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CHARACTERS AND COHOMOLOGY OF MODULES FOR AFFINE KAC-MOODY ALGEBRAS AND GENERALIZATIONS

Caracteres e cohomologia de módulos para álgebras de Kac-Moody afim e generalizações

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Characters and cohomology of modules for affine Kac-Moody algebras and generalizations

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To Beth, Emilson and Lívia.

Abstract

In this thesis we consider two main problems. The first problem concerns extensions between simple modules for current algebras associated to complex, simple, finite-dimensional Lie algebras. To begin, we compute 1-extensions between finite-dimensional simple modules, partially recovering a result due to Kodera. Then we develop a technique aimed to compute higher extensions, and which we use to compute 2-extensions between certain simple modules. Finally we prove that cohomology groups of current algebras are isomorphic to the cohomology groups of its underlying simple Lie algebra, a result stated by Feigin. This part of the thesis arises from collaboration with B. Boe, C. Drupieski and D. Nakano.

The second problem is concerned with the study of certain classes of modules for hyper algebras of current algebras. In the case that the underlying Lie algebra is simply laced, we show that local Weyl modules are isomorphic to certain Demazure modules, extending to positive characteristic a result due to Fourier-Littelmann. More generally, we extend a result of Naoi by proving that local Weyl modules admit a Demazure flag, i.e., a filtration with factors isomorphic to Demazure modules. Using this, we prove a conjecture of Jakelić-Moura stating that the character of local Weyl modules for hyper loop algebras are independent of the (algebraically closed) ground field.

Keywords: Lie algebras, Homological Algebra, Representation Theory

Resumo

Nesta tese nós estudamos dois problemas principais. O primeiro problema aborda extensões de módulos para álgebras de corrente associadas a álgebras de Lie simples, complexa e de dimensão finita. Primeiro nós calculamos 1-extensões entre módulos simples de dimensão finita dessas álgebras, recuperando parcialmente um resultado de Kodera. A seguir nós desenvolvemos uma técnica para calcular extensões mais altas entre módulos simples, com a qual nós calculamos certas 2-extensões. Por fim nós mostramos que os grupos de cohomologia da álgebra de corrente são isomorfos aos da álgebra de Lie simples associada a ela, confirmando uma afirmação de Feigin. Essa parte da tese foi desenvolvida em colaboração com B. Boe, C. Drupieski e D. Nakano.

O segundo problema aborda uma certa classe de módulos para hiperálgebras de álgebras de corrente. Quando a álgebra de Lie a qual a álgebra de corrente é associada é de tipo ADE, nós mostramos que módulos de Weyl locais são isomorfos a certos módulos de Demazure, estendendo para característica positiva um resultado de Fourier-Littelmann. Em geral, nós estendemos um resultado de Naoi, provando que módulos de Weyl locais admitem uma bandeira de Demazure, i.e., uma filtração cujos fatores são isomorfos a módulos de Demazure. Usando esse resultado, nós provamos uma conjectura de Jakelić-Moura que afirma que o caracter dos módulos de Weyl locais para hiperálgebras de laços são independentes do corpo base, desde que este seja algebricamente fechado.

Palavras-chave: Álgebras de Lie, Teoria de Representações, Álgebra Homológica

Contents

Introduction

The main theme of this thesis is representation theory of infinite-dimensional Lie algebras. This topic is related to research areas in geometry and algebra which have been attracting a great deal of attention lately, such as quantum groups, crystal basis, character theory, Kazhdan-Lusztig theory, categorification, flag varieties and quiver varieties.

This thesis is composed by two main projects. On the first one we study extensions between simple modules for current algebras. We develop a general machinery to approach this and similar problems, and compute $\operatorname{Ext}^n_{\mathfrak{g}[t]}(L_1, L_2)$ for simple $\mathfrak{g}[t]$-modules L_1, L_2 depending on n and on $\mathfrak{g}$. This topic is closely related to recent works of Kodera [**Kod10**] and Neher-Savage [**NS11**].

On the second one we study a relationship between local Weyl modules and Demazure modules for hyper algebras of current algebras. We prove a conjecture of Jakelić-Moura [**JM07**] stating that the character of local Weyl modules for hyper loop algebras are independent of the (algebraically closed) ground field. This is closely related to a conjecture posed by Chari-Pressley [**CP01**].

In the following paragraphs each of these topics will be explained in more detail.

Background

Given a Lie algebra it is interesting to describe its simple and indecomposable modules, their characters, and extensions between them. If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, then the category of finite-dimensional $\mathfrak{g}$-modules is semisimple. It means that all of its modules are completely reducible, or that any extension between its modules is isomorphic to their direct sum. In particular, finite-dimensional indecomposable $\mathfrak{g}$-modules are simple and they are parametrized by the infinite set of dominant weights associated to $\mathfrak{g}$. There is a description of these finite-dimensional simple $\mathfrak{g}$-modules via generators and relations, and their characters can be computed using Weyl's character formula.

Given an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$ and a connected, simply connected, semisimple algebraic group $G_{\mathbb{F}}$ over $\mathbb{F}$ of the same Lie type as $\mathfrak{g}$, the category of finite-dimensional $G_{\mathbb{F}}$-modules is equivalent to that of the hyper algebra $U_{\mathbb{F}}(\mathfrak{g})$. A hyper algebra $U_{\mathbb{F}}(\mathfrak{a})$ is a Hopf algebra associated to a Lie algebra $\mathfrak{a}$, similar to its universal enveloping algebra, and obtained from it by first choosing a certain $\mathbb{Z}$-form and then changing scalars from $\mathbb{Z}$ to $\mathbb{F}$. Thus, in particular, they provide a way to pass from a category of modules for a Lie algebra over $\mathbb{C}$ to its analog over $\mathbb{F}$. This process is known as *reduction modulo p*. The category of finite-dimensional $G_{\mathbb{F}}$-modules (or equivalently, of finite-dimensional $U_{\mathbb{F}}(\mathfrak{g})$-modules) is not semisimple, and the modules obtained by reduction modulo p of simple $\mathfrak{g}$-modules - called Weyl modules - provide examples of indecomposable, reducible modules. Finite-dimensional simple $G_{\mathbb{F}}$-modules are also parametrized by the set of dominant weights, but their characters are not known in general. Weyl modules, on the other hand, not only have a description via generators and relations, but their characters can also be computed using Weyl's character formula.

Associated to every finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ there is an untwisted affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$. It can be realized as the semi-direct product of a central extension

$\tilde{\mathfrak{g}} \oplus \mathbb{C}c$ of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and a derivation d of $\tilde{\mathfrak{g}} \oplus \mathbb{C}c$ (cf. [**Kum02**, 13.1]). Its importance is more noticeable in mathematical physics due to its relation to conformal field theory. A well studied category of $\hat{\mathfrak{g}}$-modules, category $\mathcal{O}$, is not semisimple, and the study of extensions of modules in $\mathcal{O}$ has been a very interesting topic of research. Similar to the case of finite-dimensional $\mathfrak{g}$-modules, simple modules in $\mathcal{O}$ are also parametrized by an infinite set, of dominant weights associated to $\hat{\mathfrak{g}}$, they can also be described by generators and relations, and their characters are given by Weyl-Kac's character formula. However, nontrivial simple modules in $\mathcal{O}$ are infinite-dimensional (cf. [**Kum02**, Chapter 2]).

In [**GL76**], Garland-Lepowsky computed the cohomology of the subalgebra $\mathfrak{g} \otimes t\mathbb{C}[t] \subset \hat{\mathfrak{g}}$, with coefficients in certain simple $\hat{\mathfrak{g}}$-modules. Their result for $\mathfrak{g} \otimes t\mathbb{C}[t]$ is similar to Kostant's Theorem on the cohomology of the nilpotent radical of a finite-dimensional semisimple Lie algebra.

In [**Kod10**], Kodera described 1-extensions between finite-dimensional simple modules for generalized current algebras, that is Lie algebras of the form $(\mathfrak{g} \otimes A)$ with A being a finitely generated commutative $\mathbb{C}$-algebra. Higher cohomology and higher extensions of these algebras have not been studied in general. However, they are of interest - they describe invariants of these algebras, such as deformations and outer automorphisms, besides parametrizing extensions. In [**FGT08**], Fishel-Grojnowski-Teleman calculated $\mathrm{H}^n(\mathfrak{g} \otimes \mathbb{C}[t]/(t^s), \mathbb{C})$ and proved, in particular, that as a $\mathbb{C}$-module it is isomorphic to $\mathrm{H}^n(\mathfrak{g}, \mathbb{C})^{\otimes s}$. More recently, Chari-Khare-Ridenour [**CKR12**] calculated higher extensions between certain finite-dimensional graded simple modules for truncated current algebras of the form $\mathfrak{g} \otimes (\mathbb{C}[t]/(t^2))$ in terms of homomorphisms of $\mathfrak{g}$-modules. The calculation of higher extensions of generalized current algebras is the subject of our first chapter.

As we pointed out above, nontrivial simple $\hat{\mathfrak{g}}$-modules are not finite-dimensional. This brings us to consider the loop algebra $\tilde{\mathfrak{g}}$, which admits nontrivial finite-dimensional modules. The category of finite-dimensional $\tilde{\mathfrak{g}}$-modules is not semisimple, and Chari-Pressley described its simple modules as tensor products of evaluation modules in [**CP86**]. They also introduced [**CP01**] global and local Weyl modules for affine Kac-Moody Lie algebras and loop algebras, in addition to defining Weyl modules for the quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ as integrable modules given via generators and relations. Moreover, in this latter paper, they conjectured that local Weyl modules were isomorphic to the limit $q \to 1$ of irreducible quantum Weyl modules, and proved it in the case $\hat{\mathfrak{g}} \cong \widehat{\mathfrak{sl}_2}$.

Chari-Pressley's conjecture boils down to the calculation of characters of local Weyl modules for loop algebras. After the works of Chari-Loktev on $\widehat{\mathfrak{sl}_n}$ [**CL06**], Feigin-Loktev on generalized current algebras [**FL04**] and Fourier-Littelmann on the relation between Demazure and local Weyl modules for current algebras $\mathfrak{g} \otimes \mathbb{C}[t]$ [**FL07**], Naoi completed a proof of Chari-Pressley's conjecture in [**Nao12**]. Using a certain decomposition of tensor products of Demazure crystals proved by Joseph [**Jos03**], Naoi extended the work of Fourier-Littelmann, showing that any local Weyl module for the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ admits a filtration whose factors are Demazure modules.

It was pointed out by Nakajima that Chari-Pressley's conjecture could be deduced from global basis theory. His proposed proof remains unpublished, and a brief sketch is given in the introduction of [**FL07**]. In the particular case when $\mathfrak{g}$ is non simply laced, the relation between local Weyl modules and Demazure modules given by [**Nao12**] also depends on the theory of global basis, although in a different manner than Nakajima's proposed proof.

In analogy with finite-dimensional simple Lie algebras, one can consider hyper algebras of certain generalized current algebras over $\mathbb{F}$ (cf. [**Gar78**, **Mit85**, **Cha13**]). Jakelić-Moura studied the category of finite-dimensional modules for hyper algebras of loop algebras $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$, and they defined local Weyl modules for these algebras [**JM07**]. They also conjectured that the character of these Weyl modules is the same as their characteristic zero counterparts, a conjecture which is similar to that of Chari-Pressley (cf. [**JM07**, Conjecture 4.7.(a)]). The study of finite-dimensional $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-modules and the proof of Jakelić-Moura's conjecture is the subject of our second chapter.

Results of the first project

The main goal of the first chapter of this thesis is to develop techniques to compute extensions of modules for current algebras. In the first section, we set up the necessary notation and state some basic facts used throughout the chapter. In Subsection 1.2.1 we compute 1-extensions between any finite-dimensional simple modules for current and loop algebras. It partially recovers in a very neat way results obtained by Kodera [**Kod10**]. Afterwards, we start working on higher extensions and obtaining previously unknown results. In Subsection 1.2.2 we compute n-extensions between two finite-dimensional simple modules for the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ which are supported at the same point. In Subsection 1.2.3 we compute 2-extensions between non-isomorphic, finite-dimensional, simple modules for the current algebra supported at the same points.

The third section is devoted to fill in the gap left in Subsection 1.2.3, namely we want to compute $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V,V)$ where V is a finite-dimensional simple $\mathfrak{g}[t]$-module supported at a pair of points $a, b \in \mathbb{C}$. In fact we first reduce the computation of $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V,V)$ to the explicit description of $\mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C})$, with $I = (t-a)(t-b) \subset \mathbb{C}[t]$, as a module for $\mathfrak{g} \times \mathfrak{g}$. In Subsection 1.3.2 we set up a spectral sequence to compute higher cohomologies of a truncation of $\mathfrak{g} \otimes I$. The reason why we truncate $\mathfrak{g} \otimes I$ is explained in Subsection 1.3.3 by the non-convergence of a different spectral sequence. In Subsection 1.3.4 we relate the cohomology of a truncation of $\mathfrak{g} \otimes I$ to that of a truncation of $\mathfrak{g} \otimes t\mathbb{C}[t]$, and in Subsection 1.3.5 we compute the second cohomology of the latter one. In Subsection 1.3.6 we recall some notation used in [**GL76**], and some of its results. We also compute particular examples of Garland-Lepowsky's results which will be useful for our calculations. In Subsection 1.3.7 we prove that the cohomology of the current algebra is isomorphic to that of the underlying semisimple Lie algebra, a claim which was made (but not proved) by Feigin [**Fei80**]. These results come together in Subsection 1.3.8 where we explicitly compute the second cohomology of a truncation of $\mathfrak{g} \otimes I$. The consequences of this description to the cohomology of $\mathfrak{g} \otimes I$ are drawn in Subsection 1.3.9, where certain composition factors of $\mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C})$ are described. This description gives us an explicit reason for the non-convergence of the spectral sequence of Subsection 1.3.3. In Subsection 1.3.10, we state a conjecture related to the second cohomology of $\mathfrak{sl}_2 \otimes I$, then partially fill in the gap left in Subsection 1.2.3, and explain how far we are from the actual proof.

Results of the second project

The goal of the second chapter is to extend the results of [**FL07**, **Nao12**] to positive characteristic and prove the conjecture of [**JM07**]. Due to extra technical difficulties which arise when dealing with hyperalgebras in positive characteristic, there are several differences between our proofs and those of Chari-Presley's conjecture. For instance, in proving the existence of Demazure flags, some of the tricks used in [**Nao12**] do not admit a hyperalgebra analogue. Our approach to overcoming these issues actually makes use of the characteristic zero version of the same statements which were proved in [**FL07**, **Nao12**]. We also need to use the fact proved in [**Mat88**, **Mat89**], which is that the characters of Demazure modules do not depend on the ground field. Our proofs require different presentations of these modules in terms of generators and relations. Technical issues for proving one of these presentations in the hyperalgebra context when $\mathfrak{g}$ is of type G_2 imposed that we restrict ourselves to characteristic at least 5 in that case. Outside type G_2, there is no restriction in the characteristic of the ground field.

The second chapter is organized as follows. We start Subsection 0.1 by fixing the notation regarding finite and affine Kac-Moody algebras and reviewing the construction of certain hyperalgebras. Next, using generators and relations, we define Weyl modules for hyper loop algebras, their graded analogues for hyper current algebras, and we delineate a useful subclass of the class of Demazure modules. We then state our main result, Theorem 2.1.2, and recall the precise statement (2.1.4) of the conjecture in [**JM07**]. Theorem 2.1.2 is stated in three parts. Part (a) extends to positive characteristic the isomorphism between local graded Weyl modules and Demazure modules for current algebras $\mathfrak{g}[t]$ with simply laced $\mathfrak{g}$. Part (b) extends to positive characteristic the existence of Demazure flags for local graded Weyl modules for the current algebra. Part (c) establishes an isomorphism between local graded Weyl modules for current algebras and a twist of certain local graded Weyl modules for hyper loop algebras by regarding the latter as modules for the hyper current algebra.

In Section 2.2 we fix some further notation and establish a few technical results needed in the proofs. Subsection 2.3.1 brings a review of the finite-dimensional representation theory of the finite type hyperalgebras which we will need, while Subsection 2.3.2 gives a very brief account of the relevant results from [**JM07**]. The local graded Weyl modules for hyper current algebras are introduced in Subsection 2.3.3. The main results of this subsection are Corollary 2.3.12, which proves that the local graded Weyl modules for $\mathfrak{g}[t]$ admit integral forms, and Theorem 2.3.13, which establishes the basic properties of the category of finite-dimensional graded modules for hyper current algebras. Assuming Theorem 2.1.2 (b), we prove (2.1.4) in Subsection 2.3.4. The proof makes use of the version of Theorem 2.1.2 in characteristic zero as well as [**Nao12**, Corollary A] (Proposition 2.3.15). In Subsection 2.3.5 we prove a second presentation of Demazure modules in terms of generators and relations, essentially by replacing a highest-weight generator by a lowest-weight one. This is the presentation which allows us to use results of [**Mat88**, **Mat89**] about the independence of the characters of Demazure modules from the ground field.

In the first three subsections of Section 2.4 we collect the results of [**Jos03**, **Jos06**] on crystal and global basis which we need to prove Theorem 2.4.5 which is an integral analogue of [**Nao12**, Corollary 4.16]. That shows the existence of higher level Demazure flags for Demazure modules when the underlying simple Lie algebra $\mathfrak{g}$ is simply laced. We remark that the proof of Theorem 2.4.5 is the only one in which the theory of global basis is used. We further remark that, in order to prove Theorem 2.1.2 (b), we only need the statement of Theorem 2.4.5 for $\mathfrak{g}$ of type A. It is interesting to observe that the only other proof relying on quantum groups is that of Theorem 2.1.2 (c) in characteristic zero (see [**FL07**, Lemma 1, Lemma 3, Equation (15)]).

Theorem 2.1.2 is proved in Section 2.5. In particular, in Subsection 2.5.2 we prove a positive characteristic analogue of [**Nao12**, Proposition 4.1], which provides a third presentation of Demazure modules in terms of generator and relations in the case that $\mathfrak{g}$ is not simply laced. This is where the restriction on the characteristic of the field in the case of G_2 appears. Parts (b) and (c) of Theorem 2.1.2 are proved in Subsections 2.5.3 and 2.5.4, respectively. They also hold for $\mathfrak{g}$ of type G_2 if we assume that we can extend the results in Subsection 2.5.2 to this case.

CHAPTER 0

Notation

In this chapter we fix some notation to be used throughout the entire thesis.

0.1. Finite type data

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The associated root system will be denoted by $R \subset \mathfrak{h}^*$. We fix a simple system $\Delta = \{\alpha_i : i \in I\} \subset R$ and denote the corresponding set of positive roots by R^+. The Borel subalgebra associated to R^+ will be denoted by $\mathfrak{b}^+ \subset \mathfrak{g}$ and the opposite Borel subalgebra will be denoted by $\mathfrak{b}^- \subset \mathfrak{g}$. We fix a Chevalley basis of the Lie algebra $\mathfrak{g}$ consisting of $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, for each $\alpha \in R^+$, and $h_i \in \mathfrak{h}$, for each $i \in I$. We also define $h_\alpha \in \mathfrak{h}, \alpha \in R^+$, by $h_\alpha = [x_\alpha^+, x_\alpha^-]$ (in particular, $h_i = h_{\alpha_i}, i \in I$) and set $R^\vee = \{h_\alpha \in \mathfrak{h} : \alpha \in R\}$. Let $(\ ,\ )$ denote the invariant symmetric bilinear form on $\mathfrak{g}$ such that $(h_\theta, h_\theta) = 2$, where θ is the highest root of $\mathfrak{g}$. Let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the linear isomorphism induced by $(\ ,\ )$ and keep denoting by $(\ ,\ )$ the non degenerate bilinear form induced by ν on $\mathfrak{h}^*$. Notice that

$$(x_\alpha^+, x_\alpha^-) = \frac{2}{(\alpha,\alpha)} \qquad \text{for all} \qquad \alpha \in R^+ \tag{0.1.1}$$

and

$$(\alpha,\alpha) = \begin{cases} 2, & \text{if } \alpha \text{ is long,} \\ 2/r^\vee, & \text{if } \alpha \text{ is short,} \end{cases} \tag{0.1.2}$$

where $r^\vee \in \{1,2,3\}$ is the lacing number of $\mathfrak{g}$. For notational convenience, set

$$r_\alpha^\vee = \frac{2}{(\alpha,\alpha)} = \begin{cases} 1, & \text{if } \alpha \text{ is long,} \\ r^\vee, & \text{if } \alpha \text{ is short.} \end{cases} \tag{0.1.3}$$

We shall need the following fact [**Car72**, Section 4.2]. Given $\alpha \in R$, let $x_\alpha = x_{\pm\alpha}^\pm$ according to whether $\alpha \in \pm R^+$. For $\alpha, \beta \in R$ let $m = \max\{n : \beta - n\alpha \in R\}$. Then, there exists $\varepsilon \in \{-1, 1\}$ such that

$$[x_\alpha, x_\beta] = \varepsilon(m+1)x_{\alpha+\beta}. \tag{0.1.4}$$

The weight lattice is defined as $P = \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z}, \forall\ \alpha \in R\}$, the dominant weights are $P^+ = \{\lambda \in P : \lambda(h_\alpha) \in \mathbb{N}, \forall\ \alpha \in R^+\}$, the coweight lattice is $P^\vee = \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{Z}, \forall\ \alpha \in R\}$, and the dominant coweights are $P^{\vee+} = \{h \in P^\vee : \alpha(h) \in \mathbb{N}, \forall\ \alpha \in R^+\}$. The fundamental weights will be denoted by $\omega_i, i \in I$. The root lattice of $\mathfrak{g}$ will be denoted by Q and we let $Q^+ = \mathbb{Z}_{\geq 0} R^+$. We consider the usual partial order on $\mathfrak{h}^*$: $\mu \leq \lambda$ if and only if $\lambda - \mu \in Q^+$. The Weyl group of $\mathfrak{g}$, denoted $\mathcal{W}$, is the subgroup of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ generated by the simple reflections $s_i, i \in I$, defined by $s_i(\mu) = \mu - \mu(h_i)\alpha_i$ for all $\mu \in \mathfrak{h}^*$. As usual, w_0 will denote the longest element in $\mathcal{W}$.

0.2. Affine type data

Consider the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, with Lie bracket given by $[x\otimes t^r, y\otimes t^s] = [x,y]\otimes t^{r+s}$, for any $x, y \in \mathfrak{g}$, $r, s \in \mathbb{Z}$. We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g}\otimes 1$ of $\tilde{\mathfrak{g}}$. The subalgebra $\mathfrak{g}[t] = \mathfrak{g}\otimes\mathbb{C}[t]$ is the current algebra of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, let $\tilde{\mathfrak{a}} = \mathfrak{a}\otimes\mathbb{C}[t,t^{-1}]$ and $\mathfrak{a}[t] = \mathfrak{a}\otimes\mathbb{C}[t]$. Let also $\mathfrak{a}[t]_\pm := \mathfrak{a}\otimes(t^{\pm 1}\mathbb{C}[t^{\pm 1}])$. In particular, as vector spaces,

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+ \qquad \text{and} \qquad \mathfrak{g}[t] = \mathfrak{n}^-[t]\oplus\mathfrak{h}[t]\oplus\mathfrak{n}^+[t].$$

The affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is the 2-dimensional extension $\hat{\mathfrak{g}} := \tilde{\mathfrak{g}}\oplus\mathbb{C}c\oplus\mathbb{C}d$ of $\tilde{\mathfrak{g}}$ with Lie bracket given by

$$[x\otimes t^r, y\otimes t^s] = [x,y]\otimes t^{r+s} + r\ \delta_{r,-s}\ (x,y)\ c, \quad [c,\hat{\mathfrak{g}}] = \{0\}, \quad \text{and} \quad [d, x\otimes t^r] = r\ x\otimes t^r$$

for any $x, y \in \mathfrak{g}, r, s \in \mathbb{Z}$. Observe that the derived subalgebra $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}},\hat{\mathfrak{g}}] = \tilde{\mathfrak{g}}\oplus\mathbb{C}c$, and we have a non split short exact sequence of Lie algebras $0\to\mathbb{C}c\to\hat{\mathfrak{g}}'\to\tilde{\mathfrak{g}}\to 0$.

Set $\hat{\mathfrak{h}}' = \mathfrak{h}\oplus\mathbb{C}c$. Notice that $\mathfrak{g}, \mathfrak{g}[t]$, and $\mathfrak{g}[t]_\pm$ remain subalgebras of $\hat{\mathfrak{g}}$. Set

$$\hat{\mathfrak{h}} = \mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d, \qquad \hat{\mathfrak{n}}^\pm = \mathfrak{n}^\pm\oplus\mathfrak{g}[t]_\pm, \qquad \text{and} \qquad \hat{\mathfrak{b}}^\pm = \hat{\mathfrak{n}}^\pm\oplus\hat{\mathfrak{h}}.$$

The root system, positive root system, and set of simple roots associated to the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^-\oplus\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}}^+$ will be denoted by $\hat{R}, \hat{R}^+$ and $\hat{\Delta}$ respectively. Let $\hat{I} = I\sqcup\{0\}$ and $h_0 = c - h_\theta$, so that $\{h_i : i\in\hat{I}\}\cup\{d\}$ is a basis of $\hat{\mathfrak{h}}$. Identify $\mathfrak{h}^*$ with the subspace $\{\lambda\in\hat{\mathfrak{h}}^* : \lambda(c) = \lambda(d) = 0\}$. Let also $\delta\in\hat{\mathfrak{h}}^*$ be such that $\delta(d) = 1$ and $\delta(h_i) = 0$ for all $i\in\hat{I}$ and define $\alpha_0 = \delta - \theta$. Then, $\hat{\Delta} = \{\alpha_i : i\in\hat{I}\}$, $\hat{R}^+ = R^+\cup\{\alpha + r\delta : \alpha\in R\cup\{0\}, r\in\mathbb{Z}_{>0}\}$, $\hat{\mathfrak{g}}_{\alpha+r\delta} = \mathfrak{g}_\alpha\otimes t^r$, if $\alpha\in R, r\in\mathbb{Z}$, and $\hat{\mathfrak{g}}_{r\delta} = \mathfrak{h}\otimes t^r$, if $r\in\mathbb{Z}\setminus\{0\}$. Observe that $\alpha(c) = 0$ for all $\alpha\in\hat{R}$. A root $\gamma\in\hat{R}$ is called real if $\gamma = (\alpha + r\delta)$ with $\alpha\in R, r\in\mathbb{Z}$, and imaginary if $\gamma = r\delta$ with $r\in\mathbb{Z}\setminus\{0\}$. Set $x^\pm_{\alpha,r} = x^\pm_\alpha\otimes t^r, h_{\alpha,r} = h_\alpha\otimes t^r, \alpha\in R^+, r\in\mathbb{Z}$, and observe that $\{x^\pm_{\alpha,r}, h_{i,r} : \alpha\in R^+, i\in I, r\in\mathbb{Z}\}$ is a basis of $\tilde{\mathfrak{g}}$. Given $\alpha\in R^+$ and $r\in\mathbb{Z}_{>0}$, set $x^+_{\pm\alpha+r\delta} = x^\pm_{\alpha,r}$, $x^-_{\pm\alpha+r\delta} = x^\mp_{\alpha,-r}$ and $h_{\pm\alpha+r\delta} = [x^+_{\pm\alpha+r\delta}, x^-_{\pm\alpha+r\delta}] = \pm h_\alpha + r r^\vee_\alpha c$.

Define also $\Lambda_i\in\hat{\mathfrak{h}}^*, i\in\hat{I}$, by the requirement $\Lambda_i(d) = 0, \Lambda_i(h_j) = \delta_{ij}$ for all $i, j\in\hat{I}$. Set $\hat{P} = \mathbb{Z}\delta\oplus(\oplus_{i\in\hat{I}}\mathbb{Z}\Lambda_i), \hat{P}^+ = \mathbb{Z}\delta\oplus(\oplus_{i\in\hat{I}}\mathbb{N}\Lambda_i), \hat{P}' = \oplus_{i\in\hat{I}}\mathbb{Z}\Lambda_i$, and $\hat{P}'^+ = \hat{P}'\cap\hat{P}^+$. Notice that

$$\Lambda_0(h) = 0 \quad \text{iff} \quad h\in\mathfrak{h}\oplus\mathbb{C}d \qquad \text{and} \qquad \Lambda_i - \omega_i = \omega_i(h_\theta)\Lambda_0 \quad \text{for all } i\in I.$$

Hence, $\hat{P} = \mathbb{Z}\Lambda_0\oplus P\oplus\mathbb{Z}\delta$. Given $\Lambda\in\hat{P}$, the number $\Lambda(c)$ is called the level of Λ. Since $\alpha(c) = 0$ for all $\alpha\in\hat{R}$, the level of Λ depends only on its class modulo the root lattice $\hat{Q}$. Set also $\hat{Q}^+ = \mathbb{Z}_{\geq 0}\hat{R}^+$ and let $\widehat{\mathcal{W}}$ denote the affine Weyl group, which is generated by the simple reflections $s_i, i\in\hat{I}$. Denote by W^1_a the subset of $\widehat{\mathcal{W}}$ consisting of minimal length left coset representatives of elements in the quotient $\mathcal{W}\backslash\widehat{\mathcal{W}}$. Finally, observe that $\{\Lambda_0,\delta\}\cup\Delta$ is a basis of $\hat{\mathfrak{h}}^*$.

CHAPTER 1

Extensions for current algebras

1.1. Preliminaries

1.1.1. Notation. Let A be a finitely-generated commutative $\mathbb{C}$-algebra, and let $\mathfrak{g}\otimes A$ be the Lie algebra with underlying vector space $\mathfrak{g}\otimes A$ and with Lie bracket defined by $[x\otimes a, y\otimes b] = [x,y]\otimes ab$. Let $\operatorname{MaxSpec} A$ be the set of maximal ideals in A. Given $\mathfrak{m} \in \operatorname{MaxSpec} A$, let $\mathrm{ev}_{\mathfrak{m}} : \mathfrak{g}\otimes A \to \mathfrak{g}$ be the Lie algebra homomorphism induced by the natural map $A \to A/\mathfrak{m} \cong \mathbb{C}$. Given a $\mathfrak{g}$-module V, let $\mathrm{ev}_{\mathfrak{m}}^* V$ be the $\mathfrak{g}\otimes A$-module obtained by pulling back the $\mathfrak{g}$-module structure map for V along $\mathrm{ev}_{\mathfrak{m}}$. For all $\mathfrak{m}, \mathfrak{m}' \in \operatorname{MaxSpec} A$, one has $\mathrm{ev}_{\mathfrak{m}}^* V(0) \cong \mathrm{ev}_{\mathfrak{m}'}^* V(0)$ as $\mathfrak{g}\otimes A$-modules.

1.1.2. Irreducible modules. Let $\mathcal{P}$ be the set of finitely-supported functions from $\operatorname{MaxSpec} A$ to P^+, that is, the set of functions $\pi : \operatorname{MaxSpec} A \to P^+$ such that $\pi(\mathfrak{m}) = 0$ for all but finitely many $\mathfrak{m} \in \operatorname{MaxSpec} A$. Then there exists a bijection between $\mathcal{P}$ and the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g}\otimes A$-modules, which associates to $\pi \in \mathcal{P}$ the isomorphism class of the $\mathfrak{g}\otimes A$-module $\mathcal{V}(\pi) := \bigotimes_{\mathfrak{m}\in \operatorname{MaxSpec} A} \mathrm{ev}_{\mathfrak{m}}^* V(\pi(\mathfrak{m}))$, where the factors in the tensor product are taken with respect to some arbitrary fixed ordering on $\operatorname{MaxSpec} A$. Note that there are only finitely-many nontrivial factors in the tensor product because $\pi(\mathfrak{m}) = 0$ for all but finitely many $\mathfrak{m} \in \operatorname{MaxSpec} A$, and that different orderings of the factors in the tensor product yield isomorphic modules.

Recall the involution $\lambda \mapsto \lambda^*$ on P^+ defined by $\lambda^* = -w_0\lambda$, where w_0 is the longest element in the Weyl group $\mathcal{W}$. Given $\pi \in \mathcal{P}$, define $\pi^* \in \mathcal{P}$ by $\pi^*(\mathfrak{m}) = \pi(\mathfrak{m})^*$. Then the dual module $\mathcal{V}(\pi)^* = \operatorname{Hom}_{\mathbb{C}}(\mathcal{V}(\pi), \mathbb{C})$ is isomorphic as a $\mathfrak{g}\otimes A$-module to $\mathcal{V}(\pi^*)$.

1.2. Extensions between simple modules

1.2.1. $\mathbf{Ext^1}$. Let $\pi, \pi' \in \mathcal{P}$. Our goal is to compute the space $\operatorname{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$. First, since π and π' are finitely-supported, there exist distinct maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n \in \operatorname{MaxSpec} A$ such that $\pi(\mathfrak{m}) = 0 = \pi'(\mathfrak{m})$ if $\mathfrak{m} \notin \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$. Then we can write $\mathcal{V}(\pi) = \bigotimes_{i=1}^n \mathrm{ev}_{\mathfrak{m}_i}^* V(\pi(\mathfrak{m}_i))$ and $\mathcal{V}(\pi') = \bigotimes_{i=1}^n \mathrm{ev}_{\mathfrak{m}_i}^* V(\pi'(\mathfrak{m}_i))$. For brevity, set $\pi_i = \pi(\mathfrak{m}_i)$ and $\pi_i' = \pi'(\mathfrak{m}_i)$.

Set $I = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n$. Then $\mathfrak{g}\otimes I$ is an ideal in $\mathfrak{g}\otimes A$ that annihilates both $\mathcal{V}(\pi)$ and $\mathcal{V}(\pi')$. By the Chinese Remainder Theorem, there exists a ring isomorphism

$$A/I \cong A/\mathfrak{m}_1 \times A/\mathfrak{m}_2 \times \cdots \times A/\mathfrak{m}_n. \tag{1.2.1}$$

Then $(\mathfrak{g}\otimes A)/(\mathfrak{g}\otimes I) \cong \mathfrak{g}\otimes (A/I)$ is isomorphic as a Lie algebra to $\bigoplus_{i=1}^n \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathfrak{g}\otimes(A/\mathfrak{m}_i)$. Observe that $\mathfrak{g}_i \cong \mathfrak{g}$ as a Lie algebra because $A/\mathfrak{m}_i \cong \mathbb{C}$. Under this identification, $\bigoplus_{i=1}^n \mathfrak{g}_i$ acts on $\mathcal{V}(\pi) = \bigotimes_{i=1}^n \mathrm{ev}_{\mathfrak{m}_i}^* V(\pi_i)$ component-wise, that is, if $x_1, \dots, x_n \in \mathfrak{g}$ and $v_1 \otimes \cdots \otimes v_n \in \mathcal{V}(\pi)$, then $(x_1, \dots, x_n).(v_1\otimes\cdots\otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes x_i.v_i \otimes \cdots \otimes v_n$.

Now consider the Lyndon–Hochschild–Serre (LHS) spectral sequence for the Lie algebra $\mathfrak{g}\otimes A$ and the ideal $\mathfrak{g}\otimes I$:

$$E_2^{i,j} = \operatorname{Ext}^i_{\mathfrak{g}\otimes A/\mathfrak{g}\otimes I}(\mathcal{V}(\pi), \operatorname{Ext}^j_{\mathfrak{g}\otimes I}(\mathbb{C}, \mathcal{V}(\pi'))) \Rightarrow \operatorname{Ext}^{i+j}_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')). \tag{1.2.2}$$

Observe that

$$E_2^{1,0} \cong \mathrm{H}^1(\mathfrak{g}^{\oplus n}, \mathcal{V}(\pi^*) \otimes \mathcal{V}(\pi')) = 0, \quad \text{and} \tag{1.2.3}$$

$$E_2^{2,0} \cong \mathrm{H}^2(\mathfrak{g}^{\oplus n}, \mathcal{V}(\pi^*) \otimes \mathcal{V}(\pi')) = 0, \tag{1.2.4}$$

which follows from the Künneth formula and from the first and second Whitehead Lemmas. Then the 5-term exact sequence of low degree terms in (1.2.2) yields the isomorphism

$$\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \cong \mathrm{Hom}_{\mathfrak{g}\otimes A/\mathfrak{g}\otimes I}(\mathcal{V}(\pi), \mathrm{H}^1(\mathfrak{g}\otimes I, \mathbb{C}) \otimes \mathcal{V}(\pi')). \tag{1.2.5}$$

For an arbitrary Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, the cohomology space $\mathrm{H}^1(\mathfrak{a}, \mathbb{C})$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}], \mathbb{C})$. Since $\mathfrak{g}$ is semisimple, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and hence $[\mathfrak{g}\otimes I, \mathfrak{g}\otimes I] = \mathfrak{g}\otimes I^2$. Then $\mathrm{H}^1(\mathfrak{g}\otimes I, \mathbb{C})$ is isomorphic as a $(\mathfrak{g}\otimes A)/(\mathfrak{g}\otimes I)$-module to $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}\otimes (I/I^2), \mathbb{C})$. Considering I as a module over A, we get by the Chinese Remainder Theorem for Modules the isomorphism

$$I/I^2 \cong I/(\mathfrak{m}_1 I) \times I/(\mathfrak{m}_2 I) \times \cdots \times I/(\mathfrak{m}_n I),$$

which is compatible with (1.2.1). Set $d_i = \dim_{A/\mathfrak{m}_i} I/(\mathfrak{m}_i I)$. Then $\mathfrak{g}\otimes (I/I^2) \cong \bigoplus_{i=1}^n \mathfrak{g}_i^{\oplus d_i}$ as a $\bigoplus_{i=1}^n \mathfrak{g}_i$-module, i.e., $\mathfrak{g}\otimes (I/I^2)$ is a direct sum of copies of the adjoint representations for the summands in $\bigoplus_{i=1}^n \mathfrak{g}_i$. Then

$$\mathrm{H}^1(\mathfrak{g}\otimes I, \mathbb{C}) \cong (\textstyle\bigoplus_{i=1}^n \mathfrak{g}_i^{\oplus d_i})^* \cong \bigoplus_{i=1}^n (\mathfrak{g}_i^*)^{\oplus d_i}, \tag{1.2.6}$$

a direct sum of copies of the coadjoint representations for the summands in $\bigoplus_{i=1}^n \mathfrak{g}_i$. Now applying the Künneth formula, we get

$$\begin{aligned}
\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{\oplus_{j=1}^n \mathfrak{g}_j}(\mathcal{V}(\pi), (\mathfrak{g}_i^*)^{\oplus d_i} \otimes \mathcal{V}(\pi')) \\
&\cong \bigoplus_{i=1}^n \left(\mathrm{Hom}_{\mathfrak{g}}(V(\pi_i), (\mathfrak{g}^*)^{\oplus d_i} \otimes V(\pi_i')) \otimes \bigotimes_{\substack{1\le j\le n \\ j\neq i}} \mathrm{Hom}_{\mathfrak{g}}(V(\pi_j), V(\pi_j')) \right) \\
&\cong \bigoplus_{i=1}^n \left(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\pi_i), V(\pi_i'))^{\oplus d_i} \otimes \bigotimes_{\substack{1\le j\le n \\ j\neq i}} \mathrm{Hom}_{\mathfrak{g}}(V(\pi_j), V(\pi_j')) \right)
\end{aligned}$$

Recall that $\mathrm{Hom}_{\mathfrak{g}}(V(\pi_j), V(\pi_j')) \cong \mathbb{C}$ if $\pi_j = \pi_j'$, and is zero otherwise. Then the above calculation shows that $\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$ is zero unless $\#\{1 \le i \le n : \pi_i \neq \pi_i'\} \le 1$.

Suppose that $\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \neq 0$. Without loss of generality, we may assume that $\pi_i = \pi_i'$ for $1 \le i \le n-1$. If $\pi_n \neq \pi_n'$, then only one summand in the decomposition of $\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$ is nonzero, and we get $\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\pi_n), V(\pi_n'))^{\oplus d_n}$. On the other hand, if $\pi_n = \pi_n'$, then $\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\pi_i), V(\pi_i))^{\oplus d_i}$.

In the special cases of current and loop algebras, $d_i = 1$ for all $1 \le i \le n$. So we have just proved the following result.

Theorem 1.2.1. Let $\pi, \pi' \in \mathcal{P}$, $\mathcal{V}(\pi) = \bigotimes_{i=1}^n \mathrm{ev}^*_{\mathfrak{m}_i} V(\pi_i(\mathfrak{m}_i))$, $\mathcal{V}(\pi') = \bigotimes_{i=1}^n \mathrm{ev}^*_{\mathfrak{m}_i} V(\pi'(\mathfrak{m}_i))$ and, for each $i = 1, \dots, n$, let $\pi_i = \pi(\mathfrak{m}_i), \pi_i' = \pi'(\mathfrak{m}_i)$ and $A = \mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$.

(*i*) If $\#\{1 \le i \le n : \pi_i \neq \pi_i'\} \ge 2$, then

$$\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) = 0.$$

(*ii*) If there exists i such that $\pi_i \neq \pi'_i$ and $\pi_j = \pi'_j$ for all $1 \leq j \leq n, j \neq i$, then

$$\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\pi_i), V(\pi'_i)).$$

(*iii*) If $\pi = \pi'$, then

$$\mathrm{Ext}^1_{\mathfrak{g}\otimes A}(\mathcal{V}(\pi), \mathcal{V}(\pi)) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\pi_i), V(\pi_i)).$$

□

The previous theorem recovers a result of Kodera (cf. [**Kod10**, Theorem 1.2]) in a more concise way. This result was also obtained by Neher-Savage in the setting of equivariant map algebras (cf. [**NS11**, Theorem 3.7, Theorem 3.9]).

1.2.2. Higher extensions between modules evaluated at zero. Assume now that $A = \mathbb{C}[t]$, so that $\mathfrak{g}\otimes A = \mathfrak{g}[t]$ is the current algebra. By [**Kum02**, Theorem E.13], there exists a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{g}[t], \mathfrak{g}; \mathbb{C}) \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{g}[t], \mathbb{C}). \tag{1.2.7}$$

The edge map $\mathrm{H}^i(\mathfrak{g}[t], \mathbb{C}) \to E_2^{0,i} = \mathrm{H}^i(\mathfrak{g}, \mathbb{C})$ of the spectral sequence is just the restriction map induced by the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}[t]$. Since this inclusion splits via the evaluation map $\mathrm{ev}_0 : \mathfrak{g}[t] \to \mathfrak{g}$, the restriction map in cohomology $\mathrm{H}^i(\mathfrak{g}[t], \mathbb{C}) \to \mathrm{H}^i(\mathfrak{g}, \mathbb{C})$ is a split surjection. It follows that $d_r^{0,\bullet} = 0, r \geq 2$ and that the space $E_2^{0,\bullet}$ of (1.2.7) consists of permanent cycles, i.e. $E_2^{0,\bullet} = E_\infty^{0,\bullet}$. Hence, using the fact that $E_2^{i,j} \cong E_2^{i,0} \otimes E_2^{0,j}$ and $d_2^{i,j}(ab) = d_2^{i,0}(a)b + (-1)^i a d_2^{0,j}(b)$ in (1.2.7), and that $d_2^{\bullet,0} = 0$, it follows that the spectral sequence collapses at the E_2-page, yielding the isomorphism

$$\mathrm{H}^n(\mathfrak{g}[t], \mathbb{C}) \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathfrak{g}[t], \mathfrak{g}; \mathbb{C}) \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}).$$

Let M be a $\mathfrak{g}[t]$-module that is finitely semisimple for $\mathfrak{g}$, that is, as a $\mathfrak{g}$-module M decomposes as a possibly-infinite direct sum of finite-dimensional irreducible modules. Then by [**Kum02**, Theorem E.13], there exists a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{g}[t], \mathfrak{g}; M) \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{g}[t], M). \tag{1.2.8}$$

Moreover, (1.2.8) is a module over (1.2.7), and $E_2^{i,j} \cong E_2^{i,0} \otimes E_{2,\mathbb{C}}^{0,j}$, where $E_{2,\mathbb{C}}^{0,\bullet}$ denotes the space $E_2^{0,\bullet}$ in (1.2.7), which consists of permanent cycles. Using the derivation property of the differential on (1.2.8), namely, $d_2^{i,j}(mr) = d_2^{i,0}(m)r + (-1)^i m d_{2,\mathbb{C}}^{0,j}(r)$, it follows that the spectral sequence (1.2.8) also collapses at the E_2-page, and hence that

$$\mathrm{H}^n(\mathfrak{g}[t], M) \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathfrak{g}[t], \mathfrak{g}; M) \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}).$$

The space $\mathrm{H}^i(\mathfrak{g}[t], \mathfrak{g}; M)$ can be rewritten by [**Lep79**, Proposition 4.11] as $\mathrm{H}^i(\mathfrak{g}[t]_+, M)^{\mathfrak{g}}$ (as can also be seen by applying an LHS spectral sequence for relative Lie algebra cohomology), so we obtain

$$\mathrm{H}^n(\mathfrak{g}[t], M) \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathfrak{g}[t]_+, M)^{\mathfrak{g}} \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}). \tag{1.2.9}$$

Now let $\lambda, \mu \in P^+$. By abuse of notation, in this section we will denote $\mathrm{ev}_0^* V(\lambda)$ and $\mathrm{ev}_0^* V(\mu)$ simply by $V(\lambda)$ and $V(\mu)$. Taking $M = V(\lambda)^* \otimes V(\mu)$ in (1.2.9), and using the fact that $\mathfrak{g}[t]_+$ acts trivially on $V(\lambda)$ and $V(\mu)$, we get

$$\mathrm{Ext}^n_{\mathfrak{g}[t]}(V(\lambda), V(\mu)) \cong \bigoplus_{i+j=n} \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathrm{H}^i(\mathfrak{g}[t]_+, \mathbb{C}) \otimes V(\mu)) \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}).$$

Using the explicit description of $\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$, described below in Theorem 1.3.7, this provides an explicit description for $\mathrm{Ext}^n_{\mathfrak{g}[t]}(V(\lambda), V(\mu))$. This description was essentially known already to Fialowski and Malikov [**FM94**, Proposition 2].

1.2.3. Higher extensions between tensor products of evaluation modules. Consider $a_1, \ldots, a_n \in \mathbb{C}$ with $a_i \neq a_j$ if $i \neq j$, $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in P^+$, and set $V = \otimes_{i=1}^n \mathrm{ev}^*_{a_i} V(\lambda_i)$ and $V' = \otimes_{i=1}^n \mathrm{ev}^*_{a_i} V(\mu_i)$. Let $I \subset \mathbb{C}[t]$ be the ideal generated by $(t-a_1)\cdots(t-a_n)$. Then (1.2.2) becomes

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathfrak{g}[t]/\mathfrak{g}\otimes I}(V, \mathrm{H}^j(\mathfrak{g}\otimes I, \mathbb{C}) \otimes V') \Rightarrow \mathrm{Ext}^{i+j}_{\mathfrak{g}[t]}(V, V').$$

We consider the terms contributing to $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V')$. First, $E_2^{2,0} = 0$ by (1.2.4). Next, $\mathrm{H}^1(\mathfrak{g}\otimes I, \mathbb{C})$ is a finite-dimensional $\mathfrak{g}[t]/(\mathfrak{g}\otimes I) \cong \mathfrak{g}^{\oplus n}$-module by (1.2.6), so $E_2^{1,1} = 0$ by the first Whitehead Lemma. Then we are left to consider the space $E_2^{0,2} = \mathrm{Hom}_{\mathfrak{g}[t]/\mathfrak{g}\otimes I}(V, \mathrm{H}^2(\mathfrak{g}\otimes I, \mathbb{C}) \otimes V')$ and its contribution to $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V')$.

Using the second Whitehead Lemma we have $E_2^{2,1} = 0$ by a similar line of reasoning as for $E_2^{1,1}$. Then the differential $d_2 : E_2^{0,2} \to E_2^{2,1}$ is zero, so we have

$$\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V') \cong \ker(d_3 : E_3^{0,2} \to E_3^{3,0}).$$

Observe that $E_3^{3,0} \cong E_2^{3,0}$ because $E_2^{1,1} = 0$. Now by the Künneth formula and the first and second Whitehead Lemmas,

$$E_2^{3,0} \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\mathfrak{g}}(V(\lambda_1), V(\mu_1)) \otimes \cdots \otimes \mathrm{Ext}^3_{\mathfrak{g}}(V(\lambda_i), V(\mu_i)) \otimes \cdots \otimes \mathrm{Hom}_{\mathfrak{g}}(V(\lambda_n), V(\mu_n)).$$

Since $V(-)$ are simple $\mathfrak{g}$-modules, $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V(\lambda)^* \otimes V(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), V(\mu)) = 0$ unless $\lambda = \mu$. Similarly, since $\mathrm{H}^\bullet(\mathfrak{g}, M) = \mathrm{H}^\bullet(\mathfrak{g}, M^{\mathfrak{g}})$ for any finite-dimensional $\mathfrak{g}$-module M by the assumption that $\mathfrak{g}$ is a simple complex Lie algebra (cf. [**CE48**, Section 24]), and since $\dim \mathrm{H}^3(\mathfrak{g}, \mathbb{C}) = 1$ also by the same assumption on $\mathfrak{g}$, we conclude that $\mathrm{Ext}^3_{\mathfrak{g}}(V(\lambda), V(\mu)) = 0$ unless $\lambda = \mu$. Thus we have shown:

Theorem 1.2.2. If $\lambda_i \neq \mu_i$ for some $i = 1, \ldots, n$, then

$$\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V') \cong \mathrm{Hom}_{\mathfrak{g}[t]/\mathfrak{g}\otimes I}(V, \mathrm{H}^2(\mathfrak{g}\otimes I, \mathbb{C}) \otimes V').$$

□

The only case which is not covered by Theorem 1.2.2 is that when $V = V'$. In this case we can rewrite $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V) \cong \mathrm{H}^2(\mathfrak{g}[t], V^* \otimes V)$ and decompose $V^* \otimes V = \bigoplus_{\lambda_i'} \mathrm{ev}^*_{a_1} V(\lambda_1') \otimes \cdots \otimes \mathrm{ev}^*_{a_n} V(\lambda_n')$, where the direct sum runs through all $\lambda_i' \in P^+$ such that $V(\lambda_i')$ appears as a $\mathfrak{g}$-composition factor of $V(\lambda_i^*) \otimes V(\mu_i)$. Then the computation of $\mathrm{Ext}^2_{\mathfrak{g}[t]}(V, V)$ boils down to the computation of $\mathrm{H}^2(\mathfrak{g}[t], \mathrm{ev}^*_{a_1} V(\lambda_1') \otimes \cdots \otimes \mathrm{ev}^*_{a_n} V(\lambda_n'))$. So, in order to compute $\mathrm{Ext}^2(V, V')$, we must study the $\mathfrak{g}^{\oplus n}$-composition factors of $\mathrm{H}^2(\mathfrak{g}\otimes I, \mathbb{C})$.

1.3. Homology and cohomology of annihilating ideals

1.3.1. Composition factors of $\mathrm{H}^n(\mathfrak{g}\otimes I, \mathbb{C})$. For any Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ and a trivial $\mathfrak{a}$-module M, it follows from the Universal Coefficient Theorem (cf. [**Wei94**, Theorem 3.6.5]) that there are isomorphisms of $\mathbb{C}$-modules $\mathrm{H}^n(\mathfrak{a}, M) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_n(\mathfrak{a}, \mathbb{C}), M)$. In particular, $\mathrm{H}^n(\mathfrak{a}, \mathbb{C}) \cong \mathrm{H}_n(\mathfrak{a}, \mathbb{C})^*$. Thus $\mathrm{H}^n(\mathfrak{a}, \mathbb{C})$ is finite-dimensional if and only if $\mathrm{H}_n(\mathfrak{a}, \mathbb{C})$ is finite-dimensional.

Recall that $\mathrm{H}_n(\mathfrak{g}\otimes I, \mathbb{C})$ is a $\mathfrak{g}[t]$-module by definition. Hence $\mathrm{H}_n(\mathfrak{g}\otimes I, \mathbb{C})$ becomes a $\mathfrak{g}$-module by restriction.

Lemma 1.3.1. Let $f = (t - a_1) \cdots (t - a_r) \in \mathbb{C}[t]$ with $a_i \neq a_j$ if $i \neq j$, and let $I = \langle f \rangle \subset \mathbb{C}[t]$. If L is a $\mathfrak{g}$-composition factor of $\mathrm{H}_n(\mathfrak{g} \otimes I, \mathbb{C})$, then $L \cong V(\alpha)$ for some $\alpha \in Q^+$ such that $\alpha \leq n\theta$.

Proof. First observe that $\mathrm{H}_n(\mathfrak{g} \otimes I, \mathbb{C})$ is a $\mathfrak{g}$-subquotient of $\Lambda^n(\mathfrak{g} \otimes I)$, since the Koszul differential is a $\mathfrak{g}[t]$- (hence $\mathfrak{g}$-) module homomorphism. Also observe that $\Lambda^n(\mathfrak{g} \otimes I)$ is a quotient of $T^n(\mathfrak{g} \otimes I)$ as a $\mathfrak{g}$-module. Then observe that $\mathfrak{g} \otimes I \cong \left(\bigoplus_{i \geq 0} \mathfrak{g} \otimes \mathbb{C}(t^i f)\right)$ as a $\mathfrak{g}$-module. Denote $\mathfrak{g} \otimes \mathbb{C}(t^i f)$ by $\mathfrak{g}_i$ and observe that $\mathfrak{g}_i$ is isomorphic to the adjoint representation of $\mathfrak{g}$. Thus $T^n(\mathfrak{g} \otimes I)$ is isomorphic to $\left(\bigoplus_{0 \leq i_1, \ldots, i_n} (\mathfrak{g}_{i_1} \otimes \cdots \otimes \mathfrak{g}_{i_n})\right)$ as a $\mathfrak{g}$-module, with $(\mathfrak{g}_{i_1} \otimes \cdots \otimes \mathfrak{g}_{i_n}) \cong T^n \mathfrak{g}$. Since $T^n(\mathfrak{g} \otimes I)$ is semisimple, it follows that $\Lambda^n(\mathfrak{g} \otimes I)$ is semisimple, $\mathrm{H}_n(\mathfrak{g} \otimes I, \mathbb{C})$ is semisimple, and every $\mathfrak{g}$-composition factor of $\mathrm{H}_n(\mathfrak{g} \otimes I, \mathbb{C})$ is a composition factor of $T^n(\mathfrak{g} \otimes I)$. Since $\mathfrak{g}$ is isomorphic to $V(\theta)$, it follows that $T^n \mathfrak{g}$ is isomorphic to $\left(\bigoplus_{\lambda \in P^+} V(\lambda)^{\varepsilon(\lambda)}\right)$, where $\varepsilon(\lambda) \geq 0$ and $\varepsilon(\lambda) \neq 0$ only if $\lambda \in Q^+$ and $\lambda \leq n\theta$. □

It follows from Lemma 1.3.1 that $\mathrm{H}_n(\mathfrak{g} \otimes I, \mathbb{C})$ is a semisimple $\mathfrak{g}$-module, has finitely many distinct $\mathfrak{g}$-composition factors and that they are all finite-dimensional.

Remark 1.3.2. In Lemma 1.3.1 we showed that the highest weights of its composition factors as a $\mathfrak{g}$-module are bounded above by $n\theta$. Suppose $I = (t-a)(t-b)$. As usual, $\mathrm{H}^n(\mathfrak{g} \otimes I, \mathbb{C})$ is a $\mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g}[t]/\mathfrak{g} \otimes I$-module, with $\mathfrak{g}$ being mapped into $\mathfrak{g} \times \mathfrak{g}$ via the diagonal map. It follows that, if one decomposes $\mathrm{H}^n(\mathfrak{g} \otimes I, \mathbb{C})$ as a $\mathfrak{g} \times \mathfrak{g}$-module, namely $\oplus_{\lambda, \mu \in P^+} V(\lambda) \boxtimes V(\mu)$, then $\lambda + \mu$ must be bounded from above by $n\theta$.

1.3.2. A spectral sequence. From now on, let $a, b \in \mathbb{C}$, $a \neq b$, $f = (t-a)(t-b)$ and $I \subset \mathbb{C}[t]$ be the nontrivial ideal generated by f. The decreasing multiplicative filtration $\mathbb{C}[t] \supset I \supset I^2 \supset I^3 \supset \cdots$ on $\mathbb{C}[t]$ induces a corresponding decreasing Lie algebra filtration $\mathfrak{g}[t] \supset \mathfrak{g} \otimes I \supset \mathfrak{g} \otimes I^2 \supset \mathfrak{g} \otimes I^3 \supset \cdots$ on $\mathfrak{g}[t]$, which induces a Lie algebra filtration on $\mathfrak{g} \otimes I/I^s$ for any $s > 1$. Then the associated graded algebras $\mathrm{gr}\, \mathfrak{g}[t] := \bigoplus_{n \geq 0} (\mathfrak{g} \otimes I^n)/(\mathfrak{g} \otimes I^{n+1})$ and $\mathrm{gr}(\mathfrak{g} \otimes I/I^s) := \bigoplus_{0<n<s} (\mathfrak{g} \otimes I^n)/(\mathfrak{g} \otimes I^{n+1})$ satisfy $\mathrm{gr}\, \mathfrak{g}[t] \cong \mathfrak{g}[t] \oplus \mathfrak{g}[t]$ and $\mathrm{gr}(\mathfrak{g} \otimes I/I^s) \cong (\mathfrak{g} \otimes (t\mathbb{C}[t]/t^s\mathbb{C}[t])) \oplus (\mathfrak{g} \otimes (t\mathbb{C}[t]/t^s\mathbb{C}[t]))$. Denote $\mathfrak{g} \otimes (t\mathbb{C}[t]/t^s\mathbb{C}[t])$ by $\mathfrak{g}[t]_s^+$.

The filtration on $\mathfrak{g} \otimes I/I^s$ defines a decreasing filtration on the exterior algebra $\Lambda^\bullet(\mathfrak{g} \otimes I/I^s)$ via

$$F^j \Lambda^n(\mathfrak{g} \otimes I/I^s) = \sum_{\substack{j \leq j_1 + j_2 + \cdots + j_n \\ 0 < j_i, \ldots, j_n < s}} (\mathfrak{g} \otimes I^{j_1}/I^s) \wedge (\mathfrak{g} \otimes I^{j_2}/I^s) \wedge \cdots \wedge (\mathfrak{g} \otimes I^{j_n}/I^s). \tag{1.3.1}$$

By definition, this filtration is *bounded above* since $F^j \Lambda^\bullet(\mathfrak{g} \otimes I/I^s) = \Lambda^\bullet(\mathfrak{g} \otimes I/I^s)$ if $j \leq n$, and *bounded below* since $F^j \Lambda^n(\mathfrak{g} \otimes I/I^s) = 0$ if $j > n(s-1)$.

Define an increasing filtration on the cochain complex $C^\bullet = \mathrm{Hom}_{\mathbb{C}}(\Lambda^\bullet(\mathfrak{g} \otimes I/I^s), \mathbb{C})$ by

$$F^j C^n = \mathrm{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{g} \otimes I/I^s)/F^{j+1}\Lambda^n(\mathfrak{g} \otimes I/I^s), \mathbb{C}).$$

This filtration is compatible with the Koszul differential on $C^\bullet$ because the filtration on $\Lambda^\bullet(\mathfrak{g} \otimes I/I^s)$ is compatible with the Koszul differential. The grading on $\mathrm{gr}(\mathfrak{g} \otimes I/I^s)$ induces an internal grading on $\Lambda^\bullet(\mathrm{gr}(\mathfrak{g} \otimes I/I^s))$. Denote the j-th graded piece of $\Lambda^\bullet(\mathrm{gr}(\mathfrak{g} \otimes I/I^s))$ by $\Lambda^\bullet(\mathrm{gr}(\mathfrak{g} \otimes I/I^s))_{(j)}$. Then

$$F^j C^n / F^{j-1} C^n \cong \mathrm{Hom}_{\mathbb{C}}(\Lambda^n(\mathrm{gr}(\mathfrak{g} \otimes I/I^s))_{(j)}, \mathbb{C}).$$

Write δ for the Koszul differential on $C^\bullet$. Then $(C^\bullet, \delta)$ is a cochain complex with differential of degree $+1$ and an increasing filtration. Set $C_n := C^{-n}$, and set $F_j C_n = F^j C^{-n}$. Then $(C_\bullet, \delta)$ is a chain complex with differential of degree -1 and an increasing filtration. This is the typical setup for a homology spectral sequence associated to a filtered differential module. By Classical Convergence

Theorem (cf. [**Wei94**, 5.5.1]) there exists a homology spectral sequence $E^r_{p,q} \Rightarrow \mathrm{H}^{-(p+q)}(\mathfrak{g}\otimes I/I^s, \mathbb{C})$, with

$$\begin{aligned} E^0_{p,q} &= F_pC_{p+q}/F_{p-1}C_{p+q} \\ &= F^pC^{-(p+q)}/F^{p-1}C^{-(p+q)} \\ &= \mathrm{Hom}_{\mathbb{C}}(\Lambda^{-(p+q)}(\mathrm{gr}(\mathfrak{g}\otimes I/I^s))_{(p)}, \mathbb{C}), \end{aligned}$$

and $E^1_{p,q} = \mathrm{H}^{-(p+q)}(\mathrm{gr}(\mathfrak{g}\otimes I/I^s), \mathbb{C})_{(p)}$. Here $\mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I/I^s), \mathbb{C})$ inherits an internal grading from the grading on $\mathrm{gr}(\mathfrak{g}\otimes I/I^s)$, and $\mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I/I^s), \mathbb{C})_{(p)}$ denotes its p-th graded component. Observe that $E^0_{p,q} = E^1_{p,q} = 0$ unless $p+q \leq 0$, $2p+q \geq 0$ and $p \geq 0$.

1.3.3. A similar, non-convergent spectral sequence. We could have considered a similar homology spectral sequence, replacing $\mathfrak{g}\otimes I/I^s$ by $\mathfrak{g}\otimes I$. In that case, the terms of its E^0-page would be given by $E^0_{p,q} \cong \mathrm{Hom}_{\mathbb{C}}(\Lambda^{-(p+q)}(\mathrm{gr}(\mathfrak{g}\otimes I))_{(p)}, \mathbb{C})$ and its E^1-page would be globally isomorphic to $\mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I), \mathbb{C})$. However, we have not said anything about whether this spectral sequence converges, or, if it does, how the limit might be related to $\mathrm{H}^\bullet(\mathfrak{g}\otimes I, \mathbb{C})$.

Lemma 1.3.3. *$E^r \cong E^1$ for all $r \geq 1$, so the spectral sequence converges to $\mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I), \mathbb{C})$.*

Proof. First observe that the action of $\mathfrak{g}[t]$ commutes with the Koszul differential on the cochain complex $C'^\bullet = \mathrm{Hom}_{\mathbb{C}}(\Lambda^\bullet(\mathfrak{g}\otimes I), \mathbb{C})$. Consider

$$\begin{gathered} F'^j\Lambda^n\mathfrak{g}\otimes I = \sum_{j\leq j_1+j_2+\cdots+j_n} (\mathfrak{g}\otimes I^{j_1})\wedge(\mathfrak{g}\otimes I^{j_2})\wedge\cdots\wedge(\mathfrak{g}\otimes I^{j_n}), \\ F'^jC'^n = \mathrm{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{g}\otimes I)/F'^{j+1}\Lambda^n(\mathfrak{g}\otimes I), \mathbb{C}), \\ F'_jC'_n = F'^jC'^{-n}. \end{gathered}$$

It then follows from the construction of the spectral sequence associated to a filtered differential module (cf. [**Wei94**, The construction 5.4.6]) that the action of $\mathfrak{g}[t]$ passes to an action on the spectral sequence. Then notice that $(\mathfrak{g}\otimes I)\cdot F'_jC'_n \subseteq F'_{j-1}C'_n$. The construction of the spectral sequence then shows that the action of $\mathfrak{g}[t]$ factors through the quotient $\mathfrak{g}[t]/(\mathfrak{g}\otimes I) \cong \mathfrak{g}\times\mathfrak{g}$. Now the E^1-page is globally isomorphic to

$$\begin{aligned} \mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I), \mathbb{C}) &\cong \mathrm{H}^\bullet(\mathfrak{g}[t]_+\times\mathfrak{g}[t]_+, \mathbb{C}) \\ &\cong \mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})\otimes\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C}), \end{aligned}$$

and this identification is compatible with the action of $\mathfrak{g}[t]/(\mathfrak{g}\otimes I) \cong \mathfrak{g}\times\mathfrak{g}$. But $\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$ is completely reducible and multiplicity free as a $\mathfrak{g}$-module (cf. Subsection 1.3.7). Since the differential d^1 on E^1 is a $\mathfrak{g}[t]/(\mathfrak{g}\otimes I) \cong \mathfrak{g}\times\mathfrak{g}$-module homomorphism, this implies that $d^1 \equiv 0$, so $E^2 \cong E^1$. Now the same reasoning implies that $d^2 \equiv 0$, and then $d^3 \equiv 0$, and so on, so that $E^i \cong E^{i+1}$ for all $i \geq 1$, and $E^1_{p,q} = E^\infty_{p,q}$. □

Recall that the adjoint action of a Lie algebra $\mathfrak{a}$ on itself induces the trivial action on the Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{a}, \mathbb{C})$ (cf. [**Kum02**, p.71]). So the action of $\mathfrak{g}[t]$ on $\mathrm{H}^\bullet(\mathfrak{g}\otimes I, \mathbb{C})$ will automatically factor through the quotient $\mathfrak{g}[t]/(\mathfrak{g}\otimes I)$. We can say at this point that $\mathrm{H}^\bullet(\mathfrak{g}\otimes I, \mathbb{C})$ *is not* isomorphic as a $\mathfrak{g}[t]/(\mathfrak{g}\otimes I) \cong \mathfrak{g}\times\mathfrak{g}$-module to $\mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I), \mathbb{C})$. Indeed, suppose this were true. Then, by Künneth formula (cf. [**Wei94**, Theorem 3.6.3]), we would have an isomorphism of $\mathfrak{g}\times\mathfrak{g}$-modules $\mathrm{H}^\bullet(\mathfrak{g}\otimes I, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})\boxtimes\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$. Now consider the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{g}[t]/(\mathfrak{g}\otimes I), \mathrm{H}^j(\mathfrak{g}\otimes I, \mathbb{C})) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{g}[t], \mathbb{C}).$$

We could then rewrite the E_2-page of the spectral sequence as

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{c+d=j} \mathrm{H}^i(\mathfrak{g}\times\mathfrak{g}, \mathrm{H}^c(\mathfrak{g}[t]_+,\mathbb{C})\boxtimes \mathrm{H}^d(\mathfrak{g}[t]_+,\mathbb{C})) \\ &\cong \bigoplus_{a+b=i}\bigoplus_{c+d=j} \mathrm{H}^a(\mathfrak{g},\mathrm{H}^c(\mathfrak{g}[t]_+,\mathbb{C}))\otimes \mathrm{H}^b(\mathfrak{g},\mathrm{H}^d(\mathfrak{g}[t]_+,\mathbb{C})) \\ &\cong \bigoplus_{a+b=i}\bigoplus_{c+d=j} (\mathrm{H}^a(\mathfrak{g},\mathbb{C})\otimes \mathrm{H}^c(\mathfrak{g}[t]_+,\mathbb{C})^{\mathfrak{g}})\otimes(\mathrm{H}^b(\mathfrak{g},\mathbb{C})\otimes \mathrm{H}^d(\mathfrak{g}[t]_+,\mathbb{C})^{\mathfrak{g}}). \end{aligned}$$

By Theorem 1.3.11, $\mathrm{H}^\bullet(\mathfrak{g}[t]_+,\mathbb{C})^{\mathfrak{g}} = \mathrm{H}^0(\mathfrak{g}[t]_+,\mathbb{C}) = \mathbb{C}$. So now $E_2^{i,j}=0$ for all $j>0$, and the spectral sequence collapses to yield the isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}[t],\mathbb{C}) \cong E_2^{\bullet,0} = \mathrm{H}^\bullet(\mathfrak{g}[t]/\mathfrak{g}\otimes I,\mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}\times\mathfrak{g},\mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g},\mathbb{C})\otimes \mathrm{H}^\bullet(\mathfrak{g},\mathbb{C}).$$

This is absurd, because by Corollary 1.3.12, $\mathrm{H}^\bullet(\mathfrak{g}[t],\mathbb{C})\cong \mathrm{H}^\bullet(\mathfrak{g},\mathbb{C})$, and $\mathrm{H}^\bullet(\mathfrak{g},\mathbb{C})$ has dimension $2^{\mathrm{rank}(\mathfrak{g})}>1$. So we cannot have $\mathrm{H}^\bullet(\mathfrak{g}\otimes I,\mathbb{C})\cong \mathrm{H}^\bullet(\mathrm{gr}(\mathfrak{g}\otimes I),\mathbb{C})$ as a $\mathfrak{g}[t]/(\mathfrak{g}\otimes I)$-module.

It turns out that we have constructed a homology spectral sequence with $E^1_{p,q} = \mathrm{H}^{-(p+q)}(\mathrm{gr}\,\mathfrak{g}\otimes I,\mathbb{C})_{(p)}$ by considering an increasing filtration F' on the chain complex ${C'}_n$ which *does not* converge to $\mathrm{H}^\bullet(\mathfrak{g}\otimes I,\mathbb{C})$. One explanation why it doesn't converge is that the filtration on ${C'}_\bullet$ is not *exhaustive*, that is, that ${C'}_\bullet$ is not equal to $\cup_i {F'}_i{C'}_\bullet$. Indeed, if $f\in\cup_i{F'}_i{C'}_n$ then $f({F'}^j\Lambda^n(\mathfrak{g}\otimes I))=0$ for some $j\geq 0$, but there exist $f\in {C'}_n$ with $f({F'}^j\Lambda^n(\mathfrak{g}\otimes I))\neq 0$ for all $j\geq 0$.

We will explicitly see, in Example 1.3.17, a composition factor of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$ which does not appear as a composition factor of $\mathrm{H}^2(\mathrm{gr}\,\mathfrak{g}\otimes I,\mathbb{C})$.

1.3.4. Back to the convergent spectral sequence. Since the filtration $F_\bullet C_\bullet$ is bounded below and bounded above (cf. (1.3.1)), the spectral sequence constructed in Subsection 1.3.2 converges to $\mathrm{H}^\bullet(\mathfrak{g}\otimes I/I^s,\mathbb{C})$. Since $\mathfrak{g}\otimes I/I^s$ is finite-dimensional and since every finite-dimensional $\mathfrak{g}\times\mathfrak{g}$-module is semisimple, it follows that $\mathrm{H}^{-n}(\mathfrak{g}\otimes I/I^s,\mathbb{C})$ is isomorphic to $(\bigoplus_{p+q=n}E^\infty_{p,q})$ as a $\mathfrak{g}\times\mathfrak{g}$-module. Using again the fact that $\mathfrak{g}\otimes I/I^s$ is finite-dimensional, it follows that $E^1_{p,q}$ is finite-dimensional. Then $E^\infty_{p,q}$, which is a subquotient of $E^1_{p,q}$, is also finite-dimensional. And using again the fact that every finite-dimensional $\mathfrak{g}\times\mathfrak{g}$-module is semisimple, it follows that $E^\infty_{p,q}$ can be identified with a $\mathfrak{g}\times\mathfrak{g}$-submodule of $E^1_{p,q}$. Thus

$$\begin{aligned} \mathrm{H}^{-n}(\mathfrak{g}\otimes I/I^s,\mathbb{C}) \cong \left(\oplus_{p+q=n}E^\infty_{p,q}\right) \subseteq \left(\oplus_{p+q=n}E^1_{p,q}\right) &\cong \left(\oplus_{p+q=n}\mathrm{H}^{-n}(\mathrm{gr}(\mathfrak{g}\otimes I/I^s),\mathbb{C})_{(p)}\right) \\ &\cong \mathrm{H}^{-n}(\mathrm{gr}(\mathfrak{g}\otimes I/I^s),\mathbb{C}). \end{aligned} \tag{1.3.2}$$

Now recall that $\mathrm{gr}(\mathfrak{g}\otimes I/I^s)$ is isomorphic to $\mathfrak{g}[t]_s^+\oplus\mathfrak{g}[t]_s^+$ as a $\mathfrak{g}\times\mathfrak{g}$-module. Then, by Künneth formula (cf. [**Wei94**, Theorem 3.6.3]), there is an isomorphism of $\mathfrak{g}\times\mathfrak{g}$-modules

$$\mathrm{H}^n(\mathrm{gr}(\mathfrak{g}\otimes I/I^s),\mathbb{C}) \cong \left(\bigoplus_{p+q=n}\mathrm{H}^p(\mathfrak{g}[t]_s^+,\mathbb{C})\boxtimes \mathrm{H}^q(\mathfrak{g}[t]_s^+,\mathbb{C})\right). \tag{1.3.3}$$

Thus it follows from (1.3.2) that, for each $n\geq 0$, $\mathrm{H}^n(\mathfrak{g}\otimes I/I^s,\mathbb{C})$ is isomorphic to a $\mathfrak{g}\times\mathfrak{g}$-submodule of $\bigoplus_{p+q=n}\mathrm{H}^p(\mathfrak{g}[t]_s^+,\mathbb{C})\boxtimes\mathrm{H}^q(\mathfrak{g}[t]_s^+,\mathbb{C})$.

1.3.5. Low degree cohomology and homology of $\mathfrak{g}[t]_s^+$. We already know that $\mathrm{H}^0(\mathfrak{g}[t]_s^+,\mathbb{C})$ is isomorphic to $\mathbb{C}$ and that

$$\mathrm{H}^1(\mathfrak{g}[t]_s^+,\mathbb{C}) \cong \left(\frac{\mathfrak{g}[t]_s^+}{\left[\mathfrak{g}[t]_s^+,\mathfrak{g}[t]_s^+\right]}\right)^* \cong \left(\mathfrak{g}\otimes\frac{t\mathbb{C}[t]/t^s\mathbb{C}[t]}{t^2\mathbb{C}[t]/t^s\mathbb{C}[t]}\right)^* \cong \mathfrak{g}^*.$$

In order to compute $\mathrm{H}^2(\mathfrak{g}[t]_s^+, \mathbb{C})$, consider the following Lyndon–Hochschild–Serre spectral sequence $E_2^{p,q} \cong \mathrm{H}^p(\mathfrak{g}[t]_s^+, \mathrm{H}^q(\mathfrak{g} \otimes (t^s\mathbb{C}[t]), \mathbb{C})) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}[t]_+, \mathbb{C})$. Since $E_2^{1,0} \cong \mathrm{H}^1(\mathfrak{g}[t]_s^+, \mathbb{C}) \cong \mathfrak{g}^* \cong \mathrm{H}^1(\mathfrak{g}[t]_+, \mathbb{C})$ and $\mathrm{H}^0(\mathfrak{g}[t]_s^+, \mathrm{H}^1(\mathfrak{g} \otimes (t^s\mathbb{C}[t]), \mathbb{C})) \cong \mathrm{H}^0(\mathfrak{g}[t]_s^+, (\mathfrak{g} \otimes (t^s\mathbb{C}[t])/(t^{2s}\mathbb{C}[t]))^*) \cong \mathfrak{g}^*$, its associated 5-term exact sequence of low degree terms yields the following exact sequence

$$0 \to \mathfrak{g}^* \to \mathrm{H}^2(\mathfrak{g}[t]_s^+, \mathbb{C}) \to \mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}). \tag{1.3.4}$$

We want to prove that the map on the right of (1.3.4) is surjective. By comparing dimensions and using the Universal Coefficient Theorem, this is equivalent to proving that

$$0 \to \mathrm{H}_2(\mathfrak{g}[t]_+, \mathbb{C}) \to \mathrm{H}_2(\mathfrak{g}[t]_s^+, \mathbb{C}) \to \mathfrak{g} \to 0 \tag{1.3.5}$$

is an exact sequence of $\mathfrak{g}$-modules. The exactness of $\mathrm{H}_2(\mathfrak{g}[t]_+, \mathbb{C}) \to \mathrm{H}_2(\mathfrak{g}[t]_s^+, \mathbb{C}) \to \mathfrak{g} \to 0$ is guaranteed by the 5-term exact sequence of low degree terms associated to the Lyndon–Hochschild–Serre spectral sequence $E^2_{p,q} \cong \mathrm{H}_p(\mathfrak{g}[t]_s^+, \mathrm{H}_q(\mathfrak{g} \otimes (t^s\mathbb{C}[t]), \mathbb{C})) \Rightarrow \mathrm{H}_{p+q}(\mathfrak{g}[t]_+, \mathbb{C})$. In order to prove the injectivity of the edge homomorphism $\mathrm{H}_2(\mathfrak{g}[t]_+, \mathbb{C}) \to \mathrm{H}_2(\mathfrak{g}[t]_s^+, \mathbb{C})$, first observe that that is an inflation map induced by the quotient map of Lie algebras $\pi : \mathfrak{g}[t]_+ \to \mathfrak{g}[t]_s^+$.

Now denote the Chevalley complex of $\mathfrak{g}[t]_+$ by $(\Lambda^\bullet\mathfrak{g}[t]_+, \partial_\bullet)$, the Chevalley complex of $\mathfrak{g}[t]_s^+$ by $\left(\Lambda^\bullet\mathfrak{g}[t]_s^+, \overline{\partial}_\bullet\right)$, and the morphism of complexes induced by π by $\Lambda^\bullet\pi : \Lambda^\bullet\mathfrak{g}[t]_+ \to \Lambda^\bullet\mathfrak{g}[t]_s^+$. Then consider the gradings on $\mathfrak{g}[t]_+$ and $\mathfrak{g}[t]_s^+$ induced by the grading on $\mathbb{C}[t]$ by powers of t. That is $\mathfrak{g}[t]_+ = \oplus_{d>0}(\mathfrak{g}[t]_+)_{(d)}$, with $(\mathfrak{g}[t]_+)_{(d)} = \mathfrak{g} \otimes (\mathbb{C}t^d)$ and $\mathfrak{g}[t]_s^+ = \oplus_{0<d<s}(\mathfrak{g}[t]_s^+)_{(d)}$, with $(\mathfrak{g}[t]_s^+)_{(d)} = \mathfrak{g} \otimes (\mathbb{C}t^d)$. This grading on $\mathfrak{g}[t]_+$ also induces a decreasing Lie algebra filtration $\mathfrak{g}[t]_+ \supset (\mathfrak{g} \otimes t^2\mathbb{C}[t]) \supset \cdots \supset (\mathfrak{g} \otimes t^m\mathbb{C}[t]) \supset \cdots$ on $\mathfrak{g}[t]_+$, and a decreasing filtration on its Chevalley complex

$$F_p\Lambda^n := \sum_{\substack{m_1+\ldots+m_n \geq p \\ m_1,\ldots,m_n \geq 1}} (\mathfrak{g} \otimes t^{m_1}\mathbb{C}[t]) \wedge \cdots \wedge (\mathfrak{g} \otimes t^{m_n}\mathbb{C}[t]) \subseteq \Lambda^n(\mathfrak{g}[t]_+).$$

Observe that the isomorphism of $\mathbb{C}$-modules $(\mathfrak{g}[t]_+) \cong (\mathfrak{g} \otimes t^s\mathbb{C}[t]) \oplus (\mathfrak{g}[t]_s^+)$ induces an isomorphism of $\mathbb{C}$-modules $\Lambda^p(\mathfrak{g}[t]_+) \cong \left(\bigoplus_{a+b=p} \Lambda^a(\mathfrak{g} \otimes t^s\mathbb{C}[t]) \otimes \Lambda^b(\mathfrak{g}[t]_s^+)\right)$. Via this latter isomorphism, $(\Lambda^p\pi)$ decomposes as $(\Lambda^p\pi)\left(\bigoplus_{a+b+1=p} \Lambda^{a+1}(\mathfrak{g} \otimes t^s\mathbb{C}[t]) \otimes \Lambda^b(\mathfrak{g}[t]_s^+)\right) = 0$ and $(\Lambda^p\pi)|_{\Lambda^p(\mathfrak{g}[t]_s^+)} = \mathrm{id}$, implying that

$$\ker(\Lambda^p\pi) \subseteq F_{s+p-1}\Lambda^p \subseteq F_s\Lambda^p. \tag{1.3.6}$$

Now observe that $\Lambda^\bullet\mathfrak{g}[t]_+$, $im(\partial_\bullet)$, $\ker(\partial_\bullet)$ and $\mathrm{H}_\bullet(\mathfrak{g}[t]_+, \mathbb{C})$ induce a grading from $\mathfrak{g}[t]_+$. Since $\mathrm{H}_p(\mathfrak{g}[t]_+, \mathbb{C})$ is finite-dimensional (cf. [**GL76**, Theorem 8.6]), it follows that there is $s > 1$, such that $\mathrm{H}_p(\mathfrak{g}[t]_+, \mathbb{C})_{(d)} = 0$ for all $d \geq s$. Thus, for each $p \geq 0$, there is $s(p) > 1$, such that, for all $s > s(p)$, $im(\partial_{p+1}) \cap (\Lambda^p(\mathfrak{g}[t]_+))_{(s)} = \ker(\partial_p) \cap (\Lambda^p(\mathfrak{g}[t]_+))_{(s)}$. Since the decreasing filtration on $\Lambda^\bullet(\mathfrak{g}[t]_+)$ is induced from its grading, it follows that

$$(F_s\Lambda^p) \cap (im(\partial_{p+1})) = (F_s\Lambda^p) \cap (\ker(\partial_p)), \forall\ s > s(p). \tag{1.3.7}$$

Proposition 1.3.4. For each $p \geq 0$, there exists $s(p) > 1$, such that the morphism of $\mathfrak{g}$-modules $\pi_* : \mathrm{H}_p(\mathfrak{g}[t]_+, \mathbb{C}) \to \mathrm{H}_p(\mathfrak{g}[t]_s^+, \mathbb{C})$ is injective for all $s > s(p)$.

Proof. Choose $s > 1$ such that $(F_s\Lambda^p) \cap (im(\partial_{p+1})) = (F_s\Lambda^p) \cap (\ker(\partial_p))$ (cf. (1.3.7)). Now suppose $\ker \pi_* \neq \{0\}$. This means that there is $\mathsf{h}_p \in \mathrm{H}_p(\mathfrak{g}[t]_+, \mathbb{C}) \setminus \{0\}$, such that $\pi_*(\mathsf{h}_p) = 0$. That is $\mathsf{h}_p = \mathsf{k}_p + im(\partial_{p+1})$, for some $\mathsf{k}_p \in \ker \partial_p \setminus im(\partial_{p+1})$, such that $(\Lambda^p\pi)(\mathsf{k}_p) \in im(\overline{\partial}_{p+1})$. Observe that $\Lambda^p\pi$ is surjective for each $p \geq 0$. Thus $im(\overline{\partial}_{p+1}) = (\Lambda^p\pi)(im(\partial_{p+1}))$. It follows that $\ker \pi_* \neq \{0\}$ is equivalent to having $\lambda_{p+1} \in (\Lambda^{p+1}(\mathfrak{g}[t]_+))$ such that $(\mathsf{k}_p - \partial_{p+1}(\lambda_{p+1})) \in \ker(\Lambda^p\pi)$. Since $(\mathsf{k}_p - \partial_{p+1}(\lambda_{p+1})) \in \ker(\partial_p)$, $\ker(\Lambda^p\pi) \subset F_s\Lambda^p$ (cf. (1.3.6)) and $(F_s\Lambda^p) \cap (im(\partial_{p+1})) = (F_s\Lambda^p) \cap (\ker(\partial_p))$ (cf. (1.3.7)), it follows that $(\mathsf{k}_p - \partial_{p+1}(\lambda_{p+1})) \in (F_s\Lambda^p) \cap (im(\partial_{p+1}))$. But this contradicts the fact that $\mathsf{k}_p \notin im(\partial_{p+1})$. □

Using Proposition 1.3.4 and the fact that the edge homomorphism $\mathrm{H}_2(\mathfrak{g}[t]_+, \mathbb{C}) \to \mathrm{H}_2(\mathfrak{g}[t]_s^+, \mathbb{C})$ is given by π_*, it follows that (1.3.5) is an exact sequence for all $s > s(2)$. Hence we proved the following result.

Theorem 1.3.5. There is $s(2) > 0$ such that $0 \to \mathfrak{g}^* \to \mathrm{H}^2(\mathfrak{g}[t]_s^+, \mathbb{C}) \to \mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}) \to 0$ is an exact sequence of $\mathfrak{g}$-modules for all $s > s(2)$. □

Thus if s is sufficiently large, $\mathrm{H}^2(\mathfrak{g}[t]_s^+, \mathbb{C})$ is isomorphic to $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}) \oplus \mathfrak{g}^*$ as a $\mathfrak{g}$-module. An explicit computation of $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C})$ is given in [**GL76**, Theorem 8.6], as we explain in 1.3.6.

In [**FGT08**], Fishel-Grojnowski-Teleman prove that $\mathrm{H}^n(\mathfrak{g} \otimes \mathbb{C}[t]/\mathfrak{g} \otimes t^s\mathbb{C}[t], \mathbb{C}) \cong \mathrm{H}^n(\mathfrak{g}, \mathbb{C})^{\otimes s}$. Our previous computation can be related to theirs via an LHS spectral sequence.

1.3.6. Cohomology of $\mathfrak{g}[t]_+$. Recall the standard realization for $\widehat{\mathfrak{g}}$ (cf. Section 0.2). In [**GL76**, §2] Garland-Lepowsky define the affine Lie algebra as $\tilde{\mathfrak{g}} \oplus \mathbb{C}c$, so what we denote by $\widehat{\mathfrak{g}}$ they denote by $\mathfrak{g}^e$ (cf. [**GL76**, p.48]). In [**GL76**, §3], the subalgebra $\mathfrak{g}_S$ of $\widehat{\mathfrak{g}}$ is our $\mathfrak{g}$, their $\mathfrak{r}$ is our $\mathfrak{g} \oplus \mathbb{C}c$, and their $\mathfrak{r}^e$ is our $\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Also recall that the fundamental dominant weights $\Lambda_0, \ldots, \Lambda_n \in \widehat{\mathfrak{h}}^*$ are defined by $\Lambda_i(h_j) = \delta_{ij}$ (Kronecker delta) and $\Lambda_i(d) = 0$. Set $P_S = \{\lambda \in \widehat{\mathfrak{h}}^* : \lambda(h_i) \in \mathbb{N}\ \forall\ i = 1, \ldots, n\} = \mathbb{C}\delta + \mathbb{C}\Lambda_0 + \sum_{i=1}^n \mathbb{N}\Lambda_i$.

Proposition 1.3.6. [**GL76**, Proposition 3.1] There is a natural bijection, denoted $\lambda \mapsto M(\lambda)$, between P_S and the set of (isomorphism classes of) finite-dimensional irreducible $\mathfrak{r}^e$-modules which are irreducible as $\mathfrak{g}$-modules. The correspondence is described as follows: The highest weight space (relative to $\mathfrak{h}$) of the $\mathfrak{g}$-module $M(\lambda)$ is $\widehat{\mathfrak{h}}$-stable, and λ is the resulting weight for the action of $\widehat{\mathfrak{h}}$.

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Then V is made an irreducible $\mathfrak{r}^e$-module by having c and d act as zero. Conversely, every finite-dimensional irreducible $\mathfrak{r}^e$-module that restricts to V as a $\mathfrak{g}$-module can be obtained as the tensor product of V and certain one-dimensional representations for $\mathbb{C}c$ and $\mathbb{C}d$.

Recall that W_a^1 is the set of minimal length left coset representatives for $\mathcal{W}$ in $\widehat{\mathcal{W}}$. Let $\hat{\rho} = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_n$, and for, $w \in \mathcal{W}$, set $w \cdot 0 = w\hat{\rho} - \hat{\rho}$. If $w_1, w_2 \in \widehat{\mathcal{W}}$ and $w_1 \cdot 0 = w_2 \cdot 0$, then $w_1 = w_2$ [**GL76**, Corollary 2.6]. If $w \in W_a^1$, then $w \cdot 0 \in P_S$ [**GL76**, Theorem 8.5]. In particular, the $M(w \cdot 0)$ for $w \in W_a^1$ are mutually non-isomorphic as $\mathfrak{r}^e$-modules.

Garland-Lepowsky compute the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$ by studying the standard Koszul complex for $\mathfrak{g}[t]_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$, computing the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}[t]_-, \mathbb{C})$, and then using an involution that maps $\mathfrak{g}[t]_-$ isomorphicaly to $\mathfrak{g}[t]_+$. The action of $\mathfrak{r}^e$ on $\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$ is then induced by the action of $\mathfrak{r}^e$ on the Koszul complex.

Theorem 1.3.7. [**Lep79**, Theorem 5.7] For each $j \geq 0$, there exists an isomorphism of $\mathfrak{r}^e$-modules

$$\mathrm{H}^j(\mathfrak{g}[t]_+, \mathbb{C}) \cong \bigoplus_{w \in W_a^1, \ell(w) = j} M(w \cdot 0).$$

Since c is central in $\widehat{\mathfrak{g}}$, it acts trivially on the Koszul complex, and hence also acts trivially on $\mathrm{H}^\bullet(\mathfrak{g}[t]_+, \mathbb{C})$. By Theorem 1.3.7, c acts trivially on $M(w \cdot 0)$ for all $w \in W_a^1$. For each $w \in W_a^1$, let $w \cdot 0 = \lambda_w - d_w\delta \in P_S$, with $\lambda_w \in P^+, d_w \geq 0$. Using Theorem 1.3.7 and the Cartan involution on $\hat{\mathfrak{g}}$ (cf. [**Lep79**, p.185,190]), it follows that, for each $j \geq 0$, there exists an isomorphism of $\mathfrak{g}$-modules

$$\mathrm{H}^j(\mathfrak{g}[t]_+, \mathbb{C}) \cong \bigoplus_{\substack{w \in W_a^1 \\ \ell(w) = j}} V(\lambda_w)^*, \tag{1.3.8}$$

with $V(\lambda_w)^*$ concentrated on t-degree d_w for each $w \in W_a^1$.

Affine Lie type	$\alpha_i(\alpha_0^\vee) \neq 0$
$\tilde{A}_1$	$i = 1$
$\tilde{A}_{n,\ n\geq 2}$	$i = 1, n$
$\tilde{B}_{n,\ n\geq 2}$	$i = 2$
$\tilde{C}_{n,\ n\geq 3}$	$i = 1$
$\tilde{D}_{n,\ n\geq 4}$	$i = 2$
$\tilde{E}_6$	$i = 2$
$\tilde{E}_7$	$i = 1$
$\tilde{E}_8$	$i = 8$
$\tilde{F}_4$	$i = 1$
$\tilde{G}_2$	$i = 2$

TABLE 1.

Example 1.3.8. In order to compute $\mathrm{H}^0(\mathfrak{g}[t]_+, \mathbb{C})$, one must consider $w \in W_a^1 \subset \widehat{\mathcal{W}}$ such that $\ell(w) = 0$. Hence w must be 1 and $\mathrm{H}^0(\mathfrak{g}[t]_+, \mathbb{C}) \cong M(1 \cdot 0) \cong \mathbb{C}$, concentrated on t-degree 0.

Example 1.3.9. In order to compute $\mathrm{H}^1(\mathfrak{g}[t]_+, \mathbb{C})$, one must consider $w \in W_a^1 \subset \widehat{\mathcal{W}}$ such that $\ell(w) = 1$. Hence $w = s_i$ for some $i = 0, 1, \dots, n$. If $i = 1, \dots, n$, then s_i is in the same coset as 1. Thus the only element in W_a^1 of length 1 must be s_0, and $\mathrm{H}^1(\mathfrak{g}[t]_+, \mathbb{C}) \cong_{\mathfrak{r}^e} M(r_0 \cdot 0)$. Since $s_0 \cdot 0 = -\alpha_0 = (\theta - \delta)$, it follows that $\mathrm{H}^1(\mathfrak{g}[t]_+, \mathbb{C}) \cong_{\mathfrak{g}} \mathfrak{g}^*$, concentrated on t-degree 1.

In order to compute $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C})$, one must consider $w \in W_a^1 \subset \widehat{\mathcal{W}}$ such that $\ell(w) = 2$. Thus w must have a reduced expression of the form $s_i s_j$, with $0 \leq i, j \leq n$. If $i = 1, \dots, n$ or $i = 0$ and $s_j s_0 = s_0 s_j$, then w is in the same coset as an element of length smaller than 2. Thus w must have the form $s_0 s_j$ with $j = 1, \dots, n$ satisfying $\alpha_j(\alpha_0^\vee) \neq 0$. Table 1 contains the roots satisfying $\alpha_i(\alpha_0^\vee) \neq 0$ in each Lie type, according to [**Bou68**, Plates I-IX].

For any $s_i \in W$, $s_0 \cdot s_i \cdot 0 = s_0 \cdot (-\alpha_i) = ((1 - \alpha_i(\alpha_0^\vee))\theta - \alpha_i) - (1 - \alpha_i(\alpha_0^\vee))\delta$. From the Dynking diagrams, it follows that $\alpha_i(\alpha_0^\vee) = -1$ if $\hat{\mathfrak{g}} \ncong \hat{\mathfrak{sl}}_2$, and $\alpha_i(\alpha_0^\vee) = -2$ if $\hat{\mathfrak{g}} \cong \hat{\mathfrak{sl}}_2$ (cf. [**Bou68**, Plates I-IX]).

Proposition 1.3.10. If $\mathfrak{g} \cong \mathfrak{sl}_2$, then $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}) \cong_{\mathfrak{g}} V(4)$, concentrated in degree 3. If $\hat{\mathfrak{g}}$ is of type $\tilde{A}_n$ with $n > 1$, then $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}) \cong_{\mathfrak{g}} V(2\theta - \alpha_1)^* \oplus V(2\theta - \alpha_n)^*$, concentrated in degree 2. In any other type, $\mathrm{H}^2(\mathfrak{g}[t]_+, \mathbb{C}) \cong_{\mathfrak{g}} V(2\theta - \alpha_i)^*$, concentrated in degree 2, with i as in Table 1. ☐

1.3.7. Restriction map. According to Feigin [**Fei80**], the restriction map

$$\mathrm{H}^\bullet(\mathfrak{g}[t], \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$$

induced by the evaluation homomorphism $\mathrm{ev}_0 : \mathfrak{g}[t] \to \mathfrak{g}$ is a ring isomorphism. He states that this result can be deduced from the calculations of [**GL76**], though he provides no details or explanation. It seems likely that Feigin's strategy would have been to take $M = \mathbb{C}$ in (1.2.9). Then the isomorphism $\mathrm{H}^\bullet(\mathfrak{g}[t], \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$ follows from showing for all $j \geq 1$ that $\mathrm{H}^j(\mathfrak{g}[t]_+, \mathbb{C})^{\mathfrak{g}} = 0$.

Recall that, by Theorem 1.3.7, c acts trivially on $M(w \cdot 0)$ for all $w \in W_a^1$, so if $w_1, w_2 \in W_a^1$ and $M(w_1 \cdot 0) \cong M(w_2 \cdot 0)$ as $\mathfrak{g}$-modules, then also $M(w_1 \cdot 0) \cong M(w_2 \cdot 0)$ as $\mathfrak{g} \oplus \mathbb{C}c$-modules. It now follows from the proof of [**Lep79**, Lemma 6.8] that the $M(w \cdot 0)$ for $w \in W_a^1$ are mutually non-isomorphic as $\mathfrak{g}$-modules.

Theorem 1.3.11. $\mathrm{H}^j(\mathfrak{g}[t]_+, \mathbb{C})^{\mathfrak{g}} = 0$ if $j \geq 1$.

Proof. By the discussion of the previous paragraph, the $M(w \cdot 0)$ for $w \in W_a^1$ are mutually non-isomorphic as $\mathfrak{g}$-modules. Thus, the trivial $\mathfrak{g}$-module occurs as a $\mathfrak{g}$-summand of $\mathrm{H}^j(\mathfrak{g}[t]_+, \mathbb{C})$ only for $j = 0$, where it corresponds to the identity element in W_a^1. $\square$

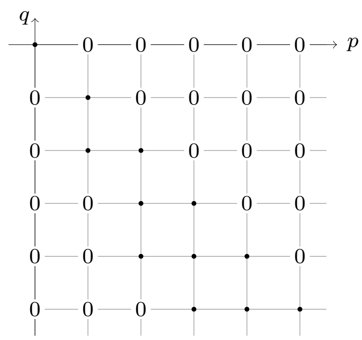
Corollary 1.3.12. There exists an isomorphism $\mathrm{H}^\bullet(\mathfrak{g}[t], \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{C})$.

1.3.8. Second cohomology of $\mathfrak{g} \otimes I/I^s$. Recall that $I = (t-a)(t-b)$. By (1.3.2) we know that $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$ is a $\mathfrak{g} \times \mathfrak{g}$-submodule of $\mathrm{H}^2(\mathrm{gr}(\mathfrak{g} \otimes I/I^s), \mathbb{C})$. Since $\mathrm{H}^2(\mathfrak{g}[t]_s^+, \mathbb{C})$ was calculated in Proposition 1.3.10, it gives us an explicit upper bound for the $\mathfrak{g} \times \mathfrak{g}$-composition factors of $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$.

In Subsection 1.3.2 we constructed a fourth-quadrant spectral sequence

$$E^1_{p,q} = \Big(\oplus_{a+b=-(p+q)} \mathrm{H}^a(\mathfrak{g}[t]_s^+, \mathbb{C}) \boxtimes \mathrm{H}^b(\mathfrak{g}[t]_s^+, \mathbb{C})\Big)_{(p)} \Rightarrow \mathrm{H}^{-(p+q)}(\mathfrak{g} \otimes I/I^s, \mathbb{C}).$$

Recall that $E^1_{p,q}$ is nonzero only if $2p + q \geq 0$, which gives us the following picture of the E^1-page:



By Example 1.3.8, $\mathrm{H}^0(\mathfrak{g}[t]_s^+, \mathbb{C}) \cong \mathbb{C}$ is concentrated in degree 0. Thus $E^1_{p,-p} = 0$ for all $p > 0$, and $E^1_{0,0} = \mathbb{C}$. By Example 1.3.9, $\mathrm{H}^1(\mathfrak{g}[t]_s^+, \mathbb{C}) \cong \mathfrak{g}^*$ is concentrated in degree 1. Thus $E^1_{p,-(p+1)} = 0$ for all $p > 1$, and $E^1_{1,-2} = \mathfrak{g}^* \oplus \mathfrak{g}^*$. It follows that $E^1_{2,-4} = E^\infty_{2,-4}$ and $E^1_{3,-5} = E^\infty_{3,-5}$, since

$$0 = E^r_{(2-r),(r-5)} \xrightarrow{d^r} E^r_{2,-4} \xrightarrow{d^r} E^r_{(r+2),-(r+3)} = 0 \quad \text{and}$$
$$0 = E^r_{(3-r),(r-6)} \xrightarrow{d^r} E^r_{3,-5} \xrightarrow{d^r} E^r_{(r+3),-(r+4)} = 0,$$

for $r \geq 1$. Hence $E^1_{2,-4} \cong \mathrm{H}^2(\mathrm{gr}(\mathfrak{g} \otimes I/I^s), \mathbb{C})_{(2)}$ and $E^1_{3,-5} \cong \mathrm{H}^2(\mathrm{gr}(\mathfrak{g} \otimes I/I^s), \mathbb{C})_{(3)}$ must appear as $\mathfrak{g} \times \mathfrak{g}$-composition factors of $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$. In particular, $\mathrm{H}^1(\mathfrak{g}[t]_s^+, \mathbb{C}) \boxtimes \mathrm{H}^1(\mathfrak{g}[t]_s^+, \mathbb{C})$, which is in degree 2, and $\mathrm{H}^2(\mathfrak{g}[t], \mathbb{C}) \boxtimes \mathbb{C}$ and $\mathbb{C} \boxtimes \mathrm{H}^2(\mathfrak{g}[t], \mathbb{C})$, which are in either degree 2 or 3, are composition factors of $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$.

Now consider the LHS spectral sequence $E_2^{p,q} \cong \mathrm{H}^p(\mathfrak{g} \otimes I/\mathfrak{g} \otimes I^s, \mathrm{H}^q(\mathfrak{g} \otimes I^s, \mathbb{C})) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g} \otimes I, \mathbb{C})$. Since $E_2^{1,0} \cong \mathrm{H}^1(\mathfrak{g} \otimes I/I^s, \mathbb{C}) \cong (\mathfrak{g} \oplus \mathfrak{g})^* \cong \mathrm{H}^1(\mathfrak{g} \otimes I, \mathbb{C})$, and $E^2_{0,1} \cong \mathfrak{g}^* \oplus \mathfrak{g}^*$, its 5-term exact sequence of low degree terms yields the following exact sequence of $\mathfrak{g} \times \mathfrak{g}$-modules

$$0 \to \mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C}) \to \mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C}). \tag{1.3.9}$$

It follows from this exact sequence that $(\mathfrak{g}^* \boxtimes \mathbb{C}) \oplus (\mathbb{C} \boxtimes \mathfrak{g}^*)$ is also a composition factor of $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$.

The discussion above shows that every composition factor of $\mathrm{H}^2(\mathrm{gr}(\mathfrak{g} \otimes I/I^s), \mathbb{C})$ is a composition factor of $\mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$, proving the following result.

Theorem 1.3.13. If $s > s(2)$, then $\mathrm{H}^2(\mathrm{gr}(\mathfrak{g} \otimes I/I^s), \mathbb{C}) \cong \mathrm{H}^2(\mathfrak{g} \otimes I/I^s, \mathbb{C})$ as $\mathfrak{g} \times \mathfrak{g}$-modules. $\square$

1.3.9. Second cohomology of $\mathfrak{g}\otimes I$. Our goal now is to give a more precise description of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$. We already know from (1.3.9) that $0\to(\mathfrak{g}\times\mathfrak{g})^*\xrightarrow{d_2}\mathrm{H}^2(\mathfrak{g}\otimes I/I^s,\mathbb{C})\xrightarrow{\inf}\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$ is an exact sequence of $\mathfrak{g}\times\mathfrak{g}$-modules. Even though the inflation map may not be surjective, its image is a $\mathfrak{g}\times\mathfrak{g}$-submodule of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$. By the first isomorphism theorem of $U(\mathfrak{g}\times\mathfrak{g})$-modules, $im(\inf)\cong\mathrm{H}^2(\mathfrak{g}\otimes I/I^s,\mathbb{C})/\ker(\inf)$, with $\ker(\inf)=im(d_2)=(\mathfrak{g}\times\mathfrak{g})^*$. Thus, by Theorem 1.3.13, (1.3.3) and Proposition 1.3.10, $\mathfrak{g}^*\boxtimes\mathfrak{g}^*$, $V(w\cdot 0)^*\boxtimes\mathbb{C}$ and $\mathbb{C}\boxtimes V(w\cdot 0)^*$, with $w\in W_a^1$, $\ell(w)=2$, are $\mathfrak{g}\times\mathfrak{g}$-composition factors of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$.

Lemma 1.3.14. If $\lambda,\mu\in P^+$ and $\lambda\neq 0$, then

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(\lambda)\boxtimes V(\mu),\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C}))\cong\mathrm{Hom}_{\mathfrak{g}}(V(\lambda),\mathrm{H}^2(\mathfrak{g}[t]_+,\mathrm{ev}_c^*V(\mu)^*)),\ \text{with } c\neq 0.$$

Proof. Consider the following Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g}[t]/\mathfrak{g}\otimes I,\mathrm{H}^q(\mathfrak{g}\otimes I,\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu)))\Rightarrow\mathrm{H}^{p+q}(\mathfrak{g}[t],\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu)).$$

First observe that $E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g}\oplus\mathfrak{g},\mathbb{C})\otimes\mathrm{H}^0(\mathfrak{g}\times\mathfrak{g},\mathrm{H}^q(\mathfrak{g}\otimes I,\mathbb{C})\otimes(V(\lambda)\boxtimes V(\mu)))$. Since $\lambda\neq 0$, the $\mathfrak{g}\times\mathfrak{g}$-module $V(\lambda)\boxtimes V(\mu)$ is nontrivial and $E_2^{p,0}=0$ for all $p\geq 0$. Since $\mathfrak{g}$ is simple, first and second Whitehead Lemmas imply that $E_2^{2,q}=E_2^{1,q}=0$ for all $q\geq 0$. Thus

$$\begin{aligned}\mathrm{H}^2(\mathfrak{g}[t],\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu))\cong E_2^{0,2}&\cong\mathrm{H}^0(\mathfrak{g}\times\mathfrak{g},\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})\otimes(V(\lambda)\boxtimes V(\mu)))\\ &\cong\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(\lambda)^*\boxtimes V(\mu)^*,\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})).\end{aligned}\tag{1.3.10}$$

Now we consider the following Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g}[t]/\mathfrak{g}\otimes(t-a),\mathrm{H}^q(\mathfrak{g}\otimes(t-a),\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu)))\Rightarrow\mathrm{H}^{p+q}(\mathfrak{g}[t],\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu)).$$

First observe that $E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g},\mathbb{C})\otimes\mathrm{H}^0(\mathfrak{g},\mathrm{H}^q(\mathfrak{g}[t]_+,\mathrm{ev}_{b-a}^*V(\mu))\otimes V(\lambda))$. Since $\lambda\neq 0$, the $\mathfrak{g}$-module $V(\lambda)$ is nontrivial and $E_2^{p,0}=0$ for all $p\geq 0$. Since $\mathfrak{g}$ is simple, first and second Whitehead Lemmas imply that $E_2^{2,q}=E_2^{1,q}=0$ for all $q\geq 0$. Thus, if we denote $c=b-a$, we have $c\neq 0$ and

$$\begin{aligned}\mathrm{H}^2(\mathfrak{g}[t],\mathrm{ev}_a^*V(\lambda)\otimes\mathrm{ev}_b^*V(\mu))\cong E_2^{0,2}&\cong\mathrm{H}^0(\mathfrak{g},\mathrm{H}^2(\mathfrak{g}[t]_+,\mathrm{ev}_c^*V(\mu))\otimes V(\lambda))\\ &\cong\mathrm{Hom}_{\mathfrak{g}}(V(\lambda)^*,\mathrm{H}^2(\mathfrak{g}[t]_+,\mathrm{ev}_c^*V(\mu))).\end{aligned}\tag{1.3.11}$$

The result follows by comparing (1.3.10) and (1.3.11). ☐

Corollary 1.3.15. Let $\lambda\in P^+$ and $\lambda\neq 0$. The $\mathfrak{g}\times\mathfrak{g}$-modules $V(\lambda)^*\boxtimes\mathbb{C}$ and $\mathbb{C}\boxtimes V(\lambda)^*$ are composition factors of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$ if, and only if, $\lambda=w\cdot 0$ for some $w\in W_a^1$ with $\ell(w)=2$. Moreover their multiplicities are 1.

Proof. By Lemma 1.3.14, $\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(\lambda)^*\boxtimes\mathbb{C},\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C}))\cong\mathrm{Hom}_{\mathfrak{g}}(V(\lambda)^*,\mathrm{H}^2(\mathfrak{g}[t]_+,\mathbb{C}))$. By (1.3.8), we have $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda)^*,\mathrm{H}^2(\mathfrak{g}[t]_+,\mathbb{C}))\cong\oplus_{\substack{w\in W_a^1\\ \ell(w)=2}}\mathrm{Hom}_{\mathfrak{g}}(V(\lambda)^*,V(w\cdot 0)^*)$. Since $V(\lambda)$ and $V(w\cdot 0)$ are simple $\mathfrak{g}$-modules, we have $\dim\mathrm{Hom}_{\mathfrak{g}}(V(\lambda)^*,V(w\cdot 0)^*)=\begin{cases}1 & \text{, if } \lambda=w\cdot 0\\ 0 & \text{, otherwise.}\end{cases}$ ☐

Lemma 1.3.16. The multiplicity of $\mathbb{C}\boxtimes\mathbb{C}$ as $\mathfrak{g}\times\mathfrak{g}$-composition factor of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$ is 1.

Proof. Consider the LHS spectral sequence $E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g}[t]/\mathfrak{g}\otimes I,\mathrm{H}^q(\mathfrak{g}\otimes I,\mathbb{C}))\Rightarrow\mathrm{H}^{p+q}(\mathfrak{g}[t],\mathbb{C})$. Observe that $E_2^{p,q}\cong\mathrm{H}^p(\mathfrak{g}\oplus\mathfrak{g},\mathbb{C})\otimes\mathrm{H}^0(\mathfrak{g}\oplus\mathfrak{g},\mathrm{H}^q(\mathfrak{g}\otimes I,\mathbb{C}))$, and that $\mathrm{H}^n(\mathfrak{g}[t],\mathbb{C})\cong\mathrm{H}^n(\mathfrak{g},\mathbb{C})$ by Corollary 1.3.12.

Since $\mathfrak{g}$ is simple, first and second Whitehead Lemmas imply that $E_2^{2,q}=E_2^{1,q}=0$ for all $q\geq 0$, $\mathrm{H}^2(\mathfrak{g},\mathbb{C})=0$, and $E_2^{3,0}=\mathrm{H}^3(\mathfrak{g}\times\mathfrak{g},\mathbb{C})\cong\mathbb{C}^2$. Since $E_\infty^{3,0}=\operatorname{coker}d_3^{0,2}:E_2^{0,2}\to E_2^{3,0}$ is a quotient of $E_2^{3,0}$ and $E_\infty^{3,0}\subseteq\mathrm{H}^3(\mathfrak{g},\mathbb{C})=\mathbb{C}$, it follows that $d_3^{0,2}:E_2^{0,2}\to E_2^{3,0}$ cannot be zero. This proves that $\dim\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(\mathbb{C}\boxtimes\mathbb{C},\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C}))\geq 1$.

In order to prove the other inequality consider the following LHS spectral sequence

$$E_2^{p,q} \cong \mathrm{H}^p(\mathfrak{g}\otimes(t-a)/\mathfrak{g}\otimes I, \mathrm{H}^q(\mathfrak{g}\otimes I,\mathbb{C})) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{g}\otimes(t-a),\mathbb{C}).$$

Observe that $E_2^{p,q} \cong \mathrm{H}^p(\mathfrak{g},\mathbb{C})\otimes \mathrm{H}^0(\mathfrak{g},\mathrm{H}^q(\mathfrak{g}\otimes I,\mathbb{C}))$ and that $\mathrm{H}^n(\mathfrak{g}\otimes(t-a),\mathbb{C}) \cong \mathrm{H}^n(\mathfrak{g}[t]_+,\mathbb{C})$. Recall from (1.3.8) that $\mathrm{H}^n(\mathfrak{g}[t]_+,\mathbb{C}) \cong \bigoplus_{\substack{w\in W_a^1\\ \ell(w)=n}} V(w\cdot 0)^*$.

Since $\mathfrak{g}$ is simple, $E_2^{3,0}\cong \mathrm{H}^3(\mathfrak{g},\mathbb{C})=\mathbb{C}$, and first and second Whitehead Lemmas imply that $E_2^{2,q}=E_2^{1,q}=0$ for all $q\geq 0$. Thus $\mathrm{H}^2(\mathfrak{g}[t]_+,\mathbb{C}) = E_\infty^{2,0} = \ker d_3^{0,2} : E_2^{0,2}\to E_2^{3,0}$, implying that $\dim E_2^{0,2} = \dim\ker d_3^{0,2} + \dim im(d_3^{0,2}) \leq \sum_{\substack{w\in W_a^1\\ \ell(w)=2}} \dim V(w\cdot 0)^* + 1$. From Corollary 1.3.15, we know that $\mathbb{C}\boxtimes V(w\cdot 0)^*$ are composition factors of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$, so $\dim E_2^{0,2} \geq \sum_{\substack{w\in W_a^1\\ \ell(w)=2}} \dim V(w\cdot 0)^*$. This proves that $\dim \mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(\mathbb{C}\boxtimes\mathbb{C}, \mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C}))\leq 1$. ☐

Example 1.3.17. Recall that $\mathrm{H}^2(\mathrm{gr}\,\mathfrak{g}\otimes I,\mathbb{C}) \cong (\mathrm{H}^2(\mathfrak{g}[t]_+,\mathbb{C})\boxtimes\mathbb{C}) \oplus (\mathfrak{g}\boxtimes\mathfrak{g})^* \oplus (\mathbb{C}\boxtimes \mathrm{H}^2(\mathfrak{g}[t]_+,\mathbb{C}))$. Thus, by Proposition 1.3.10, $\mathrm{H}^2(\mathrm{gr}\,\mathfrak{g}\otimes I,\mathbb{C})$ does not contain any trivial $\mathfrak{g}\times\mathfrak{g}$-composition factor. By Lemma 1.3.16, $(\mathbb{C}\boxtimes\mathbb{C})$ is a composition factor of $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$. This explicitly shows the difference between $\mathrm{H}^2(\mathrm{gr}\,\mathfrak{g}\otimes I,\mathbb{C})$ and $\mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})$ predicted in Subsection 1.3.3.

1.3.10. Second cohomology in rank one. From Lemma 1.3.1, it follows that the only possible composition factors of $\mathrm{H}_2(\mathfrak{sl}_2\otimes I,\mathbb{C})$ are of the form $V(\lambda)\boxtimes V(\mu)$ with $(\lambda+\mu)\in\{0,2,4\}$. From Lemma 1.3.16, it follows that $\mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})$ has $(\mathbb{C}\boxtimes\mathbb{C})$ as a composition factor with multiplicity 1. From Corollary 1.3.15, it follows that $\mathbb{C}\boxtimes V(2)$ and $V(2)\boxtimes\mathbb{C}$ are not composition factors of $\mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})$. Form Corollary 1.3.15 and Proposition 1.3.10, it follows that $\mathbb{C}\boxtimes V(4)$ and $V(4)\boxtimes\mathbb{C}$ are composition factors of $\mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})$ with multiplicity 1.

Throughout this subsection, denote $\mathfrak{sl}_2$ by $\mathfrak{g}$ and fix a Chevalley basis $\{y,h,x\}$ with $y\in\mathfrak{g}_{-\alpha}$, $x\in\mathfrak{g}_\alpha$ and $h=[x,y]\in\mathfrak{h}$.

Lemma 1.3.18. $\mathrm{Hom}_{\mathfrak{sl}_2}(V(3), \mathrm{H}_2(\mathfrak{sl}_2[t]^+, \mathrm{ev}_a^* V(1))) = (0)$ for any $a\neq 0$.

Proof. Fix a basis of weight vectors $\{v_{-1},v_1\}\subset \mathrm{ev}_a^* V(1)$ where v_{-1} has weight -1 and v_1 has weight 1. As a $\mathfrak{g}$-module, we have the following isomorphisms

$$\begin{aligned}
\left(\Lambda^2\mathfrak{g}[t]_+\otimes \mathrm{ev}_a^* V(1)\right) &\cong \left(\Lambda^2(\oplus_{i\geq 1}\mathfrak{g}\otimes\mathbb{C}t^i)\otimes V(1)\right)\\
&\cong \left(\Lambda^2(\mathfrak{g}\otimes\mathbb{C}t)\otimes V(1)\right)\oplus\left(\oplus_{i\geq 2}(\mathfrak{g}\otimes\mathbb{C}t)\otimes(\mathfrak{g}\otimes\mathbb{C}t^i)\otimes V(1)\right)\\
&\quad\oplus\left(\Lambda^2(\oplus_{i\geq 2}\mathfrak{g}\otimes\mathbb{C}t^i)\otimes V(1)\right)\\
&\cong \left(\oplus_{i\geq 1}\Lambda^2(\mathfrak{g}\otimes\mathbb{C}t^i)\otimes V(1)\right)\oplus\left(\oplus_{1\leq i<j}(\mathfrak{g}\otimes\mathbb{C}t^i)\otimes(\mathfrak{g}\otimes\mathbb{C}t^j)\otimes V(1)\right).
\end{aligned}\tag{1.3.12}$$

Thus the weight-3 subspace, $\left(\Lambda^2\mathfrak{g}[t]_+\otimes\mathrm{ev}_a^* V(1)\right)_3$, is generated by

$$\{(x\otimes t^i)\wedge(x\otimes t^j)\otimes v_{-1}, (x\otimes t^\ell)\wedge(h\otimes t^m)\otimes v_1 : 1\leq i<j; 1\leq \ell,m\}.$$

The elements of weight 3 in the kernel of $\partial_2 : (\Lambda^2\mathfrak{g}[t]_+\otimes\mathrm{ev}_a^* V(1)) \to (\mathfrak{g}[t]_+\otimes\mathrm{ev}_a^* V(1))$ are scalar multiples of $k_{i,j} = (x\otimes t^i)\wedge(x\otimes t^j)\otimes v_{-1} - ((x\otimes t^i)\wedge(h\otimes t^j)\otimes v_1 + (h\otimes t^i)\wedge(x\otimes t^j)\otimes v_1)$. Since $x\cdot k_{i,j}\neq 0$, it follows that $k_{i,j}$ do not represent highest-weight vectors of weight 3 in $\mathrm{H}_2(\mathfrak{g}[t]_+,\mathrm{ev}_a^* V(1))$. Hence $\mathrm{Hom}_{\mathfrak{g}}(V(3),\mathrm{H}_2(\mathfrak{g}[t]^+,\mathrm{ev}_a^* V(1))) = (0)$. ☐

Corollary 1.3.19. $\mathrm{Hom}_{\mathfrak{sl}_2\times\mathfrak{sl}_2}(V(1)\boxtimes V(3), \mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})) = 0$.

Proof. By Lemma 1.3.14, there is an isomorphism $\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(1)\boxtimes V(3), \mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})) \cong \mathrm{Hom}_{\mathfrak{g}}(V(1), \mathrm{H}^2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(3)^*))$. By [**Kum02**, Lemma 3.1.13.(3)] and the fact that finite-dimensional $\mathfrak{sl}_2$-modules are self-dual, $\mathrm{H}^2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(3)^*) \cong \mathrm{H}_2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(3))$ as $\mathfrak{sl}_2$-modules. By Lemma 1.3.18, $\mathrm{Hom}_{\mathfrak{g}}(V(1), \mathrm{H}_2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(3))) = 0$, finishing the proof. □

The proof of next lemma is similar to that of Lemma 1.3.18.

Lemma 1.3.20. $\mathrm{Hom}_{\mathfrak{sl}_2}(V(1), \mathrm{H}_2(\mathfrak{sl}_2[t]^+, \mathrm{ev}_a^* V(1))) = (0)$ for any $a \neq 0$.

Proof. Fix a basis of weight vectors $\{v_{-1}, v_1\} \subset \mathrm{ev}_a^* V(1)$ where v_{-1} has weight -1 and v_1 has weight 1. Using isomorphism (1.3.12), the weight-1 subspace, $\left(\Lambda^2\mathfrak{g}[t]_+ \otimes \mathrm{ev}_a^* V(1)\right)_1$, is generated by $\{(x\otimes t^i)\wedge(h\otimes t^j)\otimes v_{-1}, (x\otimes t^i)\wedge(y\otimes t^j)\otimes v_1, (h\otimes t^\ell)\wedge(h\otimes t^m)\otimes v_1 : 1\le i<j; 1\le \ell<m\}$. The elements of weight 1 in the kernel of $\partial_2 : (\Lambda^2\mathfrak{g}[t]_+\otimes \mathrm{ev}_a^* V(1)) \to (\mathfrak{g}[t]_+ \otimes \mathrm{ev}_a^* V(1))$ are scalar multiples of

$$\begin{aligned} k_{i,j} = {} & (x\otimes t^i)\wedge(h\otimes t^j)\otimes v_{-1} + (h\otimes t^i)\wedge(x\otimes t^j)\otimes v_{-1} \\ & + (x\otimes t^i)\wedge(y\otimes t^j)\otimes v_1 + (y\otimes t^i)\wedge(x\otimes t^j)\otimes v_1 - (h\otimes t^i)\wedge(h\otimes t^j)\otimes v_1. \end{aligned}$$

Since $x\cdot k_{i,j}\neq 0$, it follows that $k_{i,j}$ do not represent highest-weight vectors in $\mathrm{H}_2(\mathfrak{g}[t]_+, \mathrm{ev}_a^* V(1))$. Hence $\mathrm{Hom}_{\mathfrak{g}}(V(1), \mathrm{H}_2(\mathfrak{g}[t]^+, \mathrm{ev}_a^* V(1))) = (0)$. □

Corollary 1.3.21. $\mathrm{Hom}_{\mathfrak{sl}_2\times\mathfrak{sl}_2}(V(1)\boxtimes V(1), \mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})) = 0$.

Proof. By Lemma 1.3.14, there is an isomorphism $\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(1)\boxtimes V(1), \mathrm{H}^2(\mathfrak{g}\otimes I,\mathbb{C})) \cong \mathrm{Hom}_{\mathfrak{g}}(V(1), \mathrm{H}^2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(1)^*))$. By [**Kum02**, Lemma 3.1.13.(3)] and the fact that finite-dimensional $\mathfrak{sl}_2$-modules are self-dual, $\mathrm{H}^2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(1)^*) \cong \mathrm{H}_2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(1))$ as $\mathfrak{sl}_2$-modules. By Lemma 1.3.20, $\mathrm{Hom}_{\mathfrak{g}}(V(1), \mathrm{H}_2(\mathfrak{g}[t]^+, \mathrm{ev}_c^* V(1))) = 0$, finishing the proof. □

The only possible composition factor of $\mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})$ that we haven't yet taken care of is $\mathfrak{sl}_2\boxtimes\mathfrak{sl}_2$.

Conjecture 1.3.22. $\mathrm{Hom}_{\mathfrak{sl}_2\times\mathfrak{sl}_2}(V(2)\boxtimes V(2), \mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})) \cong \mathbb{C}$.

Assuming the conjecture, we can prove the following result.

Theorem 1.3.23. Let $a, b\in\mathbb{C}$, $\lambda,\mu\in P^+$ and $V = \mathrm{ev}_a^* V(\lambda)\otimes \mathrm{ev}_b^* V(\mu)$. Suppose $\lambda,\mu\neq 0$ and $a\neq b$. If $\lambda=\mu=1$, then $\mathrm{Ext}^2_{\mathfrak{sl}_2[t]}(V,V)$ is 2-dimensional, otherwise $\mathrm{Ext}^2_{\mathfrak{sl}_2[t]}(V,V)$ is 4-dimensional.

Proof. First rewrite $\mathrm{Ext}^2_{\mathfrak{sl}_2[t]}(V,V) = \mathrm{H}^2(\mathfrak{sl}_2[t], V^*\otimes V)$. Then decompose $V^*\otimes V$ as the $\mathfrak{sl}_2[t]$-module $\oplus_{\lambda',\mu'} \mathrm{ev}_a^* V(\lambda')\otimes \mathrm{ev}_b^* V(\mu')$, where the direct sum runs through all $\lambda',\mu'\in P^+$ such that $\lambda-\lambda', \mu-\mu'\in 2\mathbb{Z}$. By Lemma 2.3.1,

$$\mathrm{H}^2(\mathfrak{sl}_2[t], \mathrm{ev}_a^* V(\lambda')\otimes \mathrm{ev}_b^* V(\mu')) \cong \mathrm{Hom}_{\mathfrak{sl}_2[t]/\mathfrak{sl}_2\otimes I}(\mathrm{ev}_a^* V(\lambda')^*\otimes \mathrm{ev}_b^* V(\mu')^*, \mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})).$$

From the results above $\mathrm{Hom}_{\mathfrak{sl}_2[t]/\mathfrak{sl}_2\otimes I}(\mathrm{ev}_a^* V(\lambda')^*\otimes \mathrm{ev}_b^* V(\mu')^*, \mathrm{H}^2(\mathfrak{sl}_2\otimes I,\mathbb{C})) \cong \mathbb{C}$ for (λ',μ') in the set $\{(0,0),(2,2),(4,0),(0,4)\}$.

If $\lambda=\mu=1$, then $(0,0)$ and $(2,2)$ occur with multiplicity one. Otherwise each of this pairs in $\{(0,0),(2,2),(4,0),(0,4)\}$ occur with multiplicity one. □

CHAPTER 2

On Demazure and local Weyl modules for hyper current algebras

2.1. The main results

2.1.1. Integral forms. We use the following notation. Given a $\mathbb{Q}$-algebra U with unity, an element $x \in U$, and $k \in \mathbb{N}$, set

$$x^{(k)} = \frac{1}{k!}x^k \qquad \text{and} \qquad \binom{x}{k} = \frac{1}{k!}x(x-1)\cdots(x-k+1).$$

In the case $U = U(\tilde{\mathfrak{g}})$, we also introduce elements $\Lambda_{x,\pm r} \in U(\tilde{\mathfrak{g}}), x \in \mathfrak{g}, r \in \mathbb{N}$, by the following identity of power series in the variable u:

$$\Lambda_x^{\pm}(u) := \sum_{r\geq 0} \Lambda_{x,\pm r} u^r = \exp\left(-\sum_{s>0} \frac{x \otimes t^{\pm s}}{s} u^s\right).$$

Most of the time we will work with $x = h_\alpha$ for some $\alpha \in R^+$. We then simplify notation and write $\Lambda_\alpha^{\pm}(u) = \Lambda_{h_\alpha}^{\pm}(u)$ and, if $\alpha = \alpha_i$ for some $i \in I$, we simply write $\Lambda_i^{\pm}(u) = \Lambda_{h_i}^{\pm}(u)$. To shorten notation, we also set $\Lambda_x(u) = \Lambda_x^+(u)$.

Consider the $\mathbb{Z}$-subalgebra $U_{\mathbb{Z}}(\hat{\mathfrak{g}}') \subseteq U(\hat{\mathfrak{g}}')$ generated by $\{(x_{\alpha,r}^{\pm})^{(k)} : \alpha \in R^+, r \in \mathbb{Z}, k \in \mathbb{N}\}$. By [**Gar78**, Theorem 5.8], $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ is a free $\mathbb{Z}$-submodule of $U(\hat{\mathfrak{g}}')$ and satisfies $\mathbb{C} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\hat{\mathfrak{g}}') = U(\hat{\mathfrak{g}}')$. In other words, $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ is an integral form of $U(\hat{\mathfrak{g}}')$. Moreover, the image of $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ in $U(\tilde{\mathfrak{g}})$ is an integral form of $U(\tilde{\mathfrak{g}})$ denoted by $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$. For any Lie subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, set

$$U_{\mathbb{Z}}(\mathfrak{a}) = U(\mathfrak{a}) \cap U_{\mathbb{Z}}(\hat{\mathfrak{g}}), \qquad U_{\mathbb{Z}}(\tilde{\mathfrak{a}}) = U(\tilde{\mathfrak{a}}) \cap U_{\mathbb{Z}}(\tilde{\mathfrak{g}}) \qquad \text{and} \qquad U_{\mathbb{Z}}(\mathfrak{a}[t]) = U(\mathfrak{a}[t]) \cap U_{\mathbb{Z}}(\tilde{\mathfrak{g}}).$$

Notice that $U_{\mathbb{Z}}(\mathfrak{a})$ can be naturally identified with $U(\mathfrak{a}) \cap U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$. The subalgebra $U_{\mathbb{Z}}(\mathfrak{g})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ generated by $\{(x_\alpha^{\pm})^{(k)} : \alpha \in R^+, k \in \mathbb{N}\}$. The subalgebra $U_{\mathbb{Z}}(\mathfrak{n}^+)$ of $U_{\mathbb{Z}}(\mathfrak{g})$ is generated as $\mathbb{Z}$-subalgebra by the set $\{(x_\alpha^+)^{(k)} : \alpha \in R^+, k \in \mathbb{N}\}$ and, similarly, $U_{\mathbb{Z}}(\mathfrak{n}^-)$ is generated by $\{(x_\alpha^-)^{(k)} : \alpha \in R^+, k \in \mathbb{N}\}$. The subalgebra $U_{\mathbb{Z}}(\mathfrak{h})$ is generated as a $\mathbb{Z}$-subalgebra by $\left\{\binom{h_i}{k} : i \in I, k \in \mathbb{N}\right\}$. The subalgebra $U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+)$ of $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$ is generated as $\mathbb{Z}$-subalgebra by the set $\{(x_{\alpha,r}^+)^{(k)} : \alpha \in R^+, k \in \mathbb{N}, r \in \mathbb{Z}\}$ and, similarly, $U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^-)$ is generated by $\{(x_{\alpha,r}^-)^{(k)} : \alpha \in R^+, k \in \mathbb{N}, r \in \mathbb{Z}\}$, while $U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$ is generated by $\left\{\binom{h_i}{k}, \Lambda_{i,r} : i \in I, k \in \mathbb{N}, r \in \mathbb{Z}\right\}$. Observe that $U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$ is a commutative $\mathbb{Z}$-algebra. We also consider the subalgebras $U_{\mathbb{Z}}(\mathfrak{b}^-) = U_{\mathbb{Z}}(\mathfrak{n}^-)U_{\mathbb{Z}}(\mathfrak{h}), U_{\mathbb{Z}}(\mathfrak{b}^+) = U_{\mathbb{Z}}(\mathfrak{h})U_{\mathbb{Z}}(\mathfrak{n}^+)$, $U_{\mathbb{Z}}(\tilde{\mathfrak{b}}^-) = U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^-)U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$, and $U_{\mathbb{Z}}(\tilde{\mathfrak{b}}^+) = U_{\mathbb{Z}}(\tilde{\mathfrak{h}})U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+)$. Set $\hat{\mathfrak{b}}'^{\pm} = \hat{\mathfrak{h}}' \oplus \hat{\mathfrak{n}}^{\pm}$ and $U_{\mathbb{Z}}(\hat{\mathfrak{n}}^{\pm}) = U_{\mathbb{Z}}(\hat{\mathfrak{g}}) \cap U(\hat{\mathfrak{n}}^{\pm}), U_{\mathbb{Z}}(\hat{\mathfrak{h}}') = U_{\mathbb{Z}}(\hat{\mathfrak{g}}) \cap U(\hat{\mathfrak{h}}'), U_{\mathbb{Z}}(\hat{\mathfrak{b}}'^{\pm}) = U_{\mathbb{Z}}(\hat{\mathfrak{g}}) \cap U(\hat{\mathfrak{b}}'^{\pm})$.

Recall that $\mathfrak{h}[t]_{\pm} = \mathfrak{h} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]$. It follows from the above that $U_{\mathbb{Z}}(\mathfrak{h}[t]_{\pm}) := U(\mathfrak{h}[t]_{\pm}) \cap U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$ is an integral form of $U(\mathfrak{h}[t]_{\pm})$ and is generated, as a $\mathbb{Z}$-algebra with 1, by $\{\Lambda_{i,\pm r} : i \in I, r > 0\}$. In fact, it is known to be the free commutative algebra with 1 over this set. The PBW Theorem implies that multiplication establishes isomorphisms of $\mathbb{Z}$-modules

$$U_{\mathbb{Z}}(\hat{\mathfrak{g}}') \cong U_{\mathbb{Z}}(\hat{\mathfrak{n}}^-) \otimes U_{\mathbb{Z}}(\hat{\mathfrak{h}}') \otimes U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+),$$
$$U_{\mathbb{Z}}(\tilde{\mathfrak{g}}) \cong U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^-) \otimes U_{\mathbb{Z}}(\tilde{\mathfrak{h}}) \otimes U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+) \text{ and}$$
$$U_{\mathbb{Z}}(\mathfrak{g}[t]) \cong U_{\mathbb{Z}}(\mathfrak{n}[t]^-) \otimes U_{\mathbb{Z}}(\mathfrak{h}[t]) \otimes U_{\mathbb{Z}}(\mathfrak{n}[t]^+).$$

Moreover, restricted to $U_{\mathbb{Z}}(\tilde{\mathfrak{h}})$ this gives rise to an isomorphism of $\mathbb{Z}$-algebras

$$U_{\mathbb{Z}}(\tilde{\mathfrak{h}}) \cong U_{\mathbb{Z}}(\mathfrak{h}[t]_-) \otimes U_{\mathbb{Z}}(\mathfrak{h}) \otimes U_{\mathbb{Z}}(\mathfrak{h}[t]_+).$$

2.1.2. Hyperalgebras. Given a field $\mathbb{F}$, define the $\mathbb{F}$-hyperalgebra of $\mathfrak{a}$ by $U_{\mathbb{F}}(\mathfrak{a}) = \mathbb{F}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{a})$, where $\mathfrak{a}$ is any of the Lie algebras with $\mathbb{Z}$-forms defined above. Clearly, if the characteristic of $\mathbb{F}$ is zero, the algebra $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ is naturally isomorphic to $U(\tilde{\mathfrak{g}}_{\mathbb{F}})$ where $\tilde{\mathfrak{g}}_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} \tilde{\mathfrak{g}}_{\mathbb{Z}}$ and $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of the Chevalley basis of $\tilde{\mathfrak{g}}$, and similarly for all algebras $\mathfrak{a}$ we have considered. For fields of positive characteristic we just have an algebra homomorphism $U(\mathfrak{a}_{\mathbb{F}}) \to U_{\mathbb{F}}(\mathfrak{a})$ which is neither injective nor surjective. We will keep denoting by x the image of an element $x \in U_{\mathbb{Z}}(\mathfrak{a})$ in $U_{\mathbb{F}}(\mathfrak{a})$. Notice that we have $U_{\mathbb{F}}(\tilde{\mathfrak{g}}) = U_{\mathbb{F}}(\tilde{\mathfrak{n}}^-)U_{\mathbb{F}}(\tilde{\mathfrak{h}})U_{\mathbb{F}}(\tilde{\mathfrak{n}}^+)$.

Given an algebraically closed field $\mathbb{F}$, let $\mathbb{A}$ be a Henselian discrete valuation ring of characteristic zero having $\mathbb{F}$ as its residue field. Set $U_{\mathbb{A}}(\mathfrak{a}) = \mathbb{A} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{a})$ whenever $U_{\mathbb{Z}}(\mathfrak{a})$ has been defined. Clearly $U_{\mathbb{F}}(\mathfrak{a}) \cong \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{a})$. We shall also fix an algebraic closure $\mathbb{K}$ of the field of fractions of $\mathbb{A}$. For an explanation why we shall need to move from integral forms to $\mathbb{A}$-forms, see Remark 2.1.5 (and [**JM07**, Section 4C]). As mentioned in the introduction, we assume the characteristic of $\mathbb{F}$ is at least 5 if $\mathfrak{g}$ is of type G_2.

Notice that the Hopf algebra structure of the universal enveloping algebras induce such structure on the hyperalgebras. For any Hopf algebra H, denote by H^0 its augmentation ideal.

2.1.3. ℓ-weight lattice. Consider the set $\mathcal{P}_{\mathbb{F}}^+$ consisting of $|I|$-tuples $\boldsymbol{\omega} = (\boldsymbol{\omega}_i)_{i\in I}$, where $\boldsymbol{\omega}_i \in \mathbb{F}[u]$ and $\boldsymbol{\omega}_i(0) = 1$ for all $i \in I$. Endowed with coordinate-wise polynomial multiplication, $\mathcal{P}_{\mathbb{F}}^+$ is a monoid. We denote by $\mathcal{P}_{\mathbb{F}}$ the multiplicative abelian group associated to $\mathcal{P}_{\mathbb{F}}^+$ which will be referred to as the ℓ-weight lattice associated to $\mathfrak{g}$. One can describe $\mathcal{P}_{\mathbb{F}}$ in another way. Given $\mu \in P$ and $a \in \mathbb{F}^\times$, let $\boldsymbol{\omega}_{\mu,a}$ be the element of $\mathcal{P}_{\mathbb{F}}$ defined as

$$(\boldsymbol{\omega}_{\mu,a})_i(u) = (1 - au)^{\mu(h_i)} \quad \text{for all} \quad i \in I.$$

If $\mu = \omega_i$ is a fundamental weight, we simplify notation and write $\boldsymbol{\omega}_{\omega_i,a} = \boldsymbol{\omega}_{i,a}$. We refer to $\boldsymbol{\omega}_{i,a}$ as a fundamental ℓ-weight, for all $i \in I$ and $a \in \mathbb{F}^\times$. Notice that $\mathcal{P}_{\mathbb{F}}$ is the free abelian group on the set of fundamental ℓ-weights.

Let $\mathrm{wt} : \mathcal{P}_{\mathbb{F}} \to P$ be the unique group homomorphism such that $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$ for all $i \in I, a \in \mathbb{F}^\times$. Let also $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}^-$ be the unique group automorphism of $\mathcal{P}_{\mathbb{F}}$ mapping $\boldsymbol{\omega}_{i,a}$ to $\boldsymbol{\omega}_{i,a^{-1}}$ for all $i \in I, a \in \mathbb{F}^\times$. For notational convenience we set $\boldsymbol{\omega}^+ = \boldsymbol{\omega}$.

The abelian group $\mathcal{P}_{\mathbb{F}}$ can be identified with a subgroup of the monoid of $|I|$-tuples of formal power series with coefficients in $\mathbb{F}$ by identifying the rational function $(1 - au)^{-1}$ with the corresponding geometric formal power series $\sum_{n\geq 0}(au)^n$. This allows us to define an inclusion $\mathcal{P}_{\mathbb{F}} \hookrightarrow U_{\mathbb{F}}(\tilde{\mathfrak{h}})^*$. Indeed, for each $\boldsymbol{\omega}^\pm = \sum_{r\geq 0} \omega_{i,\pm r} u^r \in \mathcal{P}_{\mathbb{F}}$, we set

$$\boldsymbol{\omega}\left(\binom{h_i}{k}\right) = \binom{\mathrm{wt}(\boldsymbol{\omega})(h_i)}{k}, \quad \boldsymbol{\omega}(\Lambda_{i,r}) = \omega_{i,r}, \quad \text{for all} \quad i \in I, r, k \in \mathbb{Z}, k \geq 0,$$
$$\text{and} \quad \boldsymbol{\omega}(xy) = \boldsymbol{\omega}(x)\boldsymbol{\omega}(y), \quad \text{for all} \quad x, y \in U_{\mathbb{F}}(\tilde{\mathfrak{h}}).$$

2.1.4. Demazure and local Weyl modules. Given $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^{+}$, the local Weyl module $W_{\mathbb{F}}(\boldsymbol{\omega})$ is the quotient of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ by the left ideal generated by

$$U_{\mathbb{F}}(\tilde{\mathfrak{n}}^{+})^{0}, \quad h-\boldsymbol{\omega}(h), \quad (x_{\alpha}^{-})^{(k)} \quad \text{for all} \quad h \in U_{\mathbb{F}}(\tilde{\mathfrak{h}}),\ \alpha \in R^{+},\ k > \mathrm{wt}(\boldsymbol{\omega})(h_{\alpha}).$$

It is known that the local Weyl modules are finite-dimensional (cf. Theorem 2.3.8 (c)).

For $\lambda \in P^{+}$, the graded local Weyl module $W_{\mathbb{F}}^{c}(\lambda)$ is the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}^{c}(\lambda)$ generated by

$$U_{\mathbb{F}}(\mathfrak{n}^{+}[t])^{0}, \quad U_{\mathbb{F}}(\mathfrak{h}[t]_{+})^{0}, \quad h-\lambda(h), \quad (x_{\alpha}^{-})^{(k)}, \quad \text{for all} \quad h \in U_{\mathbb{F}}(\mathfrak{h}),\ \alpha \in R^{+},\ k > \lambda(h_{\alpha}). \tag{2.1.1}$$

Also, given $\ell \geq 0$, let $D_{\mathbb{F}}(\ell,\lambda)$ denote the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}(\ell,\lambda)$ generated by $I_{\mathbb{F}}^{c}(\lambda)$ together with

$$(x_{\alpha,s}^{-})^{(k)} \quad \text{for all} \quad \alpha \in R^{+}, s,k \in \mathbb{Z}_{\geq 0},\ k > \ \max\{0, \lambda(h_{\alpha}) - s\ell r_{\alpha}^{\vee}\}. \tag{2.1.2}$$

In particular, $D_{\mathbb{F}}(\ell,\lambda)$ is a quotient of $W_{\mathbb{F}}^{c}(\lambda)$.

Remark 2.1.1. The algebra $U_{\mathbb{F}}(\mathfrak{g}[t])$ inherits a $\mathbb{Z}$-grading from the grading on the polynomial algebra $\mathbb{C}[t]$. The ideals $I_{\mathbb{F}}^{c}(\lambda)$ and $I_{\mathbb{F}}(\ell,\lambda)$ are clearly graded and, hence, the modules $W_{\mathbb{F}}^{c}(\lambda)$ and $D_{\mathbb{F}}(\ell,\lambda)$ are graded. In [**CP01**], local Weyl modules were simply called Weyl modules, and certain infinite-dimensional modules, which were called *maximal integrable modules* and are now called *global Weyl modules*, were also defined. The modern names, local and global Weyl modules were coined by Feigin and Loktev in [**FL04**], where they introduced these modules in the context of generalized current algebras. We will not consider the global Weyl modules in this thesis.

We are ready to state the main theorem of this chapter.

Theorem 2.1.2. Let $\lambda \in P^{+}$.

(a) If $\mathfrak{g}$ is simply laced, then $D_{\mathbb{F}}(1,\lambda)$ and $W_{\mathbb{F}}^{c}(\lambda)$ are isomorphic $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules.
(b) There exist $k \geq 1$ and $\lambda_j \in P^{+}, j=1,\dots,k$, (independent of $\mathbb{F}$) such that the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module $W_{\mathbb{F}}^{c}(\lambda)$ admits a filtration $(0) = W_0 \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k = W_{\mathbb{F}}^{c}(\lambda)$, with $W_j/W_{j-1} \cong D_{\mathbb{F}}(1,\lambda_j)$.
(c) For any $a \in \mathbb{F}^{\times}$, there exists an automorphism φ_a of $U_{\mathbb{F}}(\mathfrak{g}[t])$ such that the pull-back of $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$ by φ_a is isomorphic to $W_{\mathbb{F}}^{c}(\lambda)$.

Assume the characteristic of $\mathbb{F}$ is zero. Then, part (a) of this theorem was proved in [**CP01**] for $\mathfrak{g} = \mathfrak{sl}_2$, in [**CL06**] for type A, and in [**FL07**] for types ADE. Both [**CL06**] and [**FL07**] use the $\mathfrak{sl}_2$-case (and the former exhibits an explicit basis for $W_{\mathbb{F}}^{c}(\lambda)$). Part (b) was proved in [**Nao12**]. Part (c) for simply laced $\mathfrak{g}$ was proved in [**FL07**] using part (a) (see [**FL07**, Lemma 1, Lemma 3, equation (15)]). The same proof works in the non simply laced case once part (b) is established. We will make use of Theorem 2.1.2 in the characteristic zero setting for extending it to the positive characteristic context.

We will see in Subsection 2.3.5 that the class of modules $D_{\mathbb{F}}(\ell,\lambda)$ form a subclass of the class of Demazure modules for $U_{\mathbb{F}}(\mathfrak{g}[t])$. In particular, it follows from [**Mat88**, Lemme 8] that $\dim(D_{\mathbb{F}}(\ell,\lambda))$ depends only on ℓ and λ, but not on $\mathbb{F}$ (see also the Remark on page 56 of [**Mat89**] and references therein). Together with Theorem 2.1.2(b), this implies the following corollary.

Corollary 2.1.3. For all $\lambda \in P^{+}$, we have $\dim W_{\mathbb{F}}^{c}(\lambda) = \dim W_{\mathbb{C}}^{c}(\lambda)$. $\square$

As an application of this corollary, we will prove a conjecture of [**JM07**]. The following theorem was proved in [**JM07**].

Theorem 2.1.4. Suppose $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{\times}$ and let $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, v the image of 1 in $W_{\mathbb{K}}(\boldsymbol{\omega})$, and $L_{\mathbb{A}}(\boldsymbol{\omega}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then, $L_{\mathbb{A}}(\boldsymbol{\omega})$ is a free $\mathbb{A}$-module such that $\mathbb{K} \otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega}) \cong W_{\mathbb{K}}(\boldsymbol{\omega})$. $\square$

We now recall the conjecture of [**JM07**]. Let $\boldsymbol{\varpi}$ be the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. It easily follows that $\mathbb{F}\otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega})$ is a quotient of $W_{\mathbb{F}}(\boldsymbol{\varpi})$ and, hence,

$$\dim W_{\mathbb{K}}(\boldsymbol{\omega}) \le \dim W_{\mathbb{F}}(\boldsymbol{\varpi}). \tag{2.1.3}$$

It was conjectured in [**JM07**] that

$$\mathbb{F}\otimes_{\mathbb{A}} L_{\mathbb{A}}(\boldsymbol{\omega}) \cong W_{\mathbb{F}}(\boldsymbol{\varpi}). \tag{2.1.4}$$

We will prove (2.1.4) in Subsection 2.3.4. In particular, it follows that

$$\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) = \dim W^c_{\mathbb{C}}(\lambda). \tag{2.1.5}$$

Remark 2.1.5. Recall that for all $\boldsymbol{\varpi}\in\mathcal{P}^+_{\mathbb{F}}$, there exists $\boldsymbol{\omega}\in\mathcal{P}^\times_{\mathbb{A}}$ such that $\boldsymbol{\varpi}$ is the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. This is the main reason why we consider $\mathbb{A}$-forms instead of $\mathbb{Z}$-forms. The study of finite-dimensional representations of twisted hyper loop algebras was initiated in [**BM12**]. The proof of one part of [**BM12**, Theorem 4.1] relies on (2.1.4) and, therefore, it will be completed once we finish the proof of (2.1.4).

It was also conjectured in [**JM07**] that, if $\boldsymbol{\varpi} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j,a_j}$ for some $m\ge 0, \lambda_j\in P^+, a_j\in \mathbb{F}^\times, j=1,\dots,m$, with $a_i\ne a_j$ for $i\ne j$, then

$$W_{\mathbb{F}}(\boldsymbol{\omega}) \cong \bigotimes_{j=1}^{m} W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda_j,a_j}). \tag{2.1.6}$$

In characteristic zero this was proved in [**CP01**]. It was then proved in [**Bia12**, Seção 2.4.3] that the characteristic zero case of (2.1.6) together with (2.1.4) implies (2.1.6) in general.

2.2. Further notation and technical lemmas

2.2.1. Some commutation relations. We begin recalling the following well-known relation in $U_{\mathbb{Z}}(\mathfrak{g})$

(2.2.1)
$$(x^+_\alpha)^{(l)}(x^-_\alpha)^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x^-_\alpha)^{(k-m)}\binom{h_\alpha-k-l+2m}{m}(x^+_\alpha)^{(l-m)} \quad \text{for all} \quad \alpha\in R^+, l,k\in\mathbb{Z}_{\ge 0}.$$

Since for all $\alpha\in R^+, s\in\mathbb{Z}$, the span of $x^\pm_{\alpha,\pm s}, h_\alpha$ is a subalgebra isomorphic to $\mathfrak{sl}_2$, we get the following relation in $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$

$$(x^+_{\alpha,s})^{(l)}(x^-_{\alpha,-s})^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x^-_{\alpha,-s})^{(k-m)}\binom{h_\alpha-k-l+2m}{m}(x^+_{\alpha,s})^{(l-m)}. \tag{2.2.2}$$

Next, we consider the case when the grades of the elements in the left-hand side is not symmetric.

Given $m>0$, consider the Lie algebra endomorphism τ_m of $\tilde{\mathfrak{g}}$ induced by the ring endomorphism of $\mathbb{C}[t,t^{-1}], t\mapsto t^m$. Notice that the restriction of τ_m to $\mathfrak{g}[t]$ gives rise to an endomorphism of $\mathfrak{g}[t]$. Moreover, denoting by τ_m its extension to an algebra endomorphism of $U(\tilde{\mathfrak{g}})$, notice that $U_{\mathbb{Z}}(\mathfrak{a})$ is invariant under τ_m for $\mathfrak{a}=\mathfrak{g},\mathfrak{n}^\pm,\mathfrak{h},\tilde{\mathfrak{n}}^\pm,\tilde{\mathfrak{h}},\mathfrak{n}^\pm[t],\mathfrak{h}[t],\mathfrak{h}[t]_+$. In fact $\tau_m((x^\pm_{\alpha,r})^{(k)}) = (x^\pm_{\alpha,mr})^{(k)}$ and $\tau_m(\Lambda_{\alpha,r})$ satisfies $\sum_{i\ge 0}\tau_m(\Lambda_{\alpha,r})u^r = \exp\left(-\sum_{s\ge 1}\frac{h_{\alpha,ms}}{s}u^s\right)$ for all $r,m\in\mathbb{Z}$ and $\alpha\in R^+$.

Given $\alpha\in R^+, m,s\in\mathbb{Z}, m>0$, consider the following power series:

$$X^-_{\alpha,m,s}(u) = \sum_{r=1}^{\infty} x^-_{\alpha,m(r-1)+s}\ u^r \qquad\text{and}\qquad \Lambda^\pm_{\alpha,m}(u) = \tau_m(\Lambda^\pm_\alpha(u)).$$

The next lemma is crucial in the proof that the rings $U_{\mathbb{Z}}(\mathfrak{a})$ are integral forms and will also be needed in Subsection 2.3.3.

Lemma 2.2.1. Let $\alpha \in R^+$, $k, l \geq 0, m > 0, s \in \mathbb{Z}$. Then

$$\left(x^+_{\alpha,m-s}\right)^{(l)} \left(x^-_{\alpha,s}\right)^{(k)} = (-1)^l \left((X^-_{\alpha,m,s}(u))^{(k-l)} \Lambda^+_{\alpha,m}(u)\right)_k \quad \text{mod } U_{\mathbb{Z}}(\tilde{\mathfrak{g}})U_{\mathbb{Z}}(\tilde{\mathfrak{n}}^+)^0,$$

where the subindex k denotes the coefficient of u^k of the above power series. Moreover, if $0 \leq s \leq m$, the same holds modulo $U_{\mathbb{Z}}(\mathfrak{g}[t])U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0_{\mathbb{Z}}$.

Proof. The case $m = 1, s = 0$ was proved in [**Gar78**, Lemma 7.5] (cf. [**JM07**, Equation (1-11)]). Consider the Lie algebra endomorphism $\sigma_s : \tilde{\mathfrak{sl}}_\alpha \to \tilde{\mathfrak{sl}}_\alpha$ given by $x^\pm_{\alpha,r} \mapsto x^\pm_{\alpha,r\mp s}$. The first statement of the lemma is obtained from the case $m = 1, s = 0$ by applying $(\sigma_s \circ \tau_m)$. The second statement is then clear. $\square$

Sometimes it will be convenient to work with smaller set of generators for the hyperalgebras.

Proposition 2.2.2. [**Mit85**, Corollary 4.4.12] The ring $U_{\mathbb{Z}}(\hat{\mathfrak{g}}')$ is generated by $(x^\pm_i)^{(k)}, i \in \hat{I}, k \geq 0$ and $U_{\mathbb{Z}}(\mathfrak{g})$ is generated by $(x^\pm_i)^{(k)}, i \in I, k \geq 0$. $\square$

An adaptation of the proof of Proposition 2.2.2 gives:

Lemma 2.2.3. The algebra $U_{\mathbb{Z}}(\mathfrak{n}^\pm[t])$ is generated by $(x^\pm_{i,r})^{(k)}, i \in I, k, r \geq 0$. In particular, $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$ is generated by $(x^\pm_{i,r})^{(k)}, i \in I, r, k \in \mathbb{Z}, k \geq 0$. $\square$

Combining Lemma 2.2.1 (with $m = 1, s = 0$, and $k = l \geq 1$) with Lemma 2.2.3, it is not difficult to see that $U_{\mathbb{Z}}(\mathfrak{g}[t])$ is generated by $(x^\pm_{i,r})^{(k)}, i \in I, r, k \geq 0$.

Given $\beta \in R^+$ and $r, k \in \mathbb{Z}, k \geq 0$, define the hyperdegree of $(x^\pm_{\beta,r})^{(k)}$ to be k. For a monomial of the form $(x^\pm_{\beta_1,r_1})^{(k_1)} \cdots (x^\pm_{\beta_l,r_l})^{(k_l)}$ (choice of $\pm$ fixed) define its hyperdegree to be $k_1 + \cdots + k_l$.

Lemma 2.2.4. [**Mit85**, Lemma 4.2.13] Let $r, s, k, l \in \mathbb{Z}, k, l \geq 0, \alpha, \beta \in R^+$. Then $(x^\pm_{\alpha,r})^{(k)}(x^\pm_{\beta,s})^{(l)}$ is in the $\mathbb{Z}$-span of $(x^\pm_{\beta,s})^{(l)}(x^\pm_{\alpha,r})^{(k)}$ together with monomials of hyperdegree strictly smaller than $k + l$. $\square$

2.2.2. On certain automorphisms of hyper current algebras. Let $\mathfrak{a}, \mathfrak{b}$ be such that $U_{\mathbb{Z}}(\mathfrak{a})$ have been defined. Then, given a homomorphism of $\mathbb{A}$-algebras $f : U_{\mathbb{A}}(\mathfrak{a}) \to U_{\mathbb{A}}(\mathfrak{b})$, we have an induced homomorphism $U_{\mathbb{F}}(\mathfrak{a}) \to U_{\mathbb{F}}(\mathfrak{b})$. We will now use this procedure to define certain homomorphism between hyperalgebras. As a rule, we shall use the same symbol to denote the induced homomorphism in the hyperalgebra level.

Recall that there exists a unique involutive Lie algebra automorphism ψ of $\mathfrak{g}$ such that $x^\pm_i \mapsto x^\mp_i$ and $h_i \mapsto -h_i$ for all $i \in I$. It admits a unique extension to an automorphism of $\mathfrak{g}[t]$ such that $\psi(x \otimes f(t)) = \psi(x) \otimes f(t)$ for all $x \in \mathfrak{g}, f \in \mathbb{C}[t]$. Keep denoting by ψ its extension to an automorphism of $U(\mathfrak{g}[t])$. In particular, it easily follows that

$$\psi\left((x^\pm_{\alpha,r})^{(k)}\right) = (-1)^{k(\mathrm{ht}(\alpha)-1)}(x^\mp_{\alpha,r})^{(k)} \qquad \text{for all} \quad \alpha \in R^+, r, k \geq 0. \tag{2.2.3}$$

Since $U_{\mathbb{Z}}(\mathfrak{g}[t])$ is generated by the elements $(x^\pm_{\alpha,r})^{(k)}$, it follows that the restriction of ψ to $U_{\mathbb{Z}}(\mathfrak{a})$ induces an automorphism of $U_{\mathbb{Z}}(\mathfrak{a})$, for $\mathfrak{a} = \mathfrak{g}, \mathfrak{h}, \mathfrak{g}[t], \mathfrak{h}[t], \mathfrak{h}[t]_+$. Notice that we have an inclusion $P \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(U_{\mathbb{Z}}(\mathfrak{h}), \mathbb{Z})$ determined by

$$\mu\left(\binom{h_i}{k}\right) = \binom{\mu(h_i)}{k} \quad \text{and} \quad \mu(xy) = \mu(x)\mu(y) \quad \text{for all} \quad i \in I, k \geq 0, x, y \in U_{\mathbb{Z}}(\mathfrak{h}). \tag{2.2.4}$$

Therefore,

$$\mu\left(\psi\left(\binom{h_i}{k}\right)\right) = \binom{-\mu(h_i)}{k} \quad \text{for all} \quad i \in I, k > 0, \mu \in P. \tag{2.2.5}$$

Suppose now that γ is a Dynkin diagram automorphism of $\mathfrak{g}$ and keep denoting by γ the $\mathfrak{g}$-automorphism determined by $x_i^\pm \mapsto x_{\gamma(i)}^\pm, h_i \mapsto h_{\gamma(i)}, i \in I$. It admits a unique extension to an automorphism of $\mathfrak{g}[t]$ such that $\gamma(x \otimes f(t)) = \gamma(x) \otimes f(t)$ for all $x \in \mathfrak{g}, f \in \mathbb{C}[t]$. Keep denoting by γ its extension to an automorphism of $U(\mathfrak{g}[t])$. Let γ also denote the associated automorphism of P determined by $\gamma(\omega_i) = \omega_{\gamma(i)}, i \in I$. In particular, $\gamma(\alpha_i) = \alpha_{\gamma(i)}, i \in I$. It then follows that, for each $\alpha \in R^+, k > 0$, there exist $\varepsilon_{\alpha,k}^\pm \in \{-1, 1\}$ (depending on how the Chevalley basis was chosen) such that

$$\gamma\left((x_{\alpha,r}^\pm)^{(k)}\right) = \varepsilon_{\alpha,k}^\pm (x_{\gamma(\alpha),r}^\pm)^{(k)} \qquad \text{for all} \quad r \geq 0. \tag{2.2.6}$$

This implies that the restriction of γ to $U_\mathbb{Z}(\mathfrak{a})$ induces an automorphism of $U_\mathbb{Z}(\mathfrak{a})$, for any $\mathfrak{a}$ in the set $\{\mathfrak{g}, \mathfrak{n}^\pm, \mathfrak{h}, \mathfrak{g}[t], \mathfrak{n}^\pm[t], \mathfrak{h}[t], \mathfrak{h}[t]_+\}$. It is also easy to see that

$$\mu\left(\gamma\left(\binom{h_i}{k}\right)\right) = \binom{(\gamma^{-1}(\mu))(h_i)}{k} \quad \text{for all} \quad i \in I, k > 0, \mu \in P. \tag{2.2.7}$$

We end this subsection constructing the automorphism mentioned in Theorem 2.1.2(c). Thus, let $a \in \mathbb{F}, \tilde{a} \in \mathbb{A}$ such that the image of $\tilde{a}$ in $\mathbb{F}$ is a, and $\varphi_{\tilde{a}}$ the Lie algebra automorphism of $\mathfrak{g}[t]_\mathbb{K}$ given by $x \otimes t \mapsto x \otimes (t - \tilde{a})$. Keep denoting by $\varphi_{\tilde{a}}$ the induced automorphism of $U_\mathbb{K}(\mathfrak{g}[t])$ and observe that $\varphi_{\tilde{a}}$ is the identity on $U_\mathbb{K}(\mathfrak{g})$. One easily checks that

$$\varphi_{\tilde{a}}\left((x_{\alpha,r}^\pm)^{(k)}\right) = \sum_{k_0+\cdots+k_r=k} \prod_{s=0}^{r} \tbinom{r}{s}^{k_s} (-\tilde{a})^{k_s(r-s)} (x_{\alpha,s}^\pm)^{(k_s)} \in U_\mathbb{A}(\mathfrak{g}[t]).$$

Hence, $\varphi_{\tilde{a}}$ induces an automorphism of $U_\mathbb{A}(\mathfrak{g}[t])$. Notice that, in the hyperalgebra level, we have

$$(x_{\alpha,r}^\pm)^{(k)} \mapsto \sum_{k_0+\cdots+k_r=k} \prod_{s=0}^{r} \tbinom{r}{s}^{k_s} (-a)^{k_s(r-s)} (x_{\alpha,s}^\pm)^{(k_s)}. \tag{2.2.8}$$

This justifies a change of notation from $\varphi_{\tilde{a}}$ to φ_a.

2.2.3. Subalgebras of rank 1 and 2. For any $\alpha \in R^+$, consider the Lie subalgebra of $\mathfrak{g}$ generated by $x_\alpha^\pm$ which is isomorphic to $\mathfrak{sl}_2$. Denote this subalgebra by $\mathfrak{sl}_\alpha$. Consider also $\mathfrak{n}_\alpha^\pm = \mathbb{C}x_\alpha^\pm$, $\mathfrak{h}_\alpha = \mathbb{C}h_\alpha$ and $\mathfrak{b}_\alpha^\pm = \mathbb{C}h_\alpha \oplus \mathbb{C}x_\alpha^\pm$. Notice that $U_\mathbb{Z}(\mathfrak{g}) \cap U(\mathfrak{sl}_\alpha)$ coincides with the $\mathbb{Z}$-subalgebra $U_\mathbb{Z}(\mathfrak{sl}_\alpha)$ of $U(\mathfrak{g})$ generated by $(x_\alpha^\pm)^{(k)}, k \geq 0$. Indeed, since $(x_\alpha^\pm)^{(k)}$ belong to a $\mathbb{Z}$-basis of $U_\mathbb{Z}(\mathfrak{g})$, we must have that $U_\mathbb{Z}(\mathfrak{n}^\pm) \cap U(\mathfrak{sl}_\alpha) = U_\mathbb{Z}(\mathfrak{n}_\alpha^\pm)$ where the latter is the $\mathbb{Z}$-subalgebra generated by $(x_\alpha^\pm)^{(k)}, k \geq 0$. It then suffices to show that, given $k > 0$, an element of the form $c\binom{h_\alpha}{k} \in U_\mathbb{Z}(\mathfrak{g}) \cap U(\mathfrak{sl}_\alpha)$ only if $c \in \mathbb{Z}$. Write $h_\alpha = \sum_{i \in I} m_i h_i$ with $m_i \in \mathbb{Z}_{\geq 0}, i \in I$, and recall that the m_i are relatively prime. It follows from Vandermonde's convolution formula that

$$\binom{h_\alpha}{k} = \sum_{i \in I} \binom{m_i h_i}{k} + \sum_{\boldsymbol{k}} \prod_{i \in I} \binom{m_i h_i}{k_i}$$

where the second sum is over $\boldsymbol{k} = (k_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ such that $\sum_{i \in I} k_i = k$ and $k_i < k$ for all $i \in I$. On the other hand, for all $l > 0, i \in I$, there exist $c_l \in \mathbb{Z}$ such that $\binom{m_i h_i}{l} = m_i^l \binom{h_i}{l} + \sum_{m=1}^{l-1} c_l \binom{h_i}{m}$. Hence, $\binom{h_\alpha}{k} = \sum_{i \in I} m_i^k \binom{h_i}{k} + \eta$, where η is a $\mathbb{Z}$-linear combination of elements of the form $\prod_{i \in I} \binom{h_i}{k_i}$ with $k_i < k$ for all $i \in I$. Since this is an expression for $\binom{h_\alpha}{k}$ in terms of a $\mathbb{Z}$-basis of $U_\mathbb{Z}(\mathfrak{h})$, it follows that $c\binom{h_\alpha}{k} \in U_\mathbb{Z}(\mathfrak{g}) \cap U(\mathfrak{sl}_\alpha)$ only if $cm_i^k \in \mathbb{Z}$ for all $i \in I$. Since the m_i are relatively prime, we must have $c \in \mathbb{Z}$ as claimed.

This implies that $U_{\mathbb{Z}}(\mathfrak{g})\cap U(\mathfrak{sl}_\alpha)$ is naturally isomorphic to $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ and, hence, the corresponding subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\alpha)$ of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ is naturally isomorphic to $U_{\mathbb{F}}(\mathfrak{sl}_2)$. Similarly, for any $\alpha\in R^+, r\in\mathbb{Z}$, the Lie subalgebra $\mathfrak{sl}_{\alpha,r}$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,\pm r}$ is isomorphic to $\mathfrak{sl}_2$ and $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})\cap U(\mathfrak{sl}_{\alpha,r})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\tilde{\mathfrak{g}})$ generated by $(x^{\pm}_{\alpha,\pm r})^{(k)}, k\geq 0$. We shall denote the corresponding subalgebra of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ by $U_{\mathbb{F}}(\mathfrak{sl}_{\alpha,r})$. We also consider the subalgebra $\tilde{\mathfrak{sl}}_\alpha$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,r}, r\in\mathbb{Z}$ and the subalgebra $\mathfrak{sl}_\alpha[t]$ of $\mathfrak{g}[t]$ generated by $x^{\pm}_{\alpha,r}, r\geq 0$. The corresponding subalgebras $U_{\mathbb{F}}(\tilde{\mathfrak{sl}}_\alpha)$ and $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ are naturally isomorphic to $U_{\mathbb{F}}(\tilde{\mathfrak{sl}}_2)$ and $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$.

We will also need to work with root subsystems of rank 2. Suppose $\alpha,\beta\in R^+$ form a simple system of a root subsystem R' of rank 2 and let $\mathfrak{t}$ denote a simple Lie algebra of type R'. Denote by $\mathfrak{g}_{\alpha,\beta}$ the subalgebra of $\mathfrak{g}$ generated by $x^{\pm}_\alpha$ and $x^{\pm}_\beta$, which is isomorphic to $\mathfrak{t}$. Notice that, for $r,s\in\mathbb{Z}$, the subalgebra $\mathfrak{g}^{r,s}_{\alpha,\beta}$ of $\tilde{\mathfrak{g}}$ generated by $x^{\pm}_{\alpha,\pm r}$ and $x^{\pm}_{\beta,\pm s}$ is also isomorphic to $\mathfrak{t}$. Let $U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$ be the subalgebra of $U_{\mathbb{Z}}(\mathfrak{g})$ generated by $(x^{\pm}_\alpha)^{(k)}$, $(x^{\pm}_\beta)^{(k)}$, $k\geq 0$, and $U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})$ the subalgebra of $U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$ generated by $(x^{\pm}_{\alpha,\pm r})^{(k)}, (x^{\pm}_{\beta,\pm s})^{(k)}$, $k\geq 0$. Proposition 2.2.2 implies that $U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$ and $U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})$ are naturally isomorphic to $U_{\mathbb{Z}}(\mathfrak{t})$. Recall that if $\mathfrak{a}$ is a subalgebra of $U(\tilde{\mathfrak{g}})$, then $U_{\mathbb{Z}}(\mathfrak{a}) = U(\mathfrak{a})\cap U_{\mathbb{Z}}(\tilde{\mathfrak{g}})$. As in the rank 1 case, we have:

$$U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta}) \qquad \text{and} \qquad U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta}). \tag{2.2.9}$$

Indeed, by definition, $U'_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})\subseteq U_{\mathbb{Z}}(\mathfrak{g})\cap U(\mathfrak{g}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$ and $U'_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})\subseteq U_{\mathbb{Z}}(\tilde{\mathfrak{g}})\cap U(\mathfrak{g}^{r,s}_{\alpha,\beta}) = U_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})$. In order to prove the other inclusion, let $\mathfrak{n}^{\pm}_{\alpha,\beta}$ be defined by $\mathfrak{n}^{\pm}_{\alpha,\beta} = \mathfrak{n}^{\pm}\cap\mathfrak{g}_{\alpha,\beta}$ and observe that $U_{\mathbb{Z}}(\mathfrak{n}^{\pm}_{\alpha,\beta}) = U(\mathfrak{g}_{\alpha,\beta})\cap U_{\mathbb{Z}}(\mathfrak{n}^{\pm})$. Let also $\mathfrak{h}_{\alpha,\beta} = \mathfrak{h}\cap\mathfrak{g}_{\alpha,\beta}$ and $U_{\mathbb{Z}}(\mathfrak{h}_{\alpha,\beta}) = U(\mathfrak{h}_{\alpha,\beta})\cap U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})$. To prove the first statement in (2.2.9) it remains to show that $U_{\mathbb{Z}}(\mathfrak{h})\cap U(\mathfrak{g}_{\alpha,\beta})\subseteq U_{\mathbb{Z}}(\mathfrak{h}_{\alpha,\beta})$. It suffices to show that, given $k_\alpha, k_\beta\geq 0$, an element of the form $c\binom{h_\alpha}{k_\alpha}\binom{h_\beta}{k_\beta}$ is in $U_{\mathbb{Z}}(\mathfrak{h})\cap U(\mathfrak{g}_{\alpha,\beta})$ only if $c\in\mathbb{Z}$. Notice that, if R' is of type G_2, then R is of type G_2, $\mathfrak{h}_{\alpha,\beta}=\mathfrak{h}$ and there is nothing to prove. Hence, suppose $r^\vee<3$ and write $h_\alpha = \sum_{i\in I} m_i h_i$ and $h_\beta = \sum_{i\in I} n_i h_i$ with $m_i, n_i\in\mathbb{Z}_{\geq 0}, i\in I$. Recalling that there exists $i_\alpha, i_\beta\in I$ such that $m_{i_\alpha} = n_{i_\beta} = 1$, the argument is completed similarly to what we have done in the rank 1 case. It follows from (2.2.9) that $U_{\mathbb{F}}(\mathfrak{g}_{\alpha,\beta}) = \mathbb{F}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}_{\alpha,\beta})\subseteq U_{\mathbb{F}}(\mathfrak{g})$ and $U_{\mathbb{F}}(\mathfrak{g}^{r,s}_{\alpha,\beta}) = \mathbb{F}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}^{r,s}_{\alpha,\beta})\subseteq U_{\mathbb{F}}(\tilde{\mathfrak{g}})$ are isomorphic to $U_{\mathbb{F}}(\mathfrak{t})$.

2.2.4. The algebra $\mathfrak{g}_{\rm sh}$. Another important subalgebra used in the proof of Theorem 2.1.2 is the subalgebra $\mathfrak{g}_{\rm sh}$ generated by the root vectors associated to short simple roots.

Let $\Delta_{\rm sh} = \{\alpha\in\Delta : (\alpha,\alpha)<2\}$ denote the set of simple short roots. In particular, if $\mathfrak{g}$ is simply laced, $\Delta_{\rm sh}=\emptyset$. Let $R^+_{\rm sh} = \mathbb{Z}\Delta_{\rm sh}\cap R^+$ and $R_{\rm sh} = \mathbb{Z}\Delta_{\rm sh}\cap R$ (and notice that, if $\mathfrak{g}$ is not simply laced, $R_{\rm sh}\neq\{\alpha\in R : (\alpha,\alpha)<2\}$). Set $I_{\rm sh} = \{i\in I : \alpha_i\in\Delta_{\rm sh}\}$ and define $P_{\rm sh} = \oplus_{i\in I_{\rm sh}}\mathbb{Z}\omega_i$ and $P^+_{\rm sh} = P_{\rm sh}\cap P^+$. Consider also the subalgebras $\mathfrak{h}_{\rm sh} = \oplus_{i\in I_{\rm sh}}\mathbb{C}h_i$, $\mathfrak{b}^{\pm}_{\rm sh} = \mathfrak{h}_{\rm sh}\oplus\mathfrak{n}^{\pm}_{\rm sh}$, where $\mathfrak{n}^{\pm}_{\rm sh} = \oplus_{\pm\alpha\in R^+_{\rm sh}}\mathfrak{g}_\alpha$, and $\mathfrak{g}_{\rm sh} = \mathfrak{n}^-_{\rm sh}\oplus\mathfrak{h}_{\rm sh}\oplus\mathfrak{n}^+_{\rm sh}$. Then, if $\Delta_{\rm sh}\neq\emptyset$, $\mathfrak{g}_{\rm sh}$ is a simply laced Lie subalgebra of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_{\rm sh}$ and $\Delta_{\rm sh}$ can be identified with the choice of simple roots associated to the given triangular decomposition. The subsets $Q_{\rm sh}, Q^+_{\rm sh}$, and the Weyl group $\mathcal{W}_{\rm sh}$ are defined in the obvious way. The restriction of $(\ ,\ )$ to $\mathfrak{g}_{\rm sh}$ is an invariant symmetric and non degenerate bilinear form on $\mathfrak{g}_{\rm sh}$, but the normalization is not the same as the one we fixed for $\mathfrak{g}$. Indeed, $(\alpha,\alpha) = 2/r^\vee$ for all $\alpha\in R_{\rm sh}$. The set $\{x^{\pm}_\alpha, h_i : \alpha\in R^+_{\rm sh}, i\in I_{\rm sh}\}$ is a Chevalley basis for $\mathfrak{g}_{\rm sh}$.

Observe that $U_{\mathbb{Z}}(\mathfrak{g})\cap U(\mathfrak{g}_{\rm sh})$ coincides with the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ generated by $(x^{\pm}_\alpha)^{(k)}, \alpha\in\Delta_{\rm sh}$, and, hence, Proposition 2.2.2 implies that $U_{\mathbb{F}}(\mathfrak{g}_{\rm sh})$ can be naturally identified with a subalgebra of $U_{\mathbb{F}}(\mathfrak{g})$. Similar observation apply to $U_{\mathbb{Z}}(\mathfrak{a})$ for $\mathfrak{a} = \mathfrak{n}^{\pm}_{\rm sh}, \mathfrak{h}_{\rm sh}$.

Consider the linear map $\mathfrak{h}^*\to\mathfrak{h}^*_{\rm sh}, \lambda\mapsto\bar\lambda$, given by restriction and let $i_{\rm sh}:\mathfrak{h}^*_{\rm sh}\to\mathfrak{h}^*$ be the linear map such that $i_{\rm sh}(\overline{\alpha}) = \alpha$ for all $\alpha\in\Delta_{\rm sh}$. In particular, $\overline{i_{\rm sh}(\mu)} = \mu$ for all $\mu\in\mathfrak{h}^*_{\rm sh}$. Given

$\lambda \in P$, consider the function $\eta_\lambda : P_{\rm sh} \to P$ given by

$$\eta_\lambda(\mu) = i_{\rm sh}(\mu) + \lambda - i_{\rm sh}(\overline{\lambda}). \tag{2.2.10}$$

Lemma 2.2.5. If $\lambda \in P^+, \mu \in P^+_{\rm sh}$, and $\mu \le \overline{\lambda}$, then $\eta_\lambda(\mu) \in P^+$. □

PROOF. For each $i \in I_{\rm sh}$, let $m_i \in \mathbb{Z}_{\ge 0}$ such that $\mu = \overline{\lambda} - \sum_{i\in I_{\rm sh}} m_i\overline{\alpha}_i$. In particular, $\eta_\lambda(\mu) = \lambda - \sum_{i\in I_{\rm sh}} m_i\alpha_i$. Then, for $j \in I_{\rm sh}$, we have $\eta_\lambda(\mu)(h_j) = \mu(h_j) \ge 0$ while, for $j \in I \setminus I_{\rm sh}$, we have $\eta_\lambda(\mu)(h_j) = \lambda(h_j) - \sum_{i\in I_{\rm sh}} m_i\alpha_i(h_j) \ge \lambda(h_j) \ge 0$. □

The affine Kac-Moody algebra associated to $\mathfrak{g}_{\rm sh}$ is naturally isomorphic to the subalgebra

$$\hat{\mathfrak{g}}_{\rm sh} := \mathfrak{g}_{\rm sh} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

of $\hat{\mathfrak{g}}$ and, under this isomorphism, $\hat{\mathfrak{h}}_{\rm sh}$ is identified with $\mathfrak{h}_{\rm sh} \oplus \mathbb{C}c \oplus \mathbb{C}d$. The subalgebras $\mathfrak{g}_{\rm sh}[t]$ and $\hat{\mathfrak{n}}^\pm_{\rm sh}$, as well as $\hat{P}_{\rm sh}, \hat{Q}_{\rm sh}$, etc, are defined in the obvious way. Moreover, $U_{\mathbb{F}}(\tilde{\mathfrak{g}}_{\rm sh})$ and $U_{\mathbb{F}}(\mathfrak{g}_{\rm sh}[t])$ can be naturally identified with a subalgebra of $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$.

2.3. Finite-dimensional modules

2.3.1. Modules for hyperalgebras. We now review the finite-dimensional representation theory of $U_{\mathbb{F}}(\mathfrak{g})$. If the characteristic of $\mathbb{F}$ is zero, then $U_{\mathbb{F}}(\mathfrak{g}) \cong U(\mathfrak{g}_{\mathbb{F}})$ and the results stated here can be found in [**Hum78**]. The literature for the positive characteristic setting is more often found in the context of algebraic groups, in which case $U_{\mathbb{F}}(\mathfrak{g})$ is known as the hyperalgebra or algebra of distributions of an algebraic group of the same Lie type as $\mathfrak{g}$ (cf. [**Jan03**, Part II]). A more detailed review in the present context can be found in [**JM07**, Section 2].

Let V be a $U_{\mathbb{F}}(\mathfrak{g})$-module. A nonzero vector $v \in V$ is called a weight vector if there exists $\mu \in U_{\mathbb{F}}(\mathfrak{h})^*$ such that $hv = \mu(h)v$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$. The subspace consisting of weight vectors of weight μ is called weight space of weight μ and it will be denoted by V_μ. If $V = \oplus_{\mu\in U_{\mathbb{F}}(\mathfrak{h})^*} V_\mu$, then V is said to be a weight module. If $V_\mu \ne 0$, μ is said to be a weight of V and $\mathrm{wt}(V) = \{\mu \in U_{\mathbb{F}}(\mathfrak{h})^* : V_\mu \ne 0\}$ is said to be the set of weights of V. Notice that the inclusion (2.2.4) induces an inclusion $P \hookrightarrow U_{\mathbb{F}}(\mathfrak{h})^*$. In particular, we can consider the partial order $\le$ on $U_{\mathbb{F}}(\mathfrak{h})^*$ given by $\mu \le \lambda$ if $\lambda - \mu \in Q^+$ and we have

$$(x^\pm_\alpha)^{(k)} V_\mu \subseteq V_{\mu\pm k\alpha} \quad \text{for all} \quad \alpha \in R^+, k > 0, \mu \in U_{\mathbb{F}}(\mathfrak{h})^*. \tag{2.3.1}$$

If V is a weight-module with finite-dimensional weight spaces, its character is the function $\mathrm{ch}(V) : U_{\mathbb{F}}(\mathfrak{h})^* \to \mathbb{Z}$ given by $\mathrm{ch}(V)(\mu) = \dim V_\mu$. As usual, if V is finite-dimensional, $\mathrm{ch}(V)$ can be regarded as an element of the group ring $\mathbb{Z}[U_{\mathbb{F}}(\mathfrak{h})^*]$ where we denote the element corresponding to $\mu \in U_{\mathbb{F}}(\mathfrak{h})^*$ by e^μ. By the inclusion (2.2.4) the group ring $\mathbb{Z}[P]$ can be regarded as a subring of $\mathbb{Z}[U_{\mathbb{F}}(\mathfrak{h})^*]$ and, moreover, the action of $\mathcal{W}$ on P induces an action of $\mathcal{W}$ on $\mathbb{Z}[P]$ by ring automorphisms where $w \cdot e^\mu = e^{w\mu}$.

If $v \in V$ is weight vector such that $(x^+_\alpha)^{(k)}v = 0$ for all $\alpha \in R^+, k > 0$, then v is said to be a highest-weight vector. If V is generated by a highest-weight vector, then it is said to be a highest-weight module. Similarly, one defines the notions of lowest-weight vectors and modules by replacing $(x^+_\alpha)^{(k)}$ by $(x^-_\alpha)^{(k)}$.

Theorem 2.3.1. Let V be a $U_{\mathbb{F}}(\mathfrak{g})$-module.

(a) If V is finite-dimensional, then V is a weight-module, $\mathrm{wt}(V) \subseteq P$, and $\dim V_\mu = \dim V_{\sigma\mu}$ for all $\sigma \in \mathcal{W}, \mu \in U_{\mathbb{F}}(\mathfrak{h})^*$. In particular, $\mathrm{ch}(V) \in \mathbb{Z}[P]^{\mathcal{W}}$.
(b) If V is a highest-weight module of highest weight λ, then $\dim(V_\lambda) = 1$ and $V_\mu \ne 0$ only if $\mu \le \lambda$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.

(c) For each $\lambda \in P^+$, the $U_{\mathbb{F}}(\mathfrak{g})$-module $W_{\mathbb{F}}(\lambda)$ given by the quotient of $U_{\mathbb{F}}(\mathfrak{g})$ by the left ideal $I_{\mathbb{F}}(\lambda)$ generated by

$$U_{\mathbb{F}}(\mathfrak{n}^+)^0, \quad h-\lambda(h) \quad \text{and} \quad (x_\alpha^-)^{(k)}, \quad \text{for all} \quad h \in U_{\mathbb{F}}(\mathfrak{h}), \alpha \in R^+, k > \lambda(h_\alpha),$$

is nonzero and finite-dimensional. Moreover, every finite-dimensional highest-weight module of highest weight λ is a quotient of $W_{\mathbb{F}}(\lambda)$.

(d) If V is finite-dimensional and irreducible, then there exists a unique $\lambda \in P^+$ such that V is isomorphic to the irreducible quotient $V_{\mathbb{F}}(\lambda)$ of $W_{\mathbb{F}}(\lambda)$. If the characteristic of $\mathbb{F}$ is zero, then $W_{\mathbb{F}}(\lambda)$ is irreducible.

(e) For each $\lambda \in P^+$, $\mathrm{ch}(W_{\mathbb{F}}(\lambda))$ is given by the Weyl character formula. In particular, $\mu \in \mathrm{wt}(W_{\mathbb{F}}(\lambda))$ if, and only if, $\sigma\mu \le \lambda$ for all $\sigma \in \mathcal{W}$. Moreover, $W_{\mathbb{F}}(\lambda)$ is a lowest-weight module with lowest weight $w_0\lambda$. □

Remark 2.3.2. The module $W_{\mathbb{F}}(\lambda)$ defined in Theorem 2.3.1 (c) is called Weyl module (or costandard module) of highest weight λ. The known proofs of Theorem 2.3.1 (e) make use of geometric results such as Kempf's Vanishing Theorem.

We shall need the following lemma in the proof of Lemma 2.5.5 below.

Lemma 2.3.3. Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g})$-module, $\mu \in P$, and $\alpha \in R^+$. If $v \in V_\mu \setminus \{0\}$ is such that $(x_\alpha^-)^{(k)}v = 0$ for all $k > 0$, then $\mu(h_\alpha) \in \mathbb{Z}_{\le 0}$ and $(x_\alpha^+)^{(-\mu(h_\alpha))}v \neq 0$. □

Remark 2.3.4. In characteristic zero, it is well-known that the following stronger statement holds: if $v \in V_\mu \setminus \{0\}$ is such that $\mu(h_\alpha) \in \mathbb{Z}_{\le 0}$, then $(x_\alpha^+)^{(-\mu(h_\alpha))}v \neq 0$. In positive characteristic this stronger statement is not true for all finite-dimensional representations.

We will need to consider the integral version of Weyl modules. Notice that the formulas in the definition of the inclusion $P \hookrightarrow U_{\mathbb{F}}(\mathfrak{h})^*$ also give rise to an inclusion $P \hookrightarrow U_{\mathbb{Z}}(\mathfrak{h})^* := \mathrm{Hom}_{\mathbb{Z}}(U_{\mathbb{Z}}(\mathfrak{h}), \mathbb{Z})$. Then, given a $U_{\mathbb{Z}}(\mathfrak{g})$-module V and $\mu \in P$, the weight space V_μ can be defined as before and (2.3.1) remains valid. In particular, if $V = \oplus_{\mu \in P} V_\mu$ and V_μ is a finitely generated $\mathbb{Z}$-module for all $\mu \in P$, we can define the character of V by setting $\mathrm{ch}(V)(\mu)$ as the rank of V_μ as a $\mathbb{Z}$-module. Notice that $\mathbb{F} \otimes_{\mathbb{Z}} V$ is then a $U_{\mathbb{F}}(\mathfrak{g})$-module with finite-dimensional weight spaces and $\mathrm{ch}(\mathbb{F} \otimes_{\mathbb{Z}} V)(\mu) \ge \mathrm{ch}(V)(\mu)$ for all $\mu \in P$. Moreover, equality holds if, an only if, $pT = 0$ where p is the characteristic of $\mathbb{F}$ and T is the torsion subgroup of V.

A $U_{\mathbb{Z}}(\mathfrak{g})$-module V is said to be integrable if, for all $v \in V$, there exists $m > 0$ such that $(x_\alpha^\pm)^{(k)}v = 0$ for all $\alpha \in R^+, k > m$. Proposition 2.2.2 implies that this equivalent to saying that there exists $m' > 0$ such that $(x_i^\pm)^{(k)}v = 0$ for all $i \in I, k > m'$.

Proposition 2.3.5. Let V be an integrable $U_{\mathbb{Z}}(\mathfrak{g})$-module. Then, for all $\sigma \in \mathcal{W}, \mu \in P$, there exists an isomorphism of $\mathbb{Z}$-modules $V_\mu \to V_{\sigma\mu}$.

Proof. It suffices to prove the statement with $\sigma = s_i$ for some $i \in I$. Consider the map $T: V_\mu \to V_{s_i\mu}$ given by

$$T(v) = \sum_{\substack{a,b,c \ge 0 \\ b-a-c=\mu(h_i)}} (-1)^b (x_i^+)^{(a)} (x_i^-)^{(b)} (x_i^+)^{(c)} v.$$

Proceeding as in [**Lus93**, 5.2], one shows that T is an isomorphism. □

Given $\lambda \in P^+$, let $I_{\mathbb{Z}}(\lambda) \subset U_{\mathbb{Z}}(\mathfrak{g})$ be the left ideal generated by

$$U_{\mathbb{Z}}(\mathfrak{n}^+)^0, \quad h-\lambda(h), \quad (x_\alpha^-)^{(k)}, \quad \text{for all} \quad h \in U_{\mathbb{Z}}(\mathfrak{h}),\ \alpha \in R^+,\ k > \lambda(h_\alpha),$$

and set

$$W_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}(\mathfrak{g}[t])/I_{\mathbb{Z}}(\lambda). \tag{2.3.2}$$

Theorem 2.3.6. For all $\lambda \in P^+, W_{\mathbb{Z}}(\lambda)$ is finitely generated and free as a $\mathbb{Z}$-module. Moreover, $W_{\mathbb{Z}}(\lambda) = \oplus_{\mu \leq \lambda} W_{\mathbb{Z}}(\lambda)_\mu$ and $\mathrm{ch}(W_{\mathbb{Z}}(\lambda))$ is given by the Weyl character formula.

Proof. Quite clearly $W_{\mathbb{Z}}(\lambda) = \oplus_{\mu \leq \lambda} W_{\mathbb{Z}}(\lambda)_\mu$ and $W_{\mathbb{Z}}(\lambda)_\mu$ is finitely generated for all $\mu \in P$. Moreover, standard arguments show that $W_{\mathbb{Z}}(\lambda)$ is integrable. In particular, Proposition 2.3.5 implies that $W_{\mathbb{Z}}(\lambda)_\mu \neq 0$ only if $\sigma\mu \leq \lambda$ for all $\sigma \in \mathcal{W}$. Since the set $\{\sigma\mu : \mu, \sigma\mu \leq \lambda \text{ for all } \sigma \in \mathcal{W}\}$ is finite, it follows that $W_{\mathbb{Z}}(\lambda)$ is finitely generated. An application of Lemma 2.3.9 gives an isomorphism of $U_{\mathbb{F}}(\mathfrak{g})$-module $W_{\mathbb{F}}(\lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}(\lambda)$. Since Theorem 2.3.1 (e) implies that $\mathrm{ch}(W_{\mathbb{F}}(\lambda))$ does not depend on the characteristic of $\mathbb{F}$, it follows that $W_{\mathbb{Z}}(\lambda)$ is free and have the same character as $\mathrm{ch}(W_{\mathbb{F}}(\lambda))$. □

The next lemma can be proved exactly as [**Nao12**, Lemma 4.5].

Lemma 2.3.7. Let $m_i \in \mathbb{Z}_{\geq 0}, i \in I, V$ a finite-dimensional $U_{\mathbb{F}}(\mathfrak{n}^-)$-module and suppose $v \in V$ satisfies $(x_i^-)^{(k)} v = 0$ for all $i \in I, k > m_i$. Then, given $\alpha \in R^+$, we have $(x_\alpha^-)^{(k)} v = 0$ for all $k > \sum_{i \in I} n_i m_i$ where n_i are such that $h_\alpha = \sum_{i \in I} n_i h_i$. □

2.3.2. Modules for hyper loop algebras. We now recall some basic results about the category of finite-dimensional $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-modules in the same spirit as Subsection 2.3.1. The results of this subsection can be found in [**JM07**, Section 3] and references therein.

Given a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module V and $\xi \in U_{\mathbb{F}}(\tilde{\mathfrak{h}})^*$, let

$$V_\xi = \{v \in V : \text{ for all } x \in U_{\mathbb{F}}(\tilde{\mathfrak{h}}), \text{ there exists } k > 0 \text{ such that } (x - \xi(x))^k v = 0\}.$$

We say that V is an ℓ-weight module if $V = \bigoplus_{\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}} V_{\boldsymbol{\omega}}$. In this case, regarding V as a $U_{\mathbb{F}}(\mathfrak{g})$-module, we have

$$V_\mu = \bigoplus_{\substack{\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}: \\ \mathrm{wt}(\boldsymbol{\omega}) = \mu}} V_{\boldsymbol{\omega}} \quad \text{for all} \quad \mu \in P \quad \text{and} \quad V = \bigoplus_{\mu \in P} V_\mu.$$

A nonzero element of $V_{\boldsymbol{\omega}}$ is said to be an ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$. An ℓ-weight vector v is said to be a highest-ℓ-weight vector if $U_{\mathbb{F}}(\tilde{\mathfrak{h}})v = \mathbb{F}v$ and $(x_{\alpha,r}^+)^{(k)} v = 0$ for all $\alpha \in R^+$ and all $r, k \in \mathbb{Z}, k > 0$. If V is generated by a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$, V is said to be a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$.

Theorem 2.3.8. Let V be a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module.

(a) If V is finite-dimensional, then V is an ℓ-weight module. Moreover, if V is finite-dimensional and irreducible, then V is a highest-ℓ-weight module whose highest ℓ-weight lies in $\mathcal{P}_{\mathbb{F}}^+$.
(b) If V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$, then $\dim V_{\boldsymbol{\omega}} = 1$ and $V_\mu \neq 0$ only if $\mu \leq \mathrm{wt}(\boldsymbol{\omega})$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.
(c) For each $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$, the local Weyl module $W_{\mathbb{F}}(\boldsymbol{\omega})$ is nonzero and finite-dimensional. Moreover, every finite-dimensional highest-ℓ-weight-module of highest ℓ-weight $\boldsymbol{\omega}$ is a quotient of $W_{\mathbb{F}}(\boldsymbol{\omega})$.
(d) If V is finite-dimensional and irreducible, then there exists a unique $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$ such that V is isomorphic to the irreducible quotient $V_{\mathbb{F}}(\boldsymbol{\omega})$ of $W_{\mathbb{F}}(\boldsymbol{\omega})$.
(e) For $\mu \in P$ and $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{F}}^+$, we have $\mu \in \mathrm{wt}(W_{\mathbb{F}}(\boldsymbol{\omega}))$ if and only if $\mu \in \mathrm{wt}(W_{\mathbb{F}}(\mathrm{wt}(\boldsymbol{\omega})))$, i.e. $w\mu \leq \mathrm{wt}(\boldsymbol{\omega})$, for all $w \in \mathcal{W}$. □

2.3.3. Graded modules for hyper current algebras. Recall the following elementary fact.

Lemma 2.3.9. Let A be a ring, $I \subset A$ a left ideal, $B = \mathbb{F} \otimes_{\mathbb{Z}} A$ an $\mathbb{F}$-algebra, and J the image of I in B, i.e. J is the $\mathbb{F}$-span of $\{(1 \otimes a) \in B : a \in I\}$. Then $\mathbb{F} \otimes_{\mathbb{Z}} (A/I)$ is a left B-module, J is a left ideal of B, and we have an isomorphism of left B-modules $B/J \cong \mathbb{F} \otimes_{\mathbb{Z}} (A/I)$. □

Proof. First observe that $(\mathbb{F} \otimes_{\mathbb{Z}} (A/I))$ is a left B-module, with left action linear extending $(\mu \otimes b)(\lambda \otimes \overline{a}) = (\mu\lambda) \otimes \overline{ba}$, for any $(\mu \otimes b) \in B$ and $(\lambda \otimes \overline{a}) \in (\mathbb{F} \otimes_{\mathbb{Z}} (A/I))$. Also observe that, for any $(\lambda \otimes a) \in B$, $\mu \in \mathbb{F}$ and $i \in I$, we have $(\lambda \otimes a)(\mu \otimes i) = (\lambda\mu) \otimes (ai)$. It follows from linearity on both factors and the fact that $I \subseteq A$ is a left ideal that $J \subseteq B$ is a left ideal.

Suppose $I \subseteq A$ is a left ideal and M is a cyclic left A-module given by the following short exact sequence of left A-modules $0 \to I \to A \to M \to 0$. Now consider the exact sequence obtained from this short exact sequence by applying the right-exact functor $(\mathbb{F} \otimes_{\mathbb{Z}} -)$, we get $(\mathbb{F} \otimes_{\mathbb{Z}} I) \xrightarrow{j} (\mathbb{F} \otimes_{\mathbb{Z}} A) \xrightarrow{p} (\mathbb{F} \otimes_{\mathbb{Z}} M) \longrightarrow 0$.

By hypothesis, there is an isomorphism of left B-modules $(\mathbb{F} \otimes_{\mathbb{Z}} A) \cong B$. Moreover the image of j is the $\mathbb{F}$-submodule of B generated by $\{(1 \otimes i) \in B : i \in I\}$, which is, by hypothesis, J. Since the latter sequence is exact, there is an isomorphism of left B-modules $(\mathbb{F} \otimes_{\mathbb{Z}} M) \cong B/J$. □

We shall use Lemma 2.3.9 with A being one of the integral forms so that B is the corresponding hyperalgebra.

Given $\lambda \in P^+$, let $I^c_{\mathbb{Z}}(\lambda) \subset U_{\mathbb{Z}}(\mathfrak{g}[t])$ be the left ideal generated by

$$U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0, \quad U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0, \quad h - \lambda(h), \quad (x_\alpha^-)^{(k)}, \quad \text{for all} \quad h \in U_{\mathbb{Z}}(\mathfrak{h}),\ \alpha \in R^+,\ k > \lambda(h_\alpha),$$

and set

$$W^c_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}(\mathfrak{g}[t])/I^c_{\mathbb{Z}}(\lambda).$$

Similarly, if $\ell \geq 0$ is also given, let $I_{\mathbb{Z}}(\ell, \lambda)$ be the left ideal of $U_{\mathbb{Z}}(\mathfrak{g}[t])$ generated by

$$U_{\mathbb{Z}}(\mathfrak{n}^+[t])^0, \quad U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0, \quad h - \lambda(h), \quad (x_{\alpha,s}^-)^{(k)}, \quad \text{for all} \quad h \in U_{\mathbb{Z}}(\mathfrak{h}),\ \alpha \in R^+,$$
$$s, k \in \mathbb{Z}_{\geq 0},\ k > \ \max\{0, \lambda(h_\alpha) - r_\alpha^\vee \ell s\}.$$

Then set

$$D_{\mathbb{Z}}(\ell, \lambda) = U_{\mathbb{Z}}(\mathfrak{g}[t])/I_{\mathbb{Z}}(\ell, \lambda).$$

Notice that $W^c_{\mathbb{Z}}(\lambda)$ and $D_{\mathbb{Z}}(\ell, \lambda)$ are weight modules.

Since the ideals defining $W^c_{\mathbb{F}}(\lambda)$ and $D_{\mathbb{F}}(\ell, \lambda)$ (cf. Subsection 2.1.4) are the images of $I^c_{\mathbb{Z}}(\lambda)$ and $I_{\mathbb{Z}}(\ell, \lambda)$ in $U_{\mathbb{F}}(\mathfrak{g}[t])$, respectively, an application of Lemma 2.3.9 gives isomorphisms of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules

$$W^c_{\mathbb{F}}(\lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} W^c_{\mathbb{Z}}(\lambda) \quad \text{and} \quad D_{\mathbb{F}}(\ell, \lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} D_{\mathbb{Z}}(\ell, \lambda).$$

As before, $D_{\mathbb{Z}}(\ell, \lambda)$ is a quotient of $W^c_{\mathbb{Z}}(\lambda)$ for all $\lambda \in P^+$ and all $\ell > 0$. We shall see next (Proposition 2.3.11) that the latter is a finitely generated $\mathbb{Z}$-module and, hence, so is the former. Together with Corollary 2.1.3, this implies that

$$D_{\mathbb{Z}}(\ell, \lambda) \ \text{ is a free } \mathbb{Z}\text{-module.} \tag{2.3.3}$$

We record the following elementary lemma to be used in the proof of Proposition 2.3.11.

Lemma 2.3.10. Let n be a non negative integer and, for $l \leq k$, let $[l, k] = \{m \in \mathbb{Z} : l \leq m \leq k\}$. Then,

$$\mathbb{Z}_{\geq n^2} \subseteq \bigcup_{m \geq 1} [mn, m(n+1)].$$

The proof of the next proposition is an adaptation of that of [**JM07**, Theorem 3.11].

Proposition 2.3.11. For every $\lambda \in P^+$, the $U_\mathbb{Z}(\mathfrak{g}[t])$-module $W^c_\mathbb{Z}(\lambda)$ is a finitely generated $\mathbb{Z}$-module.

Proof. First, we prove that $W^c_\mathbb{Z}(\lambda)$ is an integrable $U_\mathbb{Z}(\mathfrak{g})$-module. Let ϑ be the image of 1 in $W^c_\mathbb{Z}(\lambda)$ and observe that $W^c_\mathbb{Z}(\lambda) = U_\mathbb{Z}(\mathfrak{n}^-[t])\vartheta$. In particular, the weights of $W^c_\mathbb{Z}(\lambda)$ are bounded above by λ, which implies that $(x^+_\alpha)^{(k)}W^c_\mathbb{Z}(\lambda) = 0$ for k sufficiently large.

Let $\mathcal{T} = R^+ \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, Ξ be the set of functions $\phi : \mathbb{Z}_{>0} \to \mathcal{T}$, $\phi(j) = (\beta_j, r_j, k_j)$, such that $k_j = 0$ for all j sufficiently large. Observed that any $w \in W^c_\mathbb{Z}(\lambda)$ has the form

$$w = \sum_{\xi \in \Xi} c_\xi (x^-_{\beta_1,r_1})^{(k_1)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta. \tag{2.3.4}$$

So without loss of generality, we assume $w = (x^-_{\beta_1,r_1})^{(k_1)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta$ for some $(\beta_j, r_j, k_j) = \phi(j)$ and $\phi \in \Xi$. Given $\phi \in \Xi$ and $n > 0$ such that $k_j = 0$ for all $j > n$, denote $\vartheta_\phi = (x^-_{\beta_1,r_1})^{(k_1)} \cdots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta \in W^c_\mathbb{Z}(\lambda)$, and define the hyperdegree of ϕ (or hyperdegree of ϑ_ϕ) to be $d(\phi) = \sum_{j>0} k_j$. We will use induction on the hyperdegree of ϕ in order to prove that $(x^-_\alpha)^{(k)}\vartheta_\phi = 0$, for all $k \geq k_\phi$, for some $k_\phi > 0$.

If the hyperdegree of ϕ is 0, then $w = \vartheta$. In this case, the defining relations for $W^c_\mathbb{Z}(\lambda)$ and $W_\mathbb{Z}(\lambda)$ show that $U_\mathbb{Z}(\mathfrak{g})\vartheta$ is a quotient of $W_\mathbb{Z}(\lambda)$. Since $W_\mathbb{Z}(\lambda)$ is a finitely generated $\mathbb{Z}$-module, it follows that $U_\mathbb{Z}(\mathfrak{g})\vartheta$ is finitely generated, and that there exists $k_\vartheta > 0$, such that $(x^-_\alpha)^{(k)}\vartheta = 0$ for all $k \geq k_\vartheta$.

Now suppose the hyperdegree of ϕ is $d > 0$. By induction hypothesis, there exists $m_0 > 0$ such that $(x^-_\alpha)^{(m)}((x^-_{\beta_2,r_2})^{(k_2)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta) = 0$, for all $m \geq m_0$. Consider the left ideal $I_{\alpha,m_0} \subseteq U_\mathbb{Z}(\mathfrak{g}[t])$ generated by $\{(x^-_\alpha)^{(k)} : k \geq m_0\}$. Since the adjoint representation of $\mathfrak{g}$ is integrable, it follows that there exists $\ell_0 > 0$, such that $\mathrm{ad}^\ell(x^-_\alpha)(x^-_{\beta,r}) = 0$, for all $\ell \geq \ell_0$. This implies that there exists $k_0 > 0$ such that $(x^-_\alpha)^{(k)}(x^-_{\beta_1,r_1})^{(k_1)} \in I_{\alpha,m_0}$ for all $k_1 > 0$ and $k > k_0$. Thus

$$(x^-_\alpha)^{(k)}((x^-_{\beta_1,r_1})^{(k_1)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta) = \sum_{m \geq m_0} u_m (x^-_\alpha)^{(m)}((x^-_{\beta_2,r_2})^{(k_2)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta),$$

for some $u_m \in U_\mathbb{Z}(\mathfrak{g}[t])$. Since $(x^-_\alpha)^{(m)}((x^-_{\beta_2,r_2})^{(k_2)} \dots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta) = 0$ for all $m \geq m_0$, it follows that $(x^-_\alpha)^{(k)}\vartheta_\phi = 0$ for k sufficiently large. Thus proving that $W^c_\mathbb{Z}(\lambda)$ is an integrable $U_\mathbb{Z}(\mathfrak{g})$-module.

Proposition 2.3.5 then implies that the set of weights of $W^c_\mathbb{Z}(\lambda)$ is invariant under $\mathcal{W}$. Since the weights are bounded by λ, it follows the set of weights is contained in that of $W_\mathbb{Z}(\lambda)$ and, hence, is finite.

Since any element $w \in W^c_\mathbb{Z}(\lambda)$ has the form (2.3.4), the subgroup of $W^c_\mathbb{Z}(\lambda)$ consisting of elements of hyperdegree d and weight $\lambda \in P$ must be finitely generated. And since $\mathrm{wt}(W^c_\mathbb{Z}(\lambda))$ is a finite set, the subgroup of $W^c_\mathbb{Z}(\lambda)$ consisting of elements of hyperdegree d must be finitely generated. So, in order to show that $W^c_\mathbb{Z}(\lambda)$ is finitely generated, we only have to show that it lives in finitely many distinct hyperdegrees.

Define the exponent of $\phi \in \Xi$ as $e(\phi) = \max\{k_j : j > 0\}$, observe that $0 \leq e(\phi) \leq d(\phi)$, and set

$$\Xi_{d,e} = \{\phi \in \Xi : d(\phi) = d, e(\phi) = e\}, \qquad \Xi_d = \bigcup_{0 \leq e \leq d} \Xi_{d,e},$$

$$\Xi' = \{\phi \in \Xi : \phi(j) = (\beta_j, r_j, k_j), 0 \leq r_j < \lambda(h_{\beta_j})^2, \forall j \in \mathbb{Z}_{>0}\},$$

$$W' = \sum_{\phi \in \Xi'} \mathbb{Z}\vartheta_\phi \subseteq W^c_\mathbb{Z}(\lambda) \quad \text{and} \quad W_d = \sum_{\phi \in \Xi_d} \mathbb{Z}\vartheta_\phi \subseteq W'.$$

We will show next that $W' = W_{\mathbb{Z}}^c(\lambda)$. Observe that $\Xi = \cup_{d,e\geq 0}\Xi_{d,e}$. We will show that, for any $d, e \geq 0$, $\vartheta_\phi \in W'$ for all $\phi \in \Xi_{d,e}$ by induction on d and e. If $d(\phi) = 0$, then $\vartheta_\phi = \vartheta \in W'$. Assume that $d(\phi) > 0$ and that the statement holds for all $\phi' \in (\cup_{d<d(\phi)}\Xi_d) \cup (\cup_{e<e(\phi)}\Xi_{d(\phi),e})$. The proof splits into two cases, according to whether $e(\phi) = d(\phi)$ or not.

Suppose $e(\phi) < d(\phi)$, in which case $n > 1$. By induction hypothesis $(x^-_{\beta_2,r_2})^{(k_2)} \cdots (x^-_{\beta_n,r_n})^{(k_n)}\vartheta \in W'$. So without loss of generality, we can assume that $0 \leq r_j < \lambda(h_{\beta_j})^2$ if $j > 1$. Using Lemma 2.2.4 to commute $(x^-_{\beta_1,r_1})^{(k_1)}$ with $(x^-_{\beta_2,r_2})^{(k_2)} \cdots (x^-_{\beta_n,r_n})^{(k_n)}$, and induction hypothesis on the terms of hyperdegree strictly smaller than d, we obtain $\vartheta_\phi + W' = (x^-_{\beta_2,r_2})^{(k_2)} \cdots (x^-_{\beta_n,r_n})^{(k_n)}(x^-_{\beta_1,r_1})^{(k_1)}\vartheta + W'$. By induction hypothesis $(x^-_{\beta_1,r_1})^{(k_1)}\vartheta \in W'$, and the result follows.

Suppose $e(\phi) = d(\phi)$, in which case $\vartheta_\phi = (x^-_{\beta,r})^{(e)}\vartheta$, and suppose $r \geq \lambda(h_\beta)^2$. If $e = 1$, Lemma 2.2.1 with $l = \lambda(h_\beta)$ and $k = \lambda(h_\beta) + 1$ yields

$$(x^+_{\beta,m-s})^{(\lambda(h_\beta))}(x^-_{\beta,s})^{(\lambda(h_\beta)+1)}\vartheta = (-1)^{\lambda(h_\beta)}(X^-_{\beta,m,s}(u)\Lambda^+_{\beta,m}(u))_{\lambda(h_\beta)+1}\vartheta.$$

Since $U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0\vartheta = 0$ and $(x^-_{\beta,s})^{(k)}\vartheta = 0$ for all $k > \lambda(h_\beta), s \geq 0$, we have $(-1)^{\lambda(h_\beta)}x^-_{\beta,(m\lambda(h_\beta)+s)}\vartheta = 0$. If $e > 1$, from Lemma 2.2.1 with $l = e\lambda(h_\beta)$ and $k = e(\lambda(h_\beta) + 1)$, we obtain

$$\begin{aligned} 0 &= (x^+_{\beta,m-s})^{(e\lambda(h_\beta))}(x^-_{\beta,s})^{(e(\lambda(h_\beta)+1))}\vartheta \\ &= (-1)^{e\lambda(h_\beta)}((X^-_{\beta,m,s}(u))^{(e)}\Lambda^+_{\beta,m}(u))_{e(\lambda(h_\beta)+1)}\vartheta \\ &= (-1)^{e\lambda(h_\beta)}(W_e + x^-_{\beta,(m\lambda(h_\beta)+s)}\vartheta). \end{aligned}$$

Thus $(x^-_{\beta,m\lambda(h_\beta)+s})^{(e)}\vartheta \in \sum_{\phi'\in\Xi_d}\mathbb{Z}\vartheta_{\phi'} = W_d$. Observe that $0 \leq s \leq m$ is equivalent to $m\lambda(h_\beta) \leq m\lambda(h_\beta) + s \leq m(\lambda(h_\beta) + 1)$. Using Lemma 2.3.10 and varying s and m, we obtain that $(x^-_{\beta,r})^{(e)}\vartheta \in W'$ for all $r \geq \lambda(h_\beta)^2$, finishing the proof. $\square$

We now prove an analogue of Theorem 2.1.4 for graded local Weyl modules.

Corollary 2.3.12. Let $\lambda \in P^+$ and v be the image of 1 in $W_{\mathbb{C}}^c(\lambda)$. Then $U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is a free $\mathbb{Z}$-module of rank $\dim(W_{\mathbb{C}}^c(\lambda))$. Moreover, $U_{\mathbb{Z}}(\mathfrak{g}[t])v = \oplus_{\mu\in P}(U_{\mathbb{Z}}(\mathfrak{g}[t])v \cap W_{\mathbb{C}}^c(\lambda)_\mu)$. In particular, $U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is an integral form for $W_{\mathbb{C}}^c(\lambda)$.

Proof. To simplify notation, set $L = U_{\mathbb{Z}}(\mathfrak{n}^-)v$. Let also ϑ be as in the proof of Proposition 2.3.11. Since v satisfies the relations satisfied by ϑ, it follows that there exists an epimorphism of $U_{\mathbb{Z}}(\mathfrak{g}[t])$-modules $W_{\mathbb{Z}}(\lambda) \to L, \vartheta \mapsto v$. Since $W_{\mathbb{Z}}(\lambda)$ is finitely generated, it follows that so is L. On the other hand, since $L \subseteq W_{\mathbb{C}}^c(\lambda)$, it is also torsion free and, hence, a free $\mathbb{Z}$-module of finite-rank. Since $U_{\mathbb{Z}}(\mathfrak{n}^-)$ spans $U(\mathfrak{n}^-)$ and $W_{\mathbb{C}}^c(\lambda) = U(\mathfrak{n}^-)v$, it follows that L contains a basis of $W_{\mathbb{C}}^c(\lambda)$. This implies that the rank of L is at least $\dim(W_{\mathbb{C}}^c(\lambda))$. On the other hand, $\mathbb{C}\otimes_{\mathbb{Z}} L$ is a $\mathfrak{g}[t]$-module generated by the vector $1 \otimes v$ which satisfies the relations (2.1.1). Therefore, it is a quotient of $W_{\mathbb{C}}^c(\lambda)$. Since $\dim(\mathbb{C}\otimes_{\mathbb{Z}} L) = \operatorname{rank}(L)$, the first and the last statements follow. The second statement is clear since L is a weight module. $\square$

Consider the category $\mathcal{G}_{\mathbb{F}}$ of $\mathbb{Z}$-graded finite-dimensional representations of $U_{\mathbb{F}}(\mathfrak{g}[t])$. If V is such a module, let $V[r]$ be its r-th graded piece. Given $s \in \mathbb{Z}$, let $\tau_s(V)$ be the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module such that $\tau_s(V)[r] = V[r-s]$ for all $r \in \mathbb{Z}$. For each $U_{\mathbb{F}}(\mathfrak{g})$-module V, let $\mathrm{ev}_0(V)$ be the module in $\mathcal{G}_{\mathbb{F}}$ obtained by extending the action of $U_{\mathbb{F}}(\mathfrak{g})$ to one of $U_{\mathbb{F}}(\mathfrak{g}[t])$ on V by setting $U_{\mathbb{F}}(\mathfrak{g}[t]_+)V = 0$. For $\lambda \in P^+, r \in \mathbb{Z}$, set $V_{\mathbb{F}}(\lambda, r) = \mathrm{ev}_r(V_{\mathbb{F}}(\lambda))$ where $\mathrm{ev}_r = \tau_r \circ \mathrm{ev}_0$.

Theorem 2.3.13. Let $\lambda \in P^+$

(a) If V in $\mathcal{G}_{\mathbb{F}}$ is simple, then it is isomorphic to $V_{\mathbb{F}}(\lambda, r)$ for a unique $(\lambda, r) \in P^+ \times \mathbb{Z}$.

(b) $W^c_{\mathbb{F}}(\lambda)$ is finite-dimensional.
(c) If V is a graded finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module generated by a weight vector v of weight λ satisfying $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$, then V is a quotient of $W^c_{\mathbb{F}}(\lambda)$.

Proof. To prove part (a), suppose $V \in \mathcal{G}_{\mathbb{F}}$ is simple. If $V[r], V[s] \neq 0$ for $s < r \in \mathbb{Z}$, $(\oplus_{k \geq r} V[k])$ would be a proper submodule of V, contradicting the fact that it is simple. Thus there must exist a unique $r \in \mathbb{Z}$ such that $V[r] \neq 0$. Since $U_{\mathbb{F}}(\mathfrak{g}[t]_+)$ changes degrees, $V = V[r]$ must be a simple $U_{\mathbb{F}}(\mathfrak{g})$-module. This shows that $V \cong V_{\mathbb{F}}(\lambda, r)$ for some $\lambda \in P^+, r \in \mathbb{Z}$.

To prove part (b), observe that $W^c_{\mathbb{F}}(\lambda) \cong \mathbb{F} \otimes_{\mathbb{Z}} W^c_{\mathbb{Z}}(\lambda)$ (cf. Lemma 2.3.9). Thus the dimension of $W^c_{\mathbb{F}}(\lambda)$ must be at most the number of generators of $W^c_{\mathbb{Z}}(\lambda)$, which is proved to be finite in Proposition 2.3.11.

To prove part (c), observe that the $U_{\mathbb{F}}(\mathfrak{g})$-submodule $V' = U_{\mathbb{F}}(\mathfrak{g})v \subseteq V$ is a finite-dimensional highest-weight module of highest weight λ. Thus, by Theorem 2.3.1 (c), V' is a quotient of $W_{\mathbb{F}}(\lambda)$. The statement follows by comparing the defining relations of V and $W^c_{\mathbb{F}}(\lambda)$. □

Remark 2.3.14. Denote by v the image of 1 in $W^c_{\mathbb{F}}(\lambda)$. From the defining relations (2.1.1) it follows that $\mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}[t])v$ is a quotient of $W^c_{\mathbb{F}}(\lambda)$. It follows from Theorem 2.1.2(b) that $\mathbb{F} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}[t])v \cong W^c_{\mathbb{F}}(\lambda)$ for all $\lambda \in P^+$ (cf. Section 2.3.4 below). Moreover, since $\mathbb{F} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}(\lambda) \cong W^c_{\mathbb{F}}(\lambda)$, Theorem 2.1.2 (b) also implies that $W_{\mathbb{Z}}(\lambda)$ is free.

2.3.4. Proof of the Jakelić-Moura conjecture. We now prove (2.1.4). The argument will use Corollary 2.1.3, the characteristic zero versions of Theorem 2.1.2 (c) and (2.1.6) and the following proposition [**Nao12**, Corollary A].

Proposition 2.3.15. Let $\lambda \in P^+$. Then, $\dim W^c_{\mathbb{C}}(\lambda) = \prod_{i \in I} (\dim W^c_{\mathbb{C}}(\omega_i))^{\lambda(h_i)}$. □

We shall also need the following general construction. Given a $\mathbb{Z}_{s \geq 0}$-filtered $U_{\mathbb{F}}(\mathfrak{g}[t])$-module W, we can consider the associated graded $U_{\mathbb{F}}(\mathfrak{g}[t])$-module $\mathrm{gr}(W) = \oplus_{s \geq 0} W_s / W_{s-1}$ which has the same dimension as W. Suppose now that W is any cyclic $U_{\mathbb{F}}(\mathfrak{g}[t])$-module and fix a generator w. Then, the $\mathbb{Z}$-grading on $U_{\mathbb{F}}(\mathfrak{g}[t])$ induces a filtration on W. Namely, set w to have degree zero and define the s-th filtered piece of W by $W_s = F^s U_{\mathbb{F}}(\mathfrak{g}[t])w$ where $F^s U_{\mathbb{F}}(\mathfrak{g}[t]) = \oplus_{r \leq s} U_{\mathbb{F}}(\mathfrak{g}[t])[r]$. Then, $\mathrm{gr}(W)$ is cyclic since it is generated by the image of w in $\mathrm{gr}(W)$.

Recall the notation fixed for (2.1.4): $\boldsymbol{\omega} \in \mathcal{P}^\times_{\mathbb{A}}$, $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, $\boldsymbol{\varpi}$ is the image of $\boldsymbol{\omega}$ in $\mathcal{P}_{\mathbb{F}}$. Also recall that, using (2.1.3), (2.1.4) will be proved if we show that

$$\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) \leq \dim W_{\mathbb{K}}(\boldsymbol{\omega}).$$

Fix $w \in W_{\mathbb{F}}(\boldsymbol{\varpi})_\lambda \setminus \{0\}$. Not only w generates $W_{\mathbb{F}}(\boldsymbol{\varpi})$ as a $U_{\mathbb{F}}(\tilde{\mathfrak{g}})$-module, but it also follows from the proof of [**JM07**, Theorem 3.11] (with a correction incorporated in the proof of [**JM10**, Theorem 3.7]) that $U_{\mathbb{F}}(\mathfrak{n}^-[t])w = W_{\mathbb{F}}(\boldsymbol{\varpi})$. Hence, we can apply the general construction reviewed above to $W_{\mathbb{F}}(\boldsymbol{\varpi})$. Set $V = \mathrm{gr}(W_{\mathbb{F}}(\boldsymbol{\varpi}))$ and denote the image of w in V by v. The module V is finite-dimensional and v is a highest-weight vector of weight λ satisfying $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ (the latter follows since $\dim(V_\lambda) = 1$, V is graded, and $U_{\mathbb{F}}(\mathfrak{h}[t])$ is commutative). Hence, v satisfies the defining relations (2.1.1) of $W^c_{\mathbb{F}}(\lambda)$. In particular, we get that $\dim W_{\mathbb{F}}(\boldsymbol{\varpi}) \leq \dim W^c_{\mathbb{F}}(\lambda)$.

Since $\dim W^c_{\mathbb{F}}(\lambda) = \dim W^c_{\mathbb{K}}(\lambda)$ by Corollary 2.1.3, it now suffices to show that $\dim W^c_{\mathbb{K}}(\lambda) = \dim W_{\mathbb{K}}(\boldsymbol{\omega})$. For proving this, consider the decomposition $\boldsymbol{\omega} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j, a_j}$ where $m \geq 0, a_j \in \mathbb{K}^\times, \lambda_j \in P^+$, such that $a_i \neq a_j$ if $i \neq j$, and $\lambda = \sum_{j=1}^m \lambda_j$. By (2.1.6) (in characteristic zero) $W_{\mathbb{K}}(\boldsymbol{\omega}) \cong \otimes_{j=1}^m W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda_j, a_j})$. Theorem 2.1.2(c) (in characteristic zero) implies that $\dim W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda_j, a_j}) =$

$\dim W^c_{\mathbb{K}}(\lambda_j)$. Hence,

$$\dim W_{\mathbb{K}}(\boldsymbol{\omega}) = \prod_{j=1}^{m} \dim W^c_{\mathbb{K}}(\lambda_j) = \prod_{j=1}^{m}\prod_{i\in I} \dim W^c_{\mathbb{K}}(\omega_i)^{\lambda_j(h_i)} = \prod_{i\in I} W^c_{\mathbb{K}}(\omega_i)^{\lambda(h_i)} = \dim W^c_{\mathbb{K}}(\lambda).$$

Here, the second and last equality follow from Proposition 2.3.15 and the others are clear. This completes the proof of (2.1.4).

Notice that all equalities of dimensions proved here actually imply the corresponding equalities of characters. In particular, it follows that

$$\mathrm{ch}(W_{\mathbb{F}}(\boldsymbol{\varpi})) = \prod_{i\in I} (\mathrm{ch}(W^c_{\mathbb{C}}(\omega_i)))^{\mathrm{wt}(\boldsymbol{\varpi})(h_i)} \quad \text{for all} \quad \boldsymbol{\varpi} \in \mathcal{P}^+_{\mathbb{F}}. \tag{2.3.5}$$

2.3.5. Joseph-Mathieu-Polo relations for Demazure modules. We now explain the reason why we call $D_{\mathbb{F}}(\ell,\lambda)$ Demazure modules. We begin with the following lemma. Let γ be the Dynkin diagram automorphism of $\mathfrak{g}$ induced by w_0 and recall from Subsection 2.2.2 that it induces an automorphism of $U_{\mathbb{F}}(\mathfrak{g}[t])$ which is also denoted by γ.

Lemma 2.3.16. Let $\lambda \in P^+, \ell \geq 0$, and set $\lambda^* = -w_0\lambda$. Let W be the pull-back of $D_{\mathbb{F}}(\ell,\lambda^*)$ by γ. Then, $D_{\mathbb{F}}(\ell,\lambda) \cong W$.

Proof. Let $v \in D_{\mathbb{F}}(\ell,\lambda^*)_{\lambda^*} \setminus \{0\}$. By (2.1.1) and (2.1.2) we have

$$U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0, \qquad hv = \lambda^*(h)v, \qquad (x^-_{\alpha,s})^{(k)}v = 0,$$

for all $h \in U_{\mathbb{F}}(\mathfrak{h}), \alpha \in R^+, s,k \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda^*(h_\alpha) - s\ell r^\vee_\alpha\}$. Denote by w the vector v regarded as an element of W. Evidently, $W = U_{\mathbb{F}}(\mathfrak{g}[t])w$. Since γ restricts to automorphisms of $U_{\mathbb{F}}(\mathfrak{n}^+[t])$ and of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)$, it follows that $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 w = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 w = 0$, while (2.2.7) implies that $w \in W_\lambda$. Finally, (2.2.6) and (2.2.7) together imply that

$$(x^-_{\alpha,s})^{(k)}v = 0 \qquad \text{for all} \qquad \alpha \in R^+, s,k \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda(h_\alpha) - s\ell r^\vee_\alpha\}.$$

This shows that w satisfies the defining relations of $D_{\mathbb{F}}(\ell,\lambda)$ and, hence, there exists an epimorphism $D_{\mathbb{F}}(\ell,\lambda)$. Since $(\lambda^*)^* = \lambda$, reversing the roles of λ and λ^* we get an epimorphism on the other direction. Since these are finite-dimensional modules, we are done. □

In order to continue, we need the concepts of weight vectors, weight spaces, weight modules and integrable modules for $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$ which are similar to those for $U_{\mathbb{F}}(\mathfrak{g})$ (cf. Subsection 2.3.1) by replacing I with $\hat{I}$ and P with $\hat{P}'$. Also, using the obvious analogue of (2.2.4), we obtain an inclusion $\hat{P}' \hookrightarrow U_{\mathbb{F}}(\hat{\mathfrak{h}}')^*$. Let V be a $\mathbb{Z}$-graded $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module whose weights lie in $\hat{P}'$. As before, let $V[r]$ denote the r-th graded piece of V. For $\mu \in \hat{P}$, say $\mu = \mu' + m\delta$ with $\mu' \in \hat{P}', m \in \mathbb{Z}$, set

$$V_\mu = \{v \in V[m] : hv = \mu'(h)v \text{ for all } h \in U_{\mathbb{F}}(\hat{\mathfrak{h}}')\}.$$

If $V_\mu \neq 0$ we shall say that μ is a weight of V and let $\mathrm{wt}(V) = \{\mu \in \hat{P} : V_\mu \neq 0\}$.

We record the following partial affine analogue of Theorem 2.3.1.

Theorem 2.3.17. Let V be a graded $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module.

(a) If V is integrable, then V is a weight-module and $\mathrm{wt}(V) \subseteq \hat{P}$. Moreover, $\dim V_\mu = \dim V_{\sigma\mu}$ for all $\sigma \in \widehat{\mathcal{W}}, \mu \in \hat{P}$.

(b) If V is a highest-weight module of highest weight λ, then $\dim(V_\lambda) = 1$ and $V_\mu \neq 0$ only if $\mu \leq \lambda$. Moreover, V has a unique maximal proper submodule and, hence, also a unique irreducible quotient. In particular, V is indecomposable.

(c) Let $\Lambda \in \hat{P}^+$ and $m = \Lambda(d)$. Then, the $U_{\mathbb{F}}(\hat{\mathfrak{g}}')$-module $\hat{W}_{\mathbb{F}}(\Lambda)$ generated by a vector v of degree m satisfying the defining relations

$$U_{\mathbb{F}}(\hat{\mathfrak{n}}^+)^0 v = 0, \quad hv = \Lambda(h)v \quad \text{and} \quad (x_i^-)^{(k)} v = 0, \quad \text{for all} \quad h \in U_{\mathbb{F}}(\hat{\mathfrak{h}}'), i \in \hat{I}, k > \Lambda(h_i),$$

is nonzero and integrable. Moreover, for every positive real root α, we have

$$(x_\alpha^-)^{(k)} v = 0 \qquad \text{for all} \qquad k > \Lambda(h_\alpha). \tag{2.3.6}$$

Furthermore, every integrable highest-weight module of highest weight Λ is a quotient of $\hat{W}_{\mathbb{F}}(\Lambda)$. □

Traditionally (cf. [**FL07**,**Mat89**,**Nao12**]), given $\Lambda \in \hat{P}^+, \sigma \in \widehat{\mathcal{W}}$, the Demazure module $V_{\mathbb{F}}^\sigma(\Lambda)$ is defined as the $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-submodule generated by $\hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda}$. In particular, $V_{\mathbb{F}}^\sigma(\Lambda) \cong V_{\mathbb{F}}^{\sigma'}(\Lambda)$ if $\sigma\Lambda = \sigma'\Lambda$ for some $\sigma' \in \widehat{\mathcal{W}}$. Our focus is on the Demazure modules which are stable under the action of $U_{\mathbb{F}}(\mathfrak{g})$. Since $V_{\mathbb{F}}^\sigma(\Lambda)$ is defined as a $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-module, it is stable under the action of $U_{\mathbb{F}}(\mathfrak{g})$ if, and only if,

$$U_{\mathbb{F}}(\mathfrak{n}^-)^0 \hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda} = 0. \tag{2.3.7}$$

In particular, since $V_{\mathbb{F}}^\sigma(\Lambda)$ is an integrable $U_{\mathbb{F}}(\mathfrak{sl}_\alpha)$-module for any $\alpha \in R^+$, we have $(\sigma\Lambda)(h_\alpha) \leq 0$ for all $\alpha \in R^+$. Conversely, using the exchange condition (see [**Hum90**, Section 5.8]), it follows that, for all $i \in \hat{I}$, we have $(x_i^\varepsilon)^{(k)} \hat{W}_{\mathbb{F}}(\Lambda)_{\sigma\Lambda} = 0$ for all $k > 0$, where $\varepsilon = +$ if $\sigma\Lambda(h_i) \geq 0$ and $\varepsilon = -$ if $\sigma\Lambda(h_i) \leq 0$. This implies that, if $\sigma\Lambda(h_i) \leq 0$ for all $i \in I$, then $V_{\mathbb{F}}^\sigma(\Lambda)$ is $U_{\mathbb{F}}(\mathfrak{g})$-stable. Thus, henceforth, assume $(\sigma\Lambda)(h_i) \leq 0$ for all $i \in I$ and observe that this implies that $\sigma\Lambda$ must have the form

$$\sigma\Lambda = \ell\Lambda_0 + w_0\lambda + m\delta \qquad \text{for some} \qquad \lambda \in P^+, m \in \mathbb{Z}, \text{ and } \ell = \Lambda(c). \tag{2.3.8}$$

Conversely, given $\ell \in \mathbb{Z}_{\geq 0}, \lambda \in P^+$, and $m \in \mathbb{Z}$, since $\widehat{\mathcal{W}}$ acts simply transitively on the set of alcoves of $\hat{\mathfrak{h}}^*$ (cf. [**Hum90**, Theorem 4.5.(c)]), there exists a unique $\Lambda \in \hat{P}^+$ such that $\ell\Lambda_0 + w_0\lambda + m\delta \in \widehat{\mathcal{W}}\Lambda$. Thus, if $\sigma \in \widehat{\mathcal{W}}$ and $\Lambda \in \hat{P}^+$ are such that

$$\sigma\Lambda = \ell\Lambda_0 + w_0\lambda + m\delta, \tag{2.3.9}$$

then $V_{\mathbb{F}}^\sigma(\Lambda)$ is $U_{\mathbb{F}}(\mathfrak{g})$-stable. Henceforth, we fix $\sigma, \Lambda, w_0, \lambda$, and m as in (2.3.9). Notice that, if $\gamma = \pm\alpha + s\delta \in \hat{R}^+$ with $\alpha \in R^+$, then $\sigma\Lambda(h_\gamma) = \pm w_0\lambda(h_\alpha) + s\ell r_\alpha^\vee$.

The following lemma is a rewriting of [**Mat89**, Lemme 26] using the above fixed notation.

Lemma 2.3.18. The $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-module $V_{\mathbb{F}}^\sigma(\Lambda)$ is isomorphic to the $U_{\mathbb{F}}(\hat{\mathfrak{b}}'^+)$-module generated by a vector v of degree m satisfying the following defining relations

$$hv = \sigma\Lambda(h)v, \quad h \in U_{\mathbb{F}}(\hat{\mathfrak{h}}'), \quad U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = U_{\mathbb{F}}(\mathfrak{n}^-[t]_+)^0 v = 0 \quad \text{and}$$
$$(x_{\alpha,s}^+)^{(k)} v = 0 \quad \text{for all} \quad \alpha \in R^+,\ s \geq 0,\ k > \max\{0, -w_0\lambda(h_\alpha) - s\ell r_\alpha^\vee\}. \tag{2.3.10}$$

□

Remark 2.3.19. In [**Mat89**], Mathieu attributes Lemma 2.3.18 to Joseph and Polo. This is the reason for the title of this subsection.

Recall the functor τ_m defined in the paragraph preceding Theorem 2.3.13 and set

$$D_{\mathbb{F}}(\ell, \lambda, m) = \tau_m(D_{\mathbb{F}}(\ell, \lambda)).$$

Proposition 2.3.20. The graded $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules $V_{\mathbb{F}}^\sigma(\Lambda)$ and $D_{\mathbb{F}}(\ell, \lambda, m)$ are isomorphic.

PROOF. It suffices to prove the statement for $m = 0$ and, thus, for simplicity, we assume that this is the case. Recall the automorphism of $U_{\mathbb{F}}(\mathfrak{g}[t])$ defined in Subsection 2.2.2. We begin showing that, if W is the pull-back of $V_{\mathbb{F}}^{\sigma}(\Lambda)$ by ψ, then W is a quotient of $D_{\mathbb{F}}(\ell,\lambda^*)$, where $\lambda^* = -w_0\lambda$. Indeed, let v be as in Lemma 2.3.18 and denote by w the vector v when regarded as an element of W. Since (2.3.7) and Lemma 2.3.18 imply that $U_{\mathbb{F}}(\mathfrak{n}^-[t])^0 v = 0$ and $\psi(U_{\mathbb{F}}(\mathfrak{n}^-[t])) = U_{\mathbb{F}}(\mathfrak{n}^+[t])$, it follows that $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 w = 0$. Also, ψ restricts to an automorphism of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)$ and, hence, $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 w = 0$. Since $hv = w_0\lambda(h)v$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$, (2.2.5) implies that $hw = \lambda^*(h)w$ for all $h \in U_{\mathbb{F}}(\mathfrak{h})$. Finally, Lemma 2.3.18 and (2.2.3) imply

$$(x_{\alpha,s}^-)^{(k)} w = 0 \quad \text{for all} \quad \alpha \in R^+,\ s \geq 0,\ k > \max\{0, \lambda^*(h_\alpha) - s\ell r_\alpha^\vee\}.$$

Comparing with the defining relations of $D_{\mathbb{F}}(\ell,\lambda^*)$, this completes the proof that W is a quotient of $D_{\mathbb{F}}(\ell,\lambda^*)$. By pulling back by ψ again and using Lemma 2.3.16, we get that $V_{\mathbb{F}}^{\sigma}(\Lambda)$ is a quotient of $D_{\mathbb{F}}(\ell,\lambda)$. It now suffices to show that $\dim(D_{\mathbb{F}}(\ell,\lambda)) \leq \dim(V_{\mathbb{F}}^{\sigma}(\Lambda))$.

Conversely, let this time $v \in D_{\mathbb{F}}(\ell,\lambda^*)_{\lambda^*} \setminus \{0\}$, let W be the pull-back of $D_{\mathbb{F}}(\ell,\lambda^*)$ by ψ, and w denote v when regarded as element of W. Proceeding as above, we get that w satisfies all the defining relations of $V_{\mathbb{F}}^{\sigma}(\Lambda)$ given in Lemma 2.3.18. Hence, W is a quotient of $V_{\mathbb{F}}^{\sigma}(\Lambda)$ and, therefore, $\dim(W) \leq \dim(V_{\mathbb{F}}^{\sigma}(\Lambda))$. Since Lemma 2.3.16 implies that $\dim(W) = \dim(D_{\mathbb{F}}(\ell,\lambda))$, we are done. $\square$

The next corollary is now immediate.

Corollary 2.3.21. $D_{\mathbb{F}}(\ell,\lambda)$ is isomorphic to the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}^-(\ell,\lambda)$ generated by $h - w_0\lambda(h), h \in U_{\mathbb{F}}(\mathfrak{h})$, $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0, U_{\mathbb{F}}(\mathfrak{n}^-[t])^0$, and

$$(x_{\alpha,s}^+)^{(k)} v = 0 \quad \text{for all} \quad \alpha \in R^+,\ s \geq 0,\ k > \max\{0, -w_0\lambda(h_\alpha) - s\ell r_\alpha^\vee\}.$$

$\square$

Remark 2.3.22. Observe that this proposition says that the difference between our first definition of $D_{\mathbb{F}}(\ell,\lambda)$ and the one given by Lemma 2.3.18, lies on exchanging a "highest-weight generator" by a "lowest-weight" one. More precisely, let v be as in Lemma 2.3.18. Then, the isomorphism of Proposition 2.3.20 must send v to a nonzero element in $D_{\mathbb{F}}(\ell,\lambda)_{w_0\lambda}$. In particular, if $w \in D_{\mathbb{F}}(\ell,\lambda)_{w_0\lambda}$, it satisfies the relations listed in Lemma 2.3.18. Our proof of Proposition 2.3.20 differs from the one given in [**FL07**, Corollary 1] in characteristic zero. There, the authors show that $V_{\mathbb{F}}^{\sigma}(\Lambda)$ is a quotient of $D_{\mathbb{F}}(\ell,\lambda)$ by using that v is an extremal weight vector in $\hat{W}_{\mathbb{F}}(\Lambda)$. For the converse, they simply claim that the a vector in $D_{\mathbb{F}}(\ell,\lambda)_{w_0\lambda}$ must satisfy several relations, including (2.3.10). As we have just observed, this is true, but we do not see how to deduce it before Proposition 2.3.20 is established.

Corollary 2.3.23. Let $\mathfrak{g} = \mathfrak{sl}_2$ and the subalgebra $\mathfrak{a} = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]_+ \subseteq \mathfrak{g}[t]$. For $\ell, \lambda \in \mathbb{Z}_{\geq 0}$, let $I'_{\mathbb{F}}(\ell,\lambda)$ be the left ideal of $U_{\mathbb{F}}(\mathfrak{a})$ generated by the generators of $I_{\mathbb{F}}(\ell,\lambda)$ which lie in $U_{\mathbb{F}}(\mathfrak{a})$. Then, given $k, l, s \in \mathbb{Z}_{\geq 0}$ with $k > \max\{0, \lambda - s\ell\}$, we have

$$(x_i^+)^{(l)} (x_{i,s}^-)^{(k)} \ \in\ U_{\mathbb{F}}(\mathfrak{a}) U_{\mathbb{F}}(\mathfrak{n}^+)^0 \oplus I'_{\mathbb{F}}(\ell,\lambda) \tag{2.3.11}$$

where i is the unique element of I.

PROOF. The statement is a hyperalgebraic version of that of [**Nao12**, Lemma 4.10] and the proof is essentially the same. Namely, by using the automorphism of $\mathfrak{g}[t]$ determined by $x_{i,r}^{\pm} \mapsto x_{i,r}^{\mp}, i \in I, r \in \mathbb{Z}_{\geq 0}$, one observes that proving (2.3.11) is equivalent to proving

$$(x_i^-)^{(l)} (x_{i,s}^+)^{(k)} \ \in\ U_{\mathbb{F}}(\mathfrak{a}^-) U_{\mathbb{F}}(\mathfrak{n}^-)^0 + I''_{\mathbb{F}}(\ell,\lambda) \quad \text{for all} \quad k, l, s \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda - s\ell\}, \tag{2.3.12}$$

where $\mathfrak{a}^- = \mathfrak{n}^-[t]_+ \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]$ and $I''_{\mathbb{F}}(\ell,\lambda)$ is the left ideal of $U_{\mathbb{F}}(\mathfrak{a}^-)$ generated by the generators of $I^-_{\mathbb{F}}(\ell,\lambda)$ given in Corollary 2.3.21 which lie in $U_{\mathbb{F}}(\mathfrak{a}^-)$. Since $\mathfrak{g}[t] = \mathfrak{a}^- \oplus \mathfrak{n}^-$, the PBW Theorem implies that

$$U_{\mathbb{F}}(\mathfrak{g}[t]) = U_{\mathbb{F}}(\mathfrak{a}^-)U_{\mathbb{F}}(\mathfrak{n}^-)^0 \oplus U_{\mathbb{F}}(\mathfrak{a}^-)$$

and, hence, $(x_i^-)^{(l)}(x_{i,s}^+)^{(k)} = u + u'$ with $u \in U_{\mathbb{F}}(\mathfrak{a}^-)U_{\mathbb{F}}(\mathfrak{n}^-)^0$ and $u' \in U_{\mathbb{F}}(\mathfrak{a}^-)$. Consider the Demazure module $D_{\mathbb{F}}(\ell,\lambda)$ and let $w \in D_{\mathbb{F}}(\ell,\lambda)_{-\lambda} \setminus \{0\}$. It follows from the proof of Proposition 2.3.20 that, if $k > \max\{0, \lambda - s\ell\}$, then

$$u'w = \left((x_i^-)^{(l)}(x_{i,s}^+)^{(k)} - u\right) w = 0.$$

Since $\hat{\mathfrak{b}}'^+ = \mathfrak{a}^- \oplus \mathbb{C}c$ and $\mathfrak{a}^-$ is an ideal of $\hat{\mathfrak{b}}'^+$, it follows from Lemma 2.3.18 that $I''_{\mathbb{F}}(\ell,\lambda)$ is the annihilating ideal of w inside $U_{\mathbb{F}}(\mathfrak{a})$ and, hence, $u' \in I''_{\mathbb{F}}(\ell,\lambda)$. $\square$

2.4. Joseph's Demazure flags

2.4.1. Quantum groups. Let $\mathbb{C}(q)$ be the field of rational functions on an indeterminate q. Let also $C = (c_{ij})_{i,j\in\hat{I}}$ be the Cartan matrix of $\hat{\mathfrak{g}}$ and $d_i, i \in \hat{I}$, non negative relatively prime integers such that the matrix DC, with $D = \operatorname{diag}(d_i)_{i\in I}$, is symmetric. Set $q_i = q^{d_i}$ and, for $m, n \in \mathbb{Z}, n \geq 0$, set $[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, [n]_{q_i}! = [n]_{q_i}[n-1]_{q_i}\dots[1]_{q_i}, \left[\begin{array}{c} m \\ n \end{array}\right]_{q_i} = \frac{[m]_{q_i}[m-1]_{q_i}\dots[m-n+1]_{q_i}}{[n]_{q_i}!}$. The quantum group $U_q(\hat{\mathfrak{g}}')$ is a $\mathbb{C}(q)$-associative algebra (with 1) with generators $x_i^\pm, k_i^{\pm 1}$, $i \in \hat{I}$ subject to the following defining relations for all $i, j \in \hat{I}$:

$$k_i k_i^{-1} = 1 \quad k_i k_j = k_j k_i \quad k_i x_j^\pm k_i^{-1} = q_i^{\pm c_{ij}} x_j^\pm \quad [x_i^+, x_j^-] = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{m=0}^{1-c_{ij}} (-1)^m \left[\begin{array}{c} 1-c_{ij} \\ m \end{array}\right]_{q_i} (x_i^\pm)^{1-c_{ij}-m} x_j^\pm (x_i^\pm)^m = 0, \quad i \neq j.$$

Let $U_q(\hat{\mathfrak{n}}^\pm)$ be the subalgebra generated by $x_i^\pm, i \in \hat{I}$ and $U_q(\hat{\mathfrak{b}}^\pm)$ be the subalgebra generated by $U_q(\hat{\mathfrak{n}}^\pm)$ together with $k_i^{\pm 1}, i \in \hat{I}$.

We shall need an integral form of $U(\hat{\mathfrak{g}}')$. Let $\mathbb{Z}_q = \mathbb{Z}[q, q^{-1}]$, denote by $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^\pm)$ the $\mathbb{Z}_q$-subalgebra of $U_q(\hat{\mathfrak{n}}^\pm)$ generated by $\frac{(x_i^\pm)^m}{[m]_{q_i}!}, i \in \hat{I}, m \geq 0$, and by $U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$ the $\mathbb{Z}_q$-subalgebra of $U_q(\hat{\mathfrak{g}}')$ generated by $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^\pm)$ and $k_i, i \in \hat{I}$. Let also $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^\pm) = U_q(\hat{\mathfrak{b}}^\pm) \cap U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$. Then, $U_{\mathbb{Z}_q}(\mathfrak{a})$, $\mathfrak{a} = \hat{\mathfrak{g}}', \hat{\mathfrak{n}}^\pm, \hat{\mathfrak{b}}^\pm$, is a free $\mathbb{Z}_q$-module such that the natural map $\mathbb{C}(q) \otimes_{\mathbb{Z}_q} U_{\mathbb{Z}_q}(\mathfrak{a}) \to U_q(\mathfrak{a})$ is $\mathbb{C}(q)$-algebra isomorphism, i.e., $U_{\mathbb{Z}_q}(\mathfrak{a})$ is a $\mathbb{Z}_q$-form of $U_q(\mathfrak{a})$. Moreover, letting $\mathbb{Z}$ be a $\mathbb{Z}_q$-module where q acts as 1, there exists an epimorphism of $\mathbb{Z}$-algebras $\mathbb{Z} \otimes_{\mathbb{Z}_q} U_{\mathbb{Z}_q}(\mathfrak{a}) \to U_{\mathbb{Z}}(\mathfrak{a})$, which is an isomorphism if $\mathfrak{a} = \hat{\mathfrak{n}}^\pm$ and whose kernel is the ideal generated by $k_i - 1, i \in \hat{I}$, for $\mathfrak{a} = \hat{\mathfrak{g}}', \hat{\mathfrak{b}}^\pm$.

Given $\Lambda \in \hat{P}^+$, let $V_q(\Lambda)$ be the simple (type 1) $U_q(\hat{\mathfrak{g}}')$-module of highest weight Λ. Given a highest-weight vector $v \in V_q(\Lambda)$, set $V_{\mathbb{Z}_q}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^-)v$, which is a $\mathbb{Z}_q$-form of $V_q(\Lambda)$. Given $\sigma \in \widehat{\mathcal{W}}$ and a nonzero vector $v \in V_q(\Lambda)$ of weight $\sigma\Lambda$, set $V^\sigma_{\mathbb{Z}_q}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)v$, which is a free $\mathbb{Z}_q$-module as well as a $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-module and $\mathbb{C} \otimes_{\mathbb{Z}_q} V^\sigma_{\mathbb{Z}_q}(\Lambda) \cong V^\sigma_{\mathbb{C}}(\Lambda)$. In particular,

$$V^\sigma_{\mathbb{Z}}(\Lambda) := \mathbb{Z} \otimes_{\mathbb{Z}_q} V^\sigma_{\mathbb{Z}_q}(\Lambda) \tag{2.4.1}$$

is an integral form of $V^\sigma_{\mathbb{C}}(\Lambda)$.

2.4.2. Crystals. A normal crystal associated to the root data of $\hat{\mathfrak{g}}$ defined as a set B equipped with maps $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \varepsilon_i, \varphi_i : B \to \mathbb{Z}$, for each $i \in \hat{I}$, and $\mathrm{wt} : B \to \hat{P}$ satisfying

(1) $\varepsilon_i(b) = \max\{n : \tilde{e}_i b \neq 0\}, \varphi_i(b) = \max\{n : \tilde{f}_i b \neq 0\}$, for all $i \in \hat{I}, b \in B$;
(2) $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)(h_i)$, for all $i \in \hat{I}, b \in B$;
(3) for $b, b' \in B$, $b' = \tilde{e}_i b$ if and only if $\tilde{f}_i b' = b$;
(4) if $b \in B, i \in \hat{I}$ are such that $\tilde{e}_i b \neq 0$, then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$.

For convenience, we extend $\tilde{e}_i, \tilde{f}_i, , \varepsilon_i, \varphi_i, \mathrm{wt}$ to $B \sqcup \{0\}$ by setting them to map 0 to 0. Denote by $\mathcal{E}$ the submonoid of the monoid of maps $B \sqcup \{0\} \to B \sqcup \{0\}$ generated by $\{\tilde{e}_i : i \in \hat{I}\}$, and similarly define $\mathcal{F}$. A normal crystal is said to be of highest weight $\Lambda \in \hat{P}^+$ if there exists $b_\Lambda \in B$ satisfying

$$\mathrm{wt}(b_\Lambda) = \Lambda, \qquad \mathcal{E} b_\Lambda = \{0\}, \qquad \text{and} \qquad \mathcal{F} b_\Lambda = B.$$

Given $B' \subset B$ and $\mu \in \hat{P}$, define $B'_\mu = \{b \in B' : \mathrm{wt}(b) = \mu\}$ and define the character of B' as $\mathrm{ch}(B') = \sum_{\mu \in \hat{P}} \# B'_\mu e^\mu \in \mathbb{Z}[\hat{P}]$.

Given crystals B_1, B_2, a morphism from B_1 to B_2 is a map $\psi : B_1 \to B_2 \sqcup \{0\}$ satisfying:

(1) if $\psi(b) \neq 0$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \varepsilon_i(\psi(b)) = \varepsilon_i(b), \varphi_i(\psi(b)) = \varphi_i(b)$, for all $i \in \hat{I}$;
(2) if $\tilde{e}_i b \neq 0$, then $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$;
(3) if $\tilde{f}_i b \neq 0$, then $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

The set $B_1 \times B_2$ admits a structure of crystal denoted by $B_1 \otimes B_2$ (cf. [**Jos03**, Section 2.4]). There is, up to isomorphism, exactly one family $\{B(\Lambda) : \Lambda \in \hat{P}^+\}$ of normal highest weight crystals such that, for all $\lambda, \mu \in \hat{P}^+$, the crystal structure of $B(\lambda) \otimes B(\mu)$ induces a crystal structure on its subset $\mathcal{F}(b_\lambda \otimes b_\mu)$, the inclusion is a homomorphism of crystals, and $\mathcal{F}(b_\lambda \otimes b_\mu) \cong B(\lambda + \mu)$.

Given a crystal B and $\sigma \in \widehat{\mathcal{W}}$ with a fixed reduced expression $\sigma = s_{i_1} \dots s_{i_n}$, define

$$\mathcal{E}^\sigma = \{\tilde{e}_{i_1}^{m_1} \dots \tilde{e}_{i_n}^{m_n} : m_j \in \mathbb{N}\} \subset \mathcal{E} \quad \text{and} \quad \mathcal{F}^\sigma = \{\tilde{f}_{i_1}^{m_1} \dots \tilde{f}_{i_n}^{m_n} : m_j \in \mathbb{N}\} \subset \mathcal{F}.$$

If $B = B(\Lambda)$, $\Lambda \in \hat{P}^+$ and $\sigma \in \widehat{\mathcal{W}}$, define the Demazure subset $B^\sigma(\Lambda) = \mathcal{F}^\sigma b_\Lambda \subseteq B(\Lambda)$. Then $B^\sigma(\Lambda)$ is $\mathcal{E}$-stable, i.e., $\mathcal{E} B^\sigma(\Lambda) \subset B^\sigma(\Lambda) \sqcup \{0\}$. It was proved in [**Jos03**, Section 4.6] that $\mathrm{ch}(V_{\mathbb{C}}^\sigma(\Lambda)) = \mathrm{ch}(B^\sigma(\Lambda))$. This fact and the following theorem are the main results of [**Jos03**] that we shall need.

Theorem 2.4.1. Let $\Lambda, \mu \in \hat{P}^+$. For any $\sigma \in \widehat{\mathcal{W}}$, there exist a finite set J and elements $\sigma_j \in \widehat{\mathcal{W}}, b_j \in B^\sigma(\Lambda)$ for each $j \in J$, satisfying:

(1) $b_\mu \otimes B^\sigma(\Lambda) = \sqcup_{j \in J} B_j$ where $B_j := \mathcal{F}^{\sigma_j}(b_\mu \otimes b_j)$;
(2) $\mathcal{E}(b_\mu \otimes b_j) = \{0\}$;
(3) $\mathrm{ch}(B_j) = \mathrm{ch}(B^{\sigma_j}(\nu_j))$, where $\nu_j = \mu + \mathrm{wt}(b_j) \in \hat{P}^+$.

Remark 2.4.2. The proof of Theorem 2.4.1 establishes an algorithm to find the set J and the elements σ_j, b_j.

2.4.3. Globalizing. The theory of global basis of Kashiwara shows, in particular, that, for each $\Lambda \in \hat{P}^+$, there is a map $G : B(\Lambda) \to V_q(\Lambda)$ such that

$$V_{\mathbb{Z}_q}(\Lambda) = \bigoplus_{b \in B(\Lambda)} \mathbb{Z}_q G(b), \tag{2.4.2}$$

the weight of $G(b)$ is $\mathrm{wt}(b)$ and $G(b_\Lambda)$ is a highest-weight vector of $V_q(\Lambda)$.

Fix $\Lambda, \mu \in \hat{P}^+, \sigma \in \widehat{\mathcal{W}}$ and let $J, b_j, \sigma_j, \nu_j, j \in J$, be as in Theorem 2.4.1. Let $b \in B(\Lambda)_{\sigma\Lambda}$ and set $V_{\mathbb{Z}_q}^\sigma(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+) G(b)$. Similarly, let b'_j be the unique element of B_j such that $\mathrm{wt}(b'_j) = \sigma_j \nu_j$.

Choose a linear order on J such that $\mathrm{wt}(b_j) < \mathrm{wt}(b_k)$ only if $j > k$. For $j \in J$, let Y_j be the $\mathbb{Z}_q$-submodule of $V_q(\mu) \otimes V_q^{\sigma}(\Lambda)$ spanned by $G(b_\mu) \otimes G(b)$ with $b \in B_k, k \leq j$, and set

$$y_j = G(b_\mu) \otimes G(b'_j). \tag{2.4.3}$$

Let also $Z_j = \sum_{k \leq j} U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^-)\,(G(b_\mu) \otimes G(b_k))$. Since J is linearly ordered and finite, say $\#J = n$, identify it with $\{1, \ldots, n\}$. For convenience, set $Y_0 = \{0\}$. Observe that $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$ is a filtration of the $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-module $G(b_{\Lambda_0}) \otimes V_{\mathbb{Z}_q}^{\sigma}(\Lambda)$. The following result was proved in [**Jos06**, Corollary 5.10].

Theorem 2.4.3. Suppose $\mathfrak{g}$ is simply laced and $\mu(h_i) \leq 1$ for all $i \in \hat{I}$. Then:

(a) The $\mathbb{Z}_q$-module Y_j is $U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)$-stable for all $j \in J$.
(b) For all $j \in J$, Y_j/Y_{j-1} is isomorphic to $V_{\mathbb{Z}_q}^{\sigma_j}(\nu_j)$. In particular, Y_j/Y_{j-1} is a free $\mathbb{Z}_q$-module.
(c) For all $j \in J$, the image of $\{G(b_\mu) \otimes G(b) : b \in B_j\}$ in Y_j/Y_{j-1} is a $\mathbb{Z}_q$-basis of Y_j/Y_{j-1}.
(d) For each $j \in J$, Z_j is $U_{\mathbb{Z}_q}(\hat{\mathfrak{g}}')$-stable and $Y_j = Z_j \cap \Big(G(b_\mu) \otimes V_{\mathbb{Z}_q}^{\sigma}(\Lambda)\Big)$.

Remark 2.4.4. The above theorem was proved in [**Jos06**] for any simply-laced symmetric Kac-Moody Lie algebra. However, as pointed out in [**Nao12**, Remark 4.15], the proof also holds for $\hat{\mathfrak{sl}}_2$.

It follows from Theorem 2.4.3 and the fact that $G(b_\mu)$ is a highest-weight vector of $V_q(\Lambda)$ (2.4.2) that

$$Y_j = \sum_{k \leq j} U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+) y_j. \tag{2.4.4}$$

2.4.4. Simply laced Demazure flags. Given $\ell \geq 0, \lambda \in P^+, m \in \mathbb{Z}$, let $D_{\mathbb{F}}(\ell, \lambda, m) = \tau_m(D_{\mathbb{F}}(\ell, \lambda))$ and $D_{\mathbb{Z}}(\ell, \lambda, m) = \tau_m(D_{\mathbb{Z}}(\ell, \lambda))$.

Theorem 2.4.5. Suppose $\mathfrak{g}$ is simply laced, let $\mu \in P^+$ and $\ell' > \ell \geq 0$. Then, there exist $k > 0, \mu_1, \ldots, \mu_k \in P^+, m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}$, and a filtration of $U_{\mathbb{Z}}(\mathfrak{g}[t])$-modules $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = D_{\mathbb{Z}}(\ell, \mu)$ such that D_j and D_j/D_{j-1} are free $\mathbb{Z}$-modules for all $j = 1, \ldots, k$, and $D_j/D_{j-1} \cong D_{\mathbb{Z}}(\ell', \mu_j, m_j)$. Moreover, for all $j \in J$, there exists $\vartheta_j \in D_j$ whose image in D_j/D_{j-1} satisfies the defining relations of $D_{\mathbb{Z}}(\ell', \mu_j, m_j)$ and $D_j = \sum_{k \leq j} U_{\mathbb{Z}}(\mathfrak{n}^-[t])\vartheta_k$.

Proof. The proof follows closely that of [**Nao12**, Corollary 4.16]. First notice that it is enough to prove the theorem for $\ell' = \ell + 1$. Then let $\Lambda \in \widehat{P}^+$ and $w \in \widehat{\mathcal{W}}$ be such that $w\Lambda = \ell\Lambda_0 + w_0\mu$, and let $V_{\mathbb{Z}_q}^{w}(\Lambda) = U_{\mathbb{Z}_q}(\hat{\mathfrak{n}}^+)G(b)$ where $b \in B(\Lambda)_{w\Lambda}$.

From Subsection 2.4.3, we know that the $U_{\mathbb{Z}_q}(\hat{\mathfrak{b}}^+)$-submodule $G(b_{\Lambda_0}) \otimes V_{\mathbb{Z}_q}^{w}(\Lambda) \subseteq V_q(\Lambda_0) \otimes V_q(\Lambda)$ admits a filtration $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_k$. For each $j = 1, \ldots, k$, let $D_j = \mathbb{Z} \otimes_{\mathbb{Z}_q} Y_j$, and observe that

$$D_k = \mathbb{Z} \otimes_{\mathbb{Z}_q} \Big(G(b_{\Lambda_0}) \otimes_{\mathbb{Z}_q} V_{\mathbb{Z}_q}^{w}(\Lambda)\Big) \cong (\mathbb{Z} \otimes_{\mathbb{Z}_q} G(b_{\Lambda_0})) \otimes_{\mathbb{Z}} \Big(\mathbb{Z} \otimes_{\mathbb{Z}_q} V_{\mathbb{Z}_q}^{w}(\Lambda)\Big) \cong \mathbb{Z}_{\Lambda_0} \otimes_{\mathbb{Z}} D_{\mathbb{Z}}(\ell, \mu),$$

where $\mathbb{Z}_{\Lambda_0}$ is a $U_{\mathbb{Z}}(\hat{\mathfrak{b}}^+)$-module on which $U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+)^0$ and $U_{\mathbb{Z}}(\mathfrak{g})^0$ act trivially and $U_{\mathbb{Z}}(\hat{\mathfrak{h}})$ acts by Λ_0. Moreover, as a $\mathbb{Z}$-module it is free of rank 1. Thus D_k is isomorphic to $D_{\mathbb{Z}}(\ell, \mu)$ as a $U_{\mathbb{Z}}(\mathfrak{g}[t])$-module. It follows from Theorem 2.4.3 (d) that D_j is a $U_{\mathbb{Z}}(\mathfrak{g}[t])$-module for all $j = 1, \ldots, k$ and, hence, so is D_j/D_{j-1}. So we have a filtration of $U_{\mathbb{Z}}(\mathfrak{g}[t])$-modules $0 = D_0 \subset D_1 \subset \cdots \subset D_k = D_{\mathbb{Z}}(\ell, \mu)$.

By Theorem 2.4.3 (b), $Y_j/Y_{j-1} \cong V_{\mathbb{Z}_q}^{\sigma_j}(\nu_j)$ for some $\sigma_j \in \widehat{\mathcal{W}}$, $\nu_j \in \widehat{P}^+$. By (2.4.1) $D_j/D_{j-1} \cong V_{\mathbb{Z}}^{\sigma_j}(\nu_j)$. Thus D_j/D_{j-1} is isomorphic to $D_{\mathbb{Z}}(\ell_j, \mu_j, m_j)$ for some $\mu_j \in P^+, m_j \in \mathbb{Z}$ and $\ell_j = \nu_j(c)$

(cf. (2.3.8)). Since all the weights of $V_q(\Lambda_0)\otimes V_q(\Lambda)$ are of the form $\Lambda+\Lambda_0-\eta$ for some $\eta\in\hat{Q}^+$, and $\alpha_i(c)=0$ for all $i\in\hat{I}$, it follows that $\ell_j=\ell+1$ for all j.

Keep denoting the image of y_j in D_j by y_j (cf. (2.4.3)). It follows from (2.4.4) that $D_j=\sum_{k\le j}U_{\mathbb{Z}}(\hat{\mathfrak{n}}^+)y_j$. As in Remark 2.3.22, we now replace the "lowest-weight" generators y_j by "highest-weight generators". Thus, let b''_j be the unique element of B_j such that $\mathrm{wt}(b''_j)=w_0\sigma_j\nu_j=(\ell+1)\Lambda_0+\mu_j+m_j\delta$ and let ϑ_j be defined similarly to y_j by replacing b'_j by b''_j. ☐

The next corollary is now immediate.

Corollary 2.4.6. Let $\mathfrak{g},\mu,\ell',\ell,k,\mu_j, j=1,\dots,k$, be as in Theorem 2.4.5. Then, there exists a filtration of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules $0=D_0\subseteq D_1\subseteq\cdots\subseteq D_k=D_{\mathbb{F}}(\ell,\mu)$, such that $D_j/D_{j-1}\cong D_{\mathbb{F}}(\ell',\mu_j)$ for all $j=1,\dots,k$. ☐

Proof. Tensor the filtration of Theorem 2.4.5 with $\mathbb{F}$ over $\mathbb{Z}$ and use the fact that the characters of Demazure modules are independent of the ground ring. ☐

2.5. Proof of the main theorem

2.5.1. Isomorphism between Demazure and graded local Weyl modules. We now prove Theorem 2.1.2(a). As observed in Remark 2.1.1, $D_{\mathbb{F}}(1,\lambda)$ is a quotient of $W^c_{\mathbb{F}}(\lambda)$. To prove the converse, suppose first that we have proved Theorem 2.1.2(a) with $\mathfrak{g}=\mathfrak{sl}_2$ and let w be the image of 1 in $W^c_{\mathbb{F}}(\lambda)$. Recall that we are assuming that $\mathfrak{g}$ is simply laced. Thus, in order to show that $W^c_{\mathbb{F}}(\lambda)$ is a quotient of $D_{\mathbb{F}}(1,\lambda)$, it remains to prove that

$$(x^-_{\alpha,s})^{(k)}w=0\quad\text{for all}\quad\alpha\in R^+,\ s>0,\ k>\max\{0,\lambda(h_\alpha)-s\}.\tag{2.5.1}$$

Given $\alpha\in R^+$, consider the $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$-submodule (cf. Subsection 2.1.2) $W_\alpha\subseteq W^c_{\mathbb{F}}(\lambda)$ generated by w which is a quotient of the graded local Weyl module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ with highest weight $\lambda(h_\alpha)\omega$, where ω is the unique fundamental weight of $\mathfrak{sl}_2$. Since we are assuming that the theorem holds for $\mathfrak{sl}_2$, it follows that w must satisfy the same relations as the generator of the corresponding Demazure module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$, i.e., we have $(x^-_{\alpha,r})^{(k)}w=0$, for all $r>0$ and $k>\max\{0,\lambda(h_\alpha)-r\}$.

For $\mathfrak{g}=\mathfrak{sl}_2$, it follows from the Demazure character formula (see also the main result of [**FL06**]) that $\dim(D_{\mathbb{F}}(1,\lambda))=2^\lambda$, where we have identified P with $\mathbb{Z}$ as usual. On the other hand, it is shown in [**JM12**] that $\dim(W^c_{\mathbb{F}}(\lambda))\le 2^\lambda$, which completes the proof.

2.5.2. A smaller set of relations for non simply laced Demazure modules. In this subsection we assume $\mathfrak{g}$ is not simply laced and prove the following analogue of [**Nao12**, Proposition 4.1].

Proposition 2.5.1. For all $\lambda\in P^+$, $D_{\mathbb{F}}(1,\lambda)$ is isomorphic to the quotient of $U_{\mathbb{F}}(\mathfrak{g}[t])$ by the left ideal $I_{\mathbb{F}}(\lambda)$ generated by

$$U_{\mathbb{F}}(\mathfrak{n}^+[t])^0,\quad U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0,\quad h-\lambda(h),\quad (x_i^-)^{(k)},\quad (x^-_{\alpha,s})^{(\ell)}\tag{2.5.2}$$

for all $h\in U_{\mathbb{F}}(\mathfrak{h}), i\in I\setminus I_{\mathrm{sh}}, \alpha\in R^+_{\mathrm{sh}}, s\ge 0, k>\lambda(h_i), \ell>\max\{0,\lambda(h_\alpha)-sr^\vee\}$.

Let $w\in D_{\mathbb{F}}(1,\lambda)_\lambda\setminus\{0\}$ and let V be the $U_{\mathbb{F}}(\mathfrak{g}[t])$-module generated by a vector v with defining relations given by (2.5.2). In particular, there exists a unique epimorphism $V\to D_{\mathbb{F}}(1,\lambda)$ mapping v to w. To prove the converse, observe first that, since $(x_i^-)^{(k)}v=0$ for all $i\in I, k>\lambda(h_i)$, Lemma 2.3.7 implies that $(x^-_\alpha)^{(k)}v=0$ for all $\alpha\in R^+, k>\lambda(h_\alpha)$. In particular, V is a quotient of $W^c_{\mathbb{F}}(\lambda)$ and, hence, it is finite-dimensional. It remains to show that

$$(x^-_{\alpha,s})^{(k)}v=0\quad\text{for all}\quad\alpha\in R^+\setminus R^+_{\mathrm{sh}},\ s>0,\ k>\max\{0,\lambda(h_\alpha)-sr^\vee_\alpha\}.$$

These relations will follow from the next few lemmas.

Lemma 2.5.2. Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, let $\lambda \in P^+$, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$. If $\alpha \in R^+$ is long, then $(x^-_{\alpha,s})^{(k)} v = 0$ for all $s \geq 0, k > \max\{0, \lambda(h_\alpha) - s\}$.

Proof. Consider the subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ (see Subsection 2.2.3). By Theorem 2.3.13 (c), the submodule $W = U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])v$ is a quotient of the local graded Weyl module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$ with highest weight $\lambda(h_\alpha)$. Theorem 2.1.2 (a) implies that $W \cong D^\alpha_{\mathbb{F}}(1, \lambda(h_\alpha))$ where the latter is the corresponding Demazure module for $U_{\mathbb{F}}(\mathfrak{sl}_\alpha[t])$. In particular, v satisfies the relations (2.1.2). $\square$

Lemma 2.5.3. Assume $\mathfrak{g}$ is not of type G_2. Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, $\lambda \in P^+$, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ and $(x^-_{\alpha,s})^{(k)} v = 0$ for all $\alpha \in R^+_{\text{sh}}, k > \max\{0, \lambda(h_\alpha) - 2s\}$. Then, for every short root γ, we have $(x^-_{\gamma,s})^{(k)} v = 0$ for all $s \geq 0, k > \max\{0, \lambda(h_\gamma) - 2s\}$.

Proof. The proof will proceed by induction on $\text{ht}(\gamma)$. If $\text{ht}(\gamma) = 1$, then γ is simple and, hence, $\gamma \in R^+_{\text{sh}}$. Thus, suppose $\text{ht}(\gamma) > 1$ and that $\gamma \notin R^+_{\text{sh}}$. By [**Nao12**, Lemma 4.6], there exist $\alpha, \beta \in R^+$ such that $\gamma = \alpha + \beta$ with α long and β short. Notice that $\{\alpha, \beta\}$ form a simple system of a rank-two root subsystem. In particular, $h_\gamma = 2h_\alpha + h_\beta$ and, hence $\lambda(h_\gamma) = 2\lambda(h_\alpha) + \lambda(h_\beta)$.

Fix $s \geq 0$ and suppose first that $\lambda(h_\gamma) - 2s \geq 0$. In this case, we can choose $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$a + b = s, \qquad \lambda(h_\alpha) - a \geq 0, \qquad \text{and} \qquad \lambda(h_\beta) - 2b \geq 0.$$

Indeed, $b = \max\{0, s - \lambda(h_\alpha)\}$ and $a = s - b$ satisfy these conditions. Then, Lemma 2.5.2 implies that $(x^-_{\alpha,a})^{(k)} v = 0$ for all $k > \lambda(h_\alpha) - a$, while the induction hypothesis implies that $(x^-_{\beta,b})^{(k)} v = 0$ for all $k > \lambda(h_\beta) - 2b$. Applying Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{a,b}_{\alpha,\beta})$ (cf. Subsection 2.2.3), it follows that $(x^-_{\gamma,s})^{(k)} v = 0$ for all $k > 2(\lambda(h_\alpha) - a) + (\lambda(h_\beta) - 2b) = \lambda(h_\gamma) - 2s$.

Now suppose $\lambda(h_\gamma) - 2s \leq 0$ and notice that this implies $s - \lambda(h_\alpha) = s - \frac{1}{2}(\lambda(h_\gamma) - \lambda(h_\beta)) \geq \frac{\lambda(h_\beta)}{2} \geq 0$. We need to show that $(x^-_{\gamma,s})^{(k)} v = 0$ for all $k > 0$. Letting $a = \lambda(h_\alpha)$ and $b = s - \lambda(h_\alpha)$, we have

$$a + b = s, \qquad \lambda(h_\alpha) - a \leq 0, \qquad \text{and} \qquad \lambda(h_\beta) - 2b \leq 0.$$

Then, Lemma 2.5.2 implies that $(x^-_{\alpha,a})^{(k)} v = 0$ for all $k > 0$, while the induction hypothesis implies that $(x^-_{\beta,b})^{(k)} v = 0$ for all $k > 0$. The result follows from an application of Lemma 2.3.7 as before. $\square$

It remains to prove an analogue of Lemma 2.5.3 for $\mathfrak{g}$ of type G_2. This is much more technically complicated and will require that we assume that characteristic $\mathbb{F}$ is at least 5. For the remainder of this subsection we assume $\mathfrak{g}$ is of type G_2 and set $I = \{1, 2\}$ so that α_1 is short. Given $\gamma = s\alpha_1 + l\alpha_2 \in R^+$, set $s_\gamma = s$. Set also

$$\mathfrak{n}^+[t]_> = \bigoplus_{\gamma \in R^+} \bigoplus_{s \geq s_\gamma} \mathbb{C}x^+_{\gamma,s}, \quad \mathfrak{n}^+[t]_< = \bigoplus_{\gamma \in R^+} \bigoplus_{s=0}^{s_\gamma - 1} \mathbb{C}x^+_{\gamma,s}, \quad \mathfrak{a} = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t]_>,$$

and observe that $\mathfrak{n}^+[t]_>$ and $\mathfrak{n}^+[t]_<$ are subalgebras of $\mathfrak{n}^+[t]$ such that $\mathfrak{n}^+[t] = \mathfrak{n}^+[t]_> \oplus \mathfrak{n}^+[t]_<$. The hyperalgebras $U_{\mathbb{F}}(\mathfrak{n}^+[t]_>), U_{\mathbb{F}}(\mathfrak{n}^+[t]_<)$, and $U_{\mathbb{F}}(\mathfrak{a})$ are then defined in the usual way (see Subsection 2.1.1) and the PBW theorem implies that

$$U_{\mathbb{F}}(\mathfrak{n}^+[t]) = U_{\mathbb{F}}(\mathfrak{n}^+[t]_>) \oplus U_{\mathbb{F}}(\mathfrak{n}^+[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t]_<)^0. \tag{2.5.3}$$

We now prove a version of [**Nao12**, Lemma 4.11] for hyperalgebras.

Lemma 2.5.4. Given $\lambda \in P^+$, let $I'_{\mathbb{F}}(\lambda)$ be the left ideal of $U_{\mathbb{F}}(\mathfrak{a})$ generated by the generators of $I_{\mathbb{F}}(\lambda)$ described in (2.5.2) which lie in $U_{\mathbb{F}}(\mathfrak{a})$. Then,

$$I_{\mathbb{F}}(\lambda) \subseteq I'_{\mathbb{F}}(\lambda) \oplus U_{\mathbb{F}}(\mathfrak{a}) U_{\mathbb{F}}(\mathfrak{n}^+[t]_<)^0.$$

PROOF. Recall that $I_{\mathbb{F}}(\lambda)$ is the left ideal of $U_{\mathbb{F}}(\mathfrak{g}[t])$ generated by the set $\mathcal{I}$ whose elements are the elements in $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0$, $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0$, together with the elements

$$\binom{h_i}{l} - \binom{\lambda(h_i)}{l}, \quad (x_2^-)^{(m)}, \quad (x_{1,s}^-)^{(k)}$$
$$\text{for} \quad i \in I,\ k, l, m, s \in \mathbb{Z}_{\geq 0},\ m > \lambda(h_2),\ k > \max\{0, \lambda(h_1) - 3s\}.$$

To simplify notation, set $U_< = U_{\mathbb{F}}(\mathfrak{n}^+[t]_<)$ and $J = I'_{\mathbb{F}}(\lambda) \oplus U_{\mathbb{F}}(\mathfrak{a}) U_{\mathbb{F}}(\mathfrak{n}^+[t]_<)^0$. Observe that $U_{\mathbb{F}}(\mathfrak{a}) J \subseteq J$. Therefore, since $U_{\mathbb{F}}(\mathfrak{g}[t]) = U_{\mathbb{F}}(\mathfrak{a}) U_<$ by (2.5.3) and we clearly have $\mathcal{I} \subseteq J$, it suffices to show that

$$U_<^0\, \mathcal{I} \subseteq J.$$

We will decompose the set $\mathcal{I}$ into parts, and prove the inclusion for each part. Namely, we first decompose $\mathcal{I}$ into $(\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^+[t]) U_{\mathbb{F}}(\mathfrak{h}[t])) \sqcup (\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^-[t]))$, and then we further decompose $\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^-[t])$ into $\{(x_2^-)^{(m)} : m > \lambda(h_2)\} \sqcup \{(x_{1,s}^-)^{(k)} : s \in \mathbb{Z}_{\geq 0}, k > \max\{0, \lambda(h_1) - 3s\}\}$.

Since $\mathfrak{h}[t] \oplus \mathfrak{n}^+[t]$ is a subalgebra of $\mathfrak{g}[t]$,$U_{\mathbb{F}}(\mathfrak{n}^+[t]) U_{\mathbb{F}}(\mathfrak{h}[t]) = U_{\mathbb{F}}(\mathfrak{h}[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t])$ by PBW Theorem, and, therefore,

$$U_<^0 \left(\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^+[t]) U_{\mathbb{F}}(\mathfrak{h}[t]) \right) \subseteq U_{\mathbb{F}}(\mathfrak{h}[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t]).$$

Now, by (2.5.3), $U_{\mathbb{F}}(\mathfrak{h}[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t]) \subseteq J$, showing that $U_<^0 \left(\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^+[t]) U_{\mathbb{F}}(\mathfrak{h}[t]) \right) \subseteq J$. In particular, we have shown that

$$U_{\mathbb{F}}(\mathfrak{g}[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 \subseteq J. \tag{2.5.4}$$

It remains to show that

$$U_<^0 \left(\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}^-[t]) \right) \subseteq J.$$

We begin by proving that $U_<^0 U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq J$, where $\mathfrak{n}_2^-$ is the subalgebra spanned by x_2^-. Consider the natural Q-grading on $U_{\mathbb{F}}(\mathfrak{g}[t])$ and, for $\eta \in Q$, let $U_{\mathbb{F}}(\mathfrak{g}[t])_\eta$ denote the corresponding graded piece. Observe that $\mathfrak{m}_2 := \mathfrak{n}^+[t]_< \oplus \mathfrak{n}_2^-$ is a subalgebra of $\mathfrak{g}[t]$ and that

$$U_<^0 U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq \bigoplus_\eta U_{\mathbb{F}}(\mathfrak{m}_2)_\eta,$$

where the sum runs over $\mathbb{Z}_{>0}\alpha_1 \oplus \mathbb{Z}\alpha_2$. Together with the PBW Theorem, this implies that

$$U_<^0 U_{\mathbb{F}}(\mathfrak{n}_2^-) \subseteq U_{\mathbb{F}}(\mathfrak{n}_2^-) U_<^0 \subseteq U_{\mathbb{F}}(\mathfrak{a}) U_<^0 \subseteq J.$$

Finally, we show that $U_<^0\, \mathcal{I}_1 \subseteq J$, where $\mathcal{I}_1 = \left(\mathcal{I} \cap U_{\mathbb{F}}(\mathfrak{n}_1^-[t]) \right)$ and $\mathfrak{n}_1^-$ is the subalgebra spanned by x_1^-. Consider

$$\mathfrak{n}^+[t]_<^1 = \bigoplus_{\gamma \in R^+ \setminus \{\alpha_1\}} \bigoplus_{s=0}^{s_\gamma - 1} \mathbb{C} x_{\gamma,s}^+,$$

which is a subalgebra of $\mathfrak{n}^+[t]_<$ such that $\mathfrak{n}^+[t]_< = \mathfrak{n}_1^+ \oplus \mathfrak{n}^+[t]_<^1$, where $\mathfrak{n}_1^+ = \mathbb{C} x_1^+$. Moreover, $\mathfrak{m}_1 := \mathfrak{n}^+[t]_<^1 \oplus \mathfrak{n}_1^-[t]$ is a subalgebra of $\mathfrak{g}[t]$ such that $U(\mathfrak{m}_1)_\eta \neq 0$ only if $\eta \in \mathbb{Z}\alpha_1 \oplus \mathbb{Z}_{\geq 0}\alpha_2$ and $U(\mathfrak{m}_1)_0 = \mathbb{C}$. This implies that

$$U_{\mathbb{F}}(\mathfrak{n}^+[t]_<^1)^0 U_{\mathbb{F}}(\mathfrak{n}_1^-[t]) = U_{\mathbb{F}}(\mathfrak{n}_1^-[t]) U_{\mathbb{F}}(\mathfrak{n}^+[t]_<^1)^0.$$

Since $U^0_< = U_{\mathbb{F}}(\mathfrak{n}^+_1)U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 \oplus U_{\mathbb{F}}(\mathfrak{n}^+_1)^0$, we get

$$\begin{aligned} U^0_< \,\mathcal{I}_1 &\subseteq \left(U_{\mathbb{F}}(\mathfrak{n}^+_1)U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 + U_{\mathbb{F}}(\mathfrak{n}^+_1)^0\right)\mathcal{I}_1 \\ &\subseteq U_{\mathbb{F}}(\mathfrak{n}^+_1)U_{\mathbb{F}}(\mathfrak{n}^-_1[t])U_{\mathbb{F}}(\mathfrak{n}^+[t]^1_<)^0 + U_{\mathbb{F}}(\mathfrak{n}^+_1)^0\mathcal{I}_1 \\ &\subseteq U_{\mathbb{F}}(\mathfrak{g}[t])U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 + U_{\mathbb{F}}(\mathfrak{n}^+_1)^0\mathcal{I}_1. \end{aligned}$$

The first summand in the last line is in J by (2.5.4) while the second one is in J by Corollary 2.3.23 (with $\lambda = \lambda(h_1)$ and $\ell = 3$) together with (2.5.4). $\square$

Set $\mathfrak{h}_i = \mathbb{C}h_i, i \in I$, and $\mathfrak{b} = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t]_+ \oplus \mathfrak{h}_2 \oplus \mathfrak{n}^+[t]_>$. Observe that $\mathfrak{b}$ is an ideal of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{h}_1$. One easily checks that there exists a unique Lie algebra homomorphism $\phi : \mathfrak{b} \to \mathfrak{g}[t]$ such that

$$\phi(x^\pm_{\gamma,r}) = x^\pm_{\gamma,r\mp s_\gamma} \quad \text{for all} \quad \gamma \in R^+.$$

Moreover, ϕ is the identity on $\mathfrak{h}[t]_+ + \mathfrak{sl}_{\alpha_2}$. Also, ϕ can be extended to a Lie algebra map $\mathfrak{a} \to U(\mathfrak{g}[t])$ by setting $\phi(h_1) = h_1 - 3$ (cf. [**Nao12**, Section 4.2]). Proceeding as in Section 2.2.2, one sees that ϕ induces an algebra homomorphism $U_{\mathbb{F}}(\mathfrak{a}) \to U_{\mathbb{F}}(\mathfrak{g}[t])$ also denoted by ϕ.

We are ready to prove the analogue of Lemma 2.5.3 for type G_2.

Lemma 2.5.5. Let V be a finite-dimensional $U_{\mathbb{F}}(\mathfrak{g}[t])$-module, $\lambda \in P^+$, and suppose $v \in V_\lambda$ satisfies $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 v = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 v = 0$ and $(x^-_{1,s})^{(k)} v = 0$ for all $k > \max\{0, \lambda(h_1) - 3s\}$. Then, for every short root γ, we have $(x^-_{\gamma,s})^{(k)} v = 0$ for all $s \geq 0, k > \max\{0, \lambda(h_\gamma) - 3s\}$.

Proof. Notice that the conclusion of the lemma is equivalent to

$$(x^-_{\gamma,s})^{(k)} \in I_{\mathbb{F}}(\lambda) \quad \text{for all} \quad s \geq 0, \quad k > \max\{0, \lambda(h_\gamma) - 3s\}$$

for every short root γ. Recall that the short roots in R^+ are α_1, $\alpha := \alpha_1 + \alpha_2$ and $\vartheta := 2\alpha_1 + \alpha_2$ while the long roots are α_2, $\beta := 3\alpha_1 + \alpha_2$ and $\theta := 3\alpha_1 + 2\alpha_2$. For $\gamma = \alpha$, we have $h_\gamma = h_1 + 3h_2$ and the proof is similar to that of Lemma 2.5.3. Namely, fix $s \geq 0$ and recall that $h_\gamma = h_\alpha + 3h_\beta$. Suppose first that $\mu(h_\gamma) - 3s \geq 0$. In this case, we can choose $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$a + b = s, \qquad \mu(h_\alpha) - 3a \geq 0, \qquad \text{and} \qquad \mu(h_\beta) - b \geq 0.$$

Indeed take $a = \max\{0, s - \mu(h_\beta)\}$. Since $\alpha \in R^+_{\rm sh}$, it follows that $(x^-_{\alpha,a})^{(k)} v = 0$ for all $k > \mu(h_\alpha) - 3a$. By Lemma 2.5.2, we have $(x^-_{\beta,b})^{(k)} v = 0$ for all $k > \mu(h_\beta) - b$. Applying Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{a,b}_{\alpha,\beta})$, it follows that $(x^-_{\gamma,s})^{(k)} v = 0$ for all $k > (\mu(h_\alpha) - 3a) + 3(\mu(h_\beta) - b) = \mu(h_\gamma) - 3s$. Now, suppose $\mu(h_\gamma) - 3s \leq 0$ in which case we have to show that $(x^-_{\gamma,s})^{(k)} v = 0$ for all $k > 0$. Notice that we can choose $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$a + b = s, \qquad \mu(h_\alpha) - 3a \leq 0, \qquad \text{and} \qquad \mu(h_\beta) - b \leq 0.$$

Indeed take $a = s - \mu(h_\beta)$. Since $\alpha \in R^+_{\rm sh}$, it follows that $(x^-_{\alpha,a})^{(k)} v = 0$ for all $k > 0$. By Lemma 2.5.2, we have $(x^-_{\beta,b})^{(k)} v = 0$ for all $k > 0$. A new application of Lemma 2.3.7 completes the proof.

We shall use that the lemma holds for $\gamma = \alpha$ in the remainder of the proof. It remains to show that the lemma holds with $\gamma = \vartheta$. Notice that $h_\vartheta = 2h_1 + 3h_2$ and, thus, we want to prove that

$$(x^-_{\vartheta,s})^{(k)} \in I_{\mathbb{F}}(\lambda) \quad \text{for all} \quad s \geq 0, \quad k > \max\{0, 2\lambda(h_1) + 3\lambda(h_2) - 3s\}. \tag{2.5.5}$$

We prove (2.5.5) by induction on $\lambda(h_1)$. Following [**Nao12**], we prove the cases $\lambda(h_1) \in \{0, 1, 2\}$ and then we show that (2.5.5) for $\lambda - 3\omega_1$ in place of λ implies it for λ. To shorten notation, set $a = \lambda(h_1), b = \lambda(h_2)$.

1) Assume $a = 0$. Since $\alpha_1 \in R^+_{\rm sh}$, it follows that $(x_1^-)^{(k)}v = 0$ for all $k > 0$. By Lemma 2.5.2, we have $(x_{2,s}^-)^{(k)}v = 0$ for all $k > \max\{0, b - s\}$. Applying Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{\alpha_1,\alpha_2})$, it follows that $(x_{\vartheta,s}^-)^{(k)}v = 0$ for all $k > 3\max\{0, b - s\} = \max\{0, 2a + 3b - 3s\}$ as desired.

2) Assume $a = 1$. This time we have $(x_1^-)^{(k)}v = 0$ for all $k > 1$. We split in 3 subcases.

2.1) Suppose $b > s - 1$ and notice $2a + 3b - 3s > 0$. Lemma 2.5.2 implies $(x_{2,s}^-)^{(k)}v = 0$ for all $k > \max\{0, b - s\} = b - s$. Applying Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{\alpha_1,\alpha_2})$, it follows that $(x_{\vartheta,s}^-)^{(k)}v = 0$ for all $k > 2 + 3(b - s) = 2a + 3b - 3s$.

2.2) Suppose $b = s - 1$ in which case $2a + 3b - 3s < 0$. Notice that $h_\beta = h_1 + h_2$ and, hence, $\lambda(h_\beta) = a + b = s$. Lemma 2.5.2 then implies that $(x_{\beta,s}^-)^{(k)}v = 0$ for all $k > 0$. Notice that $\{-\alpha_1, \beta\}$ form a basis for R. Since, $(x_1^+)^{(k)}v = 0$ for all $k > 0$, Lemma 2.3.7 applied to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{0,s}_{-\alpha_1,\beta})$ implies that $(x_{\vartheta,s}^-)^{(k)}v = 0$ for all $k > 0$.

2.3) Suppose $b < s - 1$ in which case $2a + 3b - 3s < 0$. This time we apply Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{1,s-2}_{\alpha_1,\alpha_2})$. Indeed, we have $(x_{1,1}^-)^{(k)}v = 0$ for all $k > \max\{0, a - 3\} = 0$ and Lemma 2.5.2 implies that $(x_{2,s-2}^-)^{(k)}v = 0$ for all $k > \max\{0, b - (s - 2)\} = 0$. Thus, since$3(b - s) < -3$ and $a = 1$, we have $\max\{0, 2a + 3b - 3s\} = 0$ and Lemma 2.3.7 implies that $(x_{\vartheta,s}^-)^{(k)}v = 0$ for all $k > 0$.

3) Assume $a = 2$. We split in subcases as before.

3.1) If $b > s - 1$ the proof is similar to that of step 2.1.

3.2) Suppose $b = s - 1$ and notice that $2a + 3b - 3s = 1$. Hence, we want to show that (2.5.5) holds for $k > 1$. For $k > 3$ we apply Lemma 2.3.7 to the subalgebra $U_{\mathbb{F}}(\mathfrak{g}^{1,s-2}_{\alpha_1,\alpha_2})$ in a similar fashion as we did in step 2.3 (the same can be conclude using the argument from step 2.2). For $k \in \{2, 3\}$ we need our hypothesis on the characteristic of $\mathbb{F}$. Assume we have chosen the Chevalley basis so that $x_\vartheta^- = [x_1^+, x_\beta^-]$ and observe that (0.1.4) implies that $[x_1^+, x_\vartheta^-] = \pm 2x_\alpha^-$. Using this, one easily checks that

$$(x_{\vartheta,s}^-)^{(2)} = (x_1^+)^{(2)}(x_{\beta,s}^-)^{(2)} - \frac{1}{2}x_1^+(x_{\beta,s}^-)^{(2)}x_1^+ - \frac{1}{2}x_{\beta,s}^-x_{\vartheta,s}^-x_1^+ \mp x_{\beta,s}^-x_{\alpha,s}^-.$$

Using the case $\gamma = \alpha$ and Lemma 2.5.2 we see that $x_{\alpha,s}^-v = (x_{\beta,s}^-)^{(2)}v = 0$. Hence, since $2 \in \mathbb{F}^\times$, (2.5.5) holds for $k = 2$. For $k = 3$, we have $(x_{\vartheta,s}^-)^{(3)} = \frac{1}{3}x_{\vartheta,s}^-(x_{\vartheta,s}^-)^{(2)}$ and, since $3 \in \mathbb{F}^\times$, (2.5.5) also holds for $k = 3$.

3.3) If $b < s - 1$ the proof is similar to that of step 2.3.

4) Assume $a \geq 3$ and that (2.5.5) holds for $\lambda - 3\omega_1$.

4.1) Suppose $s \geq 2$ and recall the definition of the map $\phi : U_{\mathbb{F}}(\mathfrak{a}) \to U_{\mathbb{F}}(\mathfrak{g}[t])$. The induction hypothesis together with Lemma 2.5.4 implies that

$$(x_{\vartheta,s-2}^-)^{(k)} \in I'_{\mathbb{F}}(\lambda - 3\omega_1) \quad \text{for all} \quad k > \max\{0, 2a + 3b - 3s\}$$

and, therefore

$$(x_{\vartheta,s}^-)^{(k)} = \phi\left((x_{\vartheta,s-2}^-)^{(k)}\right) \in \phi(I'_{\mathbb{F}}(\lambda - 3\omega_1)) \quad \text{for all} \quad k > \max\{0, 2a + 3b - 3s\}.$$

One easily checks that ϕ sends the generators of $I'_{\mathbb{F}}(\lambda - 3\omega_1)$ to generators of $I_{\mathbb{F}}(\lambda)$, completing the proof of (2.5.5) for $s \geq 2$.

4.2) For $s = 0$, notice that $U_{\mathbb{F}}(\mathfrak{g})v$ is a quotient of $W_{\mathbb{F}}(\lambda)$, and (2.5.5) follows. Equivalently, apply Lemma 2.3.7 to $U_{\mathbb{F}}(\mathfrak{g}^{0,0}_{\alpha_1,\alpha_2}) = U_{\mathbb{F}}(\mathfrak{g})$ and the proof is similar to that of step 2.1.

4.3) If $s=1$ and $b\geq 1$, we have $2a+3b-3s>0$ and the usual application of Lemma 2.3.7 to $U_{\mathbb{F}}(\mathfrak{g}^{0,1}_{\alpha_1,\alpha_2})$ completes the proof of (2.5.5). If $s=1$ and $b=0$, we need to show that $(x^-_{\vartheta,1})^{(k)}v=0$ for $k>2a-3$.

Consider the subalgebra $U_{\mathbb{F}}(\mathfrak{sl}_\vartheta[t])\cong U_{\mathbb{F}}(\mathfrak{sl}_2[t])$ defined in Section 2.2.3. Since $\lambda(h_\vartheta)=2a$, it follows that $W:=U_{\mathbb{F}}(\mathfrak{sl}_\vartheta[t])v$ is a quotient of $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-module $W^c_{\mathbb{F}}(2a)$, where we identified the weight lattice of $\mathfrak{sl}_2$ with $\mathbb{Z}$ as usual. Since $W^c_{\mathbb{F}}(2a)\cong D_{\mathbb{F}}(1,2a)$ by Theorem 2.1.2(a), the defining relations of $D_{\mathbb{F}}(1,2a)$ imply $(x^-_{\vartheta,1})^{(k)}v=0$ for $k>2a-1$. It remains to check that $(x^-_{\vartheta,1})^{(k)}v=0$ for $k\in\{2a-2,2a-1\}$.

Suppose by contradiction that $(x^-_{\vartheta,1})^{(2a-1)}v\neq 0$ and notice that

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-1)}v=0 \quad \text{for all} \quad k>0. \tag{2.5.6}$$

Indeed,

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-1)}v\in W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)}$$

is a vector of degree $2a-1>1$ for all $k\geq 0$. By the Weyl group invariance of the character of $W^c_{\mathbb{F}}(2a)$, we know that $W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)}=0$ if $k>1$, and that $W^c_{\mathbb{F}}(2a)_{-2a-2(k-1)}$ is one-dimensional concentrated in degree zero if $k=1$. This proves (2.5.6). Then, Lemma 2.3.3 implies that

$$(x^+_{\vartheta})^{(2a-2)}(x^-_{\vartheta,1})^{(2a-1)}v\neq 0.$$

On the other hand, it follows from Lemma 2.2.1 that

$$(x^+_{\vartheta})^{(2a-2)}(x^-_{\vartheta,1})^{(2a-1)}v=x^-_{\vartheta,2a-1}v.$$

Since $2a-1\geq 2$ and $2a-3(2a-1)=-4a+3<0$, it follows from step 4.1 that $x^-_{\vartheta,2a-1}v=0$ yielding a contradiction as desired.

Similarly, assume by contradiction that $(x^-_{\vartheta,1})^{(2a-2)}v\neq 0$ and notice that

$$(x^-_{\vartheta})^{(k)}(x^-_{\vartheta,1})^{(2a-2)}v=0 \quad \text{for all} \quad k>1.$$

Suppose first that we also have $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v=0$. It then follows from Lemma 2.3.3 that

$$(x^+_{\vartheta})^{(2a-4)}(x^-_{\vartheta,1})^{(2a-2)}v\neq 0.$$

On the other hand, Lemma 2.2.1 implies that

$$(x^+_{\vartheta})^{(2a-4)}(x^-_{\vartheta,1})^{(2a-2)}v=(x^-_{\vartheta,a-1})^{(2)}v+\sum_{r=a}^{2a-2}x^-_{\vartheta,2a-2-r}x^-_{\vartheta,r}v.$$

Since $a-2\geq 2$, step 4.1 implies that $(x^-_{\vartheta,r})^{(k)}v=0$ for all $r\geq a-1, k>0$, implying that the right-hand side is zero, which is a contradiction. It remains to check the possibility that $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v\neq 0$. In this case it follows that $x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v$ is a lowest-weight vector for the algebra $U_{\mathbb{F}}(\mathfrak{sl}_\vartheta)$ and, hence, Lemma 2.3.3 implies that

$$(x^+_{\vartheta})^{(2a-2)}x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v\neq 0.$$

Using (2.2.1) we get

$$(x^+_{\vartheta})^{(2a-2)}x^-_{\vartheta}(x^-_{\vartheta,1})^{(2a-2)}v=\left(x^-_{\vartheta}(x^+_{\vartheta})^{(2a-2)}+(x^+_{\vartheta})^{(2a-3)}\right)(x^-_{\vartheta,1})^{(2a-2)}v.$$

Lemma 2.2.1 together with step 4.1 will again imply that the right-hand side is zero. This completes the proof. ∎

2.5.3. Existence of Demazure flag. If $\mathfrak{g}$ is simply laced, Theorem 2.1.2 (b) follows immediately from part (a) with $k = 1$. Thus, assume from now on that $\mathfrak{g}$ is not simply laced and recall the notation introduced in Subsection 2.2.4.

Given $\lambda \in P^+$, let $\mu = \overline{\lambda} \in P^+_{\rm sh}$ and v be the image of 1 in $W^c_{\mathbb{C}}(\lambda)$. Consider $W^{\rm sh}_{\mathbb{C}} := U(\mathfrak{g}_{\rm sh}[t])v$ and $W^{\rm sh}_{\mathbb{Z}} := U_{\mathbb{Z}}(\mathfrak{g}_{\rm sh}[t])v$. By [**Nao12**, Lemma 4.17], there is an isomorphism of $U(\mathfrak{g}_{\rm sh}[t])$-modules $W^{\rm sh}_{\mathbb{C}} \cong D_{\mathbb{C}}(1,\mu)$. By Corollary 2.3.12, $W^{\rm sh}_{\mathbb{Z}}$ is an integral form of $W^c_{\mathbb{C}}(\mu) \cong D_{\mathbb{C}}(1,\mu)$. Hence, we have an isomorphism of $U_{\mathbb{Z}}(\mathfrak{g}_{\rm sh}[t])$-modules $W^{\rm sh}_{\mathbb{Z}} \cong D_{\mathbb{Z}}(1,\mu)$.

Since $\mathfrak{g}_{\rm sh}$ is of type A, Theorem 2.4.5 implies that there are $k > 0$, $\mu_1,\dots,\mu_k \in P^+_{\rm sh}$, $m_1,\dots,m_k \in \mathbb{Z}_{\geq 0}$, and a filtration of $U_{\mathbb{Z}}(\mathfrak{g}_{\rm sh}[t])$-modules $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_k = W^{\rm sh}_{\mathbb{Z}}$, such that D_j and D_j/D_{j-1} are free $\mathbb{Z}$-modules, and $D_j/D_{j-1} \cong D_{\mathbb{Z}}(r^\vee,\mu_j,m_j)$ for all $j = 1,\dots,k$. In particular,

$$W^{\rm sh}_{\mathbb{Z}}/D_j \quad \text{is a free } \mathbb{Z}\text{-module for all } j = 0,\dots,k. \tag{2.5.7}$$

Set $\lambda_j = \eta_\lambda(\mu_j) \in P^+$ where η_λ is defined in (2.2.10), $W^j_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{g}[t])D_j$ and $W^j_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} W^j_{\mathbb{Z}}$. We have $0 = W^0_{\mathbb{F}} \subseteq W^1_{\mathbb{F}} \subseteq \cdots \subseteq W^k_{\mathbb{F}}$, and $\lambda_k = \lambda$ since $\mu_k = \mu$. Hence, we are left to show that

$$W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong D_{\mathbb{F}}(1,\lambda_j,m_j) \quad \text{for all } j = 1,\dots,k, \text{ and} \quad W^k_{\mathbb{F}} \cong W^c_{\mathbb{F}}(\lambda).$$

Notice that $W^k_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{g}[t])v$. Then, Corollary 2.3.12 implies that $W^k_{\mathbb{Z}}$ is an integral form of $W^c_{\mathbb{C}}(\lambda)$. Since $\mathbb{Z}$ is a PID and $W^k_{\mathbb{Z}}$ is a finitely generated, free $\mathbb{Z}$-module, it follows that $W^j_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite-rank for all $j = 1,\dots,k$. Set $W^j_{\mathbb{C}} = U(\mathfrak{g}[t])D_j$. It follows from [**Nao12**, Proposition 4.18] or Theorem 2.1.2 (b) (in characteristic zero) that $W^j_{\mathbb{C}}/W^{j-1}_{\mathbb{C}} \cong D_{\mathbb{C}}(1,\lambda_j,m_j)$ for all $j = 1,\dots,k$. Moreover, since $W^j_{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{Z}} W^j_{\mathbb{Z}}$, it follows that $\mathbb{C} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}}) \cong (W^j_{\mathbb{C}}/W^{j-1}_{\mathbb{C}}) \cong D_{\mathbb{C}}(1,\lambda_j,m_j)$. Therefore, $W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}}$ is a finitely generated $\mathbb{Z}$-module of rank $\dim(D_{\mathbb{C}}(1,\lambda_j,m_j))$ for all $j = 1,\dots,k$. Since $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong \mathbb{F} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}}/W^{j-1}_{\mathbb{Z}})$, it follows that

$$\dim(W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}) \geq \dim(D_{\mathbb{C}}(1,\lambda_j,m_j)) = \dim(D_{\mathbb{F}}(1,\lambda_j,m_j)).$$

Now, let $v_j \in D_j$ be as in Theorem 2.4.5, w be the image of v in $W^k_{\mathbb{F}}$, $u_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\rm sh}[t])$ be such that $v_j = u_j v$, and $w_j = u_j w$. It follows that

$$W^j_{\mathbb{Z}} = \sum_{n \leq j} U_{\mathbb{Z}}(\mathfrak{g}[t])v_n \quad \text{and} \quad W^j_{\mathbb{F}} = \sum_{n \leq j} U_{\mathbb{F}}(\mathfrak{g}[t])w_n.$$

We will show that the image $\bar{w}_j$ of w_j in $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}$ satisfies the relations described in Proposition 2.5.1, which implies that $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}}$ is a quotient of $D_{\mathbb{F}}(1,\lambda_j,m_j)$ and, hence, $W^j_{\mathbb{F}}/W^{j-1}_{\mathbb{F}} \cong D_{\mathbb{F}}(1,\lambda_j,m_j)$ for all $j = 1,\dots,k$.

By construction, v_j is a weight vector of weight λ_j and degree m_j, and so is w_j. Since $D_j/D_{j-1} \cong D_{\mathbb{Z}}(r^\vee,\mu_j,m_j)$, it follows that

$$U_{\mathbb{F}}(\mathfrak{n}^+_{\rm sh}[t])^0 \bar{w}_j = U_{\mathbb{F}}(\mathfrak{h}_{\rm sh}[t]_+)^0 \bar{w}_j = 0 \quad \text{and} \quad (x^-_{\alpha,s})^{(k)}\bar{w}_j = 0$$

for all $\alpha \in R^+_{\rm sh}, s \geq 0, k > \max\{0, \lambda(h_\alpha) - sr^\vee\}, j = 1,\dots,k$. Thus, it remains to show that

$$(x^+_{\alpha,s})^{(m)}\bar{w}_j = \Lambda_{i,r}\bar{w}_j = (x^-_i)^{(k)}\bar{w}_j = 0$$

for all $\alpha \in R^+ \setminus R^+_{\rm sh}, i \in I \setminus I_{\rm sh}, s \geq 0, r, m > 0, k > \lambda_j(h_i), j = 1,\dots,k$. Since,

$$\lambda_j + m\alpha \notin \lambda - Q^+ \qquad \text{for all} \qquad \alpha \in R^+ \setminus R^+_{\rm sh}, m > 0, \tag{2.5.8}$$

we get $(x^+_{\alpha,s})^{(m)}w_j = 0$ for all $m > 0, s \geq 0$. In particular, it follows that $\bar{w}_j$ is a highest-weight vector of weight λ_j and, hence, $(x^-_i)^{(k)}\bar{w}_j = 0$ for all $i \in I, k > \lambda(h_i)$. Finally, we show that

$$\Lambda_{i,r}\bar{w}_j = 0 \quad \text{for all} \quad i \in I \setminus I_{\rm sh}, r > 0, j = 1,\dots,k. \tag{2.5.9}$$

Observe that

$$\Lambda_{i,r} u_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\rm sh}) U_{\mathbb{Z}}(\mathfrak{h}[t]_+).$$

In particular, $\Lambda_{i,r} v_j \in W^{\rm sh}_{\mathbb{Z}} \cap W^j_{\mathbb{Z}}$. We will show that $\Lambda_{i,r} v_j \in D_{j-1}$ which implies (2.5.9). Let $y_j \in U_{\mathbb{Z}}(\mathfrak{n}^-_{\rm sh})$ be such that $\Lambda_{i,r} u_j = y_j$ modulo $U_{\mathbb{Z}}(\mathfrak{n}^-_{\rm sh}) U_{\mathbb{Z}}(\mathfrak{h}[t]_+)^0$. Thus, we want to show that

$$y_j v \in D_{j-1}. \tag{2.5.10}$$

We prove this recursively on $j = 1, \dots, k$. Notice that, since $\mathbb{C} \otimes_{\mathbb{Z}} (W^j_{\mathbb{Z}} / W^{j-1}_{\mathbb{Z}}) \cong D_{\mathbb{C}}(1, \lambda_j, m_j)$, there exists $n_j \in \mathbb{Z}_{>0}$ such that $n_j y_j v \in W^{j-1}_{\mathbb{Z}}, j = 1, \dots, k$. In particular, since $W^0_{\mathbb{Z}} = 0$ and $W^1_{\mathbb{Z}}$ is a torsion-free $\mathbb{Z}$-module, (2.5.10) follows for $j = 1$. Next, we show that (2.5.10) implies

$$W^j_{\mathbb{Z}} \cap W^{\rm sh}_{\mathbb{Z}} = D_j. \tag{2.5.11}$$

Indeed, it follows from (2.5.8) and (2.5.10) that

$$W^j_{\mathbb{Z}} = U_{\mathbb{Z}}(\mathfrak{n}^-[t]) U_{\mathbb{Z}}(\mathfrak{g}_{\rm sh}[t]) v_j + W^{j-1}_{\mathbb{Z}}.$$

Since $U_{\mathbb{Z}}(\mathfrak{h}_{\rm sh}[t]_+)^0 U_{\mathbb{Z}}(\mathfrak{n}^+_{\rm sh})^0 v_j \in D_{j-1}$ and, by induction hypothesis, $W^{j-1}_{\mathbb{Z}} \cap W^{\rm sh}_{\mathbb{Z}} = D_{j-1}$, (2.5.11) follows by observing that

$$(U_{\mathbb{Z}}(\mathfrak{n}^-[t]) v_j) \cap W^{\rm sh}_{\mathbb{Z}} \subseteq D_j$$

(which is easily verified by weight considerations). Finally, observe that, since $n_{j+1} y_{j+1} v \in W^j_{\mathbb{Z}} \cap W^{\rm sh}_{\mathbb{Z}} = D_j$, (2.5.7) implies that $y_{j+1} v \in D_j$. Thus, (2.5.11) for j implies (2.5.10) for $j+1$ and the recursive step is proved.

Remark 2.5.6. It follows from the above that $W^j_{\mathbb{F}} / W^{j-1}_{\mathbb{F}} \cong D_{\mathbb{F}}(1, \lambda_j, m_j)$ for any field $\mathbb{F}$. Hence, $W^j_{\mathbb{Z}} / W^{j-1}_{\mathbb{Z}}$ must be isomorphic to $D_{\mathbb{Z}}(1, \lambda_j, m_j)$ for all $j = 1, \dots, k$.

It remains to show that $W^k_{\mathbb{F}} \cong W^c_{\mathbb{F}}(\lambda)$. Since we have a projection $W^c_{\mathbb{F}}(\lambda) \twoheadrightarrow W^k_{\mathbb{F}}$ of $U_{\mathbb{F}}(\mathfrak{g}[t])$-modules by the universal property of $W^c_{\mathbb{F}}(\lambda)$, it suffices to show that $\dim(W^c_{\mathbb{F}}(\lambda)) \le \dim(W^k_{\mathbb{F}})$. This follows if we show that there exists a filtration $0 = \tilde{W}^0_{\mathbb{F}} \subseteq \tilde{W}^1_{\mathbb{F}} \subseteq \cdots \subseteq \tilde{W}^k_{\mathbb{F}} = W^c_{\mathbb{F}}(\lambda)$ such that $\tilde{W}^j_{\mathbb{F}} / \tilde{W}^{j-1}_{\mathbb{F}}$ is a quotient of $D_{\mathbb{F}}(1, \lambda_j, m_j)$ for all $j = 1, \dots, k$. Let w' be the image of 1 in $W^c_{\mathbb{F}}(\lambda)$, $w'_j = u_j w' \in W^c_{\mathbb{F}}(\lambda)$, $\tilde{W}^j_{\mathbb{F}} := \sum_{n \le j} U_{\mathbb{F}}(\mathfrak{g}[t]) w'_n \subseteq W^c_{\mathbb{F}}(\lambda)$, and $\bar{w}'_j$ be the image of w'_j in $\tilde{W}^j_{\mathbb{F}} / \tilde{W}^{j-1}_{\mathbb{F}}$. Observe that $\tilde{W}^k_{\mathbb{F}} = W^c_{\mathbb{F}}(\lambda)$. We need to show that $\bar{w}'_j$ satisfies the defining relations of $D_{\mathbb{F}}(1, \lambda_j)$ listed in Proposition 2.5.1. Let $\tilde{D}_j = \mathbb{F} \otimes_{\mathbb{Z}} D_j$ and $D'_j = \sum_{n \le j} U_{\mathbb{F}}(\mathfrak{g}_{\rm sh}[t]) w'_n$. Notice that D'_k is a quotient of $W^c_{\mathbb{F}}(\mu) \cong \tilde{D}_k$ and let $\pi : \tilde{D}_k \to D'_k$ be a $U_{\mathbb{F}}(\mathfrak{g}_{\rm sh}[t])$-module epimorphism such that $v_k \mapsto w'_k$ (we keep denoting the image of v_j in $\tilde{D}_j$ by v_j). In particular, $w'_j = \pi(v_j)$ and π induces an epimorphism $\tilde{D}_j \to D'_j$ for all $j = 1, \dots, k$. In particular,

$$x w'_j \in D'_{j-1} \quad \text{for all} \quad x \in U_{\mathbb{Z}}(\mathfrak{g}_{\rm sh}[t]) \quad \text{such that} \quad x v_j \in D_{j-1}.$$

This immediately implies that

$$U_{\mathbb{F}}(\mathfrak{n}^+_{\rm sh}[t])^0 \bar{w}'_j = U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 \bar{w}'_j = 0 \quad \text{and} \quad (x^-_{\alpha,s})^{(k)} \bar{w}'_j = 0$$

for all $\alpha \in R^+_{\rm sh}, s \ge 0, k > \max\{0, \lambda(h_\alpha) - s r^\vee\}$, $j = 1, \dots, k$. Note that (2.5.10) has been used here. The relations

$$(x^+_{\alpha,s})^{(m)} \bar{w}'_j = (x^-_i)^{(k)} \bar{w}'_j = 0$$

for all $\alpha \in R^+ \setminus R^+_{\rm sh}, i \in I \setminus I_{\rm sh}, s \ge 0, m > 0, k > \lambda_j(h_i), j = 1, \dots, k$ follow from (2.5.8) as before.

2.5.4. Isomorphism between local Weyl modules and graded local Weyl modules. We now prove Theorem 2.1.2 (c). Recall the definition of the automorphism φ_a of $U_{\mathbb{F}}(\mathfrak{g}[t])$ from Subsection 2.2.2. In particular, let $\tilde{a} \in \mathbb{A}^\times$ be such that its image in $\mathbb{F}$ is a. Denote by $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ the pull-back of $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$ (regarded as a $U_{\mathbb{F}}(\mathfrak{g}[t])$-module) by φ_a.

Notice that $\dim W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = \dim W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}) = \dim W_{\mathbb{K}}^c(\lambda) = \dim W_{\mathbb{F}}^c(\lambda)$. In fact, the first equality follows from (2.1.4), the second from Proposition 2.3.15 and (2.3.5) (with $\mathbb{F} = \mathbb{K}$), and the third from Corollary 2.1.3. Since $\dim \varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})) = \dim W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$, Theorem 2.1.2 (c) will be proved if we show that $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ is a quotient of $W_{\mathbb{F}}^c(\lambda)$.

Let $w \in W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})_\lambda \setminus \{0\}$ and denote by w_a the vector w when regarded as an element of $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$. Since $W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = U_{\mathbb{F}}(\mathfrak{g}[t])w$ and φ_a is an automorphism of $U_{\mathbb{F}}(\mathfrak{g}[t])$, it follows that $\varphi_a^* W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}) = U_{\mathbb{F}}(\mathfrak{g}[t])w_a$. Thus, we need to show that w_a satisfies the defining relations (2.1.1) of $W_{\mathbb{F}}^c(\lambda)$. Since φ_a fixes every element of $U_{\mathbb{F}}(\mathfrak{g})$, w_a is a vector of weight λ annihilated by $(x_\alpha^-)^{(k)}$ for all $\alpha \in R^+, k > \lambda(h_\alpha)$. Equation (2.2.8) implies that φ_a maps $U_{\mathbb{F}}(\mathfrak{n}^+[t])$ to itself and, hence, $U_{\mathbb{F}}(\mathfrak{n}^+[t])^0 w_a = 0$. Therefore, it remains to show that $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0 w_a = 0$.

For showing this, let $v \in W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}})_\lambda \setminus \{0\}$ and $L = U_{\mathbb{A}}(\mathfrak{g}[t])v$. By (2.1.4), $\mathbb{F} \otimes_{\mathbb{A}} L \cong W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a})$. In particular, the action of $U_{\mathbb{F}}(\mathfrak{h}[t]_+)^0$ on $\varphi_a^*(W_{\mathbb{F}}(\boldsymbol{\omega}_{\lambda,a}))$ is obtained from the restriction of the action of $U_{\mathbb{K}}(\mathfrak{h}[t]_+)^0$ on $\varphi_{\tilde{a}}^*(W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}))$ to $U_{\mathbb{A}}(\mathfrak{h}[t]_+)^0$. Since $U_{\mathbb{K}}(\mathfrak{h}[t]_+)$ is generated by $h_{i,r}, i \in I, r > 0$, we are left to show that $h_{i,r} v_a = 0$, where v_a is the vector v regarded as an element of $\varphi_{\tilde{a}}^*(W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}}))$.

The irreducible quotient of $W_{\mathbb{K}}(\boldsymbol{\omega}_{\lambda,\tilde{a}})$ is the evaluation module with evaluation parameter $\tilde{a}$ (cf. [**JM07**, Section 3B]). Hence, $h_{i,s}v = \tilde{a}^s \lambda(h_i) v$ for all $i \in I, s \in \mathbb{Z}$. Using this, it follows that, for all $i \in I, r > 0$, we have

$$h_{i,r} v_a = (h_i \otimes (t - \tilde{a})^r) v = \sum_{s=0}^{r} \binom{r}{s} (-\tilde{a})^s h_{i,r-s} v = \lambda(h_i) \tilde{a}^r \sum_{s=0}^{r} \binom{r}{s} (-1)^s v = 0,$$

completing the proof of Theorem 2.1.2 (c).

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