UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

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Sheaves and distributions on threefold hypersurfaces

Feixes e distribuições sobre hipersuperfícies tridimensionais

Campinas

2021

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Este trabalho corresponde à versão final da Tese defendida pelo aluno Danilo de Rezende Santiago e orientada pelo Prof. Dr. Marcos Benevenuto Jardim.

Campinas

2021

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

Sa59s

Santiago, Danilo de Rezende, 1991-
Sheaves and distributions on threefold hypersurfaces / Danilo de Rezende Santiago. – Campinas, SP : [s.n.], 2021.

Orientador: Marcos Benevenuto Jardim.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Hipersuperfícies. 2. Feixes reflexivos. 3. Distribuições de codimensão 1. 4. Mônadas (Matemática). 5. Espaço de módulos. I. Jardim, Marcos Benevenuto, 1973-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Feixes e distribuições sobre hipersuperfícies tridimensionais
Palavras-chave em inglês:
Hypersurfaces
Reflexive sheaves
Codimension one distributions
Monads (Mathematics)
Moduli spaces
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Marcos Benevenuto Jardim [Orientador]
Alana Cavalcante Felippe
Alan do Nascimento Muniz
Simone Marchesi
Maurício Barros Corrêa Júnior
Data de defesa: 29-04-2021
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0002-8707-4111
- Currículo Lattes do autor: http://lattes.cnpq.br/1197633578570969

Tese de Doutorado defendida em 29 de abril de 2021 e aprovada

pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

To my parents, Helena Rezende and Luiz Francisco.

Acknowledgements

First of all, I thank God for putting people on my academic journey who trained and encouraged me. I thank my friend, Prof. Dr. Danilo Dias, for all the guidance since my master's degree. I thank all professors of undergraduate and graduate who helped in my training, especially the supervisor Prof. Dr. Marcos Jardim for making this dream possible.

I thank my family, colleagues and friends, Fábio Rodrigues and Luiz Carlos, who welcomed me on my arrival in Campinas-Sp. I thank my friends Guilherme, Jośe Lucas, Thiago, Miqueias, Matheus and Felipe. I thank you for the discussions that Douglas, Hugo and Charles provided. I thank the professors Alana Cavalcante, Alan Muniz, Maurício Correia and Simone Marchesi for accepting the invitation to evaluate this work.

I thank my girlfriend, Elismara Sousa, for being at my side, encouraging and supporting me in all areas of my life.

I am grateful to all IMECC employees, especially to the employees of the Graduate Secretariat for being so kind and helpful to their students.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq with the process 141174/2019-6.

Resumo

Esta tese é dedicada ao estudo das distribuições genéricas de codimensão um e das folheações por curvas sobre as hipersuperfícies suaves tridimensionais. Mostramos que os feixes normais de folheações por curvas genéricas em $\mathbb{P}^3$ preenchem componentes irredutíveis dos espaços de módulos dos feixes reflexivos estáveis de posto 2 e classes de Chern prescritas. Construímos também famílias de feixes reflexivos estáveis de posto 2 sobre hipersuperfícies suaves de dimensão 3 e grau $d \in \{2, 3, 4, 5\}$ contendo as distribuições genéricas de codimensão um que preenchem componentes irredutíveis dos espaços de módulos dos feixes reflexivos estáveis de posto 2 e determinadas classes de Chern.

Estudamos também os feixes localmente livres de posto 2 e classes de Chern $c_1 = 0$ e $c_2 = d \cdot H^2$ que são dados como cohomologia de uma mônada linear sobre uma hipersuperfície suave de dimensão 3 e grau $d \geq 2$. Apresentamos uma caracterização cohomológica destes feixes como também fazemos uma descrição matricial deles utilizando representações de aljavas.

Palavras-chave: hipersuperfícies, feixes reflexivos, folheações por curvas, distribuições, mônadas, representações de aljavas, espaço de módulos.

Abstract

This thesis is dedicated to the study of generic codimension one distributions and foliations by curves on the smooth three dimensional hypersurfaces. We show that the normal sheaves of a generic foliations by curves on $\mathbb{P}^3$ fill irreducible components of the moduli spaces of the stable rank 2 reflexive sheaves with prescribed Chern classes. We also build families of the stable rank 2 reflexive sheaves on smooth threefold hypersurfaces of degree $d \in \{2, 3, 4, 5\}$ containing the generic codimension one distributions which fill an irreducible components of the moduli spaces of stable rank 2 reflexive sheaves with prescribed Chern classes.

We also study the stable rank 2 locally free sheaves and Chern classes $c_1 = 0$ e $c_2 = d \cdot H^2$ that are given as cohomology sheaves of a linear monads on a smooth hyperfurfaces of dimension 3 and degree $d \geq 2$. We present a cohomological characterization of these sheaves as we also make a matrix description of them using quiver representations.

Keywords: hypersurfaces. reflexive sheaves. foliations by curves. distributions. monads. quiver representations. moduli spaces.

Contents

Introduction

In [15] Maruyama proved that the rank r stable reflexive sheaves on a projective variety X with fixed Chern classes $c_1, \ldots, c_r$ can be parametrized by an algebraic quasi-projective variety, denoted by $\mathcal{M}_X(r; c_1, \ldots, c_r)$. Although this result has been known for almost 40 years, there are just a few concrete examples and established facts about such varieties. When $X = \mathbb{P}^3$, $r = 2$ and prescribed Chern classes, there are several works in this direction in the literature, see for example [8, 14, 21] . In this work we are particularly interested in the small degree 3-fold hypersurfaces, namely, the smooth 3-fold hypersurface of degree $d \in \{1, 2, 3, 4, 5\}$. This class includes an example of Calabi-Yau 3-folds, the 3-fold hypersurfaces of degree $d = 5$. We build concrete examples of moduli spaces of stable rank 2 reflexive sheaves on such projective varieties.

There is a connection between reflexive sheaves and the distributions and foliations on a projective variety. The mathematicians as Grassmann, Jacobi, Clebsch, Cartan and Frobenius started the study of the theory of distributions and foliations in the 19th century. They were motivated by the work due to Pfaff, who proposed a geometric approach to the study of differential equations, see [3, Chapter III]. The qualitative study of foliations induced by polynomial differential equations was investigated by Poincaré, Darboux and Painelevé. In modern terminology, this corresponds to the study of codimension one holomorphic foliations on complex projective spaces.

Techniques from algebraic geometry have been extremely useful in the study of distributions and foliations on complex projective spaces, see for instance [1, 2, 7, 22, 23, 25, 26]. From the point of view of algebraic geometry, a *foliation by curves* $\mathscr{F}$ on a smooth projective threefold X is a short exact sequence of the form

$$\mathscr{F} \quad : \quad 0 \to \mathcal{O}_X(-r - \tau_X) \xrightarrow{\sigma} TX \to N_{\mathscr{F}} \to 0 \tag{1}$$

where $N_{\mathscr{F}}$ is a torsion free sheaf called the *normal sheaf* of $\mathscr{F}$ and

$$\tau_X := \min\{t \in \mathbb{Z} \mid H^0(TX(t)) \neq 0\}.$$

The non negative integer r above is called the *degree* of $\mathscr{F}$. Note that $\mathrm{rk}(N_{\mathscr{F}}) = 2$.

The image of the morphism $\sigma^{\vee} : \Omega_X^1 \to \mathcal{O}_X(\tau_X + r)$ is the twisted ideal sheaf $I_Z(r + \tau_X)$ of a subscheme of X of dimension at most 1, called the *singular scheme* of $\mathscr{F}$ and denoted by $\mathrm{Sing}(\mathscr{F})$. Thus dualizing the sequence in display (1) we obtain

$$0 \to N_{\mathscr{F}}^{\vee} \to \Omega_X^1 \xrightarrow{\sigma^{\vee}} I_Z(r + \tau_X) \to 0, \tag{2}$$

where $N_{\mathscr{F}}^{\vee}$ is called the *conormal sheaf of* $\mathscr{F}$.

In [1], we prove in section 5 that if the singular scheme has dimension 0, then the conormal sheaves of the foliations on a smooth projective variety X of dimension 3 and Picard rank 1 are μ-stable, whenever the tangent bundle TX is μ-stable, and apply this fact to the characterization of certain irreducible components of the moduli space of rank 2 reflexive sheaves on $\mathbb{P}^3$ and on a smooth quadric hypersurface $Q_3 \subset \mathbb{P}^4$.

Main Theorem 1. *1. The moduli space of stable rank 2 sheaves on* $\mathbb{P}^3$ *with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, 3k^2 + 4k + 2, 8k^3 + 16k^2 + 12k + 4), & k \geq 1 \\ (-1, 3k^2 + k + 1, 8k^3 + 4k^2 + 2k + 1), & k \geq 0 \end{cases}$$

contains a rational irreducible component whose generic point is the normal sheaf of a generic foliation by curves on $\mathbb{P}^3$*.*

2. The moduli space of stable rank 2 sheaves on Q_3 *with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 6)H^3), & k \geq 1 \\ (-H, (3k^2 + 3k + 2)H^2, (8k^3 + 12k^2 + 8k - 2)H^3), & k \geq 0 \end{cases}$$

contains a irreducible component containing the family of the generic foliation by curves on Q_3*.*

These results are in sections 6 and 7 of article [1] and chapter 2 of this thesis.

A *generic codimension 1 distribution* $\mathscr{F}$ on a smooth projective threefold X is given by an exact sequence

$$\mathscr{F} \quad : \quad 0 \to T_{\mathscr{F}} \xrightarrow{\sigma} TX \to I_Z(r + 2) \to 0, \tag{3}$$

where $T_{\mathscr{F}}$ is a reflexive sheaf of rank 2 called of *tangent sheaf* of $\mathscr{F}$, $r := c_1(TX) - c_1(T_{\mathscr{F}}) - 2 \geq 0$ is the *degree* of $\mathscr{F}$ and I_Z is an ideal sheaf of a subscheme $Z :=$ Sing$(\mathscr{F})$, called the *singular scheme* of $\mathscr{F}$, with Sing$(\mathscr{F})$ empty or has dimension equal to zero.

In [26], it is shown that the codimension one distributions with at most isolated singularities on certain smooth projective 3-folds with Picard group rank 1 have μ-stable tangent sheaves. Moreover, the authors characterized certain irreducible components of the moduli space rank 2 reflexive sheaves on $\mathbb{P}^3$. In the following theorem, proved in Chapter 2, we characterize certain irreducible components of the moduli space rank 2 reflexive sheaves on the 3-folds smooth hypersurfaces of degree $d \in \{2, 3, 4, 5\}$.

Main Theorem 2. *1. The moduli space of stable rank 2 reflexive sheaves on Q_3 with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 6)H^3), & k \geq 1 \\ (-H, (3k^2 + 3k + 2)H^2, (8k^3 + 12k^2 + 8k - 2)H^3), & k \geq 0 \end{cases}$$

contains a irreducible component containing the family of the tangent sheaves of a generic codimension one distributions on Q_3.

2. *The moduli space of stable rank 2 reflexive sheaves on a smooth cubic threefold hypersurface X with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2 + 4k + 4)H^2, (8k^3 + 16k^2 + 16k + 10)H^3), & k \geq 1 \\ (-H, (3k^2 + 7k + 7)H^2, (8k^3 + 28k^2 + 38k + 23)H^3), & k \geq 0 \end{cases}$$

contains a irreducible component containing the family of the tangent sheaves of a generic codimension one distributions on X.

3. *The moduli space of stable rank 2 reflexive sheaves on a smooth quartic threefold hypersurface X with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2 + 8k + 11)H^2, (8k^3 + 32k^2 + 54k + 50)H^3), & k \geq 0 \\ (-H, (3k^2 + 5k + 8)H^2, (8k^3 + 20k^2 + 28k + 30)H^3), & k \geq 0 \end{cases}$$

contains a irreducible component containing the family of the tangent sheaves of a generic codimension one distributions on X.

4. *The moduli space of stable rank 2 reflexive sheaves on a smooth quintic threefold hypersurface X with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2 + 6k + 13)H^2, (8k^3 + 24k^2 + 44k + 68)H^3), & k \geq 0 \\ (-H, (3k^2 + 9k + 17)H^2, (8k^3 + 36k^2 + 74k + 97)H^3), & k \geq 0 \end{cases}$$

contains a irreducible component containing the family of the tangent sheaves of a generic codimension one distributions on X.

We wish to produce an article based on these results.

We also use two new tools, monads and quiver representations, to study a family of locally free sheaves on a 3-fold hypersurface of degree $d \geq 2$.

A *monad* over a projective variety X is a complex

$$\mathcal{M}_\bullet : \; 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of locally free sheaves A, B and C on X which is exact at A and at C. The coherent sheaf

$$E = \ker(\beta)/\operatorname{Im}(\alpha)$$

is called the *cohomology sheaf of the monad* $\mathcal{M}_\bullet$ and one also says that $\mathcal{M}_\bullet$ is a monad for E. This is one of the simplest ways of constructing sheaves, after kernels and cokernels. Some authors have presented existence conditions for monads on a large class of projective varieties, see [10, 17, 32]. In particular, [17] showed the existence of monads on a 3-fold smooth hypersurface of the form

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0, \quad c \geq 1. \tag{4}$$

Here we will study the case $c = 1$, the family of locally free sheaves on a 3-fold smooth hypersurface of degree $d \geq 2$ that arise as a cohomology sheaf of monad

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0. \tag{5}$$

Initially, we apply [20, Theorem 3.3] to obtain a cohomological characterization of these locally free sheaves. Later we use the connection between monads over a projective variety and representations of quivers, see [18, 19, 34], to give a matrix description of the locally free sheaves that are obtained from (5).

Definition 0.0.1. Let X be a 3-fold hypersurface. We say that a matrix $A \in \mathrm{Mat}_{4\times 5}(\mathbb{C})$ is *globally injective* on X if for every $(\lambda_1 : \cdots : \lambda_5) \in X$, we have

$$\sum_{i=1}^{5} \lambda_i A_i \neq 0,$$

where A_i are the columns of the matrix A. Similarly, we say that a matrix $B \in \mathrm{Mat}_{5\times 4}(\mathbb{C})$ is *globally surjective* on X if for every $(\lambda_1 : \cdots : \lambda_5) \in X$, we have

$$\sum_{i=1}^{5} \lambda_i B_i \neq 0,$$

where B_i are the lines of the matrix B.

In section 3.2, we prove:

Main Theorem 3. *There is a bijective correspondence between pairs* (A, B)*, where*

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{51} & c_{52} & c_{53} & c_{54} \end{pmatrix},$$

with A globally injective on X and B globally surjective on X and isomorphism classes of monads whose cohomology sheaf is locally free as in (5)*.*

As an application of the Ottaviani's Bertini-type theorem [12, Teorema 2.8], we finish this work showing that this family of locally free sheaves also appears as a family of distributions and foliations.

We will now give a short overview of the contents of this thesis.

Chapter 1: We introduce some preliminaries necessary through the text. In the first section we classify the cohomology rings of the invertible sheaves on 3-fold hypersurfaces. In the second section we recall the definition and we present some criteria to determine stability in the sense of Mumford-Takemoto and Gieseker-Maruyama. In the third section we introduce the concept of moduli spaces and moduli functors. In the fourth section we summarize the main facts abouts monads that will be useful in this thesis. In the fifth section we remember the definitions of quivers and their representations. In the sixth and final section we present an equivalence between the abelian category of representation of quivers and the category of the monads.

Chapter 2: This chapter is dedicated to the study of the generic codimension 1 distributions and generic foliations by curves on 3-fold hypersurfaces. To be more precise, in the first section we present the definition of a generic codimension 1 distribution on a smooth projective 3-fold and we study some properties of its tangent sheaf. In sections $2, 3, 4$ and 5 we do the proof of the **Main Theorem 2**. The sixth section is dedicated to the study of the generic foliations by curves on $\mathbb{P}^3$ and Q_3. Here it is made the proof of the **Main Theorem 1**.

Chapter 3: This chapter is dedicated to the study of linear monads on 3-fold hypersurfaces. In the first section we will give a cohomological characterization of stable rank 2 locally free sheaves on a 3-fold hypersurface of degree d with Chern classes $c_1 = 0$ and $c_2 = d \cdot L$. In the second section we apply the equivalence between quiver representations and monads on X to give a matrix description of the locally free sheaves that are obtained from monads, **Main Theorem 3**.

As an immediate consequence of the **Main Theorem 3**, we show that the family of stable rank 2 locally free sheaves on a 3-fold hypersurface of degree d with Chern classes $c_1 = 0$ and $c_2 = d \cdot L$ satisfying certain cohomological conditions has dimension 9. In the third section, we will give a sufficient condition to the family of the locally free sheaves on a smooth 3-fold hypersurface of degree $d = 3, 4, 5$ given as cohomology sheaf of the monad in display (5) to fill a irreducible component of the moduli space of stable rank 2 locally free sheaves on X with Chern class $c_1 = 0$ and $c_2 = d \cdot L$. In the last section, we will use these bundles to get examples of LCI foliations by curves on X, which are defined in the Section 2.6.

1 Preliminaries

In this chapter we recall some basic concepts and fix the notations that will be useful for the development of the work. We work over the complex numbers $\mathbb{C}$.

1.1 The Cohomology of 3-Fold Hypersurfaces

Let X be a smooth projective variety with Picard group $\mathrm{Pic}(X) = \mathbb{Z}$. Let $\mathcal{O}_X(1)$ the ample generator of $\mathrm{Pic}(X)$, and given a sheaf F on X we set $F(k) := F \otimes \mathcal{O}_X(1)^{\otimes k}$, $H^i(F(k))$ as its i-th cohomology group, $h^i(F(k))$ its dimension, i.e $\dim H^i(F(k)) = h^i(F(k))$ and $H^p_*(F) = \bigoplus_{k\in\mathbb{Z}} H^p(F(k))$.

Definition 1.1.1. A *3-fold hypersurface* $X \subset \mathbb{P}^4$ of the degree d is the zero locus of a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_4]_d$.

Remark 1.1.2. For a generic $f \in \mathbb{C}[x_0, \ldots, x_4]_d$, its zero locus is nonsingular.

Let X denote a smooth hypersurface of degree d in $\mathbb{P}^4$. Let H be the class of a hyperplane section, so that

$$\mathrm{Pic}(X) = H^2(X, \mathbb{Z}) = \mathbb{Z}H \quad .$$

It is know that the even cohomology ring $H^2(X, \mathbb{Z})$ is generated by H, $L \in H^4(X, \mathbb{Z})$ and $P \in H^6(X, \mathbb{Z})$ with the relations: $H^2 = dL,\ H.L = P,\ H^3 = dP$, see [17]. The dualizing sheaf of X is $\omega_X = \mathcal{O}_X(d-5)$.

The main goal here is to make the explicit calculations of the dimensions of the cohomology groups $H^i(\mathcal{O}_X(k))$, where $k \in \mathbb{Z}$, $0 \leq i \leq 3$ and $X = Z(f) \subset \mathbb{P}^4$ is a hypersurface of the degree $d \geq 2$. For this, we use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(k-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^4}(k) \to \mathcal{O}_X(k) \to 0, \tag{1.1}$$

which induces the long exact sequence in cohomology

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^4}(k-d)) \xrightarrow{\cdot f} H^0(\mathcal{O}_{\mathbb{P}^4}(k)) \to H^0(\mathcal{O}_X(k)) \to H^1(\mathcal{O}_{\mathbb{P}^4}(k-d)) = 0. \tag{1.2}$$

If $k < 0$,

$$h^0(\mathcal{O}_X(k)) = h^0(\mathcal{O}_{\mathbb{P}^4}(k)) - h^0(\mathcal{O}_{\mathbb{P}^4}(k-d)) = 0.$$

If $0 \leq k < d$,

$$h^0(\mathcal{O}_X(k)) = h^0(\mathcal{O}_{\mathbb{P}^4}(k)) - h^0(\mathcal{O}_{\mathbb{P}^4}(k-d)) = \binom{k+4}{k}.$$

If $k \geq d$,

$$h^0(\mathcal{O}_X(k)) = h^0(\mathcal{O}_{\mathbb{P}^4}(k)) - h^0(\mathcal{O}_{\mathbb{P}^4}(k-d)) = \binom{k+4}{k} - \binom{k-d+4}{k-d}.$$

Using the long exact sequence in cohomology derived from the sequence (1.1) and the fact that $\mathcal{O}_{\mathbb{P}^4}$ has no intermediate cohomology it follows that $\mathcal{O}_X$ also has no intermediate cohomology, i.e. $H^i(\mathcal{O}_X(k)) = 0$ for all $k \in \mathbb{Z}$ and $i = 1, 2$.

By Serre duality, see for example [24, Chapter I], being $\omega_X = \mathcal{O}_X(d-5)$ the dualizing sheaf, we have

$$h^3(\mathcal{O}_X(k)) = h^0(\mathcal{O}_X(-k+d-5)).$$

Furthemore, if $0 \leq i \leq 3$ and $k \in \mathbb{Z}$, we have

$$h^i(\mathcal{O}_X(k)) = \begin{cases} \binom{k+4}{k} - \binom{k-d+4}{k-d} & \text{if } i = 0 \text{ and } k \geq 0, \\ 1 & \text{if } i = 3 \text{ and } k = d-5, \\ \binom{-k+d-1}{-k+d-5} - \binom{-k-1}{-k-5} & \text{if } i = 3 \text{ and } k < d-5, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Note also that, from the sequence (1.2), we have the isomorphisms

$$H^0(\mathcal{O}_X(k)) \simeq \begin{cases} 0 & \text{if } k < 0, \\ H^0(\mathcal{O}_{\mathbb{P}^4}(k)) & \text{if } 0 \leq k < d, \\ H^0(\mathcal{O}_{\mathbb{P}^4}(k))/f.H^0(\mathcal{O}_{\mathbb{P}^4}(k-d)) & \text{if } k \geq d. \end{cases} \tag{1.4}$$

Whenever we want to calculate the cohomology of the tangent bundle TX of X we will use the standard normal bundle sequence

$$0 \to TX \to T\mathbb{P}^4|_X \to \mathcal{O}_X(d) \to 0, \tag{1.5}$$

see [9, Chapter V].

1.2 Stability

In this section we introduce the concept of stability of coherent sheaves on a irreducible smooth projective variety in the sense of Mumford-Takemoto and Gieseker-Maruyama. We also present some criteria to determine stability. For more details on stability in general abelian categories, see [5].

Definition 1.2.1. Let X be an irreducible smooth projective variety of dimension n and fix $\mathcal{L}$ a ample invertible sheaf with $c_1(\mathcal{L}) := H$. The slope $\mu(E)$ with respect to $\mathcal{L}$ of a torsion-free sheaf E on X with respect to $\mathcal{L}$ is defined as follows:

$$\mu(E) := \frac{c_1(E).H^{n-1}}{\mathrm{rk}(E)}.$$

We say that E is *μ-semistable* with respect to $\mathcal{L}$ if, for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, we have $\mu(F) \leq \mu(E)$.

Moreover, if for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have $\mu(F) < \mu(E)$, we say that E is *μ-stable* with respect to $\mathcal{L}$.

We have the following simple properties of stability and semistability, see [24, Lemma 1.2.4].

Lemma 1.2.2. *i) Line bundles are μ-stable.*

ii) The sum $E_1 \oplus E_2$ of two μ-semistable sheaves is μ-semistable if and only if $\mu(E_1) = \mu(E_2)$.

iii) E is μ-(semi)stable if and only if E^* is.

iv) E is μ-(semi)stable if and only if $E(k)$ is.

We present below a stability criterion for reflexive sheaves of rank 2 on a projective variety. For this we need the following definition.

Definition 1.2.3. Let E be a torsion-free sheaf of rank 2 on a projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$. Then there is a uniquely determined integer k_E such that $c_1(E(k_E)) \in \{-1, 0\}$. We set

$$E_{norm} := E(k_E)$$

and call E *normalized* if $E = E_{norm}$.

We then have the following criterion, see [24, Lemma 1.2.5].

Lemma 1.2.4. *A reflexive sheaf E of rank 2 on a smooth projective variety X with* $\mathrm{Pic}(X) = \mathbb{Z}$ *is μ-stable if and only if E_{norm} has no sections:*

$$H^0(E_{norm}) = 0.$$

If $c_1(E)$ is even, then E is μ-semi-stable if and only if

$$H^0(E_{norm}(-1)) = 0.$$

The next theorem characterizes the endomorphisms of a μ-stable locally free sheaf on a irreducible smooth projective variety, see [24, Theorem 1.2.9].

Theorem 1.2.5. *μ-stables locally free sheaves are simple.*

We now present the definition of stability of coherent sheaves on a smooth irreducible projective variety in the sense of Gieseker-Maruyama.

Let X be a smooth irreducible projective variety of dimension n. Recall that the Euler characteristic of a coherent sheaf F is

$$\chi(F) := \sum_{i=0}^{n} (-1)^i h^i(X, F),$$

where $h^i(X, F) = \dim_k H^i(X, F)$.

Definition 1.2.6. Let X be a smooth irreducible projective variety of dimension n and let H be an ample divisor on X. For a coherent sheaf F on X we set

$$P_F(m) := \frac{\chi(F \otimes \mathcal{O}_X(mH))}{\mathrm{rk}(F)}.$$

The sheaf F is GM-*semistable* with respect to the polarization H if and only

$$P_E(m) \leq P_F(m) \quad \text{for} \quad m >> 0$$

for all non-zero subsheaves $E \subset F$ with $\mathrm{rk}\, E < \mathrm{rk}\, F$; if strict inequality holds for every E then F is GM-*stable* with respect to H.

The following implications occur

$$\mu - \text{stable} \Rightarrow \text{GM} - \text{stable} \Rightarrow \text{GM} - \text{semistable} \Rightarrow \mu - \text{semistable}$$

For more details on GM-stability, see [13, 28].

1.3 The moduli spaces

Moduli spaces are geometric objects which arise from classification problems. Roughly speaking, the moduli space of stable reflexive sheaves on a smooth projective variety X is a scheme whose points are in natural bijection to isomorphism classes of stable reflexive sheaves on X. This correspondence is given in terms of representable functors.

We present below the formal definition of a moduli functor, a fine moduli space and a coarse moduli space and we gather the results on moduli spaces of reflexive sheaves on a smooth projective variety. For more details see [28, 31, 33].

1.3.1 Moduli problems

A moduli problem is essentially a classification problem: we have a collection of objects $\mathcal{A}$ with an equivalence relation $\sim$ and we want to classify these objects up to equivalence.

Let $\mathfrak{Sch}$ denote the category of schemes of finite type over $\mathbb{C}$ and let $\mathfrak{Sets}$ denote the category of sets.

Definition 1.3.1. The *functor of points* of a scheme X is a contravariant functor

$$h_X := \mathrm{Hom}(-, X) : \mathfrak{Sch} \to \mathfrak{Sets},$$

from the category of schemes to the category of sets defined by

$$\begin{aligned} h_X(Y) &:= \mathrm{Hom}(Y, X) \\ h_X(f\colon Y \to Z) &:= h_X(f) : h_X(Z) \to h_X(Y) \\ & \qquad\qquad\qquad\qquad g \mapsto g \circ f \end{aligned}$$

Furthermore, a morphism of schemes $f\colon X \to Y$ induces a natural transformation of functors $h_f\colon h_X \to h_Y$ given by

$$\begin{aligned} h_{f,Z}\colon h_X(Z) &\to h_Y(Z) \\ g &\mapsto f \circ g \end{aligned}$$

We denote by $\mathrm{Fun}(\mathfrak{Sch}^{\mathrm{op}}, \mathfrak{Sets})$ the category of the contravariant functors from schemes to sets form a category, with morphisms given by natural transformations.

The above construction can be phrased as follows: there is a functor $h : \mathfrak{Sch} \to \mathrm{Fun}(\mathfrak{Sch}^{\mathrm{op}}, \mathfrak{Sets})$ given by

$$X \to h_X; \quad (f\colon X \to Y) \to h_f\colon h_X \to h_Y.$$

Example 1.3.2. For a scheme X, we have $h_X(\operatorname{Spec}\mathbb{C}) := \operatorname{Hom}(\operatorname{Spec}\mathbb{C}, X)$ is the set of $\mathbb{C}$-points of X.

Definition 1.3.3. A contravariant functor $F : \mathfrak{Sch} \to \mathfrak{Sets}$ is called *representable* if there exists an scheme X and a natural isomorphism $F \simeq h_X$.

Definition 1.3.4. A *(naive) moduli problem* (in algebraic geometry) is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 1.3.5. Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphism of vector bundles.

Our goal is to find a scheme M whose k-points are in bijection with the set of equivalence classes $\mathcal{A}/\sim$. Furthermore, we want M to encode how these objects vary in 'families'. More precisely, we refer to $(\mathcal{A}, \sim)$ as a naive moduli problem, because there is often a natural notion of families of objects over a scheme S and an extension of $\sim$ to families over S, such that we can pullback families by morphisms $T \to S$.

Definition 1.3.6. Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then an extended moduli problem (or a moduli problem) is given by

1. Sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$, for all schemes S,

2. pullback maps $f^* : \mathcal{A}_S \to \mathcal{A}_T$, for every morphism of schemes $T \to S$,

satisfying the following properties:

(i) $(\mathcal{A}_{\operatorname{Spec}\mathbb{C}}, \sim_{\operatorname{Spec}\mathbb{C}}) = (\mathcal{A}, \sim)$;

(ii) for the identity $\mathrm{Id} : S \to S$ and any family $\mathcal{F}$ over S, we have $\mathrm{Id}^*\mathcal{F} = \mathcal{F}$;

(iii) for a morphism $f : T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$ over S, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$;

(iv) for morphisms $f : T \to S$ and $g : S \to R$, and a family $\mathcal{F}$ over R, we have an equivalence $(g \circ f)^*\mathcal{F} \sim_T f^*g^*\mathcal{F}$.

For a family $\mathcal{F}$ over S and a point $s\colon \operatorname{Spec}\mathbb{C} \to S$, we write $\mathcal{F}_s := s^*\mathcal{F}$ to denote the corresponding family over $\operatorname{Spec}\mathbb{C}$.

Lemma 1.3.7. *A moduli problem defines a functor (moduli functor)* $\mathcal{M} : \mathfrak{Sch} \to \mathfrak{Sets}$ *given by*

$$\mathcal{M}(S) := \{\textit{families over } S\}/\sim_S, \quad \mathcal{M}(f : T \to S) = f^* : \mathcal{M}(S) \to \mathcal{M}(T).$$

We will often refer to a moduli problem simply by its moduli functor. There can be several different extensions of a naive moduli problem. As it can be seen in the next example.

Example 1.3.8. Let us consider the naive moduli problem given by vector bundles on a fixed scheme X up to isomorphism. Then this can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X \times S$ flat over S, but there are two possible equivalence relations:

$$\mathcal{F} \sim'_S \mathcal{G} \iff \mathcal{F} \simeq \mathcal{G}$$
$$\mathcal{F} \sim_S \mathcal{G} \iff \mathcal{F} \simeq \mathcal{G} \otimes \pi_S^*\mathcal{L} \text{ for a line bundle } \mathcal{L} \to S,$$

where $\pi_S\colon X \times S \to S$. For the second equivalence relation, since $\mathcal{L} \to S$ is locally trivial. there is a cover S_i of S such that $\mathcal{F}|_{X\times S_i} \simeq \mathcal{G}|_{X\times S_i}$.

Definition 1.3.9. Let $\mathcal{M}\colon \mathfrak{Sch} \to \mathfrak{Sets}$ be a moduli functor. A scheme M is a *fine moduli space* for $\mathcal{M}$ if it represents $\mathcal{M}$.

The ideal situation is when there is a scheme that represents our given moduli functor, i.e. there is a fine moduli space. Unfortunately, there are many natural moduli problems which do not admit a fine moduli space; see for example [33, Example 2.21 and Example 2.22]. This motivates the following definition:

Definition 1.3.10. A *coarse moduli space* for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of functors $\eta\colon \mathcal{M} \to h_M$ such that

1. $\eta_{\operatorname{Spec}\mathbb{C}}\colon \mathcal{M}(\operatorname{Spec}\mathbb{C}) \to h_M(\operatorname{Spec}\mathbb{C})$ is bijective;
2. For any scheme N and natural transformation $\nu : \mathcal{M} \to h_N$, there exists a unique morphism of schemes $f\colon M \to N$ such that $\nu = h_f \circ \eta$, where $h_f\colon h_M \to h_N$ is the corresponding natural transformation.

1.3.2 Moduli space of reflexive sheaves

In this subsection we deal with the problem of classifying reflexive sheaves on smooth irreducible projective varieties.

Let X be a smooth, irreducible projective variety of dimension n over $\mathbb{C}$ and let H be an ample divisor on X. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant moduli functor

$$\begin{aligned} \mathcal{M}_X^{H,P}(-) : \mathfrak{Sch} &\to \mathfrak{Sets} \\ S &\mapsto \mathcal{M}_X^{H,P}(S) \end{aligned}$$

where
$\mathcal{M}_X^{H,P}(S) = \{S-$ flat families $\mathcal{F} \to X \times S$ of reflexive sheaves on X all whose fibers are μ-stable with respect to H and have Hilbert polinomial $P\}/\sim$,
with

$$\mathcal{F} \sim_S \mathcal{G} \iff \mathcal{F} \simeq \mathcal{G} \otimes \pi_S^*\mathcal{L} \quad \text{for a line bundle } \mathcal{L} \to S,$$

being $\pi_S\colon X\times S\to S$ the natural projection. And if $f: S'\to S$ is a morphism in $\mathfrak{Sch}$, let $\mathcal{M}_X^{H,P}(f)(-)$ be the map obtained by pulling-back sheaves via $= f\times id_X$:

$$\begin{aligned}\mathcal{M}_X^{H,P}(f)(-) : \mathcal{M}_X^{H,P}(S) &\to \mathcal{M}_X^{H,P}(S') \\ [\mathcal{F}] &\mapsto [f_X^*\mathcal{F}]\end{aligned}$$

In 1977, M. Maruyama proved, see [15] :

Theorem 1.3.11. *The contravariant moduli functor $\mathcal{M}_X^{H,P}(-)$ has a coarse moduli scheme $M_X^{H,P}$ which is a separated scheme and locally of finite type over $\mathbb{C}$. In addition, $M_X^{H,P}$ decomposes into a disjoint union of schemes $M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})$ where $n=\dim X$ and $M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})$ is the moduli space of rank r μ-stable with respect to H reflexive sheaves on X with Chern classes $(c_1,\dots,c_{\min(r,n)})$ up to numerical equivalence.*

The next proposition gives us an bounds to calculate the dimension of the Zariski tangent space of the moduli spaces of stable sheaves on a projective scheme X, see [13, Theorem 4.5.2].

Proposition 1.3.12. *Let X be a smooth, irreducible projective variety of dimension n and let E be a μ-stable reflexive sheaf on X with Chern classes $c_i(E)=c_i\in H^{2i}(X,\mathbb{Z})$, representing a point $[E]\in M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})$. Then the Zariski tangent space of $M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})$ at $[E]$ is canonically given by*

$$T_{[E]}M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})\simeq \operatorname{Ext}^1(E,E).$$

If $\operatorname{Ext}^2(E,E)=0$ then $M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)})$ is smooth at $[E]$. In general, there are bounds

$$\begin{aligned}\dim\operatorname{Ext}^1(E,E) &\geq \dim_{[E]} M_X^{H,P}(r;c_1,\dots,c_{\min(r,n)}) \\ &\geq \dim\operatorname{Ext}^1(E,E)-\dim\operatorname{Ext}^2(E,E).\end{aligned}$$

1.4 Monads

In this section we establish the notation and gather the most important facts about monads that will be useful through this text. Let X be a projective variety with structure sheaf $\mathcal{O}_X$ and dualizing sheaf ω_X.

Definition 1.4.1. A *monad* over a projective variety X is a complex

$$\mathcal{M}_\bullet : \; 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.6}$$

of locally free sheaves A, B and C on X which is exact at A and at C. The coherent sheaf

$$E = \ker(\beta)/\operatorname{Im}(\alpha)$$

is called the *cohomology sheaf of the monad* $\mathcal{M}_\bullet$ and one also says that $\mathcal{M}_\bullet$ is a monad for E.

The set

$$S = \{x \in X : \alpha_x \text{ is not injective}\},$$

where $\alpha_x : A_x \to B_x$ is the map induced in the stalks, is a subvariety called the *degeneration locus* of the monad $\mathcal{M}_\bullet$. Note that S is also the locus where the sheaf E is not locally-free.

Clearly, the cohomology sheaf E of a monad $\mathcal{M}_\bullet$ is always a coherent sheaf, but more can be said in particular cases. In fact, we have

Proposition 1.4.2. *Let E be the cohomology sheaf of a monad $\mathcal{M}_\bullet$.*

(1) E is locally-free if and only if the degeneration locus of $\mathcal{M}_\bullet$ is empty;

(2) E is reflexive if and only if the degeneration locus of $\mathcal{M}_\bullet$ is a subvariety of codimension at least 3;

(3) E is torsion-free if and only if the degeneration locus of $\mathcal{M}_\bullet$ is a subvariety of codimension at least 2.

Proof. This result is proved in [16, Proposition 4] when the sheaves A, B and C are given by

$$A = \mathcal{O}_X(-1)^{\oplus a}, \quad B = \mathcal{O}_X^{\oplus b} \quad \text{and} \quad C = \mathcal{O}_X(1)^{\oplus c}.$$

However the same argument is valid when A, B and C are locally free sheaves. □

Let's see the following examples:

Example 1.4.3. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d such that $(0:0:0:0:1) \notin X$. If

$$\alpha := (-x_1\, x_0\, 0\, 0) \text{ and } \beta := (x_0\, x_1\, x_2\, x_3),$$

then the cohomology sheaf of the monad

$$\mathcal{M}_\bullet :\ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

is a torsion-free sheaf but it is not a reflexive sheaf. Indeed, by Proposition 1.4.2, its degeneration locus,

$$S = \{x_0 = x_1 = 0\},$$

has codimension 2.

Example 1.4.4. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d such that $(0:0:0:0:1) \notin X$. If

$$\alpha := (-x_1\, x_0\, \ -x_4\, x_3) \text{ and } \beta := (x_0\, x_1\, x_2\, x_3),$$

then the cohomology sheaf of the monad

$$\mathcal{M}_\bullet :\ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

is a locally-free sheaf, since, by Proposition 1.4.2, its degeneration locus is empty.

As we can see, monads give us a rather simple way of obtaining new sheaves. When the sheaf we get is locally-free, we may consider its associated vector bundle, and by abuse of language we will not distinguish between one and the other.

Every monad $\mathcal{M}_\bullet$ on X can be broken down, using the fact that α is injective and β is surjective, into two short exact sequences:

$$0 \to K \to B \xrightarrow{\beta} C \to 0 \tag{1.7}$$

and

$$0 \to A \xrightarrow{\alpha} K \to E \to 0, \tag{1.8}$$

where $K := \ker \beta$ is also locally-free.

From the exact sequences above one easily deduces that if a coherent sheaf E on X is the cohomology sheaf of a monad $\mathcal{M}_\bullet$, then:

i) the Chern character of E is given by

$$ch(E) = ch(B) - ch(A) - ch(C)$$

ii) and the rank of E is given by

$$\mathrm{rk}(E) = \mathrm{rk}(B) - \mathrm{rk}(A) - \mathrm{rk}(C).$$

Remark 1.4.5. If E is the cohomology sheaf given by monad in the Example 1.4.3, we have: $ch(E) = (2, 0, -H^2, 0)$ and $\mathrm{rk}(E) = 2$.

Definition 1.4.6. To a given monad $\mathcal{M}_\bullet : \ 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$, whose cohomology E is locally free we can also associate the *dual monad*

$$\mathcal{M}_\bullet^* : \ 0 \to C^* \overset{\beta^*}{\to} B^* \overset{\alpha^*}{\to} A^* \to 0,$$

whose cohomology is precisely E^*.

A *morphism* between monads is a morphism of complexes. Two monads are isomorphic if they are isomorphic as complexes.

Definition 1.4.7. A monad

$$\mathcal{M}_\bullet : \ 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

on a projective variety X whose cohomology is E is called *Horrocks* if

(i) A and C are direct sums of invertible sheaves;

(ii) $H^1_*(B) = H^{n-1}_*(B) = 0$ and if $n \geq 4$ then $H^p_*(B) \simeq H^p_*(E)$ for $2 \leq p \leq n-2$;

if moreover the monad satisfies

(iii) no direct summand of A is isomorphic to a direct summand of B;

(iv) no direct summand of C is the image of a line subbundle of B;

then it is also called *minimal*.

Definition 1.4.8. A projective variety $X \subset \mathbb{P}^n$ of pure dimension n is arithmetically Cohen-Macaulay (ACM) if $H^p_*(\mathcal{O}_X) = 0$ for every $1 \leq p \leq n-1$ and $H^1_*(I_X) = 0$

Examples of ACM projective variety are 3-fold smooth hypersurfaces.

In [20, Theorem 3.3], we find the following correspondence between locally free sheaves on ACM varieties and classes of monads:

Theorem 1.4.9. *Let X be an ACM variety of dimension $n \geq 3$ and let E be a locally free sheaf on X. Then there is a 1-1 correspondence between collections $\{h_1, \ldots, h_r, g_1, \ldots, g_s\}$ with $h_i \in H^1(E^* \otimes \omega_X(k_i))$ and $g_j \in H^1(E(-l_j))$ for integers $k_{i's}$ and $l_{j's}$ and equivalence classes of monads for E of the form*

$$\mathcal{M}_\bullet : \ 0 \to \oplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} F \xrightarrow{\beta} \oplus_{j=1}^s \mathcal{O}_X(l_j) \to 0.$$

This correspondence is such that:

(i) $\mathcal{M}_\bullet$ is Horrocks if and only if the $g_{j's}$ generate $H^1_*(E)$ and the $h_{i's}$ generate $H^1_*(E^* \otimes \omega_X)$ as $S(X)$-modules;

(ii) $\mathcal{M}_\bullet$ is minimal Horrocks if and only if the $g_{j's}$ contitute a minimal set of generators for $H^1_*(E)$ and the $h_{i's}$ constitute a minimal set of generators for $H^1_*(E^* \otimes \omega_X)$ as $S(X)$-modules.

In [32, Theorem 3.3], we find the following theorem on the existence of monads on ACM varieties:

Theorem 1.4.10. *Let X be a variety of dimension n and let L be a line bundle on X. Suppose there is a linear sistem $V \subset H^0(L)$, with no base points, defining a morphism $X \to \mathbb{P}(V)$ whose image $X' \hookrightarrow \mathbb{P}(V)$ is a projective ACM variety. Then there exists a monad of type*

$$\mathcal{M}^*_\bullet : \ 0 \to (L^*)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L^c \to 0$$

if and only if one of following conditions holds:

i) $b \geq a + c$ and $b \geq 2c + n - 1$, *ii)* $b \geq a + c + n$.

In this work we are interested in a special type of monad, called linear.

Definition 1.4.11. A monad on X is called *linear* if it is of the following form:

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X(1)^{\oplus c} \to 0.$$

Similarly, the cohomology sheaf of a linear monad is called a *linear sheaf.*

Bundles that can be obtained as the cohomology of a linear monad are know as linear bundles. Linear monads are often used to build examples of stable bundles of rank 2 and 3 on hypersurfaces in $\mathbb{P}^4$, see [17, Main Theorem].

Theorem 1.4.12. *Let X be a 3-dimensional non-singular projective complex variety with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, where H is the class of a hyperplane section, i.e. $H = c_1(\mathcal{O}_X(1))$. Consider the following linear monad:*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad (c \geq 1)$$

whose cohomology sheaf is locally free.

i) The kernel $K = \ker \beta$ is a stable rank $c + 2$ bundle with $c_1(K) = -c.H$ and $c_2(K) = \frac{1}{2}(c^2 + c).H^2$;

ii) The cohomology $E = \ker \beta / \operatorname{Im} \alpha$ is a stable rank 2 bundle with $c_1(E) = 0$ and $c_2(E) = c.H^2$.

1.5 Representations of quivers

We start this section by recalling the definitions of quivers and their representations. For this section we will use as main references [4, 18, 30].

Definition 1.5.1. A *quiver* Q is given by a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps $h, t : Q_1 \to Q_0$ called *head* and *tail*, respectively.

Definition 1.5.2. Let Q be a quiver.

i) A *linear representation* of a quiver Q is given by $R = (\{V_i\}_{i \in Q_0}; \{f_\alpha\}_{\alpha \in Q_1})$ where V_i is a $\mathbb{C}$-vector space and $f_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ is linear.

ii) A *morphism* between two representations R and $R^{'}$ is given by $\phi = \{\phi_i\}_{i \in Q_0}$ where $\phi_i : V_i \to V_i^{'}$ is linear and for each arrow α we have $f_\alpha^{'} \phi_{t(\alpha)} = \phi_{h(\alpha)} f_\alpha$.

We denote $\mathrm{Rep}_{\mathbb{C}}\, Q$ the abelian category of the linear representations of the quiver Q.

The *algebra of the linear quiver* Q is the associative $\mathbb{C}$-algebra $\mathbb{C}Q$ determined by generators e_i, where $i \in Q_0$, and α, where $\alpha \in Q_1$ and the relations: $e_i e_j = 0$ if $i \neq j$, $e_i^2 = e_i$, $e_{t(\alpha)}\alpha = \alpha e_{h(\alpha)} = \alpha$.

From the relations above, for any arrows α, β we get $\alpha\beta = 0$ unless $h(\alpha) = t(\beta)$. Thus a product of arrows $\alpha_l \cdots \alpha_1$ is zero unless the sequence $\pi = (\alpha_1, \ldots, \alpha_l)$ is a *path*, i.e., $h(\alpha_i) = t(\alpha_{i+1})$ for $i = 1, \ldots, l-1$. We then put $s(\pi) = s(\alpha_1)$, $t(\pi) = t(\alpha_l)$ and the *length* of the path π, $l(\pi) = l$. For any vertex i we also view e_i as the *path of length* 0 at the vertex i.

Clearly the paths generate the vector space $\mathbb{C}Q$. They also are linearly independent. Consider the *path algebra* with basis the set of all paths and multiplication given by concatenation of paths. From the concept of a path algebra we get following definition of quiver with relations.

Definition 1.5.3. Let Q be a quiver.

i) A *relation* on a quiver Q is a linear combination of paths in $\mathbb{C}Q$ having a common tail and a common head and of length at least 2.

ii) A *quiver with relations* is a pair (Q, I) where Q is a quiver and I is a two-sided ideal of $\mathbb{C}Q$ generated by relations. The quotient algebra $\dfrac{\mathbb{C}Q}{I}$ is the path algebra of (Q, I).

In this work, we shall be interested in the quiver Q:

$$Q := \underset{\bullet}{-1} \quad \begin{array}{c} \xrightarrow{\alpha_1} \\ \vdots \\ \xrightarrow[\alpha_n]{} \end{array} \quad \underset{\bullet}{0} \quad \begin{array}{c} \xrightarrow{\beta_1} \\ \vdots \\ \xrightarrow[\beta_n]{} \end{array} \quad \underset{\bullet}{1} \tag{1.9}$$

with the relations $P_{ij} = \beta_i\alpha_j + \beta_j\alpha_i$, for $1 \leq i \leq j \leq n$.

A representation $R = (V_{-1}, V_0, V_1, \{f_{\alpha_i}\}, \{g_{\beta_i}\})$ of Q is said to satisfy the relations P_{ij} when $g_{\beta_i} f_{\alpha_j} + g_{\beta_j} f_{\alpha_i} = 0$.

Let $X \subset \mathbb{P}^{n-1}$ be a smooth hypersurface of degree d.

Definition 1.5.4. Let $R = (V_{-1}, V_0, V_1, \{f_{\alpha_i}\}, \{g_{\beta_i}\})$ be a representation of the quiver Q with relations P_{ij}.

i) R is globally injective on X if for every $(\lambda_1 : \cdots : \lambda_n) \in X$, $\sum \lambda_i f_{\alpha_i}$ is injective.

ii) R is globally surjective on X if for every $(\lambda_1 : \cdots : \lambda_n) \in X$, $\sum \lambda_i g_{\beta_i}$ is surjective.

1.6 Equivalence between categories of monads and quiver representations

Throughout this section Q denotes the quiver given as in display (3.2) with the relations P_{ij} and $X \subset \mathbb{P}^{n-1}$ a smooth hypersurface of degree d.

Let $\mathfrak{C}$ be the category of complexes of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X(1)^{\oplus c} \to 0,$$

regarded as a full subcategory of the category of complexes of sheaves on X. We will also denote by $\mathfrak{D}$ the abelian category of representations of the quiver Q.

Below we will present an equivalence functor $\mathbf{F}$ between $\mathfrak{C}$ and $\mathfrak{D}$. For more details on this equivalence functor see [19, 34].

Lemma 1.6.1. *Let A and B be two coherent sheaves on X. Then*

$$\mathrm{Hom}(A^a, B^b) \simeq \mathrm{Mat}_{b\times a} \otimes_{\mathbb{C}} \mathrm{Hom}(A, B),$$

where $\mathrm{Mat}_{b\times a}$ denotes the vector space of $b \times a$ matrices of complex numbers.

Proof. Consider $\phi \in \mathrm{Hom}(A^a, B^b)$ a morphism. Let $p_i : B^b \to B$ denote the projections, for every $i = 1, \ldots, b$, and $\eta_j : A \to A^a$ the inclusions, for $j = 1, \ldots, a$.

We defined $\phi_{ij} : A \to B$ by $\phi_{ij} := p_i \circ \phi \circ \eta_j$ for every $i = 1, \ldots, b$ e $j = 1, \ldots, a$. Thus, we get the matrix

$$\phi \longleftrightarrow (\phi_{ij})_{b\times a} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1a} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2a} \\ \cdots & \cdots & \ddots & \vdots \\ \phi_{b1} & \phi_{b2} & \cdots & \phi_{ba} \end{pmatrix}$$

Now, fix a basis $\gamma = \{x_1, \ldots, x_n\}$ for $\mathrm{Hom}(A, B)$. Then

$$\phi_{ij} = \lambda_1^{ij} x_1 + \lambda_2^{ij} x_2 + \cdots + \lambda_n^{ij} x_n,$$

where $\lambda_k^{ij} \in \mathbb{C}$ and hence

$$\phi = \phi_1 \otimes x_1 + \phi_2 \otimes x_2 + \cdots + \phi_n \otimes x_n,$$

where $\phi_k = (\lambda_k^{ij})_{b\times a}$ for every $k = 1, \ldots, n$. □

Remark 1.6.2. An immediate consequence of the Lemma 1.6.1 is that the morphisms α and β in the linear monads can be seen as matrices whose entries are elements of $H^0(\mathcal{O}_X(1))$, i.e. homogeneous polynomials of degree 1.

Proposition 1.6.3. *There is an equivalence of categories between $\mathfrak{C}$ and $\mathfrak{D}$. Moreover, under this equivalence, monads whose cohomology sheaf is locally free are in* $1-1$ *correspondence with globally injective and surjective representation of Q.*

Proof. Fix homogeneous coordinates $[X_1 : \ldots : X_n]$ of X, and let $\{X_1, \ldots, X_n\}$ be the corresponding basis of $H^0(\mathcal{O}_X(1))$.

By Lemma 1.6.1, we have the isomorphism

$$\mathrm{Hom}(\mathcal{O}_X(-1)^{\oplus a}, \mathcal{O}_X^{\oplus b}) \simeq \mathrm{Mat}_{b\times a} \otimes_{\mathbb{C}} H^0(\mathcal{O}_X(1)).$$

Consider the monad

$$\mathcal{M}_\bullet : 0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0.$$

As α and β can be seen as matrices whose entries are linear forms on $X_1, \ldots, X_n$ we have

$$\alpha = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n, \quad \beta = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n,$$

where $\alpha_i \in \mathrm{Mat}_{b\times a}$ and $\beta_i \in \mathrm{Mat}_{c\times b}$. So, we can set

$$\mathbf{F}(\mathcal{M}_\bullet) = (\mathbb{C}^a, \mathbb{C}^b, \mathbb{C}^c, \{\alpha_i\}, \{\beta_j\}).$$

Moreover, we have

$$\begin{aligned} \beta \circ \alpha = 0 \quad &\Longleftrightarrow \quad \sum_{i\leq j} (\beta_i \alpha_j + \beta_j \alpha_i) X_i X_j = 0 \\ &\Longleftrightarrow \quad \beta_i \alpha_j + \beta_j \alpha_i = 0, \end{aligned}$$

for $1 \leq i \leq j \leq n$. Therefore, $\mathbf{F}(\mathcal{M}_\bullet)$ satisfies the relations of Q.

Now, we consider $\phi_\bullet : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ a morphism between monads. As, by Lemma (1.6.1), $\mathrm{Hom}(\mathcal{O}_X(i)^{\oplus a}, \mathcal{O}_X(i)^{\oplus b}) \simeq \mathrm{Mat}_{b\times a}$, we can define $\mathbf{F}(\phi_\bullet)$ as being the morphism of representations obtained from the above isomorphism.

It is not difficult to see that $\mathbf{F}$ is dense, faithfull and full, i.e. $\mathbf{F}$ is an equivalence.

Note also that $\mathbf{F}(\mathcal{M}_\bullet)$ is globally surjective if and only if the morphism β is surjective and $\mathbf{F}(\mathcal{M}_\bullet)$ is globally injective if and only if the degeneration locus of $\mathcal{M}_\bullet$ is empty. Therefore, the cohomology sheaf of $\mathcal{M}_\bullet$ is locally free if and only if $\mathbf{F}(\mathcal{M}_\bullet)$ is globally injective and globally surjective. □

Remark 1.6.4.

- When $X = \mathbb{P}^3$, the category $\mathfrak{C}$ is equivalent the category of representations of quiver Q satisfing the relations P_{ij} with $n = 4$, since $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$.

- When $X \subset \mathbb{P}^4$ is a hypersurface of degree $d \geq 2$, $n = 5$ in the quiver Q, since $\dim H^0(\mathcal{O}_X(1)) = 5$.

2 Distributions and foliations on 3-fold hypersurfaces

Our main objective here is to obtain concrete examples of moduli spaces of rank 2 stable reflexive sheaves on 3-fold hypersurfaces of degree $d = 1, \ldots, 5$ with prescribed Chern classes. For this we will use the generic codimension 1 distributions and foliations by curves.

In the next section we present the concept of distributions and foliations on smooth projective threefolds. We use [1, 25, 26] as our main references.

2.1 General definitions

Let X be a smooth projective threefold X of Picard rank 1 and let $\mathcal{O}_X(1)$ denote the ample generator of $\mathrm{Pic}(X)$. A *codimension l distribution* $\mathscr{F}$ on a smooth projective threefold X is given by an exact sequence

$$\mathscr{F} \quad : \quad 0 \to T_{\mathscr{F}} \xrightarrow{\sigma} TX \to N_{\mathscr{F}} \to 0, \tag{2.1}$$

where $T_{\mathscr{F}}$ is a reflexive sheaf of rank $s := 3 - l$ and $N_{\mathscr{F}}$ is a torsion free sheaf; these are respectively called the *tangent* and *normal* sheaves of $\mathscr{F}$. If $\mathscr{F}$ is a codimension one distibution, we can rewrite the exact sequence in display (2.1) in the following manner:

$$\mathscr{F} \quad : \quad 0 \to T_{\mathscr{F}} \xrightarrow{\sigma} TX \to I_Z(r+2) \to 0, \tag{2.2}$$

where $r := C_1(TX) - c_1(T_{\mathscr{F}}) - 2 \geq 0$, called the *degree* of $\mathscr{F}$, and I_Z is an ideal sheaf of a subscheme $Z := \mathrm{Sing}(\mathscr{F})$, called the *singular scheme* of $\mathscr{F}$. In this case, $T_{\mathscr{F}}$ is a reflexive sheaf of rank 2 on X.

Codimension one distributions $\mathscr{F}$ with the property that $\mathrm{Sing}(\mathscr{F})$ is either empty or has dimension equal to zero are called *generic* because they can be defined by a general 1-form $\omega \in H^0(\Omega^1_X(t))$ for some t. Indeed, if we dualize the sequence in display (2.2), we get

$$0 \to \mathcal{O}_X(-2-r) \to \Omega^1_X \to T^\vee_{\mathscr{F}} \to 0 \tag{2.3}$$

since $\mathcal{E}xt^1(I_Z(r+2), \mathcal{O}_X) = 0$; that is, the tangent sheaf can be described as a quotient of Ω^1_X. If $X \subset \mathbb{P}^4$ is a smooth hypersurface, to according [26, Teorema 1], the tangent sheaf $T^\vee_{\mathscr{F}}$ is always a μ-stable rank 2 reflexive sheaf on X.

When $X = \mathbb{P}^3$, the generic codimension one distribution of degree r provides a family of μ-stable rank 2 reflexive with given Chern classes parametrized by open subset of $\mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(r+2))$. It was shown in [26, Theorem 4] that such families are dense within an irreducible component of the (Gieseker–Maruyama) moduli space of stable rank 2 sheaves on the projective space $\mathbb{P}^3$.

Theorem 2.1.1. *For each $r \geq 0$, $r \neq 2$, the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$ with Chern classes*

$$(c_1, c_2, c_3) = (2-r, r^2+2, r^3+2r^2+2r)$$

contains a nonsingular, rational irreducible component of dimension $(r+1)(r+3)(r+4)/2-1$ whose generic point is the tangent sheaf of a generic distribution of degree r on $\mathbb{P}^3$.

Our main goal here is to extend this theorem to a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d = 2, \ldots, 5$. To be more precise, we build families of the stable rank 2 reflexive sheaves on X containing the generic codimension one distributions which fill an irreducible components of the moduli spaces of stable rank 2 reflexive sheaves with prescribed Chern classes.

Let $\mathcal{D}(r)$ denote the family of stable rank 2 reflexive sheaves on X given by the exact sequence in display (2.3) and let $\mathcal{F}(r)$ be the family of the stable rank 2 reflexive sheaves F on X given by the exact sequence

$$0 \to \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) \xrightarrow{\sigma} \Omega^1_{\mathbb{P}^4}|_X \to F \to 0. \tag{2.4}$$

Note that each generic codimension one distribution

$$0 \to \mathcal{O}_X(-2-r) \xrightarrow{\phi} \Omega^1_X \to T^\vee_{\mathscr{F}} \to 0$$

fits into a commutative diagram as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X(-d) & = & \mathcal{O}_X(-d) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r)\oplus\mathcal{O}_X(-d) & \xrightarrow{\sigma} & \Omega^1_{\mathbb{P}^4}|_X & \longrightarrow & T^\vee_{\mathscr{F}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r) & \xrightarrow{\phi} & \Omega^1_X & \longrightarrow & T^\vee_{\mathscr{F}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2.5}$$

where φ is given by the standard normal bundle sequence

$$0 \to \mathcal{O}_X(-d) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4}|_X \to \Omega^1_X \to 0,$$

and therefore $\mathcal{D}(r) \subset \mathcal{F}(r)$.

We also have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(-d) & = & \mathcal{O}_X(-d) & & \\
 & & & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r) & \xrightarrow{\sigma_1} & \Omega^1_{\mathbb{P}^4}|_X & \longrightarrow & T_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r) & \xrightarrow{\phi} & \Omega^1_X & \longrightarrow & T^\vee_{\mathscr{F}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.6}$$

where T_X is a rank 3 reflexive sheaf on X.

In general, if $\sigma = (\sigma_1\ \sigma_2)$ is generic, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X(-d) & = & \mathcal{O}_X(-d) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle \sigma_2} & & & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) & \xrightarrow{\sigma} & \Omega^1_{\mathbb{P}^4}|_X & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X(-2-r) & \longrightarrow & G & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{2.7}
$$

where G is a rank 3 reflexive sheaf on X.

So, in order to get the dimension of the family $\mathcal{F}(r)$, we must investigate when bundle monomorphisms

$$\sigma : \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^4}|_X$$

define isomorphic quotients. Before, consider the following lemma:

Lemma 2.1.2. *The sheaf* $\Omega^1_{\mathbb{P}^4}|_X$ *is simple, i.e.* $\dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) = 1$.

Proof. Applying the functor $\operatorname{Hom}(., \Omega^1_{\mathbb{P}^4}|_X)$ to the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^4}(-d) \xrightarrow{\cdot f} \Omega^1_{\mathbb{P}^4} \to \Omega^1_{\mathbb{P}^4}|_X \to 0, \tag{2.8}$$

where $X = \{f = 0\} \subset \mathbb{P}^4$ is a hypersurface of degree d, we get

$$0 \to \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) \to \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_X) \to \cdots \tag{2.9}$$

Now, applying the functor $\operatorname{Hom}(., \Omega^1_{\mathbb{P}^4}|_X)$ to the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^4} \to 0,$$

we get $\dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_X) = 1$, since $H^0(\Omega^1_{\mathbb{P}^4}|_X(1)) = H^1(\Omega^1_{\mathbb{P}^4}|_X(1)) = 0$ and $\dim H^1(\Omega^1_{\mathbb{P}^4}|_X) = 1$. Since, by display (2.9),

$$1 \leq \dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) \leq \dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_X)$$

we conclude that $\dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) = 1$, as desired. □

Lemma 2.1.3. *Let* $\sigma, \sigma^{'} : \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^4}|_X$ *be two bundle monomorphisms such that* $\operatorname{coker}\sigma := F$ *and* $\operatorname{coker}\sigma^{'} := F^{'}$ *are reflexive sheaves.* F *and* $F^{'}$ *are isomorphic if and only if there is an automorphism* $\psi \in \operatorname{Aut}(\mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d))$ *with* $\sigma^{'} \circ \psi = \sigma$.

Proof. It is easy to see that the quotients of σ and $\sigma^{'} \circ \psi$ are isomorphic.

Conversely suppose

$$\sigma, \sigma^{'} : \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^4}|_X$$

are bundle monomorphisms and

$$g : F \to F^{'}$$

an isomorphism of the quotients.

Applying the functor $\operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, .)$ to the exact sequence

$$0 \to \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) \xrightarrow{\sigma^{'}} \Omega^1_{\mathbb{P}^4}|_X \xrightarrow{p^{'}} F^{'} \to 0,$$

we get the isomorphism

$$\operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) \simeq \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, F^{'})$$

since

$$\operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d)) \simeq H^0(T\mathbb{P}^4|_X(-2-r)) \oplus H^0(T\mathbb{P}^4|_X(-d)) = 0$$

and

$$\operatorname{Ext}^1(\Omega^1_{\mathbb{P}^4}|_X, \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d)) \simeq H^1(T\mathbb{P}^4|_X(-2-r)) \oplus H^1(T\mathbb{P}^4|_X(-d)) = 0.$$

Thus, given $\xi \in \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, F^{'})$, there exists a unique $\lambda \in \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X)$ such that $p^{'} \circ \lambda = \xi$. Being $\Omega^1_{\mathbb{P}^4}|_X$ simple, by Lemma 2.1.2, it follows that $\lambda = c \cdot \operatorname{id}$.

Therefore, as $g \circ p \in \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_X, F^{'})$, we get the following isomorphism betweeen exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) & \xrightarrow{\sigma} & \Omega^1_{\mathbb{P}^4}|_X & \xrightarrow{p} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \lambda_{g\circ p}} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d) & \xrightarrow{\sigma^{'}} & \Omega^1_{\mathbb{P}^4}|_X & \xrightarrow{p^{'}} & F^{'} & \longrightarrow & 0 \end{array}$$

that is, there is a automorphism $\psi \in \mathrm{Aut}(\mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d))$ such that $\sigma^{'} \circ \psi = \sigma$. □

As an immediate consequence of this lemma, we have:

Proposition 2.1.4. *The dimension of the family of the sheaves constructed as in* (2.4) *is given by*

$$\begin{aligned} \dim \mathcal{F}(r) &= \dim \mathrm{Hom}(\mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d), \Omega^1_{\mathbb{P}^4}|_X) \\ &- \dim \mathrm{Aut}(\mathcal{O}_X(-2-r) \oplus \mathcal{O}_X(-d)). \end{aligned}$$

The next Proposition gives us a tool in order to compute the dimension of the tangent space at a point $T^{\vee}_{\mathscr{F}}$ of the Gieseker-Maruyama moduli space of stable rank 2 reflexive sheaves on X.

Proposition 2.1.5. *Let $X \hookrightarrow \mathbb{P}^4$ be a smooth hypersurface of degree $d \in \{2,3,4,5\}$. If a sheaf $T^{\vee}_{\mathscr{F}}$ satisfies the exact sequence* (2.3)*, then:*

i) If $2 \leq d \leq 4$,

$$\dim \mathrm{Ext}^1(T^{\vee}_{\mathscr{F}}, T^{\vee}_{\mathscr{F}}) - \dim \mathrm{Ext}^2(T^{\vee}_{\mathscr{F}}, T^{\vee}_{\mathscr{F}}) = \frac{d}{6}(5-d)(11-13d+8d^2+9r^2+6r(1+d))+1;$$

ii) If $d = 5$,

$$\dim \mathrm{Ext}^1(T^{\vee}_{\mathscr{F}}, T^{\vee}_{\mathscr{F}}) = \dim \mathrm{Ext}^2(T^{\vee}_{\mathscr{F}}, T^{\vee}_{\mathscr{F}}).$$

Proof. Indeed, applying the functor $\mathrm{Hom}(.,T^{\vee}_{\mathscr{F}})$ to the exact sequence (2.3), we obtain the equality

$$\sum_{j=0}^{3} (-1)^j \dim \mathrm{Ext}^j(T^{\vee}_{\mathscr{F}}, T^{\vee}_{\mathscr{F}}) = \chi(TX \otimes T^{\vee}_{\mathscr{F}}) - \chi(T^{\vee}_{\mathscr{F}}(2+r)) \tag{2.10}$$

since $\mathrm{Ext}^i(\Omega^1_X, T^\vee_{\mathscr{F}}) \simeq H^i(TX \otimes T^\vee_{\mathscr{F}}))$ and $\mathrm{Ext}^i(\mathcal{O}_X(-2-r), T^\vee_{\mathscr{F}}) \simeq H^i(T^\vee_{\mathscr{F}}(2+r))$ for $0 \le i \le 3$.

Now, we twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X and the standard normal bundle sequence (1.5) by $\otimes T^\vee_{\mathscr{F}}$ and then taking the Euler caracteristic, we get

$$\chi(TX \otimes T^\vee_{\mathscr{F}}) - \chi(T^\vee_{\mathscr{F}}(2+r)) = \frac{1}{6}d(d-5)(11-13d+8d^2+9r^2+6r(1+d)). \quad (2.11)$$

Next, replacing (2.11) in (2.10), we obtain

$$\sum_{j=0}^{3}(-1)^j \dim \mathrm{Ext}^j(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \frac{1}{6}d(d-5)(11-13d+8d^2+9r^2+6r(1+d)).$$

By Serre duality, we have $\mathrm{Ext}^3(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) \simeq \mathrm{Hom}(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}(d-5))$, since $\omega_X \simeq \mathcal{O}_X(d-5)$. The stability of $T^\vee_{\mathscr{F}}$ implies that $\mathrm{Hom}(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}(d-5)) = 0$ for $d \in \{2,3,4\}$, since $\mu(T^\vee_{\mathscr{F}}(d-5)) < \mu(T^\vee_{\mathscr{F}})$. Thus, when $2 \le d \le 4$, we have

$$\dim \mathrm{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) - \dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \frac{1}{6}d(5-d)(11-13d+8d^2+9r^2+6r(1+d))+1.$$

When $d = 5$, we have

$$\dim \mathrm{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) - \dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 0,$$

since $\mathrm{Ext}^3(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) \simeq \mathrm{Hom}(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}})$ and $\sum_{j=0}^{3}(-1)^j \dim \mathrm{Ext}^j(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 0$.

□

Remark 2.1.6. When $d \ge 6$ it was not to calculate

$$\dim \mathrm{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) - \dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}})$$

since the argument used above to compute $\dim \mathrm{Ext}^3(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}})$ does not apply in this case.

2.1.1 Properties of the tangent sheaf

Throughout this section X denote a smooth hypersurface of degree d in $\mathbb{P}^4$. The main goal here is the study of the cohomology of the sheaf $T^\vee_{\mathscr{F}}$ of a generic codimension one distribution on X. We start with the following lemma:

Lemma 2.1.7. *If a sheaf $T_{\mathscr{F}}$ satisfies the exact sequence in display (2.3), then:*

1) $h^0(T_{\mathscr{F}}^\vee(t)) = 0$ *for* $t \leq 1$*;*

2) $h^1(T_{\mathscr{F}}^\vee) = 1$ *and* $h^1(T_{\mathscr{F}}^\vee(t)) = 0$ *for* $t \neq 0$;

3) $h^2(T_{\mathscr{F}}^\vee(t)) = h^0(\mathcal{O}_X(r-t+d-3)$ *for* $t \geq 2d-4$; *in particular,* $h^2(T_{\mathscr{F}}^\vee(t)) = 0$ *for* $t \geq r+d-3$;

4) $h^3(T_{\mathscr{F}}^\vee(t)) = 0$ *for* $t \geq d-3$.

Proof. For items 1) and 2), we consider the long exact sequence of cohomology obtained from the exact sequence in display (2.3) twisted by $\mathcal{O}_X(t)$

$$\cdots \to H^0(\Omega_X^1(t)) \to H^0(T_{\mathscr{F}}^\vee(t)) \to 0 \to H^1(\Omega_X^1(t)) \to H^0(T_{\mathscr{F}}^\vee(t)) \to 0.$$

Being $h^0(\Omega_X^1(t)) = 0$ for $t \leq 1$ follows that $h^0(T_{\mathscr{F}}^\vee(t)) = 0$ for $t \leq 1$; now, as $h^1(\Omega_X^1(t)) = 0$ for $t \neq 0$ and $h^1(\Omega_X^1) = 1$, we get the item 2).

For items 3) and 4), we consider the long exact sequence of cohomology

$$0 \longrightarrow H^2(\Omega_X^1(t)) \longrightarrow H^2(T_{\mathscr{F}}^\vee(t)) \longrightarrow H^3(\mathcal{O}_X(t-2-r)) \longrightarrow H^3(\Omega_X^1(t))$$

$$\longrightarrow H^3(T_{\mathscr{F}}^\vee(t)) \longrightarrow 0.$$

We know that $h^2(\Omega_X^1(t)) = 0$ for $t \geq 2d-4$ and $h^3(\Omega_X^1(t)) = 0$ for $t \geq d-3$. So, $h^3(T_{\mathscr{F}}^\vee(t)) = 0$ for $t \geq d-3$ and $h^2(T_{\mathscr{F}}^\vee(t)) = h^0(\mathcal{O}_X(r-t+d-3)$ for $t \geq 2d-4$, since, by Serre duality, $h^3(\mathcal{O}_X(t-2-r)) = h^0(\mathcal{O}_X(r-t+d-3))$. □

Another important lemma is:

Lemma 2.1.8. *If $T_{\mathscr{F}}$ is a sheaf satisfying the exact sequence in display (2.3) and TX is the tangent bundle on X, then*

$$H^3(TX \otimes T_{\mathscr{F}}^\vee) \simeq \operatorname{Ext}^3(\Omega_X^1, T_{\mathscr{F}}^\vee) = 0,$$

for $d = 2, \ldots, 5$.

Proof. Indeed, applying the functor $\text{Hom}(., T_{\mathscr{F}}^{\vee})$ to the exact sequence in display (2.3), we get the long exact sequence in cohomology

$$\cdots \to \text{Ext}^3(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) \to \text{Ext}^3(\Omega_X^1, T_{\mathscr{F}}^{\vee}) \to H^3(T_{\mathscr{F}}^{\vee}(2+r)) \to 0.$$

By Serre duality,

$$\text{Ext}^3(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) \simeq \text{Hom}(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}(d-5)).$$

Being $T_{\mathscr{F}}^{\vee}$ μ-stable and $\mu(T_{\mathscr{F}}^{\vee}(d-5)) < \mu(T_{\mathscr{F}}^{\vee})$ (for $d = 2, 3, 4$) follows that $\text{Ext}^3(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = 0$.

By item 4) of the Lemma (2.1.7), $H^3(T_{\mathscr{F}}^{\vee}(2+r)) = 0$ since $2+r \geq d-3$ and hence $\text{Ext}^3(\Omega_X^1, T_{\mathscr{F}}^{\vee}) = 0$.

For the case $d = 5$, suppose that $\text{Ext}^3(\Omega_X^1, T_{\mathscr{F}}^{\vee}) \neq 0$.

By Serre duality,

$$\text{Ext}^3(\Omega_X^1, T_{\mathscr{F}}^{\vee}) \simeq \text{Hom}(T_{\mathscr{F}}^{\vee}, \Omega_X^1)$$

since $\omega_X \simeq \mathcal{O}_X$. So, there is a morphism $q : T_{\mathscr{F}}^{\vee} \to \Omega_X^1$ such that $p \circ q = 1_{T_{\mathscr{F}}^{\vee}}$ because $T_{\mathscr{F}}^{\vee}$ is simple, where $p : \Omega_X^1 \to T_{\mathscr{F}}^{\vee}$ is given in exact sequence (2.3) which should split, contradicting the indecomponibility of Ω_X^1. □

2.2 Codimension 1 distributions on quadric threefolds

Let Q_3 denote a quadric threefold with ample line bundle $\mathcal{O}_{Q_3}(1)$ whose first Chern class is denoted by H, i.e. $c_1(\mathcal{O}_{Q_3}(1)) = H$. Recall that the cohomology ring $H^*(Q_3, \mathbb{Z})$ of Q_3 is generated by H, a line $L \in H^4(Q_3, \mathbb{Z})$ and a point $P \in H^6(Q_3, \mathbb{Z})$ with the relations: $H^2 = 2L,\ H.L = P,\ H^3 = 2P$, see Section 1.1.

Our main goal here is to get an analogue result to Theorem 2.1.1, when $X = Q_3$. We start the section by calculation the Chern classes of the tangent sheaf of a generic codimension 1 distribution on Q_3.

Recall that given a generic distribution $\mathscr{F}$ on Q_3, the integer $r := 1 - c_1(T_{\mathscr{F}})$ is called the *degree* of $\mathscr{F}$.

Lemma 2.2.1. *If a generic distribution $\mathscr{F}$ on Q_3 has degree $r = 2k+1$, then the normalization of the sheaf $T_{\mathscr{F}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{Q_3}(-3-3k) \xrightarrow{\sigma} \Omega^1_{Q_3}(-k) \to T^{\vee}_{\mathscr{F}}(-k) \to 0, \tag{2.12}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2+6k+4)H^2, (8k^3+24k^2+26k+6)H^3).$$

Proof. We know that $c(\Omega^1_{Q_3}(-k)) = c(T^{\vee}_{\mathscr{F}}(-k)).c(\mathcal{O}_{Q_3}(-3-3k))$. So,

$$c_1(T^{\vee}_{\mathscr{F}}(-k)) = c_1(\Omega^1_{Q_3}(-k)) - c_1(\mathcal{O}_{Q_3}(-3-3k)) = 0,$$

since $c_1(\Omega^1_{Q_3}(-k)) = c_1(\mathcal{O}_{Q_3}(-3-3k))$,

$$\begin{aligned} c_2(T^{\vee}_{\mathscr{F}}(-k)) &= c_2(\Omega^1_{Q_3}(-k)) \\ &= c_2(\Omega^1_{Q_3}) + 2c_1(\Omega^1_{Q_3})c_1(\mathcal{O}_{Q_3}(-k)) + 3c_1(\mathcal{O}_{Q_3}(-k))^2 \\ &= c_2(\Omega^1_{Q_3}) + 2(-3H)(-kH) + 3(-kH)^2 \\ &= (3k^2+6k+4)H^2, \end{aligned}$$

since $c_1(T^{\vee}_{\mathscr{F}}(-k)) = 0$ and

$$\begin{aligned} c_3(T^{\vee}_{\mathscr{F}}(-k)) &= c_3(\Omega^1_{Q_3}(-k)) - c_2(T^{\vee}_{\mathscr{F}}(-k))c_1(\mathcal{O}_{Q_3}(-3-3k)) \\ &= -8H^3 + (-kH)(4H^2) + (-kH)^2(-3H) + (-kH)^3 \\ &+ (3k^2+6k+4)H^2(3+3k)H \\ &= (8k^3+24k^2+26k+6)H^3. \end{aligned}$$

□

When $\mathscr{F}$ has degree $r = 2k$, we have:

Lemma 2.2.2. *If a generic distribution $\mathscr{F}$ on Q_3 has degree $r = 2k$, then the normalization of the sheaf $T_{\mathscr{F}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{Q_3}(-2-3k) \xrightarrow{\sigma} \Omega^1_{Q_3}(-k) \to T^{\vee}_{\mathscr{F}}(-k) \to 0, \tag{2.13}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+3k+2)H^2, (8k^3+12k^2+8k-2)H^3).$$

Note that the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k+1$ on Q_3 has dimension

$$\begin{aligned}\dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_{Q_3}(-3-3k), \Omega^1_{Q_3}(-k)) - 1 \\ &= 8k^3 + 42k^2 + 69k + 34.\end{aligned}$$

We prove the main result of this section.

Theorem 2.2.3. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on Q_3 with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 6)H^3)$$

contains a irreducible component of dimension $8k^3 + 42k^2 + 69k + 44$ containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k + 1$ on Q_3.

Proof. Initially note that, by the commutative diagram (2.5), each tangent sheaf $T^\vee_{\mathscr{F}}$ of a generic codimension one distribution $\mathscr{F}$ of degree $2k+1$ can be given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_{Q_3}(-3-3k) \oplus \mathcal{O}_{Q_3}(-2-k) \to \Omega^1_{\mathbb{P}^4}|_{Q_3}(-k).$$

By Proposition 2.1.4,

$$\begin{aligned}\dim \mathcal{F}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_{Q_3}(-3-3k) \oplus \mathcal{O}_{Q_3}(-2-k), \Omega^1_{\mathbb{P}^4}|_{Q_3}(-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{Q_3}(-3-3k) \oplus \mathcal{O}_{Q_3}(-2-k)) \\ &= 8k^3 + 42k^2 + 69k + 44,\end{aligned}$$

for $k \geq 0$. Thus, it is enough to argue that

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \dim \mathcal{F}(2k+1) = 8k^3 + 42k^2 + 69k + 44$$

for $k \geq 1$, and hence, by semicontinuity, we can conclude that

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(2k+1) = 8k^3 + 42k^2 + 69k + 44,$$

for a generic sheaf $F \in \mathcal{F}(2k+1)$. Or equivalent, we must to show that

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = \dim \mathcal{F}(2k+1) - 36k^2 - 72k - 45 = 8k^3 + 6k^2 - 3k - 1,$$

since, by Proposition 2.1.5,

$$\dim \mathrm{Ext}^1(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) - \dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 36k^2 + 72k + 45,$$

for $k \geq 0$.

Indeed, applying the functor $\mathrm{Hom}(., T_{\mathscr{F}}^\vee(-k))$ to the exact sequence in display (2.12), we get

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = \dim \mathrm{Ext}^2(\Omega^1_{Q_3}(-k), T_{\mathscr{F}}^\vee(-k)) = h^2(TQ_3 \otimes T_{\mathscr{F}}^\vee), \qquad (2.14)$$

since $H^1(T_{\mathscr{F}}^\vee(2k+3)) = H^2(T_{\mathscr{F}}^\vee(2k+3)) = 0$ by Lemma 2.1.7.

Now, we twist the standard normal bundle sequence in display (1.5) by $\otimes T_{\mathscr{F}}^\vee$ and pass to cohomology, we get the exact sequence in cohomology

$$0 \to H^2(TQ_3 \otimes T_{\mathscr{F}}^\vee) \to H^2(T\mathbb{P}^4|_{Q_3} \otimes T_{\mathscr{F}}^\vee) \to H^2(T_{\mathscr{F}}^\vee(2)) \to 0,$$

since $H^1(T_{\mathscr{F}}^\vee(2)) = 0$ by item 2) of Lemma 2.1.7 and $H^3(TQ_3 \otimes T_{\mathscr{F}}^\vee) = 0$ by Lemma 2.1.8. Thus, if $k \geq 1$, we have

$$h^2(TQ_3 \otimes T_{\mathscr{F}}^\vee) = h^2(T\mathbb{P}^4|_{Q_3} \otimes T_{\mathscr{F}}^\vee) - h^0(\mathcal{O}_{Q_3}(2k-2)). \qquad (2.15)$$

In order to compute $h^2(T\mathbb{P}^4|_{Q_3} \otimes T_{\mathscr{F}}^\vee)$, we twist the exact sequences

$$0 \to \mathcal{O}_{Q_3}(-2-k) \to T_X(-k) \to T_{\mathscr{F}}^\vee(-k) \to 0$$

and

$$0 \to \mathcal{O}_{Q_3}(-3-3k) \to \Omega^1_{\mathbb{P}^4}|_{Q_3}(-k) \to T_X(-k) \to 0$$

in the commutative diagram (2.6) by $\otimes T\mathbb{P}^4|_{Q_3}(k)$ and pass to cohomology; we get the equality

$$h^2(T\mathbb{P}^4|_{Q_3} \otimes T_{\mathscr{F}}^\vee)) = h^3(T\mathbb{P}^4|_{Q_3}(-3-2k)) = \frac{1}{3}(2k-1)(2k+1)(8k+3), \quad (2.16)$$

for $k \geq 1$, since
$h^2(T\mathbb{P}^4|_{Q_3}(-2)) = h^3(T\mathbb{P}^4|_{Q_3}(-2)) = h^2(T\mathbb{P}^4|_{Q_3} \otimes \Omega^1_{\mathbb{P}^4}|_{Q_3}) = h^3(T\mathbb{P}^4|_{Q_3} \otimes \Omega^1_{\mathbb{P}^4}|_{Q_3}) = 0$.

The equations (2.14), (2.15) and (2.16), give us

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = 8k^3 + 6k^2 - 3k - 1,$$

for $k \geq 1$, as we desired. □

Remark 2.2.4. When $k = 0$, we can still conclude that the stable rank 2 reflexive sheaves F on Q_3 given by short exact sequence

$$0 \to \mathcal{O}_{Q_3}(-3) \oplus \mathcal{O}_{Q_3}(-2) \to \Omega^1_{\mathbb{P}^4}|_{Q_3} \to F \to 0$$

are smooth points of the moduli space of stable rank 2 reflexive sheaves with Chern classes $(c_1, c_2, c_3) = (0, 4H^2, 6H^3)$ within an irreducible component of dimension 45, since $\operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = 0$ (see equations (2.14), (2.15) and (2.16)). However, these sheaves only form a family of dimension 44 within this irreducible component.

Similarly, the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k$ on Q_3 has dimension

$$\begin{aligned} \dim \mathcal{D}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_{Q_3}(-2-3k), \Omega^1_{Q_3}(-k)) - 1 \\ &= 8k^3 + 30k^2 + 33k + 9. \end{aligned}$$

By Proposition 2.1.4, the family of the stable rank 2 reflexive sheaves F on Q_3 given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_{Q_3}(-2-3k) \oplus \mathcal{O}_{Q_3}(-2-k) \to \Omega^1_{\mathbb{P}^4}|_X(-k)$$

has dimension

$$\begin{aligned} \dim \mathcal{F}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_{Q_3}(-2-3k) \oplus \mathcal{O}_{Q_3}(-2-k), \Omega^1_{\mathbb{P}^4}|_{Q_3}(-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{Q_3}(-2-3k) \oplus \mathcal{O}_{Q_3}(-2-k)) \\ &= 8k^3 + 30k^2 + 33k + 19, \end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{F}(0) = 18$.

By Proposition 2.1.5,

$$\dim \operatorname{Ext}^1(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) - \dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 36k^2 + 36k + 18,$$

for $k \geq 0$.

Moreover, doing an analogue calculation as in the proof of Theorem 2.2.3 we get

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 8k^3 - 6k^2 - 3k + 1,$$

for $k \geq 1$ and $\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 0$, if $k = 0$.

As in the case $c_1 = 0$, we have:

Theorem 2.2.5. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on Q_3 with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 3k + 2)H^2, (16k^3 + 12k^2 + 8k - 2)H^3)$$

contains a irreducible component of dimension 18 and $8k^3 + 30k^2 + 33k + 19$, for $k = 0$ and $k \geq 1$, respectively, containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k$ on Q_3. Moreover, this component is nonsingular in the case $k = 0$.

In the next section we will do an analogue study on a smooth cubic threefold.

2.3 Codimension 1 distributions on cubic threefolds

Let $X \hookrightarrow \mathbb{P}^4$ denote a smooth cubic threefold with ample line bundle $\mathcal{O}_X(1)$ whose first Chern classe is denoted by H, i.e. $c_1(\mathcal{O}_X(1)) = H$. The cohomology ring $H^*(X, \mathbb{Z})$ of X is generated by H, a line $L \in H^4(X, \mathbb{Z})$ and a point $P \in H^6(X, \mathbb{Z})$ with the relations: $H^2 = 3L,\ H.L = P,\ H^3 = 3P$, see Section 1.1.

Recall that given a generic distribution $\mathscr{F}$ on X, the integer $r := -c_1(T_{\mathscr{F}})$ is called the *degree* of $\mathscr{F}$.

The next lemma gives us the Chern classes of the tangent sheaf of a generic codimension 1 distribution on X.

Lemma 2.3.1. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k$, then the normalization of the sheaf $T_{\mathscr{F}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-2-3k) \xrightarrow{\sigma} \Omega_X^1(-k) \to T_{\mathscr{F}}^{\vee}(-k) \to 0. \tag{2.17}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2+4k+4)H^2, (8k^3+16k^2+16k+10)H^3).$$

Proof. We know that $c(\Omega_X^1(-k)) = c(T_{\mathscr{F}}^{\vee}(-k)).c(\mathcal{O}_X(-2-3k))$. So,

$$c_1(T_{\mathscr{F}}^{\vee}(-k)) = c_1(\Omega_X^1(-k)) - c_1(\mathcal{O}_X(-2-3k)) = 0,$$

since $c_1(\Omega_X^1(-k)) = c_1(\mathcal{O}_X(-2-3k)) = (-2-3k)H$,

$$\begin{aligned} c_2(T_{\mathscr{F}}^{\vee}(-k)) &= c_2(\Omega_X^1(-k)) \\ &= 4H^2 + 2(-2H)(-kH) + 3(-kH)^2 \\ &= (3k^2+4k+4)H^2, \end{aligned}$$

since $c_1(T_{\mathscr{F}}^{\vee}(-k)) = 0$ and

$$\begin{aligned} c_3(T_{\mathscr{F}}^{\vee}(-k)) &= c_3(\Omega_X^1(-k)) - c_2(T_{\mathscr{F}}^{\vee}(-k))c_1(\mathcal{O}_X(-2-3k)) \\ &= 2H^3 + (-kH)(4H^2) + (-kH)^2(-2H) + (-kH)^3 \\ &+ (3k^2+4k+4)H^2(2+3k)H \\ &= (8k^3+16k^2+16k+10)H^3. \end{aligned}$$

□

When $\mathscr{F}$ has degree $r = 2k+1$, we have:

Lemma 2.3.2. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k+1$, then the normalization of the sheaf $T_{\mathscr{F}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-4-3k) \xrightarrow{\sigma} \Omega_X^1(-1-k) \to T_{\mathscr{F}}^{\vee}(-1-k) \to 0. \tag{2.18}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+7k+7)H^2, (8k^3+28k^2+38k+23)H^3).$$

Note that the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k$ on X has dimension

$$\begin{aligned}\dim \mathcal{D}(2k) &= \dim \mathrm{Hom}(\mathcal{O}_X(-2-3k), \Omega^1_X(-k)) - 1 \\ &= 12k^3 + 42k^2 + 36k + 9.\end{aligned}$$

We prove the main result of this section.

Theorem 2.3.3. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 4k + 4)H^2, (8k^3 + 16k^2 + 16k + 10)H^3)$$

contains a irreducible component of dimension $12k^3 + 42k^2 + 36k + 49$ containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k$ on X.

Proof. Initially note that, by the commutative diagram (2.5), each tangent sheaf $T^\vee_{\mathscr{F}}$ of a generic codimension one distribution $\mathscr{F}$ of degree $2k$ can be given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_X(-2-3k) \oplus \mathcal{O}_X(-3-k) \to \Omega^1_{\mathbb{P}^4}|_X(-k).$$

By Proposition 2.1.4,

$$\begin{aligned}\dim \mathcal{F}(2k) &= \dim \mathrm{Hom}(\mathcal{O}_X(-2-3k) \oplus \mathcal{O}_X(-3-k), \Omega^1_{\mathbb{P}^4}|_X(-k)) \\ &- \dim \mathrm{Aut}(\mathcal{O}_X(-2-3k) \oplus \mathcal{O}_X(-3-k)) \\ &= 12k^3 + 42k^2 + 36k + 49,\end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{F}(0) = 44$. Thus, it is enough to argue that

$$\dim \mathrm{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \dim \mathcal{F}(2k) = 12k^3 + 42k^2 + 36k + 49$$

for $k \geq 1$, and hence, by semicontinuity, we can conclude that

$$\dim \mathrm{Ext}^1(F, F) = \dim \mathcal{F}(2k) = 12k^3 + 42k^2 + 36k + 49,$$

for a generic sheaf $F \in \mathcal{F}(2k)$. Or equivalent, we must to show that

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = \dim \mathcal{F}(2k) - 36k^2 - 48k - 45 = 12k^3 + 6k^2 - 12k + 4,$$

since, by Proposition 2.1.5,

$$\dim \operatorname{Ext}^1(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) - \dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 36k^2 + 48k + 45,$$

for $k \geq 0$.

Indeed, applying the functor $\operatorname{Hom}(., T_{\mathscr{F}}^\vee(-k))$ to the exact sequence (2.17), we get the equality

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = \dim \operatorname{Ext}^2(\Omega_X^1(-k), T_{\mathscr{F}}^\vee(-k)) = h^2(TX \otimes T_{\mathscr{F}}^\vee), \tag{2.19}$$

since $H^1(T_{\mathscr{F}}^\vee(2+2k)) = H^2(T_{\mathscr{F}}^\vee(2+2k)) = 0$ by Lemma (2.1.7).

Now, we twist the exact sequence in display (1.5) by $\otimes T_{\mathscr{F}}^\vee$ and then pass to cohomology, we have

$$h^2(TX \otimes T_{\mathscr{F}}^\vee) = h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee) - h^0(\mathcal{O}_X(2k-3)) \tag{2.20}$$

since, by Lemma 2.1.7, $h^1(T_{\mathscr{F}}^\vee(3)) = 0$ and $h^2(T_{\mathscr{F}}^\vee(3)) = h^0(\mathcal{O}_X(2k-3))$ and, by Lemma 2.1.8, $h^3(TX \otimes T_{\mathscr{F}}^\vee) = 0$.

In order to compute $h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee)$, we twist the exact sequences

$$0 \to \mathcal{O}_X(-3-k) \to T_X(-k) \to T_{\mathscr{F}}^\vee(-k) \to 0$$

and

$$0 \to \mathcal{O}_X(-2-3k) \to \Omega_{\mathbb{P}^4}^1|_X(-k) \to T_X(-k) \to 0$$

in the commutative diagram (2.6) by $\otimes T\mathbb{P}^4|_X(k)$ and then pass to cohomology, we get, for each $k \geq 1$, the equality

$$h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee) = h^3(T\mathbb{P}^4|_X(-2-2k)) = 16k^3 - 6k^2 + k - 1 \tag{2.21}$$

since $h^2(T\mathbb{P}^4|_X \otimes \Omega_{\mathbb{P}^4}^1|_X) = h^3(T\mathbb{P}^4|_X \otimes \Omega_{\mathbb{P}^4}^1|_X) = h^2(T\mathbb{P}^4|_X(-3)) = h^3(T\mathbb{P}^4|_X(-3)) = 0$.

Joining the equations (2.19), (2.20) and (2.21), we have

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 12k^3 + 6k^2 - 12k + 4,$$

for $k \geq 1$. □

Remark 2.3.4. When $k = 0$, we can still conclude that the stable rank 2 reflexive sheaves F on X given by short exact sequence

$$0 \to \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^4}|_X \to F \to 0$$

are smooth points of the moduli space of stable rank 2 reflexive sheaves with Chern classes $(c_1, c_2, c_3) = (0, 4H^2, 10H^3)$ within an irreducible component of dimension 45, since $\mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 0$ (see equations (2.19) and (2.20)). However, these sheaves only form a family of dimension 44 within this irreducible component.

Similarly, the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k+1$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k+1) &= \dim \mathrm{Hom}(\mathcal{O}_X(-4-3k), \Omega^1_X(-1-k)) - 1 \\ &= 12k^3 + 60k^2 + 87k + 39. \end{aligned}$$

By Proposition 2.1.4, the family of the stable rank 2 reflexive sheaves F on X given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-4-k) \to \Omega^1_{\mathbb{P}^4}|_X(-1-k)$$

has dimension

$$\begin{aligned} \dim \mathcal{F}(2k+1) &= \dim \mathrm{Hom}(\mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-4-k), \Omega^1_{\mathbb{P}^4}|_X(-1-k)) \\ &- \dim \mathrm{Aut}(\mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-4-k)) \\ &= 12k^3 + 60k^2 + 87k + 79, \end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{F}(1) = 78$.

By Proposition 2.1.5,

$$\dim \mathrm{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) - \dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 36k^2 + 84k + 78,$$

for $k \geq 0$.

Following the proof of Theorem 2.3.3, it is easy to show that

$$\dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 12k^3 + 24k^2 + 3k + 1,$$

if $k \geq 1$ and $\dim \mathrm{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 0$, for $k = 0$.

For the case $c_1 = -1$, we establish the following theorem:

Theorem 2.3.5. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 7k + 7)H^2, (8k^3 + 28k^2 + 38k + 23)H^3)$$

contains a irreducible component of dimension (78, if $k = 0$) $12k^3 + 60k^2 + 87k + 79$ for $k \geq 1$, containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k + 1$ on X. Moreover, this component is nonsingular in the case $k = 0$.

In the next section we will do an analogue study on a smooth quartic threefold.

2.4 Codimension 1 distributions on quartic threefolds

Throughout this section $X \hookrightarrow \mathbb{P}^4$ denotes a smooth quartic threefold with ample line bundle $\mathcal{O}_X(1)$ whose first Chern classe is denoted by H, i.e. $c_1(\mathcal{O}_X(1)) = H$. The cohomology ring $H^*(X, \mathbb{Z})$ of X is generated by H, a line $L \in H^4(X, \mathbb{Z})$ and a point $P \in H^6(X, \mathbb{Z})$ with the relations: $H^2 = 4L,\ H.L = P,\ H^3 = 4P$, see Section 1.1.

Recall that given a generic distribution $\mathscr{F}$ on X, the integer $r := -1 - c_1(T_{\mathscr{F}})$ is called the *degree* of $\mathscr{F}$.

Our first step is to calculate the Chern classes of a generic codimension 1 distribution $\mathscr{F}$ on X of odd degree, say $r = 2k + 1$.

Lemma 2.4.1. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k + 1$, then the normalization of the sheaf $T_{\mathscr{F}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-4 - 3k) \xrightarrow{\sigma} \Omega_X^1(-1 - k) \to T_{\mathscr{F}}^{\vee}(-1 - k) \to 0. \tag{2.22}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2 + 8k + 11)H^2, (8k^3 + 32k^2 + 54k + 50)H^3).$$

Proof. We know that $c(\Omega_X^1(-1 - k)) = c(T_{\mathscr{F}}^{\vee}(-1 - k)).c(\mathcal{O}_X(-4 - 3k))$. So,

$$c_1(T_{\mathscr{F}}^{\vee}(-1 - k)) = c_1(\Omega_X^1(-1 - k)) - c_1(\mathcal{O}_X(-4 - 3k)) = 0,$$

since $c_1(\Omega^1_X(-1-k)) = c_1(\mathcal{O}_X(-4-3k)) = (-4-3k)H$,

$$\begin{aligned} c_2(T^\vee_{\mathscr{F}}(-1-k)) &= c_2(\Omega^1_X(-1-k)) \\ &= 6H^2 + 2(-H)(-1-k)H + 3(-1-k)^2H^2 \\ &= (3k^2+8k+11)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{F}}(-1-k)) = 0$ and

$$\begin{aligned} c_3(T^\vee_{\mathscr{F}}(-1-k)) &= c_3(\Omega^1_X(-1-k)) - c_2(T^\vee_{\mathscr{F}}(-1-k))c_1(\mathcal{O}_X(-4-3k)) \\ &= 14H^3 + (-1-k)H(6H^2) + (-1-k)^2H^2(-H) + (-1-k)^3H^3 \\ &+ (3k^2+8k+11)H^2(4+3k)H \\ &= (8k^3+32k^2+54k+50)H^3. \end{aligned}$$

□

When $\mathscr{F}$ has degree $r = 2k$, we have:

Lemma 2.4.2. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k$, then the normalization of the sheaf $T^\vee_{\mathscr{F}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-3-3k) \xrightarrow{\sigma} \Omega^1_X(-1-k) \to T^\vee_{\mathscr{F}}(-1-k) \to 0. \tag{2.23}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+5k+8)H^2, (8k^3+20k^2+28k+30)H^3).$$

Note that the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k+1$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_X(-4-3k), \Omega^1_X(-1-k)) - 1 \\ &= 16k^3 + 76k^2 + 86k + 40, \end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{D}(1) = 39$.

We prove the main result of this section.

Theorem 2.4.3. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2+8k+11)H^2, (8k^3+32k^2+54k+50)H^3)$$

contains a irreducible component of dimension (139 *if* $k=0$*)* $16k^3+76k^2+86k+145$ *for* $k \geq 1$ *containing the family of the tangent sheaves of a generic codimension one distribution of degree* $2k+1$ *on* X*.*

Proof. Initially note that, by the commutative diagram (2.5), each tangent sheaf $T_{\mathscr{F}}^{\vee}$ of a generic codimension one distribution $\mathscr{F}$ of degree $2k+1$ can be given as the cokernel of the monomorphism

$$\sigma \,:\, \mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-5-k) \to \Omega^1_{\mathbb{P}^4}|_X(-1-k).$$

By Proposition 2.1.4,

$$\begin{aligned} \dim \mathcal{F}(2k+1) &= \dim \mathrm{Hom}(\mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-5-k), \Omega^1_{\mathbb{P}^4}|_X(-1-k)) \\ &- \dim \mathrm{Aut}(\mathcal{O}_X(-4-3k) \oplus \mathcal{O}_X(-5-k)) \\ &= 16k^3+76k^2+86k+145, \end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{F}(1) = 139$. Thus, it is enough to argue that

$$\dim \mathrm{Ext}^1(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(2k+1) = 16k^3+76k^2+86k+145, \quad (k \geq 1)$$

and

$$\dim \mathrm{Ext}^1(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(1) = 139,$$

if $k=0$, and hence, by semicontinuity, we can conclude that

$$\dim \mathrm{Ext}^1(F, F) = \dim \mathcal{F}(2k+1) = 16k^3+76k^2+86k+145, \quad (k \geq 1)$$

for a generic sheaf $F \in \mathcal{F}(2k+1)$ and

$$\dim \mathrm{Ext}^1(F, F) = \dim \mathcal{F}(1) = 139,$$

for a generic sheaf $F \in \mathcal{F}(1)$. Or equivalent, we must to show that

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(2k+1) - 24k^2 - 64k - 85 = 16k^3+52k^2+22k+60,$$

if $k \geq 1$ and

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(1) - 85 = 54,$$

since, by Proposition 2.1.5,

$$\dim \operatorname{Ext}^1(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) - \dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 24k^2 + 64k + 85,$$

for $k \geq 0$.

Indeed, applying the functor $\operatorname{Hom}(., T_{\mathscr{F}}^\vee(-1-k))$ to the exact sequence (2.22), we get the equalities

$$\begin{aligned} \dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) &= \dim \operatorname{Ext}^2(\Omega_X^1, T_{\mathscr{F}}^\vee) - \dim \operatorname{Ext}^2(\mathcal{O}_X, T_{\mathscr{F}}^\vee(3+2k)) \\ &= h^2(TX \otimes T_{\mathscr{F}}^\vee) - h^2(T_{\mathscr{F}}^\vee(3+2k)) \end{aligned} \tag{2.24}$$

if $k \geq 1$, and

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = h^2(TX \otimes T_{\mathscr{F}}^\vee) - 1 \tag{2.25}$$

if $k = 0$, since, by item 2) of the Lemma 2.1.7, $h^1(T_{\mathscr{F}}^\vee(3+2k)) = 0$, $h^2(T_{\mathscr{F}}^\vee(3)) = 1$ and, by stability of $T_{\mathscr{F}}^\vee$, $\operatorname{Ext}^3(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) \simeq \operatorname{Hom}(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee(-1)) = 0$.

Now, we twist the exact sequence (1.5) by $\otimes T_{\mathscr{F}}^\vee$ and then pass to cohomology, we have

$$h^2(TX \otimes T_{\mathscr{F}}^\vee) = h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee) - h^0(\mathcal{O}_X(2k-2)) \tag{2.26}$$

since, by Lemma 2.1.7, $h^1(T_{\mathscr{F}}^\vee(4)) = 0$, $h^2(T_{\mathscr{F}}^\vee(4)) = h^0(\mathcal{O}_X(2k-2))$ and, by Lemma 2.1.8, $h^3(TX \otimes T_{\mathscr{F}}^\vee) = 0$.

In order to compute $h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee)$, we twist the exact sequences

$$0 \to \mathcal{O}_X(-5-k) \to T_X(-1-k) \to T_{\mathscr{F}}^\vee(-1-k) \to 0$$

and

$$0 \to \mathcal{O}_X(-4-3k) \to \Omega_{\mathbb{P}^4}^1|_X(-1-k) \to T_X(-1-k) \to 0$$

in the commutative diagram (2.6) by $\otimes T\mathbb{P}^4|_X(1+k)$ and then pass to cohomology, we get, for each $k \geq 0$, the equalities

$$h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^\vee) = h^2(T\mathbb{P}^4|_X \otimes T_X) + 40 \tag{2.27}$$

since $h^2(T\mathbb{P}^4|_X(-4)) = h^3(T\mathbb{P}^4|_X \otimes T_X) = 0$ and $h^3(T\mathbb{P}^4|_X(-4)) = 40$;

$$h^2(T\mathbb{P}^4|_X \otimes T_X) = h^3(T\mathbb{P}^4|_X(-3-2k)) + 5, \tag{2.28}$$

since $h^2(T\mathbb{P}^4|_X(-3-2k)) = h^3(T\mathbb{P}^4|_X \otimes \Omega^1_{\mathbb{P}^4}|_X) = 0$ and $h^2(T\mathbb{P}^4|_X \otimes \Omega^1_{\mathbb{P}^4}|_X) = 5$.

A simple calculation shows that

$$h^3(T\mathbb{P}^4|_X(-3-2k)) = \frac{2}{3}(2k+1)(16k^2+22k+15). \tag{2.29}$$

for $k \geq 0$.

When $k = 0$, the equations (2.25), (2.26), (2.27), (2.28) and (2.29) give us

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 54.$$

When $k \geq 1$, the equations (2.24), (2.26), (2.27), (2.28) and (2.29) give us

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 16k^3 + 52k^2 + 22k + 60.$$

□

Similarly, the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_X(-3-3k), \Omega^1_X(-1-k)) - 1 \\ &= 16k^3 + 52k^2 + 22k + 14, \end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{D}(0) = 9$. By Proposition 2.1.4, the family of the stable rank 2 reflexive sheaves F on X is given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-5-k) \to \Omega^1_{\mathbb{P}^4}|_X(-1-k)$$

has dimension

$$\dim \mathcal{F}(2k) = \begin{cases} 99, & k = 0 \\ 208, & k = 1 \end{cases}$$

and, for each $k \geq 2$,

$$\begin{aligned} \dim \mathcal{F}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-5-k), \Omega^1_{\mathbb{P}^4}|_X(-1-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-5-k)) \\ &= 16k^3 + 52k^2 + 22k + 119. \end{aligned}$$

By Proposition 2.1.5,

$$\dim \mathrm{Ext}^1(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) - \dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 24k^2 + 40k + 59,$$

for $k \geq 0$.

Following the proof of Theorem 2.4.3, it is easy to show that

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = \begin{cases} 40, & k = 0 \\ 85, & k = 1 \end{cases}$$

and, for each $k \geq 2$,

$$\dim \mathrm{Ext}^2(T_{\mathscr{F}}^\vee, T_{\mathscr{F}}^\vee) = 16k^3 + 28k^2 - 18k + 60.$$

For the case $c_1 = -1$, we establish the following theorem:

Theorem 2.4.4. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 5k + 8)H^2, (8k^3 + 20k^2 + 28k + 30)H^3)$$

contains a irreducible component of dimension (99, if $k = 0$; 208, if $k = 1$) $16k^3 + 52k^2 + 22k + 119$ containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k$ on X.

In the next section we will do an analogue study on a smooth quintic threefold.

2.5 Codimension 1 distributions on quintic threefolds

Throughout this section $X \hookrightarrow \mathbb{P}^4$ denotes a smooth quintic threefold with ample line bundle $\mathcal{O}_X(1)$ whose first Chern classe is denoted by H, i.e. $c_1(\mathcal{O}_X(1)) = H$. The cohomology ring $H^*(X, \mathbb{Z})$ of X is generated by H, a line $L \in H^4(X, \mathbb{Z})$ and a point $P \in H^6(X, \mathbb{Z})$ with the relations: $H^2 = 5L$, $H.L = P$, $H^3 = 5P$, see Section 1.1.

Recall that given a generic distribution $\mathscr{F}$ on X, the integer $r := -2 - c_1(T_{\mathscr{F}})$ is called the *degree* of $\mathscr{F}$.

We start this section by calculating the Chern classes of a generic codimension 1 distribution $\mathscr{F}$ on X of even degree, say $r = 2k$.

Lemma 2.5.1. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k$, then the normalization of the sheaf $T^\vee_{\mathscr{F}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-3-3k) \xrightarrow{\sigma} \Omega^1_X(-1-k) \to T^\vee_{\mathscr{F}}(-1-k) \to 0. \qquad (2.30)$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2+6k+13)H^2, (8k^3+24k^2+44k+68)H^3).$$

Proof. We know that $c(\Omega^1_X(-1-k)) = c(T^\vee_{\mathscr{F}}(-1-k)).c(\mathcal{O}_X(-3-3k))$. So,

$$c_1(T^\vee_{\mathscr{F}}(-1-k)) = c_1(\Omega^1_X(-1-k)) - c_1(\mathcal{O}_X(-3-3k)) = 0,$$

since $c_1(\Omega^1_X(-1-k)) = c_1(\mathcal{O}_X(-3-3k)) = (-3-3k)H$,

$$\begin{aligned} c_2(T^\vee_{\mathscr{F}}(-1-k)) &= c_2(\Omega^1_X(-1-k)) \\ &= 10H^2 + 3(-1-k)^2H^2 \\ &= (3k^2+6k+16)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{F}}(-1-k)) = 0$ and

$$\begin{aligned} c_3(T^\vee_{\mathscr{F}}(-1-k)) &= c_3(\Omega^1_X(-1-k)) - c_2(T^\vee_{\mathscr{F}}(-1-k))c_1(\mathcal{O}_X(-3-3k)) \\ &= 40H^3 + (-1-k)H(10H^2) + (-1-k)^3H^3 \\ &+ (3k^2+6k+13)H^2(3+3k)H \\ &= (8k^3+24k^2+44k+68)H^3. \end{aligned}$$

□

When $\mathscr{F}$ has degree $r = 2k+1$, we have:

Lemma 2.5.2. *If a generic distribution $\mathscr{F}$ on X has degree $r = 2k+1$, then the normalization of the sheaf $T^\vee_{\mathscr{F}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_X(-5-3k) \xrightarrow{\sigma} \Omega^1_X(-2-k) \to T^\vee_{\mathscr{F}}(-2-k) \to 0. \qquad (2.31)$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+9k+17)H^2, (8k^3+36k^2+74k+97)H^3).$$

Note that the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_X(-3-3k), \Omega^1_X(-1-k)) - 1 \\ &= 20k^3 + 60k^2 - 15k + 44, \end{aligned}$$

for $k \geq 0$.

We prove the main result of this section.

Theorem 2.5.3. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 6k + 13)H^2, (8k^3 + 24k^2 + 44k + 68)H^3)$$

contains a irreducible component of dimension (198 if $k = 0$, 323 if $k = 1$) $20k^3 + 60k^2 - 15k + 268$ for $k \geq 2$, containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k$ on X.

Proof. Initially note that, by the commutative diagram (2.5), each tangent sheaf $T^\vee_{\mathscr{F}}$ of a generic codimension one distribution $\mathscr{F}$ of degree $2k$ can be given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-6-k) \to \Omega^1_{\mathbb{P}^4}|_X(-1-k).$$

By Proposition 2.1.4,

$$\begin{aligned} \dim \mathcal{F}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-6-k), \Omega^1_{\mathbb{P}^4}|_X(-1-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_X(-3-3k) \oplus \mathcal{O}_X(-6-k)) \\ &= 20k^3 + 60k^2 - 15k + 268, \end{aligned}$$

if $k \geq 2$, and

$$\dim \mathcal{F}(2k) = \begin{cases} 198, & \text{if } k = 0 \\ 323, & \text{if } k = 1 \end{cases}$$

Thus, it is enough to argue that

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \dim \mathcal{F}(2k) = 20k^3 + 60k^2 - 15k + 268, \quad (k \geq 2)$$

and

$$\dim \operatorname{Ext}^1(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(2k) = \begin{cases} 198, & \text{if } k = 0 \\ 323, & \text{if } k = 1 \end{cases}$$

and hence, by semicontinuity, we can conclude that

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(2k) = 20k^3 + 60k^2 - 15k + 268, \quad (k \geq 2)$$

and

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(2k) = \begin{cases} 198, & \text{if } k = 0 \\ 323, & \text{if } k = 1 \end{cases}$$

for a generic sheaf $F \in \mathcal{F}(2k)$. Or equivalent, we must to show that

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(2k) = 20k^3 + 60k^2 - 15k + 268,$$

if $k \geq 2$, and

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \mathcal{F}(2k) = \begin{cases} 198, & \text{if } k = 0 \\ 323, & \text{if } k = 1 \end{cases}$$

since, by Proposition 2.1.5,

$$\dim \operatorname{Ext}^1(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = \dim \operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}),$$

for $k \geq 0$.

Applying the functor $\operatorname{Hom}(., T_{\mathscr{F}}^{\vee}(-1-k))$ to the exact sequence (2.30), we get

$$\dim \operatorname{Ext}^2(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = h^2(TX \otimes T_{\mathscr{F}}^{\vee}) - h^2(T_{\mathscr{F}}^{\vee}(2+2k)) + 1 \qquad (2.32)$$

if $k \geq 0$, since, by Lemma 2.1.7, $h^1(T_{\mathscr{F}}^{\vee}(2+2k)) = 0$, by Lemma 2.1.8, $h^3(TX \otimes T_{\mathscr{F}}^{\vee}) = 0$ and $\dim \operatorname{Ext}^3(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) = 1$, because $\operatorname{Ext}^3(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee}) \simeq \operatorname{Hom}(T_{\mathscr{F}}^{\vee}, T_{\mathscr{F}}^{\vee})$.

Now, we twist the exact sequence (1.5) by $\otimes T_{\mathscr{F}}^{\vee}$ and then pass to cohomology, we have

$$h^2(TX \otimes T_{\mathscr{F}}^{\vee}) = h^2(T\mathbb{P}^4|_X \otimes T_{\mathscr{F}}^{\vee}) - h^2(T_{\mathscr{F}}^{\vee}(5)) \qquad (2.33)$$

since, by Lemma 2.1.7, $h^1(T_{\mathscr{F}}^{\vee}(5)) = 0$ and, by Lemma 2.1.8, $h^3(TX \otimes T_{\mathscr{F}}^{\vee}) = 0$.

Using the exact sequence (2.30), it is easy to see that

$$h^2(T^\vee_{\mathscr{F}}(5)) = h^0(\mathcal{O}_X(2k-3)) + 1, \tag{2.34}$$

since $h^2(\Omega^1_X(5) = 1$ and $h^2(\Omega^1_X(5) = 0$.

In order to compute $h^2(T\mathbb{P}^4|_X \otimes T^\vee_{\mathscr{F}})$, we twist the exact sequences

$$0 \to \mathcal{O}_X(-6-k) \to T_X(-1-k) \to T^\vee_{\mathscr{F}}(-1-k) \to 0$$

and

$$0 \to \mathcal{O}_X(-3-3k) \to \Omega^1_{\mathbb{P}^4}|_X(-1-k) \to T_X(-1-k) \to 0$$

in the commutative diagram (2.6) by $\otimes T\mathbb{P}^4|_X(1+k)$ and then pass to cohomology, we get, for each $k \geq 0$, the equality

$$h^2(T\mathbb{P}^4|_X \otimes T^\vee_{\mathscr{F}}) = h^3(T\mathbb{P}^4|_X(-2-2k)) + 224, \tag{2.35}$$

since $h^2(T\mathbb{P}^4|_X(t)) = 0$, for all $t \neq 0$ and $h^3(T\mathbb{P}^4|_X \otimes T_X) = 0$.

Now, we twist the exact sequence

$$0 \to T\mathbb{P}^4(-5) \to T\mathbb{P}^4 \to T\mathbb{P}^4|_X \to 0$$

by $\otimes \mathcal{O}_X(-2-2k)$ and then pass to cohomology, we get

$$h^3(T\mathbb{P}^4|_X(-2-2k)) = \frac{5}{6}(32k^3 + 36k^2 + 46k + 12), \tag{2.36}$$

for $k \geq 0$.

Joining the equations (2.32), (2.33), (2.34), (2.35) and (2.36), we get

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 20k^3 + 60k^2 - 15k + 268,$$

if $k \geq 2$, and

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \begin{cases} 198, & k = 0 \\ 323, & k = 1 \end{cases}$$

as desired. □

Similarly, the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension one distribution of degree $2k+1$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_X(-5-3k), \Omega^1_X(-2-k)) - 1 \\ &= 20k^3 + 90k^2 + 60k + 54, \end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{D}(1) = 39$. By Proposition 2.1.4, the family of the stable rank 2 reflexive sheaves F on X is given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_X(-5-3k) \oplus \mathcal{O}_X(-7-k) \to \Omega^1_{\mathbb{P}^4}|_X(-2-k)$$

has dimension

$$\dim \mathcal{F}(2k+1) = \begin{cases} 248, & \text{if } k = 0 \\ 446, & \text{if } k = 1 \end{cases}$$

and, for each $k \geq 2$,

$$\begin{aligned} \dim \mathcal{F}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_X(-5-3k) \oplus \mathcal{O}_X(-7-k), \Omega^1_{\mathbb{P}^4}|_X(-2-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_X(-5-3k) \oplus \mathcal{O}_X(-7-k)) \\ &= 20k^3 + 90k^2 + 60k + 278. \end{aligned}$$

By Proposition 2.1.5,

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}),$$

for $k \geq 0$.

Following the proof of Theorem 2.5.3, it is easy to show that

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = \begin{cases} 248, & \text{if } k = 0 \\ 446, & \text{if } k = 1 \end{cases}$$

and, for each $k \geq 2$,

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{F}}, T^\vee_{\mathscr{F}}) = 20k^3 + 90k^2 + 60k + 278.$$

For the case $c_1 = -1$, we establish the following theorem:

Theorem 2.5.4. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 9k + 17)H^2, (8k^3 + 36k^2 + 74k + 97)H^3)$$

contains a irreducible component of dimension (248 if $k = 0$, 446 if $k = 1$) $20k^3 + 90k^2 + 60k + 278$ for $k \geq 2$, containing the family of the tangent sheaves of a generic codimension one distribution of degree $2k + 1$ on X.

The next section is dedicated to the study of generic foliations by curves on the projective space $\mathbb{P}^3$ and on a smooth quadric threefold Q_3. We will use [1] as the main reference.

2.6 Generic foliations by curves on $\mathbb{P}^3$ and Q_3

Let X be a smooth projective threefold X of Picard rank 1. We set

$$\tau_X := \min\{t \in \mathbb{Z} \mid H^0(TX(t)) \neq 0\}.$$

A *foliation by curves $\mathscr{F}$ on X* is a short exact sequence of the form

$$\mathscr{F} \quad : \quad 0 \to \mathcal{O}_X(-r - \tau_X) \xrightarrow{\sigma} TX \to N_{\mathscr{F}} \to 0 \tag{2.37}$$

where $N_{\mathscr{F}}$ is a torsion free sheaf called the *normal sheaf* of $\mathscr{F}$. The non negative integer r above is called the *degree* of $\mathscr{F}$. Note that $\mathrm{rk}(N_{\mathscr{F}}) = 2$.

The image of the morphism $\sigma^\vee : \Omega^1_X \to \mathcal{O}_X(\tau_X + r)$ is the twisted ideal sheaf $I_Z(r + \tau_X)$ of a subscheme of X of dimension at most 1, called the *singular scheme* of $\mathscr{F}$ and denoted by $\mathrm{Sing}(\mathscr{F})$. Thus dualizing the sequence in display (2.37) we obtain

$$0 \to N^\vee_{\mathscr{F}} \to \Omega^1_X \xrightarrow{\sigma^\vee} I_Z(r + \tau_X) \to 0, \tag{2.38}$$

where $N^\vee_{\mathscr{F}}$ is called the *conormal sheaf of $\mathscr{F}$*. In general, the singular scheme $Z := \mathrm{Sing}(\mathscr{F})$ may contain a pure 1-dimensional subscheme of the singular scheme of $\mathscr{F}$, which is called the 1-*dimensional component*, and it is denoted by $\mathrm{Sing}_1(\mathscr{F})$.

The set of vector fields $\sigma \in H^0(TX(r + \tau_X))$ for which $\dim \operatorname{coker} \sigma^\vee = 0$ is an open subset of $\mathbb{P}(H^0(TX(r + \tau_X)))$. For this reason, foliations by curves satisfying $\dim \mathrm{Sing}(\mathscr{F}) = 0$ are called *generic*. The first part of [1, Main Theorem 1] implies that generic foliations by curves of degree r provide a family of μ-stable rank 2 reflexive with given Chern classes parametrized by and open subset of $\mathbb{P}(H^0(TX(r+\tau_X))$. Our main goal in this section is to show that such families are dense within an irreducible component of the (Gieseker–Maruyama) moduli space of stable rank 2 sheaves on the projective space $\mathbb{P}^3$ and on a smooth quadric threefold Q_3.

We start considering the case $X = \mathbb{P}^3$. Recall that a generic foliation by curves $\mathscr{F}$ on $\mathbb{P}^3$ is given by

$$\mathscr{F} \quad : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-r+1) \xrightarrow{\sigma} T\mathbb{P}^3 \to N_{\mathscr{F}} \to 0, \tag{2.39}$$

where $r \geq 0$ is the degree of $\mathscr{F}$, since $\tau_{\mathbb{P}^3} = -1$. According to [1, Theorem 5.1], the normal sheaf $N_{\mathscr{F}}$ is a μ-stable rank 2 reflexive sheaf on $\mathbb{P}^3$.

The next lemma is dedicated to the study of the cohomology of the normal sheaf of a foliation by curves in $\mathbb{P}^3$.

Lemma 2.6.1. *If $\mathscr{F}$ is a generic foliation by curves of degree r on $\mathbb{P}^3$, then:*

(1) $h^0(N_{\mathscr{F}}(t)) = 0$ *for* $t \leq -2$;

(2) $h^1(N_{\mathscr{F}}(t)) = 0$ *for all* $t \in \mathbb{Z}$;

(3) $h^2(N_{\mathscr{F}}(t)) = h^0(\mathcal{O}_{\mathbb{P}^3}(-t+r-5))$ *for* ($t \neq -4$ *and* $t \geq -5$), *moreover* $h^2(N_{\mathscr{F}}(t)) = 0$ *for* $t \geq r-4$;

(4) $h^3(N_{\mathscr{F}}(t)) = 0$ *for* $t \geq -5$.

Proof. It follows from the long exact sequence in cohomology derived from the exact sequence (2.39). □

When $\mathscr{F}$ has odd degree, say $r = 2k+1$, the normalization of the normal sheaf fits into the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-3k) \xrightarrow{\sigma} T\mathbb{P}^3(-2-k) \to N_{\mathscr{F}}(-2-k) \to 0, \tag{2.40}$$

for $k \geq 0$. Similarly, if $\mathscr{F}$ has even degree, say $r = 2k$, then the normalization of the normal sheaf fits into the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1-3k) \xrightarrow{\sigma} T\mathbb{P}^3(-2-k) \to N_{\mathscr{F}}(-2-k) \to 0, \tag{2.41}$$

where $k \geq 0$.

For the generic foliations by curves of odd degree, i.e., those given by the exact sequence in display (2.40), we show the following theorem:

Theorem 2.6.2. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$ with Chern classes*

$$(c_1, c_2, c_3) = (0, 3k^2 + 4k + 2, 8k^3 + 16k^2 + 12k + 4)$$

contains a rational irreducible component of dimension $4k^3 + 20k^2 + 31k + 14$ whose generic point is the normal sheaf of a generic foliation of degree $2k + 1$ on $\mathbb{P}^3$ given by the exact sequence in display (2.40).

Before starting the proof of this theorem, we note that the family of sheaves $N_{\mathscr{F}}$ given by the exact sequence in display (2.40), which we will denote simply by $\mathcal{G}(2k+1)$, has dimension $h^0(T\mathbb{P}^3(2k)) - 1$, since each sheaf $N_{\mathscr{F}}$ is defined by a section

$$\sigma \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2-3k), T\mathbb{P}^3(-2-k)) \simeq H^0(T\mathbb{P}^3(2k))$$

up to a scalar multiple, i.e., $\sigma \in H^0(T\mathbb{P}^3(2k))$, so we must argue that the following equality holds

$$\dim \mathrm{Ext}^1(N_{\mathscr{F}}, N_{\mathscr{F}}) = \dim \mathcal{G}(2k+1) = h^0(T\mathbb{P}^3(2k)) - 1 = 4k^3 + 20k^2 + 31k + 14,$$

for each $k \geq 0$.

Being $N_{\mathscr{F}}$ a stable rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1(N_{\mathscr{F}}) = 0$, we have

$$\dim \mathrm{Ext}^1(N_{\mathscr{F}}, N_{\mathscr{F}}) - \dim \mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) = 8c_2(N_{\mathscr{F}}) - 3 = 24k^2 + 32k + 13,$$

see [27, Proposition 3.4].

Therefore, we must to compute the dimension of $\mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}})$, showing that

$$\dim \mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) = h^0(T\mathbb{P}^3(2k)) - 24k^2 - 32k - 14 = 4k^3 - 4k^2 - k + 1.$$

Proof of Theorem 2.6.2. Applying the functor $\mathrm{Hom}(., N_{\mathscr{F}}(-2-k))$ to the exact sequence in display (2.40), we get the isomorphism

$$\mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) \simeq H^2(\Omega^1_{\mathbb{P}^3} \otimes N_{\mathscr{F}}) \tag{2.42}$$

since $h^1(N_{\mathscr{F}}(2k)) = h^2(N_{\mathscr{F}}(2k)) = 0$.

In order to compute $h^2(\Omega^1_{\mathbb{P}^3} \otimes N_{\mathscr{F}})$, we twist the dual Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3} \to 0$$

by $\otimes N_{\mathscr{F}}$ and pass to cohomology, obtaining the exact sequence

$$0 \to H^2(\Omega^1_{\mathbb{P}^3} \otimes N_{\mathscr{F}}) \to H^2(N_{\mathscr{F}}(-1)^{\oplus 4}) \to H^2(N_{\mathscr{F}}) \to 0,$$

since $H^1(N_{\mathscr{F}}) = H^3(\Omega^1_{\mathbb{P}^3} \otimes N_{\mathscr{F}}) = 0$, by Lemma 2.6.1. Thus, we get the equality

$$h^2(\Omega^1_{\mathbb{P}^3} \otimes N_{\mathscr{F}}) = 4.h^2(N_{\mathscr{F}}(-1)) - h^2(N_{\mathscr{F}}).$$

Now, using the item (3) of the Lemma 2.6.1 and the isomorphism (2.42), we get

$$\dim \mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) = 4k^3 - 4k^2 - k + 1,$$

for $k \geq 1$ and this ends the proof. $\square$

Similarly, if a foliation by curves $\mathscr{F}$ on $\mathbb{P}^3$ is given by the short exact sequence in display (2.41), i.e., it has even degree, then the normal sheaf $N_{\mathscr{F}}$ has Chern classes

$$c_1(N_{\mathscr{F}}) = -1,$$
$$c_2(N_{\mathscr{F}}) = 3k^2 + k + 1,$$
$$c_3(N_{\mathscr{F}}) = 8k^3 + 4k^2 + 2k + 1.$$

Moreover, the family of this sheaves has dimension

$$\dim \mathcal{G}(2k) = h^0(T\mathbb{P}^3(2k-1)) - 1 = 4k^3 + 14k^2 + 14k + 3.$$

Following the proof of the Theorem 2.6.2, its is easy to show that

$$\dim \mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) = 4k^3 - 6k^2 + 6k,$$

for $k \geq 0$ and hence

$$\dim \mathcal{G}(2k) = \dim \mathrm{Ext}^1(N_{\mathscr{F}}, N_{\mathscr{F}}),$$

since

$$\dim \mathrm{Ext}^1(N_{\mathscr{F}}, N_{\mathscr{F}}) - \dim \mathrm{Ext}^2(N_{\mathscr{F}}, N_{\mathscr{F}}) = 8c_2(N_{\mathscr{F}}) - 2c_1(N_{\mathscr{F}})^2 - 3 = 24k^2 + 8k + 3,$$

for $k \geq 0$, see [27, Proposition 3.4].

As in the case $c_1 = 0$, we have:

Theorem 2.6.3. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$ with Chern classes*

$$(c_1, c_2, c_3) = (-1, 3k^2 + k + 1, 8k^3 + 4k^2 + 2k + 1)$$

contains a rational, irreducible component of dimension $4k^3 + 14k^2 + 14k + 3$ whose generic point is the normal sheaf of a generic foliation of degree $2k$ on $\mathbb{P}^3$ given by the exact sequence in display (2.41).

Now, we consider the case $X = Q_3$. Recall that a generic foliation by curve $\mathscr{F}$ on Q_3 is given by

$$\mathscr{F} \quad : \quad 0 \to \mathcal{O}_{Q_3}(-r) \xrightarrow{\sigma} TQ_3 \to N_{\mathscr{F}} \to 0$$

since $\tau_{Q_3} = 0$, where $r \geq 0$ is the degree of $\mathscr{F}$. According to [1, Main Theorem 1], the normal sheaf $N_{\mathscr{F}}$ is a μ-stable rank 2 reflexive sheaf on Q_3.

When $\mathscr{F}$ has degree odd, say $r = 2k + 1$, then the normalization of the normal sheaf fits into the short exact sequence

$$0 \to \mathcal{O}_{Q_3}(-3 - 3k) \xrightarrow{\sigma} TQ_3(-2 - k) \to N_{\mathscr{F}}(-2 - k) \to 0, \tag{2.43}$$

for $k \geq 0$. Similarly, if $\mathscr{F}$ has degree even, say $r = 2k$, then the normalization of the normal sheaf fits into the short exact sequence

$$0 \to \mathcal{O}_{Q_3}(-2 - 3k) \xrightarrow{\sigma} TQ_3(-2 - k) \to N_{\mathscr{F}}(-2 - k) \to 0, \tag{2.44}$$

where $k \geq 0$.

The isomorphism $TQ_3 \simeq \Omega^1_{Q_3}(2)$ (see [29]) assures us that all generic foliation by curve on Q_3 corresponds to a generic codimention 1 distribution on Q_3. Thus, if $\mathscr{F}$ is a generic foliation by curves of odd degree, then $N_{\mathscr{F}}(-2-k)$ belongs to the family $\mathcal{F}(2k+1)$ and in this case, we have the following theorem:

Theorem 2.6.4. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on Q_3 with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 6)H^3)$$

contains a irreducible component of dimension $8k^3 + 42k^2 + 69k + 44$ containing the family of the normal sheaves of a generic foliation by curves of degree $2k+1$ on Q_3.

Similarly, if $\mathscr{F}$ is a generic foliation by curves of even degree, then $N_{\mathscr{F}}(-2-k)$ belongs to the family $\mathcal{F}(2k)$ and in this case, we have:

Theorem 2.6.5. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on Q_3 with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 3k + 2)H^2, (8k^3 + 12k^2 + 8k - 2)H^3)$$

contains a rational irreducible component of dimension 18 and $(8k^3 + 60k^2 + 66k + 38)/2$, for $k = 0$ and $k \geq 1$, respectively, containing the family of the normal sheaves of a generic foliation by curves of degree $2k$ on Q_3.

3 Monads in hypersurfaces

Trhoughout this chapter $X \subset \mathbb{P}^4$ denotes a smooth hypersurface of degree $d \geq 2$ with $(0:0:0:0:1) \notin X$. Our main goal here is to study of the locally free sheaves E on X given by linear monad

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0. \tag{3.1}$$

According to [17, Proposition 4], E is always a stable rank 2 locally free sheaf on X.

This chapter is divided as follows: In the first section we will give a cohomological characterization of such bundles. To be more precise, we will show that all stable rank 2 locally free sheaves on X with Chern classes $c_1 = 0$ and $c_2 = d \cdot L$ satisfying certain cohomological conditions are in $1-1$ correspondence with monads of the form (3.1) whose cohomology sheaf is locally free.

In the second section, we will use the equivalence between the categories of quiver representations and monads on X, given by Proposition 1.6.3, to give a matrix description of these cohomology bundles. As an application of this equivalence, we will calculate the dimension of this family. This equivalence is discussed in Section 1.6.

In the third section, we will give a sufficient condition on the family of locally free sheaves on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d \in \{3, 4, 5\}$ given as the cohomology sheaf of

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0$$

to fill a irreducible component of the moduli space of stable rank 2 locally free sheaves on X with Chern class $c_1 = 0$ and $c_2 = d \cdot L$.

In the last section, we will use these bundles to get examples of LCI foliations by curves on X, which are defined in the Section 2.6.

3.1 Cohomological Characterization

Our main goal here is to give a cohomological characterization of the locally free sheaves given by the monad in display (3.1). We started by showing the existence of such monads. Indeed, taking

$$\alpha^T := (X_1 \;\; -X_0 \;\; X_4 \;\; -X_3) \quad \text{and} \quad \beta := (X_0 \;\; X_1 \;\; X_2 \;\; X_3),$$

we have that the monad is well defined for each $p = (x_0 : x_1 : x_2 : x_3 : x_4) \in X$, the matrices α and β have maximum rank 1 and $\beta.\alpha = 0$. Moreover, its cohomology sheaf E is a rank 2 locally free sheaf on X, since its degeneration locus is empty, see Proposition 1.4.2.

In general, given $f_1, f_2, f_3, f_4 \in H^0(\mathcal{O}_{\mathbb{P}^4}(1))$ linearly independent such that $Y \cap X = \emptyset$, where $Y := Z(f_1, f_2, f_3, f_4)$ is the common zero locus, there exists a monad

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0, \tag{3.2}$$

where the morphism β is determined by the matrix:

$$\beta := (f_1 \;\; f_2 \;\; f_3 \;\; f_4)$$

and the cohomology sheaf E is a rank 2 locally free sheaf on X with Chern classes $c_1 = 0$ and $c_2 = d \cdot L$.

Our first proposition in this chapter is dedicated to the study of the cohomology of the cohomology sheaves given by the monad in display (3.2).

Proposition 3.1.1. *If E is the cohomology sheaf of the monad (3.2), then:*

i) $h^0(E(k)) = 0$ *for* $k \leq 0$;

ii) $H^1(E(k)) \simeq \mathbb{C}.f_5^{k+1}$, *for* $-1 \leq k < d-1$, *where* $\{f_1, f_2, f_3, f_4, f_5\}$ *is a basis for* $H^0(\mathcal{O}_{\mathbb{P}^4}(1))$;

iii) $h^1(E(k)) = 0$ *for* ($k \leq -2$ *or* $k \geq d-1$).

Proof. We know that the monad $\mathcal{M}_\bullet$ in (3.2) can be broken down into two short exact sequences

$$0 \to K \to \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0 \qquad (3.3)$$

and

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} K \to E \to 0, \qquad (3.4)$$

where $K := \ker \beta$ is a rank 3 locally free sheaf on X.

Now, we consider the exact sequence in display (3.4) twisted by $\mathcal{O}_X(t)$ and after we taking the long exact sequence of cohomology, we get the exact sequence

$$0 \to H^0(\mathcal{O}_X(t-1)) \to H^0(K(t)) \to H^0(E(t)) \to 0 \qquad (3.5)$$

and the isomorphism

$$H^1(E(t)) \simeq H^1(K(t)),$$

since $H^i(\mathcal{O}_X(t)) = 0$ for $i = 1, 2$ and $t \in \mathbb{Z}$.

Next, we twist the exact sequence in display (3.3) by $\mathcal{O}_X(t)$, we get the exact sequence in cohomology

$$0 \to H^0(K(t)) \to H^0(\mathcal{O}_X^{\oplus 4}(t)) \xrightarrow{H^0(\beta)} H^0(\mathcal{O}_X(t+1)) \to H^1(K(t)) \to 0$$

and hence the isomorphism

$$H^1(K(t)) \simeq \operatorname{coker} H^0(\beta).$$

The item $i)$ follows from (3.5), since $h^0(K(t)) = 0$ for $t \leq 0$.

For the item $ii)$, note that if $-1 \leq t < d-1$, then

$$\operatorname{Im} H^0(\beta) \simeq H^0(\mathcal{O}_{\mathbb{P}^4}(t+1)) / < f_5^{t+1} >$$

since $H^0(\mathcal{O}_X(t+1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^4}(t+1))$ and $H^0(\beta) = (f_1 \ f_2 \ f_3 \ f_4)$. Thus,

$$H^1(E(t)) \simeq H^1(K(t)) \simeq \operatorname{coker} H^0(\beta) \simeq < f_5^{t+1} >$$

for $-1 \leq t < d-1$.

Note that the isomorphisms

$$H^0(\mathcal{O}_X(d) \simeq \frac{\mathbb{C}[f_1, \dots, f_5]_d}{< f >} \simeq \frac{< f_1, f_2, f_3, f_4 >_d + < f_5^d >}{< f >} \simeq \frac{< f_1, f_2, f_3, f_4 >_d}{< f >}$$

e

$$\operatorname{Im} H^0(\beta) \simeq \frac{< f_1, f_2, f_3, f_4 >_d}{< f >},$$

where $X = \{f = 0\}$ and $f = a_1f_1+a_2f_2+a_3f_3+a_4f_4+f_5^d$, with $a_i \in \mathbb{C}[f_1, \dots, f_5]_{d-1}$, implies that $H^0(\beta)$ is surjective when $t = d-1$. Thus, $H^0(\beta)$ is surjective for all $t \geq d-1$ and hence $H^1(E(t)) = 0$ for $t \geq d-1$.

When $t \leq -2$, $H^1(E(t)) = 0$ since $H^0(\mathcal{O}_X(t+1)) = 0$. Thus, the item *iii)* is satisfied.

□

Remark 3.1.2. Follows from the Proposition 3.1.1 that $H^1(E(-1))$ generates $\bigoplus_{k\geq -1} H^1(E(k))$ as an $S(X)$-module, where $S(X) := \bigoplus_{k\in\mathbb{Z}} H^1(\mathcal{O}_X(k))$.

A question that arises naturally is:

Question 1. *If E is a stable rank 2 locally sheaf on X with Chern classes $c_1(E) = 0$ and $c_2(E) = d \cdot L$ such that their cohomologies satisfy $i) - iii)$ of the Proposition 3.1.1, then E can be obtained as the cohomology sheaf of the monad (3.2)?*

When $X = Q_3$ is a quadric threefold, there exists a 1-1 correspondence between stable rank 2 locally sheaf E on Q_3 with Chern classes $c_1(E) = 0$ and $c_2(E) = 2 \cdot L$ such that their cohomologies satisfy $i) - iii)$ of Proposition 3.1.1 and monads for E as in (3.2), see [11] and [20, Theorem 4.4]. Here, we will extend this result to other 3-fold hypersurfaces.

Theorem 3.1.3. *There is a 1-1 correspondence between stable rank 2 locally free sheaves E on X with Chern classes $c_1(E) = 0$ and $c_2(E) = d \cdot L$ such that their cohomologies satisfy $i) - iii)$ of Proposition 3.1.1 and monads for E*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0.$$

Proof. Suppose that E is stable rank 2 locally free sheaf E on X with Chern classes $c_1(E) = 0$ and $c_2(E) = d \cdot L$ such that their cohomologies satisfy $i) - iii)$ of Proposition 3.1.1. By [20, Theorem 3.3], E is the cohomology sheaf of the monad

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} F \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

where F is a rank 4 locally free sheaf on X with Chern classes $c_i(F) = 0$, $i = 1, 2, 3$ and $H^i(F(k)) = 0$ for $i = 1, 2$ and $k \in \mathbb{Z}$.

By Proposition 3.1.1, $h^0(E) = 0$ and $h^1(E) = 1$ which implies in $h^0(F) = 4$. So, there exists a monomorphism

$$\phi : \mathcal{O}_X^{\oplus 4} \hookrightarrow F$$

and thus $\det \phi : \det \mathcal{O}_X^{\oplus 4} \hookrightarrow \det F$ is also a monomorphism.

Since $c_1(\mathcal{O}_X^{\oplus 4}) = c_1(F) = 0$, $\det \mathcal{O}_X^{\oplus 4} = \det F = \mathcal{O}_X$ follows that $\det \phi$ is an isomorphism and thus so is ϕ. $\square$

3.2 Description of bundles through quiver representations

The main goal here is to give a description of the stable rank 2 locally free sheaves E on X with Chren classes $c_1(E) = 0$ and $c_2(E) = d \cdot L$ such that their cohomologies satisfy $i) - iii)$ of Proposition 3.1.1, that is, those given as the cohomology sheaf of the monad (3.2). For this we use the equivalence between categories given in the Proposition 1.6.3.

Consider the following representation of the quiver Q:

$$R = \ \mathbb{C} \ \begin{matrix} \xrightarrow{A_1} \\ \vdots \\ \xrightarrow[A_5]{} \end{matrix} \ \mathbb{C}^4 \ \begin{matrix} \xrightarrow{B_1} \\ \vdots \\ \xrightarrow[B_5]{} \end{matrix} \ \mathbb{C} \tag{3.6}$$

with the relations $P_{ij} = B_i A_j + B_j A_i$, where $A_j = (a_{ij})_{1 \le i \le 4}$ and $B_i = (b_{ij})_{1 \le j \le 4}$

are matrices whose entries are complex numbers. So, we can set the matrices

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{pmatrix}, \quad B := \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \\ b_{51} & b_{52} & b_{53} & b_{54} \end{pmatrix}. \tag{3.7}$$

Following the notations above, we have:

Remark 3.2.1. i) R is globally injective on X if and only if for every $(\lambda_1 : \cdots : \lambda_5) \in X$, we have

$$\sum_{i=1}^{5} \lambda_i A_i \neq 0,$$

where A_i are the columns of the matrix A. In this case, we say that A is *globally injective* on X;

ii) R is globally surjective on X if and only if for every $(\lambda_1 : \cdots : \lambda_5) \in X$, we have

$$\sum_{i=1}^{5} \lambda_i B_i \neq 0,$$

where B_i are the lines of the matrix B. In this case, we say that B is *globally surjective* on X.

In particular, we have:

Lemma 3.2.2. *If the representation R in display (3.6) is globally surjective (injective) on X, then the matrix B (A) has rank equal to 4.*

Proof. If $\operatorname{rk} B \leq 3$, there is a invertible matrix $g \in \mathrm{Mat}_{4\times 4}$ such that

$$B.g = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence there is $p = (0:0:0:a:b) \in X$ such that $aB_4 + bB_5 = 0$, this is, R is not globally surjective. Therefore $\operatorname{rk} B = 4$. □

Recall that two representations $R = (\mathbb{C}, \mathbb{C}^4, \mathbb{C}, \{A_i\}, \{B_i\})$ and $R^{'} = (\mathbb{C}, \mathbb{C}^4, \mathbb{C}, \{A_i^{'}\}, \{B_i^{'}\})$ are *isomorphic* if there is $(\lambda, g, \delta) \in \mathbb{C}^* \times \mathrm{Gl}_4 \times \mathbb{C}^*$ such that

$$A_i = g^{-1} A_i^{'} \lambda \ \ , \ \ B_i^{'} = \delta B_i g^{-1};$$

So using isomorphism of representations, we can rewritten B as follows:

Proposition 3.2.3. *If the representation R in display (3.6) is globally surjective on X, we can assume that B is of the form*

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{51} & c_{52} & c_{53} & c_{54} \end{pmatrix},$$

where $c_{5i} \in \mathbb{C}$, for $i = 1, \ldots, 4$.

Proof. By Lemma 3.2.2, $\operatorname{rk} B = 4$. So there is a invertible matrix $g \in \mathrm{Mat}_{4\times 4}$ such that

$$B.g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{51} & c_{52} & c_{53} & c_{54} \end{pmatrix}.$$

□

Our next step is to get a simpler presentation for A. For this we will use the relations P_{ij} and the Proposition 3.2.3.

Proposition 3.2.4. *If the representation R in display (3.6) is globally injective and surjective on X, then the matrix A in display (3.7) can be rewritten in the* *form*

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \end{pmatrix}.$$

Moreover, a_{i5} is determined by B_5 and A_i, for every $i = 1, \ldots, 4$.

Proof. The relations $P_{ij} = B_iA_j + B_jA_i$ imply that $a_{ij} = -a_{ji}$ for $1 \leq i, j \leq 4$, since B can be given as in Proposition 3.2.3, because B is globally surjective on X. In particular, $a_{ii} = 0$ for $i = 1, \ldots, 4$.

Now, using the relation $P_{i5} = B_5A_i + B_iA_5$, we get $a_{i5} = -B_5A_i$, for $i = 1, \ldots, 4$. □

We prove the main result of this section.

Theorem 3.2.5. *There is a bijective correspondence between pairs (A, B), where*

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{51} & c_{52} & c_{53} & c_{54} \end{pmatrix},$$

with A globally injective on X and B globally surjective on X and isomorphism classes of monads whose cohomology sheaf is locally free as in (3.2).

Proof. Given a pair (A, B), with A globally injective on X and B globally surjective on X, we can buil a representation R as in display (3.6) that is globally injective and surjective on X. By Proposition 1.6.3, this representation corresponds to a isomorphism classes of monads whose cohomology sheaf is locally free as in (3.2).

Conversely, given a monad $\mathcal{M}_\bullet$ as in (3.2) whose cohomology sheaf is locally free, by Proposition 1.6.3, it is corresponds to a globally injective and surjective representation R of Q.

Being R a globally injective and surjective representation, the result follows from the propositions (3.2.3), (3.2.4). □

Corollary 3.2.6. *The family $\mathcal{U}_X$ of the stable rank 2 locally free sheaves on X given by the monad*

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_X(-1) \stackrel{\alpha}{\to} \mathcal{O}_X^{\oplus 4} \stackrel{\beta}{\to} \mathcal{O}_X(1) \to 0$$

has dimension 9.

Proof. By Theorem 3.1.3 and Theorem 3.2.5 , each $E \in \mathcal{U}_X$ corresponds to a pair (A, B) as above. Since that pairs (A, B) and $\lambda.(A, B) := (\lambda A, \lambda B)$, $\lambda \neq 0$, define isomorphic representations, because there is $(\lambda^{-1}, \mathrm{id}, \lambda) \in \mathbb{C}^* \times \mathrm{Gl}_4 \times \mathbb{C}^*$ such that

$$A_i = \lambda A_i \lambda^{-1} \quad \text{and} \quad \lambda B_i = \lambda B_i,$$

follows that $\dim \mathcal{U}_X = 9$. □

Remark 3.2.7. If we prove that $\dim \mathrm{Ext}^1(E, E) = 9$ (see Theorem 1.3.12), then we can conclude that the family $\mathcal{U}_X$ fills a irreducible component of dimension 9 in the moduli space of stable rank 2 locally free sheaves on X with Chern class $c_1 = 0$ and $c_2 = d \cdot L$.

3.3 The moduli space $\mathcal{M}(0, 2)$

It is know that the family of the stable rank 2 locally free sheaves on a Q_3 given as cohomology sheaves of monads

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_{Q_3}(-1) \stackrel{\alpha}{\to} \mathcal{O}_{Q_3}^{\oplus 4} \stackrel{\beta}{\to} \mathcal{O}_{Q_3}(1) \to 0$$

fills an irreducible component of dimension 9 in the moduli space of stable rank 2 locally free sheaves on Q_3 with Chern class $c_1 = 0$ and $c_2 = 2 \cdot L$. Moreover, this component is nonsingular. See, for example, [6, 32].

When X has degree $d \in \{3, 4, 5\}$ we will give a sufficient condition for the family of the locally free sheaves on X given as sheaf of the monad

$$\mathcal{M}_\bullet : \; 0 \to \mathcal{O}_X(-1) \stackrel{\alpha}{\to} \mathcal{O}_X^{\oplus 4} \stackrel{\beta}{\to} \mathcal{O}_X(1) \to 0$$

to fill an irreducible component of the moduli space of stable rank 2 locally free sheaves on X with Chern class $c_1 = 0$ and $c_2 = d \cdot L$.

The next proposition gives us the expected dimension of the irreducible component of the moduli space of stable rank 2 locally free sheaves on X with given Chern classes.

Proposition 3.3.1. *If E is a rank 2 locally free sheaf on X given as cohomology sheaf of the monad in display (3.2), then*

$$\dim \operatorname{Ext}^1(E,E) - \dim \operatorname{Ext}^2(E,E) = \begin{cases} 9, & if\ d = 3 \\ 5, & if\ d = 4 \\ 0, & if\ d = 5 \end{cases}$$

Proof. The exact sequences in display (3.3) and (3.4) twisted by $\otimes E$ gives us

$$\chi(E \otimes E) = 4.\chi(E) - \chi(E(1)) - \chi(E(-1)).$$

The stability of E implies that $\dim \operatorname{Hom}(E,E) = 1$, $\dim \operatorname{Ext}^3(E,E) = 0$, when $d = 3, 4$, and $\dim \operatorname{Ext}^3(E,E) = 1$, when $d = 5$.

Now, using the isomorphisms $\operatorname{Ext}^i(E,E) \simeq H^i(E \otimes E)$ and Proposition 3.1.1, we get the desired. □

Next, we will prove two lemmas that will be useful in the proof of the main theorem of this section.

Lemma 3.3.2. *The morphism $H^1(1 \otimes \beta) : H^1(E)^{\oplus 4} \to H^1(E(1))$ in the long exact sequence in cohomology derived from the exact sequence*

$$0 \to K \otimes E \to E^{\oplus 4} \overset{1\otimes\beta}{\to} E(1) \to 0 \tag{3.8}$$

is null.

Proof. Being $H^1(1 \otimes \beta)$ induced by the morphism β which does not depend on f_5 (see monad (3.2)) follows that $H^1(1 \otimes \beta) = 0$ since $H^1(E)^{\oplus 4}$ and $H^1(E(1))$ depends only on f_5 (see Proposition 3.1.1). □

Lemma 3.3.3. *If the morphism*

$$H^2(1 \otimes \alpha) : H^2(E(-1)) \to H^2(K \otimes E)$$

in the long exact sequence in cohomology derived from the exact sequence

$$0 \to E(-1) \stackrel{1\otimes\alpha}{\to} K \otimes E \to E \otimes E \to 0 \tag{3.9}$$

is not null, then

$$h^1(E \otimes E) = h^1(K \otimes E) = 9.$$

Proof. The exact sequence in display (3.8) implies that $h^0(K \otimes E) = 0$ since $h^0(E) = 0$ because E is stable. Moreover,

$$h^1(K \otimes E) = h^0(E(1)) + 4h^1(E) = 9,$$

since $H^1(1 \otimes \beta) = 0$, by Lemma 3.3.2.

Being $H^2(1 \otimes \alpha) \neq 0$ by hypotheses, it follows that $H^2(1 \otimes \alpha)$ is a monomorphism since $h^2(E(-1)) = 1$, by Proposition 3.1.1.

Now, using the long exact sequence in cohomology derived from the exact sequence in display (3.10), we have

$$h^1(E \otimes E) = h^1(K \otimes E) = 9,$$

since $H^0(E \otimes E) \simeq H^1(E(-1))$ because $h^0(K \otimes E) = 0$.

□

When $X \subset \mathbb{P}^4$ is a smooth hypersurface of degree $d = 3, 4, 5$ we have:

Theorem 3.3.4. *If the morphism*

$$H^2(1 \otimes \alpha) : H^2(E(-1)) \to H^2(K \otimes E)$$

in the long exact sequence in cohomology derived from the exact sequence

$$0 \to E(-1) \stackrel{1\otimes\alpha}{\to} K \otimes E \to E \otimes E \to 0 \tag{3.10}$$

is not null, then the moduli space of stable rank 2 locally free sheaf on X with Chern classes $c_1 = 0$ and $c_2 = d \cdot L$ contains a irreducible component of dimension 9 whose generic point is the locally free sheaf given by monad in display (3.2). Moreover, this component is nonsingular when $d = 3$.

Proof. It follows from Lemma 3.3.3 since

$$\mathrm{Ext}^1(E,E) \simeq H^1(E\otimes E).$$

When $d=3$, the Proposition 3.3.1 implies that $\dim \mathrm{Ext}^2(E,E)=0$ and hence this component is nonsingular.

□

3.4 LCI foliations by curves on 3-fold hypersurfaces

We say that a foliation by curves

$$\mathscr{F} \quad : \quad 0 \to \mathcal{O}_X(-r-\tau_X) \stackrel{\sigma}{\to} TX \to N_{\mathscr{F}} \to 0$$

is a *local complete intersection (LCI) foliation* when the conormal sheaf $N_{\mathscr{F}}^{\vee}$ is locally free. When the conormal sheaf $N_{\mathscr{F}}^{\vee}$ splits as a sum of line bundles, we say that $\mathscr{F}$ is a *complete intersection (CI) foliation*.

The main goal this section is to use the rank 2 locally free sheaves obtained above to get examples of LCI foliations by curves on threefold hypersurfaces. Our first result, which is a generalization of [11, Proposition 1.11], we will be the key to obtain foliations by curves of degree $\deg \mathscr{F} = 2t+d-\tau_X-1$, with $t\geq 1$, where d is the degree of the hypersurface $X\subset \mathbb{P}^4$.

Proposition 3.4.1. *If E is a rank 2 locally free sheaf given as the cohomology sheaf of the monad*

$$\mathcal{M}_{\bullet} : \ 0 \to \mathcal{O}_X(-1) \stackrel{\alpha}{\to} \mathcal{O}_X^{\oplus 4} \stackrel{\beta}{\to} \mathcal{O}_X(1) \to 0,$$

where $X\subset\mathbb{P}^4$ is a smooth hypersurface, then $E(1)$ is globally generated.

Proof. We consider $K := \ker\beta$; to show that $E(1)$ is globally generated, we will argue that $K(1)$ is globally generated.

The formula

$$\wedge^q K^* \simeq \wedge^r K^* \otimes \wedge^{r-q} K \quad (r = \mathrm{rk}\, K)$$

gives us

$$K(1) \simeq K \otimes \det K^* \simeq K \otimes \wedge^3 K^* \simeq \wedge^2 K^*.$$

Being K^* globally generated as an image of $\mathcal{O}_X^{\oplus 4}$, follows that $\wedge^2 K^*$ is globally generated and hence $K(1)$ is also. □

Note that $\Omega^1_X(2)$ is globally generated, since we have epimorphisms

$$\Omega^1_{\mathbb{P}^4}(2) \twoheadrightarrow \left(\Omega^1_{\mathbb{P}^4}(2)\right)|_X \twoheadrightarrow \Omega^1_X(2)$$

and $\Omega^1_{\mathbb{P}^4}(2)$ is globally generated. We then have that $E \otimes \Omega^1_X(t+2)$ is also globally generated, for all $t \geq 1$, since $E(1)$ is globally generated by Proposition 3.4.1. By Ottaviani's Bertini-type theorem [12, Teorema 2.8], there is a monomorphism $\phi : E(-2-t) \to \Omega^1_X$ such that $\operatorname{coker}\phi$ is a torsion free sheaf of rank 1, for each $t \geq 1$, i.e there are LCI foliations by curves on X

$$\mathscr{F} \quad : \quad 0 \to E(-2-t) \xrightarrow{\phi} \Omega^1_X \to \mathscr{I}_Z(r + \tau_X) \to 0$$

with $r = c_1(\Omega^1_X) - c_1(E(-2-t)) - \tau_X = d + 2t - \tau_X - 1$, where d is the degree of X.

In short, we proved:

Theorem 3.4.2. *If E is a rank 2 locally free sheaf given as the cohomology sheaf of the monad*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

then, for every $t \geq 1$, there is a LCI foliation by curves on X of degree $r = d + 2t - \tau_X - 1$ given by

$$\mathscr{F} \quad : \quad 0 \to E(-2-t) \xrightarrow{\phi} \Omega^1_X \to \mathscr{I}_Z(r + \tau_X) \to 0,$$

where d is the degree of X.

When $t \leq -1$, we will see that there are no injective morphisms $\phi : E(-2-t) \to \Omega^1_X$:

Lemma 3.4.3. *If E is a rank 2 locally free sheaf given as cohomology sheaf of the monad*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

then there are no injective morphisms $\phi : E(-2-t) \to \Omega^1_X$ when $t \leq -1$.

Proof. First, note that $\mathrm{Hom}(E(-2-t), \Omega^1_X) \simeq H^0(E \otimes \Omega^1_X(t+2))$.

We observe that, if $d \geq 3$, the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^4}|_X \to \Omega^1_X \to 0 \tag{3.11}$$

implies that $h^0(E \otimes \Omega^1_X(t+2)) = h^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(t+2))$ when $t \leq -1$, since $h^0(E(2+t-d)) = h^1(E(2+t-d)) = 0$. Now, using the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^4}|_X \to \mathcal{O}_X(-1)^{\oplus 5} \to \mathcal{O}_X \to 0 \tag{3.12}$$

we get $h^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(t+2) = 0$, since $h^0(E(1+t)) = 0$, for $t \leq -1$. When $d = 2$, the μ-stability of Ω^1_X implies that there are no injective morphisms $\phi : E(-2-t) \hookrightarrow \Omega^1_X$ for all $t \leq -1$, since

$$\mu(E(-2-t)) = (-2-t) \geq -1 = \mu(\Omega^1_X).$$

Therefore, there are no injective morphisms $\phi : E(-2-t) \hookrightarrow \Omega^1_X$ when $t \leq -1$. □

Finally, we consider the case $t = 0$ observing that $\mathrm{Hom}(E(-2), \Omega^1_X) \simeq H^0(E \otimes \Omega^1_X(2))$.

Lemma 3.4.4. *If E is a rank 2 locally free sheaf given as cohomology sheaf of the monad*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

then every non trivial morphism $\phi : E(-2) \to \Omega^1_X$ is a monomorphism.

Proof. First, we will show that there is non trivial morphism $\phi : E(-2) \to \Omega^1_X$. Indeed, twist the exact sequence in display (3.11) by $E(2)$ and then taking the long exact sequence in cohomology, we have

$$0 \to H^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(2)) \to H^0(E \otimes \Omega^1_X(2)) \to \cdots$$

and hence $h^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(2)) \neq 0$ implies $h^0(E \otimes \Omega^1_X(2)) \neq 0$.

Now, twist the exact sequence in display (3.12) by $E(2)$ and then taking the long exact sequence in cohomology, we obtain

$$0 \to H^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(2)) \to H^0(E(1))^{\oplus 5} \to H^0(E(2)) \to \cdots$$

Note that if $5.h^0(E(1)) > h^0(E(2))$, then $h^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(2)) \neq 0$.

A simple calculation shows that $h^0(E(1)) = 5$ and $h^0(E(2)) = 21$. Thus, $h^0(E \otimes \Omega^1_{\mathbb{P}^4}|_X(2)) \geq 4$ and hence $h^0(E \otimes \Omega^1_X(2)) \geq 4$.

To finish the proof, let us consider a non trivial morphism $\phi : E(-2) \to \Omega^1_X$, and let us suppose that ϕ is not injective. Then $\ker \phi \simeq \mathcal{O}_X(-k)$ for some $k \geq 3$, since $\ker \phi$ must be a rank 1 reflexive sheaf. Thus, $\operatorname{Im} \phi \simeq I_Z(k-4)$ for some curve $Z \subset X$, since this a subsheaf of Ω^1_X. On the other hand, ϕ induces a non zero morphism $\tau \in \operatorname{Hom}(I_Z(k-4), \Omega^1_X) \simeq H^0(\Omega^1_X(4-k)) = 0$, for each $k \geq 3$, leading to a contradiction. □

Remark 3.4.5. When $t = 0$, it was not possible to decide if there is a monomorphism $\phi : E(-2) \to \Omega^1_X$ with $\operatorname{coker} \phi$ torsion free.

3.5 Codimesion 1 distributions on 3-fold hypersurfaces

The main goal this section is to use the rank 2 locally free sheaves obtained as cohomology sheaf of the monad in display (3.1) to get examples of codimension 1 distributions on threefold hypersurfaces such that its tangent sheaf is a non split locally free sheaf. If $X \subset \mathbb{P}^4$ denotes a smooth hypersurface of degree d and TX is tangent bundle, we set

$$\gamma_X := \min\{t \in \mathbb{Z} \mid TX(t) \text{ is globally genereted}\}.$$

Let E be a locally free sheaf gives as cohomology sheaf of the monad in display (3.1). Being $E(1)$ globally generated by Proposition 3.4.1, follows that $E \otimes TX(\gamma_X + t)$ is also globally generated for all $t \geq 1$. By Ottaviani's Bertini-type theorem [12, Teorema 2.8], there is a monomorphism $\phi : E(-2-t) \to \Omega^1_X$ such that coker ϕ is a torsion free sheaf of rank 1, for each $t \geq 1$, i.e, there are non generic codimesion 1 distributions on X

$$\mathscr{F} \quad : \quad 0 \to E(-\gamma_X - t) \xrightarrow{\phi} TX \to I_Z(r+2) \to 0$$

of degree $r = c_1(TX) - c_1(E(-\gamma_X - t)) - 2 = 3 - d + 2t + 2\gamma_X$, where d is the degree of X.

Therefore, we have the following theorem:

Theorem 3.5.1. *If E is a rank 2 locally free sheaf given as cohomology sheaf of the monad*

$$\mathcal{M}_\bullet : \ 0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0,$$

then, for every $t \geq 1$, there is a codimension 1 distribution on X of degree $r = 3 - d + 2t + 2\gamma_X$ such that its tangent sheaf is a non split locally free sheaf given by

$$\mathscr{F} \quad : \quad 0 \to E(-\gamma_X - t) \xrightarrow{\phi} TX \to \mathscr{I}_Z(r+2) \to 0$$

where d is the degree of X.

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