\partial \mathbf{u}}{\partial t}$, then note that

$$\begin{aligned}
\int_\tau^T \langle \mathbf{u}', \mathbf{v}\rangle dt &= \int_\tau^T \langle \theta_m \mathbf{u}', (\theta_m \mathbf{u}) * \rho_k * \rho_k\rangle dt \\
&= \int_\tau^T \langle (\theta_m \mathbf{u})' * \rho_k, (\theta_m \mathbf{u}) * \rho_k\rangle dt - \int_\tau^T \langle \theta_m' \mathbf{u}, (\theta_m \mathbf{u}) * \rho_k * \rho_k\rangle dt \\
&= -\int_\tau^T \langle \theta_m' \mathbf{u}, (\theta_m \mathbf{u}) * \rho_k * \rho_k\rangle dt \to -\int_\tau^T \theta_m \theta_m' |\mathbf{u}|_2^2 dt, \quad as\ k \to \infty.
\end{aligned}$$

From (2.32) there follows

$$\int_\tau^T \langle B(\mathbf{u}), \mathbf{v}\rangle dt \overset{k\to\infty}{\to} \int_\tau^T \theta_m^2 \langle B(\mathbf{u}), \mathbf{u}\rangle dt = 0, \quad \big(\text{since } \langle B(\mathbf{u}), \mathbf{u}\rangle = 0 \ \forall \mathbf{u} \in V_p\big).$$

From the above, we obtain that

$$\int_\tau^T (-\theta_m \theta_m') |\mathbf{u}|_2^2 dt + \int_\tau^T \theta_m^2 \langle \mathcal{X}, \mathbf{u}\rangle dt = \int_\tau^T \theta_m^2 \langle \mathbf{f}, \mathbf{u}\rangle dt. \tag{2.33}$$

By the continuity of $\mathbf{u}$ and when $m \to \infty$, we note that

$$\int_\tau^T (-\theta_m \theta_m') |\mathbf{u}|_2^2 dt \to \frac{1}{2}|\mathbf{u}(s)|^2 - \frac{1}{2}|\mathbf{u}(s_\tau)|^2 \quad \forall\ s, s_\tau.$$

As a consequence

$$\frac{1}{2}|\mathbf{u}(s)|^2 + \int_{s_\tau}^s \langle \mathcal{X}, \mathbf{u}\rangle dt = \frac{1}{2}|\mathbf{u}(s_\tau)|^2 + \int_{s_\tau}^s \langle \mathbf{f}, \mathbf{u}\rangle dt \quad \forall\ s, s_\tau. \tag{2.34}$$

Again, since $\mathbf{u} \in C([\tau, T]; H)$ then, if $s_{\tau k} \to \tau$ we have $\mathbf{u}(s_{\tau k}) \to \mathbf{u}(\tau)$ in H, therefore, we conclude

$$\mathbf{u}(s_{\tau k}) \to \mathbf{u}_\tau \quad \text{in } H. \tag{2.35}$$

We fix s in (2.34), and we take $s_\tau = s_{\tau k}$. Thanks to (2.35), we deduce

$$\frac{1}{2}|\mathbf{u}(s)|^2 + \int_\tau^s \langle \mathcal{X}, \mathbf{u}\rangle dt = \frac{1}{2}|\mathbf{u}_\tau|^2 + \int_\tau^s \langle \mathbf{f}, \mathbf{u}\rangle dt \quad \forall s. \tag{2.36}$$

For $\varphi \in L^p(\tau, T; V_p)$, we define

$$X_m^s = \int_\tau^s \langle \mathbb{T}(\mathbf{u}_m) - \mathbb{T}(\varphi), \mathbf{u}_m - \varphi \rangle dt + \frac{1}{2}|\mathbf{u}_m(s)|_2^2. \tag{2.37}$$

Recall that from (2.26) it holds

$$\mathbf{u}_m(s) \rightharpoonup \mathbf{u}(s) \quad \text{a.e.} \;\; s \in (\tau, T) \text{ in } H.$$

Observe that, if $\varphi_1, \varphi_2 \in L^p(\tau, T; V_p)$, from (2.7), we obtain the monotonicity of the operator $\mathbb{T}$

$$\int_\tau^s \langle \mathbb{T}(\varphi_1) - \mathbb{T}(\varphi_2), \varphi_1 - \varphi_2 \rangle dt \geqslant 0.$$

Hence, we deduce from (2.37) that

$$\liminf_{m\to\infty} X_m^s \geqslant \frac{1}{2}|\mathbf{u}(s)|_2^2 \quad \text{a.e. } s \in (\tau, T). \tag{2.38}$$

By re-writing X_m^s (see 2.24) and thanks to (2.27), it follows

$$X_m^s = \int_\tau^s \langle \mathbf{f}, \mathbf{u}_m \rangle dt + \frac{1}{2}|\mathbf{u}_m(\tau)|_2^2 - \int_\tau^s \langle \mathbb{T}(\mathbf{u}_m), \varphi \rangle dt - \int_\tau^s \langle \mathbb{T}(\varphi), \mathbf{u}_m - \varphi \rangle dt \to X^s,$$

where

$$X^s = \int_\tau^s \langle \mathbf{f}, \mathbf{u} \rangle dt + \frac{1}{2}|\mathbf{u}_\tau|_2^2 - \int_\tau^s \langle \mathcal{X}, \varphi \rangle dt - \int_\tau^s \langle \mathbb{T}(\varphi), \mathbf{u} - \varphi \rangle dt.$$

From (2.36) and (2.38), we obtain that

$$\int_\tau^s \langle \mathcal{X} - \mathbb{T}(\varphi), \mathbf{u} - \varphi \rangle dt \geqslant 0, \quad \forall \varphi \in L^p(\tau, T; V_p).$$

Taking $\varphi = \mathbf{u} - \lambda\phi$, where $\phi \in L^p(\tau, T; V_p)$, together with the limit $\lambda \to 0^+$, yields

$$\int_\tau^s \langle \mathcal{X} - \mathbb{T}(\mathbf{u}), \phi \rangle dt \geqslant 0 \quad \forall \phi \in L^p(\tau, T; V_p).$$

Since ϕ is arbitrary, we conclude that $\mathcal{X} = \mathbb{T}(\mathbf{u})$.

□

Lemma 2.12. ***(Energy Equality)*** *Under conditions of Theorem 2.11, any function* $\mathbf{v}$ *in the class (2.15) can be taken as a test function in the weak formulation (2.16). Consequently, the energy equality holds*

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{u}) dx = \langle \mathbf{f}, \mathbf{u} \rangle \quad a.e. \;\; t \in (\tau, T). \tag{2.39}$$

Proof. The proof of this result follows from the application of the Lemma 2.6 and the Lemma 2.8, and from a similar argument of density used to show (2.29), but with the fact that the functions of the form $\sum_{k=1}^{l} \phi_k(t)\mathbf{w}_k$ with $\phi_k \in \mathcal{D}(\mathbb{R})$, are dense in $L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$. For more information see [23, Lemma 7.3 pg. 175]. □

The following Proposition states that, if we have a sequence of weak solutions of **(LM)**, it is possible to extract a subsequence that converges to a weak solution of **(LM)**. This Proposition is going to be used to justify the existence of weak solutions of **(LM)**, built from sequences of weak solutions of **(LM)**, to show, for example, the asymptotic compactness of the process. And besides that, the proof of this Proposition allows us to have regularity of convergence in the sense of continuity, for bounded sequences of weak solutions of **(LM)**, using the Arzelà-Ascoli compactness Theorem.

Proposition 2.13. *Let $\{\mathbf{u}_m\}$ be a sequence of weak solutions to **(LM)** such that $\mathbf{u}_m(\tau) \to \mathbf{u}_\tau$ in H. Then $\{\mathbf{u}_m\}$ is bounded in the spaces (2.15), and there exists $\mathbf{u}$ such that (up to a subsequence) $\mathbf{u}_m \to \mathbf{u}$ in the sense specified in (2.27), and $\mathbf{u}$ is again a weak solution to **(LM)**.*

Proof. Using the energy equality given in Lemma 2.12, we can deduce that there exists a constant $K > 0$, such that

$$\sup_{t\in[\tau,T]} |\mathbf{u}_m(t)|_2^2 + \int_\tau^T \|\nabla \mathbf{u}_m\|_p^p dt \leqslant K.$$

From above estimate and the argument in the proof of Theorem 2.11, we deduce, up to subsequences,

$$\begin{aligned} \mathbf{u}_m &\rightharpoonup \mathbf{u} \quad \text{in} \quad L^p(\tau, T; V_p), \\ \frac{\partial \mathbf{u}_m}{\partial t} &\rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \quad \text{in} \quad L^q(\tau, T; V_p^*), \\ \mathbb{T}(\mathbf{u}_m) &\rightharpoonup \mathcal{X} \quad \text{in} \quad L^q(\tau, T; V_p^*). \end{aligned}$$

With this information, it is a standard procedure to pass to the limit in all terms as in (2.29) and obtain

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle, \tag{2.40}$$

for all $\mathbf{v} \in V_p$ and a.e. $t \in (\tau, T)$.

It remains to show that $\mathbb{T}(\mathbf{u}) = \mathcal{X}$. The key fact is that the weak solutions can be taken as a test function and the monotonicity of the stress tensor $\mathbb{S}$ will allow to show that $\mathbf{u}$ is weak solution to **(LM)**.

First observe that $\{\mathbf{u}_m\}$ is equicontinuous in V_p^* on $[\tau, T]$ and that $\{\mathbf{u}_m\}$ is bounded in $C([\tau, T]; H)$. Therefore, by the Arzelà-Ascoli Theorem, up to a subsequence, there follows that

$$\mathbf{u}_m \to \mathbf{u} \quad \text{strongly in } C([\tau,T];V_p^*).$$

From this and the boundedness of $\{\mathbf{u}_m\}$ in $C([\tau,T];H)$ we conclude that

$$\mathbf{u}_m(s) \rightharpoonup \mathbf{u}(s) \quad \text{weakly in } H \ \ \forall\ \tau \leqslant s \leqslant T. \tag{2.41}$$

Now, as in Lemma 2.12, we can replace $\mathbf{v}$ by $\mathbf{u}$ in (2.40), and then integrating between τ and T, we obtain

$$\frac{1}{2}|\mathbf{u}(T)|_2^2 + \int_\tau^T \langle \mathcal{X}, \mathbf{u}\rangle = \frac{1}{2}|\mathbf{u}(\tau)|_2^2 + \int_\tau^T \langle \mathbf{f}, \mathbf{u}\rangle. \tag{2.42}$$

Let $\phi \in L^\infty(\tau,T;H) \cap L^p(\tau,T;V_p)$ be arbitrary. Thanks to the monotonicity of the stress tensor $\mathbb{S}(\cdot)$ we have

$$\begin{aligned}
0 &\leqslant \int_{\Omega_{\tau,T}} \big(\mathbb{S}(\mathbf{e}(\mathbf{u}_m)) - \mathbb{S}(\mathbf{e}(\phi)) : \big(\mathbf{e}(\mathbf{u}_m) - \mathbf{e}(\phi)\big)\big) dxdt \\
&= \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\mathbf{u}_m)) : \mathbf{e}(\mathbf{u}_m) dxdt - \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\mathbf{u}_m)) : \mathbf{e}(\phi) dxdt \\
&\quad - \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\phi)) : \mathbf{e}(\mathbf{u}_m - \phi) dxdt.
\end{aligned}$$

By Lemma 2.12 to $\mathbf{u}_m$, we obtain

$$\int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\mathbf{u}_m)) : \mathbf{e}(\mathbf{u}_m) dxdt = \frac{1}{2}|\mathbf{u}_m(\tau)|_2^2 - \frac{1}{2}|\mathbf{u}_m(T)|_2^2 + \int_\tau^T \langle \mathbf{f}, \mathbf{u}_m\rangle dt.$$

Combining with the previous inequality, it follows

$$\begin{aligned}
\frac{1}{2}|\mathbf{u}_m(T)|_2^2 \leqslant \frac{1}{2}|\mathbf{u}_m(\tau)|_2^2 &+ \int_\tau^T \langle \mathbf{f}, \mathbf{u}_m\rangle dt - \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\mathbf{u}_m)) : \mathbf{e}(\phi) dxdt \\
&- \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\phi)) : \mathbf{e}(\mathbf{u}_m - \phi) dxdt.
\end{aligned}$$

Letting $m \to \infty$, we can pass to the limit on the right-hand side, while on the left-hand side, we employ (2.41) together with the weak lower semicontinuity of the norm, $|\mathbf{u}(T)|_2 \leqslant \liminf_{m\to\infty} |\mathbf{u}_m(T)|_2$, which preserves the inequality. Hence,

$$\frac{1}{2}|\mathbf{u}(T)|_2^2 \leqslant \frac{1}{2}|\mathbf{u}(\tau)|_2^2 + \int_\tau^T \langle \mathbf{f}, \mathbf{u}\rangle dt - \int_\tau^T \langle \mathcal{X}, \phi\rangle dt - \int_{\Omega_{\tau,T}} \mathbb{S}(\mathbf{e}(\phi)) : \mathbf{e}(\mathbf{u} - \phi) dxdt.$$

Subtracting (2.42) leads to

$$0 \leqslant \int_\tau^T \langle \mathcal{X} - \mathbb{T}(\phi), \mathbf{u} - \phi\rangle dt;$$

and we conclude that $\mathcal{X} = \mathbb{T}(\mathbf{u})$, in the same way as in Theorem 2.11. □

2.3 Uniqueness of Weak Solutions to (LM)

The proof of the next Theorem can be found in [23]. In [42] the proof is for the power $p \geqslant (n+2)/2$.

Theorem 2.14. ***(Uniqueness)*** *Given τ, T with $\tau < T$, $\mathbf{u}_\tau \in H$ and $\mathbf{f} \in L^q(\tau, T; V_p^*)$, then:*

1. *if $n = 2$, the weak solutions to **(LM)** are unique,*

2. *and, if $n = 3$, any weak solution to **(LM)** that has additional regularity*

$$\mathbf{u} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p) \tag{2.43}$$

is unique in the class of weak solutions.

Proof. Let us assume that there exist two weak solutions $\mathbf{u}$ and $\mathbf{v}$ to **(LM)**. Setting $\mathbf{w} = \mathbf{v} - \mathbf{u}$, and using $\mathbf{w}$ as a test function, we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + \langle \mathbb{T}(\mathbf{v}) - \mathbb{T}(\mathbf{u}), \mathbf{w}\rangle + \langle B(\mathbf{v}) - B(\mathbf{u}), \mathbf{w}\rangle = 0.$$

We know that

$$\langle B(\mathbf{v}) - B(\mathbf{u}), \mathbf{w}\rangle = -\int_\Omega (\mathbf{v}\otimes\mathbf{v} - \mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{w} dx.$$

On the right-hand side, we insert $\pm(\mathbf{u}\otimes\mathbf{v})$, and use integration by parts to arrive at

$$\int_\Omega (\mathbf{w}\otimes\mathbf{v}) : \nabla\mathbf{w} + (\mathbf{u}\otimes\mathbf{w}) : \nabla\mathbf{w} dx = -\int_\Omega (\mathbf{w}\otimes\mathbf{w}) : \nabla\mathbf{u} dx \leqslant \int_\Omega |\mathbf{w}|^2|\nabla\mathbf{u}| dx.$$

Using (2.7), we deduce that

$$\frac{d}{dt}|\mathbf{w}|_2^2 + \nu_1|\mathbf{e}(\mathbf{w})|_2^2 + \nu_2\|\mathbf{e}(\mathbf{w})\|_p^p \leqslant 2\int_\Omega |\mathbf{w}|^2|\nabla\mathbf{u}| dx.$$

- Case $n = 2$

  Let us consider $\varepsilon_1 > 0$. Using the Ladyzhenskaya inequality (1.4) for $n = 2$ (Theorem 1.15) to estimate

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2|\nabla\mathbf{u}| dx &\leqslant \|\mathbf{w}\|_4^2|\nabla\mathbf{u}|_2 \\ &\leqslant \widehat{c}|\mathbf{w}|_2|\nabla\mathbf{w}|_2|\nabla\mathbf{u}|_2 \\ &\leqslant \frac{\varepsilon_1}{2}|\nabla\mathbf{w}|_2^2 + \frac{\widehat{c}}{2\varepsilon_1}|\nabla\mathbf{u}|_2^2|\mathbf{w}|_2^2.\end{aligned}$$

From the previous estimate, we have

$$\frac{d}{dt}|\mathbf{w}|_2^2 + \big(\frac{2\nu_1}{c_0^2} - \varepsilon_1\big)|\nabla \mathbf{w}|_2^2 + \frac{2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{w}\|_p^p \leqslant \frac{\hat{c}}{\varepsilon_1}|\mathbf{w}|_2^2|\nabla \mathbf{u}|_2^2.$$

Choosing $\varepsilon_1 = \dfrac{\nu_1}{c_0^2}$ and integrating from τ to t, we obtain

$$|\mathbf{w}(t)|_2^2 \leqslant \frac{\hat{c}}{\varepsilon_1}\int_\tau^t |\mathbf{w}(s)|_2^2|\nabla \mathbf{u}(s)|_2^2 ds.$$

The case $n = 2$, follows by applying the Gronwall inequality.

- Case $n = 3$

Applying the Holder inequality

$$\int_\Omega |\mathbf{w}|^2|\nabla \mathbf{u}|dx \leqslant \|\mathbf{w}\|^2_{\frac{2p}{p-1}}\|\nabla \mathbf{u}\|_p.$$

Using the interpolation inequality (Theorem 1.17) for $r = \dfrac{2p}{p-1}$, $r_1 = 6$, $r_2 = 2$ and $\alpha = 3/2p$, we obtain

$$\begin{aligned}\|\mathbf{w}\|_{\frac{2p}{p-1}} &\leqslant |\mathbf{w}|_2^{\frac{2p-3}{2p}}\|\mathbf{w}\|_6^{\frac{3}{2p}}\\ &\leqslant d|\mathbf{w}|_2^{\frac{2p-3}{2p}}|\nabla \mathbf{w}|_2^{\frac{3}{2p}},\end{aligned}$$

since $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

By plugging this inequality in the previous one, we have

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2|\nabla \mathbf{u}|dx &\leqslant \|\mathbf{w}\|^2_{\frac{2p}{p-1}}\|\nabla \mathbf{u}\|_p\\ &\leqslant d|\mathbf{w}|_2^{\frac{2p-3}{p}}|\nabla \mathbf{w}|_2^{\frac{3}{p}}\|\nabla \mathbf{u}\|_p\\ &\leqslant \frac{\nu_1}{4c_0^2}|\nabla \mathbf{w}|_2^2 + k_{1,p}\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-3}}|\mathbf{w}|_2^2,\end{aligned}$$

where $k_{1,p} = \dfrac{2p-3}{2p\varepsilon^{\frac{2p-3}{2p}}}d^{\frac{2p}{2p-3}}$, with $\varepsilon = \big(\dfrac{p\nu_1}{6c_0^2}\big)^{3/2p}$.

Therefore

$$|\mathbf{w}(t)|_2^2 \leqslant k_{1,p}\int_\tau^t \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}|\mathbf{w}(s)|_2^2 ds.$$

We conclude by applying the Gronwall inequality, since $\displaystyle\int_\tau^t \|\nabla \mathbf{u}(t)\|_p^{\frac{2p}{2p-3}}dt < \infty$.

□

Remark 2.15. *The condition $p \geqslant (n+2)/2$ implies that $\mathbf{u} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$, (for the case $n = 3$), from which the uniqueness of solution follows by Theorem 2.14.*

Remark 2.16. *Observe that, by a little modification of the proof of Theorem 2.14, we can have an expression as follows*

$$|\mathbf{u}(t) - \mathbf{v}(t)|_2 \leqslant K(p)|\mathbf{u}_\tau - \mathbf{v}_\tau|_2 \quad \forall t \geqslant \tau,$$

*where $K(p)$ is a positive constant, and $\mathbf{u}$ and $\mathbf{v}$ are two weak solutions to problem **(LM)** with initial conditions $\mathbf{u}_\tau$ and $\mathbf{v}_\tau$, respectively.*

*This is a very important fact, since weak solutions of problem **(LM)** satisfying assumptions of Theorem 2.14, define a continuous process on H.*

2.4 A Regularity Result

For this section, we assume that $\mathbb{S}$ has a potential, that is, $\mathbb{S}$ satisfies the conditions of the Definition 1.48. Moreover, we will assume more regularity on the external force $\mathbf{f}$.

We denote by $\mathcal{WS}$ the set of all weak solutions $\mathbf{u}$ to **(LM)** such that

- $\mathbf{u}$ can be approximated by a sequence $\{\mathbf{u}_m\}$ of smoothly regular functions with $\mathbf{u}_m$ satisfying the weak formulation (2.16) for each $m \in \mathbb{N}$,
- the sequence $\{\mathbf{u}_m\}$ converges to $\mathbf{u}$ in the sense of (2.27),
- $\mathbf{u}_m(\tau) \to \mathbf{u}(\tau)$ in H, and
- $\dfrac{\partial \mathbf{u}_m}{\partial t}$ can be taken as test function in the weak formulation (2.16) for each $m \in \mathbb{N}$.

Remark 2.17. *Observe that for $p \geqslant 1 + 2n/(n+2)$ the set $\mathcal{WS} \neq \varnothing$, since it contains the weak solution built from a sequence of Galerkin approximations.*

Proposition 2.18. *Given $T, \tau \in \mathbb{R}$ with $\tau < T$, $\mathbf{u}_\tau \in H$ and $\mathbf{f} \in L^2(\tau, T; L^2(\Omega)^n)$. We consider two cases: when $n = 2$ and $p > 2$ and when $n = 3$ and $p \geqslant 12/5$. Then, in both cases, any weak solution $\mathbf{u} \in \mathcal{WS}$, associated to the initial condition $\mathbf{u}_\tau$, satisfy:*

$$\mathbf{u} \in L^\infty(\tau + \varepsilon, T; V_p) \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^2(\tau + \varepsilon, T; H),$$

for all $\varepsilon > 0$ such that $\tau + \varepsilon \leqslant T$.

Proof. Since $\mathbf{u} \in \mathcal{WS}$ there exists a sequence $\{\mathbf{u}_m\}$ such that we can use $\dfrac{\partial \mathbf{u}_m}{\partial t}$ as a test function in the weak formulation. So, we have

$$\left|\frac{\partial \mathbf{u}_m}{\partial t}\right|_2^2 + \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}_m))e_{ij}\Big(\frac{\partial \mathbf{u}_m}{\partial t}\Big)dx + \int_\Omega u_{mj}\frac{\partial u_{mi}}{\partial x_j}\frac{\partial u_{mi}}{\partial t}dx = \Big(\mathbf{f}(t), \frac{\partial \mathbf{u}_m}{\partial t}\Big). \tag{2.44}$$

Observe that, as \$ is a potential

$$\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}_m))e_{ij}(\frac{d\mathbf{u}_m}{dt})dx = \int_\Omega \frac{\partial\Phi}{\partial e_{ij}}(\mathbf{e}(\mathbf{u}_m))e_{ij}(\frac{\partial\mathbf{u}_m}{\partial t})dx = \int_\Omega \frac{\partial\Phi}{\partial e_{ij}}(\mathbf{e}(\mathbf{u}_m))\frac{\partial}{\partial t}(e_{ij}(\mathbf{u}_m))dx$$
$$= \frac{d}{dt}\int_\Omega \Phi(\mathbf{e}(\mathbf{u}_m))dx = \frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1.$$

Now, from (2.44) and the Hölder inequality, it follows that

$$\begin{aligned}\frac{1}{2}\left|\frac{\partial\mathbf{u}_m}{\partial t}\right|_2^2 + \frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1 &\leqslant |\mathbf{f}(t)|_2^2 + \int_\Omega |\mathbf{u}_m|^2|\nabla\mathbf{u}_m|^2dx \\ &\leqslant |\mathbf{f}(t)|_2^2 + c_7\|\mathbf{u}_m\|_{1,p}^2\|\mathbf{u}_m\|_{2p/(p-2)}^2.\end{aligned} \tag{2.45}$$

Now, we are going to consider three cases: for $n = 3$ with $12/5 \leqslant p < 3$, for $n = 3$ with $p \geqslant 3$ and for $n = 2$ with $p > 2$.

Case 1 $n = 3$ with $12/5 \leqslant p < 3$:

We know that $\mathbf{u}_m = \mathbf{u}_m(t) \in V_p \cap H$. By the interpolation inequality (Theorem 1.17), we have

$$\|\mathbf{u}_m\|_{2p/(p-2)} \leqslant d\|\mathbf{u}_m\|_{1,p}^{\frac{6}{5p-6}}|\mathbf{u}_m|_2^{\frac{5p-12}{5p-6}},$$

where d is the constant of interpolation that depends on Ω and p.

By Lemma 1.50 we deduce that, there exist positive constants c_8 and c_9 such that

$$c_8\|\mathbf{u}_m\|_{1,p}^p \leqslant \|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1 \leqslant c_9(1 + \|\mathbf{u}_m\|_{1,p}^p). \tag{2.46}$$

The right-hand side of (2.45) can be estimated as

$$\begin{aligned}\|\mathbf{u}_m\|_{1,p}^2\|\mathbf{u}_m\|_{2p/(p-2)}^2 &\leqslant d^2\|\mathbf{u}_m\|_{1,p}^p\|\mathbf{u}_m\|_{1,p}^{p(16-5p)/(5p-6)}|\mathbf{u}_m|_2^{2(5p-12)/(5p-6)} \\ &\leqslant \frac{d^2}{c_8}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1\|\mathbf{u}_m\|_{1,p}^{p(16-5p)/(5p-6)}|\mathbf{u}_m|_2^{2(5p-12)/(5p-6)}.\end{aligned} \tag{2.47}$$

From (2.45), (2.46) and (2.47), we have

$$\frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_m(t)))\|_1 \leqslant |\mathbf{f}(t)|_2^2 + \frac{c_7d^2}{c_8}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1\|\mathbf{u}_m\|_{1,p}^{p(16-5p)/(5p-6)}|\mathbf{u}_m|_2^{2(5p-12)/(5p-6)}. \tag{2.48}$$

From this inequality, in particular we obtain

$$\begin{aligned}\|\Phi(\mathbf{e}(\mathbf{u}_m(t)))\|_1 \leqslant{}& \|\Phi(\mathbf{e}(\mathbf{u}_m(s)))\|_1 + \int_\tau^t |\mathbf{f}(\theta)|_2^2d\theta \\ &+ \frac{c_7d^2}{c_8}\int_\tau^t \|\Phi(\mathbf{e}(\mathbf{u}_m(\theta)))\|_1\|\mathbf{u}_m(\theta)\|_{1,p}^{p(16-5p)/(5p-6)}|\mathbf{u}(\theta)|_2^{2(5p-12)/(5p-6)}d\theta.\end{aligned}$$

for all $s \in [\tau, t]$.

Now, applying the Gronwall inequality, there follows

$$\begin{aligned}\|\Phi(\mathbf{e}(\mathbf{u}_m(t)))\|_1 \leqslant& \left(\|\Phi(\mathbf{e}(\mathbf{u}_m(s)))\|_1 + \int_\tau^t |\mathbf{f}(\theta)|_2^2 d\theta\right)\\ &\times \exp\left(\frac{c_7 d^2}{c_8}\int_\tau^t \|\mathbf{u}_m(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} |\mathbf{u}_m(\theta)|_2^{\frac{2(5p-12)}{5p-6}} d\theta\right)\end{aligned} \tag{2.49}$$

for all $t \geqslant s \geqslant \tau$.

Given $\varepsilon > 0$ such that $\tau + \varepsilon \leqslant T$ and integrating in s, from τ to $\tau + \varepsilon$,

$$\begin{aligned}\|\Phi(\mathbf{e}(\mathbf{u}_m(t)))\|_1 \leqslant& \left(\frac{1}{\varepsilon}\int_\tau^t \|\Phi(\mathbf{e}(\mathbf{u}_m(s)))\|_1 ds + \int_\tau^t |\mathbf{f}(\theta)|_2^2 d\theta\right)\\ &\times \exp\left(\frac{c_7 d^2}{c_8}\int_\tau^t \|\mathbf{u}_m(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} |\mathbf{u}_m(\theta)|_2^{\frac{2(5p-12)}{5p-6}} d\theta\right),\end{aligned}$$

for all $t \geqslant \tau + \varepsilon$.

Observe that all terms of the right-hand side of the above inequality are bounded. In fact: from Theorem 2.13 we have that the sequence $\{\mathbf{u}_m\}$ is bounded in $L^p(\tau, T; V_p) \cap L^\infty(\tau, T; H)$, and to conclude, only remains to prove that $\int_\tau^t \|\mathbf{u}_m(r)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} dr < \infty$ and $(5p-12)/(5p-6) \geqslant 0$, what is true since $p \geqslant 12/5$.

Case 2 $n = 3$ with $p \geqslant 3$:

Since the imbedding $W^{1,2}(\Omega)^n \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^n$ is true for $\dfrac{2p}{p-2} \leqslant \dfrac{2n}{n-2}$, which is equivalent to $p \geqslant n$, we can estimate (2.45)

$$\frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1 \leqslant |\mathbf{f}(t)|_2^2 + \int_\Omega |\mathbf{u}_m|^2 |\nabla \mathbf{u}_m|^2 dx \leqslant |\mathbf{f}(t)|_2^2 + c_7 d^2 \|\mathbf{u}_m\|_{1,p}^2 \|\mathbf{u}_m\|_{1,2}^2. \tag{2.50}$$

Let us denote by $\mathcal{U} := 1 + \|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1$, and observe that $\dfrac{d}{dt}\mathcal{U} = \dfrac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_m))\|_1$. Therefore, from (2.46) and the above estimate, we obtain that

$$\frac{d}{dt}\mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U}^{4/p}, \tag{2.51}$$

Now, if $4 \leqslant p$, then $\mathcal{U}^{4/p} \leqslant \mathcal{U}$, since $\mathcal{U} \geqslant 1$, with this and (2.51)

$$\frac{d}{dt}\mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U},$$

By applying the Gronwall inequality we conclude the proof as in the previous case.

Now, if $3 \leqslant p < 4$, let us consider $\mu = (2p-4)/p$, then $\mu \in [\frac{2}{3}, 1)$. Multiplying by $\mathcal{U}^{\mu-1}$ to (2.51), we obtain that

$$\mu \frac{d}{dt}(\mathcal{U}^\mu) \leqslant |\mathbf{f}(t)|_2^2 \mathcal{U}^{\mu-1} + \widetilde{C}_2 \mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U},$$

since $\mathcal{U}^{\mu-1} \leqslant 1$.

We conclude the boundedness of $\mathcal{U}$ as before.

Case 3 $n = 2$ and $p > 2$:

Observe that $2p/(p-2) > 2$, therefore by Theorem 1.15, we get

$$\|\mathbf{u}_m\|_{2p/(p-2)} \leqslant c_1(p)|\mathbf{u}_m|_2^{2/p}|\nabla\mathbf{u}_m|_2^{(p-2)/p} \tag{2.52}$$

Integrating from s to t with $s \in [\tau, t]$ in (2.45) and using inequality (2.52), we have

$$\|\Phi(\mathbf{e}(\mathbf{u}_m(t)))\|_1 \leqslant \|\Phi(\mathbf{e}(\mathbf{u}_m(s)))\|_1 + \int_\tau^t |\mathbf{f}(r)|_2^2 dr + \widehat{C}_7 \int_\tau^t \|\mathbf{u}_m\|_{1,p}^{(4p-4)/p}|\mathbf{u}_m|_2^{4/p} dr.$$

Again, we conclude applying the Gronwall inequality as in the previous cases, since $(4p-4)/p \leqslant p$.

From (2.46) we conclude that for $\varepsilon > 0$ with $\tau + \varepsilon \leqslant T$ the sequence $\{\mathbf{u}_m\}$ is bounded in $L^\infty(\tau+\varepsilon, T; V_p)$ and from (2.27), $\mathbf{u}_m \rightharpoonup \mathbf{u}$ weakly in $L^p(\tau, T; V_p)$. Moreover, by (2.18) $\mathbf{u} \in C([\tau, T]; H)$. Therefore, applying Proposition 1.12 for $X = V_p$ and $Y = H$, we obtain that

$$\|\mathbf{u}(r)\|_{1,p} \leqslant \liminf_{m\to\infty} \|\mathbf{u}_m\|_{L^\infty(\tau+\varepsilon,T;V_p)},$$

for all $r \in [\tau+\varepsilon, T]$, from which we have $\mathbf{u} \in L^\infty(\tau+\varepsilon, T; V_p)$. By Proposition 1.13 follows that $\mathbf{u} \in C_w((\tau, T]; V_p)$. It remains only to prove that $\dfrac{\partial \mathbf{u}}{\partial t} \in L^2(\tau+\varepsilon, T; H)$, but this is immediate consequence of (2.45). □

2.5 Existence of Pullback Attractors

2.5.1 Pullback attractor in H

In order to study asymptotic behavior of solutions in H, we consider the following assumptions on the stress tensor $\mathbb{S}$, power p and external force $\mathbf{f}$:

(I) The stress tensor $\mathbb{S}$ satisfies the conditions given in (2.7),

(II) $p \geqslant 1 + 2n/(n+2)$,

(III) $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$.

Observe that, by Theorem 2.11, conditions **(I)**, **(II)** and **(III)** guarantee existence of weak solutions for the system **(LM)**, but not the uniqueness. We denote by $\Phi(\tau, \mathbf{u}_\tau)$ the set of weak solutions to **(LM)** in $[\tau, +\infty)$ with initial datum $\mathbf{u}_\tau \in H$.

By Theorem 2.11 the set $\Phi(\tau, \mathbf{u}_\tau)$ is not empty. Besides that, we can define a multi-valued map $U(\cdot,\cdot) : \mathbb{R}^2_d \times H \to \mathcal{P}(H)$, given by

$$U(t,\tau)\mathbf{u}_\tau = \{\mathbf{u}(t) : \mathbf{u} \in \Phi(\tau, \mathbf{u}_\tau)\} \quad \mathbf{u}_\tau \in H \;\text{ and }\; (t,\tau) \in \mathbb{R}^2_d. \tag{2.53}$$

Theorem 2.19. *Under assumptions **(I)**,**(II)** and **(III)**, the multi-valued map $U(\cdot,\cdot)$ defined in (2.53) is a strict multi-valued process in H.*

Proof. It follows from Theorem 2.11 and the fact that any weak solution to **(LM)** is continuous. □

Proposition 2.20. *Consider that assumptions **(I)**,**(II)** and **(III)** are satisfied. Let $\{\mathbf{u}_\tau^m\} \subset H$ and $\mathbf{u}_\tau \in H$ be such that $\mathbf{u}_\tau^m \to \mathbf{u}_\tau$ in H. Then, for any sequence $\{\mathbf{u}_m\}$, where $\mathbf{u}_m \in \Phi(\tau, \mathbf{u}_\tau^m)$, for all $m \in \mathbb{N}$, there exist a subsequence of $\{\mathbf{u}_m\}$ (relabeled the same) and $\mathbf{u} \in \Phi(\tau, u_\tau)$, such that*

$$\mathbf{u}_m(s) \to \mathbf{u}(s) \quad \text{strongly in } H \ \ \forall s \geqslant \tau. \tag{2.54}$$

Proof. Given any $\tau < T$, observe that by Proposition 2.13 the sequence $\{\mathbf{u}_m\}$ is bounded in $L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ and the sequence $\{\dfrac{\partial \mathbf{u}_m}{\partial t}\}$ is bounded in $L^q(\tau, T; V_p^*)$.

Therefore, there exist a subsequence of $\{\mathbf{u}_m\}$ (relabeled the same) and $\mathbf{u} \in L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ with $\dfrac{\partial \mathbf{u}}{\partial t} \in L^q(\tau, T; V_p^*)$ such that

$$\begin{aligned}
&\mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u} && *\text{-weakly in } L^\infty(\tau, T; H),\\
&\mathbf{u}_m \rightharpoonup \mathbf{u} && \text{weakly in } L^p(\tau, T; V_p),\\
&\frac{\partial \mathbf{u}_m}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} && \text{weakly in } L^q(\tau, T; V_p^*),\\
&\mathbf{u}_m \to \mathbf{u} && \text{strongly in } L^2(\tau, T; H).
\end{aligned}$$

Again by Proposition 2.13, we have that $\mathbf{u} \in \Phi(\tau, \mathbf{u}_\tau)$.

It remains to prove (2.54). Observe that $\{\mathbf{u}_m\}$ is equicontinuous in V_p^* on $[\tau, T]$ and that $\{\mathbf{u}_m\}$ is bounded in $C([\tau, T]; H)$. Therefore, by the Arzelà-Ascoli Theorem, up to a subsequence, there follows that

$$\mathbf{u}_m \to \mathbf{u} \quad \text{strongly in } C([\tau, T]; V_p^*).$$

From this and the boundedness of $\{\mathbf{u}_m\}$ in $C([\tau, T]; H)$ we conclude that

$$\mathbf{u}_m(s) \rightharpoonup \mathbf{u}(s) \quad \text{weakly in } H \ \ \forall\ \tau \leqslant s \leqslant T. \tag{2.55}$$

Now, since the following estimate

$$|\mathbf{z}(r)|^2 \leqslant |\mathbf{z}(s)|^2 + 2\int_s^r \langle \mathbf{f}(\theta), \mathbf{z}(\theta)\rangle d\theta, \quad \tau \leqslant s \leqslant r \leqslant T.$$

holds for $\mathbf{z} = \mathbf{u}_m$ and $\mathbf{z} = \mathbf{u}$, it follows that the functions $J_m, J : [\tau, T] \to \mathbb{R}$ defined by

$$\begin{aligned}
J_m(r) &= |\mathbf{u}_m(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}_m(\theta)\rangle d\theta,\\
J(r) &= |\mathbf{u}(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta,
\end{aligned}$$

are non-increasing and continuous, and satisfy

$$J_m(r) \to J(r) \text{ a.e. } r \in (\tau, T).$$

To prove that $J_m(r) \to J(r)$ $\forall r \in [\tau, T]$ consider a fixed $t^* \in (\tau, T]$ and an increasing sequence $t_k \uparrow t^*$ such that $J_m(t_k) \to J(t_k)$ for all $k \geqslant 1$. Thus, for any $\epsilon > 0$ there exist $M, K > 0$ such that

$$|J(t_k) - J(t^*)| \leqslant \frac{\epsilon}{2} \quad \text{for } k \geqslant K,$$
$$|J_m(t_K) - J(t_K)| \leqslant \frac{\epsilon}{2} \quad \text{for } m \geqslant M.$$

Since J_m is a non-increasing function, we have that

$$J_m(t^*) - J(t^*) \leqslant |J_m(t_K) - J(t_K)| + |J(t_K) - J(t^*)| \leqslant \epsilon$$

for all $m \geqslant M$, and consequently $\limsup_{m\to\infty} J_m(t^*) \leqslant J(t^*)$. Taking into account that

$$\int_\tau^{t^*} \langle \mathbf{f}(\theta), \mathbf{u}_m(\theta)\rangle d\theta \to \int_\tau^{t^*} \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta,$$

we deduce that $\limsup_{m\to\infty} |\mathbf{u}_m(t^*)| \leqslant |\mathbf{u}(t^*)|$. Hence, by the weak convergence (2.55) we conclude that (2.54) holds for all $s \in [\tau, T]$. By increasing intervals and a diagonal argument we see that, for a suitable subsequence, (2.54) holds true for any $s \geqslant \tau$. The proof is complete. □

Corollary 2.21. *Consider that assumptions **(I)**,**(II)** and **(III)** are satisfied. The multi-valued process $U(\cdot,\cdot)$ is upper-semicontinuous with closed values.*

Proof. We proceed by contradiction. So, let suppose that the multi-valued process $U(\cdot,\cdot)$ is not upper-semicontinuous. Thus, there exist $(t,\tau) \in \mathbb{R}^2_d$, a neighbourhood $\mathcal{N}(U(t,\tau)\mathbf{u}_\tau)$ and a sequence $\{\mathbf{z}_k\}$ which fulfil that each $\mathbf{z}_k \in U(t,\tau)\mathbf{u}^k_\tau$, where $\mathbf{u}^k_\tau \to \mathbf{u}_\tau$ in $(L^2(\Omega))^n$, but for all $k \in \mathbb{N}$ $\mathbf{z}_k \notin \mathcal{N}(U(t,\tau)\mathbf{u}_\tau)$. Since each $\mathbf{z}_k \in U(t,\tau)\mathbf{u}^k_\tau$, there exists $\mathbf{u}_k \in \Phi(\tau, \mathbf{u}^k_\tau)$ such that $\mathbf{z}_k = \mathbf{u}_k(t)$. Now, by applying Proposition 2.20, we deduce that there exists a subsequence of $\{\mathbf{u}_k(t)\}$ (relabeled the same) which converges to a function $\mathbf{u}(t) \in U(t,\tau)\mathbf{u}_\tau$. This is contradictory because $\mathbf{z}_k \notin \mathcal{N}(U(t,\tau)\mathbf{u}_\tau)$ for all $k \in \mathbb{N}$. It follows from Proposition 2.13 that the process is closed. □

Definition 2.22. ***(Universe in** H**)*** *Given $\eta > 0$, we denote by $\mathcal{D}^H_\eta$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that*

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau} \sup_{\mathbf{v}\in D(\tau)} |\mathbf{v}|_2^2\right) = 0. \tag{2.56}$$

Denote by $\mathcal{D}^H_F$ the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Remark 2.23. *Observe that $\mathcal{D}_F^H \subset \mathcal{D}_\eta^H$ and that $\mathcal{D}_\eta^H$ is inclusion-closed.*

Definition 2.24. *Given X a Banach space, $q \in [1, \infty]$ and $\eta > 0$, let us denote by $\mathcal{I}_X^{q,\eta}$ the set of all $\mathbf{f} \in L^q_{loc}(\mathbb{R}; X)$ such that*

$$\int_{-\infty}^{0} e^{\eta s} \|\mathbf{f}(s)\|_X^q ds < \infty. \tag{2.57}$$

Remark 2.25. *Observe that the condition (2.57) is equivalent to*

$$\int_{-\infty}^{t} e^{\eta s} \|\mathbf{f}(s)\|_X^q ds < \infty,$$

for all $t \in \mathbb{R}$.

For the next Lemma, we are going to distinguish two cases in relation to p, since the choice of η will depend on this. Although we are with hypothesis **(II)**, p can be greater or equal than 2, therefore we will treat two cases, when $p > 2$ and when $p = 2$.

Lemma 2.26. *Consider $p > 2$ and $\mathbf{f}$ fulfilling **(I)**,**(II)**,**(III)** with $q = p/(p-1)$ and a constant $\eta > 0$. Then, there exist positive constants $c_{\nu_2,p}$ and $\widehat{C}_1$ such that any weak solution to **(LM)** satisfies*

$$\begin{aligned} &|\mathbf{u}(t;\tau,\mathbf{u}_\tau)|_2^2 \leqslant \mathcal{R}^2_{1,p>2}(t;\tau,\eta) && \forall t \geqslant \tau, \\ &\int_\tau^t \|\nabla \mathbf{u}(\theta;\tau,\mathbf{u}_\tau)\|_p^p d\theta \leqslant \mathcal{R}_{2,p>2}(t;\tau,\eta) && \forall t \geqslant \tau. \end{aligned} \tag{2.58}$$

where

$$\mathcal{R}^2_{1,p>2}(t;\tau,\eta) = e^{-\eta(t-\tau)}|\mathbf{u}_\tau|_2^2 + c_{\nu_2,p}\int_\tau^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^q ds + \frac{\widehat{C}_1}{\eta}$$

$$\mathcal{R}_{2,p>2}(t;\tau,\eta) = \frac{\tilde{c}_0^p}{c_2\nu_2}\mathcal{R}^2_{1,p>2}(t;\tau,\eta) + \frac{\tilde{c}_0^p c_{\nu_2,p}}{c_2\nu_2}\int_\tau^t \|\mathbf{f}(s)\|_*^q ds.$$

Proof. From the energy equality (2.12), and Korn and Young inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}|_2^2 + \frac{c_2\nu_1}{c_0^2}|\nabla \mathbf{u}|_2^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{u}\|_p^p \leqslant \frac{1}{q\epsilon^q}\|\mathbf{f}\|_*^q + \frac{\epsilon^p}{p}\|\nabla \mathbf{u}\|_p^p.$$

Choosing $\epsilon^p/p = c_2\nu_2/(2\tilde{c}_0^p)$ and denoting $c_{\nu_2,p} = 2/(q\epsilon^q)$, after the Poincaré inequality

$$\frac{d}{dt}|\mathbf{u}|_2^2 + \frac{2c_2\nu_1\lambda_1}{c_0^2}|\mathbf{u}|_2^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{u}\|_p^p \leqslant c_{\nu_2,p}\|\mathbf{f}\|_*^q \ a.e.t > \tau. \tag{2.59}$$

- Case $\eta \leqslant 2c_2\nu_1\lambda_1 c_0^{-2}$:

  From (2.59) we have

  $$\frac{d}{dt}|\mathbf{u}|_2^2 + \eta|\mathbf{u}|_2^2 + (\frac{2c_2\nu_1\lambda_1}{c_0^2} - \eta)|\mathbf{u}|_2^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{u}\|_p^p \leqslant c_{\nu_2,p}\|\mathbf{f}\|_*^q \ a.e.t > \tau.$$

  Multiplying by $e^{\eta t}$ and integrating from τ to t, we obtain

  $$|\mathbf{u}(t)|_2^2 \leqslant e^{-\eta(t-\tau)}|\mathbf{u}(\tau)|_2^2 + c_{\nu_2,p}\int_\tau^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^q ds.$$

- **Case** $\eta > 2c_2\nu_1\lambda_1 c_0^{-2}$:

  Denote $0 < \beta = \eta - \dfrac{2c_2\nu_1\lambda_1}{c_0^2}$. Consider also C_I a constant of the embedding $W_0^{1,p}(\Omega)^n \subset L^2(\Omega)^n$, i.e. $|\mathbf{u}|_2 \leqslant C_I\|\nabla\mathbf{u}\|_p$. Then the Young inequality yields

$$|\mathbf{u}|_2^2 \leqslant \frac{\gamma^{p/2}}{p/2}\|\nabla\mathbf{u}\|_p^p + \frac{(p-2)C_I^{2p/(p-2)}}{p\gamma^{p/(p-2)}}.$$

  Putting $\dfrac{\gamma^{p/2}}{p/2} = \dfrac{c_2\nu_2}{2\tilde{c}_0^p\beta}$ we gain

$$\beta|\mathbf{u}|_2^2 \leqslant \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla\mathbf{u}\|_p^p + \widehat{C}_1,$$

  where $\widehat{C}_1 = \dfrac{(p-2)C_I^{2p/(p-2)}\beta}{p\gamma^{p/(p-2)}}$. Then (2.59) reduces to

$$\frac{d}{dt}|\mathbf{u}|_2^2 + \eta|\mathbf{u}|_2^2 + \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla\mathbf{u}\|_p^p \leqslant c_{\nu_2,p}\|\mathbf{f}\|_*^q + \widehat{C}_1.$$

  Multiplying by $e^{\eta t}$ and integrating from τ to t, we obtain

$$|\mathbf{u}(t)|_2^2 \leqslant e^{-\eta(t-\tau)}|\mathbf{u}(\tau)|_2^2 - \frac{e^{-\eta(t-\tau)}}{\eta} + c_{\nu_2,p}\int_\tau^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^q ds + \frac{\widehat{C}_1}{\eta}.$$

  From the previous cases, the first estimate in (2.58) is deduced.

  Now, from (2.59), we infer that

$$\int_\tau^t \|\nabla\mathbf{u}(s)\|_p^p ds \leqslant \frac{\tilde{c}_0^p}{c_2\nu_2}|\mathbf{u}(\tau)|_2^2 + \frac{\tilde{c}_0^p c_{\nu_2,p}}{c_2\nu_2}\int_\tau^t \|\mathbf{f}(s)\|_*^q ds.$$

From this, we obtain the second estimate in (2.58). □

The case $p = 2$ is simpler, nevertheless we include it for the sake of completeness. Observe that from (2.7), for $p = 2$ it holds that $c_2 = 1$ and $\nu_2 = 0$ in (2.8).

Lemma 2.27. *Assume **(I)**,**(II)**,**(III)** hold for $p = q = 2$,. Then, for any $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$ there exists a constant $\beta > 0$ such that any weak solution to **(LM)** satisfies*

$$|\mathbf{u}(t)|_2^2 \leqslant e^{-\eta(t-\tau)}|\mathbf{u}(\tau)|_2^2 + \frac{\lambda_1}{\beta}\int_\tau^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^2 ds. \tag{2.60}$$

Proof. The energy equality (2.12) and the Korn inequality yield

$$\frac{d}{dt}|\mathbf{u}|_2^2 + \frac{2\nu_1}{c_0^2}|\nabla\mathbf{u}|_2^2 \leqslant 2\langle\mathbf{f},\mathbf{u}\rangle.$$

Fix a value $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$. Arranging coefficients in the Cauchy inequality in the right-hand side, we obtain

$$\frac{d}{dt}|\mathbf{u}|_2^2 + \frac{\eta}{\lambda_1}|\nabla\mathbf{u}|_2^2 \leqslant \frac{\lambda_1}{\beta}\|\mathbf{f}\|_*^2,$$

where $\beta := 2\nu_1\lambda_1 c_0^{-2} - \eta > 0$. By the Poincaré inequality and the Gronwall inequality, (2.60) holds. □

From now on we assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$, and satisfies $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$ for some $\eta > 0$, where $\mathcal{I}_*^{q,\eta} := \mathcal{I}_{V_p^*}^{q,\eta}$.

Corollary 2.28. *If there exists $\eta > 0$ (if $p = 2$, $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$) such that $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$, then the multi-valued process $U(\cdot,\cdot)$ on H, defined in (2.53), has a pullback $\mathcal{D}_\eta^H$-absorbing family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ with $B_0(t) = \overline{B}_H(0, \mathcal{R}_{p>2}(t))$, where*

$$\mathcal{R}_{p>2}^2(t) = 1 + \frac{C_1}{\eta} + c_{\nu_2,p}\int_{-\infty}^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^q ds$$

(if $p = 2$, $B_0(t) = \overline{B}_H(0, \mathcal{R}_{p=2}(t))$). Moreover, $\widehat{B}_0 \in \mathcal{D}_\eta^H$.

Proof. It follows directly from Lemma 2.26 for the case $p > 2$ and from Lemma 2.27 for the case $p = 2$. □

Lemma 2.29. *Consider $p > 2$ and suppose that there exist $\eta > 0$ such that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$. Then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \leqslant \tau_1(\widehat{D}, t)$, $\mathbf{u}_\tau \in D(\tau)$ and $\mathbf{u} \in \Phi(\tau, \mathbf{u}_\tau)$, it holds*

$$\begin{cases} |\mathbf{u}(r;\tau,\mathbf{u}_\tau)|_2 \leqslant \rho_{1,p>2}(t) & \text{for all } r \in [t-3, t], \\ \int_{r-1}^r \|\nabla\mathbf{u}(\theta;\tau,\mathbf{u}_\tau)\|_p^p d\theta \leqslant \rho_{2,p>2}(t) & \text{for all } r \in [t-2, t], \end{cases} \tag{2.61}$$

where

$$\rho_{1,p>2}^2(t) = 1 + \frac{\widehat{C}_1}{\eta} + c_{\nu_2,p}e^{\eta(3-t)}\int_{-\infty}^t e^{\eta s}\|\mathbf{f}(s)\|_*^q ds,$$

$$\rho_{2,p>2}(t) = \frac{\tilde{c}_0^p}{c_2\nu_2}\rho_{1,p>2}^2(t) + \frac{\tilde{c}_0^p c_{\nu_2,p}}{c_2\nu_2}\int_{t-3}^t \|\mathbf{f}(s)\|_*^q ds.$$

Proof. Fix $t \in \mathbb{R}$. From (2.58) there exists $\tau_1(\widehat{D}, t) < t - 3$ such that

$$e^{-\eta(t-\tau)}|\mathbf{u}_\tau|_2^2 \leqslant 1 \quad \forall \tau \leqslant \tau_1(\widehat{D}, t).$$

Therefore it follows from (2.58) that

$$|\mathbf{u}(r;\tau,\mathbf{u}_\tau)|_2 \leqslant \rho_{1,p>2}^2(t) \quad \text{for all } r \in [t-3,t],\ \tau \leqslant \tau_1(\widehat{D}, t) \ \text{ with } \ \mathbf{u}_\tau \in D(\tau),$$

where

$$\rho_{1,p>2}^2(t) = 1 + \frac{\widehat{C}_1}{\eta} + c_{\nu_2,p}e^{\eta(3-t)}\int_{-\infty}^t e^{\eta s}\|\mathbf{f}(s)\|_*^q ds.$$

Now, from the energy equality (Lemma 2.12), we deduce that

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}(t)|_2^2 + c_1\nu_1|\mathbf{e}(\mathbf{u})|_2^2 + c_2\nu_2\|\mathbf{e}(\mathbf{u})\|_p^p \leqslant \langle \mathbf{f}(t), \mathbf{u}(t)\rangle. \tag{2.62}$$

Applying the Korn and Young inequalities with $\varepsilon > 0$, it follows that

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + (\frac{2c_2\nu_2}{\tilde{c}_0^p} - \frac{2\varepsilon^p}{p})\|\nabla\mathbf{u}(t)\|_p^p \leqslant \frac{2}{q\varepsilon^q}\|\mathbf{u}(t)\|_*^q.$$

We choose $\varepsilon^p = \dfrac{pc_2\nu_2}{2\tilde{c}_0^p}$ and integrating from $r-1$ to r inequality (2.62), we get that

$$\frac{c_2\nu_2}{\tilde{c}_0^p}\int_{r-1}^r \|\nabla\mathbf{u}(s)\|_p^p ds \leqslant |\mathbf{u}(r-1)|_2^2 + c_{\nu_2,p}\int_{r-1}^r \|\mathbf{f}(s)\|_*^q ds.$$

Therefore, we deduce that

$$\int_{r-1}^r \|\nabla\mathbf{u}(\theta;\tau,\mathbf{u}_\tau)\|_p^p d\theta \leqslant \rho_{2,p>2}(t), \quad \text{for all } r \in [t-2,t],\ \tau \leqslant \tau_1(\widehat{D},t),\ \mathbf{u}_\tau \in \widehat{D},$$

where

$$\rho_{2,p>2}(t) = \frac{\tilde{c}_0^p}{c_2\nu_2}\rho_{1,p>2}^2(t) + \frac{\tilde{c}_0^p c_{\nu_2,p}}{c_2\nu_2}\int_{t-3}^t \|\mathbf{f}(s)\|_*^q ds.$$

□

Lemma 2.30. *Consider $p = 2$ and suppose that there exists $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$ such that $\mathbf{f} \in L^2_{loc}(\mathbb{R}; V_2^*)$ satisfies $\mathbf{f} \in \mathcal{I}_*^{2,\eta}$. Then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D},t) < t-3$, such that for any $\tau \leqslant \tau_1(\widehat{D},t)$, $u_\tau \in D(\tau)$ and $\mathbf{u}_\tau \in \Phi(\tau,\mathbf{u}_\tau)$ it holds*

$$\begin{cases} |\mathbf{u}(r;\tau,\mathbf{u}_\tau)|_2 \leqslant \rho_{1,p=2}(t) & \textit{for all } r \in [t-3,t], \\ \displaystyle\int_{r-1}^r |\nabla\mathbf{u}(\theta;\tau,\mathbf{u}_\tau)|_2^2 d\theta \leqslant \rho_{2,p=2}(t) & \textit{for all } r \in [t-2,t], \end{cases}$$

where

$$\rho_{1,p=2}^2(t) = 1 + \frac{\lambda_1}{\beta}e^{\eta(3-t)}\int_{-\infty}^t e^{\eta s}\|\mathbf{f}(s)\|_*^2 ds,$$

$$\rho_{2,p=2}(t) = \frac{\lambda_1}{\eta}\rho_{1,p=2}^2(t) + \frac{\lambda_1^2}{\beta\eta}\int_{t-3}^t \|\mathbf{f}(s)\|_*^2 ds,$$

where $\beta > 0$ is given in the Lemma 2.27.

Proof. The first estimate follows from (2.60). For the second one, from Lemma 2.27, we obtain that

$$\frac{d}{dt}|\mathbf{u}|_2^2 + \frac{\eta}{\lambda_1}|\nabla\mathbf{u}|_2^2 \leqslant \frac{\lambda_1}{\beta}\|\mathbf{f}\|_*^2.$$

Integrating from $r-1$ to r the previous inequality, we have that

$$\frac{\mu}{\lambda_1}\int_{r-1}^r |\nabla\mathbf{u}(s)|_2^2 ds \leqslant |\mathbf{u}(r-1)|_2^2 + \frac{\lambda_1}{\beta}\int_{r-1}^r \|\mathbf{f}(s)\|_*^2 ds.$$

Therefore, we deduce

$$\int_{r-1}^{r} |\nabla \mathbf{u}(s)|_2^2 ds \leqslant \rho_{2,p=2}(t), \quad \textit{for all } r \in [t-1,t], \; \tau \leqslant \tau_1(\widehat{D},t), \; \mathbf{u}_\tau \in \widehat{D},$$

where

$$\rho_{2,p=2}(t) = \frac{\lambda_1}{\eta}\rho_{1,p=2}^2(t) + \frac{\lambda_1^2}{\beta\eta}\int_{t-3}^{t} \|\mathbf{f}(s)\|_*^2 ds.$$

□

Proposition 2.31. *Consider $p \geqslant 2$ and suppose that there exists $\eta > 0$ (for the case $p = 2$, $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$), such that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$. Then, the process $U(\cdot,\cdot)$ is pullback $\widehat{B}_0$-asymptotically compact.*

Proof. We prove the result for $p > 2$, since the case $p = 2$ is analogous. Let $t \in \mathbb{R}$, and let us consider the sequence $\{\mathbf{u}_{\tau_m}\}$ with $\mathbf{u}_{\tau_m} \in B_0(\tau_n)$, where $\{\tau_m\} \subset (-\infty, t)$ satisfies that $\tau_m \to -\infty$ as $m \to \infty$ and denote by $\mathbf{u}_m = \mathbf{u}_m(\cdot; \tau_m, \mathbf{u}_{\tau_m})$ any sequence of weak solutions with $\mathbf{u}_m \in \Phi(\tau_m, \mathbf{u}_{\tau_m})$ for each $m \in \mathbb{N}$. We will prove that the sequence $\{\mathbf{u}_m\}$ is relatively compact in H.

It follows from Lemma 2.29 that there exists $m_0(t)$ such that $\tau_m < t - 2$ for $m \geqslant m_0(t)$ and

$$|\mathbf{u}_m(r)|_2 \leqslant \rho_{1,p>2}(t) \quad \forall r \in [t-2,t], \; \forall m \geqslant m_0(t). \tag{2.63}$$

$$\int_{r-1}^{r} \|\mathbf{u}_m(s)\|_{V_p}^p ds \leqslant \rho_{2,p>2}(t) \quad \forall r \in [t-2,t], \; \forall m \geqslant m_0(t).$$

Furthermore the sequence of time derivatives $\{\dfrac{\partial \mathbf{u}_m}{\partial t}\}$ is bounded in $L^q(t-2,t;V_p^*)$. Then, using in particular the Aubin-Lions compactness Theorem, there exists an element $\mathbf{u} \in L^\infty(t-2,t;H) \cap L^p(t-2,t;V_p)$ with $\dfrac{\partial \mathbf{u}}{\partial t} \in L^q(t-2,t;V_p^*)$, such that for a subsequence (relabeled the same) the following convergences hold

$$\begin{cases} \mathbf{u}_m \overset{*}{\rightharpoonup} \mathbf{u} \quad *\text{-weak in } \; L^\infty(t-2,t;H), \\ \mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weak in } \; L^p(t-2,t;V_p), \\ \dfrac{\partial \mathbf{u}_m}{\partial t} \rightharpoonup \dfrac{\partial \mathbf{u}}{\partial t} \quad \text{weak in } \; L^q(t-2,t;V_p^*), \\ \mathbf{u}_m \to \mathbf{u} \quad \text{strongly in } \; L^2(t-2,t;H), \\ \mathbf{u}_m(s) \to \mathbf{u}(s) \quad \text{in } \; H \text{ a.e. } \; s \in (t-2,t). \end{cases} \tag{2.64}$$

Observe also that $\mathbf{u} \in C([t-2,t];H)$. Moreover, for any sequence $\{t_m\}$ in $[t-2,t]$, with $t_m \to t^*$, one has

$$\mathbf{u}_m(t_m) \rightharpoonup \mathbf{u}(t^*) \quad \text{in } \; H. \tag{2.65}$$

Indeed, we know that

$$(\mathbf{u}_m(r), \phi) - (\mathbf{u}_m(s), \phi) = \int_s^r \langle \frac{\partial \mathbf{u}^m(\theta)}{\partial t}, \phi \rangle d\theta \quad \forall s, r \in [t-2, t], \ \phi \in V_p.$$

This, jointly with the fact that $\{\frac{\partial \mathbf{u}_m}{\partial t}\}$ is bounded in $L^q(t-2, t; V_p^*)$, we obtain

$$\|\mathbf{u}_m(r) - \mathbf{u}_m(s)\|_* \leqslant C|r-s|^{1/p} \quad \forall s, r \in [t-2, t].$$

Therefore $\{\mathbf{u}_m\}$ is a equicontinuous sequence in $C([t-2,t]; V_p^*)$. Since $H \hookrightarrow\hookrightarrow V_p^*$, follow from Arzelá-Ascoli Theorem that there exists a subsequence (relabeled the same) of $\{\mathbf{u}_m\}$ and $\tilde{\mathbf{u}} \in C([t-2,t]; V_p^*)$, such that

$$\mathbf{u}_m \to \tilde{\mathbf{u}} \quad \text{in} \ \ C([t-2,t]; V_p^*).$$

From (2.64) there exists $S \subset [t-1, t]$ with $|S| = 0$ such that $\mathbf{u}_m(s) \to \mathbf{u}(s)$ in H for all $s \in [t-2, t] \backslash S$. Now, let $s \in [t-2, t] \backslash S$ be, as $H \hookrightarrow V_p^*$, from the above it follows that

$$\|\mathbf{u}_m(s) - \mathbf{u}(s)\|_* \leqslant C|\mathbf{u}_m(s) - \mathbf{u}(s)|_2 \to 0, \quad m \to \infty.$$

Thus, from the uniqueness of the limit, we have that

$$\tilde{\mathbf{u}}(s) = \mathbf{u}(s) \quad \forall s \in [t-2, t] \backslash S,$$

and therefore, we conclude that

$$\mathbf{u}_m \to \mathbf{u} \quad \text{in} \ \ C([t-2,t]; V_p^*).$$

Now, by (2.63), we know that $\{\mathbf{u}_m(t_m)\}$ in bounded in H, then, there exists $\mathbf{v} \in H$, such that $\mathbf{u}_m(t_m) \rightharpoonup \mathbf{v}$ in H. We will prove that $\mathbf{v} = \mathbf{u}(t)$: Let $\phi \in V_p$

$$\begin{aligned}(\mathbf{u}_m(t_m) - \mathbf{u}(t^*), \phi) &= (\mathbf{u}_m(t_m) - \mathbf{u}_m(t^*), \phi) + (\mathbf{u}_m(t^*) - \mathbf{u}(t^*), \phi) \\ &\leqslant \|\mathbf{u}_n(t_m) - \mathbf{u}_m(t^*)\|_* \|\phi\|_{1,p} + \|\mathbf{u}_m(t^*) - \mathbf{u}(t^*)\|_* \|\phi\|_{1,p}.\end{aligned}$$

Since $\{\mathbf{u}_m\}$ is equicontinuous and $\mathbf{u}_m \to \mathbf{u}$ in $C([t-2,t]; V_p^*)$, we obtain that

$$\mathbf{u}_m(t_m) \rightharpoonup \mathbf{u}(t^*) \quad \text{in} \ \ V_p^*.$$

But, we know that $\mathbf{u}_m(t_m) \rightharpoonup \mathbf{v}$ in H, so we conclude (2.65) is true.

Then, from Proposition 2.13 $\mathbf{u}(\cdot)$ is also a weak solution to **(LM)** on $(t-2, t)$.

By the energy equality and hypothesis **(I)** on potential $\mathbb{S}$, we obtain

$$|\mathbf{z}(r)|_2^2 \leqslant |\mathbf{z}(s)|_2^2 + 2\int_s^r \langle \mathbf{f}(\theta), \mathbf{z}(\theta) \rangle d\theta$$

for all $t-1 \leqslant s \leqslant r \leqslant t$, where $\mathbf{z} = \mathbf{u}_m$ or $\mathbf{z} = \mathbf{u}$.

The maps $J_m, J : [t-2, t] \to \mathbb{R}$ defined by

$$J_m(r) = |\mathbf{u}_m(r)|_2^2 - 2\int_{t-2}^{r} \langle \mathbf{f}(\theta), \mathbf{u}_m(\theta)\rangle d\theta$$
$$J(r) = |\mathbf{u}(r)|_2^2 - 2\int_{t-2}^{r} \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta,$$

are non-increasing and continuous, and satisfy

$$J_m(r) \to J(r) \quad \text{a.e. } r \in (t-2, t). \tag{2.66}$$

We will use the functionals J_m and J to deduce that $\mathbf{u}_m \to \mathbf{u}$ in $C([t-1, t]; H)$. If this no true, then there exist $\varepsilon^* > 0$ a subsequence and $\{t_m\} \subset [t-1, t]$, with $t_m \to t^*$, such that

$$|\mathbf{u}_m(t_m) - \mathbf{u}(t^*))|_2 \geqslant \varepsilon^* \quad \forall m \geqslant m_0(t). \tag{2.67}$$

Let us fix $\varepsilon > 0$. Observe that $t^* \in [t-1, t]$ and therefore, by (2.66) and the continuity and non-increasing character of J, there exists $t-2 < \hat{t}_\varepsilon < t^*$ such that

$$\lim_{m\to\infty} J_m(\hat{t}_\varepsilon) = J(\hat{t}_\varepsilon), \tag{2.68}$$

and

$$0 \leqslant J(\hat{t}_\varepsilon) - J(t^*) \leqslant \varepsilon. \tag{2.69}$$

As $t_m \to t^*$, there exists m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m \geqslant m_\varepsilon$. Then, by (2.69)

$$\begin{aligned} J_m(t_m) - J(t^*) &\leqslant J_m(\hat{t}_\varepsilon) - J(t^*) \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + |J(\hat{t}_\varepsilon) - J(t^*)| \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + \varepsilon \end{aligned}$$

for all $m \geqslant m_\varepsilon$, and consequently, by (2.68), $\limsup_{m\to\infty} J_m(t_m) \leqslant J(t^*) + \varepsilon$. Thus, as $\varepsilon > 0$ is arbitrary, we deduce that

$$\limsup_{m\to\infty} J_m(t_m) \leqslant J(t^*). \tag{2.70}$$

Taking into account that $t_m \to t^*$ and

$$\int_{t-2}^{t_m} \langle \mathbf{f}(\theta), \mathbf{u}_m(\theta)\rangle d\theta \to \int_{t-2}^{t^*} \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta,$$

from (2.70) we deduce that $\limsup_{m\to\infty} |\mathbf{u}^m(t_m)|_2 \leqslant |\mathbf{u}(t^*)|_2$. This last inequality and (2.65), imply that $\mathbf{u}_m(t_m) \to \mathbf{u}(t^*)$ strongly in H, which is contradiction with (2.67). We have thus proved that $\mathbf{u}_m \to \mathbf{u}$ in $C([t-1, t]; H)$. We obtain in particular that $\mathbf{u}_m(t) \to \mathbf{u}(t)$ in H. □

Theorem 2.32. *Under assumptions **(I)**, **(II)** and **(III)** and assume that there exists $\eta > 0$ (for $p = 2$, $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$) such that $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$. Then, there exist the minimal pullback $\mathcal{D}_F^H$-attractor $\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\eta^H$-attractor $\mathcal{A}_{\mathcal{D}_\eta^H} = \{\mathcal{A}_{\mathcal{D}_\eta^H}(t) : t \in \mathbb{R}\}$ for the process $U : \mathbb{R}_d^2 \times H \to \mathcal{P}(H)$. The minimal pullback $\mathcal{D}_\eta^H$-attractor belongs to $\mathcal{D}_\eta^H$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^H}(t) \subset B_0(t) = \overline{B}_H(0, \mathcal{R}_{p>2}(t)) \quad \forall t \in \mathbb{R}. \tag{2.71}$$

Proof. The existence of pullback attractor for the multi-valued process $U(\cdot,\cdot)$ in the universe $\mathcal{D}_\eta^H$ follows from Theorem 1.42, and the existence of pullback attractor in the universe $\mathcal{D}_F^H$ with the inclusion (2.71) is given by Corollary 1.44. □

Remark 2.33. *If $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies that*

$$\sup_{s \leqslant 0} \left(e^{-\eta s} \int_{-\infty}^{s} e^{\eta r} \|\mathbf{f}(r)\|_*^q dr \right) < \infty, \tag{2.72}$$

we guarantee that for all $T \in \mathbb{R}$, $\bigcup_{t \leqslant T} B_0(t)$ is a bounded subset of H, (see $\rho_{1,p>2}$ in Lemma 2.29). Therefore, it follows from Remark 1.45 that

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^H}(t). \tag{2.73}$$

Remark 2.34. *Observe that, if $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies that $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$ for some $\eta > 0$, then $\mathbf{f} \in \mathcal{I}_*^{q,\mu}$ for all $\mu \in (\eta, +\infty)$, (for the case $p = 2$, for all $\mu \in (\eta, 2\nu_1\lambda_1 c_0^{-2})$). Hence, it holds that $\mathcal{D}_\eta^H \subset \mathcal{D}_\mu^H$. Thus, for all $\mu \in (\eta, +\infty)$ there exists the corresponding minimal pullback $\mathcal{D}_\mu^H$-attractor, $\mathcal{A}_{\mathcal{D}_\mu^H}$.*

Since $\mathcal{D}_\eta^H \subset \mathcal{D}_\mu^H$, it follow from Theorem 1.46 that, for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_\eta^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t)$ for all $\mu \in (\eta, +\infty)$.

Moreover, if $\mathbf{f}$ satisfies (2.72), then

$$\sup_{s \leqslant 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu r} \|\mathbf{f}(r)\|_*^q dr \right) < \infty \quad \text{for all } \mu \in (\eta, +\infty),$$

therefore, by (2.73) we conclude that

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) \quad \text{for all } \mu \in (\eta, +\infty).$$

2.5.2 Pullback Attractor in V_p

In order to prove the existence of a pullback attractor in V_p, we have to assume some hypotheses on the stress tensor $\mathbb{S}$, the power p and the external force $\mathbf{f}$:

(2.I) The stress tensor $\mathbb{S}$ has a potential, i.e., $\mathbb{S}$ satisfies the conditions of the Definition 1.48,

(2.II) If $n = 3$ we consider $p \geqslant 5/2$ and if $n = 2$ we consider $p > 2$,

(2.III) $\mathbf{f} \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega)^n)$, observe that this implies that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$.

By Lemma 1.49 the stress tensor $\mathbb{S}$ satisfies the p-coercivity and $(p-1)$-growth conditions. Thus, all the results established concerning existence, uniqueness, regularity of weak solutions and existence of pullback attractors in H to **(LM)** are valid when $\mathbb{S}$ is a potential.

Condition **(2.II)** guarantees that system **(LM)** has a unique weak solution (Theorem 2.14 for $p \geqslant (n+2)/2$). This will allow us obtain any estimates for $\mathbf{u}$ through the Galerkin sequence. If we assume the hypotheses of Proposition 2.18 for $n = 3$, i.e. $p \geqslant 12/5$, the uniqueness of weak solutions is not guaranteed. But, as in the case of pullback attractor in H, we could define a multi-valued process on V_p. The difficulty here is that “at the moment” we cannot ensure well properties for this process; for example, the upper semi-continuity, which we require to show existence of pullback attractor associated to a multi-valued process.

Now, by Proposition 2.18, all the weak solutions of problem **(LM)**, $\mathbf{u} = \mathbf{u}(t; \tau, \mathbf{u}_\tau)$, belong to the space $L^\infty(\tau + \varepsilon, t; V_p)$, for all $\tau < t$ and any $\varepsilon > 0$ with $\tau + \varepsilon < t$. Moreover, we can define the process $U(\cdot, \cdot)$ on V_p for each $(t, \tau) \in \mathbb{R}^2_d$. We observe that conditions **(2.I)** and **(2.III)** will allow us to estimate $\dfrac{\partial \mathbf{u}}{\partial t}$ in the space $L^\infty(t-1, t; H)$ for any t large enough with respect to τ, to show that the process $U(\cdot, \cdot)$ defined on V_p is asymptotically compact.

Definition 2.35. ***(Universe in*** V_p***)*** *We will denote by* $\mathcal{D}^{H,V_p}_\eta$ *the class of all families* $\hat{D}_{V_p}$ *of elements of* $\mathcal{P}(V_p)$ *of the form* $\hat{D}_{V_p} = \{D(t) \cap V_p : t \in \mathbb{R}\}$*, where* $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}^H_\eta$*. And we will denote by* $\mathcal{D}^{V_p}_F$ *the class of all families* $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ *with* D *a fixed nonempty bounded subset of* V_p*.*

Remark 2.36. *Observe that* $\mathcal{D}^{V_p}_F \subset \mathcal{D}^{H,V_p}_\eta$ *and that* $\mathcal{D}^{H,V_p}_\eta$ *is inclusion-closed.*

Remark 2.37. *We can apply Proposition 1.13 to ensure that the map* $U(\cdot, \cdot) : \mathbb{R}^2_d \times V_p \to V_p$ *defined by*

$$U(t, \tau)\mathbf{u}_\tau = \mathbf{u}(t; \tau, \mathbf{u}_\tau), \tag{2.74}$$

where $\mathbf{u}$ *is the weak solution to* ***(LM)****, is well-defined for all* $(t, \tau) \in \mathbb{R}^2_d$*. Indeed, consider* $X = V_p$ *and* $Y = H$ *then, we know that the weak solution* $\mathbf{u}$ *of problem* ***(LM)*** *is in* $C([\tau, T]; H)$ *and, by Proposition 2.18,* $\mathbf{u} \in L^\infty(\tau + \varepsilon, T; V_p)$ *for all* $\varepsilon > 0$ *with* $\tau + \varepsilon \leqslant T$*. Therefore, it follows from Proposition 1.13 that* $\mathbf{u} \in C_w((\tau, T]; V_p)$ *and* $\mathbf{u}(t)$ *is defined on* V_p*, for all* $t \in (\tau, T]$*. If the initial condition* $\mathbf{u}_\tau \in V_p$*, then the process* $U(\cdot, \cdot)$ *is defined for all* $t \in [\tau, T]$*.*

Proposition 2.38. *Assume that **(2.I)** and **(2.II)** are satisfied and suppose that* $\mathbf{f} \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n)$*. Then, the map defined by (2.74),* $U(\cdot,\cdot) : \mathbb{R}^2_d \times V_p \to V_p$ *is a closed process on* V_p*.*

Proof. From Remark 2.37 and Theorem 2.14, we have that $U(\cdot,\cdot) : \mathbb{R}^2_d \times V_p \to V_p$ is a well-defined. Next, we prove that $U(\cdot,\cdot)$ is a closed process. Given $t \in \mathbb{R}$ with $t \geqslant \tau$, suppose that $\{\mathbf{u}^k_\tau\}$ is a sequence in V_p, with $\mathbf{u}^k_\tau \to \mathbf{u}_\tau$ in V_p, as $k \to \infty$, and also assume that $U(t,\tau)\mathbf{u}^k_\tau = \mathbf{u}(t;\tau;\mathbf{u}^k_\tau) \to \mathbf{y}$ in V_p, as $k \to \infty$. We will show that $\mathbf{y} = U(t,\tau)\mathbf{u}_\tau$. In effect, we know that the process $U(\cdot,\cdot) : \mathbb{R}^2_d \times H \to H$ is a closed process in H (as $p \geqslant \dfrac{n+2}{2}$ the process is continuous, see Remark 2.16). Therefore $U(t,\tau)\mathbf{u}^k_\tau \to U(t,\tau)\mathbf{u}_\tau$ in H, as $k \to \infty$, and as $V_p \hookrightarrow H$ then, by the uniqueness of limit, we have that $\mathbf{y} = U(t,\tau)\mathbf{u}_\tau$. □

Corollary 2.39. *Assume that **(2.I)** and **(2.II)** are satisfied and suppose that there exists* $\eta > 0$ *such that* $\mathbf{f} \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n) \cap \mathcal{I}^{q,\eta}_*$*. Then the family*

$$\widehat{B}_{0,V_p} = \{B_{0,V_p}(t) = \overline{B}_H(0, \mathcal{R}_{p>2}(t)) \cap V_p : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}^{H,V_p}_\eta$ *and satisfies that for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}^H_\eta$*, there exists* $\tau(\widehat{D},t) < t$ *such that*

$$U(t,\tau)D(\tau) \subset B_{0,V_p}(t) \quad \textit{for all} \quad \tau \leqslant \tau(\widehat{D},t).$$

In particular, the family $\widehat{B}_{0,V_p}$ *is pullback* $\mathcal{D}^{H,V_p}_\eta$*-absorbing for the process* $U : \mathbb{R}^2_d \times V_p \to V_p$*.*

Proof. Follows directly from the Corollary 2.28 □

The next lemma is based on estimates given in Proposition 2.18 (high-regularity result). The demonstration will be presented only for the case $n = 3$ with $p \geqslant 5/2$, but for the case $n = 2$ with $p > 2$ it is also valid.

Lemma 2.40. *Consider assumptions **(2.I)** and **(2.II)**, and suppose that there exists* $\eta > 0$ *such that* $\mathbf{f} \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n) \cap \mathcal{I}^{q,\eta}_*$*. Then for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^H_\eta$*, there exists* $\tau_1(\widehat{D},t) < t - 3$*, such that for any* $\tau \leqslant \tau_1(\widehat{D},t)$ *and* $\mathbf{u}_\tau \in D(\tau)$*, holds*

$$\|\mathbf{u}(r;\tau,\mathbf{u}_\tau)\|_{1,p} \leqslant \rho_3(t) \quad \textit{for all} \quad r \in [t-2,t], \tag{2.75}$$

$$\int_{r-1}^{r} |\frac{\partial \mathbf{u}}{\partial t}(\theta;\tau,\mathbf{u}_\tau)|_2^2 d\theta \leqslant \rho_4(t) \quad \textit{for all} \quad r \in [t-1,t], \tag{2.76}$$

where

$$\rho_3(t) = \max\{\mathcal{R}_1(t), \mathcal{R}_2(t), \mathcal{R}_3(t)\},$$

with $\mathcal{R}_1(t)$*,* $\mathcal{R}_2(t)$ *and* $\mathcal{R}_3(t)$ *given in (2.78), (2.80) and (2.81) respectively, and*

$$\rho_4(t) = 2c_9(1 + \rho_3^p(t))\big[1 + \frac{dc_7c_9}{c_8}\big(\rho_{1,p>2}(t)\big)^{\frac{2(5p-12)}{5p-6}}\big(\rho_3(t)\big)^{\frac{p(16-5p)}{5p-6}}\big] + \int_{t-3}^{t} |\mathbf{f}(\theta)|_2^2 d\theta.$$

Proof. Let $\{\mathbf{u}_k\}$ be the Galerkin approximation of $\mathbf{u}$.

- Case $n = 3$, when $5/2 \leqslant p < 3$:

  By estimate (2.49) given in Proposition 2.18, we have that

$$\|\Phi(\mathbf{e}(\mathbf{u}_k(r)))\|_1 \leqslant \big(\|\Phi(\mathbf{e}(\mathbf{u}_k(s)))\|_1 + \int_{r-1}^{r} |\mathbf{f}(\theta)|_2^2 d\theta\big) \\ \times \exp\big(\frac{c_7 d^2}{c_8}\int_{r-1}^{r} \|\mathbf{u}_k(\theta)\|_{1,p}^{\frac{p(16-5p)}{(5p-6)}} |\mathbf{u}_k(\theta)|_2^{\frac{2(5p-12)}{(5p-6)}} d\theta\big)$$

  for all $r \in [t-2, t]$ and $s \in [r-1, r]$.

  Observe that the constants do not depend on k. Integrating this last inequality for s between $r-1$ and r, we obtain

$$\|\Phi(\mathbf{e}(\mathbf{u}_k(r)))\|_1 \leqslant \big(\int_{r-1}^{r} \|\Phi(\mathbf{e}(\mathbf{u}_k(s)))\|_1 ds + \int_{r-1}^{r} |\mathbf{f}(\theta)|_2^2 d\theta\big) \\ \times \exp\big(\frac{c_7 d^2}{c_8}\int_{r-1}^{r} \|\mathbf{u}_k(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} |\mathbf{u}_k(\theta)|_2^{\frac{2(5p-12)}{5p-6}} d\theta\big).$$

  By Lemma 2.29 and (2.46), we know that

$$\int_{r-1}^{r} \|\Phi(\mathbf{e}(\mathbf{u}_k(s)))\|_1 ds \leqslant c_9(1 + \rho_{2,p>2}(t)), \tag{2.77}$$

  for all $r \in [t-2, t]$, $\tau \leqslant \tau_1(\widehat{D}, t)$, $\mathbf{u}_\tau \in \widehat{D}$.

  Again from Lemma 2.29 and the Young inequality

$$\int_{r-1}^{r} \|\mathbf{u}_k(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} |\mathbf{u}_k(\theta)|_2^{\frac{2(5p-12)}{5p-6}} d\theta \leqslant [\rho_{1,p>2}(t)]^{\frac{2(5p-12)}{5p-6}} \int_{r-1}^{r} \|\mathbf{u}_k(\theta)\|_{1,p}^{\frac{p(16-5p)}{(5p-6)}} d\theta \\ \leqslant [\rho_{1,p>2}(t)]^{\frac{2(5p-12)}{5p-6}} [\rho_{2,p>2}(t)]^{\frac{16-5p}{5p-6}},$$

  for all $r \in [t-2, t]$, $\tau \leqslant \tau_1(\widehat{D}, t)$, $\mathbf{u}_\tau \in \widehat{D}$.

  From this last inequality and (2.77), we obtain that

$$\|\mathbf{u}_k(r)\|_{1,p} \leqslant \mathcal{R}_1(t),$$

  for all $r \in [t-2, t]$, $\tau \leqslant \tau_1(\widehat{D}, t)$ and $\mathbf{u}_\tau \in \widehat{D}$, where

$$\mathcal{R}_1^p(t) = \frac{1}{c_8}\big[(c_9(1 + \rho_{2,p>2}(t)) + \int_{t-3}^{t} |\mathbf{f}(\theta)|_2^2 d\theta\big] \\ \times \exp\Big\{\frac{d^2 c_7}{c_8}\big(\rho_{1,p>2}(t)\big)^{\frac{2(5p-12)}{(5p-6)}} \big(\rho_{2,p>2}(t)\big)^{\frac{16-5p}{5p-6}}\Big\}. \tag{2.78}$$

- Case $n = 3$, when $p \geqslant 3$:

By estimate (2.50) given in Proposition 2.18 and the embedding $W^{1,2}(\Omega)^n \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^n$ (which is true for $\dfrac{2p}{p-2} \leqslant \dfrac{2n}{n-2} \Leftrightarrow p \geqslant n$) we have that

$$\frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_k))\|_1 \leqslant |\mathbf{f}(t)|_2^2 + \int_\Omega |\mathbf{u}_k|^2|\nabla \mathbf{u}_k|^2 dx \leqslant |\mathbf{f}(t)|_2^2 + c_7 d^2 \|\mathbf{u}_k\|_{1,p}^2 \|\mathbf{u}_k\|_{1,2}^2.$$

Let us denote by $\mathcal{U} := 1 + \|\Phi(\mathbf{e}(\mathbf{u}_k))\|_1$, and observe that $\dfrac{d}{dt}\mathcal{U} = \dfrac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_k))\|_1$. Therefore, from (2.46) and the above estimate, we obtain that

$$\frac{d}{dt}\mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U}^{4/p}. \tag{2.79}$$

Now, if $4 \leqslant p$, then $\mathcal{U}^{4/p} \leqslant \mathcal{U}$, since $\mathcal{U} \geqslant 1$, with this and (2.79)

$$\frac{d}{dt}\mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U}.$$

Integrating from r to s, with $r \in [t-2,t]$ and $s \in [r-1,r]$, we have

$$\mathcal{U}(r) \leqslant \mathcal{U}(s) + \int_{r-1}^r |\mathbf{f}(\theta)|_2^2 d\theta + \widetilde{C}_2 \int_{r-1}^r \mathcal{U}(\theta) d\theta.$$

Integrating this last inequality in s between $r-1$ and r, we get

$$\mathcal{U}(r) \leqslant \int_{r-1}^r \mathcal{U}(s) ds + \int_{r-1}^r |\mathbf{f}(\theta)|_2^2 d\theta + \widetilde{C}_2 \int_{r-1}^r \mathcal{U}(\theta) d\theta.$$

Therefore, by Lemma 2.29 and (2.46), it follows that

$$\|\mathbf{u}_k\|_{1,p} \leqslant \mathcal{R}_2(t),$$

for all $r \in [t-2,t]$, $\tau \leqslant \tau_1(\widehat{D},t)$ and $\mathbf{u}_\tau \in \widehat{D}$, where

$$\mathcal{R}_2^p(t) = \frac{(1+\widetilde{C}_2)}{c_8} + \frac{(1+\widetilde{C}_2)c_9}{c_8}(1+\rho_{2,p>2}(t)) + \int_{t-3}^t |\mathbf{f}(\theta)|_2^2 d\theta. \tag{2.80}$$

Now, if $3 \leqslant p < 4$, let us consider $\mu = (2p-4)/p$, then $\mu \in \left[\dfrac{2}{3}, 1\right)$. Multiplying by $\mathcal{U}^{\mu-1}$ to (2.79), we obtain that

$$\mu \frac{d}{dt}(\mathcal{U}^\mu) \leqslant |\mathbf{f}(t)|_2^2 \mathcal{U}^{\mu-1} + \widetilde{C}_2 \mathcal{U} \leqslant |\mathbf{f}(t)|_2^2 + \widetilde{C}_2 \mathcal{U},$$

since $\mathcal{U}^{\mu-1} \leqslant 1$.

Similarly, as in the last case, we have that

$$\mu \mathcal{U}^\mu(r) \leqslant (\mu + \widetilde{C}_2) \int_{r-1}^r \mathcal{U}(s) ds + \int_{t-3}^t |\mathbf{f}(s)|_2^2 ds,$$

for all $r \in [t-2,t]$.

Thus, of this last inequality, we obtain

$$\|\mathbf{u}_k\|_{1,p} \leqslant \mathcal{R}_3(t),$$

for all $r \in [t-2, t]$, $\tau \leqslant \tau_1(\widehat{D}, t)$ and $\mathbf{u}_\tau \in \widehat{D}$, where

$$\mathcal{R}_3^p(t) = \left(\frac{\mu + \widetilde{C}_2}{c_8} + \frac{c_9}{\mu}(1 + \rho_{2,p>2}(t)) + \frac{1}{\mu}\int_{t-3}^{t} |\mathbf{f}(s)|_2^2 ds \right)^{1/\mu}. \tag{2.81}$$

Therefore, from (2.78), (2.80) and (2.81), we conclude that

$$\|\mathbf{u}_k(r)\|_{1,p} \leqslant \rho_3(t),$$

for all $r \in [t-2, t]$, $\tau \leqslant \tau_1(\widehat{D}, t)$ and $\mathbf{u}_\tau \in \widehat{D}$, where

$$\rho_3(t) = \max\{\mathcal{R}_1(t), \mathcal{R}_2(t), \mathcal{R}_3(t)\}.$$

Finally, from (2.45) we know that

$$\frac{1}{2}\left|\frac{\partial \mathbf{u}_k}{\partial t}\right|_2^2 + \frac{d}{dt}\|\Phi(\mathbf{e}(\mathbf{u}_k))\|_1 \leqslant |\mathbf{f}(t)|_2^2 + c_7\|\mathbf{u}_k\|_{1,p}^2\|\mathbf{u}_k\|_{2p/(p-2)}^2.$$

Therefore, based on this last estimate and the previous cases, we conclude the proof. □

The proof of next Lemma is based on a technique used in [59], which, formally, consists in deriving the equation and multiplying by $\dfrac{\partial \mathbf{u}}{\partial t}$ obtaining a new formulation and with this, to obtain estimates for $\dfrac{\partial \mathbf{u}}{\partial t}$ in the uniform norm.

Lemma 2.41. *Under assumptions **(2.I)**, **(2.II)** and **(2.III)**, suppose that there exists $\eta > 0$ such that $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$. Then for any $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \leqslant \tau_1(\widehat{D}, t)$ and $\mathbf{u}_\tau \in D(\tau)$, it holds*

$$\left|\frac{\partial \mathbf{u}}{\partial t}(r; \tau, \mathbf{u}_\tau)\right|_2 \leqslant \rho_5(t) \quad \textit{for all} \;\; r \in [t-1, t],$$

where

$$\rho_5^2(t) = \rho_4(t)\left\{2 + c_2\left[\rho_3(t)\right]^{\frac{2p}{2p-3}}\right\} + \int_{t-2}^{t} \left|\frac{\partial \mathbf{f}}{\partial t}(s)\right|_2^2 ds.$$

Proof. We will make formal calculations that can be justified by using the Galerkin approximations, since $\mathbf{u}$ does not have enough regularity.

By differentiating of (2.3) in time, we get

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}_x\left(\partial_D^2 \Phi(\mathbf{e}(\mathbf{u}))\mathbf{e}(\frac{\partial \mathbf{u}}{\partial t})\right) + \mathrm{div}_x\left(\frac{\partial \mathbf{u}}{\partial t} \otimes \mathbf{u} + \mathbf{u} \otimes \frac{\partial \mathbf{u}}{\partial t}\right) + \nabla_x\left(\frac{\partial P}{\partial t}\right) = \frac{\partial \mathbf{f}}{\partial t}.$$

Multiplying the above equality by $\dfrac{\partial \mathbf{u}}{\partial t}$, integrating over Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+\int_\Omega \partial_D^2\Phi(\mathbf{e}(\mathbf{u}))\mathbf{e}(\frac{\partial \mathbf{u}}{\partial t}):\mathbf{e}(\frac{\partial \mathbf{u}}{\partial t})dx+\int_\Omega\left(\frac{\partial \mathbf{u}}{\partial t}\otimes\frac{\partial \mathbf{u}}{\partial t}\right):\nabla_x\mathbf{u}dx=(\frac{\partial \mathbf{f}}{\partial t},\frac{\partial \mathbf{u}}{\partial t}), \quad (2.82)$$

since

$$\int_\Omega\left(\frac{\partial \mathbf{u}}{\partial t}\otimes\mathbf{u}\right):\nabla\frac{\partial \mathbf{u}}{\partial t}=0,$$

and

$$\int_\Omega\left(\mathbf{u}\otimes\frac{\partial \mathbf{u}}{\partial t}\right):\nabla\frac{\partial \mathbf{u}}{\partial t}=-\int_\Omega\left(\frac{\partial \mathbf{u}}{\partial t}\otimes\frac{\partial \mathbf{u}}{\partial t}\right):\nabla_x\mathbf{u}.$$

Using (2.82) and the properties of Φ, we have that

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+\nu_1\int_\Omega(1+\mu|\mathbf{e}(\mathbf{u})|)^{p-2}\left|\mathbf{e}(\frac{\partial \mathbf{u}}{\partial t})\right|^2dx\leqslant\frac{1}{2}\left|\frac{\partial \mathbf{f}}{\partial t}\right|_2^2+\frac{1}{2}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2-\int_\Omega\frac{\partial \mathbf{u}}{\partial t}\otimes\frac{\partial \mathbf{u}}{\partial t}:\nabla_x\mathbf{u}dx.$$

From this, it follows that

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+\nu_1\left|\mathbf{e}(\frac{\partial \mathbf{u}}{\partial t})\right|_2^2\leqslant\frac{1}{2}\left|\frac{\partial \mathbf{f}}{\partial t}\right|_2^2+\frac{1}{2}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+\|\mathbf{u}\|_{1,p}\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{2q}^2,$$

with $\dfrac{1}{p}+\dfrac{1}{q}=1$.

By the interpolation inequality (Theorem 1.17) for $n=3$ (for $n=2$ we use the Ladyzhenskaya inequality, Theorem 1.15), we have $\|\frac{\partial \mathbf{u}}{\partial t}\|_{2q}\leqslant d|\frac{\partial \mathbf{u}}{\partial t}|_2^{(2p-3)/2p}\|\frac{\partial \mathbf{u}}{\partial t}\|_{1,2}^{3/2p}$. Thus, the Korn and the Young inequalities give us

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+c_1\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{1,2}^2\leqslant\frac{1}{2}\left|\frac{\partial \mathbf{f}}{\partial t}\right|_2^2+\frac{1}{2}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2+\frac{c_2}{2}\|\mathbf{u}\|_{1,p}^{2p/(2p-3)}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_2^2.$$

Integrating from s to r, with $\tau\leqslant r-1\leqslant s\leqslant r$, we have

$$\left|\frac{\partial \mathbf{u}}{\partial t}(r)\right|_2^2\leqslant\left|\frac{\partial \mathbf{u}}{\partial t}(s)\right|_2^2+\int_{r-1}^r\left|\frac{\partial \mathbf{f}}{\partial t}(\theta)\right|_2^2d\theta+\int_{r-1}^r\left|\frac{\partial \mathbf{u}}{\partial t}(\theta)\right|_2^2d\theta+c_2\int_{r-1}^r\|\mathbf{u}(\theta)\|_{1,p}^{2p/(2p-3)}\left|\frac{\partial \mathbf{u}}{\partial t}(\theta)\right|_2^2d\theta,$$

for all $r-1\leqslant s\leqslant r$.

Integrating this last inequality for s between $r-1$ and r, we obtain

$$\left|\frac{\partial \mathbf{u}}{\partial t}(r)\right|_2^2\leqslant 2\int_{r-1}^r\left|\frac{\partial \mathbf{u}}{\partial t}(s)\right|_2^2ds+\int_{r-1}^r\left|\frac{\partial \mathbf{f}}{\partial t}(\theta)\right|_2^2d\theta+c_2\int_{r-1}^r\|\mathbf{u}(\theta)\|_{1,p}^{2p/(2p-3)}\left|\frac{\partial \mathbf{u}}{\partial t}(\theta)\right|_2^2d\theta.$$

By Lemma 2.40, we conclude the proof. □

Theorem 2.42. *Consider assumptions **(2.I)**, **(2.II)** and **(2.III)**, and suppose that there exists $\eta>0$ such that $\mathbf{f}\in\mathcal{I}_*^{q,\eta}$. Then, the process $U(\cdot,\cdot)$ is pullback $\mathcal{D}_\eta^{H,V_p}$-asymptotically compact.*

Proof. Given $\widehat{D} \in \mathcal{D}_\eta^{H,V_p}$ and $t \in \mathbb{R}$, we will see that, $\{U(t,\tau_m)\mathbf{u}_{\tau_m}\}$ is relatively compact in V_p, where $\mathbf{u}_{\tau_m} \in D(\tau_m)$ for each $m \in \mathbb{N}$ and $\tau_m \to -\infty$. Let us denote by $\mathbf{u}_m := \mathbf{u}_m(t) := U(t,\tau_m)\mathbf{u}_{\tau_m}$. Next, we prove that the sequence $\{\mathbf{u}_m\}$ has a Cauchy subsequence in V_p.

From Lemma 2.40, we know that $\{\mathbf{u}_m\}$ is relatively compact in H. Without loss of generality, we assume that $\{\mathbf{u}_m(t)\}$ is a Cauchy sequence in H. In the following, we prove that $\{\mathbf{u}_m\}$ is a Cauchy sequence in V_p.

Now, we deduce from (2.7) that there exists $\delta > 0$ such that

$$\big(\mathbb{S}(\mathbf{e}(\mathbf{u})) - \mathbb{S}(\mathbf{e}(\mathbf{v})), \mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{v})\big) \geqslant \delta \|\mathbf{u} - \mathbf{v}\|_{1,p}^p,$$

for all $\mathbf{u}, \mathbf{v} \in V_p$ (p-coercivity of $\mathbb{S}$).

Using the p-coercivity of $\mathbb{S}$, Lemma 2.12, Lemma 2.29, Lemma 2.40 and Lemma 2.41, we have that

$$\begin{aligned}
\delta \|\mathbf{u}_k - \mathbf{u}_l\|_{1,p}^p &\leqslant \big(\mathbb{S}(\mathbf{e}(\mathbf{u}_k)) - \mathbb{S}(\mathbf{e}(\mathbf{u}_l)), \mathbf{e}(\mathbf{u}_k) - \mathbf{e}(\mathbf{u}_l)\big) \\
&= \Big(\frac{\partial \mathbf{u}_k}{\partial t} - \frac{\partial \mathbf{u}_l}{\partial t}, \mathbf{u}_k - \mathbf{u}_l\Big) + \int_\Omega \big(\mathbf{u}_k \otimes \mathbf{u}_k - \mathbf{u}_l \otimes \mathbf{u}_l\big) : \nabla(\mathbf{u}_k - \mathbf{u}_l) dx \\
&\leqslant \left|\frac{\partial \mathbf{u}_k}{\partial t} - \frac{\partial \mathbf{u}_l}{\partial t}\right|_2 |\mathbf{u}_k - \mathbf{u}_l|_2 + \int_\Omega |\mathbf{u}_k - \mathbf{u}_l| |\nabla \mathbf{u}_k| dx \\
&\leqslant \left|\frac{\partial \mathbf{u}_k}{\partial t} - \frac{\partial \mathbf{u}_l}{\partial t}\right|_2 |\mathbf{u}_k - \mathbf{u}_l|_2 + \|\mathbf{u}_k\|_{1,p} \|\mathbf{u}_k - \mathbf{u}_l\|_{2q}^2 \\
&\leqslant 2\rho_5(t) |\mathbf{u}_k - \mathbf{u}_l|_2 + d\rho_3(t) |\mathbf{u}_k - \mathbf{u}_l|_2^{(2p-3)/p} \|\mathbf{u}_k - \mathbf{u}_l\|_{1,2}^{3/p} \\
&\leqslant 2\rho_5(t) |\mathbf{u}_k - \mathbf{u}_l|_2 + d|\Omega|^{3p-6/p^2} \rho_3(t)^{(p^2+6)/p^2} |\mathbf{u}_k - \mathbf{u}_l|_2^{(2p-3)/p}.
\end{aligned}$$

Observe that we have used the interpolation inequality (Theorem 1.17) for $n = 3$ to estimate $\|\mathbf{u}_k - \mathbf{u}_l\|_{2q}$. But, for $n = 2$ it is also valid, since we can use the Ladyzhenskaya inequality (Theorem 1.15).

Therefore, this last estimate combined with the fact that $\{\mathbf{u}_m\}$ is a Cauchy sequence in H, allow us to conclude that $\{\mathbf{u}_m\}$ is a Cauchy sequence in V_p. □

Theorem 2.43. *Consider assumptions* ***(2.I)***, ***(2.II)*** *and* ***(2.III)***, *and suppose that there exists $\eta > 0$ such that $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$. Then, there exist the minimal pullback $\mathcal{D}_F^{V_p}$-attractor $\mathcal{A}_{\mathcal{D}_F^{V_p}} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\eta^{H,V_p}$-attractor $\mathcal{A}_{\mathcal{D}_\eta^{H,V_p}} = \{\mathcal{A}_{\mathcal{D}_\eta^{H,V_p}}(t) : t \in \mathbb{R}\}$ for the closed process $U : \mathbb{R}_d^2 \times V_p \to V_p$ defined in (2.74). The minimal pullback $\mathcal{D}_\eta^{H,V_p}$-attractor belongs to $\mathcal{D}_\eta^{H,V_p}$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{V_p}}(t) \subset \mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^{H,V_p}}(t) \quad \forall t \in \mathbb{R}, \tag{2.83}$$

where $\mathcal{A}_{\mathcal{D}_F^H}$ and $\mathcal{A}_{\mathcal{D}_\eta^H}$ are the respectively the minimal pullback $\mathcal{D}_F^H$-attractor and the minimal pullback $\mathcal{D}_\eta^H$-attractor for the "multi-valued" process $U : \mathbb{R}_d^2 \times H \to H$, whose existence is guaranteed by Theorem 2.32.

Proof. The existence of pullback attractor for the closed process $U(\cdot,\cdot)$ on V_p, in the universe $\mathcal{D}_\eta^{H,V_p}$ follows from Theorem 1.33, and the existence of pullback attractor in the universe $\mathcal{D}_F^{V_p}$ with the inclusion (2.83) are given by Corollary 1.44 and Theorem 1.46. □

Remark 2.44. *Observe that by (2.83), in particular, the following pullback attraction result in V_p holds:*

$$\lim_{\tau\to-\infty} dist_{V_p}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and any } \widehat{D} \in \mathcal{D}_\eta^H.$$

Remark 2.45. *If $\mathbf{f} \in \mathcal{I}_*^{q,\eta}$ satisfies that*

$$\sup_{s\leqslant 0}\left(e^{-\eta s}\int_{-\infty}^{s} e^{\eta r}\|\mathbf{f}(r)\|_*^q dr\right) < \infty,$$

then

$$\mathcal{A}_{\mathcal{D}_F^{V_p}}(t) = \mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^{H,V_p}}(t) \quad \forall t \in \mathbb{R},$$

where the equality $\mathcal{A}_{\mathcal{D}_F^{V_p}}(t) = \mathcal{A}_{\mathcal{D}_F^H}(t)$ is a consequence of Theorem 1.46.

3 A Class of Incompressible non-Newtonian Fluid with Delay

A delay partial differential equation is an equation which involves: at least two independent variables, an unknown function of the independent variables, the behavior of the unknown function at some prior values of the independent variables and partial derivatives of the unknown function with respect to the independent variables.

Thus, a delay partial differential equation, in contrast with a partial differential equation, depends not only on the solution at a present moment but also on the solution at some past times. If, in addition, the equation depends on the derivatives of the solution at some past times, then it is called a neutral delay partial differential equation.

Delay partial differential equations are also called partial functional differential equations as their unknown solutions are used in these equations as functional arguments.

In this chapter we are going to present a mathematical model with delay for incompressible non-Newtonian fluids. We will study the existence and uniqueness of weak solutions. Moreover, we will prove the existence of pullback attractors for the multi-valued process defined from the weak solution in the Banach spaces $C_H = C([-h,0];H)$ and $M_H^2 = H \times L^2(-h,0;H)$, with $h > 0$.

3.1 Statement of the Problem

Let $\Omega \subset \mathbb{R}^n$, with $n = 2$ or $n = 3$, be an open bounded domain with regular boundary $\partial\Omega$. Given τ and T with $\tau < T$, we consider following system of partial differential equations with delay and Dirichlet boundary condition for incompressible non-Newtonian fluids, to which we will refer by **(LMD)**:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}_x \mathbb{S}(\mathbf{e}(\mathbf{u})) + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \pi &= \mathbf{f}(t) + \mathbf{g}(t, \mathbf{u}_t) \text{ in } \Omega_{\tau,T}, && (3.1)\\
\mathrm{div}_x \mathbf{u} &= 0 \quad \text{in} \quad \Omega_{\tau,T}, && (3.2)\\
\mathbf{u} &= 0 \quad \text{on} \quad (\tau,T) \times \partial\Omega, && (3.3)\\
\mathbf{u}(\tau,x) &= \mathbf{u}^\tau(x) \quad \text{with} \quad x \in \Omega, && (3.4)\\
\mathbf{u}(\tau+t,x) &= \phi(t,x) \quad \text{in} \quad \Omega_h, && (3.5)
\end{aligned}
$$

where $\Omega_{\tau,T} = (\tau,T) \times \Omega$, $\Omega_h = (-h,0) \times \Omega$ with $h > 0$, $\mathbf{u} = (u_1, \cdots, u_n)$ is the fluid flow velocity vector field, $\mathbf{u}_t(s) = \mathbf{u}(t+s)$ for any $s \in (-h,0)$, $\mathbf{u}^\tau$ is the velocity of fluid at the initial time $t = \tau$, π is the pressure, $\mathbf{g}(t, \mathbf{u}_t)$ is the delay term, ϕ is the initial condition with memory, $\mathbf{f}$ is an external force and $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ is the tensor stress satisfying

$$\begin{aligned} \mathbb{S}(\mathbf{0}) &= \mathbf{0}, \\ \big(\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})\big) : (\mathbf{A} - \mathbf{B}) &\geqslant \nu_1\big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2}|\mathbf{A} - \mathbf{B}|^2, \\ |\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})| &\leqslant c_1\nu_1\big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2}|\mathbf{A} - \mathbf{B}|, \end{aligned} \tag{3.6}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n^2}_{sym}$ and $p \geqslant 2$.

We denote by C_H the Banach space $C([-h, 0]; H)$, with norm

$$\|\phi\|_{C_H} = \sup_{s \in [-h,0]} |\phi(s)|_2 \qquad \phi \in C_H$$

and by L^2_H the Hilbert space $L^2(-h, 0; H)$, with norm

$$\|\phi\|^2_{L^2_H} = \int_{-h}^{0} |\phi(s)|_2^2 ds \qquad \phi \in L^2_H.$$

In order to state the problem in the correct framework, let us first establish suitable assumptions on the delay term $\mathbf{g}$.

In a general way, let X and Y be two separable Banach space and

$$\mathbf{g} : [\tau, T] \times C([-h, 0]; X) \to Y \tag{3.7}$$

such that the following conditions hold,

(3.I) For all $\xi \in C([-h, 0]; X)$, the mapping $t \in [\tau, T] \to \mathbf{g}(t, \xi) \in Y$ is measurable.

(3.II) For each $t \in [\tau, T]$, $\mathbf{g}(t, 0) = 0$.

(3.III) There exists $L_{\mathbf{g}} > 0$ such that, for all $s \in [\tau, T]$, for any $\xi, \eta \in C([-h, 0]; X)$,

$$\|\mathbf{g}(s, \xi) - \mathbf{g}(s, \eta)\|_Y \leqslant L_{\mathbf{g}}\|\xi - \eta\|_{C([-h,0];X)}.$$

(3.IV) There exists $C_{\mathbf{g}} > 0$ such that, for all $t \in [\tau, T]$, for any $\mathbf{u}, \mathbf{v} \in C([\tau - h, T]; X)$,

$$\int_{\tau}^{t} \|\mathbf{g}(s, \mathbf{u}_s) - \mathbf{g}(s, \mathbf{v}_s)\|_Y^2 ds \leqslant C_{\mathbf{g}}^2 \int_{\tau-h}^{t} \|\mathbf{u}(s) - \mathbf{v}(s)\|_X^2 ds.$$

Observe that conditions **(3.I)**-**(3.III)** above imply that, given $\mathbf{u} \in C([\tau - h, T]; X)$, the function $\mathbf{g}_{\mathbf{u}} : t \in [\tau, T] \to Y$ defined by $\mathbf{g}_{\mathbf{u}}(t) = \mathbf{g}(t, \mathbf{u}_t)$, $\forall t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; Y)$. Then, thanks to **(3.IV)**, the mapping

$$\mathcal{G} : \mathbf{u} \in C([\tau - h, T]; X) \to \mathbf{g}_{\mathbf{u}} \in L^2(\tau, T; Y)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$, which is uniformly continuous from $L^2(\tau-h,T;X)$ into $L^2(\tau,T;Y)$. From now on, we will write $\mathbf{g}(t,\mathbf{u}_t) = \widetilde{\mathcal{G}}(t)$ for each $\mathbf{u} \in L^2(\tau-h,T;X)$, and thus for all $t \in [\tau,T]$ and any $\mathbf{u},\mathbf{v} \in L^2(\tau-h,T;X)$, it holds,

$$\int_\tau^t \|\mathbf{g}(s,\mathbf{u}_s) - \mathbf{g}(s,\mathbf{v}_s)\|_Y^2 ds \leqslant C_{\mathbf{g}}^2 \int_{\tau-h}^t \|\mathbf{u}(s)-\mathbf{v}(s)\|_X^2 ds.$$

The delay term $\mathbf{g}$ in **(LMD)** is defined by

$$\mathbf{g} : [\tau,T] \times C([-h,0];H) \to L^2(\Omega)^n \tag{3.8}$$

satisfying hypotheses **(3.I)**-**(3.IV)** with $X = H$, $Y = L^2(\Omega)^n$.

An example of delay term verifying hypotheses **(3.I)**-**(3.IV)** can be constructed as follows, see [13, Section 3].

Let $G : [\tau,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function satisfying $G(t,\mathbf{0}) = \mathbf{0}$ for all $t \in [\tau,T]$, and assume that there exists $L_1 > 0$ such that

$$|G(t,\mathbf{x}) - G(t,\mathbf{y})| \leqslant L_1|\mathbf{x}-\mathbf{y}| \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n.$$

Let $\delta(t)$ be a function such that $\delta \in C^1([\tau,T])$, $\delta(t) \geqslant 0$ for all $t \in [\tau,T]$, $h = \max_{t\in[\tau,T]} \delta(t) > 0$ and $\delta_* = \max_{t\in[\tau,T]} \delta'(t) < 1$. Then $\mathbf{g}(t,\mathbf{u}_t) = G(t,\mathbf{u}(t-\delta(t)))$ satisfies hypotheses **(3.I)**-**(3.IV)** with $X = H$, $Y = L^2(\Omega)^n$ and $C_{\mathbf{g}}^2 = \dfrac{L_1^2}{1-\delta_*}$.

3.2 Existence of Weak Solutions to (LMD)

For the following definition of weak solution to **(LMD)**, as in Section 2.1, we consider the operators $B : L^p(\tau,T;V_p) \cap L^\infty(\tau,T;H) \to L^q(\tau,T;V_p^*)$ and $\mathbb{T} : L^p(\tau,T;V_p) \to L^q(\tau,T;V_p^*)$ given by Definitions 2.5 and 2.7, respectively, where $q = p/(p-1)$.

Definition 3.1. *By a weak solution to **(LMD)** we understand a function $\mathbf{u}$, belonging to the class*

$$\mathbf{u} \in L^2(\tau-h,T;H) \cap L^p(\tau,T;V_p) \cap L^\infty(\tau,T;H) \quad \text{with} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^q(\tau,T;V_p^*), \tag{3.9}$$

which satisfies the weak formulation

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathbb{T}(\mathbf{u}(t)),\mathbf{v}\rangle + \langle B(\mathbf{u}(t)),\mathbf{v}\rangle = \langle \mathbf{f}(t),\mathbf{v}\rangle + (\mathbf{g}(t,\mathbf{u}_t),\mathbf{v}), \tag{3.10}$$

for all $\mathbf{v} \in V_p$ and a.e. $\tau \leqslant t \leqslant T$, and

$$\mathbf{u}(\tau) = \mathbf{u}^\tau \quad \text{and} \quad \mathbf{u}(\tau+s) = \phi(s) \quad s \in (-h,0). \tag{3.11}$$

Remark 3.2. *By Theorem 1.10, the weak solution has a representative in the class*

$$\mathbf{u} \in C([\tau, T]; H). \tag{3.12}$$

whereby (3.11) makes sense. Besides, for any functions $\mathbf{u}$, $\mathbf{v}$, *belonging to the class (3.9), it holds*

$$(\mathbf{u}(t_2), \mathbf{v}(t_2)) - (\mathbf{u}(t_1), \mathbf{v}(t_1)) = \int_{t_1}^{t_2} \left(\left\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right\rangle + \left\langle \frac{\partial \mathbf{v}}{\partial t}, \mathbf{u} \right\rangle \right) dt. \tag{3.13}$$

And further observe that, if $\phi \in C([-h, 0]; H)$ *then* $\mathbf{u} \in C([\tau - h, T]; H)$.

We only sketch the proof of the next Theorem about the existence of weak solution of **(LMD)** since it is similar to the proof of Theorem 2.11.

Theorem 3.3. ***(Existence)*** *Let us consider* τ, T *with* $\tau < T$, $\mathbf{u}_\tau \in H$, $\phi \in L^2(-h, 0; H)$, $\mathbf{f} \in L^q(\tau, T; V_p^*)$, *and assume that*

$$\mathbf{g} : [\tau, T] \times C([-h, 0]; H) \to L^2(\Omega)^n,$$

satisfies hypotheses ***(3.I)-(3.IV)*** *with* $X = H$, $Y = L^2(\Omega)^n$.

Therefore, if $p \geqslant 1 + 2n/(n+2)$, *then there exists at least one weak solution of problem* ***(LMD)***.

Proof. Let $\{\mathbf{w}_j\}_{j=1}^{\infty}$ be the basis defined in (2.21), which is orthonormal in H and orthogonal in V^s with $s > n/2 + 1$.

- **Galerkin system and a priori estimates**

  Let us define $\mathbf{u}^m(t, x) = \sum_{r=1}^{m} \gamma_r^m(t) \mathbf{w}_r$, where the coefficients $\gamma_r^m(t)$ solve the so-called Galerkin system

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}^m(t), \mathbf{w}_j) + \langle \mathbb{T}(\mathbf{u}^m(t)), \mathbf{w}_j \rangle + \langle B(\mathbf{u}^m(t)), \mathbf{w}_j \rangle = \langle \mathbf{f}(t), \mathbf{w}_j \rangle \\ \qquad\qquad + (\mathbf{g}(t, \mathbf{u}_t^m), \mathbf{w}_j) \qquad\qquad \text{for } 1 \leqslant j \leqslant m, \\ \\ \mathbf{u}^m(\tau) = P^m \mathbf{u}^\tau \ \text{ and } \ \mathbf{u}^m(\tau + t) = P^m \phi(t) \ \text{ with } \ t \in (-h, 0). \end{cases} \tag{3.14}$$

  Here, P^m is the orthogonal continuous projector of H onto the linear hull of the first m eigenvectors $\mathbf{w}_j$, $j = 1, ..., m$, therefore

$$\begin{array}{l} P^m \mathbf{u}^\tau \to \mathbf{u}^\tau \ \text{ in } \ L^2(\Omega), \\ P^m \phi(t) \to \phi(t) \ \text{ in } \ L^2(-h, 0; H). \end{array} . \tag{3.15}$$

  Observe that (3.14) is a system of delay ordinary differential equations in the unknown $\gamma^m(t) = (\gamma_1^m, ..., \gamma_m^m)$. By Theorem 1.23 it has one solution defined on an interval $(\tau - h, t_m)$ with $\tau < t_m \leqslant T$. The priori estimates below us to show that $t_m = T$.

We multiply the jth equation of the Galerkin system (3.14) by $\gamma_j^m(t)$ and add the equations. The result can be written in the form

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}^m(t)|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u}^m)) : \mathbf{e}(\mathbf{u}^m)dx = \langle \mathbf{f}(t), \mathbf{u}^m\rangle + (\mathbf{g}(t, \mathbf{u}_t^m), \mathbf{u}^m), \tag{3.16}$$

because $\int_\Omega u_j^m \frac{\partial u_i^m}{\partial x_j} u_i^m dx = 0$.

Using the fact that $|P^m\mathbf{u}^\tau|_2 \leqslant |\mathbf{u}^\tau|_2$, $\int_{-h}^0 |P^m\phi(s)|_2^2 ds \leqslant \|\phi\|_{L_H^2}^2$ and the hypotheses on $\mathbf{g}$, it follows in the same way as in (2.25) that

$$\begin{aligned} &\{\mathbf{u}^m\}_{m=1}^\infty \text{ is bounded in } L^2(\tau - h, T; H),\\ &\{\mathbf{u}^m\}_{m=1}^\infty \text{ is bounded in } L^\infty(\tau, T; H),\\ &\{\mathbf{u}^m\}_{m=1}^\infty \text{ is bounded in } L^p(\tau, T; V_p). \end{aligned} \tag{3.17}$$

From (3.14), we deduce that $\left\{\frac{\partial \mathbf{u}^m}{\partial t}\right\}_{m=1}^\infty$ is bounded in $L^q(\tau, T; V_p^*)$.

- **Limiting process**

From (3.17), **(3.IV)**, the Alaoglu Theorem 1.5, the Aubin-Lions compactness result (Theorem 1.11) there follows, up to a subsequence, that

$$\begin{aligned} &\mathbf{u}^m \to \mathbf{u}, && \text{in } L^2(\tau - h, T; H),\\ &\mathbf{u}^m \overset{*}{\rightharpoonup} \mathbf{u}, && \text{in } L^\infty(\tau, T; H),\\ &\mathbf{u}^m \rightharpoonup \mathbf{u}, && \text{in } L^p(\tau, T; V_p), \text{ and a.e. in } \Omega_{\tau,T},\\ &\frac{\partial \mathbf{u}^m}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t}, && \text{in } L^q(\tau, T; V_p^*),\\ &\mathbb{T}(\mathbf{u}^m) \rightharpoonup \mathcal{X}, && \text{in } L^q(\tau, T; V_p^*),\\ &\mathbf{g}(t, \mathbf{u}_t^m) \to \mathbf{g}(t, \mathbf{u}_t), && \text{in } L^2(\tau, T; L^2(\Omega)^n). \end{aligned} \tag{3.18}$$

These convergences allow to pass to the limit in (3.14) and obtain

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle + (\mathbf{g}(t, \mathbf{u}_t), \mathbf{v}) \quad \forall \mathbf{v} \in V_p. \tag{3.19}$$

It is standard to prove that

$$\mathbf{u}(\tau) = \mathbf{u}_\tau \qquad \text{and} \qquad \mathbf{u}(\tau + t) = \phi(t) \text{ with } t \in (-h, 0).$$

To finish the proof we have to show that $\mathbb{T}(\mathbf{u}) = \mathcal{X}$. This is done exactly as in the proof of Theorem 2.11, noting that

$$\int_\tau^T (\mathbf{g}(t, \mathbf{u}_t^m), \mathbf{u}^m)dt \overset{m\to\infty}{\rightarrow} \int_\tau^T (\mathbf{g}(t, \mathbf{u}_t), \mathbf{u})dt.$$

Indeed, we have that

$$
\begin{aligned}
&|\int_\tau^T (\mathbf{g}(t,\mathbf{u}_t^m),\mathbf{u}^m)dt - \int_\tau^T (\mathbf{g}(t,\mathbf{u}_t),\mathbf{u})dt| \\
&\leqslant \int_\tau^T |(\mathbf{g}(t,\mathbf{u}_t^m) - \mathbf{g}(t,\mathbf{u}_t),\mathbf{u}^m)|dt + \int_\tau^T |(\mathbf{g}(t,\mathbf{u}_t),\mathbf{u}^m - \mathbf{u})|dt \\
&\leqslant \|\mathbf{g}(t,\mathbf{u}_t^m) - \mathbf{g}(t,\mathbf{u}_t)\|_{L^2(\tau,T;H)}\|\mathbf{u}^m\|_{L^2(\tau,T;H)} \\
&+ \|\mathbf{g}(t,\mathbf{u}_t)\|_{L^2(\tau,T;H)}\|\mathbf{u}^m - \mathbf{u})\|_{L^2(\tau,T;H)} \to 0
\end{aligned}
$$

as $m \to \infty$.

□

Lemma 3.4. ***(Energy Equality)*** *Under conditions of Theorem 3.3 and the hypotheses on* $\mathbf{g}$*, any function* $\mathbf{v}$ *in the class (3.9) can be taken as a test function in the weak formulation (3.10). Consequently, the energy equality*

$$
\frac{1}{2}\frac{d}{dt}|\mathbf{u}|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{u})dx = \langle \mathbf{f},\mathbf{u}\rangle + (\mathbf{g}(t,\mathbf{u}_t),\mathbf{u}), \tag{3.20}
$$

holds a.e. $t \in (\tau, T)$.

Proof. The proof is similar that proof of Lemma 2.12. □

The importance of the following Proposition will be clear in Section 3.4, about existence of pullback attractors, since it will allow us to show that the multi-valued processes, defined from weak solutions, are closed.

Proposition 3.5. *Let* $\{\mathbf{u}^m\}$ *be a sequence of weak solution to* ***(LMD)*** *associated to the initial conditions* $(\mathbf{u}^m(\tau),\phi^m)$*, such that* $\phi^m \to \phi$ *in* $L^2(-h,0;H)$ *and* $\mathbf{u}^m(\tau) \to \mathbf{u}^\tau$ *in* H*. Then,* $\{\mathbf{u}^m\}$ *is bounded in the spaces (3.9), and there exists* $\mathbf{u}$ *such that (up to a subsequence)* $\mathbf{u}^m \to \mathbf{u}$ *in the sense specified in (3.18), and* $\mathbf{u}$ *is again a weak solution to* ***(LMD)*** *associated to the initial condition* $(\mathbf{u}^\tau,\phi)$ *.*

Proof. The proof is similar that proof of Proposition 2.13. □

3.3 Uniqueness of Weak Solution to **(LMD)**

Let us denote by $M_H^2 = H \times L_H^2$ the Hilbert space whose elements are of the form $(\mathbf{v},\phi) \in M_H^2$, with $\mathbf{v} \in H$ and $\phi \in L_H^2 = L^2(-h,0;H)$, with norm defined by

$$
\|(\mathbf{v},\phi)\|_{M_H^2}^2 = |\mathbf{v}|_2^2 + \|\phi\|_{L_H^2}^2 \quad \text{for} \quad (\mathbf{v},\phi) \in M_H^2.
$$

Theorem 3.6. ***(Uniqueness)*** *Let us consider* $\mathbf{f} \in L^q(\tau, T; V_p^*)$, $\mathbf{g} : \mathbb{R} \times C_H \to L^2(\Omega)^n$ *satisfying* ***(3.I)-(3.IV)****, defined in (3.8). Given* $(\mathbf{v}^\tau, \phi_1)$ *and* $(\mathbf{u}^\tau, \phi_2)$ *in* M_H^2*, let us denote by* $\mathbf{v}(\cdot) = \mathbf{v}(\cdot; \tau, (\mathbf{v}^\tau, \phi_1))$ *and* $\mathbf{u}(\cdot) = \mathbf{u}(\cdot; \tau, (\mathbf{u}^\tau, \phi_2))$ *two weak solutions to* ***(LMD)****, with initial conditions* $(\mathbf{v}^\tau, \phi_1)$ *and* $(\mathbf{u}^\tau, \phi_2)$*, respectively. Then, there exist positive constants* K_1, K_2 *and* K_3 *such that*

(i) *If* $n = 2$*:*

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \left(K_1 + K_2 |\nabla \mathbf{u}(s)|_2^2\right) ds \right\}, \quad (3.21)$$

for all $\tau \leqslant t \leqslant T$*, where* $\widetilde{C}_{\mathbf{g}} = \max\{1, K_1\},$*.*

(ii) *If* $n = 3$ *and* $\mathbf{u}, \mathbf{v} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$*, then*

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\}, \quad (3.22)$$

for all $\tau \leqslant t \leqslant T$*.*

In particular, weak solutions to ***(LMD)*** *are unique.*

Proof. Setting $\mathbf{w} = \mathbf{v} - \mathbf{u}$, and by Lemma 3.4, using $\mathbf{w}$ as a test function in the weak formulation (3.10), we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + \langle \mathbb{T}(\mathbf{v}) - \mathbb{T}(\mathbf{u}), \mathbf{w} \rangle + \langle B(\mathbf{v}) - B(\mathbf{u}), \mathbf{w} \rangle = (\mathbf{g}(t, \mathbf{v}_t) - \mathbf{g}_2(t, \mathbf{u}_t), \mathbf{w}).$$

In the same way as in Theorem 2.14 and using (2.7), we thus have

$$\frac{d}{dt}|\mathbf{w}|_2^2 + 2c_2\nu_1 |\mathbf{e}(\mathbf{w})|_2^2 + 2c_2\nu_2 \|\mathbf{e}(\mathbf{w})\|_p^p \leqslant 2\int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| dx + 2(\mathbf{g}(t, \mathbf{v}_t) - \mathbf{g}(t, \mathbf{u}_t), \mathbf{w}).$$

- We consider first $n = 2$. Let $\varepsilon_1 > 0$ to be chosen. We use the Ladyzhenskaya inequality (1.4), to estimate

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| dx &\leqslant \|\mathbf{w}\|_{L^4(\Omega)}^2 |\nabla \mathbf{u}|_2 \\ &\leqslant \widehat{c} |\mathbf{w}|_2 |\nabla \mathbf{w}|_2 |\nabla \mathbf{u}|_2 \\ &\leqslant \frac{\varepsilon_1}{2} |\nabla \mathbf{w}|_2^2 + \frac{\widehat{c}^2}{2\varepsilon_1} |\nabla \mathbf{u}|_2^2 |\mathbf{w}|_2^2.\end{aligned}$$

For any $\varepsilon_2 > 0$ and from the previous estimate, we have

$$\begin{aligned}\frac{d}{dt}|\mathbf{w}|_2^2 + \left(\frac{2c_2\nu_1\lambda_1}{c_0^2} - \varepsilon_1\lambda_1 - \varepsilon_2\right)|\mathbf{w}|_2^2 + \frac{2c_2\nu_2}{\tilde{c}_0^p}\|\nabla w\|_p^p \leqslant{}& \frac{\widehat{c}^2}{\varepsilon_1}|\mathbf{w}|_2^2 |\nabla \mathbf{w}|_2^2 \\ &+ \frac{1}{\varepsilon_2}|\mathbf{g}_2(t, \mathbf{v}_t) - \mathbf{g}_2(t, \mathbf{u}_t)|_2^2.\end{aligned}$$

We choose $\varepsilon_1 = \dfrac{c_2\nu_1}{c_0^2}$ and $\varepsilon_2 = \dfrac{c_2\nu_1\lambda_1}{c_0^2}$. Integrating from τ to t, and hypothesis **(3.IV)**, there follows

$$|\mathbf{w}(t)|_2^2 \leqslant |\mathbf{w}(\tau)|_2^2 + K_2\int_\tau^t |\mathbf{w}(s)|_2^2|\nabla\mathbf{u}(s)|_2^2 ds + K_1\int_{\tau-h}^t |\mathbf{w}(s)|_2^2 ds,$$

where $K_1 = \dfrac{C_{\mathbf{g}}^2}{\varepsilon_2}$ and $K_2 = \dfrac{c_0^2\widehat{c}^2}{c_2\nu_1}$. Therefore, we have that

$$|\mathbf{w}(t)|_2^2 \leqslant |\mathbf{w}(\tau)|_2^2 + K_1\|\phi_1-\phi_2\|_{L_H^2}^2 + K_2\int_\tau^t |\mathbf{w}(s)|_2^2|\nabla\mathbf{u}(s)|_2^2 ds + K_1\int_\tau^t |\mathbf{w}(s)|_2^2 ds. \tag{3.23}$$

Applying the Gronwall inequality, we conclude that

$$|\mathbf{v}(t)-\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}}\|(\mathbf{v}^\tau,\phi_1)-(\mathbf{u}^\tau,\phi_2)\|_{M_H^2}^2 \exp\left\{\int_\tau^t \big(K_1 + K_2|\nabla\mathbf{u}(s)|_2^2\big)ds\right\}, \tag{3.24}$$

where $\widetilde{C}_{\mathbf{g}} = \max\{1, K_1\}$. Thus, **(i)** is proved.

- Now we consider $n = 3$. Applying the Hölder inequality we have

$$\int_\Omega |\mathbf{w}|^2|\nabla\mathbf{u}|dx \leqslant \|\mathbf{w}\|_{L^{\frac{2p}{p-1}}(\Omega)}^2\|\nabla\mathbf{u}\|_p. \tag{3.25}$$

Using the interpolation inequality (Theorem 1.17) for $r = \dfrac{2p}{p-1}$, $r_1 = 6$, $r_2 = 2$ and $\alpha = 3/2p$, we obtain

$$\begin{aligned}\|\mathbf{w}\|_{L^{\frac{2p}{p-1}}(\Omega)} &\leqslant |\mathbf{u}|_2^{\frac{2p-3}{2p}}\|\mathbf{u}\|_6^{\frac{3}{2p}}\\ &\leqslant d|\mathbf{u}|_2^{\frac{2p-3}{2p}}|\nabla\mathbf{u}|_2^{\frac{3}{2p}},\end{aligned}$$

since $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$.

From this it follows that

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2|\nabla\mathbf{u}|dx &\leqslant \|\mathbf{w}\|_{L^{\frac{2p}{p-1}}(\Omega)}^2\|\nabla\mathbf{u}\|_p\\ &\leqslant d|\mathbf{w}|_2^{\frac{2p-3}{p}}|\nabla\mathbf{w}|_2^{\frac{3}{p}}\|\nabla\mathbf{u}\|_p\\ &\leqslant \frac{\nu_1}{2c_0^2}|\nabla\mathbf{w}|_2^2 + K_3\|\nabla\mathbf{u}\|_p^{\frac{2p}{2p-3}}|\mathbf{w}|_2^2,\end{aligned}$$

where $K_3 = \dfrac{(2p-3)d^{\frac{2p}{2p-3}}}{2p\varepsilon^{\frac{2p}{2p-3}}}$, with $\varepsilon = \big(\dfrac{p\nu_1}{2c_0^2}\big)^{3/2p}$.

Thus we obtain

$$|\mathbf{w}(t)|_2^2 \leqslant |\mathbf{w}(\tau)|_2^2 + K_3\int_\tau^t \|\nabla\mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}|\mathbf{w}(s)|_2^2 ds + K_1\int_{\tau-h}^t |\mathbf{w}(s)|_2^2 ds. \tag{3.26}$$

We know that $\int_\tau^t \|\nabla \mathbf{u}(t)\|_p^{\frac{2p}{2p-3}} dt < \infty$. Then, applying Gronwall inequality, we conclude that

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_g \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\},$$

for all $\tau \leqslant t \leqslant T$.

□

3.4 Existence of Pullback Attractors in C_H and M_H^2

In this section we are interested in analysing the asymptotic behaviour of the weak solutions of the system **(LMD)** on the Banach spaces C_H and M_H^2. Since the proof of the results on C_H and M_H^2 are similar, both cases will be treated in the same statements.

An important point to take into account is that we consider the "minimum" hypotheses to obtain the existence of pullback attractors, that are:

(A) The tensor stress $\mathbb{S}$ satisfies (2.7) and $p \geqslant 1 + 2n/(n+2)$,

(B) $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$,

(C) $\mathbf{g} : [\tau, T] \times C([-h, 0]; H) \to L^2(\Omega)^n$ satisfying **(3.I)**-**(3.IV)**, given by (3.7), with $X = H$ and $Y = L^2(\Omega)^n$ with uniform constants $C_\mathbf{g}$ and $L_\mathbf{g}$ for any $-\infty < \tau \leqslant T < \infty$.

Under conditions **(A)**-**(C)**, Theorem 3.3 guarantees the existence of weak solutions for problem **(LMD)**, but not the uniqueness. Analogously to Section 2.5.1 (pullback attractor in H) let us define the sets: $\Phi_{C_H}(\tau; (\phi(0), \phi))$ as the set of all weak solutions to **(LMD)** defined on $[\tau - h, \infty)$ with initial condition $\phi \in C_H$ and by $\Phi_{M_H^2}(\tau; (\mathbf{u}^\tau, \phi))$ the set of all weak solutions to **(LMD)** defined (a.e.) on $(\tau - h, \infty)$ with initial condition $(\mathbf{u}^\tau, \phi) \in M_H^2$.

Now, let us define the bi-parametric families of mappings $U(\cdot,\cdot) : \mathbb{R}_d^2 \times C_H \to \mathcal{P}(C_H)$, and $S(\cdot,\cdot) : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$, given by

$$U(t,\tau)\phi = \left\{\mathbf{u}_t(\cdot; \tau, \phi(0), \phi) : \mathbf{u} \in \Phi_{C_H}(\tau; (\phi(0), \phi))\right\} \tag{3.27}$$

with $\phi \in C_H$ and $(t, \tau) \in \mathbb{R}_d^2$, and

$$S(t,\tau)(\mathbf{u}^\tau, \phi) = \left\{\left(\mathbf{u}(t; \tau, \mathbf{u}^\tau, \phi), \mathbf{u}_t(\cdot; \tau, \mathbf{u}^\tau, \phi)\right) : \mathbf{u} \in \Phi_{M_H^2}(\tau; (\mathbf{u}^\tau, \phi))\right\} \tag{3.28}$$

with $(\mathbf{u}^\tau, \phi) \in M_H^2$ and $(t, \tau) \in \mathbb{R}_d^2$.

Proposition 3.7. *Consider hypotheses **(A)**-**(C)**. Let* $\{\phi^m\} \subset C_H$ *and* $\phi \in C_H$ *be such that* $\phi^m \to \phi$ *in* C_H*. Then, for any sequence* $\{\mathbf{u}^m\}$*, where* $\mathbf{u}^m \in \Phi_{C_H}(\tau; (\phi^m(0), \phi^m))$*, for all* $m \in \mathbb{N}$*, there exist a subsequence of* $\{\mathbf{u}^m\}$ *(relabeled the same) and* $\mathbf{u} \in \Phi(\tau, (\phi(0), \phi))$*, such that*

$$\mathbf{u}_s^m \to \mathbf{u}_s \quad \text{strongly in } C_H \;\; \forall s \geqslant \tau. \tag{3.29}$$

Proof. The proof is similar to that of Proposition 2.20. But now we introduce the energy functionals

$$J(r) = |\mathbf{u}(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta - \tilde{C}r \tag{3.30}$$

and

$$J_m(r) = |\mathbf{u}^m(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}^m(\theta)\rangle d\theta - \tilde{C}r, \tag{3.31}$$

for all $r \geqslant \tau$ and some constant $\tilde{C} > 0$ given later in (3.52).

With this, we can prove that $\mathbf{u}^m(r) \to \mathbf{u}(r)$ strongly in H for all $r \geqslant \tau$. Since $\phi^m \to \phi$ in C_H, we conclude that, for each $s \geqslant \tau$ fixed and for any $\theta \in [-h, 0]$

$$\mathbf{u}_s^m(\theta) = \mathbf{u}^m(s+\theta) \to \mathbf{u}(s+\theta) = \mathbf{u}_s(\theta) \qquad \text{strongly in } H.$$

Thus, we conclude that $\mathbf{u}_s^m \to \mathbf{u}_s$ strongly in C_H, since $[-h, 0]$ is compact. □

Proposition 3.8. *Consider hypotheses **(A)**-**(C)**. Let* $\{(\mathbf{u}^{\tau m}, \phi^m)\} \subset M_H^2$ *and* $(\mathbf{u}^\tau, \phi) \in M_H^2$ *be such that* $(\mathbf{u}^{\tau m}, \phi^m) \to (\mathbf{u}^\tau, \phi)$ *in* M_H^2*. Then, for any sequence* $\{\mathbf{u}^m\}$*, where* $\mathbf{u}^m \in \Phi_{M_H^2}(\tau; (\mathbf{u}^{\tau m}, \phi^m))$*, for all* $m \in \mathbb{N}$*, there exist a subsequence of* $\{\mathbf{u}^m\}$ *(relabeled the same) and* $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$*, such that*

$$\begin{cases} \mathbf{u}^m(s) \to \mathbf{u}(s) & \text{strongly in } H, \\ \mathbf{u}_s^m \to \mathbf{u}_s & \text{strongly in } L_H^2, \end{cases} \tag{3.32}$$

for all $s \geqslant \tau$.

Proof. As in Proposition 3.7, we consider the energy functionals

$$J(r) = |\mathbf{u}(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta - \tilde{C}r, \tag{3.33}$$

and

$$J_m(r) = |\mathbf{u}^m(r)|^2 - 2\int_\tau^r \langle \mathbf{f}(\theta), \mathbf{u}^m(\theta)\rangle d\theta - \tilde{C}r, \tag{3.34}$$

for all $r \geqslant \tau$.

Therefore, as in the above Proposition we have that for all $s \geqslant \tau$, $\mathbf{u}^m(s) \to \mathbf{u}(s)$ strongly in H. Since $\phi^m \to \phi$ in L^2_H, we deduce that for each $s \geqslant \tau$ fixed

$$\begin{aligned}\int_{-h}^{0} |\mathbf{u}^m_s(r) - \mathbf{u}_s(r)|_2^2 dr &= \int_{s-h}^{s} |\mathbf{u}^m(r) - \mathbf{u}(r)|_2^2 dr \\ &\leqslant \int_{\tau-h}^{\tau} |\mathbf{u}^m(r) - \mathbf{u}(r)|_2^2 dr + \int_{\tau}^{s} |\mathbf{u}^m(r) - \mathbf{u}(r)|_2^2 dr \\ &= \|\phi^m - \phi\|^2_{L^2_H} + \int_{\tau}^{s} |\mathbf{u}^m(r) - \mathbf{u}(r)|_2^2 dr \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

We conclude that

$$\mathbf{u}^m_s \to \mathbf{u}_s \quad \text{strongly in } L^2_H.$$

□

Corollary 3.9. *Consider hypotheses **(A)-(C)**. Then, $U(\cdot,\cdot)$ is an upper-semicontinuous multi-valued process with closed values.*

Proof. It follows by Proposition 3.5 that $U(\cdot,\cdot)$ is a multi-valued process with closed values. We suppose that the multi-valued process $U(\cdot,\cdot)$ is not upper-semicontinuous. Therefore, there exist $(t,\tau) \in \mathbb{R}^2_d$, a neighbourhood $\mathcal{N}(U(t,\tau)\phi)$ and a sequence $\{\mathbf{z}^k\}$ which fulfils that each $\mathbf{z}^k \in U(t,\tau)\phi^k$, where $\phi^k \to \phi$ in C_H, but for all $k \in \mathbb{N}$ $\mathbf{z}^k \notin \mathcal{N}(U(t,\tau)\phi)$. Since each $\mathbf{z}^k \in U(t,\tau)\phi^k$, there exists $\mathbf{u}^k \in \Phi_{C_H}(\tau;(\phi^k(0),\phi^k))$ such that $\mathbf{z}^k = \mathbf{u}^k_t$. Now, applying Proposition 3.7, we deduce that there exists a subsequence of $\{\mathbf{u}^k_t\}$ (relabeled the same) which converges to a function $\mathbf{u}_t \in U(t,\tau)\phi$. This is contradictory because $\mathbf{z}^k \notin \mathcal{N}(U(t,\tau)\phi)$ for all $k \in \mathbb{N}$. □

Corollary 3.10. *Under the hypotheses of Theorem 3.3. Then, $S(\cdot,\cdot)$ is an upper-semicontinuous multi-valued process with closed values.*

Proof. It follows by Proposition 3.5 that $S(\cdot,\cdot)$ is a multi-valued process with closed values. We suppose that the multi-valued process $S(\cdot,\cdot)$ is not upper-semicontinuous. Therefore, there exist $(t,\tau) \in \mathbb{R}^2_d$, a neighbourhood $\mathcal{N}(S(t,\tau)(\mathbf{u}^\tau,\phi))$ and a sequence $\{\mathbf{z}^k\}$ which fulfils that each $\mathbf{z}^k \in S(t,\tau)(u^{\tau_k},\phi^k)$, where $(\mathbf{u}^{\tau_k},\phi^k) \to (\mathbf{u}^\tau,\phi)$ in M^2_H, but for all $k \in \mathbb{N}$ $\mathbf{z}^k \notin \mathcal{N}(S(t,\tau)(\mathbf{u}^\tau,\phi))$. Since each $\mathbf{z}^k \in S(t,\tau)(u^{\tau_k},\phi^k)$, there exists $\mathbf{u}^k \in \Phi_{M^2_H}(\tau;(\mathbf{u}^{\tau_k},\phi^k))$ such that $\mathbf{z}^k = (\mathbf{u}^k(t),\mathbf{u}^k_t)$. Now, applying Proposition 3.8, we deduce that there exists a subsequence of $\{(\mathbf{u}^k(t),\mathbf{u}^k_t)\}$ (relabeled the same) which converges to a pair $(\mathbf{u}(t),\mathbf{u}_t) \in S(t,\tau)(\mathbf{u}^\tau,\phi)$. This is contradictory because $\mathbf{z}^k \notin \mathcal{N}(S(t,\tau)(\mathbf{u}^\tau,\phi)$ for all $k \in \mathbb{N}$. □

In addition to hypotheses **(3.I)-(3.IV)** on the function $\mathbf{g} : [\tau,T]\times C([-h,0];H) \to L^2(\Omega)^n$, we also consider

(3.V) There exists a value $\eta > 0$ such that for all $\mathbf{u} \in L^2(\tau - h, t; H)$,

$$\int_\tau^t e^{\eta s}|\mathbf{g}(s, \mathbf{u}_s)|_2^2 ds \leqslant C_\mathbf{g}^2 \int_{\tau-h}^t e^{\eta s}|\mathbf{u}(s)|_2^2 ds.$$

Remark 3.11. *We denote by* $\overline{\eta} := c_2\nu_1\lambda_1 c_0^{-2}$*. Then,* η *satisfying hypothesis **(3.V)** can take the values* $0 < \eta \leqslant \overline{\eta}$ *or* $\eta > \overline{\eta}$*. As in the section 2.5, for* $p = 2$ *we assume hypothesis **(3.V)** for some* $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2}]$*.*

In the next Lemma we are going to suppose the existence of some $\eta > 0$ satisfying the hypothesis **(3.V)** and we will analyse in two cases, when $\eta \in (0, \overline{\eta}]$ and when $\eta \in (\overline{\eta}, +\infty)$.

Lemma 3.12. *Let* $p > 2$*. Assume that hypotheses **(A)-(C)** and **(3.I)-(3.V)**, for the delay term* $\mathbf{g}$*, are satisfied. Then, for any* $(\mathbf{u}^\tau, \phi) \in M_H^2$*, there exist positive constants,* $\widetilde{C}_{1,\mathbf{g}}$ *and* $\widehat{C}_1$*, such that any weak solution* $\mathbf{u} \in \Phi_{M_H^2}(\tau; (\mathbf{u}^\tau, \phi))$ *to **(LMD)** satisfies the following estimate for all* $\tau \leqslant t$*:*

$$|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{1,\mathbf{g}} e^{-(\eta - \frac{C_\mathbf{g}^2}{\eta})(t-\tau)}\|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + \int_\tau^t e^{-(\eta - \frac{C_\mathbf{g}^2}{\eta})(t-s)}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q)ds. \quad (3.35)$$

Proof. From energy equality (3.20) and (2.7), we deduce

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}(t)|_2^2 + c_2\nu_1|\mathbf{e}(\mathbf{u})|_2^2 + c_2\nu_2\|\mathbf{e}(\mathbf{u})\|_p^p \leqslant \langle \mathbf{f}(t), \mathbf{u}(t)\rangle + (\mathbf{g}(t, \mathbf{u}_t), \mathbf{u}).$$

From the Korn, the Young and the Poincaré inequalities we obtain

$$\begin{aligned} \frac{d}{dt}|\mathbf{u}(t)|_2^2 + 2\overline{\eta}|\mathbf{u}|_2^2 + 2\eta_2\|\nabla\mathbf{u}\|_p^p &\leqslant 2\|\mathbf{f}(t)\|_*\|\nabla\mathbf{u}\|_p + 2|\mathbf{g}(t, \mathbf{u}_t)|_2|\mathbf{u}|_2 \\ &\leqslant \frac{2^q}{\varepsilon_1^q q}\|\mathbf{f}(t)\|_*^q + \frac{\varepsilon_1^p}{p}\|\nabla\mathbf{u}\|_p^p + \frac{1}{\varepsilon_2}|\mathbf{g}(t, \mathbf{u}_t)|_2^2 + \varepsilon_2|\mathbf{u}|_2^2, \end{aligned}$$

where $\overline{\eta} = c_2\nu_1\lambda_1 c_0^{-2}$, $\eta_2 = \dfrac{c_2\nu_2}{\tilde{c}_0^p}$ and $\varepsilon_1, \varepsilon_2 > 0$ to be chosen.

Choosing $\varepsilon_1 = \eta_2^{1/p} p$ and $\varepsilon_2 = \overline{\eta}$, there follows

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + \overline{\eta}|\mathbf{u}|_2^2 + \eta_2\|\nabla\mathbf{u}\|_p^p \leqslant K_4\|\mathbf{f}(t)\|_*^q + \frac{1}{\overline{\eta}}|\mathbf{g}(t, \mathbf{u}_t)|_2^2, \quad (3.36)$$

where $K_4 = \dfrac{2^q}{\eta_2^{q/p} p^q q}$.

To finish the proof, we consider two cases: when $0 < \eta \leqslant \overline{\eta}$ and when is $\eta > \overline{\eta}$.

- Case $0 < \eta \leqslant \overline{\eta}$:

Multiplying (3.36) by $e^{\eta t}$, integrating from τ to t and, using hypothesis **(3.V)**, we have

$$e^{\eta t}|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{1,g} e^{\eta\tau}\|(\mathbf{u}^\tau,\phi)\|^2_{M_H^2} + \int_\tau^t e^{\eta s} K_4 \|\mathbf{f}(s)\|_*^q ds + \frac{C_{\mathbf{g}}^2}{\overline{\eta}}\int_\tau^t e^{\eta s}|\mathbf{u}(s)|_2^2 ds,$$

where $\widetilde{C}_{1,\mathbf{g}} = \max\{1, \frac{C_{\mathbf{g}}^2}{\overline{\eta}}\}$. Applying the Gronwall inequality, give us

$$|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{1,g} e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}})(t-\tau)}\|(\mathbf{u}^\tau,\phi)\|^2_{M_H^2} + K_4\int_\tau^t e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}})(t-s)}\|\mathbf{f}(s)\|_*^q ds.$$

- Case $\eta > \overline{\eta}$:

Denote $0 < \beta := \eta - \dfrac{c_2\nu_1\lambda_1}{c_0^2}$. Consider also C_I the constant of the embedding $W_0^{1,p}(\Omega)^n \subset L^2(\Omega)^n$, i.e. $|\mathbf{u}|_2 \leqslant C_I\|\nabla\mathbf{u}\|_p$. Then the Young inequality yields

$$|\mathbf{u}|_2^2 \leqslant \frac{\gamma^{p/2}}{p/2}\|\nabla\mathbf{u}\|_p^p + \frac{(p-2)C_I^{2p/(p-2)}}{p\gamma^{p/(p-2)}}.$$

Putting $\dfrac{\gamma^{p/2}}{p/2} = \dfrac{c_2\nu_2}{2\tilde{c}_0^p\beta}$ we gain

$$\beta|\mathbf{u}|_2^2 \leqslant \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla\mathbf{u}\|_p^p + \widehat{C}_1,$$

where $\widehat{C}_1 = \dfrac{(p-2)C_I^{2p/(p-2)}\beta}{p\gamma^{p/(p-2)}}$. Then (3.36) reduces to

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + \eta|\mathbf{u}|_2^2 + \frac{\eta_2}{2}\|\nabla\mathbf{u}\|_p^p \leqslant K_4\|\mathbf{f}(t)\|_*^q + \frac{1}{\overline{\eta}}|\mathbf{g}(t,\mathbf{u}_t)|_2^2 + \widehat{C}_1.$$

Multiplying by $e^{\eta t}$, integrating from τ to t, and using hypothesis **(3.V)** in (3.36), we have

$$e^{\eta t}|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{1,\mathbf{g}} e^{\eta\tau}\|(\mathbf{u}^\tau,\phi)\|^2_{M_H^2} + \int_\tau^t e^{\eta s}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q) ds + \frac{C_{\mathbf{g}}^2}{\overline{\eta}}\int_\tau^t e^{\eta s}|\mathbf{u}(s)|_2^2 ds,$$

where $\widetilde{C}_{1,\mathbf{g}} = \max\{1, \frac{C_{\mathbf{g}}^2}{\overline{\eta}}\}$. Applying the Gronwall inequality, we arrive at

$$|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{1,\mathbf{g}} e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}})(t-\tau)}\|(\mathbf{u}^\tau,\phi)\|^2_{M_H^2} + \int_\tau^t e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}})(t-s)}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q) ds \quad \forall \tau \leqslant t.$$

□

Remark 3.13. *For the case $p = 2$, we can assume that $c_2 = 1$ and $\nu_2 = 0$. With this, as in Lemma 3.12, for some $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2}]$ satisfying hypothesis **(3.V)**, we obtain*

$$|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{2,\mathbf{g}} e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}_2})(t-\tau)}\|(\mathbf{u}^\tau,\phi)\|^2_{M_H^2} + K_5\int_\tau^t e^{-(\eta-\frac{C_{\mathbf{g}}^2}{\overline{\eta}_2})(t-s)}\|\mathbf{f}(s)\|_*^2 ds, \tag{3.37}$$

where $\overline{\eta}_2 = 2\nu_1\lambda_1 c_0^{-2}$, $\widetilde{C}_{2,\mathbf{g}} = \max\{1, \frac{C_{\mathbf{g}}^2}{\overline{\eta}_2}\}$ and $K_5 > 0$.

Since in the asymptotic pullback analysis, we study the behavior of solutions when the initial time τ tends to $-\infty$ it is necessary, for the case $p > 2$, that $\eta > \frac{C_g^2}{\overline{\eta}}$, see (3.35). Besides that, in this case, there is no restrictions for constants $C_{\mathbf{g}}$ and $\overline{\eta}$. On the other hand, for the case $p = 2$, as we saw in Remark 3.13, we obtain an estimate for $|\mathbf{u}(t)|_2$ for $\eta \in (0, \overline{\eta}_2]$, see (3.37). Therefore, for the study of asymptotic pullback behavior in this case, we have to assume $\eta > \frac{C_{\mathbf{g}}}{\overline{\eta}_2}$ for some $\eta \in (0, \overline{\eta}_2]$, and this is possible whenever $\overline{\eta}_2 > C_{\mathbf{g}}$. To simplify the calculations, we will denote by $\sigma_\eta := \eta - \frac{C_g^2}{\overline{\eta}}$.

In the remainder of this section, we are going to consider the case $p > 2$ because the case $p = 2$ is analogous.

Lemma 3.14. *Let be $p > 2$. Assume that hypotheses **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \frac{C_{\mathbf{g}}^2}{\overline{\eta}}$ are satisfied. Then, given $t \in \mathbb{R}$ with $t \geqslant \tau$ we have:*

(i) *If $(\mathbf{u}^\tau, \phi) \in M_H^2$, any $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ satisfies the following estimate in L_H^2*

$$\|\mathbf{u}_t\|_{L_H^2}^2 \leqslant e^{\sigma_\eta h} \widetilde{C}_{2,\mathbf{g}} e^{-\sigma_\eta (t-\tau)} \|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + h e^{\sigma_\eta h} \int_\tau^t e^{-\sigma_\eta (t-s)} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|_*^q) ds, \tag{3.38}$$

where $\widetilde{C}_{2,\mathbf{g}} = 1 + h\widetilde{C}_{1,\mathbf{g}}$.

(ii) *If $\phi \in C_H$, any $\mathbf{u} \in \Phi_{C_H}(\tau; (\phi(0), \phi))$ satisfies the following estimate in C_H*

$$\|\mathbf{u}_t\|_{C_H}^2 \leqslant e^{-\sigma_\eta h} \widetilde{C}_{3,\mathbf{g}} e^{-\sigma_\eta (t-\tau)} \|\phi\|_{C_H}^2 + e^{\sigma_\eta h} \int_\tau^t e^{-\sigma_\eta (t-s)} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|_*^q) ds \tag{3.39}$$

where $\widetilde{C}_{3,\mathbf{g}} = 1 + (1+h)\widetilde{C}_{1,\mathbf{g}}$.

Proof. **(i)** From (3.35) for $\tau + h \leqslant t$, we obtain

$$|\mathbf{u}(t+r)|_2^2 \leqslant \widetilde{C}_{1,\mathbf{g}} e^{\sigma_\eta (t+r-\tau)} \|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + \int_\tau^{t+r} e^{-\sigma_\eta (t+r-s)} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|_*^q) ds,$$

for all $r \in (-h, 0)$.

Thus, integrating in r between $-h$ and 0, we have that

$$\|\mathbf{u}_t\|_{L_H^2}^2 \leqslant h\widetilde{C}_{1,\mathbf{g}} e^{\sigma_\eta h} e^{-\sigma_\eta (t-\tau)} \|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + h e^{\sigma_\eta h} \int_\tau^t e^{-\sigma_\eta (t-s)} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|_*^q) ds, \tag{3.40}$$

for all $t \geqslant \tau + h$.

Now, suppose that $\tau \leqslant t \leqslant \tau + h$. Then

$$\begin{aligned}
e^{\sigma_\eta(t-h)}\|\mathbf{u}_t\|^2_{L^2_H} dr &\leqslant \int_{-h}^{0} e^{\sigma_\eta(t+r)}|\mathbf{u}(t+r)|_2^2 dr = \int_{t-h}^{t} e^{\sigma_\eta r}|\mathbf{u}(r)|_2^2 dr \\
&= \int_{t-h}^{\tau} e^{\sigma_\eta r}|\mathbf{u}(r)|_2^2 dr + \int_{\tau}^{t} e^{\sigma_\eta r}|\mathbf{u}(r)|dr \\
&\overset{(3.35)}{\leqslant} e^{\sigma_\eta \tau}\|\phi\|^2_{L^2_H} + h\widetilde{C}_{1,\mathbf{g}} e^{\sigma_\eta \tau}\|(\mathbf{u}^\tau,\phi)\|^2_{M^2_H} + h\int_\tau^t e^{\sigma_\eta s}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q)ds.
\end{aligned}$$

Therefore, from both estimates above we conclude that

$$\|\mathbf{u}_t\|^2_{L^2_H} \leqslant e^{\sigma_\eta h}\widetilde{C}_{2,\mathbf{g}} e^{-\sigma_\eta(t-\tau)}\|(\mathbf{u}^\tau,\phi)\|^2_{M^2_H} + he^{\sigma_\eta h}\int_\tau^t e^{-\sigma_\eta(t-s)}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q)ds,$$

for all $\tau \leqslant t$, where $\widetilde{C}_{2,\mathbf{g}} = 1 + h\widetilde{C}_{1,\mathbf{g}}$.

(ii) Let be $t \geqslant \tau$ and $s \in [-h, 0]$, and suppose that $t + s \geqslant \tau$. Then from (3.35) we have

$$\begin{aligned}
|\mathbf{u}(t+s)|_2^2 &\leqslant \widetilde{C}_{1,\mathbf{g}} e^{-\sigma_\eta(t+s-\tau)}\|(\mathbf{u}^\tau,\phi)\|^2_{M^2_H} + \int_\tau^{t+s} e^{-\sigma_\eta(t+s-\theta)}(\widehat{C}_1 + K_4\|\mathbf{f}(\theta)\|_*^q)d\theta \\
&\leqslant (1+h)e^{-\sigma_\eta h}\widetilde{C}_{1,\mathbf{g}} e^{-\sigma_\eta(t-\tau)}\|\phi\|^2_{C_H} + e^{\sigma_\eta h}\int_\tau^t e^{-\sigma_\eta(t-\theta)}(\widehat{C}_1 + K_4\|\mathbf{f}(\theta)\|_*^q)d\theta.
\end{aligned}$$

Now, suppose that $t + s \leqslant \tau$, and observe that

$$e^{\sigma_\eta(t+h)}|\mathbf{u}(t+s)|_2^2 \leqslant e^{\sigma_\eta(t+s)}|\mathbf{u}(t+s)|_2^2 \leqslant e^{\sigma_\eta \tau}\|\phi\|^2_{C_H}.$$

Therefore, from last inequalities we conclude that

$$\|\mathbf{u}_t\|^2_{C_H} \leqslant e^{-\sigma_\eta h}\widetilde{C}_{3,\mathbf{g}} e^{-\sigma_\eta(t-\tau)}\|\phi\|^2_{C_H} + e^{\sigma_\eta h}\int_\tau^t e^{-\sigma_\eta(t-\theta)}(\widehat{C}_1 + K_4\|\mathbf{f}(\theta)\|_*^q)d\theta$$

where $\widetilde{C}_{3,\mathbf{g}} := 1 + (1+h)\widetilde{C}_{1,\mathbf{g}}$.

□

Definition 3.15. ***(Universe in*** C_H***)*** *Given $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that*

$$\lim_{\tau \to -\infty}\left(e^{\sigma\tau}\sup_{\mathbf{v}\in D(\tau)}\|\mathbf{v}\|^2_{C_H}\right) = 0.$$

And we will denote by $\mathcal{D}_F(C_H)$ the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H.

Definition 3.16. ***(Universe in*** M^2_H***)*** *Given $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(M^2_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M^2_H)$ such that*

$$\lim_{\tau \to -\infty}\left(e^{\sigma\tau}\sup_{(\mathbf{v},\phi)\in D(\tau)}\|(\mathbf{v},\phi)\|^2_{M^2_H}\right) = 0.$$

And we will denote by $\mathcal{D}_F(M^2_H)$ the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of M^2_H.

In the following, we will assume that there exists $\eta > \dfrac{C_g^2}{\overline{\eta}}$ satisfying the hypothesis **(3.V)** and

$$\int_{-\infty}^{0} e^{\sigma_\eta s} \|\mathbf{f}(s)\|_*^q ds < \infty, \tag{3.41}$$

or in other words $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\eta}$.

Remark 3.17. *Observing that $\mathbf{f} \in L^q_{loc}(\mathbb{R}, V_p^*)$, assumption (3.41) is equivalent to*

$$\int_{-\infty}^{t} e^{\sigma_\eta s} \|\mathbf{f}(s)\|_*^q ds < \infty \quad \forall t \in \mathbb{R}.$$

Corollary 3.18. *Let be $p > 2$. Under the hypotheses **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \dfrac{C_\mathbf{g}^2}{\overline{\eta}}$ are satisfied and assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies (3.41). Then, the families $\widehat{B}_{\sigma_\eta, C_H} = \{B_{\sigma_\eta, C_H}(t) : t \in \mathbb{R}\}$ with $B_{\sigma_\eta, C_H}(t) = \overline{B}_{C_H}(0, \mathcal{R}(t))$, and $\widehat{B}_{\sigma_\eta, M_H^2} = \{B_{\sigma_\eta, M_H^2}(t) : t \in \mathbb{R}\}$ with $B_{\sigma_\eta, M_H^2}(t) = \overline{B}_{M_H^2}(0, \mathcal{R}(t))$, are pullback $\mathcal{D}_{\sigma_\eta}(C_H)$ -absorbing for the process $U(\cdot,\cdot)$ and pullback $\mathcal{D}_{\sigma_\eta}(M_H^2)$ -absorbing for the process $S(\cdot,\cdot)$, respectively, where*

$$\mathcal{R}^2(t) = 1 + (1+h)e^{\sigma_\eta h} \int_{-\infty}^{t} e^{-\sigma_\eta (t-\theta)} (\widehat{C}_1 + K_4 \|\mathbf{f}(\theta)\|_*^q) d\theta. \tag{3.42}$$

Proof. It follows directly from Lemmas 3.12 and 3.14. □

Lemma 3.19. *Let be $p > 2$. Under the hypotheses **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \dfrac{C_\mathbf{g}^2}{\overline{\eta}}$ are satisfied and assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies (3.41). Then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_{\sigma_\eta}(C_H)$ ($\widehat{D} \in \mathcal{D}_{\sigma_\eta}(M_H^2)$) there exists $\tau_1(\widehat{D}, t, h) < t - h - 2$ such that for all $\tau \leqslant \tau_1(\widehat{D}, t, h)$ and any $\phi^\tau \in D(\tau)$ ($(\mathbf{u}^\tau, \phi^\tau) \in D(\tau)$) it holds*

$$\begin{aligned} |\mathbf{u}(r; \tau, (\phi^\tau(0), \phi^\tau))|_2 &\leqslant \varrho_1(t) \quad \forall r \in [t-h-2, t], \\ \int_{r-1}^{r} \|\nabla \mathbf{u}(\theta; \tau, (\phi^\tau(0), \phi^\tau))\|_p^p d\theta &\leqslant \varrho_2(t) \quad \forall r \in [t-h-1, t], \end{aligned} \tag{3.43}$$

where

$$\varrho_1^2(t) = 1 + e^{-\sigma_\eta (t-2h-2)} \int_{-\infty}^{t} e^{\sigma_\eta s} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|_*^q) ds, \tag{3.44}$$

$$\varrho_2(t) = (\frac{1}{\eta_2} + \frac{L_g^2}{\overline{\eta}\eta_2}) \varrho_1^2(t) + \frac{K_4}{\eta_2} \int_{t-h-2}^{t} \|\mathbf{f}(s)\|_*^q ds. \tag{3.45}$$

Proof. From Lemma 3.12, choose $\tau_1(\widehat{D}, t) < t - h - 2$ such that

$$\widetilde{C}_{1,g} e^{-\sigma_\eta (t-\tau)} \|(\phi^\tau(0), \phi^\tau)\|_{M_H^2}^2 < 1,$$

for all $\phi^\tau \in D(\tau)$, $((\mathbf{u}^\tau, \phi^\tau) \in D(\tau))$, with $\tau \leqslant \tau_1(\widehat{D}, t)$. Thus, we obtain the first estimate.

Now, observe that $\mathbf{u} \in C([t-h-1,t];H)$. Therefore, integrating between $r-1$ and r (3.36), with $r \in (t-h-1,t)$ and using hypothesis **(3.III)**, we have

$$\eta_2 \int_{r-1}^{r} \|\nabla \mathbf{u}(s)\|_p^p ds \leqslant |\mathbf{u}(r-1)|_2^2 + K_4 \int_{r-1}^{r} \|\mathbf{f}(s)\|_*^q ds + \frac{L_g^2}{\overline{\eta}} \int_{r-1}^{r} |\mathbf{u}_s|_{C_H}^2 ds.$$

Therefore, from (3.44), we conclude

$$\int_{r-1}^{r} \|\nabla \mathbf{u}(s)\|_p^p ds \leqslant (\frac{1}{\eta_2} + \frac{L_g^2}{\overline{\eta}\eta_2})\varrho_1^2(t) + \frac{K_4}{\eta_2} \int_{t-h-2}^{t} \|\mathbf{f}(s)\|_*^q ds,$$

for all $r \in [t-h-1,t]$, $\tau \leqslant \tau_1(\widehat{D},t)$ and $\phi^\tau \in D(\tau)$ (for all $(\mathbf{u}^\tau,\phi^\tau) \in D(\tau)$). □

Theorem 3.20. *Let be $p > 2$. Under the hypothesis **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \dfrac{C_{\mathbf{g}}^2}{\overline{\eta}}$ are satisfied and assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R};V_p^*)$ satisfies (3.41). Then, the processes $U(\cdot,\cdot)$ and $S(\cdot,\cdot)$ are pullback $\widehat{B}_{\sigma_\eta,C_H}$-asymptotically compact and pullback $\widehat{B}_{\sigma_\eta,M_H^2}$-asymptotically compact respectively.*

Proof. Given $t_0 \in \mathbb{R}$, and any sequence $\{\phi^m\}$ with $\phi^m \in B_{\sigma_\eta,C_H}(\tau_m)$ (or $(\mathbf{u}^{\tau_m},\phi^m) \in B_{\sigma_\eta,M_H^2}(\tau_m)$), where $\tau_m \to -\infty$ as $m \to \infty$. Denote by $\mathbf{u}^m = \mathbf{u}^m(\cdot;\tau_m,\phi^m(0),\phi^m)$ any sequence of weak solutions to **(LMD)** with $\mathbf{u}^m \in \Phi_{C_H}(\tau^m;(\phi^m(0),\phi))$ (and denote by $\mathbf{u}^m = \mathbf{u}^m(\cdot;\tau_m,\mathbf{u}^{\tau_m},\phi^m)$ any sequence of weak solution to **(LMD)** with $\mathbf{u}^m \in \Phi_{M_H^2}(\tau^m;(\mathbf{u}^{\tau_m},\phi))$). We will prove that the sequence $\{\mathbf{u}^m_{t_0}\}$ is relatively compact in C_H, $(\{(\mathbf{u}^m(t_0),\mathbf{u}^m_{t_0})\}$ is relatively compact in M_H^2).

It follows from Lemma 3.19 that there exists $m_0(t_0,h)$ such that

$$|\mathbf{u}^m(r)|_2 \leqslant \varrho_1(t_0) \quad \forall r \in [t_0-h-2,t_0]\ \forall m \geqslant m_0(t_0,h), \tag{3.46}$$

$$\int_{r-1}^{r} \|\mathbf{u}^m(s)\|_{1,p}^p ds \leqslant \varrho_2(t_0) \quad \forall r \in [t_0-h-1,t_0]\ \forall m \geqslant m_0(t_0,h). \tag{3.47}$$

On the other hand, for each $m \geqslant m_0(t_0,h)$, the function $\mathbf{u}^m$ is a weak solution to **(LMD)** on $[t_0-h-1,t_0]$. Thus, from (3.46) and (3.47), there exist a subsequence (relabeled the same) and a function $\mathbf{u}$ such that

$$\begin{aligned}
&\mathbf{u}^m \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{weakly-}* \text{ in } L^\infty(t_0-h-1,t_0;H),\\
&\mathbf{u}^m \rightharpoonup \mathbf{u} \quad \text{weakly in } L^p(t_0-h-1,t_0;V_p),\\
&\frac{\partial \mathbf{u}^m}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } L^q(t_0-h-1,t_0;V_p^*),\\
&\mathbf{u}^m \to \mathbf{u} \quad \text{strongly in } L^2(t_0-h-1,t_0;H),\\
&\mathbf{u}^m(t) \to \mathbf{u}(t) \quad \text{a.e. } t \in (t_0-h-1,t_0) \text{ in } H.
\end{aligned} \tag{3.48}$$

Observe also that $\mathbf{u} \in C([t_0-h-1,t_0];H)$, and that for any sequence $\{t_m\} \in [t_0-h-1,t_0]$ with $t_m \to t^*$, one has

$$\mathbf{u}^m(t_m) \to \mathbf{u}(t^*) \quad \text{in } V_p^*. \tag{3.49}$$

Moreover, by **(3.III)** and Lemma 3.19, we obtain

$$\int_{t_0-h-1}^{t_0} |\mathbf{g}(s, \mathbf{u}_s^m)|_2^2 ds \leqslant L_{\mathbf{g}}^2 (h+1)\varrho_2(t_0).$$

Thus, eventually extracting a subsequence, there exists $\xi \in L^2(t_0 - h - 1, t_0; L^2(\Omega)^n)$ such that

$$\mathbf{g}(\cdot, \mathbf{u}_\cdot^m) \rightharpoonup \xi \quad \text{weakly in } \; L^2(t_0 - h - 1, t_0; L^2(\Omega)^n),$$

and observe that, again by **(3.III)** and Lemma 3.19, we get

$$\begin{aligned} &\int_s^t |\mathbf{g}(s, \mathbf{u}^m(s))|_2^2 ds \leqslant C(t-s), \\ &\int_s^t |\xi(r)|_2^2 dr \leqslant \liminf_{m\to\infty} \int_s^t |\mathbf{g}(s, \mathbf{u}^m(s))|_2^2 ds \leqslant C(t-s), \end{aligned} \tag{3.50}$$

for all $t_0 - h - 1 \leqslant s \leqslant t \leqslant t_0$, where $C = L_{\mathbf{g}}^2 \varrho_2(t_0)$. Then, in a standard way, one can prove that $\mathbf{u}(\cdot)$ is a weak solution to the problem

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \operatorname{div}_x(\mathbb{S}(\mathbf{e}(\mathbf{v}))) + \operatorname{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi = \mathbf{f}(t) + \xi(t) \quad \text{in } \; (t_0-h-1, t_0)\times\Omega, \\ \operatorname{div}_x \mathbf{v} = 0 \quad \text{in } \; (t_0-h-1, t_0)\times\Omega, \\ \mathbf{v} = 0 \quad \text{on } \; (t_0-h-1, t_0)\times\partial\Omega, \\ \mathbf{v}(t_0-h-1, x) = \mathbf{u}(t_0-h-1, x), \quad x\in\Omega. \end{cases} \tag{3.51}$$

By the energy equality, Lemma 3.4, and (3.50), we obtain

$$\frac{1}{2}|\mathbf{z}(t)|_2^2 \leqslant \frac{1}{2}|\mathbf{z}(s)|_2^2 + \int_s^t \langle \mathbf{f}(r), \mathbf{z}(r)\rangle ds + \tilde{C}(t-s), \tag{3.52}$$

for all $t_0 - h - 1 \leqslant s \leqslant t \leqslant t_0$, where $\tilde{C} = C\overline{\eta}^{-1}$, and $\mathbf{z} = \mathbf{u}^m$ or $\mathbf{z} = \mathbf{u}$.

Then, the maps $J_m, J : [t_0 - h - 1, t_0] \to \mathbb{R}$ defined by

$$\begin{aligned} J_m(t) &= \frac{1}{2}|\mathbf{u}^m(t)|_2^2 - \int_{t_0-h-1}^t \langle \mathbf{f}(r), \mathbf{u}^m(r)\rangle dr - \tilde{C}t \\ J(t) &= \frac{1}{2}|\mathbf{u}(t)|_2^2 - \int_{t_0-h-1}^t \langle \mathbf{f}(r), \mathbf{u}(r)\rangle dr - \tilde{C}t, \end{aligned}$$

are non-increasing and continuous, and satisfy

$$J_m(t) \to J(t) \quad \text{a.e. } \; t\in(t_0-h-1, t_0). \tag{3.53}$$

We can use the functionals J_m and J to deduce that $\mathbf{u}^m \to \mathbf{u}$ in $C([t_0-h, t_0]; H)$. If this is not, then there exist $\varepsilon^* > 0$, and subsequences $\{\mathbf{u}^m\} \subset \{\mathbf{u}^m\}_{m\geqslant m_0(t_0,h)}$ and $\{t_m\} \subset [t_0 - h, t_0]$, with $t_m \to t^*$, such that

$$|\mathbf{u}^m(t_m) - \mathbf{u}(t^*))|_2 \geqslant \varepsilon^* \quad \forall m. \tag{3.54}$$

Let us fix $\varepsilon > 0$. Observe that $t^* \in [t_0 - h, t_0]$ and therefore, by (3.53) and the continuity and non-increasing character of J, there exists $t_0 - h - 1 < \hat{t}_\varepsilon < t^*$ such that

$$\lim_{m\to\infty} J_m(\hat{t}_\varepsilon) = J(\hat{t}_\varepsilon), \tag{3.55}$$

and

$$0 \leqslant J(\hat{t}_\varepsilon) - J(t^*) \leqslant \varepsilon. \tag{3.56}$$

Since $t_m \to t^*$, there exists m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m \geqslant m_\varepsilon$. Then, by (3.56)

$$\begin{aligned} J_m(t_m) - J(t^*) &\leqslant J_m(\hat{t}_\varepsilon) - J(t^*) \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)|_2 + |J(\hat{t}_\varepsilon) - J(t^*)|_2 \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)|_2 + \varepsilon \end{aligned}$$

for all $m \geqslant m_\varepsilon$, and consequently, by (3.55), $\limsup_{m\to\infty} J_m(t_m) \leqslant J(t^*) + \varepsilon$. Thus, as $\varepsilon > 0$ is arbitrary, we deduce that

$$\limsup_{m\to\infty} J_m(t_m) \leqslant J(t^*). \tag{3.57}$$

Taking into account that $t_m \to t^*$ and

$$\int_{t_0-h-1}^{t_m} \langle \mathbf{f}(r), \mathbf{u}^m(r)\rangle dr \to \int_{t_0-h-1}^{t^*} \langle \mathbf{f}(r), \mathbf{u}(r)\rangle dr,$$

from (3.57) we deduce that $\limsup_{m\to\infty} |\mathbf{u}^m(t_m)|_2 \leqslant |\mathbf{u}(t^*)|_2$. This last inequality and (3.49), imply that $\mathbf{u}^m(t_m) \to \mathbf{u}(t^*)$ strongly in H, which is contradiction with (3.54).

We have thus prove that $\mathbf{u}^m \to \mathbf{u}$ in $C([t_0-h, t_0]; H)$ and we obtain in particular that:

$$\begin{cases} \mathbf{u}^m_{t_0} \to \mathbf{u}_{t_0} \quad \text{in } C([-h,0]; H), \\ \mathbf{u}^m(t_0) \to \mathbf{u}(t_0) \quad \text{in } H, \\ \mathbf{u}^m_{t_0} \to \mathbf{u}_{t_0} \quad \text{in } L^2(-h,0; H), \end{cases} \tag{3.58}$$

This finishes the proof. □

Theorem 3.21. *Let be $p > 2$. Under the hypotheses **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \dfrac{C_{\mathbf{g}}^2}{\eta}$ are satisfied and assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies (3.41). Then, there exist the minimal pullback $\mathcal{D}_F(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H)} = \{\mathcal{A}_{\mathcal{D}_F(C_H)}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)} = \{\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) : t \in \mathbb{R}\}$ for the multi-valued process $U : \mathbb{R}^2_d \times C_H \to \mathcal{P}(C_H)$. The minimal pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-attractor belongs to $\mathcal{D}_{\sigma_\eta}(C_H)$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \subset \overline{B}_{C_H}(0, \mathcal{R}(t)) \quad \forall t \in \mathbb{R}. \tag{3.59}$$

Proof. The existence of pullback attractors for the multi-valued process $U(\cdot,\cdot)$ in the universes $\mathcal{D}_{\sigma_\eta}(C_H)$ and $\mathcal{D}_F(C_H)$ follows from Theorem 1.42. The inclusions (3.59) are given by the Corollary 1.44. □

Theorem 3.22. *Let be $p > 2$. Under the hypotheses **(A)-(C)** and **(3.I)-(3.V)** for some $\eta > \dfrac{C_{\mathbf{g}}^2}{\eta}$ are satisfied and assume that $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies (3.41). Then, there exist the minimal pullback $\mathcal{D}_F(M_H^2)$-attractor $\mathcal{A}_{\mathcal{D}_F(M_H^2)} = \{\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_{\sigma_\eta}(M_H^2)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)} = \{\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t) : t \in \mathbb{R}\}$ for the multi-valued process $S : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$. The minimal pullback $\mathcal{D}_{\sigma_\eta}(M_H^2)$-attractor belongs to $\mathcal{D}_{\sigma_\eta}(M_H^2)$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t) \subset \overline{B}_{M_H^2}(0, \mathcal{R}(t)) \quad \forall t \in \mathbb{R}. \tag{3.60}$$

Proof. The existence of pullback attractors for the multi-valued process $S(\cdot,\cdot)$ in the universes $\mathcal{D}_{\sigma\eta}(M_H^2)$ and $\mathcal{D}_F(M_H^2)$ follows from Theorem 1.42. The inclusions (3.60) are given by the Corollary 1.44. □

Remark 3.23. *If $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies that*

$$\sup_{s \leqslant 0} \left(e^{-\sigma_\eta s} \int_{-\infty}^{s} e^{\sigma_\eta r} \|\mathbf{f}(r)\|_*^q dr \right) < \infty, \tag{3.61}$$

we guarantee that for all $T \in \mathbb{R}$, $\bigcup_{t \leqslant T} \overline{B}_{C_H}(0, \mathcal{R}(t))$ and $\bigcup_{t \leqslant T} \overline{B}_{M_H^2}(0, \mathcal{R}(t))$ are bounded subsets of C_H and M_H^2 respectively (see estimate (3.42)) since

$$\sup_{s \leqslant 0} \left(e^{-\sigma_\eta s} \int_{-\infty}^{s} e^{\sigma_\eta r} \left(\widehat{C}_1 + \|\mathbf{f}(r)\|_*^q\right) dr \right) < \infty. \tag{3.62}$$

Therefore, from Remark 1.45 we obtain that

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \quad \text{and} \quad \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t). \tag{3.63}$$

Remark 3.24. *Observe that, if $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ satisfies that $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\eta}$ for some $\eta > 0$, then $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\mu}$ for all $\mu \in (\eta, +\infty)$, and besides that, $\mathcal{D}_{\sigma_\eta}(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H)$ and $\mathcal{D}_{\sigma_\eta}(M_H^2) \subset \mathcal{D}_{\sigma_\mu}(M_H^2)$. Thus, for all $\mu \in (\eta, +\infty)$ there exists the corresponding minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H)$-attractor, $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$.*

Since $\mathcal{D}_{\sigma_\eta}(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H)$ and $\mathcal{D}_{\sigma_\eta}(M_H^2) \subset \mathcal{D}_{\sigma_\mu}(M_H^2)$, there follow from Theorem 1.46 that, for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)$ and $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(M_H^2)}(t)$ for all $\mu \in (\eta, +\infty)$.

Moreover, if $\mathbf{f}$ satisfies (3.61), then

$$\sup_{s \leqslant 0} \left(e^{-\sigma_\mu s} \int_{-\infty}^{s} e^{\sigma_\mu r} \|\mathbf{f}(r)\|_*^q dr \right) < \infty \quad \text{for all } \mu \in (\eta, +\infty),$$

therefore, by (3.63) we get

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \textit{for all } \mu \in (\eta, +\infty).$$

and

$$\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(M_H^2)}(t) \quad \textit{for all } \mu \in (\eta, +\infty).$$

Remark 3.25. *We consider the canonical injection $j : C_H \to M_H^2$ defined by $j(\phi) = (\phi(0), \phi)$. Thus, we can identify*

$$j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \quad \textit{and} \quad j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(M_H^2)}(t),$$

for all $t \in \mathbb{R}$.

It is simple to show that $j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t)) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(M_H^2)}(t)$. Then, to get the other inclusion we use the fact that, $j \in \mathcal{L}(C_H, M_H^2)$ with $\|j\|_{\mathcal{L}(C_H,M_H^2)} \leqslant (1+h)^{1/2}$. Then, for $\tau \leqslant t-h$

$$\begin{aligned}
&dist_{M_H^2}\big(S(t,\tau)D(\tau), j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t))\big)\\
&= dist_{M_H^2}\big(S(t,\tau+h)\big(S(\tau+h,\tau)D(\tau)\big), j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t))\big)\\
&= dist_{M_H^2}\big(S(t,\tau+h)\big(j(\pi_{L_H^2}(S(\tau+h,\tau)D(\tau))\big), j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t))\big)\\
&= dist_{M_H^2}\big(S(t,\tau+h)\big(j(D^h(\tau))\big), j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t))\big)\\
&= dist_{M_H^2}\big(j\big(U(t,\tau+h)(D^h(\tau))\big), j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t))\big)\\
&\leqslant (1+h)^{1/2} dist_{C_H}\big(U(t,\tau+h)(D^h(\tau)), \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t)\big),
\end{aligned}$$

where $\pi_{L_H^2} : M_H^2 \to L_H^2$ with $\pi_{L_H^2}(\mathbf{u}, \phi) = \phi$, for all $(\mathbf{u}, \phi) \in M_H^2$, and $D^h(s) = \pi_{L_H^2}(S(s+h,s)D(s))$ for all $s \in \mathbb{R}$. Observe that $\widehat{D}^h \in \mathcal{D}_{\sigma_\eta}(C_H)$. Then, since $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}$ is pull-back $\mathcal{D}_{\sigma_\eta}(C_H)$-attracting, from previous inequality we obtain that $j(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)})$ is pullback $\mathcal{D}_{\sigma_\eta}(M_H^2)$-attracting in $\mathcal{D}_{\sigma_\eta}(M_H^2)$. Thus, since $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t)$ is the minimal closed set that pullback attracts any family $\widehat{D} \in \mathcal{D}_{\sigma_\eta}(M_H^2)$, we conclude that $j\big(\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t)\big) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(M_H^2)}(t)$ for all $t \in \mathbb{R}$.

3.5 Continuity of the Processes $U(\cdot,\cdot)$ and $S(\cdot,\cdot)$

Remember that in Section 3.4, about the assumptions given in Theorem 3.3, we define the process $U(\cdot,\cdot) : \mathbb{R}_d^2 \times C_H \to \mathcal{P}(C_H)$, and $S(\cdot,\cdot) : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$, given by

$$U(t,\tau)\phi = \big\{\mathbf{u}_t(\cdot;\tau,\phi(0),\phi) : \mathbf{u} \in \Phi_{C_H}(\tau;(\phi(0),\phi))\big\}$$

with $\phi \in C_H$ and $(t,\tau) \in \mathbb{R}_d^2$, and

$$S(t,\tau)(\mathbf{u}^\tau,\phi) = \big\{\big(\mathbf{u}(t;\tau,\mathbf{u}^\tau,\phi), \mathbf{u}_t(\cdot;\tau,\mathbf{u}^\tau,\phi)\big) : \mathbf{u} \in \Phi_{M_H^2}(\tau;(\mathbf{u}^\tau,\phi))\big\}$$

with $(\mathbf{u}^\tau, \phi) \in M_H^2$ and $(t, \tau) \in \mathbb{R}_d^2$.

Observe that under assumptions given in Theorem 3.6, the process $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ are uni-valued, i.e.: the processes are reduced to $U(\cdot, \cdot) : \mathbb{R}_d^2 \times C_H \to C_H$, and $S(\cdot, \cdot) : \mathbb{R}_d^2 \times M_H^2 \to M_H^2$, given by

$$U(t, \tau)\phi = \mathbf{u}_t(\cdot; \tau, \phi(0), \phi) \tag{3.64}$$

with $\phi \in C_H$ and $(t, \tau) \in \mathbb{R}_d^2$, and

$$S(t, \tau)(\mathbf{u}^\tau, \phi) = \big(\mathbf{u}(t; \tau, \mathbf{u}^\tau, \phi), \mathbf{u}_t(\cdot; \tau, \mathbf{u}^\tau, \phi)\big) \tag{3.65}$$

with $(\mathbf{u}^\tau, \phi) \in M_H^2$ and $(t, \tau) \in \mathbb{R}_d^2$.

The following results show that the processes defined in (3.64) and (3.65) are continuous processes.

Lemma 3.26. *Let us consider $\mathbf{f} \in L^q(\tau, T; V_p^*)$, and $\mathbf{g} : \mathbb{R} \times C_H \to L^2(\Omega)^n$ satisfying **(3.I)-(3.IV)** defined in (3.8). Given $(\mathbf{v}^\tau, \phi_1)$ and $(\mathbf{u}^\tau, \phi_2)$ in M_H^2, let us denote by $\mathbf{v}(\cdot) = \mathbf{v}(\cdot; \tau, (\mathbf{v}^\tau, \phi_1))$ and $\mathbf{u}(\cdot) = \mathbf{u}(\cdot; \tau, (\mathbf{u}^\tau, \phi_2))$ two weak solutions to **(LMD)**, with initial conditions $(\mathbf{v}^\tau, \phi_1)$ and $(\mathbf{u}^\tau, \phi_2)$, respectively. Then*

(i) *If $n = 2$:*

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + k_2 |\nabla \mathbf{u}(s)|_2^2\big) ds \right\}, \tag{3.66}$$

for all $\tau \leqslant t \leqslant T$, where $\widetilde{C}_{\mathbf{g}} = \max\{1, \dfrac{c_0^2 C_{\mathbf{g}}}{\nu_1 \lambda_1}\}$, $K_1 = \dfrac{c_0^2 C_{\mathbf{g}}}{\nu_1 \lambda_1}$ and $k_2 = \dfrac{c_0^2 \hat{c}}{\nu_1}$. Therefore, it also holds

$$\|\mathbf{v}_t - \mathbf{u}_t\|_{C_H}^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_2 |\nabla \mathbf{u}(s)|_2^2\big) ds \right\}, \tag{3.67}$$

for all $\tau + h \leqslant t \leqslant T$.

(ii) *If $n = 3$ and $\mathbf{u}, \mathbf{v} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$, then*

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}, \tag{3.68}$$

for all $\tau \leqslant t \leqslant T$, where $K_3 = \dfrac{(2p-3) d^{\frac{2p}{2p-3}}}{2p \varepsilon^{\frac{2p}{2p-3}}}$, with $\varepsilon = \big(\dfrac{p\nu_1}{2c_0^2}\big)^{3/2p}$. Therefore, it also holds

$$\|\mathbf{v}_t - \mathbf{u}_t\|_{C_H}^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}, \tag{3.69}$$

for all $\tau + h \leqslant t \leqslant T$.

Proof. For **(i)**, inequalities (3.66) and (3.68) are consequence of Theorem 3.6.

Now, if $t \geqslant \tau + h$ then for any $s \in (-h, 0)$ we have that $t + s \geqslant \tau$. thus by (3.66) for $t + s$, we obtain that

$$|\mathbf{v}(t+s) - \mathbf{u}(t+s)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^{t+s} \left(K_1 + K_2 |\nabla \mathbf{u}(\theta)|_2^2\right) d\theta \right\}$$
$$\leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^{t} \left(K_1 + K_2 |\nabla \mathbf{u}(\theta)|_2^2\right) d\theta \right\},$$

since $t \geqslant t + s \geqslant \tau$. Due to $\mathbf{u}, \mathbf{v} \in C([\tau, T]; H)$, we can take the maximum on $s \in [-h, 0]$ and we conclude that

$$\|\mathbf{v}_t - \mathbf{u}_t\|_{C_H}^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi_1) - (\mathbf{u}^\tau, \phi_2)\|_{M_H^2}^2 \exp\left\{ \int_\tau^{t} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\},$$

for all $\tau + h \leqslant t \leqslant T$. Thus, (3.67) is proved.

In the same way we demonstrate case **(ii)**. □

Theorem 3.27. *Under the hypotheses of Lemma 3.26, the applications $U(t, \tau) : C_H \to C_H$ and $S(t, \tau) : M_H^2 \to M_H^2$ are continuous for each $(t, \tau) \in \mathbb{R}_d^2$.*

Proof. We prove this Theorem for $n = 3$, since for $n = 2$ is analogous.

From the uniqueness of solution, we obtain that $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ are processes. To prove the continuity of both processes, we will use the Lemma 3.26.

- $U(t, \tau) : C_H \to C_H$ is continuous for each $(t, \tau) \in \mathbb{R}_d^2$:

Let be $\psi, \phi \in C_H$, and consider the solutions $\mathbf{u}(\cdot) = \mathbf{u}(\cdot; \tau, (\psi(0), \psi))$, $\mathbf{v}(\cdot) = \mathbf{v}(\cdot; \tau, (\phi(0), \phi))$ to **(LMD)**. We deduce from (3.68) that

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_g \|(\phi(0), \phi) - (\psi(0), \psi)\|_{M_H^2}^2 \exp\left\{ \int_\tau^{t} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\}$$
$$= \widetilde{C}_g \left(|\phi(0) - \psi(0)|_2^2 + \|\phi - \psi\|_{L_H^2}^2\right) \exp\left\{ \int_\tau^{t} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\}$$
$$\leqslant \widetilde{C}_g \left((1+h)\|\phi - \psi\|_{C_H}^2\right) \exp\left\{ \int_\tau^{t} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\} \quad t \geqslant \tau.$$

For $s \in [-h, 0]$ and $t \geqslant \tau$, we consider two cases. The first case is when $t + s \geqslant \tau$, and the second case is when $s + t \leqslant \tau$. Then, for the first case, we use the inequality above

$$|\mathbf{v}(t+s) - \mathbf{u}(t+s)|_2^2 \leqslant \widetilde{C}_g \left((1+h)\|\phi - \psi\|_{C_H}^2\right) \exp\left\{ \int_\tau^{t+s} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\}$$
$$\leqslant \widetilde{C}_g \left((1+h)\|\phi - \psi\|_{C_H}^2\right) \exp\left\{ \int_\tau^{t} \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\}.$$

For the second case ($s + t \leqslant \tau$), it is direct to see

$$
\begin{aligned}
|\mathbf{v}(t+s) - \mathbf{u}(t+s)|_2^2 &= |\psi(s+t-\tau) - \phi(s+t-\tau)|_2^2 \\
&\leqslant \|\psi - \phi\|_{C_H}^2 \\
&\leqslant \widetilde{C}_g\big((1+h)\|\phi-\psi\|_{C_H}^2\big) \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}.
\end{aligned}
$$

Thus, it follows from the inequalities above that, for all $\tau \leqslant t$ and $s \in [-h, 0]$

$$
|\mathbf{v}(t+s) - \mathbf{u}(t+s)|_2^2 \leqslant \widetilde{C}_g\big((1+h)\|\phi-\psi\|_{C_H}^2\big) \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}.
$$

By applying the maximum in s, we obtain

$$
\|\mathbf{v}_t - \mathbf{u}_t\|_{C_H}^2 \leqslant \widetilde{C}_g\big((1+h)\|\phi-\psi\|_{C_H}^2\big) \exp\left\{ \int_{\tau-h}^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}, \quad t \geqslant \tau.
$$

Therefore, we conclude that $U(t,\tau)$ is continuous for all $\tau \leqslant t$.

- $S(t,\tau) : M_H^2 \to M_H^2$ is continuous for each $(t,\tau) \in \mathbb{R}_d^2$:

Let $(\mathbf{u}^\tau, \psi), (\mathbf{v}^\tau, \phi) \in M_H^2$, and consider the solutions $\mathbf{u}(\cdot) = \mathbf{u}(\cdot;\tau,(\mathbf{u}^\tau,\psi))$, $\mathbf{v}(\cdot) = \mathbf{v}(\cdot;\tau,(\mathbf{v}^\tau,\phi))$ to **(LMD)**. We deduce from (3.69) that, for $\tau + h \leqslant t$

$$
\begin{aligned}
\|\mathbf{v}_t - \mathbf{u}_t\|_{L_H^2}^2 &= \int_{-h}^0 |\mathbf{v}(t+\theta) - \mathbf{u}(t+\theta)|_2^2 d\theta \\
&\leqslant h\widetilde{C}_g \|(\mathbf{v}^\tau,\psi) - (\mathbf{u}^\tau,\phi)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}.
\end{aligned}
$$

On the other hand, if $\tau \leqslant t \leqslant \tau + h$ we deduce

$$
\begin{aligned}
&\|\mathbf{v}_t - \mathbf{u}_t\|_{L_H^2}^2 = \int_{-h}^0 |\mathbf{v}(t+\theta) - \mathbf{u}(t+\theta)|_2^2 d\theta = \int_{t-h}^t |\mathbf{v}(r) - \mathbf{u}(r)|_2^2 dr \\
&= \int_{t-h}^\tau |\mathbf{v}(r) - \mathbf{u}(r)|_2^2 dr + \int_\tau^t |\mathbf{v}(r) - \mathbf{u}(r)|_2^2 dr \leqslant \int_{-h}^0 |\psi(r) - \phi(r)|_2^2 dr + \int_\tau^t |\mathbf{v}(r) - \mathbf{u}(r)|_2^2 dr \\
&\leqslant \|\psi - \phi\|_{C_H}^2 + \int_\tau^t \widetilde{C}_g \|(\mathbf{w}^\tau,\psi) - (\mathbf{u}^\tau,\phi)\|_{M_H^2}^2 \exp\left\{ \int_\tau^r \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\} dr \\
&\leqslant \|\psi - \phi\|_{C_H}^2 + h\widetilde{C}_g \|(\mathbf{w}^\tau,\psi) - (\mathbf{u}^\tau,\phi)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\}.
\end{aligned}
$$

Thus, we have for all $\tau \leqslant t$

$$
\begin{aligned}
\|\mathbf{v}_t - \mathbf{u}_t\|_{L_H^2}^2 &\leqslant \|\psi - \phi\|_{C_H} + h\widetilde{C}_g \|(\mathbf{w}^\tau,\psi) - (\mathbf{u}^\tau,\phi)\|_{M_H^2}^2 \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\} \\
&\leqslant \big((h\widetilde{C}_g + 1)\|\psi - \phi\|_{C_H}^2 + h|\mathbf{v}^\tau - \mathbf{u}^\tau|_2^2\big) \exp\left\{ \int_\tau^t \big(K_1 + K_3\|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\big) ds \right\},
\end{aligned}
$$

and the continuity of $S(t,\tau)$ follows immediately from (3.22) and this inequality. □

Remark 3.28. *The results established in this section can be seen as more general results given in section 3.3, about the uniqueness of weak solution to **(LMD)**. In summary we have shown continuity in relation to the initial conditions.*

Remark 3.29. *As last observation to end this section, the existence of pullback attractors in the Banach space $C_H = C([-h, 0]; H)$ and the Hilbert space $M_H^2 = H \times L^2(-h, 0; H)$ with $h > 0$, established in Theorem 3.21 and Theorem 3.22, is also valid for continuous processes $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ (see books [17, 45, 54]).*

4 Final comments and future proposals

In this section we start by collecting, in a didactic way, all the results shown in this work. The symbol ✓ indicates that it has been proved and χ nothing is affirmed.

Tables 1 and 2 summarize the results established in Sections 2.2, 2.3 and 2.4 on the existence, uniqueness and regularity of weak solution to **(LM)**.

Table 1 – Existence, uniqueness and regularity of weak solutions to **(LM)** for $n = 2$

$n = 2$	values of p	
Solution features	$p = 2$	$(2, +\infty)$
Existence	✓	✓
Uniqueness	✓	✓
Higher regularity	χ	✓

Table 2 – Existence, uniqueness and regularity of weak solutions to **(LM)** for $n = 3$

$n = 3$	values of p			
Solution features	$[2, 11/5)$	$[11/5, 12/5)$	$[12/5, 5/2)$	$[5/2, \infty)$
Existence	χ	✓	✓	✓
Uniqueness	χ	χ	χ	✓
Higher regularity	χ	χ	✓	✓

Tables 3 and 4 give a summary of the results established in Subsections 2.5.1 and 2.5.2 on the existence of pullback attractors in H and V_p.

Table 3 – Existence of pullback attractors to **(LM)** for $n = 2$

$n = 2$	values of p	
Pullback Attractor	$p = 2$	$(2, +\infty)$
H	✓	✓
V_p	χ	✓

Table 4 – Existence of pullback attractors to **(LM)** for $n = 3$

$n = 3$	values of p			
Pullback Attractor	$[2, 11/5)$	$[11/5, 12/5)$	$[12/5, 5/2)$	$[5/2, \infty)$
H	χ	✓	✓	✓
V_p	χ	χ	χ	✓

Remark 4.1. *A relevant fact is that it seems feasible to show existence of pullback attractors in V_p for the multi-valued case, i.e., when $p \geqslant 1 + 2n/(n+2)$. The main obstacle in this case was the upper-semicontinuity of the multi-valued process. This is an entirely*

topological problem, but the viability is suggested, since the weak solutions are weakly continuous on V_p.

Tables 5 and 6 furnish a summary of the results established in Sections 3.2 and 3.3 on the existence and uniqueness of weak solution to **(LMD)**.

Table 5 – Existence and uniqueness of weak solution to **(LMD)** for $n = 2$

$n = 2$	values of p
Solution features	$[2, +\infty)$
Existence	✓
Uniqueness	✓

Table 6 – Existence and uniqueness of weak solution to **(LMD)** for $n = 3$

$n = 3$	values of p			
Solution features	$[2, 11/5)$	$[11/5, 12/5)$	$[12/5, 5/2)$	$[5/2, \infty)$
Existence	χ	✓	✓	✓
Uniqueness	χ	χ	χ	✓

Tables 7 and 8 compile the results established in Section 3.4 on the existence of pullback attractors in $C_H = C([-h, 0]; H)$, $M_H^2 = H \times L^2(-h, 0; H)$ and $M_{V_p}^p = V_p \times L^p(-h, 0; V_p)$, with $h > 0$.

Table 7 – Existence of pullback attractors to **(LMD)** for $n = 2$

$n = 2$	values of p
Pullback Attractor	$[2, +\infty)$
C_H and M_H^2	✓
$M_{V_p}^p$	χ

Table 8 – Existence of pullback attractors to **(LMD)** for $n = 3$

$n = 3$	values of p			
Pullback Attractor	$[2, 11/5)$	$[11/5, 12/5)$	$[12/5, 5/2)$	$[5/2, \infty)$
C_H and M_H^2	χ	✓	✓	✓
$M_{V_p}^p$	χ	χ	χ	χ

Although, in all this work we have never mentioned the Banach space $M_{V_p}^p$, it is conveniently introduced in this chapter in order to formulate new problems.

Remark 4.2. *As in the case of regularity study of the pullback attractors on V_p for the system **(LM)**, we can think of an equivalent analysis for the system with delay **(LMD)**. From the existence of pullback attractors in M_H^2 build and analyze the existence of pullback attractors in $M_{V_p}^p$.*

We finish this section commenting on some future proposals. But before that, let us make a brief introduction about fractal dimension.

The geometry of the attractors for autonomous and non-autonomous systems can be complex and difficult to describe. Therefore, it is useful to have quantitative characterizations of such geometric objects. Maybe the most basic characterization of this type is the dimension of the attractor.

The treatment of this concept is necessarily abstract, since the application of the results generally makes use of certain properties of differentiability and which need to be carefully reviewed in each particular application.

There are many possible definitions of dimension, but in the field of dynamic systems the most used are the Hausdorff dimension and the Fractal dimension (also called box-counting dimension).

We will focus on the fractal dimension for two reasons: it always provides an upper limit to the Hausdorff dimension, and it is known that any set with a finite fractal dimension can be introduced into a finite dimensional Euclidean space using a linear application, whose inverse is Hölder continuous. This result is not true if we assume that the Hausdorff dimension is finite; see [17, pg. 72].

The fractal dimension of a compact subset K of a metric space X is defined by

$$dim_B(K) = \limsup_{r \to 0} \frac{\log N_X(K,r)}{-\log r},$$

where $N_X(K,r)$ is the minimum number of balls of radius r, centered at some point K, which covers K.

With this short introduction on the concept of fractal dimension for compact subsets in metric spaces, and with the analysis made in this work, it is natural to raise and discuss the following problems:

- To show results of regularity of the pullback attractors of H and V_p for other powers p for the system **(LM)** and analyse their fractal dimensions.
- To show higher regularity of different families of pullback attractors on C_H, M_H^2 and $M_{V_p}^p$ for the system **(LMD)** and analyse their fractal dimensions.
- To obtain results for tempered behavior of pullback attractors in both cases.

Bibliography

[1] ARINO, O., HBID, M., AND AIT DADS, E., Eds. *Delay Differential Equations and Applications*, vol. 205. Springer, Dordrecht, 2002. Citado na página 17.

[2] BABIN, A. V., AND VISHIK, M. I. *Attractors of Evolutions Equations*, vol. 25. North-Holland Amsterdam, London. New York. Tokyo, 1992. Citado 3 vezes nas páginas 20, 26, and 31.

[3] BALACHANDRAN, B., KALMÁR-NAGY, T., AND GILSINN, D., Eds. *Delay Differential Equations: Recent Advances and New Directions*, 1 ed. Springer US, 2009. Citado 2 vezes nas páginas 17 and 18.

[4] BIRD, R., ARMSTRONG, R., AND HASSAGER, O. *Principles of Non-Newtonian Fluid Mechanics*. Wiley, New York, 1987. Citado na página 36.

[5] BIRKHOFF, G. D. *Dynamical Systems*. American Mathematical Society, 1966. Citado na página 12.

[6] BOYER, F., AND FABRIE, P. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, 1 ed., vol. 183 of *Appl. Math. Sci.* Springer-Verlag New York, 2013. Citado na página 26.

[7] BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer-Verlag New York, USA, 2010. Citado na página 20.

[8] CARABALLO, T., HERRERA-COBOS, M., AND MARÍN-RUBIO, P. Robustness of nonautonomous attractors for a family of nonlocal reaction-diffusion equations without uniqueness. *Nonlinear Dyn. 84* (2016), 35–50. Citado 3 vezes nas páginas 31, 35, and 36.

[9] CARABALLO, T., HERRERA-COBOS, M., AND MARÍN-RUBIO, P. Asymptotic behaviour of nonlocal p-Laplacian reaction–diffusion problems. *J. Math. Anal. Appl. 459*, 2 (2018), 997 – 1015. Citado na página 36.

[10] CARABALLO, T., ŁUKASZEWICZ, G., AND REAL, J. Pullback attractors for asymptotically compact non-autonomous dynamical systems. *Nonlinear Anal. 64*, 3 (2006), 484 – 498. Citado na página 15.

[11] CARABALLO, T., ŁUKASZEWICZ, G., AND REAL, J. Pullback attractors for non-autonomous 2D-Navier–Stokes equations in some unbounded domains. *Comptes Rendus Mathematique 342*, 4 (2006), 263 – 268. Citado na página 15.

[12] Caraballo, T., Marín-Rubio, P., and Robinson, J. C. A comparison between two theories for multi-valued semiflows and their asymptotic behaviour. *Set-Valued Analysis* (2003), 297–322. Citado na página 31.

[13] Caraballo, T., and Real, J. Navier-Stokes equations with delays. *Proc. Math. Phys. Eng. Sci. 457* (2001), 2441–2453. Citado 3 vezes nas páginas 18, 31, and 83.

[14] Caraballo, T., and Real, J. Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays. *Proc. Math. Phys. Eng. Sci. 459* (2003), 3181–3194. Citado na página 18.

[15] Caraballo, T., and Real, J. Attractors for 2D Navier-Stokes models with delays. *J. Differential Equations 205* (2004), 271–297. Citado na página 18.

[16] Carlson, J., Jaffe, A., and Wiles, A. *The Millennium Prize Problems*. Clay Math. Inst., Massachusetts, 2006. Citado na página 14.

[17] Carvalho, A. N., Langa, J. A., and Robinson, J. C. *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, 1 ed., vol. 182 of *Applied Mathematical Sciences*. Springer-Verlag New York, 2013. Citado 6 vezes nas páginas 13, 15, 31, 34, 105, and 108.

[18] Cheng-Ku, Z., Mei-Hua, Y., and Chun-You, S. The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction–diffusion equations. *J. Differential Equations 223* (2006), 367–399. Citado na página 31.

[19] Cheskidov, A., and Foias, C. On global attractors of the 3D Navier-Stokes equations. *J. Differential Equations 231* (2006), 714–754. Citado na página 15.

[20] Cioranescu, D., Girault, V., and Rajagopal, K. *Mechanics and Mathematics of Fluids of the Differential Type*, 1 ed., vol. 35 of *Advances in Mechanics and Mathematics*. Springer International Publishing, 2016. Citado na página 36.

[21] Du, Q., and Gunzburger, M. D. Analysis of a Ladyzhenskaya model for incompressible viscous flow. *J. Math. Anal. Appl. 155*, 1 (1991), 21 – 45. Citado na página 15.

[22] Evans, L. C. *Partial Differential Equations*. American Mathematical Society, Providence, R.I., 2010. Citado 2 vezes nas páginas 20 and 25.

[23] Feireisl, E., and Pražák, D. *Asymptotic Behavior of Dynamical Systems in Fluid Mechanics*, vol. 4. AIMS Ser. Appl. Math., Springfield, USA, 2010. Citado 10 vezes nas páginas 14, 15, 16, 19, 20, 27, 36, 44, 55, and 57.

[24] Foias, C., Manley, O., Rosa, R., and Temam, R. *Navier-Stokes Equations and Turbulence*, vol. 83. Encyclopedia Math. Appl., Cambrige University Press, Cambrige, 2001. Citado na página 15.

[25] Friedman, A. *Partial Differential Equations.* Dover Publications, Mineola, New York, 2008. Citado 2 vezes nas páginas 20 and 27.

[26] Galdi, G., and Nečas, J., Eds. *Recent Developments in Theoretical Fluid Mechanics: Winter School, Paseky, 1992.* Chapman & Hall/CRC. Taylor & Francis, 1993. Citado na página 41.

[27] García-Luengo, J., Marín-Rubio, P., and Real, J. H^2-boundedness of the pullback attractors for non-autonomous 2D Navier-Stokes equations in bounded domains. *Nonlinear Anal. 74*, 14 (2011), 4882–4887. Citado na página 15.

[28] García-Luengo, J., Marín-Rubio, P., and Real, J. Pullback attractors in V for non-autonomous 2D-Navier-Stokes equations and their tempered behaviour. *J. Differential Equations 252*, 8 (2012), 4333–4356. Citado 4 vezes nas páginas 15, 26, 31, and 33.

[29] García-Luengo, J., Marín-Rubio, P., and Real, J. Pullback attractors for 2D Navier-Stokes equations with delays and their regularity. *Adv. Nonlinear Stud. 13* (2013), 331–357. Citado na página 18.

[30] García-Luengo, J., Marín-Rubio, P., and Real, J. Some new regularity results of pullback attractors for 2D Navier-Stokes equations with delays. *Commun. Pure Appl. Anal. 14*, 5 (2015), 1603–1621. Citado 3 vezes nas páginas 18, 35, and 36.

[31] García-Luengo, J., Marín-Rubio, P., Real, J., and Robinson, J. C. Pullback attractors for the non-autonomous 2D Navier-Stokes equations for minimally regular forcing. *Discrete Contin. Dyn. Syst. 34*, 1 (2014), 203–227. Citado na página 15.

[32] Haitao, S., and Hongqing, W. Pullback attractors of nonautonomous reaction–diffusion equations. *J. Math. Anal. Appl. 325* (2007), 1200–1215. Citado na página 31.

[33] Harvill, L. R. Vito Volterra and viscoelasticity. *J. Franklin Inst. 282*, 2 (1966), 85 – 91. Citado na página 18.

[34] Kapustyan, A. V., Melnik, V. S., Valero, J., and Yasinsky, V. V. *Global Attractors of Multivalued Dynamical Systems and Evolution Equations without Uniqueness.* Kyiv, Naukova Dumka, 2008. Citado na página 13.

[35] Kloeden, P., and Rasmussen, M. *Nonautonomous Dynamical Systems*, vol. 176. Amer. Math. Soc. Colloq. Publ., 2011. Citado na página 13.

[36] Kolmanovskii, V., and Myshkis, A. *Introduction to the Theory and Applications of Functional Differential Equations (Mathematics and Its Applications)*, 1 ed. Mathematics and Its Applications volume 463. Springer, 1999. Citado na página 17.

[37] Ladyzhenskaya, O. A. New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems. *Proc. Steklov Inst. Math. 102* (1967), 95–118. Citado na página 15.

[38] Ladyzhenskaya, O. A. On modifications of the Navier-Stokes equations for large gradients of the velocities (Russian). English translation in: Boundary value problems of mathematical physics and related aspects of function theory, part ii, pp. 57-69, consultants bureau, New York, 1970. *Nauka, Leningrad. Otdel 7* (1968), 126–154. Citado na página 15.

[39] Ladyzhenskaya, O. A. *The Mathematical Theory of Viscous Incompressible Flow*, vol. 25. Gordon and Breach, New York, 1969. Citado 4 vezes nas páginas 15, 36, 41, and 44.

[40] Ladyzhenskaya, O. A. *Mathematical Problems in the Dynamics of Viscous Incompressible Fluids.* Uspekhi Mat. Nauk, Moscow, 1970. Citado 2 vezes nas páginas 36 and 44.

[41] Ladyzhenskaya, O. A., and Serëgin, G. On semigroups generated by initial-boundary value problems describing two-dimensional visco-plastic flows. *American Mathematical Society Transl. 164*, 2 (1995), 99–123. Citado 2 vezes nas páginas 15 and 44.

[42] Lions, J. L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires.* Dunod, Paris, 1969. Citado 6 vezes nas páginas 14, 15, 19, 20, 49, and 57.

[43] Liu, L., Caraballo, T., and Fu, X. Exponential stability of an incompressible non-Newtonian fluid with delay. *Discrete Contin. Dyn. Syst. 23*, 10 (2018), 4285–4303. Citado na página 18.

[44] Málek, J., and Nečas, J. A finite-dimensional attractor for three-dimensional flow of incompressible fluids. *J. Differential Equations 127* (1996), 498–518. Citado 2 vezes nas páginas 16 and 46.

[45] Málek, J., Nečas, J., Rokyta, M., and Ružička, M. *Weak and Measure-Valued Solutions to Evolutionary PDEs.* Chapman & Hall/CRC, London, 1996. Citado 16 vezes nas páginas 14, 15, 16, 20, 24, 25, 26, 27, 28, 29, 36, 37, 40, 41, 49, and 105.

[46] Málek, J., and Pražák, D. Finite fractal dimension of the global attractor for a class of non-Newtonian fluids. *Appl. Math. Lett. 13* (2000), 105–110. Citado na página 16.

[47] Marín-Rubio, P., and Real, J. On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems. *Nonlinear Anal. 71* (2009), 3956–3963. Citado na página 35.

[48] Marín-Rubio, P., and Robinson, J. C. Attractors for the three-dimensional stochastic Navier-Stokes equations. *Stoch. Dyn. 3* (2003), 279–297. Citado na página 15.

[49] Melnik, V. S., and Valero, J. On attractors of multi-valued semi-flows and differential inclusions. *Set-Valued Var. Anal. 6* (1998), 83–111. Citado na página 13.

[50] Munkres, J. *Topology*, 2 ed. Prentice Hall, 2000. Citado 2 vezes nas páginas 20 and 24.

[51] Picard, E. La mathématique dans ses rapports avec la physique. *Il Nuovo Cimento (1901-1910) 16*, 1 (Dec 1908), 165–183. Citado na página 17.

[52] Poincaré, H. *Les Méthodes Nouvelles de la Mécanica Céleste*. Gauthier-Villars et fils, Paris, 1892. Citado na página 12.

[53] Pražák, D., and Žabenský, J. On the dimension of the attractor for a perturbed 3D Ladyzhenskaya model. *Cent. Eur. J. Math. 11*, 7 (2013), 1264–1282. Citado na página 16.

[54] Robinson, J. C. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge Texts Appl. Math., 2001. Citado 3 vezes nas páginas 13, 14, and 105.

[55] Rosa, R. The global attractor for the 2D Navier-Stokes flow on some unbounded domains. *Nonlinear Anal. 32* (1998), 71–85. Citado na página 15.

[56] Temam, R. *Infinite-Dimensional Dynamical Systems*, 1 ed. Applied Mathematical Sciences. Springer, 1997. Citado 2 vezes nas páginas 13 and 14.

[57] Vishik, M. I., and Chepyzhov, V. V. Trajectory and global attractors of three-dimensional Navier–Stokes systems. *Math. Notes 2* (2002), 177–193. Citado na página 15.

[58] Volterra, V. *Theory of Functionals and of Integral and Integro-Differential Equations*. Blackie & Son, Ltd., 1930. Citado 2 vezes nas páginas 17 and 18.

[59] Yang, M., Sun, C., and Zhong, C. Global attractors for p-Laplacian equation. *J. Math Anal. Appl. 327* (2007), 1130–1142. Citado na página 77.

[60] Yang, X.-G., Feng, B., Wang, S., Lu, Y., and Ma, T. F. Pullback dynamics of 3D Navier-Stokes equations with nonlinear viscosity. *Nonlinear Anal. Real World Appl. 48* (2019), 337–361. Citado 2 vezes nas páginas 16 and 42.

[61] Yongjun, L., and Chengkui, Z. Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction–diffusion equations. *Appl. Math. Comput. 190* (2007), 1020–1029. Citado na página 31.

[62] Zhao, C., Liu, G., and An, R. Global well-posedness and pullback attractors for an incompressible non-Newtonian fluid with infinite delays. *Differ. Equ. Dyn. Syst. 25* (2017), 39–64. Citado na página 18.

[63] Zhao, C., Liu, G., and Wang, W. Smooth pullback attractors for a non-autonomous 2D non-Newtonian fluid and their tempered behaviors. *J. Math. Fluid Mech. 16* (2014), 243–262. Citado na página 16.

[64] Zhao, C., and Zhou, S. Pullback attractors for a non-autonomous incompressible non-Newtonian fluid. *J. Differential Equations 238* (2007), 394–425. Citado na página 16.

[65] Zhao, C., and Zhou, S. Pullback trajectory attractors for evolution equations and application to 3D incompressible non-Newtonian fluid. *Nonlinearity 21* (2008), 1691–1717. Citado na página 16.

[66] Zhao, C., Zhou, S., and Li, Y. Existence and regularity of pullback attractors for an incompressible non-Newtonian fluid with delays. *Quart. Appl. Math. 67* (2009), 503–540. Citado na página 18.