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JHOAN SEBASTIAN BAEZ ACEVEDO

Lagrangian submanifolds in coadjoint orbits provided with symplectic invariant structures

Subvariedades Lagrangianas em órbitas coadjuntas munidas de estruturas simpléticas invariantes

Campinas

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Jhoan Sebastian Baez Acevedo

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Supervisor: Luiz Antonio Barrera San Martin

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Prof(a). Dr(a). LUIZ ANTONIO BARRERA SAN MARTIN

Prof(a). Dr(a). LINO ANDERSON DA SILVA GRAMA

Prof(a). Dr(a). VIVIANA JORGELINA DEL BARCO

Prof(a). Dr(a). ELIZABETH TEREZINHA GASPARIM

Prof(a). Dr(a). RYUICHI FUKUOKA

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"All that you touch
And all that you see
All that you taste
All you feel
And all that you love
And all that you hate
All you distrust
All you save
And all that you give
And all that you deal
And all that you buy
Beg, borrow or steal
And all you create
And all you destroy
And all that you do
And all that you say
And all that you eat
And everyone you meet
And all that you slight
And everyone you fight
And all that is now
And all that is gone
And all that's to come
And everything under the sun is in tune
But the sun is eclipsed by the moon"
(Eclipse, Pink Floyd)

Resumo

Neste trabalho encontramos subvariedades Lagrangianas de órbitas coadjuntas semisimples em dois casos. Para o primeiro, o caso compacto, que são as chamadas variedades flag generalizadas, provamos que as variedades flag reais podem ser vistas como subvariedades lagrangianas (na verdade 'tight' infinitesimais) com respeito da forma simplética Konstant–Kirillov–Souriau e uma classificação completa foi obtida. Para o segundo caso, o caso complexo, provamos que as órbitas de formas reais são subvariedades Lagrangianas com respeito da forma simplética Hermitiana, onde aplicamos um difeomorfismo de deformação entre a órbita coadjunta semisimples clássica e a órbita coadjunta de um produto semi-direto dado por uma decomposição de Cartan. Além do mais, usando essa deformação construímos seções lagrangianas com respeito à forma Hermitiana.

Palavras-chave: Espaços homogêneos, Estruturas simpléticas, Variedades bandeira, Subvariedades Lagrangeanas.

Abstract

In this work, we found some Lagrangian submanifolds of the coadjoint semisimple orbit in two cases. For the first one, the compact case, also known as the generalized flag manifolds, we prove that the real flags can be seen as (infinitesimally tight) Lagrangian submanifolds with respect to the Konstant–Kirillov–Souriau symplectic form and we give a complete classification. And for the second one, the complex case, we prove that the orbits of real forms are Lagrangian submanifolds with respect to the Hermitian symplectic form, where we apply a coadjoint orbit's diffeomorphic deformation between the classical semisimple case and the semi-direct product given by a Cartan decomposition. Furthermore, using that deformation we build some Lagrangian sections with respect to the Hermitian form.

Keywords: Homogeneous spaces, Symplectic structures, Flag manifolds, Lagrangian submanifolds.

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Introduction

The main theme of this thesis is the study of Lagrangian submanifolds in symplectic homogeneous spaces. Symplectic homogeneous manifolds have been studied by many authors, see for instance [32] for a classification of compact symplectic homogeneous manifolds. In particular, we focus on the (co)adjoint[1] orbits of semisimple Lie groups. However, finding Lagrangian submanifolds is a classic research topic in symplectic geometry that goes back to Darboux's theorem about the existence of trivializing neighborhoods that preserve the symplectic form and therefore the existence of local Lagrangian submanifolds. We study some applications of the semisimple Lie theory to symplectic geometry, in particular to find Lagrangian submanifolds on adjoint orbits. Our motivation to study Lagrangian submanifolds and their classification comes from questions related to the homological mirror symmetry conjecture and especially from concepts of objects and morphisms in the so called Fukaya–Seidel categories, which are generated by Lagrangian vanishing cycles (and their thimbles) with prescribed behavior inside of symplectic fibrations (see [10] and [12]).

The symplectic structures and Lagrangian submanifolds of the coadjoint orbit was studied and developed by renowned mathematicians such as Kirillov, Arnold, Kostant, and Souriau in the early to mid 1960s, although it had important roots going back to the work of Lie, Borel, and Weyl. (See [3], [18], [19], [20], [21], [30]). Alternatively, there are several theories and applications to physics using general reduction theory, as in [7], [22], [23] and [24], among others.

For instance, the most studied symplectic form in such manifolds is the Konstant–Kirillov–Souriau symplectic form (briefly KKS), which for adjoint orbits of compact Lie groups (better known as flag manifolds) is the only possible Ad-invariant symplectic form. In Chapter 1, we give a preliminary chapter where we introduce all the relevant elements and notations for this thesis, such as semisimple Lie theory and symplectic structures studied: the KKS form and the Hermitian symplectic form.

Taking in emphasis the recent papers [8] and [13], where the authors characterized

[1] We write (co)adjoints because in the semisimple case those orbits can be identified.

the isotropic submanifolds through the moment map of a Hamiltonian action, which for the flag manifolds is the adjoint action. Then, with that tool in Chapter 2, we can characterize the complex flag manifolds that admit as Lagrangian submanifold a real flag manifold. To prove that, we endow the complex flag manifolds with the KKS symplectic form and look at compact orbits of the real forms of the complex group. We provide a classification of the complex flag manifolds and real forms having Lagrangian compact orbits. This is done in a case by case analysis via the Satake diagrams of the real forms. The result is presented at Table 1 at the end of Section 2.2 and the case by case proof is done in Subsection 2.2.1. All these results are part of the paper [5], where we focus on the construction of Lagrangian submanifolds determined by real forms. In particular, we prove that the real flag manifolds can be seen as infinitesimally tight submanifolds of the correspondent complex flag manifold. In order not to deviate the objective of the thesis, the discussion regarding this type of Lagrangian submanifolds is given in Appendix A.

To conclude the Chapter 2, we describe the Lagrangian orbits given by the adjoint action of U with respect to the Hermitian form on $\mathrm{Ad}\,(G) \cdot H$, where G is a complex semisimple Lie group and U its compact form.

Additionally, in [6] we follow the next construction: let $\mathfrak{g}$ be a non-compact semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{a} \subset \mathfrak{s}$ maximal Abelian. In the underlying vector space $\mathfrak{g}$ there is another Lie algebra structure $\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times_{\mathrm{ad}} \mathfrak{s}$ given by the semi-direct product defined by the adjoint representation of $\mathfrak{k}$ in $\mathfrak{s}$, which is viewed as an Abelian Lie algebra. Let $G = \mathrm{Aut}_0\mathfrak{g}$ be the adjoint group of $\mathfrak{g}$ (identity component of the automorphism group) and put $K = \exp \mathfrak{k} \subset G$. The semi-direct product $K_{\mathrm{ad}} = K \times_{\mathrm{Ad}} \mathfrak{s}$ obtained by the adjoint representation of K in $\mathfrak{s}$ has Lie algebra $\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times_{\mathrm{ad}} \mathfrak{s}$, that orbit was studied in Chapter 3. Then, we consider coadjoint orbits for both Lie algebras $\mathfrak{g}$ and $\mathfrak{k}_{\mathrm{ad}}$. These orbits are submanifolds of $\mathfrak{g}^*$ that we identify with $\mathfrak{g}$ via the Cartan–Killing form of $\mathfrak{g}$, so that the orbits are seen as submanifolds of $\mathfrak{g}$. These are just the adjoint orbits for the Lie algebra $\mathfrak{g}$ while for $\mathfrak{k}_{\mathrm{ad}}$ they are the orbits in $\mathfrak{g}$ of the representation of K_{ad} obtained by transposing its coadjoint representation. The orbits through $H \in \mathfrak{g}$ are denoted by $\mathrm{Ad}\,(G) \cdot H$ and $K_{\mathrm{ad}} \cdot H$, respectively.

We consider the orbits through $H \in \mathfrak{a} \subset \mathfrak{s}$. In this case the compact orbit

$\mathrm{Ad}(K) \cdot H$ (contained in $\mathfrak{s}$) is a flag manifold of $\mathfrak{g}$, say $\mathbb{F}_H$. In [11] it was proved that $\mathrm{Ad}(G) \cdot H$ is diffeomorphic to the cotangent bundle $T^*\mathbb{F}_H$ of $\mathbb{F}_H = \mathrm{Ad}(K) \cdot H$. We prove here that the same happens to the semi-direct product orbit $K_{\mathrm{ad}} \cdot H$ (as foreseen by [17]). So that $\mathrm{Ad}(G) \cdot H$ and $K_{\mathrm{ad}} \cdot H$ diffeomorphic to each other.

In Chapter 4, we define a deformation $\mathfrak{g}_r$ of the original Lie algebra $\mathfrak{g}$. The deformation is parameterized by $r > 0$ and satisfies $\mathfrak{g}_1 = \mathfrak{g}$. For each r the Lie algebra $\mathfrak{g}_r$ is isomorphic to $\mathfrak{g}$ (hence semisimple) and $\mathfrak{g}_r = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition as well with $\mathfrak{k}$ a subalgebra of $\mathfrak{g}_r$. Furthermore as $r \to \infty$ the Lie algebra $\mathfrak{k}_{\mathrm{ad}}$ is recovered. (The deformation amounts essentially to change the brackets $[X, Y]$, $X, Y \in \mathfrak{s}$, by $(1/r)[X, Y]$ and keeping the other brackets unchanged.) A Lie algebra $\mathfrak{g}_r$, $r > 0$, has its own automorphism group whose identity component is denoted by G_r. Thus the adjoint orbits in $\mathfrak{g}_r$ are $\mathrm{Ad}(G_r) \cdot H$ and by the isomorphism $\mathfrak{g}_r \approx \mathfrak{g}$ it follows that $\mathrm{Ad}(G_r) \cdot H$ is diffeomorphic to $\mathrm{Ad}(G) \cdot H$ and hence to the cotangent space $T^*\mathbb{F}_H$. Thus, the Lie algebra deformation yields a continuous one parameter family of embeddings of $T^*\mathbb{F}_H$ into the vector space underlying $\mathfrak{g}$. The family is parameterized in $(0, +\infty]$, where $+\infty$ is the embedding given by the semi-direct product orbit $K_{\mathrm{ad}} \cdot H$.

The example with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, presented in Subsection 4.1.1, is elucidative of this deformation. In $\mathfrak{sl}(2, \mathbb{R}) \approx \mathbb{R}^3$ the semi-direct product orbit is the cylinder $x^2 + y^2 = 1$ while the adjoint orbit is the one-sheet hyperboloid $x^2 + y^2 - z^2 = 1$. In the deformation, the adjoint orbit in $\mathfrak{g}_r$ is the hyperboloid $x^2 + y^2 - z^2/r = 1$ that converges to the cylinder as $r \to +\infty$. The hyperboloids as well as the cylinder are unions of straight lines in $\mathbb{R}^3$ crossing the circle $x^2 + y^2 = 1$ with $z = 0$. As it is well known the hyperboloids are obtained by twisting the generatrices of the cylinder.

This picture of "twisting generatrices" holds in a general Lie algebra $\mathfrak{g}$: the semi-direct product orbit $K_{\mathrm{ad}} \cdot H$ has the cylindrical shape

$$K_{\mathrm{ad}} \cdot H = \bigcup_{X \in \mathrm{Ad}(K)H} (X + \mathrm{ad}(X)\mathfrak{s})$$

where $\mathrm{ad}(X)\mathfrak{s}$ is a subspace of $\mathfrak{k}$. While the adjoint orbit $\mathrm{Ad}(G) \cdot H$ has the hyperboloid shape

$$\mathrm{Ad}(G) \cdot H = \bigcup_{k \in K} \mathrm{Ad}(k)\left(H + \mathfrak{n}_H^+\right)$$

where $\mathfrak{n}_H^+$ is the nilpotent subalgebra which is the sum of the eigenspaces of $\mathrm{ad}(H)$

associated to positive eigenvalues. The deformation of $\mathfrak{g}$ into $\mathfrak{g}_r$ has the effect of twisting the generatrix $H + \mathrm{ad}\,(H)\,\mathfrak{s} \subset \mathfrak{k}$ into $H + \mathfrak{n}^+_{r,H}$, where $\mathfrak{n}^+_{r,H}$ is the nilpotent Lie subalgebra of $\mathfrak{g}_r$ defined the same way as $\mathfrak{n}^+$ from the adjoint $\mathrm{ad}_r\,(H)$ of H in $\mathfrak{g}_r$. The deformation of orbits allows to transfer geometric properties from semi-direct product orbit $K_{\mathrm{ad}} \cdot H$ to the adjoint orbit $\mathrm{Ad}\,(G) \cdot H$. This can be useful since in several aspects the geometry of $K_{\mathrm{ad}} \cdot H$ is more manageable than that of $\mathrm{Ad}\,(G) \cdot H$.

In that chapter, we apply this transfer approach to adjoint orbits in a complex semisimple Lie algebra $\mathfrak{g}$. In the complex case $\mathfrak{g}$ is endowed with a Hermitian metric

$$\mathcal{H}_\tau\,(X, Y) = \langle X, Y \rangle + i\Omega\,(X, Y)$$

where $\langle \cdot, \cdot \rangle$ is an inner product and Ω is a symplectic form. The form Ω restricts to symplectic forms on the orbit $\mathrm{Ad}\,(G) \cdot H$ since this is a complex submanifold. Since it is not immediate that the form in $U_{\mathrm{ad}} \cdot H$ is symplectic, we prove that the restriction of Ω to the semi-direct product orbit $U_{\mathrm{ad}} \cdot H$ (because $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ is the Cartan decomposition of $\mathfrak{g}$ complex semisimple) is also a symplectic form. By construction the diffeomorphisms between the coadjoint orbits are symplectomorphisms. Based on these facts, in Section 4.2, we construct Lagrangian submanifolds in $U_{\mathrm{ad}} \cdot H$ and then transport them to $\mathrm{Ad}_r\,(G) \cdot H$ through the deformation.

1 Adjoint orbits of semisimple Lie groups

The first chapter of this thesis will be focused on studying the (co)adjoint orbits of semisimple Lie groups and the symplectic structures they admit. In order to do that, we based on the following references [4], [15], [26], [28] and [29]. In the first part we study the compact case, also known as 'generalized flag manifolds', for which we provide the classical constructions and the respective differences between the real and the complex situations. Then, we focus on the non-compact case, in particular on the case presented in [11]. Finally, we state the symplectic forms on adjoint orbits that we will study throughout this thesis, i.e., the KKS and the Hermitian symplectic forms.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The **adjoint orbit** of G passing through $H \in \mathfrak{g}$ is the set

$$M_H = \mathrm{Ad}\,(G) \cdot H = \{\mathrm{Ad}\,(g) \cdot H : \ g \in G\} \subset \mathfrak{g}.$$

Let $K_H = \{g \in G : \ \mathrm{Ad}\,(g) \cdot H = H\}$ be the isotropy subgroup of H of the adjoint action of G. Then M_H is diffeomorphic to the homogeneous space G/K_H, because the action of G on M_H is transitive. The tangent space of $\mathrm{Ad}(G) \cdot H$ at $b_H = 1 \cdot K_H$ is given by

$$T_{b_H}\,(\mathrm{Ad}(G) \cdot H) = \{\mathrm{ad}(X) \cdot H : \ X \in \mathfrak{g}\}.$$

Analogously, the **coadjoint action** of G on the dual space $\mathfrak{g}^*$ is given by

$$\mathrm{Ad}^*(g) \cdot \alpha = \alpha \circ \mathrm{Ad}(g^{-1}), \quad g \in G, \ \alpha \in \mathfrak{g}^*.$$

That orbit can be identified as an homogeneous space G/Z_α, where Z_α is a closed subgroup

$$Z_\alpha = \{g \in G : \ \alpha \circ \mathrm{Ad}(g^{-1}) = \alpha\}.$$

The tangent space of $\mathrm{Ad}^*(G) \cdot \alpha$ at $b_\alpha = 1 \cdot Z_\alpha$ is given by

$$T_{b_\alpha}\,(\mathrm{Ad}^*(G) \cdot \alpha) = \{\mathrm{ad}^*(X) \cdot \alpha : \ X \in \mathfrak{g}\},$$

where $\mathrm{ad}^*(X) \cdot \alpha = -\alpha \circ \mathrm{ad}(X)$.

The coadjoint orbit $\mathrm{Ad}^*(G)\cdot\alpha$ admits as symplectic form the **Konstant–Kirillov–Souriau** form (briefly KKS form), given by

$$\omega_\beta\left(\widetilde{X}(\beta),\widetilde{Y}(\beta)\right)=\beta[X,Y] \quad X,Y\in\mathfrak{g},\ \beta\in\mathrm{Ad}^*(G)\cdot\alpha \tag{1.1}$$

where $\widetilde{X}=\mathrm{ad}^*(X)$. In this situation, the action of G on $\mathfrak{g}^*$ by the representation Ad^* is Hamiltonian with respect to ω, such that:

- If $X\in\mathfrak{g}$, then the induced field $\widetilde{X}$ is a Hamiltonian field which is determined by the Hamiltonian function $f_X:\mathrm{Ad}^*(G)\cdot\alpha\to\mathbb{R}$, such that

$$f_X(\beta)=\beta(X) \qquad \beta\in\mathrm{Ad}^*(G)\cdot\alpha.$$

- $\mu=\mathrm{id}$ is the moment map, which is Ad^*-equivariant.

In fact, we are interested in studying the adjoint orbits of semisimple Lie groups. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$, take the isomorphism $\Gamma:\mathfrak{g}\to\mathfrak{g}^*$ given by

$$\Gamma(X)(\cdot)=\langle X,\cdot\rangle_{\mathfrak{g}},$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is the Cartan–Killing form of $\mathfrak{g}$. This isomorphism interchanges the adjoint and the coadjoint representations, that is $\Gamma\,\mathrm{Ad}(g)=\mathrm{Ad}^*(g)\Gamma$ for all $g\in G$, which allows us to transport the symplectic form ω on coadjoint orbits to the symplectic form $\Gamma^*\omega$ on adjoint orbits. Therefore the adjoint action of G is a Hamiltonian action, where the induced vector fields are $\widetilde{X}=\mathrm{ad}(X)$ with Hamiltonian function $H_X(\cdot)=\langle X,\cdot\rangle$. Hence, the adjoint orbit $\mathrm{Ad}(G)\cdot H$ admits the KKS symplectic form, given by

$$\omega_x\left(\widetilde{X}(x),\widetilde{Y}(x)\right)=\langle x,[X,Y]\rangle_{\mathfrak{g}} \qquad X,Y\in\mathfrak{g},\quad x\in\mathrm{Ad}(G)\cdot H. \tag{1.2}$$

Other properties of this type of manifolds will depend on the properties of the Lie algebra $\mathfrak{g}$ or the Lie group G. In particular, we will see what happens in the compact and the non-compact cases.

1.1 Compact case

Let U be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$. The adjoint orbit of U is known as flag manifolds, which can be defined as follows:

Definition 1.1. *Let $\mathfrak{g} = \mathfrak{u}_{\mathbb{C}}$ be a semisimple non-compact Lie algebra and take G a connected Lie group with Lie algebra $\mathfrak{g}$. The flag manifold $\mathbb{F}_H$ is a homogeneous space*

$$\mathbb{F}_H = G/P_H,$$

where P_H is a parabolic subgroup of G, which is determined by $H \in \mathfrak{g}$, that can be chosen in the closure of a positive Weyl chamber of $\mathfrak{g}$.

The construction of the parabolic subgroup P_H and the choice of the compact group U (such that the flag manifold can be seen as an adjoint orbit) will depend on the fact that $\mathfrak{g}$ is a real or complex Lie algebra. Next we will present the constructions using tools of semisimple Lie theory.

1.1.1 Complex flag manifolds

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Given $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, set the following

- Π a root system, such that there exist $H_\alpha \in \mathfrak{h}$ given by
$$\alpha(H) = \langle H_\alpha, H\rangle_{\mathfrak{g}} \quad \forall H \in \mathfrak{h},\ \alpha \in \Pi,$$
where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is the Cartan–Killing form of $\mathfrak{g}$.
- Σ a simple root system, such that Π^+ is the set of positive roots on Π and $\{H_\alpha :\ \alpha \in \Sigma\}$ is a basis of $\mathfrak{h}$.
- $\mathfrak{a}^+$ its corresponding positive Weyl chamber, given by
$$\mathfrak{a}^+ = \{H \in \mathfrak{h} :\ \alpha(H) > 0\ \forall \alpha \in \Sigma\}.$$

Therefore,

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha,$$

the root space decomposition, such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} :\ [H, X] = \alpha(H)\cdot X\ \forall H \in \mathfrak{h}\}$$

is a root space of $\alpha \in \Pi$. Then the **Borel subalgebra** $\mathfrak{b}$ is the maximal solvable subalgebra, given by

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha.$$

Definition 1.2. *A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called* **parabolic** *when $\mathfrak{p}$ contains a Borel subalgebra.*

If $H \in \mathrm{cl}(\mathfrak{a}^+)$, then $\Theta_H \subset \Sigma$ is defined by

$$\Theta_H = \{\alpha \in \Sigma : \ \alpha(H) = 0\}, \tag{1.3}$$

that is, the set of simple roots vanishing at H. Conversely, given $\Theta \subset \Sigma$ we can choose an element $H \in \mathrm{cl}(\mathfrak{a}^+)$, such that $\Theta = \Theta_H$, where Θ_H is as in equation 1.3. In particular

$$H = \sum_{\beta \in \Sigma \backslash \langle \Theta \rangle} H_\beta.$$

Then the parabolic subalgebra generated by H (or by $\Theta_H \subset \Sigma$) is defined by

$$\mathfrak{p}_H = \mathfrak{h} \oplus \sum_{\alpha(H) \geqslant 0} \mathfrak{g}_\alpha, \tag{1.4}$$

or equivalently

$$\mathfrak{p}_H = \mathfrak{h} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \langle \Theta \rangle^-} \mathfrak{g}_\alpha, \tag{1.5}$$

such that $\langle \Theta \rangle$ is the set of roots determined by linear combinations of elements on Θ and $\langle \Theta \rangle^{\pm} = \langle \Theta \rangle \cap \Pi^{\pm}$.

Remark 1.3. *In some papers, the parabolic subalgebra defined in the equation 1.5 is denoted by $\mathfrak{p}_\Theta$.*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The **parabolic subgroup** P_H is the normalizer of $\mathfrak{p}_H$ in G, that is

$$P_H = \{g \in G : \mathrm{Ad}(g) \cdot \mathfrak{p}_H = \mathfrak{p}_H\}.$$

Then the complex flag manifold associated to H is $\mathbb{F}_H$, and it is given by the quotient G/P_H.

Furthermore, we will see that the complex flag manifold can be seen as an adjoint orbit of a compact Lie group. For instance, choose a Weyl basis given by H_α for $\alpha \in \Sigma$ and $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Pi$, which satisfies

- $[X_\alpha, X_{-\alpha}] = H_\alpha$,

- $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, such that $m_{\alpha,\beta} = 0$ when $\alpha + \beta$ is not a root and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$.

If $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$, we have that

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{iH_\alpha, A_\alpha, S_\alpha : \ \alpha \in \Pi^+\}$$

is a compact real form of $\mathfrak{g}$.

Let $U = \exp \mathfrak{u}$ be a compact real form of G, then we set

$$U_H = P_H \cap U.$$

The adjoint action of the Lie group U is transitive on $\mathbb{F}_H$ with isotropy subgroup at H is U_H and therefore we have that

$$\mathbb{F}_H \simeq U/U_H \simeq \mathrm{Ad}(U) \cdot H.$$

Additionally, we will denote by $b_H = 1 \cdot U_H$ the origin of the flag $\mathbb{F}_H$, then its tangent space at b_H is

$$T_{b_H}\mathbb{F}_H = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha : \alpha(H) > 0\} = \sum_{\alpha(H)>0} \mathfrak{u}_\alpha$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha\}$.

1.1.2 Real flag manifolds

Let $\mathfrak{g}$ be a semisimple non-compact real Lie algebra. To see the construction of real flags we will consider the following elements of real semisimple Lie theory:

- Let θ be a **Cartan involution**, that is, its associated bilinear form

  $$B_\theta(X, Y) = -\langle X, \theta Y\rangle_{\mathfrak{g}} \quad X, Y \in \mathfrak{g}$$

  is an inner product on $\mathfrak{g}$, where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is the Cartan–Killing form of $\mathfrak{g}$. The Cartan involution induces a **Cartan decomposition** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, such that

  $$\mathfrak{k} = \{X \in \mathfrak{g} : \ \theta X = X\} \quad \text{and} \quad \mathfrak{s} = \{Y \in \mathfrak{g} : \ \theta Y = -Y\}.$$

  Then, the subspaces $\mathfrak{k}$ and $\mathfrak{s}$ are orthogonal with respect to B_θ and the Cartan–Killing form. In fact, $\mathfrak{k}$ is called the compact component of Cartan decomposition, but $\mathfrak{k}$ is not necessarily compact.

Furthermore, we define the maps $\kappa : \mathfrak{g} \to \mathfrak{k}$ and $\sigma : \mathfrak{g} \to \mathfrak{s}$, given by

$$\kappa(X) = \frac{X + \theta X}{2} \quad \text{and} \quad \sigma(X) = \frac{X - \theta X}{2}$$

the parallel projections of $\mathfrak{k}$ and $\mathfrak{s}$, respectively.

- Let $\mathfrak{a} \subset \mathfrak{s}$ be the **maximal Abelian subalgebra**. Then there is $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$. Moreover, $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

  Given a pair $(\theta, \mathfrak{a})$, take Π the set of roots associated to $(\theta, \mathfrak{a})$, where the roots are linear functionals $\alpha : \mathfrak{a} \to \mathbb{R}$, so that for any $\alpha \in \Pi$ there exists $H_\alpha \in \mathfrak{a}$ such that

$$B_\theta(H_\alpha, H) = \alpha(H), \quad H \in \mathfrak{a}.$$

  Those roots can be seen as a restriction of roots on $\mathfrak{h}_{\mathbb{C}}$ (see Chapter 14 of [28]).

- The Weyl chambers associated to $(\theta, \mathfrak{a})$ are the connected components of $\{H \in \mathfrak{a} : \alpha(H) \neq 0, \forall \alpha \in \Pi\}$. Choosing one of them as the **positive Weyl chamber** $\mathfrak{a}^+$, we can define the set of positive roots associated with respect to $\mathfrak{a}^+$ as $\Pi^+ = \{\alpha \in \Pi : \alpha|_{\mathfrak{a}^+} > 0\}$ and we can define

$$\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n}^- = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha},$$

  where $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ and $\theta \mathfrak{n} = \mathfrak{n}^-$. Consequently, there exists Σ a simple root system associated to $\mathfrak{a}^+$, such that $\{H_\alpha \in \mathfrak{a} : \alpha \in \Sigma\}$ is a basis of $\mathfrak{a}$.

  Moreover, $\mathfrak{s} = \mathfrak{a} \oplus \sigma(\mathfrak{n})$ is a B_θ-orthogonal decomposition.

The trio $(\theta, \mathfrak{a}, \mathfrak{a}^+)$ is called admissible trio of $\mathfrak{g}$, then we have the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

called the **Iwasawa decomposition**. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (for example, $G = \mathrm{Aut}_0 \mathfrak{g}$). If K, A and N are connected subgroups generated by $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively, then G is diffeomorphic to $K \times A \times N$, the decomposition $G = KAN$ is called the **global Iwasawa decomposition**.

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, then

$$\mathfrak{n}_H^+ = \sum_{\alpha(H) > 0} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n}_H^- = \sum_{\alpha(H) < 0} \mathfrak{g}_\alpha. \tag{1.6}$$

Remark 1.4. *When $H \in \mathfrak{a}^+$, that is, H is a regular element, then we have that $\mathfrak{n} = \mathfrak{n}_H^+$ and $\mathfrak{n}^- = \mathfrak{n}_H^-$. In some papers, the subalgebra $\mathfrak{n}_H^+$ is denoted by $\mathfrak{n}_\Theta^+$, where $\Theta = \Theta_H$.*

Let $\mathfrak{g}(H)$ be a semisimple algebra generated by $\mathfrak{n}_H^+$ and $\mathfrak{n}_H^-$, as we saw above $\theta|_{\mathfrak{g}(H)}$ is a Cartan involution of $\mathfrak{g}(H)$, and has the Cartan decomposition $\mathfrak{g}(H) = \mathfrak{k}(H) \oplus \mathfrak{s}(H)$. Let $\mathfrak{a}(H)$ be a subalgebra generated by $\{H_\alpha : \ \alpha(H) \neq 0\}$, that is $\mathfrak{a}(H) = \mathfrak{g}(H) \cap \mathfrak{a}$ is its maximal Abelian subalgebra and there exists $\mathfrak{a}_H$, such that $\mathfrak{a}_H = \mathfrak{a} \ominus \mathfrak{a}(H)$. Then we can define

$$\mathfrak{k}_H = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_H) \tag{1.7}$$

the centralizer of $\mathfrak{a}_H$ in $\mathfrak{k}$.

Definition 1.5. *Let $(\theta, \mathfrak{a}, \mathfrak{a}^+)$ be an admissible trio of $\mathfrak{g}$, then the* **parabolic subalgebra associated to** *$H \in \mathrm{cl}(\mathfrak{a}^+)$ is*

$$\mathfrak{p}_H = \mathfrak{k}_H \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Moreover, if G is a connected Lie group with Lie algebra $\mathfrak{g}$, the **parabolic subgroup associated to** *H is the normalizer of $\mathfrak{p}_H$ in G.*

By the global Iwasawa decomposition $G = K \cdot A \cdot N$, we have that

$$K_H = \{k \in K : \ \mathrm{Ad}(k)|_{\mathfrak{a}_H} = \mathrm{id}_{\mathfrak{a}_H}\} \tag{1.8}$$

thus

$$P_H = K_H \cdot A \cdot N. \tag{1.9}$$

Therefore

$$G/P_H = \frac{K \cdot A \cdot N}{K_H \cdot A \cdot N} \simeq K/K_H,$$

and we can conclude that

$$K/K_H \simeq \mathrm{Ad}(K) \cdot H.$$

Remark 1.6. *Given $\widetilde{H} \in \mathfrak{s}$, we have that $\mathrm{Ad}(K) \cdot \widetilde{H} \cap \mathrm{cl}(\mathfrak{a}^+) \neq \emptyset$, and as the action of K is transitive, we can choose an element $H \in \mathrm{cl}(\mathfrak{a}^+)$ which determines the same manifold.*

Remark 1.7. *We will denote by $\mathbb{F}_H$ the flag manifolds passing through $H \in \mathrm{cl}(\mathfrak{a}^+)$ when there is not confusion about what compact group is acting on, in the other cases we will denote it as an adjoint orbit. In fact, in Section 2.2 we only denote by $\mathbb{F}_H$ the complex flags.*

1.2 Non-compact case

Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra. The adjoint orbit of a connected Lie group G with Lie algebra $\mathfrak{g}$ has additional structures to be described by some tools of Lie theory. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and its respective Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, which induces a global Iwasawa decomposition $G = KAN$. If $\mathfrak{a}^+$ is a positive Weyl chamber and $H \in \mathrm{cl}(\mathfrak{a}^+)$, then

$$\mathfrak{g} = \mathfrak{n}_H^+ \oplus \mathfrak{z}_H \oplus \mathfrak{n}_H^-,$$

where $\mathfrak{z}_H$ is the centralizer of H in $\mathfrak{g}$. Thus

$$Z_H = \{g \in G : \ \mathrm{Ad}(g) \cdot H = H\}$$

is the centralizer in G of H, whose Lie algebra is $\mathfrak{z}_H$. Therefore

$$\mathrm{Ad}(G) \cdot H = G/Z_H.$$

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, the adjoint orbit $\mathrm{Ad}(G) \cdot H$ is a C^∞-vector bundle over $\mathbb{F}_H = \mathrm{Ad}(K) \cdot H$ isomorphic to the cotangent bundle $T^*\mathbb{F}_H$. Moreover, the diffeomorphism is

$$\iota : \mathrm{Ad}(G) \cdot H \to T^*\mathbb{F}_H$$

such that

i. ι is equivariant with respect to the action of K, that is, for all $k \in K$,

$$\iota \circ \mathrm{Ad}(k) = \widetilde{k} \circ \iota$$

where $\widetilde{k}$ is the lifting to $T^*\mathbb{F}_H$ of the action of k on $\mathbb{F}_H$.

ii. The pullback of the canonical symplectic form on $T^*\mathbb{F}_H$ by ι is the KKS form on the adjoint orbit.

The diffeomorphism $\iota : \mathrm{Ad}(G) \cdot H \to T^*\mathbb{F}_H$ was proved on [11] as follows:

- Firstly, the authors proved that $\mathrm{Ad}(G) \cdot H$ is diffeomorphic to a vector bundle $\mathcal{V} \to K/K_H$ associated to the principal bundle $K \to K/K_H$, determined by the representation of K_H.

- After that, the authors proved that $\mathcal{V} \to K/K_H$ is isomorphic to $T^*\mathbb{F}_H$.

We are interested in the construction of the diffeomorphism between $\mathrm{Ad}(G) \cdot H$ and the C^∞-vector bundle over $\mathbb{F}_H$, because in chapter 4 we use a variation of this difeomorphism for our deformation diffeomorphism.

For that, the structure of vector bundle is associated with the principal bundle $K \to K/K_H$ with structural group K_H. Moreover, the adjoint representation of K_H on $\mathfrak{g}$ leaves invariant the subspace $\mathfrak{n}_H^+$, because if $k \in K_H$ then $\mathrm{Ad}(k)$ commutes with $\mathrm{ad}(H)$, and consequently $\mathrm{Ad}(k)$ takes eigenspaces of $\mathrm{ad}(H)$ into the same eigenspaces. Therefore $\mathrm{Ad}(k)$ leaves invariant $\mathfrak{n}_H^+$, then the restriction of Ad defines a representation ρ of K_H on $\mathfrak{n}_H^+$, and we can define the vector bundle $K \times_\rho \mathfrak{n}_H^+$ associated to the principal bundle $K \to K/K_H$. In fact, we have to emphasize the following details.

To begin with, the elements of $K \times_\rho \mathfrak{n}_H^+$ are equivalent classes of pairs (k, X) of the equivalence relation

$$(ka, \rho(a^{-1})X) \sim (k, X), \quad a \in K_H.$$

Then the group K acts on $K \times_\rho \mathfrak{n}_H^+$ by left translations.

Furthermore, the adjoint orbit $\mathrm{Ad}(G) \cdot H$ is a union of affine subspaces, that is the consequence of the global Iwasawa decomposition $G = KAN$, where $AN \subset P_H$ and the adjoint action of P_H on H is given by $\mathrm{Ad}(P_H) \cdot H = H + \mathfrak{n}_H^+$. Thus

$$\begin{aligned} \mathrm{Ad}(G) \cdot H &= \mathrm{Ad}(K)\left(H + \mathfrak{n}_H^+\right) \\ &= \bigcup_{k \in K} \mathrm{Ad}(k)\left(H + \mathfrak{n}_H^+\right). \end{aligned}$$

Therefore the diffeomorphism between $\mathrm{Ad}(G) \cdot H$ and $K \times_\rho \mathfrak{n}_H^+$ may be stated as follows.

Proposition 1.8. *The map $\gamma : \mathrm{Ad}(G) \cdot H \to K \times_\rho \mathfrak{n}_H^+$ defined by*

$$Y = \mathrm{Ad}(k)(H + X) \mapsto (k, X) \in K \times_\rho \mathfrak{n}_H^+$$

is a diffeomorphism satisfying

1. *γ is equivariant with respect to the actions of K.*

2. *γ maps fibers onto fibers.*

3. *γ maps the orbit $\mathrm{Ad}(K)\cdot H$ onto the zero section of $K\times_\rho \mathfrak{n}_H^+$.*

By the map γ we endow $\mathrm{Ad}(G)\cdot H$ with a structure of a vector bundle coming from $K\times_\rho \mathfrak{n}_H^+$, and its fibers are the affine subspaces $\mathrm{Ad}(k)(H+\mathfrak{n}_H^+)$ that have the vector space structure of $\mathrm{Ad}(k)(\mathfrak{n}_H^+)$. To conclude, we want to see an example that will be fundamental in the chapter 4.

Example 1.9. *Let $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$ with basis $\{H,S,A\}$ given by*

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad S=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

such that $[H,S]=2A$, $[H,A]=2S$ and $[A,S]=2H$.

This algebra is isomorphic to $\mathbb{R}^3$ and can be expressed in coordinates as $(x,y,z)=xH+yS+zA$ and has a Cartan decomposition $\mathfrak{k}\oplus\mathfrak{s}$ given by $\mathfrak{k}=\langle A\rangle\simeq\mathfrak{so}(2)$ and $\mathfrak{s}=\langle H,S\rangle$. The adjoint representation with respect to the basis $\{H,S,A\}$ are given by

$$t\cdot\mathrm{ad}(H)=\begin{pmatrix}0&0&0\\0&0&2t\\0&2t&0\end{pmatrix}\qquad t\cdot\mathrm{ad}(S)=\begin{pmatrix}0&0&-2t\\0&0&0\\-2t&0&0\end{pmatrix}$$

$$t\cdot\mathrm{ad}(A)=\begin{pmatrix}0&2t&0\\-2t&0&0\\0&0&0\end{pmatrix}.$$

Then by exponentials we have that

$$e^{t\cdot\mathrm{ad}(S)}=\begin{pmatrix}\cosh(2t)&0&-\sinh(2t)\\0&1&0\\-\sinh(2t)&0&\cosh(2t)\end{pmatrix},$$

$$e^{t\cdot\mathrm{ad}(H)}=\begin{pmatrix}1&0&0\\0&\cosh(2t)&\sinh(2t)\\0&\sinh(2t)&\cosh(2t)\end{pmatrix},$$

$$e^{t\cdot\mathrm{ad}(A)}=\begin{pmatrix}\cos(2t)&\mathrm{sen}(2t)&0\\-\mathrm{sen}(2t)&\cos(2t)&0\\0&0&1\end{pmatrix},$$

therefore, the adjoint orbit of $G=\mathrm{Sl}(2,\mathbb{R})$ passing through $X\in\mathfrak{s}$ is an hyperboloid.

1.3 Symplectic structures

As we have seen above, the adjoint orbit of a semisimple Lie group admits the KKS form. One of the most important results for adjoint orbits is the following (see [2]):

Theorem 1.10 (Konstant–Kirillov–Souriau). *Let $M = U/K$ be an homogeneous space of a compact semisimple Lie group U with Lie algebra $\mathfrak{u}$. If M admits a symplectic form ω, then K is the centralizer of some element $H \in \mathfrak{u}$ and hence, M can be identified with the adjoint orbit $\mathrm{Ad}(U) \cdot H$. Moreover, the form ω is the KKS form given by (1.2).*

Therefore, in the adjoint orbits of compact Lie groups there is only one possible symplectic form. When $\mathfrak{g}$ is a complex semisimple Lie algebra and G a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, the adjoint orbit of G admits at least two symplectic structures such as the KKS form and the imaginary part of the Hermitian metric $\mathcal{H}_\tau$ (see [28]), which is a symplectic form on $\mathfrak{g}$ (see [1] and [9]) and its restriction on $\mathrm{Ad}\,(G) \cdot H$ is a symplectic form. This symplectic form was studied on [10] and [11].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{u}$ its compact real form with Cartan involution τ, such that

$$\mathcal{H}_\tau(X,Y) = -\langle X, \tau Y\rangle_{\mathfrak{g}} \quad X, Y \in \mathfrak{g}$$

is a Hermitian form of $\mathfrak{g}$, where $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is the complex Cartan–Killing form of $\mathfrak{g}$. The imaginary part of $\mathcal{H}_\tau$ will be denoted by Ω^τ, that is

$$\Omega^\tau(\cdot,\cdot) = \mathrm{im}\,(\mathcal{H}_\tau(\cdot,\cdot)) \tag{1.10}$$

and will be called **symplectic Hermitian** form determined by τ, then

$$\begin{aligned} 2\mathrm{Re}\,(\mathcal{H}_\tau(X,Y)) &= 2\mathrm{Re}\,(-\langle X, \tau Y\rangle_{\mathfrak{g}}) \\ &= -2\mathrm{Re}\,(\langle X, \tau Y\rangle_{\mathfrak{g}}) \\ &= -\langle X, \tau Y\rangle_{\mathbb{R}}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is the Cartan–Killing form of the realification (sometimes called decomplexification) of $\mathfrak{g}$. As we have

$$B_\tau(X,Y) = -\langle X, \tau Y\rangle_{\mathbb{R}},$$

the inner product is identified with $2\mathrm{Re}\left(\mathcal{H}_\tau(X,Y)\right)$. Moreover, the multiplication by i on $\mathfrak{g}$ is an isometry with respect to B_τ, as follows

$$\begin{aligned} B_\tau(iX,iY) &= -\langle iX, \tau(iY)\rangle_{\mathbb{R}} \\ &= \langle iX, i\tau Y\rangle_{\mathbb{R}} \\ &= 2\mathrm{Re}\langle iX, i\tau Y\rangle_{\mathfrak{g}} \\ &= -2\mathrm{Re}\langle X, \tau Y\rangle_{\mathfrak{g}} \\ &= -\langle X, \tau Y\rangle_{\mathbb{R}} \\ &= B_\tau(X,Y). \end{aligned}$$

Hence

$$\begin{aligned} B_\tau(iX,Y) &= -\langle iX, \tau Y\rangle_{\mathbb{R}} \\ &= -i\langle X, \tau Y\rangle_{\mathbb{R}} \\ &= -2i\left(\mathrm{Re}\langle X, \tau Y\rangle_{\mathfrak{g}}\right) \\ &= 2\left(\mathrm{im}(-\langle X, \tau Y\rangle_{\mathfrak{g}})\right). \end{aligned}$$

Therefore, up to a factor of $\frac{1}{2}$ the symplectic Hermitian form is

$$\Omega^\tau(X,Y) = B_\tau(iX,Y) = -\langle iX, \tau Y\rangle_{\mathbb{R}}. \tag{1.11}$$

As $\mathfrak{g}$ is a complex Lie algebra, the tangent spaces $T_x\left(\mathrm{Ad}(G)\cdot H\right)$ of $\mathrm{Ad}(G)\cdot H$ are complex subspaces of $\mathfrak{g}$, since if $[A,x]$ is a tangent vector then $i[A,x]=[iA,x]$ is also a tangent vector. This implies that each adjoint orbit $\mathrm{Ad}(G)\cdot H$ is a complex manifold, as it is endowed with an almost complex structure (multiplication by i in each tangent space) which is integrable, simply because this almost complex structure is the restriction of a complex structure on $\mathfrak{g}$, because the Nijenhuis tensor vanishes (for instance, see [29, Chapter 14]). Then the pullback of the symplectic form Ω^τ by the inclusion $\mathrm{Ad}(G)\cdot H \hookrightarrow \mathfrak{g}$ defines a symplectic form on $\mathrm{Ad}(G)\cdot H$.

2 Lagrangian submanifolds given by Lie group actions

This chapter aims to study Lagrangian submanifolds given by the action of a Lie subgroups $L \subset G$ on the adjoint orbit $\mathrm{Ad}(G) \cdot H$. This method was studied on [8] where the authors found all Lagrangian submanifolds of $\mathbb{C}P^n$ (minimal flag manifold) determined by actions of compact Lie groups, and it was studied on [13] where the authors proved that there is a Lagrangian submanifold given by the diagonal action on the product of flag manifolds if and only if the product of flag manifolds is $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$, where Θ^* is the dual of Θ. To begin, we prove that the real flag manifolds can be seen as Lagrangian submanifolds of their respective complex flag and we give the complete classification of complex flag manifolds that admit, as Lagrangian submanifold, each real flag manifold determined by the different symmetric pairs, as we can see in [5]. After that, we prove that the complex flag manifolds are Lagrangian submanifolds of adjoint orbits of complex semisimple Lie groups with respect to the Hermitian symplectic form, as we can see in [6].

2.1 Isotropic orbits

In this section, we establish the tools that we use throughout this chapter to find Lagrangian submanifolds.

Let (M, ω) be a connected symplectic manifold and $G \times M \to M$ a Hamiltonian action of a Lie group G on M. If $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ its dual space, then there exists a smooth map $\mu : M \to \mathfrak{g}^*$, called the moment map, such that for all $X \in \mathfrak{g}$

$$dH_X = \iota_{\widetilde{X}}\omega,$$

where $\widetilde{X}$ is the Hamiltonian vector field associated to X given by

$$\widetilde{X}(x) = \frac{d}{dt}(e^{tX} \cdot x)\Big|_{t=0} \qquad x \in M,$$

and $H_X : M \to \mathbb{R}$ its Hamiltonian function, that satisfies

$$H_X(x) = \mu(x)(X) \qquad x \in M.$$

Furthermore, the moment map μ is called Ad^*-equivariant, when

$$\mu(g \cdot x) = \mathrm{Ad}^*(g)\mu(x) \qquad g \in G,\ x \in M.$$

We will assume that the moment map μ is Ad^*-equivariant. Let L be a Lie subgroup of G with Lie algebra $\mathfrak{l}$. Our interest is to describe those orbits $L \cdot x$, $x \in M$, of L that are Lagrangian submanifolds of M, or more generally isotropic. The following arguments use the moment map μ to give necessary and sufficient conditions for the orbit $L \cdot x$ to be isotropic. For $X, Y \in \mathfrak{g}$ we have

$$\widetilde{X} \cdot H_Y = \omega(\widetilde{X}, \widetilde{Y}) = -\widetilde{Y} \cdot H_X, \tag{2.1}$$

because if $p \in M$, then

$$\begin{aligned} \widetilde{X} \cdot H_Y(p) &= \widetilde{X}_p \cdot H_Y \\ &= (dH_Y)_p \left(\widetilde{X}_p\right) \\ &= \omega\left(\widetilde{Y}_p, \widetilde{X}_p\right) = -\widetilde{Y} \cdot H_X(p). \end{aligned}$$

Therefore, for $x \in M$, we have that

$$\widetilde{Y} \cdot H_X(x) = -\mu(x)\left([Y, X]\right), \tag{2.2}$$

because

$$\begin{aligned} \widetilde{Y} \cdot H_X(x) &= \left.\frac{d}{dt}\right|_{t=0} H_X\left(e^{tY}(x)\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mu\left(e^{tY} \cdot x\right)(X) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}\left(e^{tY}\right)^* \mu(x)(X) \\ &= \left(\mathrm{ad}(Y)^* \mu(x)\right)(X) \\ &= -\mu(x)[Y, X]. \end{aligned}$$

By (2.1) and (2.2)

$$\omega\left(\widetilde{X}(x), \widetilde{Y}(x)\right) = 0 \quad \text{iff} \quad \mu(x)\left([Y, X]\right) = 0.$$

On the other hand, the tangent space of $L \cdot x$ at x is given by

$$T_x(L \cdot x) = \{\widetilde{X}(x) : \ X \in \mathfrak{l}\},$$

then ω_x vanishes on $T_x(L \cdot x)$ if and only if $\mu(x)([Y, X]) = 0$ for all $X, Y \in \mathfrak{l}$.

Proposition 2.1. *An orbit Lx is isotropic if and only if $\mu(x)$ belongs to the annihilator $(\mathfrak{l}')^0$ of the derived algebra $\mathfrak{l}'$ of $\mathfrak{l}$.*

Proof. Take $y \in L \cdot x$, as seen above $T_y(L \cdot x) = T_y(L \cdot y)$ is isotropic if and only if $\mu(y)[Y, X] = 0$ for all $X, Y \in \mathfrak{l}$. But, as $[Y, X] \in \mathfrak{l}'$, then

$$\mu(y)[Y, X] = 0 \quad \text{iff} \quad \mu(y) \in (\mathfrak{l}')^\circ,$$

or equivalently $\mu(x) \in (\mathfrak{l}')^\circ$, because $y = g \cdot x$ for some $g \in L$.

Therefore $\mu(y) = \mathrm{Ad}(g)^*\mu(x)$ and $\mu(y)$ annihilates $\mathfrak{l}'$ if and only if

$$\mu(x)\left(\mathrm{Ad}(g) \cdot \mathfrak{l}'\right) = \mu(x) \cdot (\mathfrak{l}') = 0,$$

because $\mathfrak{l}'$ is invariant by automorphism on $\mathfrak{l}$. □

Remark 2.2. *If G be a compact Lie group. Then, there exists an invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$, and the moment map can be interpreted as a map with values in $\mathfrak{g}$. For instance, the set $(\mathfrak{l}')^\circ$ becomes the orthogonal complement of $\mathfrak{l}'$ with respect to the inner product. Therefore $L \cdot x$ is isotropic if and only if $\mu(x) \in (\mathfrak{l}')^\perp$, with respect to the inner product $(\cdot, \cdot)$.*

When G is a semisimple Lie group, we can replace the inner product by the Cartan–Killing form.

As we saw in Section 1.2, if U is a compact semisimple Lie group with Lie algebra $\mathfrak{u}$, then the adjoint orbits of U in $\mathfrak{u}$ are the flags of the complex group $U_{\mathbb{C}}$ which has the Lie algebra $\mathfrak{u}_{\mathbb{C}}$.

The Konstant–Kirillov–Souriau (KKS) symplectic form on $\mathrm{Ad}(U) \cdot H$ is given by

$$\omega_x\left(\widetilde{X}(x), \widetilde{Y}(x)\right) = \langle x, [X, Y]\rangle_{\mathfrak{u}} \quad X, Y \in \mathfrak{u}, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$ is the Cartan–Killing form on $\mathfrak{u}$ and $\widetilde{X} = \mathrm{ad}(X)$ is the Hamiltonian vector field of $H_X(x) = \langle x, X\rangle_{\mathfrak{u}}$. Therefore, the moment map μ of the U-adjoint action is the identity map, which is (evidently) equivariant.

Then we can use the Proposition 2.1 to determine some isotropic submanifolds of $\mathrm{Ad}(U) \cdot H$.

Example 2.3. *Choose* $\mathfrak{u} = \mathfrak{su}(3)$ *and* $H = i\,\mathrm{diag}(2, -1 - 1)$*. Then*

$$\mathfrak{u}_H = \{X \in \mathfrak{u} : \ [H, X] = 0\} = \left\{ \begin{pmatrix} -it & 0 \\ 0 & A \end{pmatrix} : \ A \in \mathfrak{u}(2), \quad \text{and} \quad t = \mathrm{tr}\, A \right\},$$

is a Lie subalgebra of $\mathfrak{u}$*. Therefore*

$$(\mathfrak{u}'_H)^{\perp} = \left\{ \begin{pmatrix} 2it & z & w \\ -\overline{z} & -it & 0 \\ -\overline{w} & 0 & -it \end{pmatrix} : \ t \in \mathbb{R} \ e \ z, w \in \mathbb{C} \right\}. \tag{2.4}$$

Let $H_0 = \mathrm{diag}\,\{1, 0, -1\}$ *be a regular element on* $\mathfrak{u}$*. Then* $(\mathfrak{u}'_H)^{\perp} \cap \mathrm{Ad}\,(U)\,(iH_0)$ *is the set of matrices*

$$X = \begin{pmatrix} 2it & z & w \\ -\overline{z} & -it & 0 \\ -\overline{w} & 0 & -it \end{pmatrix}$$

with eigenvalues 0 *and* $\pm i$*. Since* $\mathrm{Ad}\,(U)\,(iH_0) = \mathbb{F}_3(1, 2)$ *is the set of matrices in* $\mathfrak{su}\,(3)$ *which have the same eigenvalues as* iH_0*. As 0 is an eigenvalue of* $X \in (\mathfrak{u}'_H)^{\perp} \cap \mathrm{Ad}\,(U)\,(iH_0)$*, we have that*

$$\det X = -it\left(2t^2 + |w|^2 + |z|^2\right)$$

which vanishes if and only if $t = 0$*, because* $t \in \mathbb{R}$*. Thus, the characteristic polynomial of* X *becomes*

$$\lambda^3 + \left(|w|^2 + |z|^2\right)\lambda.$$

This implies that $|w|^2 + |z|^2 = 1$*, then* $(\mathfrak{u}'_H)^{\perp} \cap \mathrm{Ad}\,(U)\,(iH_0)$ *is formed by the matrices satisfying*

$$|w|^2 + |z|^2 = 1,$$

describing the sphere S^3 *in the space of matrices* $\simeq \mathbb{C}^2$*. And*

$$\dim S^3 = 3 = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{F}_3(1, 2),$$

that is, S^3 *is a Lagrangian submanifold of* $\mathbb{F}_3(1, 2)$*.*

The Lagrangian submanifold S^3 in the example above can be seen as a real flag of $\mathfrak{su}(1, 2)$. In particular, it is the classic example of the Lagrangian immersion of real flags in complex flags. In the following subsection we will classify all those possible immersions.

2.2 Lagrangian immersion of real flags on complex flags

Let U be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$ and $\mathfrak{k} \subset \mathfrak{u}$ a Lie subalgebra. We say that $(\mathfrak{u}, \mathfrak{k})$ is a **symmetric pair** if

$$[\mathfrak{k}, \mathfrak{k}^{\perp}] \subset \mathfrak{k}^{\perp} \quad \text{and} \quad [\mathfrak{k}^{\perp}, \mathfrak{k}^{\perp}] \subset \mathfrak{k},$$

where $\perp$ is with respect to the Cartan–Killing form on $\mathfrak{u}$.

In particular, given the symmetric pair $(\mathfrak{u}, \mathfrak{k})$ and $K = \langle \exp \mathfrak{k} \rangle$, then U/K is a symmetric space. The **dual symmetric pair** is $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a non-compact semisimple Lie algebra (real form of $\mathfrak{u}^{\mathbb{C}}$) with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, such that $\mathfrak{s} = i\mathfrak{k}^{\perp} \subset \mathfrak{u}^{\mathbb{C}}$.

By the section 1.1.2, the K-isotropy representation orbits on $\mathfrak{s}$ (or $\mathfrak{k}^{\perp}$) are the flag manifolds of $\mathfrak{g}$. For $H \in \mathfrak{k}^{\perp}$ we have the usual construction of Lagrangian immersion of real flags on the (corresponding) complex flag in the following sense: Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal Abelian subalgebra, then there exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, such that $\mathfrak{a} \subset \mathfrak{h}$ and $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$ (in fact, $\mathfrak{a} \subset \mathfrak{h}_{\mathbb{R}}$).

Thus if $H \in \mathfrak{a}$:

$$K/K_H = \mathrm{Ad}(K) \cdot H \subset \mathrm{Ad}(U) \cdot iH = U/U_H = \mathbb{F}_H. \tag{2.5}$$

Therefore the flags of $\mathfrak{g}$ are determined by the adjoint action of K through H and are immersed in the flags of $\mathfrak{g}_{\mathbb{C}}$ given by the adjoint action of U through iH. Moreover, as $\mathfrak{u}$ is compact we have that $(\mathfrak{k}')^{\circ}$ corresponds to the orthogonal complement of $\mathfrak{k}'$ with respect to the invariant scalar product of $\mathfrak{u}$. Then we can conclude:

Proposition 2.4. *Given a symmetric pair $(\mathfrak{u}, \mathfrak{k})$ and $H \in \mathfrak{a} \subset i\mathfrak{k}^{\perp}$, the real flag manifold $\mathrm{Ad}(K) \cdot H$ is a Lagrangian submanifold of $\mathbb{F}_H$ with respect to the KKS form.*

Proof. Since $\mathfrak{k}' \subset \mathfrak{k}$, then $\mathfrak{k}^{\perp} \subset (\mathfrak{k}')^{\perp}$ and $\mathrm{Ad}(K) \cdot H \subset \mathfrak{k}^{\perp} = i\mathfrak{s}$, then $\mathrm{Ad}(K) \cdot H \cap (\mathfrak{k}')^{\perp} \neq \varnothing$ and by Proposition 2.1 the adjoint K-orbit (real flag) is an isotropic submanifold.

Furthermore, if $b_H = 1 \cdot K$, we have that

$$\dim \left(T_{b_H} \mathrm{Ad}(K) \cdot H\right) = \dim \left(\sum_{\alpha(H)<0} \mathfrak{g}_{\alpha} \right) = \# \left\{ \alpha \in \Pi_{\mathbb{C}} : \ \alpha(H) < 0 \right\},$$

and as the root spaces of $\mathfrak{g}_{\mathbb{C}}$ are 1-dimensional complex spaces (i.e., 2-dimensional real spaces), then

$$2\dim_{\mathbb{R}}\left(\mathrm{Ad}(K)\cdot H\right) = \dim_{\mathbb{R}}(\mathbb{F}_H).$$

Hence $\mathrm{Ad}(K)\cdot H$ is a Lagrangian submanifold of $\mathbb{F}_H$. □

Now, our interest is to determine the complex flags of $\mathfrak{g}_{\mathbb{C}}$ that admit, as Lagrangian submanifold, a real flag given by the action of $K = \langle \exp \mathfrak{k} \rangle$ for the symmetric pair $(\mathfrak{u}, \mathfrak{k})$. Take $\mathfrak{a} \subset \mathfrak{s}$ a maximal Abelian subalgebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$. Take $\Pi_{\mathbb{C}}$ the set of roots of $\mathfrak{h}_{\mathbb{C}}$ such that the roots of $\mathfrak{a}$ are the restrictions on $\mathfrak{h}_{\mathbb{C}}$. If θ is a Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, then exists an involutive extension of θ in $\mathfrak{g}_{\mathbb{C}}$, which will also be denoted by θ.

Therefore as we can seen in [28], the restriction of $\Pi_{\mathbb{C}}$ on $\mathfrak{a}$ is given by

$$P = \frac{1}{2}\left(1 - \theta^*\right), \quad \text{where} \quad \theta^*\alpha = \alpha \circ \theta.$$

Let $\Pi_{\mathrm{im}} \subset \Pi_{\mathbb{C}}$ be the set of **imaginary roots**, such that $\alpha \in \Pi_{\mathrm{im}}$ if and only if $P(\alpha) = 0$. Set $\Pi_{\mathrm{co}} = \Pi_{\mathbb{C}} \backslash \Pi_{\mathrm{im}}$, then the set of restricted roots is given by $P(\Pi_{\mathrm{co}})$. Given a adequate proper order (with respect to the lexicographic order in $\mathfrak{a}^*$), take Σ_{im} the system of imaginary simple roots and Σ_{co} its complement such that the projection of Σ_{co} on $\mathfrak{a}^*$ is a **system of restricted roots** Σ and $\mathfrak{a}^+$ the positive Weyl chamber of $\mathfrak{g}$ determined by Σ.

For $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$

$$\Theta_H = \{\beta \in \Sigma : \ \beta(H) = 0\} \subset \Sigma.$$

Define $\widetilde{\Theta}_H \subset \Sigma_{\mathbb{C}}$, given by

$$\widetilde{\Theta}_H = P^{-1}(\Theta_H) \cup \Sigma_{\mathrm{im}}, \tag{2.6}$$

i.e., $\widetilde{\Theta}_H$ is determined by the **Satake diagram** of $\mathfrak{g}$ (see [28]).

Proposition 2.5. $\widetilde{\Theta}_H = \{\alpha \in \Sigma_{\mathbb{C}} : \ \alpha(H) = 0\}$.

Proof. If $H \in \mathfrak{a}$, then for all $\alpha \in \Sigma_{\mathbb{C}}$

$$\theta^*\alpha(H) = \alpha \circ \theta(H) = -\alpha(H), \tag{2.7}$$

because $\theta|_{\mathfrak{s}} = -\,\mathrm{id}$. Also, if $\alpha \in \Sigma_{\mathrm{im}}$, then $\theta^*\alpha = \alpha$, and by (2.7) we have that $\alpha(H) = 0$, therefore it is enough to see for roots in Σ_{co}. If $\alpha \in P^{-1}(\Theta_H)$, then $(\alpha - \theta^*\alpha)(H) = 0$ implies that $\alpha(H) = \theta^*\alpha(H)$, and by (2.7) we have that $\alpha(H) = 0$. Thus $\widetilde{\Theta}_H \subseteq \{\alpha \in \Sigma_{\mathbb{C}} : \ \alpha(H) = 0\}$.

Conversely, if $\alpha \in \Sigma_{\mathrm{co}}$ such that $\alpha(H) = 0$, then $\theta^*\alpha(H) = -\alpha(H) = 0$, thus $P(\alpha)(H) = 0$ and implies that $P(\alpha) \in \Theta_H$, i.e. $\alpha \in P^{-1}(\Theta_H)$. □

Therefore,

Theorem 2.6. *Given a symmetric pair $(\mathfrak{u}, \mathfrak{k})$, the complex flags of $\mathfrak{u}_{\mathbb{C}}$ of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ admit, as Lagrangian submanifold, the real flag of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}^{\perp}$ of type $\Theta \subset \Sigma$ if and only if*

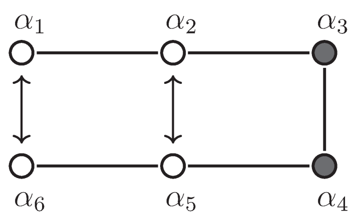
$$\widetilde{\Theta} = P^{-1}(\Theta) \cup \Sigma_{\mathrm{im}}.$$

That is, $\widetilde{\Theta}$ is determined by the Satake diagram of $\mathfrak{g}$.

In particular,

Corollary 2.7. *A maximal flag $\mathbb{F}$ of $\mathfrak{g}_{\mathbb{C}}$ admits a real flag $\mathrm{Ad}(K) \cdot H$ as Lagrangian submanifold if and only if $\Sigma_{\mathrm{im}} = \varnothing$ and $\varnothing = \Theta_H$.*

Example 2.8. *Let $\mathfrak{u} = \mathfrak{su}(7)$, $\mathfrak{k} = S(\mathfrak{u}(2) \times \mathfrak{u}(5))$ and $\mathfrak{g} = \mathfrak{su}(2,5)$ that determine the symmetric pair $(\mathfrak{u}, \mathfrak{k})$ and its respective dual symmetric pair $(\mathfrak{g}, \mathfrak{k})$. The Satake diagram of $\mathfrak{su}(2,5)$ is*



By Theorem 2.6, the flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ that admit as Lagrangian submanifold a real flag of type $\Theta \subset \Sigma = \{\beta_1 = P(\alpha_1) = P(\alpha_6), \beta_2 = P(\alpha_2) = P(\alpha_5)\}$ are

- *If $\Theta_0 = \varnothing$, then $\widetilde{\Theta}_0 = \Sigma_{\mathrm{im}} = \{\alpha_3, \alpha_4\}$.*
- *If $\Theta_1 = \{\beta_1\}$, then $\widetilde{\Theta}_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$.*
- *If $\Theta_2 = \{\beta_2\}$, then $\widetilde{\Theta}_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.*

Analogously, this is equivalent to that given in the table 1, for $n = 7$:

- $\widetilde{\Theta}_0 = \Sigma_{\mathbb{C}} \backslash \{\alpha_1, \alpha_2, \alpha_{n-2}, \alpha_{n-1}\}$,
- $\widetilde{\Theta}_1 = \Sigma_{\mathbb{C}} \backslash \{\alpha_2, \alpha_{n-2}\}$,
- $\widetilde{\Theta}_2 = \Sigma_{\mathbb{C}} \backslash \{\alpha_1, \alpha_{n-1}\}$.

Hence, using the Satake diagrams we can determine which are the complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$, for which there exists Θ such that Theorem 2.6 is satisfied.

Corollary 2.9. *The complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ admits as Lagrangian submanifold a real flag given by the K-adjoint orbit if and only if $\widetilde{\Theta}$ appears in Table 1.*

The proof of this result is given in the following subsection. For that we will use a convenient notation of partitioning an integer, that is, we define $\flat(n)$ for $n \in \mathbb{N}$, as the set of ordered l-tuples of integers $(n_1, \ldots, n_l)$ such that $0 < n_1 < \cdots < n_l \leqslant n$, for example:

$$\flat(3) = \{(1), (2), (3), (1,2), (1,3), (2,3), (1,2,3)\}.$$

Using this notation, we build the Table 1.

Remark 2.10. *In [5] we prove that this Lagrangian submanifolds are infinitesimally tight submanifolds. Tight submanifolds is detailed in Appendix A.*

2.2.1 Case by case

By the Satake diagrams, we can determine all the complex flags that admit the Lagrangian immersion of the corresponding real flag, given by the possible symmetric pairs. We will see the construction of the table 1, where for normal cases (also known as split Lie algebras): AI, CI, $G2$, $F4I$, $E6I$, $E7I$ and $E8I$ all possible $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ are admissible.

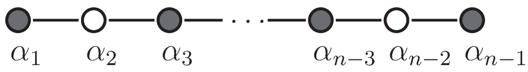
2.2.1.1 Classical algebras

Type AII

In this case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$, such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2n, \mathbb{C})$. Then the Satake diagram is

	$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{g}$	$\mathfrak{k}$	Flags type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$
A	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	All possibilities
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2n_1},\ldots,\alpha_{2n_j} : (n_1,\ldots,n_j) \in \flat(n-1)\right\}$
	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(k,n-k)$, $k < n$	$S\left(\mathfrak{u}(k)\times\mathfrak{u}(n-k)\right)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j},\alpha_{n-n_j},\ldots,\alpha_{n-n_1} : (n_1,\ldots,n_j) \in \flat(k)\right\}$
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}(n,n)$	$S\left(\mathfrak{u}(n)\times\mathfrak{u}(n)\right)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j},\alpha_n,\alpha_{2n-n_j},\ldots,\alpha_{2n-n_1} : (n_1,\ldots,n_j) \in \flat(n-1)\right\}$, or $\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j},\alpha_{2n-n_j},\ldots,\alpha_{2n-n_1} : (n_1,\ldots,n_j) \in \flat(n-1)\right\}$
B	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(n,n+1)$	$\mathfrak{so}(n)\oplus\mathfrak{so}(n+1)$	All possibilities
		$\mathfrak{so}(k,2n+1-k)$, $k \leqslant n$	$\mathfrak{so}(k)\oplus\mathfrak{so}(2n+1-k)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j} : (n_1,\ldots,n_j) \in \flat(k)\right\}$
C	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(n)$	All possibilities
		$\mathfrak{sp}(k,n-k)$, $k \leqslant n-k$	$\mathfrak{sp}(k)\times\mathfrak{sp}(n-k)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2n_1},\ldots,\alpha_{2n_j} : (n_1,\ldots,n_j) \in \flat(k)\right\}$
D	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n)\oplus\mathfrak{so}(n)$	All possibilities
		$\mathfrak{so}(k,2n-k)$, $k < n-1$	$\mathfrak{so}(k)\oplus\mathfrak{so}(2n-k)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j} : (n_1,\ldots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}(n-1,n+1)$,	$\mathfrak{so}(n-1)\oplus\mathfrak{so}(n+1)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j},\alpha_{n-1},\alpha_n : (n_1,\ldots,n_j) \in \flat(n-2)\right\}$, or $\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{n_1},\ldots,\alpha_{n_j} : (n_1,\ldots,n_j) \in \flat(n-2)\right\}$
		$\mathfrak{so}^*(2n)$, n even	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2n_1},\ldots,\alpha_{2n_j} : (n_1,\ldots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}^*(2n)$, n odd	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2n_1},\ldots,\alpha_{2n_j},\alpha_{n-1},\alpha_n : (n_1,\ldots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$, or $\Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2n_1},\ldots,\alpha_{2n_j} : (n_1,\ldots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$
E	E_6	E_6^6	$\mathfrak{sp}(4)$	All possibilities
		E_6^2	$\mathfrak{su}(2)\oplus\mathfrak{su}(6)$	See subsection 2.2.1.2
		E_6^{-14}	$\mathfrak{so}(10)\oplus\mathbb{R}$	See subsection 2.2.1.2
		E_6^{-26}	F_4	See subsection 2.2.1.2
	E_7	E_7^7	$\mathfrak{su}(8)$	All possibilities
		E_7^{-5}	$\mathfrak{su}(2)\oplus\mathfrak{so}(12)$	See subsection 2.2.1.2
		E_7^{-25}	$E_6\oplus\mathbb{R}$	See subsection 2.2.1.2
	E_8	E_8^8	$\mathfrak{so}(16)$	All possibilities
		E_8^{-24}	$\mathfrak{su}(2)\oplus E_7$	See subsection 2.2.1.2
F	F_4	F_4^4	$\mathfrak{su}(2)\oplus\mathfrak{sp}(3)$	All possibilities
		F_4^{-20}	$\mathfrak{so}(9)$	See subsection 2.2.1.2
G	G_2	G_2^2	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	All possibilities

Table 1 – Complex flags that admit a Lagrangian immersion of the real flag determined by the action of $K = \exp\mathfrak{k}$.



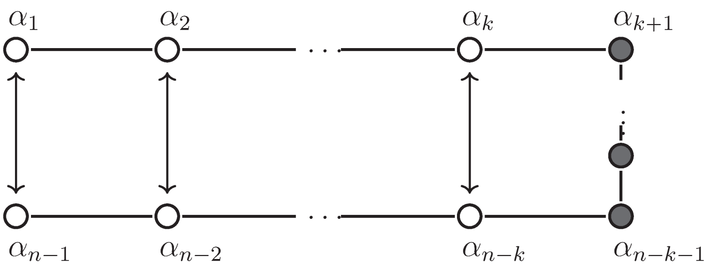
As $\Sigma_{\text{im}} = \{\alpha_{2j-1} : \ 1 \leqslant j \leqslant n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : \ 1 \leqslant j \leqslant n-1\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : \ (s_1, \ldots, s_l) \in \flat\,(n-1)\}. \tag{2.8}$$

Type $AIII$

In this case $\mathfrak{g} = \mathfrak{su}(k, n-k)$

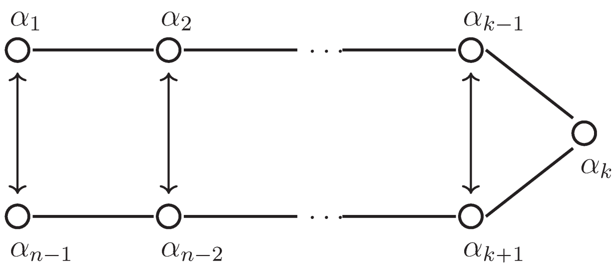
- If $k < n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : \ k < j < n-k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}) : \ 1 \leqslant j \leqslant k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : \ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.9}$$

- If $k = n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}), \ \beta_k = P(\alpha_k) : \ 1 \leqslant j \leqslant k-1\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

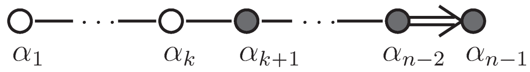
$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : \ (s_1, \ldots, s_l) \in \flat\,(k-1)\}, \tag{2.10}$$

or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : \ (s_1, \ldots, s_l) \in \flat\,(k-1)\}. \tag{2.11}$$

Type B

In this case $\mathfrak{g} = \mathfrak{so}(k, 2n+1-k)$, then the Satake diagram is



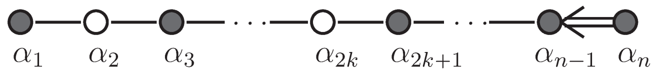
As $\Sigma_{\mathrm{im}} = \{\alpha_j : \ k < j \leqslant n\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : \ 1 \leqslant j \leqslant k\}$. If $k = n$ then $\mathfrak{g}$ is normal, but in general the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l} : \ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.12}$$

Type CII

In this case $\mathfrak{g} = \mathfrak{sp}(k, n-k)$.

- If $k < n-k$, the Satake diagram is



As $\Sigma_{\mathrm{im}} = \{\alpha_{2j-1}, \alpha_q : \ 1 \leqslant j \leqslant k, \ q > 2k\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : \ 1 \leqslant j \leqslant k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : \ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.13}$$

- If $n = 2m$ and $k = m$, the Satake diagram is

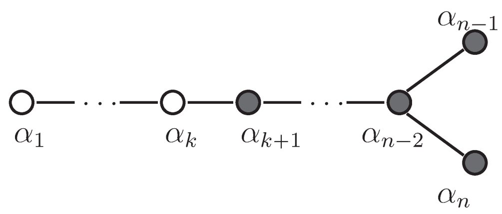
As $\Sigma_{\mathrm{im}} = \{\alpha_{2j-1} : \ 1 \leqslant j \leqslant m\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : \ 1 \leqslant j \leqslant m\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : \ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.14}$$

Type DI

In this case $\mathfrak{g} = \mathfrak{so}(k, 2n-k)$.

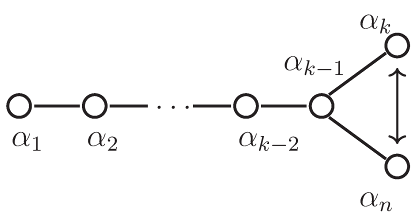
- If $k = n$ then $\mathfrak{g}$ is a normal form.
- If $k < n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : \ j > k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : \ 1 \leqslant j \leqslant k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l} : \ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.15}$$

- If $k = n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j), \ \beta_k = P(\alpha_k) = P(\alpha_n) : \ 1 \leqslant j < k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l} : \ (s_1, \ldots, s_l) \in \flat\,(k-1)\}, \tag{2.16}$$

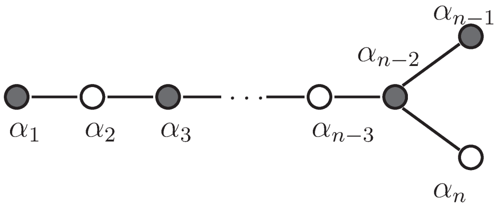
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_n : \ (s_1, \ldots, s_l) \in \flat\,(k-1)\}. \tag{2.17}$$

Type DII

In this case $\mathfrak{g} = \mathfrak{so}^*(2n)$.

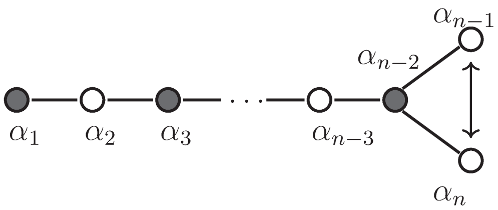
- If n is even, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j :\ j \text{ is odd}\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) :\ 1 \leqslant j \leqslant n\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} :\ (s_1, \ldots, s_l) \in \flat\,(k)\}. \tag{2.18}$$

- If n is odd, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j :\ j \text{ is odd and } j < n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}),\ \beta_k = P(\alpha_{n-1}) = P(\alpha_n) :\ 1 \leqslant j \leqslant k,\ k = (n-1)/2\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l} :\ (s_1, \ldots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}, \tag{2.19}$$

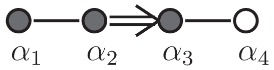
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \backslash \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l}, \alpha_{n-1}, \alpha_n :\ (s_1, \ldots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}. \tag{2.20}$$

2.2.1.2 Exceptional algebras

Type $F4II$

In this case $\mathfrak{g} = F_4^{-20}$, then the Satake diagram is

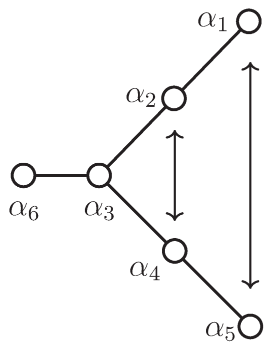


Therefore the only non-trivial possibility of $\widetilde{\Theta}$ that satisfies the Theorem 2.6 is

$$\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3\} = \Sigma_{\text{im}}. \tag{2.21}$$

Type $E6II$

In this case $\mathfrak{g} = E_6^2$, then the Satake diagram is

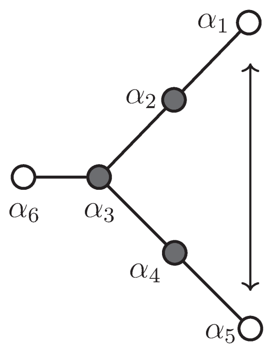


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \varnothing$,
- $\widetilde{\Theta} = \{\alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_3\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Type $E6III$

In this case $\mathfrak{g} = E_6^{-14}$, then the Satake diagram is

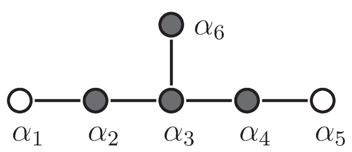


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}$.

Type $E6IV$

In this case $\mathfrak{g} = E_6^{-26}$, then the Satake diagram is

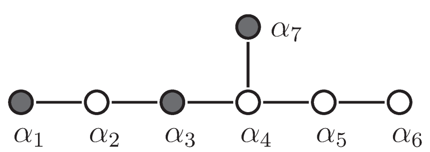


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3 \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \}$.

Type $E7II$

In this case $\mathfrak{g} = E_7^{-5}$, then the Satake diagram is

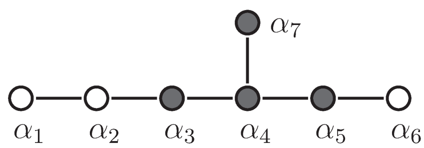


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$.
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$.

Type $E7III$

In this case $\mathfrak{g} = E_7^{-25}$, then the Satake diagram is

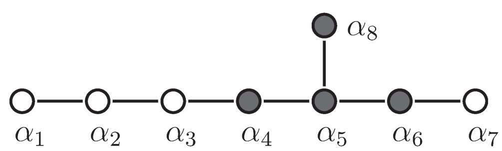


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\alpha_7\}$.

Type $E8II$

In this case $\mathfrak{g} = E_8^{-24}$, then the Satake diagram is



Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 2.6 are:

- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

2.3 Isotropic submanifolds on complex adjoint orbit

As we saw in Section 1.3, the adjoint orbit of a complex semisimple Lie group G with Lie algebra $\mathfrak{g}$ admits at least two symplectic structures. Let ω be a KKS symplectic form on $\mathrm{Ad}(G) \cdot H$, where H can be chosen in a positive Weyl chamber of $\mathfrak{g}$. To apply the Proposition 2.1, we have the following subgroups that are isotropic submanifolds:

- Let $L = U$ be a compact form of G, with Lie algebra $\mathfrak{u}$. Then U has isotropic orbit on $\mathrm{Ad}(G) \cdot H$ if and only if $H \in \mathfrak{s}$, since $(\mathfrak{u}')^{\perp} = \mathfrak{u}^{\perp} = i\mathfrak{u} = \mathfrak{s}$.

  Therefore, the orbit of U is isotropic on $\mathrm{Ad}(G) \cdot H$, for all $H \in \mathrm{cl}(\mathfrak{a}^+)$.

- Let $L = P$ be a parabolic subgroup of G, determined by the parabolic subalgebra $\mathfrak{p}$. Then, there exist $\Theta \subset \Sigma$ such that $P = P_\Theta$ and $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_\Theta^+$, such that $(\mathfrak{p}')^{\perp} = \mathfrak{h} \oplus \mathfrak{n}_\Theta^-$.

  Therefore, the orbit of P is isotropic on $\mathrm{Ad}(G) \cdot H$, for all $H \in \mathrm{cl}(\mathfrak{a}^+)$.

In fact, the problem is to determine the subgroups such that the orbits are Lagrangian. Next, we show that the adjoint action of U is Lagrangian with respect to the Hermitian symplectic form.

2.3.1 Adjoint action with respect to the Hermitian form

In this subsection we will apply the Proposition 2.1 to find Lagrangian submanifolds on the adjoint orbit with respect to the Hermitian symplectic form, when G is a complex semisimple Lie group. In fact, the action of U is symplectic in relation to Ω^τ because the adjoint representation of $\mathfrak{u}$ is anti-symmetric with respect to B_τ. We can describe the action in terms of the moment map in $\mathfrak{g}$, and we can specify it in an adjoint orbit. As we demonstrated in Section 1.3, the Hermitian symplectic form on $\mathfrak{g}$ can be seen as:

$$\Omega^\tau(X,Y) = B_\tau(iX,Y) = -\langle iX, \tau Y\rangle_{\mathbb{R}}.$$

For this, we will describe the action in terms of the moment map in $\mathfrak{g}$ and then specifying for the adjoint orbits. So for $A \in \mathfrak{u}$, define the bilinear form:

$$\beta_A(X,Y) = \Omega^\tau\left(\mathrm{ad}(A)\cdot X, Y\right) = B_\tau\left(i\,\mathrm{ad}(A)\cdot X, Y\right),$$

which we can see is symmetric:

$$\begin{aligned}
\beta_A(Y,X) &= B_\tau\left(i\,\mathrm{ad}(A)\cdot Y, X\right)\\
&= B_\tau\left(\mathrm{ad}(A)\cdot iY, X\right)\\
&= -B_\tau\left(iY, \mathrm{ad}(A)\cdot X\right)\\
&= B_\tau\left(Y, i\,\mathrm{ad}(A)\cdot X\right)\\
&= B_\tau\left(Y, \mathrm{ad}(A)\cdot iX\right)\\
&= B_\tau\left(\mathrm{ad}(A)\cdot iX, Y\right)\\
&= \beta_A(X,Y),
\end{aligned}$$

because $\mathrm{ad}(A)$ is anti-symmetric with respect to B_τ. Then define the quadratic form

$$Q(X) = \beta_A(X,X) = \Omega^\tau\left(\mathrm{ad}(A)X, X\right).$$

Proposition 2.11. *If $A \in \mathfrak{u}$ then $\mathrm{ad}(A)$ is a Hamiltonian field with Hamiltonian function $\frac{1}{2}Q(x)$.*

Proof. Let $\alpha(t)$ be any curve, then

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}Q\left(\alpha(t)\right)\right) &= \frac{d}{dt}\left(\frac{1}{2}\beta_A\left(\alpha(t),\alpha(t)\right)\right)\\
&= \beta_A\left(\alpha'(t),\alpha(t)\right)\\
&= \Omega^\tau\left(\mathrm{ad}(A)\cdot\alpha'(t),\alpha(t)\right).
\end{aligned}$$

therefore a vector field $x \mapsto \mathrm{ad}(A) \cdot x$ is Hamiltonian with function $\dfrac{1}{2}Q(x)$. $\square$

From this Hamiltonian function we can write the moment map $\mu : \mathfrak{g} \to \mathfrak{u}$, for $A \in \mathfrak{u}$:

$$\langle \mu(x), A \rangle_{\mathfrak{u}} = \frac{1}{2}Q(x), \tag{2.22}$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$ is the Cartan–Killing form of $\mathfrak{u}$. Therefore we have

$$\begin{aligned}
\langle \mu(x), A \rangle_{\mathfrak{u}} &= \frac{1}{2}Q(x) \\
&= \frac{1}{2}\Omega^{\tau}\left(\mathrm{ad}(A) \cdot x, x\right) \\
&= \frac{1}{2}B_{\tau}\left(i\,\mathrm{ad}(A) \cdot x, x\right) \\
&= -\frac{1}{2}\langle i\,\mathrm{ad}(A) \cdot x, \tau x \rangle_{\mathfrak{g}} \\
&= \frac{1}{2}\langle \mathrm{ad}(A) \cdot x, i\tau x \rangle_{\mathfrak{g}} \\
&= \frac{1}{2}\langle [x, A], \tau i x \rangle_{\mathfrak{g}} \\
&= \frac{1}{2}\langle A, [\tau i x, x] \rangle_{\mathfrak{g}},
\end{aligned}$$

Hence $\mu(x)$ is the orthogonal projection on $\mathfrak{u}$ of $[\tau i x, x]$, that is

$$\begin{aligned}
\mu(x) &= \frac{1}{2}\left([\tau i x, x] + \tau[\tau i x, x]\right) \\
&= \frac{1}{2}\left([\tau i x, x] + [x, i\tau x]\right) \\
&= \frac{1}{2}\left([\tau i x, x] - [x, \tau i x]\right) \\
&= [\tau i x, x] \in \mathfrak{u}.
\end{aligned}$$

Corollary 2.12. *The moment map μ for the adjoint action of U in $\mathfrak{g}$ (and thus for the action in each orbit $\mathrm{Ad}(G) \cdot H$) is given to $A \in \mathfrak{u}$ by*

$$\mu(x) = [\tau i x, x] = -i[\tau x, x] \in \mathfrak{u} \qquad x \in \mathfrak{g}.$$

From this expression for μ and [13, Prop. 4], it follows that the orbit $\mathrm{Ad}(U) \cdot x$ is isotropic for Ω^{τ} if and only if $[\tau x, x] = 0$, since $\mathfrak{u}$ is semisimple. Put another way, $\mathrm{Ad}(U) \cdot x$ is isotropic if and only if x commutes with τx.

Example 2.13. *Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we have*

$$\tau x = -x^{*} = \overline{x}^{T},$$

therefore the isotropic orbits are the orbits of normal transformations.

One case where the adjoint orbit $\mathrm{Ad}(U) \cdot H$ is isotropic is when $H \in \mathfrak{s} = i\mathfrak{u}$. In this case, $\mathrm{Ad}(U) \cdot H = \mathbb{F}_H$ is a flag manifold of $\mathfrak{g}$. Moreover, we have $\dim\left(\mathrm{Ad}(G) \cdot H\right) = 2\dim \mathbb{F}_H$, hence $\mathbb{F}_H$ is Lagrangian submanifold of $\mathrm{Ad}(G) \cdot H$ with respect to Ω^τ. Then

Theorem 2.14. *The only isotropic* $\mathrm{Ad}(U)$*-orbit in* $\mathrm{Ad}(G) \cdot H$ *is the flag manifold* $\mathbb{F}_H$*, it is the only orbit with dimension less or equal to* $\frac{1}{2}\dim\left(\mathrm{Ad}(G) \cdot H\right)$*.*

In fact, it is a Lagrangian submanifold.

Proof. It should be proved that if $0 \neq X \in \mathfrak{n}_H^+$, then the isotropy subgroup U_{H+X} on $H + X$ has a strictly smaller dimension than the dimension of U_H on H, as this shows that

$$\dim \mathrm{Ad}(U)(H + X) > \dim \mathbb{F}_H = \frac{1}{2}\dim\left(\mathrm{Ad}(G) \cdot H\right).$$

For this it is observed that if

$$\mathrm{Ad}(u)(H + X) = \mathrm{Ad}(u) \cdot H + \mathrm{Ad}(u) \cdot X = H + X$$

then $\mathrm{Ad}(u) \cdot H = H$ and $\mathrm{Ad}(u) \cdot X = X$. The first equality means that $U_{H+X} \subset U_H$. Take the torus $T_H = \mathrm{cl}\{e^{itH} : \ t \in \mathbb{R}\}$ which has dimension greater than 0, then $T_H \subset U_H$ but $\mathrm{Ad}(v) \cdot X \neq X$ for some $v \in T_H$ since $\mathrm{Ad}(T_H)$ has no fixed points in $\mathfrak{n}_H^+$, because the eigenvalues of $\mathrm{ad}(H)$ in $\mathfrak{n}_H^+$ are strictly positive.

This shows that $\mathrm{ad}(iH)$ is not in the isotropy algebra $H + X$ and in consequence $\dim U_{H+X} < \dim U_H$. $\square$

3 Coadjoint orbits of semi-direct products and their symplectic structures

This chapter presents one of our main results, with the objective of constructing a wider variety of Lagrangian submanifolds of any adjoint semisimple orbit. For this, we are going to change the usual structure of semisimple Lie algebras, i.e., with a new Lie bracket given by a convenient semi-direct product determined by a Cartan decomposition of this Lie algebra. This construction was inspired by [17].

In this way, the first part of this chapter is focused on the general construction of coadjoint orbits of this semi-direct structure. After that, we adapt those general results to the mentioned semi-direct product given by a Cartan decomposition. Therefore, given $\mathfrak{g}$ a semisimple Lie algebra, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, we can determine two types of coadjoint orbits: The usual semisimple and the semi-direct orbit. Finally, we study the Hermitian symplectic form in the semi-direct orbit for a complex semisimple case. These results are part of [6].

3.1 Coadjoint semi-direct orbit

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and take a representation $\rho : G \to \mathrm{Gl}(V)$ on a vector space V (with $\dim V < \infty$). The infinitesimal representation of $\mathfrak{g}$ on $\mathfrak{gl}(V)$ is also going to be denoted by ρ. The vector space V can be seen as an Abelian Lie group (or Abelian Lie algebra). In this way, we can take the semi-direct product $G \times_\rho V$ which is a Lie group whose underlying manifold is the cartesian product $G \times V$. This group is going to be denoted by G_ρ and its Lie algebra $\mathfrak{g}_\rho$ is the semi-direct product

$$\mathfrak{g}_\rho = \mathfrak{g} \times_\rho V.$$

The vector space of $\mathfrak{g}_\rho$ is $\mathfrak{g} \times V$ with bracket

$$[(X, v), (Y, w)] = ([X, Y], \rho(X) w - \rho(Y) v).$$

Our purpose is to describe the coadjoint orbit on the dual $\mathfrak{g}_\rho^*$ of $\mathfrak{g}_\rho$. To begin with, let's see how to determine the ρ-adjoint representation $\mathrm{ad}_\rho(X, v)$, where $(X, v) \in \mathfrak{g} \times_\rho V$.

Thus, take a basis of $\mathfrak{g} \times V$ denoted by $\mathcal{B} = \mathcal{B}_{\mathfrak{g}} \cup \mathcal{B}_V$ with $\mathcal{B}_{\mathfrak{g}} = \{X_1, \ldots, X_n\}$ and $\mathcal{B}_V = \{v_1, \ldots, v_d\}$ basis of $\mathfrak{g}$ and V, respectively. On this basis the matrix of $\mathrm{ad}_\rho (X, v)$ is given by

$$[\mathrm{ad}_\rho (X, v)]_{\mathcal{B}} = \begin{pmatrix} \mathrm{ad}\,(X) & 0 \\ A\,(v) & \rho\,(X) \end{pmatrix}, \tag{3.1}$$

where $\mathrm{ad}\,(X)$ is the adjoint representation of $\mathfrak{g}$ while for each $v \in V$, $A\,(v)$ is the linear map $\mathfrak{g} \to V$ defined by

$$A\,(v)\,(X) = \rho\,(X)\,(v)\,.$$

The dual space $\mathfrak{g}_\rho^*$ can be identified with $\mathfrak{g}^* \oplus V^*$, where $\mathfrak{g}^*$ is immersed on $(\mathfrak{g} \times V)^*$ by extensions of linear functionals on $\mathfrak{g}$ to $\mathfrak{g} \times V$ by the zero functional on V (in the same way, V^* is immersed on $(\mathfrak{g} \times V)^*$). Therefore, the dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \mathcal{B}_{\mathfrak{g}}^* \cup \mathcal{B}_V^*$, where $\mathcal{B}_{\mathfrak{g}}^*$ and $\mathcal{B}_V^*$ are the dual basis of $\mathcal{B}_{\mathfrak{g}}$ and $\mathcal{B}_V$, respectively. Then, the coadjoint representation $\mathrm{ad}_\rho^*(X, v)$, for $(X, v) \in \mathfrak{g}_\rho$, with respect to $\mathcal{B}^*$, is transpose with a negative sign on the off diagonal term of 3.1, that is

$$\left[\mathrm{ad}_\rho^*\,(X, v)\right]_{\mathcal{B}^*} = \begin{pmatrix} \mathrm{ad}^*\,(X) & -A\,(v)^* \\ 0 & \rho^*\,(X) \end{pmatrix}. \tag{3.2}$$

In this matrix, ad^* is the coadjoint representation of $\mathfrak{g}$, ρ^* is the dual representation of ρ, that is

$$\rho^*\,(X)\,\alpha = -\alpha \circ \rho\,(X)\,, \quad \alpha \in V^*,\ X \in \mathfrak{g}$$

and $A(v)^* : V^* \to \mathfrak{g}^*$ is the transpose of $A(v)$ for $v \in V$, which by the above equation can be seen as follows:

$$A\,(v)^*\,(\alpha)\,(X) = \alpha\,(A\,(v)\,(X)) = \alpha\,(\rho\,(X)\,(v)) = -\rho^*\,(X)\,(\alpha)\,(v)\,.$$

The adjoint representation Ad_ρ and coadjoint representation Ad_ρ^* of G_ρ are obtained by exponentials of representations in $\mathfrak{g}_\rho$. In particular, the following matrices are obtained (on the basis $\mathcal{B}$ and $\mathcal{B}^*$):

$$\left[e^{t\,\mathrm{ad}_\rho(0,v)}\right]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ tA\,(v) & 1 \end{pmatrix}, \qquad \left[e^{t\,\mathrm{ad}_\rho^*(0,v)}\right]_{\mathcal{B}^*} = \begin{pmatrix} 1 & -tA\,(v)^* \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

On the other hand, for $g \in G$ the restriction of $\mathrm{Ad}_\rho\,(g)$ to V coincides with $\rho\,(g)$ and the restriction of $\mathrm{Ad}_\rho^*\,(g)$ to V^* coincides with $\rho^*\,(g)$, where we are seeing V and V^* as subspaces of $\mathfrak{g}_\rho = \mathfrak{g} \oplus V$ and $\mathfrak{g}_\rho^* = \mathfrak{g}^* \oplus V^*$, respectively.

To describe the map $A(v)^*$ it is convenient to define the momentum map of the representation ρ.

Definition 3.1. *The* **momentum map of the representation** ρ *is the map*

$$\mu_\rho = \mu : V \otimes V^* \to \mathfrak{g}^*,$$

given by

$$\mu\left(v \otimes \alpha\right)\left(X\right) = \alpha\left(\rho\left(X\right)v\right) \qquad v \in V,\ \ \alpha \in V^*,\ \ X \in \mathfrak{g}. \tag{3.4}$$

Then

$$A\left(v\right)^* \alpha = \mu\left(v \otimes \alpha\right) \in \mathfrak{g}^*,$$

because we have the following identifications

$$A\left(v\right)^*\left(\alpha\right)\left(X\right) = \alpha\left(A\left(v\right)\left(X\right)\right) = \alpha\left(\rho\left(X\right)\left(v\right)\right) = \mu\left(v \otimes \alpha\right)\left(X\right).$$

Lemma 3.2. *The momentum map is equivariant, in the sense that we can exchange the representation $\rho \otimes \rho^*$ with the coadjoint representation, i.e., for $g \in G$, $v \in V$ and $\alpha \in V^*$*

$$\mu\left(\left(\rho(g)v\right) \otimes \left(\rho^*(g)\alpha\right)\right) = \mathrm{Ad}^*_\rho(g) \cdot \mu(v \otimes \alpha). \tag{3.5}$$

Proof. Let $g \in G$, as the restriction of $\mathrm{Ad}_\rho\left(g\right)$ and $\mathrm{Ad}^*_\rho\left(g\right)$ to V and V^* coincides with $\rho(g)$ and $\rho^*\left(g\right)$, respectively. For $X \in \mathfrak{g}$

$$\begin{aligned}
\mathrm{Ad}^*_\rho\left(g\right)\left(\mu(v \otimes \alpha)\right)\left(X\right) &= \left(A(v)^*\alpha\right)\left(\mathrm{Ad}_\rho\left(g^{-1}\right)\right)\left(X\right) \\
&= \alpha \cdot A(v)\left(\mathrm{Ad}_\rho\left(g^{-1}\right)\left(X\right)\right) \\
&= \alpha \cdot \rho\left(\mathrm{Ad}_\rho\left(g^{-1}\right)\left(X\right)\right)\left(v\right) \\
&= \alpha\left(\mathrm{Ad}_\rho\left(g^{-1}\right)\right)\left(\rho(X)\,\mathrm{Ad}_\rho(g)(v)\right) \\
&= \mathrm{Ad}^*_\rho(g)(\alpha)\left(\rho(X)\left(\mathrm{Ad}_\rho(g)(v)\right)\right) \\
&= \rho^*(g)(\alpha)\left(\rho(X)\left(\rho(g)(v)\right)\right) \\
&= \mu\left(\left(\rho(g)v\right) \otimes \left(\rho^*(g)\alpha\right)\right)\left(X\right). \quad \square
\end{aligned}$$

Additionally, the momentum map μ is bilinear in $V \times V^*$, then setting $\alpha \in V^*$ implies that the map $\mu_\alpha : V \to \mathfrak{g}^*$ given by $\mu_\alpha(v) = \mu\left(v \otimes \alpha\right)$ is a linear map and its image $\mu_\alpha\left(V\right)$ is a subspace of $\mathfrak{g}^*$. Let $\alpha \in V^*$, the coadjoint orbit of G_ρ through α will be denoted by

$$G_\rho \cdot \alpha := \mathrm{Ad}^*_\rho\left(G_\rho\right) \cdot \alpha,$$

that is, we are identifying the coadjoint representation of G_ρ as a G_ρ-action on $\mathfrak{g}_\rho^*$. The following proposition shows that the coadjoint orbit for $\alpha \in V^*$ is the union of subspaces $\mu_\beta(V)$.

Proposition 3.3. *For $\alpha \in V^*$, the coadjoint orbit is given by*

$$G_\rho \cdot \alpha = \bigcup_{\beta \in \rho^*(G)\alpha} \mu_\beta(V) \times \{\beta\} \subset \mathfrak{g}^* \times V^*,$$

and writing $\mathfrak{g}^* \times V^*$ as $\mathfrak{g}^* \bigoplus V^*$ (with the proper identifications)

$$G_\rho \cdot \alpha = \bigcup_{\beta \in \rho^*(G)\alpha} \beta + \mu_\beta(V),$$

where $\beta + \mu_\beta(V)$ is an affine subspace of $\mathfrak{g}^* \bigoplus V^*$.

Proof. Firstly, if $g \in G$ we can then identify $\mathrm{Ad}_\rho^*(g)$ with $\rho^*(g)$ on the subspace $V^* \subset \mathfrak{g}^* \times V^*$. Therefore $\rho^*(G)\,\alpha \subset \mathrm{Ad}_\rho^*(G_\rho)\,\alpha$, and as we saw before

$$\left[e^{t\,\mathrm{ad}_\rho^*(0,v)}\right]_{\mathcal{B}} = \begin{pmatrix} 1 & -tA(v)^* \\ 0 & 1 \end{pmatrix}$$

which shows that if $\beta \in V^* \subset \mathfrak{g}^* \bigoplus V^* = \mathfrak{g}^* \times V^*$, then

$$e^{t\,\mathrm{ad}_{\mathfrak{g}_\rho}^*(0,v)}\beta = \beta - tA(v)^*(\beta)$$

which in terms of the momentum map is

$$\beta - tA(v)^*(\beta) = \beta - \mu_\beta(tv).$$

Varying $v \in V$, shows that the affine subspace $\beta + \mu_\beta(V)$ is contained in the coadjoint orbit of β, for $\beta \in V^*$. Next to the fact that $\rho^*(G) \cdot \alpha \subset \mathrm{Ad}_\rho^*(G_\rho) \cdot \alpha$, we conclude that

$$\bigcup_{\beta \in \rho^*(G)\alpha} \beta + \mu_\beta(V) \subset G_\rho \cdot \alpha.$$

Conversely, if $g \in G$ and $\beta \in V^*$

$$\begin{aligned} \mathrm{Ad}_\rho^*(g)(\beta + \mu_\beta(V)) &= \rho^*(g)\beta + \mathrm{Ad}_\rho^*(g)\,\mu_\beta(V) \\ &= \rho^*(g)\beta + \mu_{\rho^*(g)\beta}(V) \end{aligned}$$

where the last equality is a consequence of the fact that μ is equivariant. For $h \in G_\rho$, there are $g \in G$ and $v \in V$, such that

$$\mathrm{Ad}_\rho^*(h)\,\alpha = \mathrm{Ad}_\rho^*(g)\,\mathrm{Ad}_\rho^*\left(e^{t(0,v)}\right)\alpha.$$

As $\mathrm{Ad}_\rho^*\left(e^{t(0,v)}\right)\alpha \in \alpha + \mu_\alpha(V)$ implies $\mathrm{Ad}_\rho^*(h)\,\alpha \in \rho^*(g)\,\alpha + \mu_{\rho^*(g)\alpha}(V)$. $\square$

The action of G_ρ is obviously transitive on $G_\rho \cdot \alpha$, then it is an homogeneous space given by

$$G_\rho \cdot \alpha = G_\rho / Z_\rho(\alpha) \quad \text{with} \quad Z_\rho(\alpha) = \{(g, v) \in G_\rho : \ (g, v) \cdot \alpha = \alpha\},$$

the isotropy subgroup at $\alpha \in V^* \subset \mathfrak{g}_\rho$, with Lie algebra

$$\mathfrak{z}_\rho(\alpha) = \{(X, v) \in \mathfrak{g}_\rho : \ \mathrm{ad}^*_\rho(X, v) \cdot \alpha = 0\}.$$

Then in terms of the basis $\mathcal{B}^*$

$$\mathrm{ad}^*_\rho(X, v) \cdot \alpha = -\mu_\alpha(v) + \ \rho^*(X)\alpha,$$

therefore

$$\mathrm{ad}^*_\rho(X, v) \cdot \alpha = 0 \ \Leftrightarrow \ \rho^*(X)\alpha = 0 \ \text{ and } \ \mu_\alpha(v) = 0.$$

Thus

$$\mathfrak{z}_\rho(\alpha) = \{X \in \mathfrak{g} : \ \rho^*(X)\alpha = 0\} \times \ker \mu_\alpha, \tag{3.6}$$

and

$$Z_\rho(\alpha) = \{g \in G : \ \rho^*(g)\alpha = \alpha\} \times \ker \mu_\alpha. \tag{3.7}$$

As $T_\alpha(G_\rho \cdot \alpha) \simeq \mathfrak{g}_\rho / \mathfrak{z}_\rho(\alpha)$, for any $(X, v) \in \mathfrak{g}_\rho$ we can compute a vector field $\widetilde{(X, v)}$ at $\xi = \beta + \mu_\beta(w) \in G_\rho \cdot \alpha$ with $\beta = \rho^*(g)\alpha$, $g \in G$ and $w \in V$, that is

$$\begin{aligned}
\widetilde{(X, v)}_\xi &= \frac{d}{dt} \left(\exp_\rho t(X, v)\right) \cdot \xi \, \bigg|_{t=0} \\
&= \frac{d}{dt} \mathrm{Ad}^*_\rho \left(e^{tX}, v\right) \cdot \xi \, \bigg|_{t=0} \\
&= \xi \circ \mathrm{ad}_\rho(X, v) \\
&= -\mathrm{ad}^*_\rho(X, v) \cdot \xi.
\end{aligned}$$

Hence

$$T_\alpha(G_\rho \cdot \alpha) = \{-\mathrm{ad}^*_\rho(X, v) \cdot \alpha : \ (X, v) \in \mathfrak{g}_\rho\}.$$

By Proposition 3.3, the coadjoint orbit $G_\rho \cdot \alpha$ is the union of vector spaces and fibers over $\rho^*(G)\, x$ of the representation ρ^*. This union is disjoint because given $\xi \in (\beta + \mu_\beta(V)) \cap (\gamma + \mu_\gamma(V))$ then

$$\xi = \beta + X = \gamma + Y \qquad X = \mu_\beta(v), \ Y = \mu_\gamma(w)$$

with $X, Y \in \mathfrak{g}$. Since the sum $\mathfrak{g}_\rho^* = \mathfrak{g}^* \bigoplus V^*$ is direct, it follows that $\beta = \gamma$ and $X = Y$. Therefore there is a fibration

$$G_\rho \cdot \alpha \to \rho^* (G) \alpha,$$

such that an element $\xi = \beta + X \in \beta + \mu_\beta (V)$ associates $\beta \in \rho^* (G) \alpha$, and its fibers are vector spaces. The following proposition shows that this fibration is the cotangent space of $\rho^* (G) \alpha$.

Proposition 3.4. *$G_\rho \cdot \alpha$ is diffeomorphic to the cotangent bundle $T^* (\rho^* (G) \alpha)$ of $\rho^* (G) \alpha$, by the diffeomorphism*

$$\phi : G_\rho \cdot \alpha \to T^* (\rho^* (G) \alpha),$$

that satisfies

$$\phi (\beta + \mu_\beta (V)) = T^*_\beta (\rho^* (G) \alpha), \quad \beta \in \rho^* (G) \alpha.$$

The restriction of ϕ to a fiber $\beta + \mu_\beta (V)$ is given by a linear isomorphism

$$\mu_\beta (V) \to T^*_\beta (\rho^* (G) \alpha).$$

Proof. Take $\xi \in G_\rho \cdot \alpha$, there is a unique $\beta \in \rho^* (G) \alpha$, such that $\xi \in \mu_\beta (V)$, then there is $v \in V$ with $\xi = \beta + \mu (v \otimes \beta)$. The vector $v \in V$ defines a linear functional f_v on V^*, and of course their respective restriction to $T_\beta (\rho^* (G) \alpha)$, therefore $f_v \in T^*_\beta (\rho^* (G) \alpha)$. Set

$$\phi (\xi) = f_v \in T^*_\beta (\rho^* (G) \alpha), \qquad \xi = \beta + \mu (v \otimes \beta).$$

A map ϕ is a linear injective map and the linear map $\mu (v \wedge w) \mapsto f_v$ is surjective. Furthermore, the restriction of ϕ to a fiber $w + \mu_w (V)$ is given by the isomorphism: $\mu_w (V) \to T^*_w (\rho (G) x)$. It follows that ϕ is a bijection.

Finally ϕ is diffeomorphism because both ϕ and ϕ^{-1} are differentiable as follows by construction: ϕ is the identity map at the base of the bundles of $\rho^* (G) x$ and ϕ is linear on the fibers. $\square$

In fact, the diffeomorphism ϕ of the Proposition 3.4 can be seen as a symplectomorphism. In the coadjoint orbit $G_\rho \cdot \alpha$ we can define the Konstant–Kirillov–Souriau (KKS) symplectic form, denoted by ω and defined as

$$\omega_\xi \left(\widetilde{(X_1, v_1)}_\xi, \widetilde{(X_2, v_2)}_\xi \right) = \xi \cdot [(X, w), (Y, z)]_\rho \quad (X_j, v_j) \in \mathfrak{g}_\rho,\ \xi \in G_\rho \cdot \alpha,$$

where $\widetilde{(X,v)} = \mathrm{ad}^*_\rho(X,v)$ is the Hamiltonian vector field of the function $H_{(X,v)} : M \to \mathbb{R}$ given by

$$H_{(X,v)}(\xi) = \xi(X,v) \quad (X,v) \in \mathfrak{g}_\rho.$$

In the same way for the cotangent bundle $T^*(\rho^*(G)\alpha)$ we can define the canonical symplectic form $\widetilde{\omega}$. The following proposition shows that these symplectic forms are related by ϕ.

Proposition 3.5. *Let ω and $\widetilde{\omega}$ be the KKS symplectic form on $G_\rho \cdot \alpha$ and the canonical symplectic form on $T^*(\rho(G)\alpha)$, respectively. If ϕ is the diffeomorphism of the Proposition 3.4, then $\phi^*\omega = \Omega$. In other words, the diffeomorphism ϕ is symplectic.*

The best way to relate these symplectic forms is through the action of the semi-direct product $G_\rho = G \times V$ on the cotangent bundle of $\rho^*(G)\alpha$. This action is described in Proposition 3.6 (in a general case), the action of G_ρ on $T^*(\rho(G)\alpha)$ is Hamiltonian and then it defines a moment map

$$m : T^*(\rho^*(G)\alpha) \to \mathfrak{g}^*_\rho.$$

The construction of m shows that it is the inverse of the diffeomorphism ϕ of the Proposition 3.4. Moreover, m is equivariant, that is, it interchanges the actions on $T^*(\rho(G)\alpha)$ and the adjoint orbit, which implies that m is a symplectic morphism.

Representations and symplectic geometry

Let $M \subset W$ be an immersed submanifold of the vector space W (real, that is, $W = \mathbb{R}^N$). The cotangent bundle $\pi : T^*M \to M$ is provided with the canonical symplectic form ω. Given a function $f : TM \to \mathbb{R}$ denote by X_f the corresponding Hamiltonian field, such that $df(\cdot) = \omega(X_f, \cdot)$. If $\alpha \in W^*$, the height function $f_\alpha : M \to \mathbb{R}$ is given by

$$f_\alpha(x) = \alpha(x)$$

and also denote by f_α its lifting $f_\alpha \circ \pi$ which is constant on the fibers of π. Denote by X_α the Hamiltonian field of this function. Since f_α is constant in the fibers, the field X_α is vertical and the restriction to the fiber T^*_xM is constant in the direction of the vector $(df_\alpha)_x \in T^*_xM$. Furthermore, if $\alpha, \beta \in W^*$, the vector fields X_α and X_β commutes. In terms of the action of Lie groups and algebras, the commutativity $[X_\alpha, X_\beta] = 0$ means that the map $\alpha \mapsto X_\alpha$ is an infinitesimal action of W^*, seen as an Abelian Lie algebra. This infinitesimal action can be extended to an action of W^* (seen as an Abelian Lie group because the fields X_α are complete).

Now, let $R : L \to \mathrm{Gl}(W)$ be a representation of the Lie group L on W and take an L-orbit given by $M = \{R(g)\,x : g \in L\}$. The action of G on M lifts to an action in the cotangent bundle T^*M by linearity. If $\mathfrak{l}$ is the Lie algebra of L, then the infinitesimal action of $\mathfrak{l}$ in the orbit M is given by the fields $y \in M \mapsto R(X)\,y$, where $X \in \mathfrak{l}$ and $R(X)$ also denotes the infinitesimal representation associated to R. The infinitesimal action of the lifting in T^*M is given by $X \in \mathfrak{l} \mapsto H_X$, where H_X is the Hamiltonian field on T^*M, such that the Hamiltonian function is $F_X : T^*M \to \mathbb{R}$ given by

$$F_X(\alpha) = \alpha(R(X)\,y) \qquad \alpha \in T_y^*M.$$

The actions of L and W^* in T^*M are going to define an action of the semi-direct product $L \times W^*$, defined by the dual representation R^*. The action of $L \times W^*$ on T^*M is Hamiltonian in the sense that the corresponding infinitesimal action of $\mathfrak{l} \times W^*$ is formed by Hamiltonian fields. When we have a Hamiltonian action we can define its moment map (See [29, Section 14.4]). In this case, a map

$$m : T^*M \to (\mathfrak{l} \times W^*)^* = \mathfrak{l}^* \times W.$$

In the action on T^*M, the field induced by $X \in \mathfrak{l}$ is the Hamiltonian field H_X of the function $F_X(\alpha) = \alpha(R(X)\,y)$, while the field induced by $\alpha \in W^*$ is the Hamiltonian field of the function f_α. So if $\gamma \in T_y^*M$, $y \in M \subset W$ then for $X \in \mathfrak{l}$ and $\alpha \in W^*$

$$m(\gamma)(X) = \gamma(R(X)\,y) \quad \text{and} \quad m(\gamma)(\alpha) = \alpha(y)$$

The first term coincides with the momentum $\mu : W \otimes W^* \to \mathfrak{l}^*$ of the representation R, that is, $m(\gamma) = \mu(y \otimes \overline{\gamma})$ such that the restriction of $\overline{\gamma} \in W^*$ to the tangent space T_yM is equal to γ. The second term shows that the linear functional $m(\gamma)$ restricted to W^* is exactly y. Consequently,

Proposition 3.6. *The moment map $m : T^*M \to \mathfrak{l}^* \times W = \mathfrak{l}^* \oplus W$ is given by*

$$m(\gamma_y) = \mu(y \otimes \overline{\gamma}) + y,$$

where $\gamma_y \in T_y^*M$ and $\overline{\gamma} \in W^*$, such that its restriction to $T_yM = \{R(X)\,y : X \in \mathfrak{l}^*\}$ is equal to γ.

3.1.1 Compact case

Let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$ and take a representation $\rho : U \to \mathrm{Gl}(V)$. We will denote by U_ρ the semi-direct Lie group $U \times_\rho V$, with Lie algebra $\mathfrak{u}_\rho = \mathfrak{u} \times_\rho V$. Since U is compact, then V admits a U-invariant inner product $\langle \cdot, \cdot \rangle$ when V is a real vector space and an Hermitian inner product when V is a complex vector space. That inner product allows us to identify V with V^* by

$$v \in V \mapsto \langle v, \cdot \rangle \in V^*,$$

and we can also identify ρ with ρ^* by

$$\rho^*(X)(v)(w) = -\langle \rho(X)w, v\rangle \quad v, w \in V,\ X \in \mathfrak{u}.$$

Now, analogously to the discussion for the general case we can characterize the coadjoint orbit of U_ρ in terms of the momentum map $\mu : V \otimes V \to \mathfrak{u}^*$ given by

$$\mu\,(v \otimes w)\,(X) = \langle \rho\,(X)\,v, w\rangle, \quad v, w \in V,\ X \in \mathfrak{u}.$$

Again, as U is compact we have that $\rho(u)$ is an isometry for all $u \in U$, then $\rho(X)$ is an skew-symmetric linear map with respect to $\langle \cdot, \cdot \rangle$ for all $X \in \mathfrak{u}$ and we have

$$\mu(v \otimes w)(X) = \langle \rho(X)v, w\rangle = -\langle \rho(X)w, v\rangle = -\mu(w \otimes v)(X),$$

that is, μ is anti-symmetric. Therefore, the momentum map μ is defined in the exterior product $\wedge^2 V = V \wedge V$. Furthermore, the compact Lie algebra $\mathfrak{u}$ admits an ad-invariant inner product such that we can identify $\mathfrak{u}^*$ with $\mathfrak{u}$, then

$$\mu : V \wedge V \to \mathfrak{u}.$$

Similarly, the dual $\mathfrak{u}^* \times V^*$ of $\mathfrak{u}_\rho = \mathfrak{u} \times V$ is identified by its inner product which is a direct sum of ad-invariant inner products of $\mathfrak{u}$ and V. In that identification, the coadjoint representation of $\mathfrak{u}$ can be seen as the adjoint representation of $\mathfrak{u}$ because its inner product is $\mathfrak{u}$-invariant, but the inner product of V is not invariant under the adjoint representation of V, then the coadjoint representation of that Abelian algebra is the transpose of its adjoint representation. This means that the coadjoint representation of $\mathfrak{u} \times V$ is written in $\mathfrak{u} \times V$ as type matrices on orthonormal bases:

$$\mathrm{ad}^*_\rho (X, v) = \begin{pmatrix} \mathrm{ad}\,(X) & -A\,(v) \\ 0 & \rho\,(X) \end{pmatrix} \qquad X \in \mathfrak{u},\ v \in V,$$

where for each $v \in V$, $A\,(v) : V \to \mathfrak{u}$ can be identified by $A\,(v)\,(w) = \mu\,(v \wedge w)$.

Then the representations Ad_ρ and Ad^*_ρ of U_ρ are obtained by exponentials of representations in $\mathfrak{u}_\rho$, take $v \in V \subset \mathfrak{u} \times V$ and by Theorem 3.3

$$U_\rho \cdot v := \mathrm{Ad}^*_\rho(U_\rho) \cdot v = \bigcup_{w \in \rho(U) v} w + A(w)\,(V)\,.$$

3.1.2 Examples on $\mathfrak{so}(n) \times \mathbb{R}^n$ and $\mathfrak{u}(n) \times \mathbb{C}^n$

We will see some interesting examples of semi-direct coadjoint orbits in order to compare them with the usual orbits. To begin with, take the canonical representation of $\mathfrak{u} = \mathfrak{so}\,(n)$ in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. The momentum map with values in $\mathfrak{u}$ is given by

$$\mu\,(v \wedge w)\,(B) = \langle Bv, w \rangle \qquad B \in \mathfrak{so}\,(n)\,,$$

and as we know the invariant inner product on $\mathfrak{so}(n)$ is

$$(A, B) = \mathrm{tr}\ AB^T = -\mathrm{tr}\ AB.$$

To describe the orbit, take the isomorphism $I : \wedge^2 \mathbb{R}^n \to \mathfrak{so}\,(n)$ given by

$$I\,(v \wedge w)\,(x) = \langle v, x \rangle w - \langle w, x \rangle v,$$

which satisfies

$$I\,(v \wedge w)^T = -I\,(v \wedge w) = I\,(w \wedge v)\,.$$

If $A \in \mathfrak{so}(n)$ we have

$$\begin{aligned} I\,(v \wedge w)\left(A^T x\right) &= \langle v, A^T x \rangle w - \langle w, A^T x \rangle v \\ &= \langle Av, x \rangle w - \langle Aw, x \rangle v. \end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis on $\mathbb{R}^n$, then

$$\begin{aligned}
\operatorname{tr}\left(I(v \wedge w) A^T\right) &= \sum_i \langle I(v \wedge w) A^T e_i, e_i \rangle \\
&= \sum_i \langle Av, e_i \rangle \langle w, e_i \rangle - \sum_i \langle Aw, e_i \rangle \langle v, e_i \rangle \\
&= \langle Av, \sum_i \langle w, e_i \rangle e_i \rangle - \langle Aw, \sum_i \langle v, e_i \rangle e_i \rangle \\
&= \langle Av, w \rangle - \langle Aw, v \rangle \\
&= 2\langle Av, w \rangle.
\end{aligned}$$

Therefore identifying V^* with $V = \mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and $\mathfrak{so}(n)^*$ with $\mathfrak{so}(n)$ by $\frac{1}{2}(\cdot, \cdot)$, the momentum map is $\mu(v \wedge w) = I(v \wedge w)$, that is

$$\mu(v \wedge w)(x) = \langle v, x \rangle w - \langle w, x \rangle v \qquad \mu(v \wedge w) \in \mathfrak{so}(n).$$

For simplicity of notation, we will denote $I(w \wedge v)$ for $v, w \in \mathbb{R}^n$ as $v \wedge w$. If v and w are $n \times 1$ column vectors, we have

$$v \wedge w = vw^T - wv^T$$

which is an $n \times n$ matrix.

As we saw above, the coadjoint representation of $\mathfrak{so}(n) \times \mathbb{R}$ is given by

$$\operatorname{ad}^*_\rho(B, v) = \begin{pmatrix} \operatorname{ad}(B) & -A(v) \\ 0 & B \end{pmatrix} \qquad B \in \mathfrak{so}(n), \ v \in \mathbb{R}^n,$$

where for each $v \in \mathbb{R}^n$, $A(v) : \mathbb{R}^n \to \mathfrak{so}(n)$ is the map

$$A(v)(w) = vw^T - wv^T.$$

The representation $\mathfrak{so}(n) \times \mathbb{R}^n$ defines a representation of the semi-direct product $U_\rho = \mathrm{SO}(n) \times \mathbb{R}^n$ on $\mathfrak{so}(n) \times \mathbb{R}^n$ by exponentials. As discussed earlier a U_ρ-orbit of $v \in \mathbb{R}^n \subset \mathfrak{so}(n) \times \mathbb{R}^n$ is given by

$$\bigcup_{w \in \mathcal{O}} w + A(w)(\mathbb{R}^n) \qquad \mathcal{O} = \mathrm{SO}(n) \cdot v.$$

In this case, the orbits of $\mathrm{SO}(n)$ in $\mathbb{R}^n$ are the $(n-1)$-dimensional spheres centered at the origin.

Example 3.7. *For $n = 2$, we have that $\mathfrak{so}(2) \times \mathbb{R}^2$ is isomorphic (as vector space) with $\mathbb{R}^3$ and for all $w \in \mathbb{R}^2$ the image $A(w)(\mathbb{R}^2) = \mathfrak{so}(2)$, therefore the coadjoint semi-direct orbits are the circular cylinders with axis on the line generated by $\mathfrak{so}(2)$ in $\mathfrak{so}(2) \times \mathbb{R}^2 \approx \mathbb{R}^3$.*

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{so}(n)$. We can induce the canonical representation of $\mathfrak{h}$ in $\mathbb{R}^n$ as a restriction on $\mathfrak{so}(n)$, then

$$\mu(v \wedge w)(B) = \langle Bv, w\rangle \qquad B \in \mathfrak{h},$$

because the inner product of $\mathbb{R}^n$ is invariant by $\mathfrak{h}$. The trace form $-\mathrm{tr}\, AB$ provides (by restriction) an inner invariant product in $\mathfrak{h}$, that allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Let $p : \mathfrak{so}(n) \to \mathfrak{h}$ be the orthogonal projection in relation with the trace form. By the identification above of $\mathfrak{h}^*$ and $\mathfrak{h}$ we can define the $\mathfrak{h}$-momentum map

$$\mu_{\mathfrak{h}} : \wedge^2\mathbb{R}^n \to \mathfrak{h} \quad \text{given by} \quad \mu_{\mathfrak{h}} = p \circ \mu,$$

where μ is the momentum map of $\mathfrak{so}(n)$. Then

$$\mu_{\mathfrak{h}}(v \wedge w) = p(vw^T - wv^T).$$

For $\mathfrak{u}(n)$, we can take a canonical representation in $\mathbb{C}^n = \mathbb{R}^{2n}$ and see $\mathfrak{u}(n)$ as an immersed subalgebra of $\mathfrak{so}(2n)$ by matrices $2n \times 2n$ of the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \qquad A^T = -A, \; B^T = B.$$

Then $\mathfrak{h} = \mathfrak{u}(n)$ and $p : \mathfrak{so}(2n) \to \mathfrak{u}(n)$ is the orthogonal projection with respect to the trace form. Hence the momentum map is

$$\mu_{\mathfrak{u}(n)}(z \wedge w) = p(zw^T - wz^T), \quad z, w \in \mathbb{R}^{2n}.$$

3.2 Coadjoint semi-direct orbit given by a Cartan decomposition

In this section, we will apply the results of the section 3.1 in the structure of any semisimple non-compact Lie algebra determined by a given Cartan decomposition. With this, we will determine a new coadjoint orbit of a semisimple algebra called the semi-direct orbit.

Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ given by the Cartan involution θ. As $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$, the subalgebra $\mathfrak{k}$ can be represented on $\mathfrak{s}$ by the adjoint representation, we can define the semi-direct product

$$\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times_{\rho} \mathfrak{s} \quad \text{where} \quad \rho = \mathrm{ad}|_{\mathfrak{k}},$$

where $\mathfrak{s}$ can be seen as an Abelian algebra. This is a new Lie algebra structure on the same vector space $\mathfrak{g}$ where the brackets $[X, Y]$ are the same when X or Y are in $\mathfrak{k}$, but the bracket changes when $X, Y \in \mathfrak{s}$. The identification between $\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times \mathfrak{s}$ and its dual $\mathfrak{k}^*_{\mathrm{ad}} = \mathfrak{k}^* \times \mathfrak{s}^*$ is given by the inner product

$$B_\theta(X, Y) = -\langle X, \theta Y\rangle \quad X, Y \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle$ is the Cartan–Killing form of $\mathfrak{g}$ and we have that $\mathrm{ad}(A)$ is skew-symmetric with respect to B_θ for $A \in \mathfrak{k}$ while $\mathrm{ad}(B)$ is symmetric with respect to B_θ for $B \in \mathfrak{s}$. Then in this situation the momentum map is given by

$$\mu(X \wedge Y)(A) = B_\theta(\mathrm{ad}(A) X, Y) \qquad A \in \mathfrak{k}; \; X, Y \in \mathfrak{s},$$

the second part of that equality is

$$B_\theta([A, X], Y) = -B_\theta([X, A], Y) = -B_\theta(A, [X, Y]) = -\langle A, [X, Y]\rangle$$

because $[X, Y] \in \mathfrak{k}$. Therefore the momentum map of the adjoint representation of $\mathfrak{k}$ on $\mathfrak{s}$ can be identified as

$$\mu(X \wedge Y) = [X, Y] \in \mathfrak{k} \qquad X, Y \in \mathfrak{s},$$

where $[\cdot, \cdot]$ is the usual bracket of $\mathfrak{g}$. Therefore, the coadjoint representation of the semi-direct product $\mathfrak{k} \times \mathfrak{s}$ in an orthonormal basis is given by

$$\mathrm{ad}^*_\rho(X, Y) = \begin{pmatrix} \mathrm{ad}(X) & \widehat{A}(Y) \\ 0 & \mathrm{ad}(X) \end{pmatrix} \qquad X \in \mathfrak{k}, \; Y \in \mathfrak{s} \tag{3.8}$$

where for each $Y \in \mathfrak{s}$, we have that

$$\widehat{A}(Y) : \mathfrak{s} \to \mathfrak{k} \quad \text{is the map} \quad \widehat{A}(Y)(Z) = [Y, Z].$$

Let G be a connected semisimple Lie group with lie algebra $\mathfrak{g}$ and take $K \subset G$ the subgroup given by $K = \langle \exp \mathfrak{k}\rangle$. The semi-direct product of K and $\mathfrak{s}$ will be denoted by

$$K_{\mathrm{ad}} = K \times_{\mathrm{Ad}} \mathfrak{s}.$$

The coadjoint orbit of $\widetilde{X} \in \mathfrak{s} \subset \mathfrak{k} \times \mathfrak{s}$ is the union of the fibers $\widehat{A}(Y)(\mathfrak{s})$ with Y passing through the K-coadjoint orbit of $\widetilde{X}$ in $\mathfrak{s}$. As $\widehat{A}(Y)(Z) = [Y, Z]$, then $\widehat{A}(Y)(\mathfrak{s}) = \mathrm{ad}(Y)(\mathfrak{s})$ where ad is the adjoint representation in $\mathfrak{g}$. To detail the coadjoint orbits of the semi-direct product, take a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$. The $\mathrm{Ad}(K)$-orbits in $\mathfrak{s}$ are passing through $\mathfrak{a}$, thus are the flags on $\mathfrak{g}$. Take a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. If $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$ then the orbit $\mathrm{Ad}(K)H$ is the flag manifold $\mathbb{F}_H$. By Proposition 3.4, the K_{ad}-orbit in $H \in \mathrm{cl}(\mathfrak{a}^+)$ is diffeomorphic to the cotangent bundle of $\mathbb{F}_H$, thus the K_{ad}-orbit itself is the union of the fibers $\mathrm{ad}(Y)(\mathfrak{s})$, with $Y \in \mathbb{F}_H$. In conclusion

$$K_{\mathrm{ad}} \cdot H = \bigcup_{Y \in \mathbb{F}_H} Y + \mathrm{ad}(Y)(\mathfrak{s}). \tag{3.9}$$

In this union the fiber over H is $H + \mathrm{ad}(H)(\mathfrak{s})$ with $\mathrm{ad}(H)(\mathfrak{s}) \subset \mathfrak{k}$. With the notations above this subspace of $\mathfrak{k}$ is given by

$$\mathrm{ad}(H)(\mathfrak{s}) = \sum_{\alpha(H)>0} \mathfrak{k}_\alpha.$$

Example 3.8. *Take $\mathfrak{sl}(2,\mathbb{R})$ with basis $\{H, S, A\}$ given by*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and coordinates $(x, y, z) = xH + yS + zA$. The Cartan decomposition $\mathfrak{k} \oplus \mathfrak{s}$ is given by $\mathfrak{k} = \mathfrak{so}(2) = \mathrm{span}\{A\}$ and $\mathfrak{s} = \mathrm{span}\{H, S\}$.

The adjoint representation of $\mathfrak{so}(2)$ in $\mathfrak{s}$ coincides with its canonical representation in $\mathbb{R}^2$. Hence, the coadjoint orbits of the semi-direct product are the cylinders $x^2 + y^2 = r$, with $r > 0$ and on the z-axis (generated by A) the orbits degenerate into points.

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, as $K_{\mathrm{ad}} \cdot H = K_{\mathrm{ad}}/Z_\rho(H)$ we have that

$$\ker \mu_H = \ker \mathrm{ad}(H)|_{\mathfrak{s}} \quad \text{then} \quad \mathfrak{z}_\rho(H) = \mathfrak{z}_{\mathfrak{k}}(H) \oplus \ker \mathrm{ad}(H)|_{\mathfrak{s}},$$

and implies that

$$Z_\rho(H) = K_H \oplus \ker \mathrm{ad}(H)|_{\mathfrak{s}} \tag{3.10}$$

is the isotropy subgroup at H.

If $X \in \mathfrak{g}$, we have that

$$\begin{aligned}\widetilde{X} = \mathrm{ad}^*_\rho(X) &= \begin{pmatrix} \mathrm{ad}\,(\kappa(X)) & \mathrm{ad}(\sigma(X)) \\ 0 & \mathrm{ad}(\kappa(X)) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{ad}\,(\kappa(X)) & 0 \\ \mathrm{ad}(\sigma(X)) & \mathrm{ad}(\kappa(X)) \end{pmatrix}^T \\ &= \left[\mathrm{ad}_\rho(X)\right]^T.\end{aligned}$$

In fact

$$\mathrm{ad}^*_\rho(X)\cdot H = \mathrm{ad}(\kappa(X))\cdot H + \mathrm{ad}(\sigma(X))\cdot H = \mathrm{ad}(X)(H),$$

then

$$T_H(K_{\mathrm{ad}}\cdot H) = \{\widetilde{X}_H = \mathrm{ad}(X)\cdot H : \ X \in \mathfrak{g}\},$$

and

$$T_p(K_{\mathrm{ad}}\cdot H) = \{\widetilde{X}_p = [\mathrm{ad}_\rho(X)]^T\cdot p : \ X \in \mathfrak{g}\}, \quad p \in K_{\mathrm{ad}}\cdot H.$$

Furthermore, we can define the Konstant–Kirillov–Souriau symplectic form at $x \in K_{\mathrm{ad}}\cdot H$

$$\omega_x\left(\widetilde{X}_x, \widetilde{Y}_x\right) = B_\theta\left(x, [X,Y]_\rho\right) \quad X, Y \in \mathfrak{g}.$$

3.2.1 Hermitian symplectic form on $U_{\mathrm{ad}}\cdot H$

Now, we will see the Hermitian symplectic form on coadjoint semi-direct orbits. For that, as $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ is a Cartan decomposition with Cartan involution τ, for $\mathfrak{g}$ semisimple complex Lie algebra. If $U \subset G$ is the compact subgroup with Lie algebra $\mathfrak{u}$. Then, we will denote by U_{ad} its respective semi-direct product (described in Section 3.2 for the general case).

If $H \in \mathfrak{s} = i\mathfrak{u}$, then its semi-direct orbit is denoted by $U_{\mathrm{ad}}\cdot H$.

To begin with, we will prove that the restriction of Ω^τ is a symplectic form on $U_{\mathrm{ad}}\cdot H$.

Proposition 3.9. *The form Ω^τ of $\mathfrak{g}$ restricted to $U_{\mathrm{ad}}\cdot H$ is a symplectic form, for $H \in \mathrm{cl}(\mathfrak{a}^+)$.*

Proof. The restriction is a closed 2-form because it is the pull-back of the imaginary part of $\mathcal{H}_\tau$ by inclusion. Hence, it remains to be seen that the restriction is a non-degenerate 2-form. Take a semi-direct coadjoint orbit

$$\mathcal{O} = \bigcup_{Y \in \mathrm{Ad}(U)H} (Y + \mathrm{ad}\,(Y)\,(i\mathfrak{u}))\,, \qquad H \in \mathrm{cl}\left(\mathfrak{a}^+\right).$$

The tangent space to a fiber $Y + \mathrm{ad}\,(Y)\,(i\mathfrak{u})$ is $\mathrm{ad}\,(Y)\,(i\mathfrak{u})$ which is a subspace of $\mathfrak{u}$, and a Lagrangian subspace of $\mathfrak{g}$. Hence the tangent spaces to the fibers are isotropic subspaces for the restriction of Ω^τ. The dimension of a fiber is half the dimension of the total orbit. Therefore, by Proposition 3.10 to prove that the restriction of Ω^τ is non-degenerate, it is enough to show that the tangent spaces to fibers are maximal isotropic. Take an element $\xi = H + X$ in the fiber over the origin H with $X \in \mathrm{ad}\,(H)\,(i\mathfrak{u})$. In terms of root spaces

$$\mathrm{ad}\,(H)\,(i\mathfrak{u}) = \sum_{\alpha \in \Pi} \mathfrak{u}_\alpha$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u}$. The tangent space $T_\xi\mathcal{O}$ of the orbit $\mathcal{O}$ in $\xi = H + X$ is generated by this vertical space $\mathrm{ad}\,(H)\,(i\mathfrak{u})$ and by the vectors $\mathrm{ad}\,(A)\,\xi$, with $A \in \mathfrak{u}$, such that

$$[A, H + X] = [A, H] + [A, X] \qquad X \in \mathrm{ad}\,(H)\,(i\mathfrak{u}) \subset \mathfrak{u}.$$

The component $[A, X] \in \mathfrak{u}$, so if $v \in \mathrm{ad}\,(H)\,(i\mathfrak{u})$ is a vector of the vertical tangent space then

$$\Omega^\tau\,(v, [A, H] + [A, X]) = \Omega^\tau\,(v, [A, H])$$

since the Hermitian form $\mathcal{H}_\tau$ is real in $\mathfrak{u}$, that is, $\mathfrak{u}$ is a Lagrangian subspace for Ω^τ. Then, to show that the tangent space to the fiber is maximal isotropic it must be shown that given $[A, H]$ with $A \in \mathfrak{u}$, there is an element v of the tangent space to the fiber such that $\Omega^\tau\,(v, [A, H]) \neq 0$. Now, the subspace

$$\{[A, H] : A \in \mathfrak{u}\}$$

is nothing less than the tangent space to the orbit $\mathrm{Ad}\,(U) \cdot H$ and is given by $\mathrm{ad}\,(H)\,(\mathfrak{u}) = i\,\mathrm{ad}\,(H)\,(i\mathfrak{u})$. Therefore, it all comes down to verify that given $Z \in \mathrm{ad}\,(H)\,(\mathfrak{u}) \subset \mathfrak{s} = i\mathfrak{u}$, $Z \neq 0$, there is $v \in \mathrm{ad}\,(H)\,(i\mathfrak{u})$ such that $\Omega^\tau\,(v, Z) \neq 0$. But this is immediate as $\mathcal{H}_\tau\,(Z, Z) > 0$ since $\mathcal{H}_\tau$ is positively defined in $\mathfrak{s}$. Hence if $v = iZ \in \mathrm{ad}\,(H)\,(i\mathfrak{u})$ then

$$\mathcal{H}_\tau\,(v, Z) = \mathcal{H}_\tau\,(iZ, Z) = i\mathcal{H}_\tau\,(Z, Z)$$

is imaginary and $\neq 0$ which means that $\Omega^{\tau}(v, Z) \neq 0$.

In short, it was shown that (the restriction of) Ω^{τ} is a symplectic form along the fiber $H + \mathrm{ad}(H)(i\mathfrak{u})$ over the origin H. In the other fibers the result is obtained by using the fact that Ω^{τ} is invariant by U and taking into account that the fiber over $Y = \mathrm{Ad}(u) \cdot H$, $u \in U$, is given by $\mathrm{Ad}(u)(H + \mathrm{ad}(H)(i\mathfrak{u}))$. □

Skew-symmetric bilinear form

Let V be a vector space (over $\mathbb{R}$ and $\dim V < \infty$) and ω a skew-symmetric bilinear form in V. The radical R^{ω} of ω is given by

$$R^{\omega} = \{v \in V : \forall w \in V,\ \omega(v, w) = 0\}.$$

By definition, ω is non-degenerate if and only if $R^{\omega} = \{0\}$. In this case $\dim V$ is even and ω is called a linear symplectic form.

Proposition 3.10. *ω is non-degenerate if and only if there is a maximal isotropic subspace W, with $2\dim W = \dim V$.*

Proof. As it is well known, if ω is a symplectic form then the dimension of the maximal isotropic subspaces (Lagrangian subspaces) is half the dimension of V. Furthermore, every isotropic subspace is contained in some Lagrangian subspace. For the converse, take the quotient space V/R^{ω} and define the form $\overline{\omega}$ in V/R^{ω} by $\overline{\omega}(\overline{v}, \overline{w}) = \omega(v, w)$ which is a skew-symmetric bilinear form in V/R^{ω}. The radical $R^{\overline{\omega}}$ of $\overline{\omega}$ vanishes, because if $\overline{v} \in R^{\overline{\omega}}$ then $\omega(v, w) = \overline{\omega}(\overline{v}, \overline{w}) = 0$, for all $w \in V$. Hence if ω is not identically null, then $\overline{\omega}$ is a symplectic form.

Now let $W \subset V$ be an isotropic subspace. So, the projection $\overline{W} \subset V/R^{\omega}$ is isotropic subspace for $\overline{\omega}$. If W is maximal isotropic then $R^{\omega} \subset W$ and as follows from the definition, $\overline{W}$ is maximal isotropic and therefore $\dim V/R^{\omega} = 2\dim \overline{W}$. In this case $\dim W = \dim \overline{W} + \dim R^{\omega}$, then

$$\begin{aligned} 2\dim W &= 2\dim \overline{W} + 2\dim R^{\omega} = \dim V - \dim R^{\omega} + 2\dim R^{\omega} \\ &= \dim V + \dim R^{\omega}. \end{aligned}$$

Hence, if ω is degenerate then $\dim R^{\omega} > 0$ and in consequence $2\dim W > \dim V$. □

4 Deformations of coadjoint orbits

In this chapter, inspired by the comparison between the semisimple and semi-direct orbits on $\mathfrak{sl}(2,\mathbb{R})$, we provide a diffeomorphic deformation between these orbits, as can be seen in [6]. In addition, we applied this deformation to find Lagrangian submanifolds of the adjoint semisimple orbit with respect to the Hermitian symplectic form.

4.1 Deformations' diffeomorphism

Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra, we will provide a compatible structure that allows us to deform the adjoint orbit of G in the coadjoint orbit of K_{ad}. The two orbit structures of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ (semisimple and semi-direct product) through $\mathfrak{s}$ are diffeomorphic to the cotangent bundle of the Flags manifolds of $\mathfrak{g}$ and therefore diffeomorphic from each other.

As previously stated, the idea is based on the case $\mathfrak{sl}(2,\mathbb{R})$ (Example 3.8), where we obtain a cylinder as an orbit, while in the usual case the result is an hyperboloid.

4.1.1 Deformed orbits of $\mathfrak{sl}(2,\mathbb{R})$

In this section, we will deform the classic adjoint semisimple orbit of $\mathfrak{sl}(2,\mathbb{R})$ into the coadjoint semi-direct orbit studied in Section 3.2. In Example 3.8 was shown that the semi-direct orbit of $\mathfrak{sl}(2,\mathbb{R})$ given by the Cartan decomposition $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{s}$ is the cylinder $x^2 + y^2 = 1$ (which in $\mathfrak{sl}(2,\mathbb{R})$ is the only one except up to conjugation), while the coadjoint semisimple orbit is an hyperboloid $x^2 + y^2 - z^2 = 1$.

The idea is to deform the semisimple orbit into a family of hyperboloids $x^2 + y^2 - z^2/r^2 = 1$, such that when $r \to \infty$ converges to the cylinder $x^2 + y^2 = 1$. Let $\mathfrak{sl}(2,\mathbb{R}) \approx \mathbb{R}^3$, such that in coordinates we can identify it as $(x, y, z) = xH + yS + zA$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As we saw in Example 3.8, the coadjoint semi-direct orbits are cylinders

$x^2 + y^2 = r$ with $r > 0$ and the axis z points. But, the coadjoint semisimple orbits are the hyperboloids $x^2 + y^2 - z^2 = r$, with $r > 0$.

In both cases, the orbits passing through H and S have the property that $r = 1$, and both surfaces can be seen as the union of straight lines. In the cylinder, we have that the lines are vertical and obtained by rotations with respect to the axis z of the vertical straight line passing through H, and the hyperboloid is obtained by rotations of the lines $H + \{(x, y, z) : z = y\}$.

Let $\mathsf{H}(\varepsilon)$ be the family of surfaces parameterized by $\varepsilon \geqslant 0$ of the form

$$x^2 + y^2 - \varepsilon^2 z^2 = 1.$$

That family can be seen as a continuous "deformation" of $\mathsf{H}(0)$ (semi-direct orbit) into $\mathsf{H}(1)$ (semisimple orbit). For $\varepsilon > 0$, the hyperboloid $\mathsf{H}(\varepsilon)$ is the union of lines obtained by the rotation of the line

$$H + \left\{ (x, y, z) : z = \frac{1}{\varepsilon^2} y \right\}.$$

Next, the hyperboloids $\mathsf{H}(\varepsilon)$ will be seen in terms of the coadjoint representation with the aim of generalizing them to the usual semisimple Lie algebras. For this we first note that the vertical lines in the cylinder $x^2 + y^2 = 1$ are the orbits of Abelian algebras generated by linear transformations in 3.8.

On the other hand, the line $H + \{(x, y, z) : z = y\}$ generates (by rotations) the other lines of the hyperboloid $x^2 + y^2 - z^2 = 1$ can bee seen as a nilpotent (in this situation, Abelian) algebra orbit. In fact, we have that

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

then $\mathrm{ad}(X)^2 = 0$, $[H, X] = 2X$ and that implies that $e^{t\,\mathrm{ad}(X)} H = H - 2tX$. As $2X = S + A$ (in coordinates is $(0, 1, 1)$) consequently it follows that $\{e^{t\,\mathrm{ad}(X)} H : t \in \mathbb{R}\}$ coincides with the line $H + \{(x, y, z) : z = y\}$. Another lines that can generate the hyperboloid $\mathsf{H}(1)$ are elements in $\{e^{t\,\mathrm{ad}(X^k)} \left(H^k\right) : t \in \mathbb{R}\}$ with $k \in \mathrm{SO}(2)$, where $X^k = \mathrm{Ad}(k) X$ and $H^k = \mathrm{Ad}(k) H$.

The surfaces $\mathsf{H}(\varepsilon)$, for $\varepsilon > 0$ can be seen as coadjoint orbits of the following Lie algebras, over the same vector space $\mathfrak{g} = \mathfrak{so}(2) \oplus \mathfrak{s}$. Set $r > 0$, the linear transformation

$T_r : \mathfrak{g} \to \mathfrak{g}$ is given by

$$T_r(A) = rA, \quad T_r(S) = S \ \text{ e } \quad T_r(H) = H$$

and we can define

$$[X, Y]_r = T_r \left[T_r^{-1} X, T_r^{-1} Y \right]. \tag{4.1}$$

By Lemma 4.1, the equation 4.1.2 is a Lie bracket. Furthermore, the linear transformation

$$T_r : (\mathfrak{g}, [\cdot, \cdot]) \to (\mathfrak{g}, [\cdot, \cdot]_r)$$

is a Lie algebra isomorphism and satisfies

$$\langle X, Y \rangle_r = \langle T_r^{-1} X, T_r^{-1} Y \rangle$$

where $\langle \cdot, \cdot \rangle_r$ and $\langle \cdot, \cdot \rangle$ are the Cartan–Killing forms of $(\mathfrak{g}, [\cdot, \cdot]_r)$ and $(\mathfrak{g}, [\cdot, \cdot])$, respectively. Thus

$$\langle H, H \rangle_r = 4 \qquad \langle S, S \rangle_r = 4 \qquad \langle A, A \rangle_r = -\frac{4}{r^2}.$$

Hence, if $X = xH + yS + zA$ then

$$\langle X, X \rangle_r = 4x^2 + 4y^2 - \frac{4}{r^2} z^2,$$

and we have that $\langle X, X \rangle_r = \dfrac{1}{4}$ is the hyperboloid $\mathsf{H}(\varepsilon)$ with $\varepsilon = \dfrac{1}{r}$.

In the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_r)$ the brackets between the basis elements $\{H, S, A\}$ are given by

$$[H, S]_r = 2rA \qquad [H, A]_r = \frac{2}{r} S \qquad [S, A]_r = -\frac{2}{r} H.$$

Therefore, the r-adjoint representation ad_r on $(\mathfrak{g}, [\cdot, \cdot]_r)$ can be represented in terms of the basis $\{H, S, A\}$ by matrices

$$\mathrm{ad}_r(H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2/r \\ 0 & 2r & 0 \end{pmatrix},$$

$$\mathrm{ad}_r(S) = \begin{pmatrix} 0 & 0 & -2/r \\ 0 & 0 & 0 \\ -2r & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_r(A) = \begin{pmatrix} 0 & 2/r & 0 \\ -2/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To see the r-coadjoint representation ad_r of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_r)$, its dual Lie algebra $\mathfrak{g}^*$ will be identified with $\mathfrak{g}$ by the inner product $\frac{1}{4}B_\theta$. That normalization of B_θ can be done because the basis $\{H, S, A\}$ in relation to that inner product is orthonormal.

Then the r-coadjoint representation ad_r^* of $(\mathfrak{g}, [\cdot,\cdot]_r)$ is the transpose of the matrices above, that is

$$\mathrm{ad}_r^*(H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2r \\ 0 & 2/r & 0 \end{pmatrix},$$

$$\mathrm{ad}_r^*(S) = \begin{pmatrix} 0 & 0 & -2r \\ 0 & 0 & 0 \\ -2/r & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_r^*(A) = \begin{pmatrix} 0 & -2/r & 0 \\ 2/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And its exponentials are given by

$$e^{t\,\mathrm{ad}_r^*(S)} = \begin{pmatrix} \cosh(2t) & 0 & -r\sinh(2t) \\ 0 & 1 & 0 \\ -\frac{1}{r}\sinh(2t) & 0 & \cosh(2t) \end{pmatrix},$$

$$e^{t\,\mathrm{ad}_r^*(H)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(2t) & r\sinh(2t) \\ 0 & \frac{1}{r}\sinh(2t) & \cosh(2t) \end{pmatrix},$$

$$e^{t\,\mathrm{ad}_r^*(A)} = \begin{pmatrix} \cos\left(\frac{2t}{r}\right) & -\mathrm{sen}\left(\frac{2t}{r}\right) & 0 \\ \mathrm{sen}\left(\frac{2t}{r}\right) & \cos\left(\frac{2t}{r}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

from this we can see the r-adjoint orbit passing through H or S is the hyperboloid denoted by $\mathsf{H}\left(\frac{1}{r}\right)$.

4.1.2 General construction

Geometrically, the aim of the construction given in Section 4.1.1 can be interpreted as follows: Lie algebras $\mathfrak{g}_r$ corresponds to different normalization of the metrics on the symmetric space G/K. We can see that in the limit $r \to \infty$, we get the flat symmetric space with the isometry group K_{Ad}. Moreover, the Lie algebras $\mathfrak{g}_r$ of the family (including $\mathfrak{g}_\infty = \mathfrak{k}_{\mathrm{ad}}$) parameterized the embeddings of the orbits $T^*\mathbb{F}_H$ into the Lie algebra $\mathfrak{g}$. In this sense, we attribute the term 'deformation' to the diffeomorphism determined by this variation in the metric, that will be defined in this section. In fact, a deformation is a 1-parameter family of immersed manifolds in a given semisimple Lie algebra $\mathfrak{g}$.

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a non-compact semisimple Lie algebra, fixing $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ a Cartan decomposition, with θ its Cartan involution. Set $r > 0$ and define the linear map

$$T_r : \mathfrak{g} \to \mathfrak{g} \quad \text{such that} \quad T_r(X) = rX \quad \forall X \in \mathfrak{k} \quad \text{and} \quad T_r(Y) = Y \quad \forall Y \in \mathfrak{s},$$

which induces the Lie bracket

$$[X, Y]_r = T_r \left[T_r^{-1}X, T_r^{-1}Y\right],$$

such that $(\mathfrak{g}, [\cdot,\cdot]_r)$ is a Lie algebra. In general we have:

Lemma 4.1. *For $r > 0$, denote by $\mathfrak{g}_r$ the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_r)$ and by $\langle \cdot,\cdot \rangle_r$ its Cartan–Killing form. Then*

1. *$T_r : \mathfrak{g} \to \mathfrak{g}_r$ is an isomorphism of Lie algebras.*
2. *$\langle X, Y\rangle_r = \langle T_r^{-1}X, T_r^{-1}Y\rangle$ for all $X, Y \in \mathfrak{g}$.*

Proof. The isomorphism between those Lie algebras is immediate because

$$[T_rX, T_rY]_r = T_r \left[T_r^{-1}T_rX, T_r^{-1}T_rY\right] = T_r [X, Y],$$

that is, T_r is a homomorphism of Lie algebras. Denote by $\mathrm{ad}_r : \mathfrak{g}_r \to \mathfrak{g}_r$ the r-adjoint representation, given by

$$\mathrm{ad}_r(X)(Y) = [X, Y]_r \quad X, Y \in \mathfrak{g}_r.$$

In relation to the Cartan–Killing form, as T_r is an isomorphism, then

$$\mathrm{ad}_r(X) = T_r \circ \mathrm{ad}\left(T_r^{-1}X\right) \circ T_r^{-1}.$$

Hence

$$\langle X, X\rangle_r = \mathrm{tr}\,(\mathrm{ad}_r(X))^2 = \mathrm{tr}\left(\mathrm{ad}\left(T_r^{-1}X\right)\right)^2 = \langle T_r^{-1}X, T_r^{-1}X\rangle$$

showing the last statement. □

Now, an interesting question is how this transformation affects our given Cartan decomposition. The following Lemma shows that the Cartan decomposition doesn't change:

Lemma 4.2. *The map $\widetilde{\theta} = T_r \circ \theta \circ T_r^{-1}$ is a Cartan involution and it gives us the same Cartan decomposition (as sets), i.e., $\mathfrak{g}_r = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition.*

Proof. As B_θ is an inner product, then

$$\begin{aligned} B_{\widetilde{\theta}}(X,Y) &= -\langle X, \widetilde{\theta}Y\rangle_r \\ &= -\langle T_r^{-1}X, T_r^{-1}\widetilde{\theta}Y\rangle \\ &= -\langle T_r^{-1}X, T_r^{-1}T_r\theta T_r^{-1}Y\rangle \\ &= B_\theta(T_r^{-1}X, T_r^{-1}Y) \end{aligned}$$

is an inner product because T_r^{-1} is an isomorphism. Furthermore, by definition of T_r and θ we have that

$$\mathfrak{k} = \{X \in \mathfrak{g}_r : \widetilde{\theta}X = X\} \quad \text{e} \quad \mathfrak{s} = \{Y \in \mathfrak{g}_r : \widetilde{\theta}Y = -Y\},$$

then $\widetilde{\theta}$ determines the same Cartan decomposition. □

As a consequence of the previous Lemma, the fact that $\mathfrak{g}_r$ has the same Cartan decomposition and the structure of the bracket restricted to $\mathfrak{s}$ implies that we can choose the same maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ and the same simple root system Σ, and also the same root system Π and positive Weyl chamber $\mathfrak{a}^+$. But the root spaces are going to change, and will be called the r-root spaces.

Then, for $\alpha \in \Pi$, the corresponding r-root space is defined by

$$\mathfrak{g}_\alpha^r = \{X \in \mathfrak{g}_r :\ \mathrm{ad}_r(H)X = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\alpha$ is the usual root space (of $\mathfrak{g}$). Then, we can define the map $\psi_r : \mathfrak{g} \to \mathfrak{g}$ given by

$$\psi_r(Z) = Z + \frac{r-1}{r+1}\theta Z, \tag{4.2}$$

such that $\mathfrak{g}^r_\alpha = \psi_r(\mathfrak{g}_\alpha)$, for all $\alpha \in \Pi$.

The construction of ψ_r is obtained by the following argument: suppose that $\mathfrak{g}$ is a split semisimple Lie algebra and take a Weyl basis $H_\alpha \in \mathfrak{h}$, $X_\beta \in \mathfrak{g}_\beta$, for $\alpha \in \Sigma$ and $\beta \in \Pi$, such that $\theta X_\alpha = -X_{-\alpha}$. Then, we want to build the correspondent X^r_α that spans $\mathfrak{g}^r_\alpha$. If $H \in \mathfrak{a}$:

$$\begin{aligned}
\mathrm{ad}_r(H)X_\alpha &= [H, X_\alpha]_r \\
&= T_r[T_r^{-1}H, T_r^{-1}X_\alpha] \\
&= T_r\left[H, T_r^{-1}\left(\kappa\left(X_\alpha\right) + \sigma\left(X_\alpha\right)\right)\right] \\
&= \frac{1}{r}T_r[H, \kappa\left(X_\alpha\right)] + T_r[H, \sigma\left(X_\alpha\right)] \\
&= \frac{1}{r}[H, \kappa\left(X_\alpha\right)] + r[H, \sigma\left(X_\alpha\right)] \\
&= \mathrm{ad}(H)\left(\frac{1+r^2}{2r}X_\alpha + \frac{r^2-1}{2r}X_{-\alpha}\right).
\end{aligned}$$

As $\alpha(H)X_\alpha = \mathrm{ad}(H)X_\alpha$, we have that:

$$\mathrm{ad}_r(H)X_\alpha = \alpha(H)\left(\frac{1+r^2}{2r}X_\alpha - \frac{r^2-1}{2r}X_{-\alpha}\right),$$

analogously

$$\mathrm{ad}_r(H)X_{-\alpha} = \alpha(H)\left(\frac{r^2-1}{2r}X_\alpha - \frac{r^2+1}{2r}X_{-\alpha}\right).$$

Hence,

$$\mathrm{ad}_r(H)\left(X_\alpha + \frac{1-r}{r+1}X_{-\alpha}\right) = \alpha(H)\left(X_\alpha + \frac{1-r}{r+1}X_{-\alpha}\right), \tag{4.3}$$

then

$$X^r_\alpha = X_\alpha + \frac{1-r}{r+1}X_{-\alpha}.$$

In terms of the Cartan involution θ, we have

$$X^r_\alpha = X_\alpha + \frac{1-r}{r+1}X_{-\alpha} = X_\alpha + \frac{r-1}{r+1}\theta X_\alpha,$$

such that $\widetilde{\theta}X^r_\alpha = -X^r_{-\alpha}$.

This construction can be generalized for all non-compact semisimple algebra in the following sense: If $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{a}$

$$\begin{aligned}
\mathrm{ad}_r(H)(\psi_r X) &= [H, X]_r + \left(\frac{r-1}{r+1}\right)[H, \theta X]_r \\
&= \left[H, \frac{1}{r}\kappa(X) + r\sigma(X)\right] + \frac{r-1}{r+1}\left[H, \frac{1}{r}\kappa(\theta X) + r\sigma(\theta X)\right] \\
&= \left(\left(\frac{1}{2r} + \frac{r}{2}\right) + \frac{r-1}{r+1}\left(\frac{1}{2r} - \frac{r}{2}\right)\right)[H, X] \\
&\quad \left(\left(\frac{1}{2r} - \frac{r}{2}\right) + \frac{r-1}{r+1}\left(\frac{1}{2r} + \frac{r}{2}\right)\right)[H, \theta X],
\end{aligned}$$

where

$$\left(\frac{1}{2r} + \frac{r}{2}\right) + \frac{r-1}{r+1}\left(\frac{1}{2r} - \frac{r}{2}\right) = 1,$$

and

$$\left(\frac{1}{2r} - \frac{r}{2}\right) + \frac{r-1}{r+1}\left(\frac{1}{2r} + \frac{r}{2}\right) = \frac{r-1}{r+1}.$$

Then

$$\mathrm{ad}_r(H)\left(\psi_r X\right) = \alpha(H)\left(\psi_r X\right).$$

As it is known, the importance of determining the root spaces is that we can describe the adjoint orbits by nilpotent subalgebras.

If $\mathfrak{n}^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$, we have:

$$\mathfrak{n}_r^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha^r = \sum_{\alpha>0} \psi_r\left(\mathfrak{g}_\alpha\right) = \psi_r\left(\mathfrak{n}^+\right),$$

thus, given $H \in \mathrm{cl}(\mathfrak{a}^+)$

$$\mathfrak{n}_H^+ = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_{r,H}^+ = \psi_r\left(\mathfrak{n}_H^+\right).$$

Let G_r be a connected Lie group with Lie algebra $\mathfrak{g}_r$, that is, G_r is semisimple and diffeomorphic to G, whose adjoint orbits are manifolds in $\mathfrak{g}$. The r-adjoint representation of G_r is going to be defined (identified) by:

$$\mathrm{Ad}_r(G) \cdot H = \mathrm{Ad}_r(K)\left(H + \psi_r\left(\mathfrak{n}_H^+\right)\right).$$

Now, let's see some results that will allow us to describe the r-adjoint orbit:

Lemma 4.3. *For $r > 0$, $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$, then*

$$\mathrm{Ad}_r\left(e^{tX}\right) \cdot Y = \mathrm{Ad}\left(e^{\frac{t}{r}X}\right) \cdot Y.$$

Proof. For $r > 0$, $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$, then

$$\mathrm{ad}_r(X) \cdot Y = T_r\left[\frac{1}{r}X, \frac{1}{r}\kappa(Y) + \sigma(Y)\right] = \frac{1}{r}\,\mathrm{ad}(X) \cdot Y,$$

where σ and κ are the projections onto $\mathfrak{s}$ and $\mathfrak{k}$, respectively.

Inductively $\mathrm{ad}_r^k(X) \cdot Y = \dfrac{1}{r^k}\,\mathrm{ad}^k(X) \cdot Y$, then

$$\sum_{k>0} \frac{t^k\,\mathrm{ad}_r^k(X)}{k!}Y = \sum_{k>0} \frac{\left(\frac{t}{r}\right)^k \mathrm{ad}^k(X)}{k!}Y = \mathrm{Ad}\left(e^{\frac{t}{r}X}\right) \cdot Y. \qquad \square$$

Therefore, given $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$ and $X \in \mathfrak{k}$:

$$\mathrm{Ad}_r\left(e^{tX}\right) \cdot H = \mathrm{Ad}\left(e^{\frac{t}{r}X}\right) \cdot H,$$

that is, they determine the same flag manifold. In addition, given $\alpha \in \Pi$

$$\begin{aligned}
\mathrm{Ad}_r\left(e^{tX}\right) \cdot X_\alpha^r &= e^{\frac{t}{r}\mathrm{ad}(X)}\left(\psi_r\left(X_\alpha\right)\right) \\
&= \sum_{k>0} \frac{\left(\frac{t}{r}\right)^k \mathrm{ad}^k(X)}{k!}\left(\psi_r\left(X_\alpha\right)\right) \\
&= \psi_r\left(\sum_{k>0} \frac{\left(\frac{t}{r}\right)^k \mathrm{ad}^k(X)}{k!}\left(X_\alpha\right)\right) \\
&= \psi_r\left(\mathrm{Ad}\left(e^{\frac{t}{r}X}\right) \cdot X_\alpha\right).
\end{aligned}$$

Thus, for all $k \in K$

$$\mathrm{Ad}_r(k)\left(\mathfrak{n}_{r,H}^+\right) = \mathrm{Ad}_r(k) \cdot \psi_r\left(\mathfrak{n}_H^+\right) = \psi_r\left(\mathrm{Ad}(k)\left(\mathfrak{n}_H^+\right)\right). \tag{4.4}$$

Hence, we conclude:

Proposition 4.4. *For $r > 0$ and $H \in \mathrm{cl}\left(\mathfrak{a}^+\right)$*

$$\mathrm{Ad}_r(G) \cdot H = \bigcup_{k \in K} \mathrm{Ad}(k)\left(H + \psi_r\left(\mathfrak{n}_H^+\right)\right),$$

that is, the r-adjoint orbit of G is a r-deformation of the adjoint orbit of G.

Remark 4.5. *If $\mathfrak{g}$ is a semisimple complex Lie algebra, then $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition with τ its Cartan decomposition and $\mathfrak{a} = i\mathfrak{h}_{\mathbb{R}}$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. In this situation*

$$\psi_r(X) = X + \left(\frac{r-1}{r+1}\right)\tau X.$$

Therefore, the Proposition 4.4 is true for the complex case.

In addition, the representation of K_H in $\mathfrak{g}_r$ makes invariant the subspace $\mathfrak{n}^+_{r,H}$, because if $k \in K$, then $\mathrm{Ad}_r(k)$ commutes with $\mathrm{ad}_r(H)$. Therefore, $\mathrm{Ad}_r(k)$ takes eigenspaces of $\mathrm{ad}_r(H)$ into the same eigenspaces. Thus, we can induce the representation ρ_r of K_H in $\mathfrak{n}^+_{r,H}$, and by (4.4), we have

$$\psi_r\left(\rho(k)\cdot X\right) = \rho_r(k)\cdot\psi_r(X) \qquad k \in K_H,\ X \in \mathfrak{n}^+_H,$$

where ρ is the representation in the case $r = 1$ (that is, the usual representation Ad). Therefore,

$$K \times_{\rho_r} \mathfrak{n}^+_{r,H} = K \times_\rho \psi_r\left(\mathfrak{n}^+_H\right).$$

So we will induce a diffeomorphism between $\mathrm{Ad}(G)\cdot H$ and $\mathrm{Ad}_r(G)\cdot H$ using the following map (this construction was proved in the Proposition 2.4 of [11])

$$\gamma_r : \mathrm{Ad}_r(G)\cdot H \to K \times_\rho \psi_r\left(\mathfrak{n}^+_H\right),$$

such that

$$Y = \mathrm{Ad}_r(k)(H+X) \mapsto (k, X) \in K \times_\rho \psi_r\left(\mathfrak{n}^+_H\right)$$

is a diffeomorphism that satisfies:

1. γ_r X is equivariant with respect to the action of K.
2. γ_r leads fibers into fibers.
3. γ_r leads the orbit $\mathrm{Ad}_r(K)\cdot H$ in the null section of $K \times_\rho \psi_r\left(\mathfrak{n}^+_H\right)$.

It's easy to see that $(d\gamma_r)_x = \mathrm{id}$, for $x = \mathrm{Ad}_r(k)(H+Y) \in \mathrm{Ad}_r(G)\cdot H$.

Furthermore, diffeomorphism γ_r is defined by the vector bundle $K \times_\rho \psi_r \left(\mathfrak{n}_H^+\right)$ associated with the main bundle $K \to K/K_H$ of $\mathrm{Ad}_r(G) \cdot H$, which is a homogeneous space. Using this diffeomorphism for $r > 0$ we define the map $\widetilde{\psi}_r$ as follows:

$$\begin{tikzcd} \mathrm{Ad}(G) \cdot H \arrow[r, "\widetilde{\psi}_r"] \arrow[d, "\gamma"'] & \mathrm{Ad}_r(G) \cdot H \arrow[d, "\gamma_r"] \\ K \times_\rho \mathfrak{n}_H^+ \arrow[r, "\psi_r"] & K \times_\rho \psi_r \left(\mathfrak{n}_H^+\right) \end{tikzcd} \tag{4.5}$$

which is a diffeomorphism, because ψ_r is linear (in the complex is the sum of linear and anti-linear maps) and γ_r is a diffeomorphism, as seen above. We conclude

$$\left(d\widetilde{\psi}\right)_x = \psi_r, \qquad x \in \mathrm{Ad}_r(G) \cdot H.$$

Hence, joining these constructions, we conclude that

Theorem 4.6. *Let $H \in \mathrm{cl}(\mathfrak{a}^+)$ and $r > 0$, then the manifolds $\mathrm{Ad}(G) \cdot H$ and $\mathrm{Ad}_r(G) \cdot H$ are diffeomorphic by $\widetilde{\psi}_r$.*

In addition, our intention is to take the diffeomorphism $\widetilde{\psi}_r$ to a similar diffeomorphism between $\mathrm{Ad}_r(G) \cdot H$ and $K_{\mathrm{ad}} \cdot H$. To do this define the map

$$\psi : \mathfrak{g} \to \mathfrak{g} \quad \text{given by} \quad \psi(X) = X + \theta X, \tag{4.6}$$

and notice that when $r \to \infty$

$$\psi_r \to \psi.$$

Lemma 4.7. *For the map $\psi = 2\kappa$ defined in (4.6), it holds that:*

1. *The image of ψ is in $\mathfrak{k}$ and its kernel in $\mathfrak{s}$.*

2. *Let $X \in \mathfrak{k}$, we have $\psi \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ \psi$.*

3. *If $X \in \mathfrak{k}$ and $\alpha \in \Pi$, then*

$$\mathrm{Ad}\left(e^{tX}\right) \psi X_\alpha = \psi\left(\mathrm{Ad}\left(e^{tX}\right) X_\alpha\right).$$

Proof. Item 1 is immediate from the definition of κ.

2. Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$

$$\begin{aligned}\psi[X,Y] &= [X,Y] + \theta[X,Y] \\ &= [X,Y] + [\theta X, \theta Y] \\ &= [X,Y] + [X, \theta Y] \\ &= [X, \psi Y].\end{aligned}$$

3. Let $X \in \mathfrak{k}$ and $\alpha \in \Pi$, note that for $Y \in \mathfrak{g}$

$$\mathrm{ad}(Y) \cdot \theta X_\alpha = [Y, \theta X_\alpha] = \theta[\theta Y, X_\alpha] = \theta\,\mathrm{ad}(\theta Y) X_\alpha,$$

inductively, we have $\mathrm{ad}^k Y \cdot \theta X_\alpha = \theta\,\mathrm{ad}^k(\theta Y) \cdot X_\alpha$, then

$$\begin{aligned}e^{t\,\mathrm{ad}(X)}\left(\theta X_\alpha\right) &= \sum_{k>0} \frac{t^k\,\mathrm{ad}^k(X)}{k!} \cdot \theta X_\alpha \\ &= \sum_{k>0} \frac{t^k \theta\,\mathrm{ad}^k(\theta X)}{k!} \cdot X_\alpha \\ &= \theta\left(\sum_{k>0} \frac{t^k\,\mathrm{ad}^k(X)}{k!} \cdot X_\alpha\right) \\ &= \theta \cdot e^{t\,\mathrm{ad}(X)}\left(X_\alpha\right),\end{aligned}$$

because $\theta X = X$ and we have

$$\begin{aligned}\mathrm{Ad}\left(e^{tX}\right) \cdot \psi X_\alpha &= e^{t\,\mathrm{ad}(X)}\left(X_\alpha\right) + \theta \cdot e^{t\,\mathrm{ad}(X)}\left(X_\alpha\right) \\ &= \psi\left(e^{t\,\mathrm{ad}(X)} \cdot X_\alpha\right) \\ &= \psi\left(\mathrm{Ad}\left(e^{tX}\right) X_\alpha\right).\end{aligned}$$

□

Hence, we can define

$$\mathrm{Ad}_\infty(G) \cdot H := \bigcup_{k \in K} \mathrm{Ad}(k)\left(H + \psi\left(\mathfrak{n}_H^+\right)\right), \tag{4.7}$$

when $r \to \infty$ we have

$$\mathrm{Ad}_r(G) \cdot H \to \mathrm{Ad}_\infty(G) \cdot H.$$

It is convenient to define the ∞-root spaces by $\mathfrak{g}_\alpha^\infty = \psi\left(\mathfrak{g}_\alpha\right)$, and consequently

$$\mathfrak{n}_{\infty,H}^+ = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha^\infty = \sum_{\alpha(H)>0} \psi\left(\mathfrak{g}_\alpha\right) = \psi\left(\mathfrak{n}_H^+\right).$$

Therefore, analogous to the diffeomorphism γ_r, define

$$\gamma_\infty : \mathrm{Ad}_\infty(G) \cdot H \to K \times_\rho \psi\left(\mathfrak{n}_H^+\right),$$

such that

$$Y = \mathrm{Ad}(k)(H + X) \mapsto (k, X) \in K \times_\rho \psi(\mathfrak{n}_H^+).$$

So we have that the map γ_∞ is a diffeomorphism for vector bundles. The map γ_∞ is well defined as a consequence of ψ, the bijectivity is a consequence of the way in which the manifold $\mathrm{Ad}(G)_\infty \cdot H$ was defined, and the differentiability is given by the idea of making $r \to \infty$, in the diffeomorphism γ_r. Then, we can define the diffeomorphism $\widetilde{\psi}$, given by:

$$\begin{tikzcd} \mathrm{Ad}(G) \cdot H \arrow[r, "\widetilde{\psi}"] \arrow[d, "\gamma"'] & \mathrm{Ad}_\infty(G) \cdot H \arrow[d, "\gamma_\infty"] \\ K \times_\rho \mathfrak{n}_H^+ \arrow[r, "\psi"] & K \times_\rho \psi\left(\mathfrak{n}_H^+\right) \end{tikzcd} \tag{4.8}$$

in the same way as for $\widetilde{\psi}_r$. So using $r \to \infty$

$$\left(d\gamma_\infty\right)_x = \mathrm{id} \quad \text{and} \quad \left(d\widetilde{\psi}\right)_x = \psi.$$

Now, we can conclude that

Theorem 4.8. *The manifolds $\mathrm{Ad}_\infty(G) \cdot H$ and $\mathrm{Ad}_r(G) \cdot H$ are diffeomorphic for $r > 0$, the diffeomorphisms are given by $\widetilde{\psi}$ and $\widetilde{\psi}_r$ defined in (4.5) and (4.8).*

Moreover

$$K_{\mathrm{ad}} \cdot H = \bigcup_{k \in K} \mathrm{Ad}(k)\,(H) + \left[\mathrm{Ad}(k) \cdot H, \mathfrak{s}\right],$$

the fiber in H is $H + [H, \mathfrak{s}]$, but $\mathfrak{s} = \mathfrak{a} \oplus \sigma(\mathfrak{n})$, then $[H, \mathfrak{s}] = [H, \sigma(\mathfrak{n})]$ because $[H, \mathfrak{a}] = 0$. In addition, for $\alpha \in \Pi^+$

$$[H, X_\alpha] = \alpha(H) X_\alpha,$$

such that

- $\alpha(H) = 0$ if $\alpha \notin \langle \Theta_H \rangle^+$, then $\alpha(H)\mathfrak{g}_\alpha = 0$, for $\alpha \notin \langle \Theta_H \rangle^+$.
- $\alpha(H) > 0$ if $\alpha \in \langle \Theta_H \rangle^+$, then $\alpha(H)\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$, for $\alpha \in \langle \Theta_H \rangle^+$.

Thus $[H, \sigma(\mathfrak{n})] = \frac{1}{2}([H, X_\alpha] - [H, \theta X_\alpha])$, but

$$[H, \theta X_\alpha] = \theta[\theta H, X_\alpha] = \theta[-H, X_\alpha] = -\theta[H, X_\alpha],$$

because $H \in \mathfrak{a} \subset \mathfrak{s}$, then

$$\begin{aligned}[H, \sigma(X_\alpha)] &= \frac{1}{2}([H, X_\alpha] + \theta[H, X_\alpha]) \\ &= \frac{1}{2}\psi([H, X_\alpha]) \\ &= \frac{1}{2}\psi(\alpha(H) \cdot X_\alpha),\end{aligned}$$

and as $[H, X_\alpha] \neq 0$ if and only if $\alpha \in \langle \Theta_H \rangle^+$, we have

$$[H, \mathfrak{s}] = \psi\left(\mathfrak{n}_H^+\right).$$

So the fibers in H of $K_{\mathrm{ad}} \cdot H$ and $\mathrm{Ad}_\infty(G) \cdot H$ coincide. In an equivalent way, we can identify the other fibers of these spaces for each $k \in K$.

$$\mathrm{Ad}(k)(H) + \underbrace{[\mathrm{Ad}(k) \cdot H, \mathfrak{s}]}_{\in \mathfrak{k}} \mapsto \mathrm{Ad}(k)(H) + \underbrace{\psi\left(\mathrm{Ad}(k) \cdot \mathfrak{n}_H^+\right)}_{\in \mathfrak{k}}.$$

So we can identify the manifolds $K_{\mathrm{ad}} \cdot H$ and $\mathrm{Ad}_\infty(G) \cdot H$ that are diffeomorphic from the bundle $T^*\mathbb{F}_H$. We can conclude:

Corollary 4.9. *The adjoint orbit $\mathrm{Ad}(G) \cdot H$ deforms in $K_{\mathrm{ad}} \cdot H$, by $\widetilde{\psi}$.*

4.1.3 Symplectomorphism of deformation

To conclude, we will see that the diffeomorphisms given in (4.5) and (4.8) are symplectomorphisms with respect to the Hermitian symplectic form. For that, consider the following proposition

Proposition 4.10. *For $r > 0$ we have*

$$\widetilde{\psi}_r^*(\Omega^\tau) = \Omega^\tau,$$

that is $\widetilde{\psi}_r$ is symplectomorphism for $r > 0$.

Proof. If $x \in \mathrm{Ad}(G) \cdot H$ and $\widetilde{x} = \widetilde{\psi}_r(x)$, then

$$\begin{aligned}\left(\widetilde{\psi}_r^* \cdot \Omega^\tau\right)_x (X, Y) &= \Omega^\tau_{\widetilde{x}}\left((d\widetilde{\psi}_r)_x X, (d\widetilde{\psi}_r)_x Y\right) \\ &= \Omega^\tau_{\widetilde{x}}(\psi_r X, \psi_r Y),\end{aligned}$$

for $X, Y \in \mathfrak{n}_H \simeq T_x \operatorname{Ad}(G)\cdot H$ and their corresponding $\psi_r(X), \psi_r(Y)$ in $\mathfrak{n}_{r,H} \simeq T_{\tilde{x}} \operatorname{Ad}_r(G)\cdot H$, as seen above $\psi_r(X_\alpha) = X_\alpha^r$ are the generators for $\alpha \in \langle \Theta_H \rangle$. □

Similarly, taking $r \to \infty$, in the manifold $\operatorname{Ad}_\infty(G)\cdot H$ we have

$$\widetilde{\psi}^* (\Omega^\tau) = \Omega^\tau,$$

where Ω^τ is a symplectic form of $\operatorname{Ad}_\infty(G)\cdot H$, because as seen above and by Proposition 3.9 it coincides with $U_{\mathrm{ad}} \cdot H$. Then, we have that

Theorem 4.11. *The manifolds $U_{\mathrm{ad}} \cdot H$ and $\operatorname{Ad}(G)\cdot H$ are symplectomorphic, with respect to the symplectic form Ω^τ.*

4.2 Lagrangian submanifolds

In this section, we will apply the results obtained Chapter 3 to find some isotropic and Lagrangian submanifolds of adjoint orbits with respect to the Hermitian symplectic form.

4.2.1 Lagrangian sections

We will build some Lagrangian submanifolds given by sections of $U_{\mathrm{ad}} \cdot H$, and we want to transport them to the adjoint orbits by the symplectomorphism $\widetilde{\psi}$.

The restriction of $\mathcal{H}_\tau$ on $\mathfrak{s} = i\mathfrak{u}$ is the Cartan–Killing form, which is an inner product on $\mathfrak{s}$ and induces a U-invariant Riemannian metric on $\operatorname{Ad}(U)H$.

Proposition 4.12. *Given $Y \in \mathfrak{s} = i\mathfrak{u}$ and $Z \in \operatorname{Ad}(U)\cdot H$, suppose that $Y \in T_Z \operatorname{Ad}(U)\cdot H$. Then $iY \in \operatorname{ad}(Z)(i\mathfrak{u})$, that is, $Z + iY$ is in the fiber over Z of the semi-direct coadjoint orbit.*

Proof. Take first $Z = H \in \mathfrak{a} \subset \mathfrak{s}$. Then,

$$T_H \operatorname{Ad}(U)\cdot H = \sum_{\alpha(H)>0} \mathfrak{s}_\alpha,$$

while

$$iT_H \operatorname{Ad}(U)\cdot H = \sum_{\alpha(H)>0} \mathfrak{u}_\alpha = \operatorname{ad}(Z)(i\mathfrak{u}).$$

By these expressions, it is immediate that iY is tangent to the fiber if Y is tangent to the orbit $\mathrm{Ad}\,(U)\cdot H$. For $Z=\mathrm{Ad}\,(u)\cdot H$, $u\in U$, we get the same result applying $\mathrm{Ad}\,(u)$. □

A vector field in the orbit $\mathrm{Ad}\,(U)\,H$ is a map $x\mapsto Y\,(x)\in\mathfrak{s}$ that takes values in the tangent space to x. By the Proposition 4.12, $iY\,(x)\in\mathfrak{u}$ is tangent to the fiber over the semi-direct orbit. Thus, given a vector field Y in $\mathrm{Ad}\,(U)\cdot H$, the vector field $iY\,(x)$ is defined in the semi-direct orbit, such that on the fiber $x+\mathrm{ad}\,(x)\,(\mathfrak{s})$ is a constant field.

Proposition 4.13. *Let $Y=\mathrm{grad}f$ be a gradient field in $\mathrm{Ad}\,(U)\cdot H$. Thus iY is the Hamiltonian vector field of the function $\widetilde{f}=f\circ\pi$, with respect to the symplectic form Ω^τ.*

Proof. If W is a vertical vector then $d\widetilde{f}\,(W)=0$ and $\Omega^\tau\,(W,iY\,(x))=0$, because both W and $iY\,(x)$ are in $\mathfrak{u}$. On the other hand, take a vector of type $[A,x+X]=[A,x]+[A,X]$, with $X\in\mathrm{ad}\,(x)\,(i\mathfrak{u})\subset\mathfrak{u}$ (these vectors, together with the vertical space, generate the tangent space as in Proposition 3.9). The component $[A,X]\in\mathfrak{u}$, so that $d\widetilde{f}\,([A,X])=0$ and $\Omega^\tau\,([A,X]\,,iY\,(x))=0$.

Since the component $v=[A,x]$ is the tangent space to x, hence

$$d\widetilde{f}\,(v)=df\,(v)=\langle Y\,(x)\,,v\rangle,$$

because $Y=\mathrm{grad}f$. But,

$$\Omega^\tau\,(iY\,(x)\,,v)=\mathcal{H}_\tau\,(iY\,(x)\,,v)=i\langle Y\,(x)\,,v\rangle,$$

because in this sequence of equality all terms are purely imaginary. Consequently, for vectors of type $w=[A,x+X]=[A,x]+[A,X]$, holds $d\widetilde{f}\,(w)=\Omega^\tau\,(iY\,(x)\,,v)$, as this equality is also true for vertical vectors, it is shown that $iY\,(x)$ is the Hamiltonian vector field of $\widetilde{f}$. □

Corollary 4.14. *Let Y be a gradient field on the flag manifold $\mathbb{F}_H=\mathrm{Ad}\,(U)\cdot H$, and for $t\in\mathbb{R}$ we define the map*

$$\sigma_{tY}\,(x)=x+tiY\,(x)\,.$$

This map is a section of $U_{\mathrm{ad}}\cdot H$. Then, the image of σ_{tY} is a Lagrangian submanifold of $U_{\mathrm{ad}}\cdot H$, with respect to the symplectic form Ω^τ.

Proof. By Proposition 4.13, $iY(x)$ is a Hamiltonian vector field that is constant in each fiber, which means that if σ_{tY} is its flow, then

$$\sigma_{tY}(x+X) = x + X + tiY(x).$$

In particular, the image of σ_{tY} on $\mathbb{F}_H$ (0-section) is a Lagrangian submanifold because the 0-section is Lagrangian. □

Denote by L_{tY} the image of section σ_{tY}, which is Lagrangian submanifold of $U_{\mathrm{ad}} \cdot H$. The next step is to find the tangent space to the section $x \mapsto x + iY(x)$. If $iY(x)$ is a section of $U_{\mathrm{ad}} \cdot H \to \mathbb{F}_H$, then the tangent space $T_x\mathbb{F}_H$ is generated by the vectors $\widetilde{A}(x) = [A, x]$ with $A \in \mathfrak{u}$. Therefore, to determine the space tangent to the section we have to compute the differential of Y in the direction of $\widetilde{A}(x) = [A, x]$. By the formula of the Lie bracket of vector fields we have $dY_x\left(\widetilde{A}(x)\right) = \left[Y, \widetilde{A}\right](x) + d\widetilde{A}_x(Y(x))$ and since $\widetilde{A}(x) = [A, x]$ is a linear field it follows that

$$dY_x\left(\widetilde{A}(x)\right) = \left[Y, \widetilde{A}\right](x) + [A, Y(x)]. \tag{4.9}$$

Multiplying this differential by i and adding the base vector, we get a vector tangent to the image of the section as

$$[A, x] + i\left[Y, \widetilde{A}\right](x) + i[A, Y(x)] \qquad A \in \mathfrak{u}.$$

These vectors are in fact tangent to the orbit $U_{\mathrm{ad}} \cdot H$ because $\left[Y, \widetilde{A}\right](x) \in T_x\mathbb{F}_H$ and, therefore, $i\left[Y, \widetilde{A}\right](x)$ is tangent to the fiber over x. The sum

$$[A, x] + i[A, Y(x)] = [A, x + iY(x)]$$

is tangent to the orbit because $A \in \mathfrak{u}$. This last equality is written as $\mathrm{ad}(A)(\sigma_Y(x))$ where $\sigma_Y(x) = x + iY(x)$ is the section defined by the field Y. Thus, the tangent vectors to the section σ_Y are

$$\mathrm{ad}(A)(\sigma_Y(x)) + i\left[Y, \widetilde{A}\right](x) \qquad A \in \mathfrak{u}.$$

This proves the following characterization of the tangent spaces to the sections.

Proposition 4.15. *The tangent space to L_{tY} on the section $\sigma_{tY}(x) = x + itY(x)$ of $U_{\mathrm{ad}} \cdot H \to \mathbb{F}_H$ is generated by*

$$[A, x] + ti\left[Y, \widetilde{A}\right](x) + ti[A, Y(x)] = \mathrm{ad}(A)(\sigma_{tY}(x)) + ti\left[Y, \widetilde{A}\right](x),$$

with $A \in \mathfrak{u}$.

By the construction of the symplectomorphism $\widetilde{\psi}$, we conclude that

Corollary 4.16. *The manifolds* $\widetilde{\psi}^{-1}(L_{tY})$ *are Lagrangian submanifolds of* $\mathrm{Ad}(G)\cdot H$ *with respect to the symplectic form* Ω^{τ}.

4.2.2 Real immersion

Let $\mathfrak{g}$ be a real semisimple non-compact Lie algebra, such that is a real form of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{u}$ a compact real form of $\mathfrak{g}_{\mathbb{C}}$ with Cartan involution τ, such that

$$\mathfrak{g} = \underbrace{(\mathfrak{g} \cap \mathfrak{u})}_{\mathfrak{k}} \oplus \underbrace{(\mathfrak{g} \cap i\mathfrak{u})}_{\mathfrak{s}} \tag{4.10}$$

is a Cartan decomposition of $\mathfrak{g}$.

Lemma 4.17. *The restriction of* $\mathcal{H}_{\tau}$ *to* $\mathfrak{g}$ *is real.*

Proof. For $X, Y \in \mathfrak{g}$, there are $X_1, Y_1 \in \mathfrak{g} \cap \mathfrak{u}$ and $X_2, Y_2 \in \mathfrak{g} \cap i\mathfrak{u}$ such that $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. Then

$$\tau X_1 = X_1, \quad \tau X_2 = -X_2, \quad \tau Y_1 = Y_1, \quad \tau Y_2 = -Y_2.$$

As we have that

$$\mathcal{H}_{\tau}(X, Y) = -\langle X, \tau Y\rangle_{\mathbb{C}} = -\langle X_1 + X_2, Y_1 - Y_2\rangle_{\mathbb{C}}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the Cartan–Killing form of $\mathfrak{g}_{\mathbb{C}}$, then

$$\mathcal{H}_{\tau}(X, Y) = -\langle X_1, Y_1\rangle_{\mathbb{C}} - \langle X_2, Y_1\rangle_{\mathbb{C}} + \langle X_1, Y_2\rangle_{\mathbb{C}} + \langle X_2, Y_2\rangle_{\mathbb{C}}. \tag{4.11}$$

However

$$\begin{aligned}
\mathcal{H}_{\tau}(Y, X) &= -\langle Y, \tau X\rangle_{\mathbb{C}} \\
&= -\langle Y_1 + Y_2, X_1 - X_2\rangle_{\mathbb{C}} \\
&= -\overline{\langle X_1 - X_2, Y_1 + Y_2\rangle_{\mathbb{C}}} \\
&= -\overline{\langle X_1, Y_1\rangle_{\mathbb{C}}} - \overline{\langle X_1, Y_2\rangle_{\mathbb{C}}} + \overline{\langle X_2, Y_1\rangle_{\mathbb{C}}} + +\overline{\langle X_2, Y_2\rangle_{\mathbb{C}}},
\end{aligned}$$

as $\mathcal{H}_{\tau}$ is an Hermitian form, we have that $\mathcal{H}_{\tau}(X, Y) = \overline{\mathcal{H}_{\tau}(Y, X)}$, thus,

$$\langle X_2, Y_1\rangle_{\mathbb{C}} = \langle X_1, Y_2\rangle_{\mathbb{C}},$$

and by equation 4.11, we have that

$$\mathcal{H}_\tau(X,Y) = -\langle X_1, Y_1\rangle_{\mathbb{C}} + \langle X_2, Y_2\rangle_{\mathbb{C}},$$

but $X_1, Y_1, iX_2, iY_2 \in \mathfrak{u}$, and the restriction of $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ to $\mathfrak{u}$ is negative-definite, we can conclude that $\mathcal{H}_\tau|_{\mathfrak{g}}$ is real. $\square$

Corollary 4.18. $\Omega_\tau|_{\mathfrak{g}} \equiv 0$.

Moreover, let $G^{\mathbb{C}}$ be a Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then,

Proposition 4.19. *Given any submanifold M of $\mathrm{Ad}_r(G^{\mathbb{C}})\cdot H$ or $U_{\mathrm{ad}}\cdot H$ contained on $\mathfrak{g}$. Then M is an isotropic submanifold of $\mathrm{Ad}_r(G^{\mathbb{C}})\cdot H$ or $U_{\mathrm{ad}}\cdot H$, respectively.*

Now, our purpose is to apply the last results for a non-trivial immersion on the coadjoint semi-direct orbit to find some Lagrangian submanifolds. With the Cartan decomposition of $\mathfrak{g}$ given in (4.10), then

$$\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times_{\mathrm{ad}} \mathfrak{s} \subseteq \mathfrak{u} \times_{\mathrm{ad}} i\mathfrak{u} = \mathfrak{u}_{\mathrm{ad}}.$$

Take $K = \langle \exp \mathfrak{k}\rangle$, then $K_{\mathrm{ad}}\cdot H$ is an immersed submanifold on $U_{\mathrm{ad}}\cdot H$, for $H \in \mathfrak{a}$. Moreover,

$$T_x K_{\mathrm{ad}}\cdot H \subseteq \mathfrak{k}_{\mathrm{ad}} \qquad \forall x \in K_{\mathrm{ad}}\cdot H,$$

where $\mathfrak{k}_{\mathrm{ad}}$ can be identified with $\mathfrak{g}$ as a vector space and by Corollary 4.18, the restriction of $\mathcal{H}_\tau$ to $\mathfrak{g}$ is real, thus,

$$\Omega_\tau|_{\mathfrak{k}_{\mathrm{ad}}} \equiv 0.$$

Therefore, $K_{\mathrm{ad}}\cdot H$ is an isotropic submanifold of $U_{\mathrm{ad}}\cdot H$, we want to see that $K_{\mathrm{ad}}\cdot H$ is a Lagrangian submanifold of $U_{\mathrm{ad}}\cdot H$, as we can see in the following example.

Example 4.20. *For $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{k} = \mathfrak{so}(2)$ and $\mathfrak{u} = \mathfrak{su}(2)$. Given*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a},$$

we have that $K_{\mathrm{ad}}\cdot H$ (cylinder) is a 2 dimensional isotropic submanifold of $U_{\mathrm{ad}}\cdot H$, a 4-dimensional manifold.

Hence, the cylinder $K_{\mathrm{ad}}\cdot H$ is a Lagrangian submanifold of $U_{\mathrm{ad}}\cdot H$.

Let σ be an **anti-linear involutive conjugation** on $\mathfrak{g}_{\mathbb{C}}$, such that $\mathfrak{g}$ is the subspace of fixed points of σ, that is

$$\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} : \sigma(X) = X\}.$$

Thence, if we have that $\mathcal{A} := \{X \in U_{\mathrm{ad}} \cdot H : \ \sigma(X) = X\}$ coincides with $K_{\mathrm{ad}} \cdot H$, then we can conclude that $K_{\mathrm{ad}} \cdot H$ is a Lagrangian submanifold of $U_{\mathrm{ad}} \cdot H$, with respect to the Hermitian symplectic form, for $H \in \mathfrak{a}$.

As $K_{\mathrm{ad}} \cdot H$ is contained on $\mathfrak{g}$ and it is a submanifold of $U_{\mathrm{ad}} \cdot H$, we have that $K_{\mathrm{ad}} \cdot H \subseteq \mathcal{A}$. For the opposite inclusion, by equation 3.9 we have that

$$U_{\mathrm{ad}} \cdot H = \bigcup_{Y \in \mathrm{Ad}(U)\cdot H} Y + \mathrm{ad}(Y)\,(i\mathfrak{u})\,,$$

then given an element $x \in U_{\mathrm{ad}} \cdot H$ implies that

$$x = \underbrace{Y}_{\in i\mathfrak{u}} + \underbrace{[Y, iZ]}_{\in \mathfrak{u}}, \quad \text{where} \quad Y = \mathrm{Ad}(u) \cdot H,\ u \in U,\ Z \in \mathfrak{u}.$$

As $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{s}$, we have the following possibilities:

- Take $X \in \mathfrak{k}$, then $e^{tX} \in U$

$$\begin{aligned}
\sigma(x) &= \sigma\underbrace{\left(\mathrm{Ad}(e^{tX}) \cdot H\right)}_{\in \mathfrak{s}} + \sigma\left(i[\mathrm{Ad}(e^{tX}) \cdot H, Z]\right)\\
&= \mathrm{Ad}(e^{tX}) \cdot H - i\left([\sigma\,\mathrm{Ad}(e^{tX}) \cdot H, \sigma Z]\right)\\
&= \mathrm{Ad}(e^{tX}) \cdot H - i\left([\mathrm{Ad}(e^{tX}) \cdot H, \sigma Z]\right),
\end{aligned}$$

  if $Z \in \mathfrak{k}$, we have that $\sigma(Z) = Z$ and if $Z \in i\mathfrak{s}$, we have that $\sigma(Z) = -Z$, then $\sigma(x) = x$ if and only if $Z \in i\mathfrak{s}$.

  Thus, x is a fixed point if and only if $x \in K_{\mathrm{ad}} \cdot H$.

- Take $X \in i\mathfrak{s}$, then $e^{tX} \in U$

$$\begin{aligned}
\sigma(x) &= \sigma\underbrace{\left(\mathrm{Ad}(e^{tX}) \cdot H\right)}_{\in i\mathfrak{k}} + \sigma\left(i[\mathrm{Ad}(e^{tX}) \cdot H, Z]\right)\\
&= -\,\mathrm{Ad}(e^{tX}) \cdot H - i\left([\sigma\,\mathrm{Ad}(e^{tX}) \cdot H, \sigma Z]\right)\\
&= -\,\mathrm{Ad}(e^{tX}) \cdot H + i\left([\mathrm{Ad}(e^{tX}) \cdot H, \sigma Z]\right),
\end{aligned}$$

  for $Z \in \mathfrak{u}$, we have that $\sigma(x) \neq x$, then in this case it is impossible to have fixed points.

- Any other possible choice of $X \in \mathfrak{u}$, we do not have fixed points because it would be a combination of the cases above.

Therefore, $\mathcal{A} = K_{\mathrm{ad}} \cdot H$, and $K_{\mathrm{ad}} \cdot H$ is the set of fixed points of σ, its dimension is half the dimension of $U_{\mathrm{ad}} \cdot H$. Hence,

Proposition 4.21. *For $H \in \mathfrak{a}$, the coadjoint orbit $K_{\mathrm{ad}} \cdot H$ is a Lagrangian submanifold of $U_{\mathrm{ad}} \cdot H$, with respect to the Hermitian symplectic form.*

By Corollary 4.9, the coadjoint orbit K_{ad} deforms into $\mathrm{Ad}(G) \cdot H$ and U_{ad} deforms into $\mathrm{Ad}(G^{\mathbb{C}}) \cdot H$. Then, we can conclude that

Corollary 4.22. *For $H \in \mathfrak{a}$, the orbit $\mathrm{Ad}(G) \cdot H$ is a Lagrangian submanifold of $\mathrm{Ad}(G^{\mathbb{C}}) \cdot H$, with respect to the Hermitian symplectic form.*

Furthermore, the coadjoint orbit $U_{\mathrm{ad}} \cdot H$ is invariant by automorphism of $\mathfrak{u}$, because any automorphism of $\mathfrak{u}$ leaves invariant its Cartan subalgebra (see [28] or [31]). Given $k \in \mathrm{Aut}(\mathfrak{k})$ we know that the k-action on $\mathfrak{g}$ leaves invariant the Cartan decomposition of $\mathfrak{g}$, its maximal Abelian subalgebra and $\mathfrak{u}$ (because $\mathfrak{k}$ is contained in $\mathfrak{u}$). If exp is the exponential between the Lie algebra $\mathfrak{u}$ and the Lie group $\mathrm{Aut}(\mathfrak{u})$, then for any $X \in i\mathfrak{s}$ we have that $\mathfrak{g}^{tX} = \exp(tX) \cdot \mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$ with Cartan decomposition $\mathfrak{g}^{tX} = \mathfrak{k}^{tX} \oplus \mathfrak{s}^{tX}$. Take G^{tX} a Lie group with Lie algebra $\mathfrak{g}^{tX} \subset \mathfrak{u}$, then we can conclude that

Corollary 4.23. *For $X \in i\mathfrak{s} \subset \mathfrak{u}$, the adjoint orbit $\mathrm{Ad}(G^{tX}) \cdot \widetilde{H}$, where $\widetilde{H} = \exp(tX) \cdot H$ is a Lagrangian submanifold of $\mathrm{Ad}(G^{\mathbb{C}}) \cdot H$, with respect to the Hermitian symplectic form.*

In fact, we have associated a family of Lagrangian submanifolds of U determined by $\mathfrak{g}$, and given by the $i\mathfrak{s}$-conjugated real forms of $\mathfrak{g}$.

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APPENDIX A – Tight submanifolds

This appendix is dedicated to studying the locally, global and infinitesimally tight Lagrangian submanifolds on adjoint orbits. Such class of submanifolds was introduced in 1991, by Y.-G. Oh (see [25]). In this paper the author studied the notion of tightness of closed Lagrangian submanifolds in compact Hermitian symmetric spaces and it was defined as follows:

Definition A.1. *Let (M, ω, J) be a Hermitian symmetric space of a compact type and $\mathcal{L}$ be a closed embedded Lagrangian submanifold of M. Then $\mathcal{L}$ is said to be* **globally tight** *(resp.* **tight***) if it satisfies*

$$\#\left(\mathcal{L} \cap g \cdot \mathcal{L}\right) = \mathrm{SB}\left(\mathcal{L}, \mathbb{Z}_2\right)$$

for any isometry $g \in G$ (resp. close to identity) such that $\mathcal{L}$ transversely intersects with $g \cdot \mathcal{L}$. Here $\mathrm{SB}\left(\mathcal{L}, \mathbb{Z}_2\right)$ denotes the sum of $\mathbb{Z}_2$-Betti numbers of $\mathcal{L}$.

In that paper, the author showed that the standard $\mathbb{RP}^n$ inside $\mathbb{CP}^n$ is tight and has the least volume among all its Hamiltonian deformations. Then the concept of tightness has applications to the problem of Hamiltonian volume minimization. Furthermore, the famous Arnold—Givental conjecture predicts that the number of intersection points of a Lagrangian $\mathcal{L}$ and its image $\phi(\mathcal{L})$ by the flow of a Hamiltonian vector field can be estimated from below by the sum of its $\mathbb{Z}_2$-Betti numbers, that is:

$$|\mathcal{L} \cap \phi(\mathcal{L})| \geqslant \sum b_k(\mathcal{L}; \mathbb{Z}_2).$$

Then the concepts of tightness address those Lagrangians which attain the lower bound, and are therefore of general interest in symplectic geometry. Besides, Oh posed the following open problem:

Problem. *Classify all possible tight Lagrangian submanifolds in other Hermitian symmetric spaces. Are the real forms on them the only possible tight Lagrangian submanifolds?*

This problem is strictly related to our result given in the Section 2.2, where we have proven that the real flags can be seen as Lagrangian submanifolds of the complex flag manifolds, that is, its respective real form is Lagrangian. Then, our goal in this appendix is to see that the Oh's conjecture is satisfied for the complex flag manifolds case.

With the same aim, Iriyeh and Sakai studied and classified the Tight submanifolds on $S^2 \times S^2$ (see [16]), where the authors proved that if $\mathcal{L}$ is a closed embedded tight Lagrangian surface on $S^2 \times S^2$, then $\mathcal{L}$ must be one of the following cases:

- $\mathcal{L} = \{(x, -x) \in S^2 \times S^2 : \ x \in S^2\}$ (global tight submanifold).
- $\mathcal{L} = S^1(a) \times S^1(b) \subset S^2 \times S^2$, where $S^1(a)$ stands for the round circle with radius $0 < a \leqslant 1$. (locally tight submanifold).

This is a particular case of tight submanifolds in the product of flag manifolds, that was studied in [13]. In this paper, the authors showed that a product of flag manifolds $\mathbb{F}_{\Theta_1} \times \mathbb{F}_{\Theta_2}$ admits a Lagrangian orbit by the diagonal action (and shifted diagonal action) if and only if $\Theta_2 = \Theta_1^*$, that is $\Theta_2 = \sigma\Theta_1$ where σ is the symmetry of the Dynkin diagram given by $\sigma = -w_0$ and w_0 is the main involution (element of greatest length) of the Weyl group $\mathcal{W}$. Such Lagrangian orbit is given by the graph of

$$-\operatorname{id} : \operatorname{Ad}(U)(iH) \to \operatorname{Ad}(U)(i\sigma(H)),$$

or the graph of $-\operatorname{Ad}(m)$, $m \in U$ for the shifted diagonal action.

But perhaps the most important fact given in that paper was the concept of infinitesimally Tight submanifold, where the authors prove that the Lagrangian orbits given by the diagonal (shifted diagonal) action are infinitesimally tight.

This new concept is defined as follows:

Definition A.2. *Let $\mathcal{L}$ in $M = G/H$ be a submanifold. An element $X \in \mathfrak{g} = \operatorname{Lie}(G)$ is called* **transversal** *to $\mathcal{L}$ if it satisfies the following two conditions:*

1. *for any $x \in \mathcal{L}$, if $\widetilde{X}(x) \in T_xN$ then $\widetilde{X}(x) = 0$, and*
2. *the set*
$$f_N(X) = \left\{x \in N : \ 0 = \widetilde{X}(x) \in T_xN\right\}$$
is finite.

That is, $\widetilde{X}$ is only tangent to $\mathcal{L}$ at most at finitely many points where it vanishes.

A Lagrangian submanifold $\mathcal{L}$ in $M = G/H$ is called **infinitesimally tight** if the equality

$$\#\left(f_{\mathcal{L}}(X)\right) = \mathrm{SB}\left(\mathcal{L}, \mathbb{Z}_2\right)$$

holds for any $X \in \mathfrak{g}$ such that $\widetilde{X}$ is transversal to $\mathcal{L}$.

Example A.3. *As was showed in Example 2.3, the sphere $\mathcal{L} = S^3$ given by the action of $U_H \simeq U(2)$ is a Lagrangian submanifold of $\mathbb{F}_3(1,2)$. We have that*

$$(\mathfrak{u}_H)^{\perp} = \left\{ X_\beta = \begin{pmatrix} 0 & \beta \\ -\overline{\beta}^T & 0 \end{pmatrix} : \ \beta = (z_1, z_2) \in \mathbb{C}^2 \right\},$$

then $\mathcal{L} = \mathrm{Ad}(U)(iH_0) \cap (\mathfrak{u}_H)^{\perp}$, where $H_0 = \mathrm{diag}\{1, 0, -1\}$.

If $X_\beta, X_\gamma \in (\mathfrak{u}_H)^{\perp}$, we have that

$$[X_\beta, X_\gamma] = \begin{pmatrix} i\ \mathrm{Im}\gamma\overline{\beta}^T & 0 \\ 0 & -\overline{\beta}^T\gamma + \overline{\gamma}^T\beta \end{pmatrix}.$$

Given $x = X_\gamma \in \mathcal{L}$ and $X \in \mathfrak{u}$ then $\widetilde{X}(x) = \mathrm{ad}(X)(X_\gamma)$, take the decomposition $X = Y + X_\beta$, for $Y \in \mathfrak{u}_H$ and $X_\beta \in (\mathfrak{u}_H)^{\perp}$. Then, X is transversal if $\widetilde{X}_\beta(x) = 0$ and $[Y, X_\beta] = 0$, because the singularities of $\widetilde{X}_\beta$ on S^3 are the elements in the set $\mathbb{R}X_\beta \cap S^3$ (a finite set).

As we only have 2 points (antipodals) in $\mathbb{R}X_\beta \cap S^3$, and the sum of $\mathbb{Z}_2$-Betti numbers of S^3 is 2, we can conclude that S^3 is an infinitesimally tight submanifold of $\mathbb{F}_3(1,2)$.

Furthermore, in [13] was given the following Theorem:

Theorem A.4. *Let $M = G/H$ a homogeneous space with a G-invariant symplectic form ω. Then a Lagrangian submanifold $L \subset M$ is infinitesimally tight if and only if L is locally tight.*

Thus, the manifold S^3 is a local tight submanifold of $\mathbb{F}_3(1,2)$.

To generalize the Example A.3 for the complex flag manifolds given in Table 1, we need to compute the sum of $\mathbb{Z}_2$-Betti numbers of the real flag manifolds and to find the transversal elements.

To begin with, by the paper [27] we have the following:

Let $H \in \mathrm{cl}(\mathfrak{a}^+)$, the $\mathbb{Z}_2$-homology of $\mathrm{Ad}(K) \cdot H$ is freely generated by the **Schubert cells** $\mathcal{S}_{[w]}^{\Theta_H}$, for $[w] \in \mathcal{W}/\mathcal{W}_{\Theta_H}$. Therefore

$$\mathrm{SB}\left(\mathrm{Ad}(K) \cdot H, \mathbb{Z}_2\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right). \tag{A.1}$$

And, as $\mathrm{Ad}(K) \cdot H \subseteq \mathfrak{s} = i\mathfrak{k}^{\perp}$, for $x \in \mathrm{Ad}(K) \cdot H$ we have that:

$$T_x\left(\mathrm{Ad}(K) \cdot H\right) = \left\{\widetilde{A}(x) : \; A \in \mathfrak{k}\right\},$$

where $\widetilde{A} = \mathrm{ad}(A)$. Then

- If $X \in \mathfrak{k}^{\perp}$, then $\widetilde{X} = \mathrm{ad}(X)$ is a Hamiltonian field of the function $H_X = \langle X, x\rangle$. Thus the singularities of X are the singularities of H_X, and their number is finite, if and only if X is regular.

  Therefore, the transversal elements are the regular elements X, and they satisfies

  $$\#\left(f_{\mathrm{Ad}(K)\cdot H}(X)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

- If $Y \in \mathfrak{k}$, then $\widetilde{Y}$ is tangent, thus it cannot be transversal.

- If $Z = X + Y$ for $X \in \mathfrak{k}^{\perp}$ and $Y \in \mathfrak{k}$, then $\widetilde{Z}(x) \notin T_x \, \mathrm{Ad}(K) \cdot H$ if $\widetilde{X}(x) \neq 0$, so for Z to have singularity in x we need that $\widetilde{X}(x) = \widetilde{Y}(x) = 0$ in a finite quantity. But this only happens for X regular, such that $[X, Y] = 0$. Thus:

  $$\#\left(f_{\mathrm{Ad}(K)\cdot H}(Z)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

Hence:

Theorem A.5. *The real flags are infinitesimally Tight submanifolds of their corresponding complex flag, given in the Table 1.*

Therefore, as a consequence of Theorem A.4:

Corollary A.6. *The real flags are local Tight submanifolds of their corresponding complex flag.*

Remark A.7. *There are several articles related to this kind of Lagrangian submanifold. For instance, in [14], for any irreducible compact homogeneous Kähler manifold were classified the compact tight Lagrangian submanifolds which have the $\mathbb{Z}_2$-homology of a sphere. In this article, the authors gave a brief discussion about the tight Lagrangian submanifolds on the complex flag manifolds.*