MATHEUS CORREIA DOS SANTOS

ELLIPTIC EQUATIONS WITH NONLINEAR GRADIENT TERMS AND FRACTIONAL DIFFUSION EQUATIONS

EQUAÇÕES ELÍPTICAS COM TERMOS GRADIENTES NÃO LINEARES E EQUAÇÕES DE DIFUSÃO FRACIONÁRIAS

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Instituto de Matemática, Estatística
e Computação Científica

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Orientador: **Lucas Catão de Freitas Ferreira**

Coorientadores: **Marcelo da Silva Montenegro**
José Antonio Carrillo de la Plata

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Abstract

We analyse two problems in this work. In the first part we study the existence of solutions to a semilinear elliptic equation in the whole space and with dependence on the gradient and where no restriction is imposed on the behavior of the nonlinearity at infinity. We prove that there exists a solution which is locally unique and inherits many of the symmetry properties of the nonlinearity. Positivity and asymptotic behavior of the solution are also addressed. Our results can be extended to other domains like half-space and exterior domains and also to some fractional operators. For the second part, we analyse the asymptotic behavior of solutions to the one dimensional fractional version of the porous medium equation introduced by Caffarelli and Vázquez and where the pressure is obtained as the inverse of the fractional Laplacian of the density. Due to the convexity of the kernel of the Riesz potential in one dimension, we show that the entropy associated with the equation is displacement convex and satisfies a functional inequality involving also entropy dissipation and the Euclidean transport distance. An argument by approximation shows that this functional inequality is enough to deduce the exponential convergence, in the entropy level, of solutions to the unique steady state. A new interpolation inequality is also proved in order to obtain the exponential decay also in L^p spaces.

Keywords: Semilinear elliptic equation; Existence of solutions; Asymptotic behavior of solutions; Fractional Laplacian; Optimal Transport.

Resumo

Analisaremos dois problemas neste trabalho. Na primeira parte, estudaremos a existência de soluções para uma equação elíptica semilinear no espaço euclidiano todo e com dependência do gradiente e onde nenhuma restrição é imposta sobre o comportamento da não linearidade no infinito. Provaremos que existe uma solução que é localmente única e que herda muitas das propriedades de simetria da não linearidade. A positividade da solução e seu comportamento assintótico também são analisados. Os resultados obtidos também podem ser estendidos para outros casos como o de domínios exteriores ou o semiespaço e também para alguns operadores fracionários. Na segunda parte, analisaremos o comportamento assintótico das soluções da versão fracionária unidimensional da equações de meios porosos introduzida por Caffarelli e Vázquez e onde a pressão é obtida como a inversa do laplaciano fracionário da densidade. Devido à convexidade do núcleo do potencial de Riesz em dimensão um, mostraremos que a entropia associada à equação é displacement convex e satisfaz uma desigualdade funcional envolvendo a dissipação da entropia e a distância de transporte euclidiana. Um argumento por aproximação mostra que essa desigualdade funcional é suficiente para deduzir que a entropia das soluções converge exponencialmente para a entropia do estado estacionário. Também provaremos uma nova desigualdade de interpolação que permitirá obter a convergência exponencial das soluções em espaços L^p.

Keywords: Equações semilineares elípticas; Existência de soluções; Comportamento assintótico de soluções; Laplaciano fracionário; Transporte Ótimo.

Contents

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Notation

E_k	space $L^\infty(dx) \cap L^\infty(\|x\|^k dx)$
F_k	subspace of $C^1(\mathbb{R}^d)$ where $u, \partial_{x_i} u \in E_k$
$k_{d,s}$	kernel of the Riesz potential of order $2s$ in $\mathbb{R}^d$
$\mathcal{N}(u)$	Riesz potential of order 1 of u in $\mathbb{R}^d$
$C^\alpha(\mathbb{R}^d)$	bounded Hölder continuous functions of order α
$[u]_{C^\alpha}$	Hölder seminorm of order α
$C^{k,\alpha}(\mathbb{R}^d)$	subspace of $C^k(\mathbb{R}^d)$ s.t. the k-th derivatives are C^α
$C_b(\mathbb{R}^d)$	space of continuous bounded functions on $\mathbb{R}^d$
$H^\gamma(\mathbb{R}^d)$	fractional Sobolev space of order γ
$[u]_{H^\gamma}$	fractional Sobolev seminorm of order γ
$\mathcal{M}_+(\mathbb{R}^d)$	space of Borelian measures on $\mathbb{R}^d$
$\mathcal{P}(\mathbb{R}^d)$	space of probability measures on $\mathbb{R}^d$
$\mathcal{P}_2(\mathbb{R}^d)$	space of $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \|x\|^2 d\rho < \infty$
$\mathcal{P}_{ac}(\mathbb{R}^d)$	space of $\rho \in \mathcal{P}(\mathbb{R}^d)$ which are absolutely continuous w.r.t Lebesgue measure
$T\#\rho$	push-forward of ρ through T
$\rho_n \rightharpoonup \rho$	convergence of ρ_n to ρ in $P_2(\mathbb{R})$ under the topology generated by $C_b(\mathbb{R})$
$W_2(\mu, \nu)$	Euclidean Wasserstein distante between μ and ν
$\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{I}$	functionals defined on $\mathcal{P}_{2,ac}(\mathbb{R})$
$(-\Delta)^{-s} u$	inverse of the fractional Laplacian of u
T_μ^ν	optimal transport map between μ and ν

Introduction

This work is devoted to the analysis of two different types of partial differential equations involving nonlinearities which arise from many applications such as conformal geometry, Chern-Simons-Higgs theory, stochastic control theory, long-range diffusive phenomena, convective process and so on.

The first model which shall be studied in Chapter 1 is a semilinear elliptic equation where no restriction is imposed on the behavior of the nonlinearity at infinity. More explicitly we will study the existence, symmetry and asymptotic behavior of solutions to the following nonlinear elliptic PDE:

$$\Delta u + g(x, u, \nabla u) = 0 \quad \text{in} \ \ \mathbb{R}^n \tag{0.0.1}$$

$$u \to 0 \quad \text{as } |x| \to \infty, \tag{0.0.2}$$

with $n \geq 3$ and where $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ verifies $g(x, 0, 0) \not\equiv 0$ and belong to a large class of nonlinear functions which include, for example, polynomial and exponential type growths on u or ∇u.

If on one hand the literature about problems with polynomial behavior is wide and very well understood in many cases, on the other hand the same is not true for the exponential case since many embedding results become hard to apply when dealing with this type of nonlinearity, specially in this case where the domain is the whole space $\mathbb{R}^n$. And in spite of this apparent lack of results, exponential-type nonlinearities appear naturally in many contexts like, as said before, in geometry or in Chern-Simons gauge theory.

Nonlinear gradient terms also appear naturally in models connected with convective processes, in the physical theory of growth and roughening of surfaces or in stochastic control theory. These nonlinearities involving the gradient introduce new difficulties when combined with unbounded domains and strong-growth nonlinearities, preventing the use of variational and sub-super solutions methods, Ladyzenskaya-Ural'tseva conditions, Banach fixed point theorem in Sobolev spaces, implicit function theorem, compactness arguments, and Leray-Schauder theory, among others.

In smooth bounded domains $\Omega \subset \mathbb{R}^n$, there is a rich literature for (0.0.1)-(0.0.2) with general conditions on $g(x, u, \nabla u)$ for existence of solutions, including polynomial or exponential-type growths. In this case existence results have been studied by means of different approaches involving the aforementioned arguments and techniques. For that matter, the reader is referred to [5, 4, 36, 35, 50, 48, 66, 67] and their bibliographies. As pointed out in [35] and [48], the use of techniques based on maximum principles in most cases imposes that the nonlinearity grows at most quadratically in ∇u. This kind of restriction appears in the works [5, 18, 40, 60], and was overcame in [68] for a logistic equation with $|\nabla u|^q$ with $q > 1$ and in bounded domains by combining bifurcation methods and C^1-*a priori* bounds.

For the case of explosive boundary conditions, that is $u \to \infty$ as $x \to \partial\Omega$ (or as $|x| \to \infty$), existence of solutions for (0.0.1) have been addressed in bounded domains Ω and in $\mathbb{R}^n$ by considering at most polynomial growth at infinity on the gradient ∇u (see e.g. [2], [53], and [42] in $\mathbb{R}^n$). For example, the authors of [42] assumed $-g(u, \nabla u) = f_1(u) \pm f_2(\nabla u)$ with increasing continuous f and g having at most power growth at infinity and $g(x, 0, 0) = f_1(0) = f_2(0) = 0$. We also mention the work [3] for existence of distributional solutions in $\mathbb{R}^n$ with polynomial growth on both u and ∇u, and without prescribing conditions on u as $|x| \to \infty$.

Even when g is independent of ∇u, the problem (0.0.1)-(0.0.2) in the whole space $\mathbb{R}^n$ with exponential-type growths on u has been considered in dimension $n = 2$ in the majority of papers. Usually it is used Trudinger-Moser type inequalities and variational methods for proving existence of solutions (see e.g. results of [82] with $n = 2$ and its references). In the case of bounded domains, a well known problem arises particularly when

$$g(x, u, \nabla u) = \lambda V(x)e^u, \tag{0.0.3}$$

which was studied e.g. in [36, 65, 72, 80] (see also their references) with V being a positive bounded smooth function, where the parameter λ is assumed to be positive and sufficiently small.

One of the goals of the work on chapter 1 is to provide existence results by using a relatively simpler strategy but new for this prototypical situation. We will overcome this problem on arbitrary growth at infinity and on the noncompactness of the domain by looking for controlled solutions u in the space F_k (defined on Section 1.2) which already have a good decay in $|u|$ and also in $|\nabla u|$. This choice on the space of functions is going to be enough to prove that the functional associated with the nonlinearity g is an operator on F_k. A further smallness condition is imposed on $g(x, ., .)$ in order to make the operator a contraction on a subset of F_k. The solution is then obtained as a fixed point of this operator. It is worthy of note that this existence result is only local. There are no signs about when this solution might be unique or about any kind of multiplicity.

Several symmetry and asymptotic behavior results are also addressed, showing that the solutions inherit many of the properties of the nonlinearity g.

This first part was done under the supervision of Prof. Lucas C. F. Ferreira and Prof. Marcelo Montenegro at the State University of Campinas - Unicamp and was funded with scholarships from Capes and CNPq. The results will also appear in [43].

The Chapter 2 is dedicated to the analysis of the long-time asymptotics of the nonlinear nonlocal equation

$$\partial_t \rho = \nabla \cdot \Big(\rho (\nabla (-\Delta)^{-s} \rho + \lambda x) \Big), \qquad \lambda > 0,\ x \in \mathbb{R}^d\,, \tag{0.0.4}$$

obtained from the fractional version of the porous medium equation introduced by Caffarelli and Vázquez [20, 21]

$$\partial_\tau u = \nabla \cdot \left(u \nabla (-\Delta)^{-s} u \right), \tag{0.0.5}$$

by passing to self-similar variables.

The equation (0.0.5) is one of the two fractional variations of the classical porous medium equation and the existence of solutions was first studied by Caffarelli and Vázquez in [20]. In that work, the authors proved that whenever an initial data u_0 belongs to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with the following decay:

$$0 \leqslant u_0(y) \leqslant A e^{-a|y|}\ , \quad \text{for some } A, a > 0\ , \tag{0.0.6}$$

then there exist a weak solution u such that $u \in C([0,\infty); L^1(\mathbb{R}^n))$. The following other properties were also obtained in [21, 20, 19]:

- The mass of any solution is preserved, i.e.,

$$\int_{\mathbb{R}^d} u(\tau, y)\, dy = \int_{\mathbb{R}^d} u_0(y)\, dy\ , \quad \text{for all} \quad t > 0\ ;$$

- Regularity: the weak solutions are $C^\alpha(\mathbb{R}^d)$ for some $\alpha < 1$;

- The sign is conserved: if $u_0(y) \geqslant 0$ for all $y \in \mathbb{R}^d$ then $u(\tau, y) \geqslant 0$ for all $\tau > 0$ and $y \in \mathbb{R}^d$;

- The positivity is conserved: if $u_0(y_0) > 0$ for some y_0 then $u(\tau, y_0) > 0$ for all $\tau > 0$.

- Compactness of the support: if supp u_0 is compact then supp $u(\tau, .)$ is also compact for all $\tau > 0$.

- Exponential decay: If u_0 satisfies the condition (0.0.6), then there exist a function $C = C(t)$, which is increasing when $1/2 \leqslant s \leqslant 1$ and constant when $0 < s < 1/2$, such that

$$u(\tau, y) \leqslant Ae^{C(\tau)\tau - a|y|}\ ;$$

- There exist constants C, α_1 and α_2 depending on d and s such that

$$\sup_{y\in\mathbb{R}^d} |u(\tau, y)| \leqslant \frac{C}{\tau^\alpha} \|u_0\|_{L^1}^\gamma\ , \quad \text{for all} \ \ \tau > 0$$

- Stationary solution: for each initial mass $m := \int \rho_0(x)dx$ of the rescaled equation (0.0.4) there exists only one stationary solution $\rho_{m,\infty}$. This solution is C^{1-s} with compact support and the solution $\rho(t, x)$ with $\rho(0, .) = \rho_0$ satisfies

$$\|\rho(t, .) - \rho_{m,\infty}\|_{L^1}\ , \|\rho(t, .) - \rho_{m,\infty}\|_{L^\infty} \to 0 \text{ as } t \to \infty\ .$$

- Asymptotic behavior: for each initial mass $M := \int u_0(y)dy$ there exists a weak solution $U_M(\tau, y)$ of (0.0.5) such that, for every solution $u(\tau, y)$ with $u(0, .) = u_0$, we have

$$\|u(\tau, .) - U_M(\tau, .)\|_{L^1} \to 0 \quad \text{and} \ \ \tau^\alpha \|u(\tau, .) - U_M(\tau, .)\|_{L^\infty} \to 0$$

as $\tau \to \infty$.

In spite of the asymptotic behavior given by the last two items above, no rate of convergence was established for these results in [21] and thus, we shall show in the Chapter 2 that in some cases we actually have an exponential decay for the difference $\|u(\tau) - U_M(\tau)\|_{L^p}$ and $\|\rho(t) - \rho_{m,\infty}\|_{Ł^p}$.

Due to the divergence structure of the equation (0.0.4), it is possible to identify its solutions (at least when $d = 1$) as gradient flows of the associate entropy $\mathcal{E}$ on the space $\mathcal{P}_2(\mathbb{R})$ of probability measures when this one is equipped with the Wasserstein distance. This idea was presented by Otto in [63] and has been extensively studied and used to obtain quantitative and qualitative properties of solutions to equations of this kind. With respect to long time behavior, there is two approaches which has appeared in the literature and both take advantage of the entropy functional associated to the equation. In our case we can define the entropy of a solution ρ in the instant t as

$$\mathcal{E}(\rho(t)) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{(-\Delta)^{-s}\rho(t,x) + \lambda|x|^2\right\}\rho(t,x)\; dx \;,$$

and its dissipation as

$$\mathcal{I}(\rho) = \int_{\mathbb{R}^d} \rho \left|\nabla\left((-\Delta)^{-s}\rho + \frac{\lambda}{2}|x|^2\right)\right|^2 dx \;,$$

It is known that rates of convergence for the functional $\mathcal{E}$ lead to rates in some L^p spaces, as one can see by results like Csiszár-Kullback-Pinsker Inequality and its variants. Thus, in order to estimate the behavior of $\mathcal{E}$ we can proceed in two ways: for the first one, known as Bakry-Émery method and used in [8], we first note that, at least formally, for sufficiently smooth solutions ρ we have

$$\frac{d}{dt}\left(\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_{m,\infty})\right) = -\mathcal{I}(\rho(t)) \;, \tag{0.0.7}$$

and also

$$\frac{d}{dt}\mathcal{I}(\rho(t)) = -2\lambda\mathcal{I}(\rho(t)) - \mathcal{R}(\rho(t)) \tag{0.0.8}$$

with $\mathcal{R} \geqslant 0$. Therefore this last relation together with the Gronwall's Inequality imply that

$$\mathcal{I}(\rho(t)) \leqslant \mathcal{I}(\rho_0)e^{-2\lambda t} \;,$$

and thus $\mathcal{I}(\rho(t)) \to 0$ as $t \to \infty$. Now, integrating (0.0.8) on $[t, \infty)$ and using (0.0.7) we obtain

$$
\begin{aligned}
-\mathcal{I}(\rho(t)) &= \int_t^\infty \frac{s}{ds}\mathcal{I}(\rho(s))\, ds \leqslant -2\lambda \int_t^\infty \mathcal{I}(\rho(s))\, ds \\
&= 2\lambda \int_t^\infty \frac{s}{ds}\mathcal{E}(\rho(s))\, ds = 2\lambda \left(\mathcal{E}(\rho_{m,\infty}) - \mathcal{E}(\rho(t))\right) \, .
\end{aligned} \tag{0.0.9}
$$

i.e.,

$$
\mathcal{E}(\rho) - \mathcal{E}(\rho_{m,\infty}) \leqslant \frac{1}{2\lambda}\mathcal{I}(\rho). \tag{0.0.10}
$$

This inequality and the relation (0.0.7) imply

$$
\frac{d}{dt}\left(\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_{m,\infty})\right) \leqslant -2\lambda \left(\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_{m,\infty})\right) \tag{0.0.11}
$$

and, by Gronwall again, we obtain that

$$
\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_{m,\infty}) \leqslant e^{-2\lambda t}\left(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_{m,\infty})\right) \, .
$$

Another way of obtaining this decay is trying to establish the inequality (0.0.10) directly and so applying it to the relation (0.0.7) to obtain (0.0.11). This approach usually requires less regularity from the solutions and can be done by optimal transportation methods since what we want is a functional inequality for measures. In the Chapter 2 we shall use this second approach and we shall obtain rigorously a generalized version of (0.0.10) for a good set of measures in $\mathcal{P}_2(\mathbb{R})$ and apply it to the solutions of (0.0.4). The inequality (0.0.10) appears in [21] as an open question about the spectral gap. We also prove that the decay in the entropy level implies a rate of convergence of the solutions towards the stationary state in some L^p spaces.

The inequality (0.0.10) is usually called, in the context of optimal transport, log-Sobolev inequality in the linear diffusion case or generalized log-Sobolev inequalities otherwise. In particular, it becomes the logarithmic Sobolev inequality [46] for linear Fokker-Planck equation [7, 28, 73], and a special family of Gagliardo-Nirenberg inequalities for nonlinear Fokker-Planck equations with porous medium type diffusion [37, 26, 29]. This inequality is closely related with the notion of displacement convexity of functionals over $\mathcal{P}_2(\mathbb{R})$ which will be defined and explored in the Sections 2.2 and 2.3.

All the results proved in this work about the rate of convergence of solution to (0.0.4) are valid only in dimension 1. This restriction is due to the fact that we need to use the convexity of the kernel $c_{d,s}|x|^{2s-d}$, used in the definition of $(-\Delta)^{-s}$, in order to obtain a generalized version of the log-Sobolev inequality. It is worth of note to say that the problem of obtaining any rate for the long-time asymptotic behavior of $\rho(t,.)$ is still an open problem.

This second work was done under the supervision of Prof. José A. Carrillo at Imperial College London during a PhD Sandwich Program from Nov/2013 till Oct/2014 and was funded with a Capes/Science Without Borders scholarship. The results will also appear in [30].

Chapter 1

Semilinear elliptic equations in $\mathbb{R}^n$ with arbitrary growth

We analyse the existence, symmetry and asymptotic behavior of the solutions to a nonlinear elliptic equation in $\mathbb{R}^n$ where the nonlinear term $g(x, u, \nabla u)$ can depend on the unknown u and its first order derivatives ∇u. No restriction is imposed on the behavior of g at infinity except in the variable x, and thus, our results cover nonlinearities with arbitrary growth in u and ∇u, including in particular exponential type behavior. Using a fixed point argument we obtain a solution u that is locally unique, C^1 and with polynomial decay which inherits many of the symmetry properties of g. Positivity and asymptotic behavior of the solution are also addressed. The techniques, and even most of the arguments used in our results, can also be applied to the case where the domain is the half-space or an exterior domain and also to the case involving certain ranges of the fractional Laplacian.

1.1 Introduction

In this chapter we analyse the existence, symmetry and asymptotic behavior of solutions to the following nonlinear elliptic PDEs

$$\Delta u + g(x, u, \nabla u) = 0 \quad \text{in } \mathbb{R}^n \tag{1.1.1}$$

$$u \to 0 \quad \text{as } |x| \to \infty, \tag{1.1.2}$$

with $n \geq 3$ and $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ verifying $g(x, 0, 0) \not\equiv 0$ and belonging to a large class of nonlinear functions which include, for example, polynomial and exponential type growths on u or ∇u. Since we are interested in g depending on u and ∇u, we write $g(x, z, p)$ for $z \in \mathbb{R}$, $p \in \mathbb{R}^n$ and the gradient of g with respect to the $(n + 1)$-last variables will be denoted by $\nabla_{(z,p)} g(x, z, p)$. Throughout the paper, we frequently consider (1.1.1) with either $g(x, u, \nabla u)$, $g(x, u, |\nabla u|)$, $g(x, |u|, \nabla u)$, or $g(x, |u|, |\nabla u|)$ with the same hypotheses on g, except for the symmetry results.

Exponential-type nonlinearities appear naturally in many contexts like: conformal geometry and the prescribed curvature problem in 2 dimensions, where one is interested on determining the class of functions $K : M \to \mathbb{R}$ on the manifold M with curvature k such that the problem

$$-\Delta u = K e^{2u} + k \ , \quad \text{on } M \ ,$$

admits a solution and hence, this solution leads to metric on M which is pointwise conformal to the original one and has K as its Gaussian curvature (see [34], [33], [49]); condensate or multivortex solutions of the (2+1)-dimensional Chern-Simons gauge theory, where one is interested on the existence and multiplicity of solutions to the equation

$$\Delta u = \frac{4}{\kappa^2} e^u (e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_j \ , \quad \text{on } \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z} \ ,$$

where κ is a constant called Chern-Simons coupling parameter and δ_j are the Dirac measures at the prescribed zeros p_j of the Higgs scalar (see [22, 31, 71, 75]);

On the other hand, nonlinear gradient terms appear naturally in models connected with convective processes, in the physical theory of growth and roughening of surfaces or in stochastic control theory. One of the most widely studied examples is the following equation

$$-\Delta u = g(u)|\nabla u|^2 + \lambda f(x) \ , \quad \text{on } \mathbb{R}^d \ , \tag{1.1.3}$$

where $\lambda > 0$, g is a positive continuous function and f is a positive measurable one. The parabolic version of this equation was proposed by Kardar, Parisi and Zhang in [47] as a model for the evolution of the profile of a growing interface, which appears in the growth of smoke, flame fronts or tumors,

$$\partial_t u = \nu \Delta u + \mu |\nabla u|^2 + f(x) \ .$$

Also, for suitable choices of g and f, one can view (1.1.3) as the equation for the stationary states of the following model

$$\partial_t u = \varepsilon \Delta u + |\nabla u|^2$$

which is the viscosity approximation of some Hamilton-Jabobi type equations (see the standard reference [58]). The classical references in the treatment of (1.1.3) are [55] and [51], while many other results in that direction were also obtained by L. Boccardo, F. Murat, A. Porreta, J.-P. Puel and others.

As we said in the Introduction, the combination of the exponential growth, gradient terms and noncompact domains let the problem very hard to handle with standard methods since the natural spaces where they look for solutions are L^p spaces and generalizations. Here we will overcome all these difficulties by searching for solutions in a space where the functions have an polynomial decay which will be enough to control the nonlinearity.

The organization of this chapter is as follows. We first present an integral equivalent form for the problem (1.1.1)-(1.1.2) in the Section 1.2 together with the spaces where we shall look for the solutions. Due to this integral formulation, we need to prove some lemmas about convolutions and regularity of the Newtonian potential that will be used in the proof of the main theorem. On Section 1.3 we state our main result and we prove it by using a fixed point argument in the spaces defined on section 1.2. Two concrete examples where our hypothesis are satisfied are presented, and finally we show that the solutions obtained for this method inherit many of the properties of the nonlinearity g, like the positivity and symmetry, as well as the asymptotic behavior of u and its gradient. It is worthy of note to point out that by slight modifications on the proofs, our approach can be employed for other unbounded domains like half-space and exterior domains, with either Dirichlet or Neumann homogeneous boundary conditions.

1.2 Integral Formulation, Lemmas and Known Results

This section we present the integral formulation, using Green's function, which will be used to prove the existence of solution to (1.1.1)-(1.1.2). We shall also prove some Lemmas about the Newtonian potential on the spaces used in the next section and remind some well known results about regularity of elliptic equations. Let us start defining the following weighted spaces: for a fixed $k \in \mathbb{R}$, let

$$E_k \equiv \left\{ u \text{ measurable } \;:\; \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (1+|x|)^k |u(x)| < \infty \right\}$$

and

$$F_k \equiv \left\{ u \in C^1(\mathbb{R}^n) \;:\; \sup_{x\in\mathbb{R}^n} (1+|x|)^k \left(|u(x)| + |\nabla u(x)|\right) < \infty \right\},$$

which are Banach spaces with respective norms

$$\|u\|_{E_k} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (1+|x|)^k |u(x)|$$

and

$$\|u\|_{F_k} = \sup_{x\in\mathbb{R}^n} (1+|x|)^k \left(|u(x)| + |\nabla u(x)|\right).$$

Spaces like above with the homogeneous weight $|x|^k$ have been used in [44] to treat the equation $\Delta u + u|u|^{p-2} + V(x)u + f(x) = 0$ for $p > n/(n-2)$ with $n \geq 3$.

As we will see in the proof of Theorem 1.3.3, the choice of a proper value for k in the above spaces depends uniquely on the spaces where the function $x \mapsto g(x,0,0)$ is defined and how $|\nabla_{(z,p)} g(\cdot, u, \nabla u)|$ behaves with $|(u, \nabla u)|$.

The problem (1.1.1)-(1.1.2) is formally equivalent to the following integral equation

$$u(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\, g(y, u(y), \nabla u(y))\, dy, \tag{1.2.1}$$

where ω_n is the area of the unit sphere. Therefore, it will be convenient for our purposes to denote the Newtonian potential of a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{N}(f)(x) := \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\, f(y)\, dy,$$

and consider the nonlinear integral operator

$$\mathcal{B}(u)(x) := \mathcal{N}(g(\cdot, u, \nabla u))(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\, g(y, u(y), \nabla u(y))\, dy,$$

acting in the space F_k.

Therefore, in order to solve the mild version (1.2.1) of (1.1.1)-(1.1.2) we just need to look for fixed points of the operator $\mathcal{B}$ in some appropriate subset of F_k.

We start by analyzing an integral that will be useful for our needs.

Lemma 1.2.1. *Let* $\alpha, \beta > 0$ *and* $0 < n - \alpha < \beta$, *then*

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^\alpha} \frac{1}{(1+|y|)^\beta} \, dy < \infty.$$

Proof. Let us define, for every $x \in \mathbb{R}^n$, the functions $k_x(y) = |x-y|^{-\alpha}$ and $r(y) = (1+|y|)^{-\beta}$. Now, using the simplest rearrangement inequality theorem in [57, p. 82], one has

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^\alpha} \frac{1}{(1+|y|)^\beta} \, dy = \int_{\mathbb{R}^n} k_x(y) r(y) \, dy \leqslant \int_{\mathbb{R}^n} k_x^*(y) r^*(y) \, dy \tag{1.2.2}$$

where k_x^* and r^* are the symmetric-decreasing rearrangements of k_x and r respectively. For k_x^* we have that

$$k_x^*(y) = \int_0^\infty \chi_{\{|k_x|>t\}^*}(y) \, dt \ ,$$

where $\{|k_x| > t\}^*$ is the ball centered at the origin with the same measure as $\{|k_x| > t\}$. Thus, we can compute

$$|k_x(y)| > t \Leftrightarrow |x-y|^{-\alpha} > t \Leftrightarrow y \in B(x, t^{-1/\alpha}) \ .$$

and conclude that $\{|k_x| > t\}^* = B(0, t^{-1/\alpha})$. Hence

$$\begin{aligned} \int_0^\infty \chi_{\{|k_x|>t\}^*}(y) \, dt = \int_0^\infty \chi_{B(0,t^{-1/\alpha})}(y) \, dt &= \int_0^{|y|^{-\alpha}} 1 \, dt \\ = \frac{1}{|y|^\alpha} \end{aligned}$$

and $k_x^*(y) = |y|^{-\alpha}$ for all y and independently of x. In the same way we can compute $r^*(y)$ and obtain that $r^*(y) = r(y)$ for all $y \in \mathbb{R}^n$. Therefore, from (1.2.2) we have

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^\alpha} \frac{1}{(1+|y|)^\beta} \, dy \leqslant \int_{\mathbb{R}^n} \frac{1}{|y|^\alpha} \frac{1}{(1+|y|)^\beta} \, dy \ , \quad \text{for all } x \in \mathbb{R}^n \ ,$$

which is finite, due to the conditions on α and β. $\square$

The following convolution lemma will be useful for some estimates and its proof can be found in [57, p. 124].

Lemma 1.2.2. *Let* $0<\alpha,\beta<n$ *with* $0<\alpha+\beta<n$. *Then*

$$\int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \frac{1}{|x-y|^{n-\beta}}\, dy = \frac{C(\alpha,\beta,n)}{|x|^{n-\alpha-\beta}}$$

where $C(\alpha,\beta,n)=\frac{c_\alpha c_\beta c_{n-\alpha-\beta}}{c_{\alpha+\beta}c_{n-\alpha}c_{n-\beta}}$ *and* $c_\gamma=\pi^{-\gamma/2}\Gamma(\frac{\gamma}{2})$.

Let us recall the following version of the Dominated Convergence Theorem which will be used in the proof of the next lemma.

Lemma 1.2.3. *Let* f_n, f, g_n *and* g *be measurable functions on* $\mathbb{R}^n$ *such that* $f_n \to f$ *a.e. and* $g_n \to g$ *a.e. as well. If* $|f_n| \leqslant g_n$ *a.e and* $g_n \in L^1(\mathbb{R}^n)$ *for all* n, *and* $\int g_n \to \int g$ *as* $n\to\infty$ *then,* $\int f_n \to \int f$.

The next result gives the necessary regularity we will need for the Newtonian potential of a function in the space E_k.

Lemma 1.2.4. *Let* $0<k<n-2$ *and* $f\in E_{k+2}$. *Then* $\mathcal{N}(f)\in F_k$ *and there exists a constant* $C_k>0$ *satisfying*

$$\|\mathcal{N}(f)\|_{F_k} \leqslant C_k \|f\|_{E_{k+2}} \quad , \quad \text{for all } f\in E_{k+2}. \tag{1.2.3}$$

Proof. First we show that $\mathcal{N}(f)\in C^1(\mathbb{R}^n)$. For fixed $x,z\in\mathbb{R}^n$ with $|z|=1$ and $0<t<1$, we define the function $h_y(s)=|x-y+sz|^{2-n}$ on $[0,t]$. Note that h_y is differentiable on $[0,t]$ if and only if $y\notin L:=\{x+sz \mid s\in[0,t]\}$. If this is the case, we may write

$$h_y'(s)=(2-n)\frac{z\cdot(x-y+sz)}{|x-y+sz|^n} \quad , \quad \text{for all } s\in(0,t).$$

By Mean Value Theorem, for each $y\in\mathbb{R}^n\backslash L$ there exists $t_y\in(0,t)$ such that

$$\frac{h_y(t)-h_y(0)}{t}=(2-n)\frac{z\cdot(x-y+t_y z)}{|x-y+t_y z|^n}. \tag{1.2.4}$$

Since L is a measure-zero set, we may write

$$\frac{\mathcal{N}(f)(x+tz)-\mathcal{N}(f)(x)}{t} \quad = \quad \frac{1}{(n-2)w_n}\int_{\mathbb{R}^n\backslash L}\left(\frac{h_y(t)-h_y(0)}{t}\right) f(y)\, dy$$

$$= \quad -\frac{1}{w_n}\int_{\mathbb{R}^n} \frac{z\cdot(x-y+t_y z)}{|x-y+t_y z|^n} f(y)\ dy.$$

For each $y \in \mathbb{R}^n \backslash L$, let H_t be the function

$$H_t(y) = -\frac{1}{w_n}\frac{z\cdot(x-y+t_y z)}{|x-y+t_y z|^n} f(y)\ ,$$

where $t_y \in (0,t)$ and satisfies (1.2.4). In spite of the fact that t_y may be not unique, the definition of $H_t(y)$ ensures that a different t satisfying (1.2.4) gives the same value to the expression of $H_t(y)$. Thus H_t is well defined. Furthermore, we have that $H_t \to H_0$ a.e in $\mathbb{R}^n$ as $t \to 0$. Note that

$$|H_t(y)| \leqslant \frac{1}{w_n}\frac{|f(y)|}{|x-y+t_y z|^{n-1}} \leqslant G_t(y)\ , \tag{1.2.5}$$

where

$$G_t(y) = \frac{\|f\|_{E_{k+2}}}{w_n}\frac{1}{|x-y+t_y z|^{n-1}}\frac{1}{(1+|y|)^{k+2}} \in L^1(\mathbb{R}^n)\ ,$$

by Lemma 1.2.1. We also have

$$G_t(y) \to G_0(y) \ \text{ a.e. in } \mathbb{R}^n \ \text{ and } \ \int_{\mathbb{R}^n} G_t(y)\ dy = \int_{\mathbb{R}^n} \widetilde{G}_t(y)\ dy \tag{1.2.6}$$

as $t \to 0$, where

$$\begin{aligned} \widetilde{G}_t(y) \quad &= \quad \frac{\|f\|_{E_{k+2}}}{w_n}\frac{1}{|y|^{n-1}}\frac{1}{(1+|x+t_y z-y|)^{k+2}} \\ &\leqslant \quad \frac{\|f\|_{E_{k+2}}}{w_n}\frac{1}{|y|^{n-1}}\frac{C_1}{(1+|y|)^{k+2}} \in L^1(\mathbb{R}^n)\ , \end{aligned} \tag{1.2.7}$$

where C_1 depends on x but not on t since we took $t < 1$. Therefore, since $\widetilde{G}_t(y) \to \widetilde{G}_0(y)$ a.e in $\mathbb{R}^n$ as $t \to 0$, we have from (1.2.6), (1.2.7) and the Dominated Convergenge Theorem that

$$\int_{\mathbb{R}^n} G_t(y)\ dy = \int_{\mathbb{R}^n} \widetilde{G}_t(y)\ dy \to \int_{\mathbb{R}^n} \widetilde{G}_0(y)\ dy = \int_{\mathbb{R}^n} G_0(y)\ dy. \tag{1.2.8}$$

Then, from (1.2.5) and (1.2.8) and the Lemma 1.2.3, we conclude that

$$\lim_{t\to 0^+}\frac{\mathcal{N}(f)(x+tz)-\mathcal{N}(f)(x)}{t} = \lim_{t\to 0^+}\int_{\mathbb{R}^n} H_t(y)\ dy = \int_{\mathbb{R}^n} H_0(y)\ dy.$$

Thus,

$$\nabla\mathcal{N}(f)(x)\cdot z = -\frac{1}{w_n}\int_{\mathbb{R}^n}\frac{z\cdot(x-y)}{|x-y|^n}f(y)\,dy \quad , \qquad \text{for all } \ |z|=1,$$

$$\nabla\mathcal{N}(f)(x) = -\frac{1}{w_n}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}f(y)\,dy.$$

For a fixed $x_0 \in \mathbb{R}^n$ we have

$$|\nabla\mathcal{N}(f)(x_0)-\nabla\mathcal{N}(f)(x)| \ \leqslant \ \int_{\mathbb{R}^n}\frac{1}{w_n}\left|\frac{x_0-y}{|x_0-y|^n}-\frac{x-y}{|x-y|^n}\right||f(y)|\,dy$$

and the continuity of $\nabla\mathcal{N}(f)$ in x_0 follows from the same arguments as above applied to the new functions

$$\begin{aligned}
H_x(y) \ &:= \ \frac{1}{w_n}\left|\frac{x_0-y}{|x_0-y|^n}-\frac{x-y}{|x-y|^n}\right||f(y)|;\\
G_x(y) \ &:= \ \frac{\|f\|_{E_{k+2}}}{w_n}\left(\frac{1}{|x_0-y|^{n-1}}+\frac{1}{|x-y|^{n-1}}\right)\frac{1}{(1+|y|)^{k+2}};\\
\widetilde{G}_x(y) \ &:= \ \frac{\|f\|_{E_{k+2}}}{w_n}\frac{1}{|y|^{n-1}}\left(\frac{1}{(1+|x_0-y|)^{k+2}}+\frac{1}{(1+|x-y|)^{k+2}}\right)
\end{aligned}$$

and the estimate

$$\widetilde{G}_x(y) \leqslant \frac{C\,\|f\|_{E_{k+2}}}{w_n|y|^{n-1}(1+|x_0-y|)^{k+2}} \in L^1(\mathbb{R}^n)\ , \quad \text{if} \quad |x-x_0|<\frac{1}{2}.$$

For the existence of C_k satisfying (1.2.3), we first note from the definition of $\|.\|_{F_k}$ that

$$\begin{aligned}
\|\mathcal{N}(f)\|_{F_k} &= \sup_{x\in\mathbb{R}^n}\Big\{(1+|x|)^k|f(x)|+(1+|x|)^k|\nabla f(x)|\Big\}\\
&\leqslant \sup_{x\in\mathbb{R}^n}(1+|x|)^k|f(x)|+\sup_{x\in\mathbb{R}^n}(1+|x|)^k|\nabla f(x)|\\
&= \|\mathcal{N}(f)\|_{E_k}+\|\nabla\mathcal{N}(f)\|_{E_k}\ .
\end{aligned}$$

Let us then estimate these two terms. Beginning with the first one, we have that for every $0<k<n-2$, we can apply the Lemma 1.2.2 with $\alpha=2$ and $\beta=n-k-2$ and obtain, for every $x\in\mathbb{R}^n$,

$$|\mathcal{N}(f)(x)| \ \leqslant \ \frac{1}{(n-2)w_n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-2}}|f(y)|\,dy$$

$$
\begin{aligned}
&= \frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{|y|^{k+2}}{|x-y|^{n-2}} \frac{|f(y)|}{|y|^{k+2}}\, dy \\
&\leqslant \frac{1}{(n-2)w_n} \sup_{y\in\mathbb{R}^n} \left(|y|^{k+2}|f(y)|\right) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^{k+2}}\, dy \\
&= \frac{C(n-k-2,2,n)}{(n-2)w_n} \sup_{y\in\mathbb{R}^n} \left(|y|^{k+2}|f(y)|\right) \frac{1}{|x|^k} \\
&\leqslant \frac{C(n-k-2,2,n)}{(n-2)w_n} \|f\|_{E_{k+2}} \frac{1}{|x|^k} \\
&=: L_k \|f\|_{E_{k+2}} \frac{1}{|x|^k}.
\end{aligned}
$$

Now, using the Lemma 1.2.1 with $\alpha = n-2$ and $\beta = k+2$ we obtain

$$
\begin{aligned}
|\mathcal{N}(f)(x)| &\leqslant \frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} |f(y)|\, dy \\
&= \frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{(1+|y|)^{k+2}}{|x-y|^{n-2}} \frac{|f(y)|}{(1+|y|)^{k+2}}\, dy \\
&\leqslant \left(\frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|)^{k+2}}\, dy\right) \|f\|_{E_{k+2}} \\
&\leqslant \left(\frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{1}{|y|^{n-2}} \frac{1}{(1+|y|)^{k+2}}\, dy\right) \|f\|_{E_{k+2}} \\
&=: M_k \|f\|_{E_{k+2}}.
\end{aligned}
$$

Therefore, for every $x \in \mathbb{R}^n$,

$$
\begin{aligned}
(1+|x|)^k|\mathcal{N}(f)(x)| &\leqslant 2^k \left(|\mathcal{N}(f)(x)| + |x|^k|\mathcal{N}(f)(x)|\right) \\
&\leqslant 2^k(M_k + L_k) \|f\|_{E_{k+2}},
\end{aligned}
$$

which implies that

$$
\|\mathcal{N}(f)\|_{E_k} \leqslant 2^k(M_k + L_k) \|f\|_{E_{k+2}} .
$$

The estimates for the term $\|\nabla\mathcal{N}(f)\|_{E_k}$ are similar but we will include them here just for the sake of completeness. In this case for $0 < k < n-2$, we can apply Lemma 1.2.2 with $\alpha = 1$ and $\beta = n-k-2$ and obtain, for every $x \in \mathbb{R}^n$,

$$
|\nabla\mathcal{N}(f)(x)| \leqslant \frac{1}{w_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} |f(y)|\, dy
$$

$$
\begin{aligned}
&= \frac{1}{w_n}\int_{\mathbb{R}^n} \frac{|y|^{k+2}}{|x-y|^{n-1}}\frac{|f(y)|}{|y|^{k+2}}\,dy \\
&\leqslant \frac{1}{w_n}\sup_{y\in\mathbb{R}^n}\left(|y|^{k+2}|f(y)|\right)\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1}}\frac{1}{|y|^{k+2}}\,dy \\
&= \frac{C(n-k-2,1,n)}{w_n}\sup_{y\in\mathbb{R}^n}\left(|y|^{k+2}|f(y)|\right)\frac{1}{|x|^{k+1}} \\
&\leqslant \frac{C(n-k-2,1,n)}{w_n}\,\|f\|_{E_{k+2}}\,\frac{1}{|x|^{k+1}} \\
&=: \widetilde{L}_k\,\|f\|_{E_{k+2}}\,\frac{1}{|x|^{k+1}}.
\end{aligned}
$$

Applying Lemma 1.2.1 with $\alpha = n-1$ and $\beta = k+2$, we conclude

$$
\begin{aligned}
|\nabla\mathcal{N}(f)(x)| &\leqslant \frac{1}{w_n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1}}|f(y)|\,dy \\
&= \frac{1}{w_n}\int_{\mathbb{R}^n}\frac{(1+|y|)^{k+2}}{|x-y|^{n-1}}\frac{|f(y)|}{(1+|y|)^{k+2}}\,dy \\
&\leqslant \left(\frac{1}{w_n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1}}\frac{1}{(1+|y|)^{k+2}}\,dy\right)\|f\|_{E_{k+2}} \\
&\leqslant \left(\frac{1}{w_n}\int_{\mathbb{R}^n}\frac{1}{|y|^{n-1}}\frac{1}{(1+|y|)^{k+2}}\,dy\right)\|f\|_{E_{k+2}} \\
&=: \widetilde{M}_k\,\|f\|_{E_{k+2}}.
\end{aligned}
$$

Thus we obtain that for every $x\in\mathbb{R}^n$,

$$
\begin{aligned}
(1+|x|)^{k+1}|\nabla\mathcal{N}(f)(x)| &\leqslant 2^{k+1}\left(|\nabla\mathcal{N}(f)(x)| + |x|^{k+1}|\nabla\mathcal{N}(f)(x)|\right) \\
&\leqslant 2^{k+1}(\widetilde{M}_k+\widetilde{L}_k)\,\|f\|_{E_{k+2}}.
\end{aligned}
$$

Therefore, taking the sup in the above expression yields

$$
\|\nabla\mathcal{N}(f)\|_{E_{k+1}} \leqslant 2^{k+1}(M_k+L_k)\,\|f\|_{E_{k+2}} \ .
$$

We conclude by putting all these estimates together:

$$
\begin{aligned}
\|\mathcal{N}(f)\|_{F_k} &\leqslant \|\mathcal{N}(f)\|_{E_k} + \|\nabla\mathcal{N}(f)\|_{E_k} \\
&\leqslant \|\mathcal{N}(f)\|_{E_k} + 2\,\|\nabla\mathcal{N}(f)\|_{E_{k+1}}
\end{aligned}
$$

$$\leqslant 2^k(M_k + L_k)\,\|f\|_{E_{k+2}} + 2^{k+2}(\widetilde{M}_k + \widetilde{L}_k)\,\|f\|_{E_{k+2}}$$
$$\leqslant 2^{k+2}(\widetilde{M}_k + \widetilde{L}_k + M_k + L_k)\,\|f\|_{E_{k+2}}$$

and thus we can take $C_k = 2^{k+2}(M_k + L_k + \widetilde{M}_k + \widetilde{L}_k)$ for the constant in the theorem.

□

To finish this section, we shall include here the following two regularity theorems for elliptic equations whose proof can be found in [45].

Lemma 1.2.5 (L^p Regularity)**.** *Let $v \in W^{1,2}(\Omega)$ be a weak solution of $\Delta v = f$ in a domain $\Omega \subseteq \mathbb{R}^n$ with $f \in L^p(\Omega)$ for some $1 < p < \infty$. Then $v \in W^{2,p}_{loc}(\Omega)$.*

Lemma 1.2.6 (C^α Regularity)**.** *Let $k \geqslant 0$, $0 < \alpha < 1$ and $1 < p < \infty$. If $\Omega \subseteq \mathbb{R}^n$ be a $C^{k+2,1}$ domain, $f \in C^{k,\alpha}(\overline{\Omega})$ and $v \in W^{2,p}_{loc}(\Omega)$ is a weak solution to $\Delta v = f$ in Ω, then $v \in C^{k+2,\alpha}(\overline{\Omega})$.*

1.3 Existence and symmetries

The fixed point method which will be used in this section needs an appropriate choice of a subset $A \subseteq F_k$ where the operator $\mathcal{B}$ can be seen as a contraction, and this contraction depends uniquely on how large the functions $g(.,0,0)$ and $\nabla_{(z,p)}g(.,z,p)$ are. Therefore, we shall solve the problem (1.2.1) under the following hypotheses on g:

H1) $g(x,\cdot,\cdot)$ belongs to $C^1((\mathbb{R}\times\mathbb{R}^n)\backslash(0,0)) \cap C(\mathbb{R}\times\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$;

H2) There exists $0 < k < n-2$ such that the function $x \mapsto g(x,0,0)$ belongs to E_{k+2};

H3) For the same k in (**H2**), there exists $\delta > 0$ such that

$$\sup_{0<\|w\|_{F_k}\leq\delta} \left\|\nabla_{(z,p)}g(\cdot,w,\nabla w)\right\|_{E_2} < \infty,$$

and a further smallness condition on this sup.

Note that, by the choice of spaces we are dealing with, it follows from this approach that the solution given by this method is already $C^1(\mathbb{R}^n)$.

Remark 1.3.1. *For $w \in F_k$, $(w, \nabla w) \equiv 0$ iff $w \equiv 0$. In spite of the fact that $g(x, \cdot, \cdot)$ is not differentiable at the point $(0,0)$, we are assuming with $(\mathbf{H3})$ that $\nabla_{(z,p)}(x, \cdot, \cdot)$ is bounded near to the origin. Notice that the supremum of $\|.\|_{E_2}$ in $(\mathbf{H3})$ is computed by excluding $w \equiv 0$.*

The assumptions $(\mathbf{H1})$, $(\mathbf{H2})$ and $(\mathbf{H3})$ cover many types of nonlinearities with strong growth and gradient dependence. In what follows, we give some examples.

Example 1.3.2. *Recall first that (1.1.1) is also being defined with u or ∇u replaced respectively by $|u|$ or $|\nabla u|$ in the arguments of g.*

- $g(x, u, |\nabla u|) = \lambda V(x)e^u + \mu W(x)e^{|\nabla u|}$ *or* $\lambda V(x)e^{e^{\cdots^{e^u}}} + \mu W(x)e^{e^{\cdots^{e^{|\nabla u|}}}}$, *for every* $V, W \in E_{k+2}$ *with* $0 < k < n-2$, *and* $\lambda, \mu \in \mathbb{R}$;
- $g(x, |u|, |\nabla u|) = W(x)e^{|u|^{m_1}+|\nabla u|^{m_2}}$, $|u|^{m_1} + W(x)e^{|\nabla u|^{m_2}}$, $W(x)e^{|u|^{m_1}} + |\nabla u|^{m_2}$, $W(x)e^{|u|^{m_1}}|\nabla u|^{m_2} + f$ *or* $W(x)|u|^{m_1}e^{|\nabla u|^{m_2}} + f$, *for* $m_1, m_2 > 1$ *and* $W, f \in E_{k+2}$ *with* $0 < k < n-2$;
- $g(x, |u|, |\nabla u|) = e^{|u|^{m_1}+|\nabla u|^{m_2}} - 1 + f(x)$ *or* $g(x, u, \nabla u) = e^{e^{(|u|^{m_1}+|\nabla u|^{m_2})}} - 1 + f(x)$, *for* $m_1, m_2 > 1$ *and* $f \in E_{k+2}$ *with* $0 < k < n-2$;
- $g(x, |u|, |\nabla u|) = |u|^{m_1} + |\nabla u|^{m_2} + f(x)$ *or* $|u|^{m_1}|\nabla u|^{m_2} + f(x)$, *for* $m_1, m_2 > 1$ *and* $f \in E_{k+2}$ *with* $0 < k < n-2$.

We will show existence of solutions for (1.1.1)-(1.1.2) in $\mathbb{R}^n$ with $n \geq 3$ and conditions on g (see $(\mathbf{H1})$-$(\mathbf{H3})$) covering polynomial and exponential type growths on u and ∇u, see Examples 1.3.2, 1.3.6 and 1.3.7. In particular, since $g(x, 0, 0)$ does not need to be continuous, the nonlinearity (0.0.3) can be treated with singular potentials V (non-continuous and bounded) and $|\lambda|$ close to zero, including also negative values (see Example 1.3.6 below).

From now on we assume that $n \geqslant 3$ and that $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies $(\mathbf{H1})$-$(\mathbf{H3})$. We begin with existence and local uniqueness of solutions for the integral equation (1.2.1).

Theorem 1.3.3. *There exists a constant $Q_k > 0$ such that if $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies **(H1)**-**(H3)** for some $0 < k < n-2$ and the following inequalities*

$$\sup_{0<\|w\|_{F_k}\leq\varepsilon} \left\|\nabla_{(z,p)}g(., w, \nabla w)\right\|_{E_2} < Q_k$$

and

$$\|g(.,0,0)\|_{E_{k+2}} \leqslant \varepsilon Q_k,$$

are satisfied for some $\varepsilon > 0$, *then the integral equation* (1.2.1) *has a unique solution* $u \in F_k$ *with* $\|u\|_{F_k} \leqslant \varepsilon$, *which is in particular a weak solution for* (1.1.1)-(1.1.2). *Furthermore, if* $g \in C^{m,\alpha}_{loc}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ *for an integer* $m \geq 0$ *with* $0 < \alpha < 1$, *then* $u \in C^{m+2,\alpha}_{loc}(\mathbb{R}^n)$ *and* u *verifies* (1.1.1)-(1.1.2) *classically.*

Remark 1.3.4. *In the statement of Theorem 1.3.3, the constant* Q_k *can be taken as* $\frac{1}{2C_k}$ *where* C_k *is as in Lemma 1.2.4 in the previous section, as we can see from the proof below. In fact, in view of the proof of Lemma 1.2.4, it is possible to estimate* C_k *and* Q_k *explicitly.*

Proof. Let $x \in \mathbb{R}^n$, $(x, z_1, p_1), (x, z_2, p_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $[(z_1, p_1), (z_2, p_2)]$ be the closed line segment between (z_1, p_1) and (z_2, p_2) in $\mathbb{R}^{n+1}$. If $(0, 0) \notin [(z_1, p_1), (z_2, p_2)]$ then, from the hypothesis (**H1**), we have

$$|g(x, z_1, p_1) - g(x, z_2, p_2)| \leqslant \sup_{(z,p)\in[(z_1,p_1),(z_2,p_2)]} |\nabla_{(z,p)} g(x, z, p)||(z_1 - z_2, p_1 - p_2)| \,.$$

Now, if $(0, 0) \in [(z_1, p_1), (z_2, p_2)]$, then we have that $|(z_1, p_1)| + |(z_2, p_2)| = |(z_1, p_1) - (z_2, p_2)|$ and, by (**H1**)

$$\begin{aligned}
|g(x, z_1, p_1) - g(x, z_2, p_2)| &\leqslant |g(x, z_1, p_1) - g(x, 0, 0)| + |g(x, 0, 0) - g(x, z_2, p_2)| \\
&\leqslant \sup_{(z,p)\in[(z_1,p_1),(0,0))} |\nabla_{(z,p)} g(x, z, p)||(z_1, p_1)| \\
&\qquad + \sup_{(z,p)\in((0,0),(z_2,p_2)]} |\nabla_{(z,p)} g(x, z, p)||(z_2, p_2)| \\
&\leqslant \sup_{(z,p)\in[(z_1,p_1),(z_2,p_2)]\setminus(0,0)} |\nabla_{(z,p)} g(x, z, p)|(|(z_1, p_1)| + |(z_2, p_2)|) \\
&= \sup_{(z,p)\in[(z_1,p_1),(z_2,p_2)]\setminus(0,0)} |\nabla_{(z,p)} g(x, z, p)||(z_1 - z_2, p_1 - p_2)| \,.
\end{aligned}$$

Thus, for a fixed $\delta > 0$ and $u, v \in F_k$ with $0 < \|u\|_{F_k}, \|v\|_{F_k} < \delta$, we write $(u, \nabla u) = (u(x), \nabla u(x))$ to obtain

$$\begin{aligned}
|g(x, u, \nabla u) - g(x, v, \nabla v)| &\leqslant \sup_{(z,p)\in[(u,\nabla u),(v,\nabla v)]\setminus(0,0)} |\nabla_{(z,p)} g(x, z, p)||(u - v, \nabla u - \nabla v)| \\
&\leqslant \sup_{0<\|w\|_{F_k}\leqslant\delta} |\nabla_{(z,p)} g(x, w, \nabla w)||(u - v, \nabla u - \nabla v)| \,,
\end{aligned}$$

because for every (z,p) with $|z|, |p_i| < \delta$, $i = 1, \ldots, n$, we can find a function $w \in F_k$ such that $\|w\|_{F_k} < \delta$ and $(z,p) = (w, \nabla w)$ at a point x. Hence, we have that

$$(1+|x|)^{k+2}|g(x,u,\nabla u) - g(x,v,\nabla v)| \leqslant \sup_{0<\|w\|_{F_k}\leqslant\delta} (1+|x|)^2|\nabla_{(z,p)}g(x,w,\nabla w)|(1+|x|)^k|(u-v,\nabla u - \nabla v)| \ ,$$

and by (**H3**), it follows that

$$\|g(.,u,\nabla u) - g(.,v,\nabla v)\|_{E_{k+2}} \leqslant \sup_{0<\|w\|_{F_k}\leqslant\delta} \left\|\nabla_{(z,p)}g(.,w,\nabla w)\right\|_{E_2} \|u-v\|_{F_k} \ .$$

Now, take $Q_k = \frac{1}{2C_k}$ where C_k is the constant given by the Lemma 1.2.4 and, by hypothesis, let $\varepsilon > 0$ be such that

$$G_\varepsilon := \sup_{0<\|w\|_{F_k}\leqslant\varepsilon} \left\|\nabla_{(z,p)}g(.,w,\nabla w)\right\|_{E_2} < \frac{1}{2C_k} \ .$$

We will show that $\mathcal{B}$ is a contraction in the ball $A_\varepsilon = \{u \in F_k : \ \|u\|_{F_k} \leqslant \varepsilon\}$. Fix $u, v \in A_\varepsilon$ and note that

$$\mathcal{B}(u) - \mathcal{B}(v) = \mathcal{N}(g(.,u,\nabla u) - g(.,v,\nabla v)) \ ,$$

so we can use Lemma 1.2.4 and estimate

$$\begin{aligned}
\|\mathcal{B}(u) - \mathcal{B}(v)\|_{F_k} &= \|\mathcal{N}(g(.,u,\nabla u) - g(.,v,\nabla v))\|_{F_k} \\
&\leqslant C_k \|g(.,u,\nabla u) - g(.,v,\nabla v)\|_{E_{k+2}} \\
&\leqslant C_k G_\varepsilon \|u-v\|_{F_k} \\
&\leqslant \frac{1}{2}\|u-v\|_{F_k} \ .
\end{aligned}$$

Thus for $u \in A_\varepsilon$ and $v = 0$ in the above inequality, we have

$$\begin{aligned}
\|\mathcal{B}(u)\|_{F_k} &\leqslant \|\mathcal{B}(u) - \mathcal{B}(0)\|_{F_k} + \|\mathcal{B}(0)\|_{F_k} \\
&\leqslant \frac{1}{2}\|u\|_{F_k} + \|\mathcal{N}(g(.,0,0))\|_{F_k} \\
&\leqslant \frac{1}{2}\|u\|_{F_k} + C_k \|g(.,0,0)\|_{E_{k+2}}
\end{aligned}$$

$$\leqslant \frac{\varepsilon}{2} + C_k \frac{\varepsilon}{2C_k} = \varepsilon \ ,$$

which shows that $\mathcal{B}(A_\varepsilon) \subseteq A_\varepsilon$. Therefore $\mathcal{B}$ is a contraction in A_ε and the result follows by applying the Banach fixed point theorem.

The regularity of u will follow from the fact that u is also a solution in the weak sense to (1.1.1) and the classical regularity results. Indeed, since $u \in C^1(\mathbb{R}^n)$, $g(., u, \nabla u) \in L^\infty(\mathbb{R}^n)$ and the kernel $|x|^{2-n}$ is locally integrable, we have for all $\phi \in C_c^\infty(\mathbb{R}^n)$ that,

$$\begin{aligned}
\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, dx &= -\int_{\mathbb{R}^n} u(x) \Delta \phi(x) \, dx \\
&= -\frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} g(y, u(y), \nabla u(y)) \Delta \phi(x) \, dy dx \\
&= -\int_{\mathbb{R}^n} g(y, u(y), \nabla u(y)) \left(\frac{1}{(n-2)w_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Delta \phi(x) \, dx \right) dy \\
&= \int_{\mathbb{R}^n} g(y, u(y), \nabla u(y)) \phi(y) \, dy.
\end{aligned}$$

Therefore, u is a weak solution. Moreover, u is a weak solution of (1.1.1) on every ball Ω in $\mathbb{R}^n$ and $u \in W^{1,2}(\Omega)$. Therefore, since we have $g(., u, \nabla u) \in L^s(\Omega)$ for all $s > 1$ and for every ball Ω, we can use Lemma 1.2.5 and conclude that $u \in W^{2,s}(\Omega)$ for every $s > 1$. Therefore, for $s > n$ we have the embedding $W^{2,s}(\alpha) \hookrightarrow C^{1,\alpha}(\Omega)$, for $\alpha = 1 - \frac{n}{s}$ and we conclude that $u \in C^{1,\gamma}(\Omega)$ for every $0 < \gamma < 1$.

Now, if $g \in C^\alpha_{loc}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ then $g(., u, \nabla u) \in C^{0,\alpha\gamma}(\Omega)$ for all $0 < \gamma < 1$ and, by Lemma 1.2.6, we have that $u \in C^{2,\alpha\gamma}(\Omega)$. Hence $g(., u, \nabla u) \in C^\alpha(\Omega)$ and we can perform the previous argument once more and conclude that $u \in C^{2,\alpha}(\Omega)$.

Therefore, if $g \in C^{1,\alpha}_{loc}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ we now have that $g(., u, \nabla u) \in C^{1,\alpha\gamma}(\Omega)$ and we can conclude that $u \in C^{3,\alpha}(\Omega)$.

Repeating the argument above we can infer that whenever $g \in C^{m,\alpha}_{loc}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ with $m \geqslant 1$ then $g(., u, \nabla u) \in C^{m,\alpha}(\Omega)$ and, by Lemma 1.2.6, we have that $u \in C^{m+2,\alpha}(\Omega)$ for every ball Ω. In view of the fact that u is a solution of (1.1.1) in the sense of distributions and $u \in F_k \cap C^{m+2,\alpha}_{loc}(\mathbb{R}^n)$, then u is a classical solution of (1.1.1)-(1.1.2).

□

Remark 1.3.5. *The fixed point theorem applied above gives an iterative method to construct* *the solution u, which is the limit in the norm $\|.\|_{F_k}$ of the following sequence*

$$u_1 = \mathcal{B}(0) = \mathcal{N}(g(.,0,0)) \quad and \quad u_m = \mathcal{B}(u_{m-1}) \ , \quad m \in \mathbb{N}.$$

Moreover, all elements of this sequence verify $\|u_m\|_{F_k} \leq \varepsilon$.

In the sequel we present two examples.

Example 1.3.6. *Let $Q_k = \frac{1}{2C_k}$ where C_k is as in Lemma 1.2.4 (see Remark 1.3.4). Let λ and μ be real parameters and let*

$$g(x,u,\nabla u) = \lambda V(x) e^u + \mu W(x) e^{|\nabla u|},$$

where $V, W \in E_{k+2}$ for some $0 < k < n-2$. The case $\mu = 0$ is the so-called Liouville equation which arises, as pointed out above, in many physical situations and has produced a rich mathematical theory when $n = 2$ (see e.g. [10], [41], [65], [72]). Here we solve the problem for all dimension $n \geq 3$. We have that

$$\begin{aligned} &(1+|x|)^2 |\nabla_{(z,p)} g(x,w,\nabla w)| = \\ &= \left(\left(|\lambda|(1+|x|)^2 |V(x)| e^{w(x)} \right)^2 + \left(|\mu|(1+|x|)^2 |W(x)| e^{|\nabla w(x)|} \right)^2 \right)^{1/2} \\ &\leqslant \left(|\lambda| \, \|V\|_{E_{k+2}} + |\mu| \, \|W\|_{E_{k+2}} \right) e^{\|w\|_{F_k}}, \end{aligned}$$

for all $0 \neq w \in F_k$, and

$$\begin{aligned} (1+|x|)^{k+2} |g(x,0,0)| &= |\lambda|(1+|x|)^{k+2} |V(x)| + |\mu|(1+|x|)^{k+2} |W(x)| \\ &\leqslant |\lambda| \, \|V\|_{E_{k+2}} + |\mu| \, \|W\|_{E_{k+2}} \, . \end{aligned}$$

Then, Theorem 1.3.3 allows us to solve the problem of the present example if we can find $\varepsilon > 0$ such that

$$\left(|\lambda| \, \|V\|_{E_{k+2}} + |\mu| \, \|W\|_{E_{k+2}} \right) e^{\varepsilon} \leqslant \frac{1}{2C_k}$$

and

$$|\lambda| \, \|V\|_{E_{k+2}} + |\mu| \, \|W\|_{E_{k+2}} \leqslant \frac{\varepsilon}{2C_k}$$

We see from these inequalities that if we take

$$\varepsilon = 2C_k\left(|\lambda|\,\|V\|_{E_{k+2}} + |\mu|\,\|W\|_{E_{k+2}}\right)\ ,$$

the problem will have a solution if λ and μ are such that

$$2C_k\left(|\lambda|\,\|V\|_{E_{k+2}} + |\mu|\,\|W\|_{E_{k+2}}\right)e^{2C_k\left(|\lambda|\|V\|_{E_{k+2}}+|\mu|\|W\|_{E_{k+2}}\right)} \leqslant 1$$

The continuous dependence of the solution with respect to λ and μ follows by using that the solution u satisfies

$$\|u\|_{F_k} \leqslant \varepsilon.$$

This means that the equation $\Delta u + \lambda V(x)e^u + \mu W(x)e^{|\nabla u|} = 0$ has a bounded solution in $\mathbb{R}^n$ if the parameters $|\lambda|$ and $|\mu|$ are small enough, regardless the sign of λ, μ, V, W, and allowing to consider non-continuous coefficients V and W.

Example 1.3.7. *According to Remark 1.3.4, let us take $Q_k = \frac{1}{2C_k}$ where C_k is as in Lemma 1.2.4. Take g of the form $g(x, z, p_1, \dots, p_n) = h(x, z^{r_0}, p_1^{r_1} \dots, p_n^{r_n})$, where $r_i > 1$ for all i. If $r = \min\{r_0, \dots, r_n\}$ and $k = \frac{2}{r-1}$, suppose that $h(x, 0, 0) \in E_{k+2}$ and there exists $m > 0$ such that $\nabla_{(z,p)}h(x, w, \nabla w) \in E_m$ for all $w \in F_k$. Then, differentiating we obtain*

$$\nabla_{(z,p)}g(x, z, p) = \left(r_0 z^{r_0-1}\partial_z h,\ r_1 p_1^{r_1-1}\partial_{p_1}h,\ \dots,\ r_n p_n^{r_n-1}\partial_{p_n}h\right).$$

If $w \in F_k$ with $\|w\|_{F_k} \leqslant 1$ then

$$\begin{aligned}
&(1+|x|)^2|\nabla_{(z,p)}g(x, w, \nabla w)| \leqslant \\
&\leqslant \left|\left(r_0\left[(1+|x|)^{\frac{2}{r_0-1}}|w|\right]^{r_0-1}|\partial_z h(x, w, \nabla w)|, \dots, r_n\left[(1+|x|)^{\frac{2}{r_n-1}}|w|\right]^{r_n-1}|\partial_{p_n}h(x, w, \nabla w)|\right)\right| \\
&\leqslant R\left|\left(\|w\|_{F_{\frac{2}{r_0-1}}}^{r_0-1}\left\|\nabla_{(z,p)}h(., w, \nabla w)\right\|_{E_m}, \dots, \|w\|_{F_{\frac{2}{r_n-1}}}^{r_n-1}\left\|\nabla_{(z,p)}h(., w, \nabla w)\right\|_{E_m}\right)\right| \\
&\leqslant \sqrt{n+1}R\,\|w\|_{F_{\frac{2}{r-1}}}^{r-1}\left\|\nabla_{(z,p)}h(., w, \nabla w)\right\|_{E_m},
\end{aligned}$$

where $R = \max\{r_0, \ldots, r_n\}$. Thus, for $\varepsilon \leqslant 1$,

$$\sup_{\|w\|_{F_{\frac{2}{r-1}}} \leq \varepsilon} \left\| \nabla_{(z,p)} g(\cdot, w, \nabla w) \right\|_{E_2} \leqslant \sqrt{n+1} R \varepsilon^{r-1} \sup_{\|w\|_{F_k} \leq \varepsilon} \left\| \nabla_{(z,p)} h(\cdot, w, \nabla w) \right\|_{E_m}.$$

If h is such that $\|h(.,0,0)\|_{E_{k+2}} \leq \frac{\varepsilon}{2C_k}$ and

$$\sqrt{n+1} R (2C_k)^r \|h(.,0,0)\|_{E_{k+2}}^{r-1} \sup_{\|w\|_{F_k} \leq \varepsilon} \left\| \nabla_{(z,p)} h(\cdot, w, \nabla w) \right\|_{E_m} < 1,$$

then there exists a solution $u \in F_k$ such that $\|u\|_{F_k} \leqslant 2C_k \|h(\cdot, 0, 0)\|_{E_{k+2}}$.

A natural question is whether u presents qualitative properties according to g. The following results show that the solution obtained by the previous theorem inherits indeed many properties from the nonlinearity g.

Theorem 1.3.8. *Under the hypotheses of Theorem 1.3.3, the solution u satisfies:*

(i) If $g \geqslant 0$, then $u \geqslant 0$;

(ii) If $g(x,z,p) \geqslant 0$ for all $(x,z,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, with $g(x,z,p) \not\equiv 0$ when $|(z,p)|_{\mathbb{R}\times\mathbb{R}^n} \leq \varepsilon$, then $u > 0$;

(iii) u is radially symmetric provided that $g(\cdot, z, p)$ is radially symmetric for each fixed $(z,p) \in \mathbb{R} \times \mathbb{R}^n$ such that $|(z,p)|_{\mathbb{R}\times\mathbb{R}^n} \leq \varepsilon$.

Proof. The item (i) follows from the fact that the Newtonian potential of a nonnegative function is nonnegative. To prove item (ii), notice that $\|u\|_{F_k} \leq \varepsilon$ implies that $|(u(x), \nabla u(x))|_{\mathbb{R}\times\mathbb{R}^n} \leq \varepsilon$, for all $x \in \mathbb{R}^n$. It follows that $g(x, u(x), \nabla u(x)) \not\equiv 0$, and then $u = \mathcal{N}(g(x, u, \nabla u))$ is positive. To establish item (iii), recall first that the solution u is the limit under the norm $\|.\|_{F_k}$ of the sequence u_m (see Remark 1.3.5). Notice that u_1 is radially symmetric if and only if $g(x,0,0)$ is radially symmetric. Since $\|u_1\|_{F_k} \leq \varepsilon$, we have that $|(u_1(x), \nabla u_1(x))|_{\mathbb{R}\times\mathbb{R}^n} \leq \varepsilon$, for all $x \in \mathbb{R}^n$, and then $u_2 = \mathcal{N}(g(x, u_1, \nabla u_1))$ is radially symmetric provided that u_1 is radially symmetric. By induction, u_m is radially symmetric. Since the convergence in F_k preserves radial symmetry, we conclude that u is radially symmetric. □

More results about symmetry as in item (iii) of Theorem 1.3.8 can be proved by considering orthogonal transformations in the space. Let $\mathcal{G}$ be a subset of the orthogonal matrix group $\mathcal{O}(n)$ of $\mathbb{R}^n$. We say that a function u is symmetric under the action of $\mathcal{G}$ when

$u(x) = u(Tx)$, for all $T \in \mathcal{G}$. Similarly we say that u is antisymmetric under the action of $\mathcal{G}$ when $u(x) = -u(Tx)$, for all $T \in \mathcal{G}$.

Theorem 1.3.9. *Assume the hypotheses of Theorem 1.3.3 and let u be the solution given by it. Let $\mathcal{G}$ be a subset of $\mathcal{O}(n)$ and suppose that by the action of $\mathcal{G}$, the function $g = g(x, z, p)$ satisfies*

(i) g is symmetric in x and p. Then u is symmetric under $\mathcal{G}$;

(ii) g is antisymmetric in p. Then $u \equiv 0$;

(iii) g is antisymmetric in x, even in z (i.e. $g(\cdot, z, \cdot) = g(\cdot, -z, \cdot)$) and symmetric in p. Then u is antisymmetric.

Proof. (i) Given $T \in \mathcal{G}$, we have that $g(Tx, 0, 0) = g(x, 0, 0)$, then

$$
\begin{aligned}
u_1(Tx) &= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|Tx - y|^{n-2}} g(y, 0, 0)\, dy \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - T^{-1}y|^{n-2}} g(y, 0, 0)\, dy \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - z|^{n-2}} g(Tz, 0, 0)\, dz \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - z|^{n-2}} g(z, 0, 0)\, dz = u_1(x)
\end{aligned}
$$

by the change of variables $y = Tz$. Thus, u_1 is symmetric under $\mathcal{G}$.

To prove that u_2 is symmetric, notice that $\nabla u_1(x) = \nabla(u_1(Tx)) = T^\top \cdot \nabla u_1(Tx)$. We compute

$$
\begin{aligned}
u_2(Tx) &= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|Tx - y|^{n-2}} g(y, u_1(y), \nabla u_1(y))\, dy \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - T^{-1}y|^{n-2}} g(y, u_1(y), \nabla u_1(y))\, dy \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - z|^{n-2}} g(Tz, u_1(Tz), \nabla u_1(Tz))\, dz \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - z|^{n-2}} g(Tz, u_1(z), T \cdot \nabla u_1(z))\, dz \\
&= \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{1}{|x - z|^{n-2}} g(z, u_1(z), \nabla u_1(z))\, dz = u_2(x),
\end{aligned}
$$

by the symmetry of g. Then u_2 is symmetric as well. Using an induction argument, we see that u_m is symmetric under $\mathcal{G}$, for all $m \in \mathbb{N}$. Since u is the limit of u_m in the norm of F_k, it preserves the symmetry.

(ii) Since g antisymmetric in p, then $g(x,0,0) = g(x,0,T0) = -g(x,0,0)$ implies $g(.,0,0) \equiv 0$. Therefore, the fixed point of $\mathcal{B}$ is $u \equiv 0$.

(iii) One has $g(Tx,0,0) = -g(x,0,0)$, and the computations above give us $u_1(Tx) = -u_1(x)$. Thus, it follows for u_2

$$\begin{aligned} u_2(Tx) &= \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-2}}g(Tz,u_1(Tz),\nabla u_1(Tz))\,dz \\ &= \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-2}}g(Tz,-u_1(z),T\cdot\nabla u_1(z))\,dz \\ &= -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-2}}g(z,u_1(z),\nabla u_1(z))\,dz = -u_2(x). \end{aligned}$$

By induction one has $u_m(Tx) = -u_m(x)$. Therefore, one concludes that u is antisymmetric. □

It follows from the definition of the space F_k that the solution given by Theorem 1.3.3 satisfies $u = \mathcal{O}((1+|x|)^{-k})$ and $\nabla u = \mathcal{O}((1+|x|)^{-k})$ as $|x| \to \infty$, if $g(x,0,0) = \mathcal{O}((1+|x|)^{-k-2})$. In the next theorem, we improve this behavior by assuming a natural condition, namely if $g(x,0,0) = o((1+|x|)^{-k-2})$ then the solution u and its gradient are $o((1+|x|)^{-k})$ as well.

The following lemma was proved in [44] and it will be necessary for the next theorem.

Lemma 1.3.10. *Let $0 < k < n-2$. If $f \in E_{k+2}$, then*

$$\limsup_{|x|\to\infty} |x|^k |\mathcal{N}(f)(x)| \leqslant L_k \limsup_{|x|\to\infty} |x|^{k+2} |f(x)|\,,$$

where $L_k = \frac{C(n-k-2,1,n)}{(n-2)w_n}$.

Theorem 1.3.11. *Let g be as in Theorem 1.3.3. If $\lim\limits_{|x|\to\infty}(1+|x|)^{k+2}|g(x,0,0)| = 0$, then*

$$\lim_{|x|\to\infty}(1+|x|)^k\left(|u(x)| + |\nabla u(x)|\right) = 0. \tag{1.3.1}$$

Proof. First recall that the solution given by Theorem 1.3.3 satisfies $\|u\|_{F_k} \leq \varepsilon$. Note also that, by the proof of Theorem 1.3.3, if $u \in F_k$, then $g(x,u,\nabla u) \in E_{k+2}$ and therefore $\nabla u = \nabla\mathcal{N}(g(x,u,\nabla u)) \in E_{k+1}$. Thus, one concludes that

$$\limsup_{|x|\to\infty}(1+|x|)^k|\nabla u(x)| = \limsup_{|x|\to\infty}\frac{(1+|x|)^{k+1}|\nabla u(x)|}{1+|x|} \leqslant \limsup_{|x|\to\infty}\frac{\|\nabla u\|_{E_{k+1}}}{1+|x|} = 0.$$

Splitting the expression (1.3.1) into two ones, one only needs to check $\lim\limits_{|x|\to\infty}(1+|x|)^k|u(x)| = 0$. For that matter, one estimates

$$\begin{aligned}
|g(x,u,\nabla u)| &\leqslant |g(x,u,\nabla u) - g(x,0,0)| + |g(x,0,0)| \\
&\leqslant \sup_{0<\|w\|_{F_k}\leq\varepsilon}|\nabla_{(z,p)}g(x,w,\nabla w)||(u,\nabla u)| + |g(x,0,0)|.
\end{aligned}$$

Using the hypotheses, one has

$$\limsup_{|x|\to\infty}|x|^{k+2}|g(x,u,\nabla u)| \leqslant \limsup_{|x|\to\infty}|x|^{k+2}\sup_{0<\|w\|_{F_k}\leq\varepsilon}|\nabla_{(z,p)}g(x,w,\nabla w)||(u,\nabla u)|.$$

By Lemma 1.3.10, one concludes

$$\begin{aligned}
\limsup_{|x|\to\infty}|x|^k|u(x)| &= \limsup_{|x|\to\infty}|x|^k|\mathcal{B}(u)| \\
&= \limsup_{|x|\to\infty}|x|^k|\mathcal{N}(g(x,u,\nabla u))| \\
&\leqslant L_k\limsup_{|x|\to\infty}|x|^{k+2}|g(x,u,\nabla u)| \\
&\leqslant L_k\limsup_{|x|\to\infty}|x|^{k+2}\sup_{0<\|w\|_{F_k}\leq\varepsilon}|\nabla_{(z,p)}g(x,w,\nabla w)||(u,\nabla u)| \\
&\leqslant L_k\sup_{0<\|w\|_{F_k}\leq\varepsilon}\left\|\nabla_{(z,p)}g(.,w,\nabla w)\right\|_{E_2}\limsup_{|x|\to\infty}|x|^k|(u,\nabla u)| \\
&\leqslant L_kG_\varepsilon\limsup_{|x|\to\infty}|x|^k(|u(x)|+|\nabla u(x)|) \\
&\leqslant L_kG_\varepsilon\left(\limsup_{|x|\to\infty}|x|^k|u(x)| + \limsup_{|x|\to\infty}|x|^k|\nabla u(x)|\right) \\
&\leqslant L_kG_\varepsilon\limsup_{|x|\to\infty}|x|^k|u(x)|
\end{aligned}$$

and, since $L_k G_\varepsilon \leqslant C_k G_\varepsilon < \frac{1}{2}$, the result follows.

$\square$

Remark 1.3.12. *It is worthy of note that many results in this chapter remain true in problems where the associated Green's function $G(x,y)$ satisfies the following estimate:*

$$|G(x,y)| \leqslant \frac{C}{|x-y|^{n-2s}},$$

for some constant $C = C(n,s)$ with $1/2 < s \leqslant 1$. This includes cases like bounded domains with Dirichlet's condition and also fractional Laplacians.

Chapter 2

Asymptotic Behavior for the 1D Porous Medium Equation with Fractional Pressure

We analyze the rate of convergence towards the stationary state of solutions to the one dimensional fractional version of the porous medium equation, where the pressure is obtained as the inverse of the fractional Laplacian of the density. Using self-similar variables and the convexity of the interaction potential in dimension one, it is possible to show that the associated entropy is displacement convexity and therefore, satisfies a functional inequality originated from optimal transport theory which involves also the entropy dissipation and the Euclidean transport distance. An argument by approximation on the equation assures that this functional inequality is enough to deduce the exponential convergence of solutions to the unique steady state.

2.1 Introduction

In this chapter, we analyse the long-time asymptotics of the nonlinear nonlocal equation

$$\partial_t \rho = \nabla \cdot \Big(\rho(\nabla(-\Delta)^{-s}\rho + \lambda x)\Big), \qquad \lambda > 0,\ x \in \mathbb{R}^d\,, \tag{2.1.1}$$

obtained from the fractional version of the porous medium equation introduced by Caffarelli and Vázquez [20, 21]

$$\partial_\tau u = \nabla \cdot (u \nabla (-\Delta)^{-s} u)\,, \tag{2.1.2}$$

by passing to self-similar variables. Indeed, by adding the Fokker-Planck confining term $\nabla \cdot (xu)$, solutions to (2.1.1) will characterize the long-time asymptotic behaviour of solutions to (2.1.2). This connection will be further explained below.

The equation (2.1.2) is one of the two fractional variations of the classical porous medium equation

$$\partial_\tau u = \Delta u^m \tag{2.1.3}$$
$$= \nabla \cdot (u \nabla p)\,, \quad p = \frac{m}{m-1} u^{m-1} \tag{2.1.4}$$

for $m > 1$. The first version can be recovered from (2.1.3) by replacing the ordinary Laplacian by its fractional version

$$\partial_\tau u = -(-\Delta)^s u^m$$

and appears, for example, in stochastic equations when jump processes are introduced into the modelling of heat conduction, known as anomalous diffusion, (see [1, 79, 81]). The mathematical theory behind it can be checked in the surveys [76, 77] and references therein.

The other version, which is the one of our interest here, follows from (2.1.4) and can be viewed as a continuity equation, $\partial_\tau u + \nabla \cdot (u\mathbf{v}) = 0$, for a density or concentration $u(\tau, y)$ with velocity $\mathbf{v} = -\nabla p$, where the velocity potential or pressure p is related to u by the inverse of a fractional Laplacian operator $p = \frac{m}{m-1}(-\Delta)^{-s} u^{m-1}$, $0 < s < 1$. The standard porous medium equation is recovered for $s = 0$ and the equation (2.1.2) when $m = 2$. This model appears when one has nonlocal effects as long-range diffusive interactions and it has been studied by an extensive list of authors.

We assume that the unknown $u(\tau, y)$, representing a density or concentration, is defined for $y \in \mathbb{R}^d$ and $\tau > 0$ and supply initial data $u(y, 0) = u_0(y)$, a nonnegative mass distribution in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We also point out that the pressure can be represented as

$$p = (-\Delta)^{-s} u = k_{d,s} * u,$$

with the singular convolution kernel

$$k_{d,s}(y) = c_{d,s}|y|^{2s-d}, \qquad c_{d,s} = \frac{s2^{-2s}\Gamma(d/2-s)}{\pi^{d/2}\Gamma(1+s)}, \tag{2.1.5}$$

and $0 < s < \min(1, d/2)$, called the Riesz potential of u as in the standard textbooks [52, 70]. This representation also makes sense for $s = d/2$ with the logarithm kernel

$$k_{d,\frac{d}{2}}(y) = -2^{1-d}\pi^{-d/2}\Gamma(d/2)^{-1}\log|y|$$

(see [23, 54] in one dimension) and for $1/2 < s < 1$ in one dimension with the negative coefficient $c_{1,s}$ and the positive exponent $2s-1$ in $k_{d,s}(y)$. As a result, the kernel $k_{d,s}(y)$ does not necessarily decay to zero at infinity in the last two cases, but the magnitude of the gradient $\nabla k_{d,s}(y)$ does. When the kernel $k_{d,s}(y)$ is replaced by a less singular radially symmetric function, the same equation appeared in granular flow [12, 74, 56, 25] and biological swarming [61, 14, 13].

To describe the long time behaviour of solutions to (2.1.2), it is more convenient to study the corresponding transformed equation (2.1.1) as discussed in [26, 21], by defining

$$\rho(t,x) := (1+\tau)^{\alpha}u(\tau,y), \tag{2.1.6}$$

with the similarity variables $x = y(1+\tau)^{-\beta}$ and $t = \log(1+\tau)$. The exponents α and β can be determined from dimensional analysis and the mass conservation [11], which are given by

$$\alpha = \frac{d}{d+2-2s}, \quad \beta = \frac{1}{d+2-2s}. \tag{2.1.7}$$

In this way, the rescaled density $\rho(t,x)$ satisfies (2.1.1) with $\lambda = \beta = 1/(d+2-2s)$. We will keep $\lambda > 0$ arbitrary in (2.1.1) as a parameter to characterize the convexity of the energy defined below and the convergence rate to the steady state later on. As a result, the long time behaviour of the original density $u(\tau,y)$ is completely specified if we establish the behavior of $\rho(t,x)$ with $\lambda = \beta$. Furthermore, due to the change of variables $t = \log(1+\tau)$, any convergence of $u(\tau, \,.\,)$ as $t \to \infty$ is going to be slower than the one for $\rho(t, \,.\,)$.

Let us point out that the fractional porous medium equation (2.1.1) can be viewed as a particular case of a much more general aggregation-diffusion equation [25, 14, 9] written as

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \Big(\mathcal{W} * \rho + V + U'(\rho) \Big) \right), \qquad x \in \mathbb{R}^d\,, \tag{2.1.8}$$

where $V, W : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $U : [0, \infty) \to \mathbb{R}$ and we recover (2.1.1) for $V(x) = \frac{\lambda}{2}|x|^2$ and $W(x) = c_{d,s}|x|^{2s-d}$, $0 < s < 1$ and $U = 0$.

During the past fifteen years, several important techniques [63, 26, 37, 25, 78, 6] have been developed for the convergence of linear or nonlinear Fokker-Planck equations to their steady states with sharp rate. These techniques can also be employed to prove the convergence of solutions of (2.1.1) as $t \to \infty$ by realizing that the free energy $\mathcal{E}(\rho)$ defined as

$$\begin{aligned} \mathcal{E}(\rho) &= \frac{1}{2} \int_{\mathbb{R}^d} \Big\{ (-\Delta)^{-s} \rho(x) + \lambda |x|^2 \Big\} \rho(x)\, dx \\ &= \frac{c_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho(x)\rho(y)}{|x-y|^{d-2s}}\, dydx + \lambda \int_{\mathbb{R}^d} \frac{|x|^2}{2} \rho(x)\, dx, \end{aligned} \tag{2.1.9}$$

is a Lyapunov functional for $0 < s < \min(1, d/2)$. One can similarly define the Lyapunov functional for $1/2 \le s < 1$ in one dimension, assuming that ρ satisfies a growth condition at infinity, namely $\rho \log |x| \in L^1(\mathbb{R})$ if $s = 1/2$ and $\rho |x|^{2s-1} \in L^1(\mathbb{R})$ if $1/2 < s < 1$. In fact, (2.1.1) is a gradient flow of the free energy functional (2.1.9) with respect to the Euclidean transport distance in the metric space of probability measures [6, 24].

The basic properties of the energy $\mathcal{E}(\rho)$ and its dissipation $\mathcal{I}(\rho)$ defined below, together with the long-time asymptotics of solutions to (2.1.1), are already derived in [21]. More precisely, along the evolution governed by (2.1.1), one can obtain the formal relation

$$\frac{d}{dt} \mathcal{E}\Big(\rho(t,.)\Big) = -\mathcal{I}\Big(\rho(t,.)\Big), \tag{2.1.10}$$

where we denote by $\mathcal{I}(\rho)$ the entropy production or entropy dissipation of $\mathcal{E}$ given by

$$\mathcal{I}(\rho) = \int_{\mathbb{R}^d} \rho \left| \nabla \xi \right|^2 dx\ , \quad \text{with} \quad \xi = \frac{\delta \mathcal{E}}{\delta \rho} = (-\Delta)^{-s} \rho + \frac{\lambda}{2} |x|^2.$$

Using this relation, the solution of (2.1.1) is shown to converge towards a function ρ_∞ (which coincides with the respective stationary state) in [21], but no rate of convergence is obtained. To be more precise, they show that solutions of the fractional porous medium equation (2.1.1) satisfy the energy inequality

$$\mathcal{E}\Big(\rho(t,\cdot)\Big) + \int_0^t \mathcal{I}\Big(\rho(\tau,\cdot)\Big)d\tau \le \mathcal{E}\Big(\rho(0,\cdot)\Big)$$

that is enough to conclude the converge of $\rho(t,x)$ to the steady state $\rho_\infty(x)$.

Let us now say a few words about the function ρ_∞ of (2.1.1) and its characterizations. The existence and uniqueness of the steady state were initially characterized in [21] by the following obstacle problem for the pressure $P = (-\Delta)^{-s}\rho_\infty$

$$\begin{aligned} P(x) &\geqslant \Phi(x) && \text{for } x \in \mathbb{R}^d\ , \\ (-\Delta)^s P(x) &\geqslant 0 && \text{for } x \in \mathbb{R}^d\ , \\ (-\Delta)^s P(x) &= 0 && \text{for } P(x) > \Phi(x)\ , \end{aligned}$$

where the obstacle is the quadratic function $\Phi(x) = C_* - \frac{\lambda}{2}|x|^2$. The self-similar solution of (2.1.2) were also obtained and given by

$$u(\tau,y) = (1+\tau)^{-d/(d+2-2s)}\rho_\infty\Big(y(1+\tau)^{-1/(d+2-2s)}\Big).$$

The explicit expression for ρ_∞ was then obtained by Biler, Imbert and Karch in [15, 16] for even more general nonlinear dependence of the pressure $p = (-\Delta)^{-s}u^{m-1}$, $m > 1$. In case $m = 2$ of our interest here, they obtained that

$$\rho_\infty(x) = K_{d,s}\Big(R^2 - |x|^2\Big)_+^{1-s} \tag{2.1.11}$$

where

$$K_{d,s} = \frac{2^{2s-1}\Gamma(d/2+1)}{\Gamma(2-s)\Gamma(d/2+1-s)}\lambda\,.$$

The radius R of the support is determined by the conservation of mass, that is,

$$1 = \int_{\mathbb{R}^d} u(\tau,y)dy = \frac{2^{2s}\pi^{d/2}\Gamma(d/2+1)\lambda}{(d+2-2s)\Gamma(d/2+1-s)^2}R^{d+2-2s}. \tag{2.1.12}$$

The expression (2.1.11) allow us to check directly that ρ_∞ is in fact the minimum for the energy $\mathcal{E}$ by a recent result of Chafaï, Gozlan and Zitt in [32, Theorem 1.2] where it was proved that $\mathcal{E}$ restricted to $\mathcal{P}(\mathbb{R}^d)$ is strictly convex in the classic sense for $0 < s < \min(1, d/2)$, and it has a unique compactly supported minimizer ρ_∞ characterized by

$$(-\Delta)^{-s}\rho_\infty(x) + \lambda\frac{|x^2|}{2} = C_* \,, \quad \text{for all } \; x \in \text{supp }(\rho_\infty) \tag{2.1.13a}$$

$$(-\Delta)^{-s}\rho_\infty(x) + \lambda\frac{|x^2|}{2} \geqslant C_* \,, \quad \text{a.e. } \mathbb{R}^d \,, \tag{2.1.13b}$$

for some constant C_* determined by the total mass. This formulation is equivalent to the obstacle problem in [21]. Using the following relation (see [15, 16])

$$\begin{aligned}(-\Delta)^{-s}(R^2 - |x|^2)_+^{1-s} &= \frac{2^{-2s}\Gamma(2-s)\Gamma(d/2-s)}{\Gamma(d/2)}\left(R^2 - \frac{d-2s}{d}|x|^2\right)\\ &= \frac{\lambda}{2K_{d,s}}\left(\frac{d}{d-2s}R^2 - |x|^2\right) \quad , \quad \text{for all } |x| \leqslant R,\end{aligned} \tag{2.1.14}$$

it is easy to verify that $\rho_\infty = K_{d,s}(R^2 - |x|^2)_+^{1-s}$ is indeed the minimizer for $\mathcal{E}$ for $0 < s < \min(1, d/2)$. Similar computations can be done in the range $1/2 \leq s < 1$, see [23, 9] for instance.

In this chapter, we will focus on obtaining the sharp convergence rate for the solutions of the Cauchy problem for (2.1.1) towards the equilibrium ρ_∞, for all $0 < s < 1/2$ in one dimension, although many of the calculations presented also hold in general dimensions. In the particular case of $s = 1/2$ in one dimension, the kernel is given by the logarithmic potential and it was treated in [23], see also [54] for related functional inequalities. In fact, it is shown in [23] that the energy $\mathcal{E}(\rho)$ is displacement convex, which can not be derived directly from the criteria given in the seminal paper by McCann [59]. We will take advantage of these techniques in [23] to prove certain functional inequalities, in particular the HWI inequalities as introduced in [64] (also obtained in [54] for the logarithmic case $s = 1/2$). This displacement convexity and related inequalities are then used to show the convergence towards equilibrium in one dimension, through the exponential decay of the transport distances and the relative energy, for general $s \in (0, 1)$. Roughly speaking, the strategy will be as follow: since we already have the relation (2.1.10) for the solutions of (2.1.1), we will prove that the following generalized log-Sobolev inequality

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \frac{1}{2\lambda}\mathcal{I}(\rho)$$

holds for a class of probability measures ρ. This inequality appears in [21] as an open problem with some of its equivalences. Hence, the result will follow from the Gronwall's inequality since, for any solution $\rho(t,x)$ satisfying an initial condition $\rho(0,.) = \rho_0$, we have

$$\frac{d}{dt}\left(\mathcal{E}(\rho(t,.)) - \mathcal{E}(\rho_\infty)\right) = -\mathcal{I}(\rho(t,.)) \leqslant -2\lambda\left(\mathcal{E}(\rho(t,.)) - \mathcal{E}(\rho_\infty)\right) \ ,$$

and therefore

$$\mathcal{E}(\rho(t,.)) - \mathcal{E}(\rho_\infty) \leqslant e^{-2\lambda t}\left(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\right) \ .$$

This inequality is the main result of this chapter and will be proved in Theorem 2.4.1.

The organization of this chapter is as follows. We first remind the reader in Section 2.2 about the basics of optimal transport theory and we prove two lemmas about the energies $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ that will be used in the later sections. Section 2.3 will be devoted to the functional inequalities that we will prove in one dimension. In fact, in order to obtain our main result we will follow closely the strategy developed for nonlinear diffusion equations in [8, 7, 26, 37, 29, 25] to reduce the prove of the convergence to the proof of a Log-Sobolev type inequality. This inequality is then proved as a consequence of the HWI inequality which crucially uses the displacement convexity of the functionals involved. Finally, on section 2.4 we obtain the rate of convergence towards equilibrium of the solutions to (2.1.1) by an approximation method using the construction of solutions in [20]. This convergence is proved for the energy of solutions and for the spaces L^1 and L^2.

Finally, we point out that the problem of sharp convergence rates in several space dimensions is still open. Moreover, it could be interesting to prove or disprove analogous functional inequalities involving nonlocal operators in several space dimensions corresponding to the ones established here in one dimension; see more comments at the end of Section 2. New techniques or inequalities have to be developed. Showing asymptotic convergence when the confining term $\nabla \cdot (\lambda x \rho)$ is replace by the general drift $\nabla \cdot (\rho \nabla V)$ is another interesting problem, see [32, 25].

2.2 Optimal transport results

We use optimal transport techniques to prove the Log-Sobolev, the Talagrand, and the HWI inequalities for both the energies $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ for smooth probability measures $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$. Therefore, in order to make this chapter more self-contained, this section will be devoted to review some results in optimal transport theory that shall be required later. The section finishes with some simple but new properties about the above mentioned energies.

Firstly, let us define the metric that is going to be used on the probability space. The Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$ is defined for any $\rho_1, \rho_2 \in \mathcal{P}_2(\mathbb{R})$ by

$$W_2(\rho_1, \rho_2) := \left(\inf_{\pi \in \Pi(\rho_1,\rho_2)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y) \right)^{\frac{1}{2}},$$

where $\Pi(\rho_1, \rho_2)$ be the set of all nonnegative Radon measures π on $\mathbb{R}\times\mathbb{R}$ such that

$$\pi(A \times \mathbb{R}) = \rho_1(A) \quad \text{and} \quad \pi(\mathbb{R} \times B) = \rho_2(B), \quad \text{for all } A, B \subseteq \mathbb{R}. \tag{2.2.1}$$

If a measure π satisfies (2.2.1) we say that it has marginals (projections) ρ_1 and ρ_2. The infimum in the definition of W_2 is actually a minimum, and in the case we are dealing with in this chapter, it is unique and can be characterized by the following:

Theorem 2.2.1 (see, for example, [78] for a proof)**.** *Given $\rho_1, \rho_2 \in \mathcal{P}_2(\mathbb{R})$ with ρ_1 absolutely continuous with respect to the Lebesgue measure, there exists a Borel map $T_{\rho_1}^{\rho_2} : \mathbb{R} \to \mathbb{R}$ such that $T_{\rho_1}^{\rho_2} \# \rho_1 = \rho_2$, i.e.,*

$$\int_{\mathbb{R}} \varphi(x) \, d\rho_2(x) = \int_{\mathbb{R}} \varphi(T_{\rho_1}^{\rho_2}(x)) \, d\rho_1(x), \ \textit{for every } \rho_2\textit{-integrable function } \varphi,$$

and $T_{\rho_1}^{\rho_2}$ also satisfies

$$W_2(\rho_1, \rho_2) = \left(\int_{\mathbb{R}} |x - T_{\rho_1}^{\rho_2}(x)|^2 \, d\rho_1(x) \right)^{\frac{1}{2}},$$

It is well known that the optimal map $T_{\rho_1}^{\rho_2}$ is nondecreasing on $\mathbb{R}$ and increasing on supp (ρ_1). Furthermore, if F_1 and F_2 are, respectively, the cumulative distribution functions of ρ_1 and ρ_2, defined by

$$F_i(x) := \int_{-\infty}^{x} d\rho_i = \rho_i\Big((-\infty, x]\Big),$$

then we can write $T_{\rho_1}^{\rho_2}$ in terms of them, that is $T_{\rho_1}^{\rho_2}(x) = F_2^{-1} \circ F_1(x)$.

One can use the previous theorem to define a very convenient set of curves in $\mathcal{P}_{2,ac}(\mathbb{R})$ in the following way: given $\rho_0, \rho_1 \in \mathcal{P}_{2,ac}(\mathbb{R})$ and $T_{\rho_0}^{\rho_1}$ the optimal map such that $T_{\rho_0}^{\rho_1} \# \rho_0 = \rho_1$, let us write

$$\rho_t := \Big((1-t)\mathrm{Id} + t T_{\rho_0}^{\rho_1} \Big) \# \rho_0, \quad \text{for all } \ t \in [0,1].$$

The curve $(\rho_t)_{0\leqslant t\leqslant 1}$ is called displacement interpolation between ρ_0 and ρ_1. It is not hard to see that ρ_t also belongs to $\mathcal{P}_{2,ac}(\mathbb{R})$ for all $t \in [0,1]$, and with this one can define the following:

Definition 2.2.2. *For a functional $\mathcal{F} : \mathcal{P}_{2,ac}(\mathbb{R}) \to (-\infty, +\infty]$, we are going to say that*

- *$\mathcal{F}$ is displacement convex if, given any $\rho_0, \rho_1 \in \mathcal{P}_{2,ac}(\mathbb{R})$ with $(\rho_t)_{0\leqslant t\leqslant 1}$ their displacement interpolation, the function $t \mapsto \mathcal{F}(\rho_t)$ is convex on $[0,1]$.*

- *$\mathcal{F}$ is strictly displacement convex if $t \mapsto \mathcal{F}(\rho_t)$ is strictly convex on $[0,1]$.*

- *$\mathcal{F}$ is γ-displacement convex, for some $\gamma \in \mathbb{R}$, if for all $t \in [0,1]$*

$$(1-t)\mathcal{F}(\rho_0) + t\mathcal{F}(\rho_1) - \mathcal{F}(\rho_t) \geqslant \gamma \frac{t(1-t)}{2} W_2(\rho_0, \rho_1)^2,$$

  or equivalentely

$$\frac{d^2}{dt^2}\mathcal{F}(\rho_t) \geqslant \gamma W_2(\rho_0, \rho_1)^2.$$

Some of the functionals we are going to use in the rest of the chapter are combinations of the following ones:

$$\mathcal{U}(\rho) = \int_{\mathbb{R}} \rho(x) \log \rho(x)\ dx\ , \tag{2.2.2}$$

$$\mathcal{V}(\rho) = \frac{1}{2}\int_{\mathbb{R}} x^2\ d\rho(x) \ \text{ and} \tag{2.2.3}$$

$$\mathcal{W}_s(\rho) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{c_{1,s}}{|x-y|^{1-2s}}\ d\rho(x)\ d\rho(y)\ , \tag{2.2.4}$$

where we are committing an abuse of notation by identifying every absolutely continuous measure with its density. So we shall write $d\rho(x)$ and $\rho(x)dx$ meaning the same thing throughout the rest of the chapter. Therefore, we can write the energy functional $\mathcal{E}$ as

$$\mathcal{E}(\rho) = \lambda\mathcal{V}(\rho) + \mathcal{W}_s(\rho)$$

and besides this one, we are also going to need the following approximate functional: for every $\varepsilon > 0$ and every measure $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$ we define:

$$\mathcal{E}_\varepsilon(\rho) := \mathcal{E}(\rho) + \varepsilon\, \mathcal{U}(\rho)\ ,$$

and associated with it we have a respective energy dissipation

$$\mathcal{I}_\varepsilon(\rho) := \int_{\mathbb{R}} \left|\partial_x(-\partial_{xx})^{-s}\rho(x) + \lambda x + \varepsilon\partial_x \log\rho(x)\right|^2 d\rho(x) \ .$$

These two functionals are associated to the regularized equation (2.4.2) in the next section.

The metric W_2 on $\mathcal{P}_2(\mathbb{R})$ is related with a very known notion of convergence. We say that the a sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_2(\mathbb{R})$ is narrowly convergent to $\rho \in \mathcal{P}(\mathbb{R})$ (denoted by $\rho_n \rightharpoonup \rho$) if

$$\lim_{n\to\infty} \int \varphi(x)\ d\rho_n(x) = \int \varphi(x)\ d\rho(x)\ , \tag{2.2.5}$$

for all $\varphi \in C_b(\mathbb{R})$, the space of bounded and continuous functions.

With this definition in hand, we can characterize the notion of convergence in $\mathcal{P}_2(\mathbb{R})$:

Theorem 2.2.3. *The pair $(\mathcal{P}_2(\mathbb{R}), W_2)$ is a complete metric space and the convergence under the distance W_2 is stronger than the convergence in the narrow sense. In fact, the following three facts are equivalent for any $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_2(\mathbb{R})$ and $\rho \in \mathcal{P}(\mathbb{R})$:*

- $W_2(\rho_n, \rho) \to 0$ *as* $n \to +\infty$;
- $\rho_n \rightharpoonup \rho$ *and*
$$\lim_{n\to\infty} \int x^2\ d\rho_n(x) = \int x^2\ d\rho(x);$$
- $\rho_n \rightharpoonup \rho$ *and*
$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} x^2\ d\rho_n(x) = 0. \tag{2.2.6}$$

The convergence in the narrow sense can be characterized using the notion of tightness of a set of measures. We say that a set $S \subseteq \mathcal{P}(\mathbb{R})$ is tight if the following condition holds:

$$\text{for all } \varepsilon > 0 \quad \text{there exists a compact } K_\varepsilon \subseteq \mathbb{R} \ \text{ such that } \sup_{\mu\in S} \mu(\mathbb{R}\backslash K_\varepsilon) \leqslant \varepsilon. \tag{2.2.7}$$

The condition (2.2.7) is equivalent to the following integral and easier-to-verify condition: there exists $\varphi : \mathbb{R} \to [0, \infty]$ such that

$$\text{for all } c > 0,\ \ \{x \in \mathbb{R} \mid \varphi(x) \leqslant c\} \ \text{ is compact and } \sup_{\mu\in S} \int_{\mathbb{R}} \varphi(x)\ d\mu(x) < \infty \tag{2.2.8}$$

Now we can state the following Euclidean version of the Prokhorov's compactness theorem:

Theorem 2.2.4 (Prokhorov)**.** *A set $S \subseteq \mathcal{P}(\mathbb{R})$ is tight if and only if it is relatively compact (under the narrow convergence).*

For a detailed proof of the above theorems and generalizations, the reader may check the standard references [6], [78] (for the results involving the Wasserstein distance) and [38] (for a proof of the Prokhorov's Theorem).

Let us now prove a lemma that shall be used in the last section for the convergence in entropy of the solutions of the approximate problems. The proof uses similar arguments given in the Theorem 1.4 of [69].

Lemma 2.2.5. *The entropy $\mathcal{E}_\varepsilon$ is narrow lower semicontinuous and λ-displacement convex, for all $\varepsilon, \lambda > 0$. In the case $\lambda = 0$, $\mathcal{E}_\varepsilon$ is strictly displacement convex.*

Proof. We already know from [59] that the functional $\mathcal{U}$ is narrow lower semicontinuous and also strictly displacement convex, so we just need to show the result for $\mathcal{E}$. For this, firstly let us show the semicontinuity by writing the energy in the following way:

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} F(x,y)\, d\rho(x)d\rho(y),$$

where

$$F(x,y) = \begin{cases} \dfrac{\lambda}{4}(x^2+y^2) + \dfrac{c_{1,s}}{2}\dfrac{1}{|x-y|^{1-2s}}\,, & \text{if } x \neq y \\ +\infty \,, & \text{if } x = y \end{cases}.$$

Since F is non-negative and smooth outside the diagonal $x = y$, we can find a sequence $\{F_k\}_{k\in\mathbb{N}} \subset C_b(\mathbb{R}^2)$ such that $F_k(x,y) \nearrow F(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Therefore, if $\{\rho_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_{2,ac}(\mathbb{R})$ is such $\rho_n \rightharpoonup \rho$ for some $\rho \in \mathcal{P}(\mathbb{R})$, by the monotone convergence theorem and the fact that $\rho_n \times \rho_n \rightharpoonup \rho \times \rho$, we have that

$$\begin{aligned}\mathcal{E}(\rho) &= \int F(x,y)\, d\rho(x)d\rho(y) = \lim_{k\to\infty}\int F_k(x,y)\, d\rho(x)d\rho(y) \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int F_k(x,y)\, d\rho_n(x)d\rho_n(y) \leqslant \liminf_{n\to\infty}\int F(x,y)\, d\rho_n(x)d\rho_n(y) \\ &= \liminf_{n\to\infty}\mathcal{E}(\rho_n)\,.\end{aligned}$$

Now, for the convexity property, we also know from [59] or [78] that the functional $\lambda\mathcal{V}(\rho)$ is λ-displacement convex. Therefore we just need to show that $\mathcal{W}$ is displacement convex. For this, let $\rho_0, \rho_1 \in \mathcal{P}_{2,ac}(\mathbb{R})$ and, to simplify the notation, $\theta := T_{\rho_0}^{\rho_1}$ the optimal map given by (2.2.1) such that $\theta\#\rho_0 = \rho_1$ and let $(\rho_t)_{0\leqslant t\leqslant 1}$ be their displacement interpolant. To simplify the notation, let us call

$$k_s(x) = \frac{c_{1,s}}{|x|^{1-2s}}$$

the kernel of the Riesz potential. Then we have the following for each $t \in [0,1]$

$$\begin{aligned}\mathcal{W}(\rho_t) &= \frac{1}{2}\int\int k_s(x-y)\, d\rho_t(x)d\rho_t(y)\\ &= \frac{1}{2}\int\int k_s((1-t)(x-y)+t(\theta(x)-\theta(y)))\, d\rho_0(x)d\rho_0(y).\end{aligned} \tag{2.2.9}$$

Also by (2.2.1) we know that θ is nondecreasing on $\mathbb{R}$ and increasing outside a set N such that $\rho_0(N) = 0$. Therefore, for all $x, y \in N^c$ with $x \neq y$ we have that $x - y$ and $\theta(x) - \theta(y)$ are both either on the negative or on the positive semi-line, where we can use the convexity of k_s and write

$$k_s((1-t)(x-y)+t(\theta(x)-\theta(y))) \leqslant (1-t)k_s(x-y)+tk_s(\theta(x)-\theta(y)). \tag{2.2.10}$$

The only care we need to take is to avoid the singularity of the function k_s by proving that

$$(1-t)(x-y)+t(\theta(x)-\theta(y)) \neq 0,$$

for all $t \in [0,1]$ and forall $x, y \in N^c$ with $x \neq y$. For this, let us suppose that there exist $t_* \in (0,1]$ and $x_*, y_* \in N^c$ with $x_* \neq y_*$ such that

$$(1-t_*)(x_*-y_*)+t_*(\theta(x_*)-\theta(y_*)) = 0, \tag{2.2.11}$$

then we obtain that

$$\frac{\theta(x_*)-\theta(y_*)}{x_*-y_*} = -\frac{1-t_*}{t_*} \leqslant 0,$$

which contradicts the strict monotonicity of θ. If $t_* = 0$ then the only way to have (2.2.11) is if $x_* = y_*$. Finally, from (2.2.9) and (2.2.10) we can deduce

$$\begin{aligned}\mathcal{W}(\rho_t) &\leqslant \frac{(1-t)}{2}\int\int k_s(x-y)\ d\rho_0(x)d\rho_0(y) + \frac{t}{2}\int\int k_s(\theta(x)-\theta(y))\ d\rho_0(x)d\rho_0(y)\\ &= (1-t)\mathcal{W}(\rho_0) + t\mathcal{W}(\rho_1)\end{aligned}$$

which gives the displacement convexity of the functional $\mathcal{W}$. □

The next lemma shows that the functional $\mathcal{E}_\varepsilon$ also admits a unique minimum. This result is going to allow us to obtain the same desired asymptotic behavior for the solutions of the approximate problem.

Lemma 2.2.6. *Let $\varepsilon, \lambda > 0$ and $\varpi_\varepsilon := \inf\{\mathcal{E}_\varepsilon(\rho) \mid \rho \in \mathcal{P}_2(\mathbb{R})\}$. Then:*

1) ϖ_ε is finite;

2) There exists a unique $\rho^\varepsilon_\infty \in \mathcal{P}_{2,ac}(\mathbb{R})$ such that $\mathcal{E}_\varepsilon(\rho^\varepsilon_\infty) = \varpi_\varepsilon$.

Proof. Note that for every $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$ we have the following estimate:

$$\begin{aligned}\varepsilon\mathcal{U}(\rho) + \frac{\lambda}{2}\mathcal{V}(\rho) &= \varepsilon\int \rho(x)\log\rho(x)\ dx + \frac{\lambda}{4}\int x^2\rho(x)\ dx\\ &= \varepsilon\int\rho(x)\left\{\log\rho(x) + \frac{\lambda}{4\varepsilon}x^2\right\}\ dx\\ &= \varepsilon\int\rho(x)\log\left(\frac{\rho(x)}{e^{-\frac{\lambda}{4\varepsilon}x^2}}\right)\ dx\\ &= \varepsilon M\int\frac{\rho(x)}{e^{-\frac{\lambda}{4\varepsilon}x^2}}\log\left(\frac{\rho(x)}{e^{-\frac{\lambda}{4\varepsilon}x^2}}\right)\frac{e^{-\frac{\lambda}{4\varepsilon}x^2}}{M}\ dx,\end{aligned} \tag{2.2.12}$$

where $M := \int e^{-\frac{\lambda}{4\varepsilon}x^2}\ dx = \sqrt{\frac{4\pi\varepsilon}{\lambda}}$. Therefore, since $s \mapsto s\log s$ in convex, we can use the Jensen's Inequality on the last expression and obtain

$$\begin{aligned}\varepsilon M\int\frac{\rho(x)}{e^{-\frac{\lambda}{4\varepsilon}x^2}}\log\left(\frac{\rho(x)}{e^{-\frac{\lambda}{4\varepsilon}x^2}}\right)\frac{e^{-\frac{\lambda}{4\varepsilon}x^2}}{M}\ dx &\geqslant \varepsilon M\left(\int\frac{\rho(x)}{M}dx\right)\log\left(\int\frac{\rho(x)}{M}\ dx\right)\\ &= \varepsilon\log\frac{1}{M} = \varepsilon\log\frac{\lambda}{4\pi\varepsilon}.\end{aligned} \tag{2.2.13}$$

In particular, this also implies that for all $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$:

$$\mathcal{E}_\varepsilon(\rho) > \varepsilon \mathcal{U}(\rho) + \frac{\lambda}{2}\mathcal{V}(\rho) \geqslant \varepsilon \log \frac{\lambda}{2\pi\varepsilon} > \varepsilon \log \frac{\lambda}{4\pi\varepsilon};,$$

which proves the item *(i)*.

Now, let $\{\rho_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_{2,ac}(\mathbb{R})$ such that $\mathcal{E}_\varepsilon(\rho_n) \leqslant \varpi + \frac{1}{n}$. By the inequalities (2.2.12) and (2.2.13) above, we have the following estimate for the second moments

$$\begin{aligned}
0 &\leqslant \frac{\lambda}{4}\int x^2\rho_n(x)\,dx \\
&= \frac{\lambda}{2}\mathcal{V}(\rho_n) \\
&\leqslant \varepsilon\mathcal{U}(\rho_n) + \frac{\lambda}{2}\mathcal{V}(\rho_n) - \varepsilon\log\frac{\lambda}{4\pi\varepsilon} + \frac{\lambda}{2}\mathcal{V}(\rho_n) \\
&< \mathcal{E}_\varepsilon(\rho_n) - \varepsilon\log\frac{\lambda}{4\pi\varepsilon} \\
&\leqslant \varpi + 1 - \varepsilon\log\frac{\lambda}{4\pi\varepsilon}.
\end{aligned}$$

Therefore we have that the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ satisfies

$$\sup_{n\in\mathbb{N}} \int x^2\rho_n(x)\,dx < \infty$$

and, for the condition (2.2.8) together with the Prokhorov's Theorem 2.2.4, we can extract a subsequence, still denoted by $\{\rho_n\}_{n\in\mathbb{N}}$, such that $\rho_n \rightharpoonup \rho_*$, for some $\rho_* \in \mathcal{P}(\mathbb{R})$. It is easy to check that $\rho_* \in \mathcal{P}_2(\mathbb{R})$ for, by the Lemma 2.2.5, we know that

$$\mathcal{E}_\varepsilon(\rho_*) \leqslant \liminf_{n\in\mathbb{N}} \mathcal{E}_\varepsilon(\rho_n) = \varpi < \infty,$$

which means that $\rho_* \in D(\mathcal{E}_\varepsilon) \subseteq \mathcal{P}_2(\mathbb{R})$ and that ρ_* is a minimum for $\mathcal{E}_\varepsilon$.

The uniqueness of the ground state follows from the Lemma 2.2.5 for, if ρ_*^1 and ρ_*^2 are two different minimums in $\mathcal{P}_{2,ac}(\mathbb{R})$ to $\mathcal{E}_\varepsilon$ we can define $\rho_{\frac{1}{2}} = (\frac{1}{2}Id + \frac{1}{2}\theta)\#\rho_*^1$, where $\theta := T_{\rho_*^1}^{\rho_*^2}$, the optimal transport map $T_{\rho_*^1}^{\rho_*^2}\#\rho_*^1 = \rho_*^2$. Therefore, by the λ-displacement convexity of $\mathcal{E}_\varepsilon$ we have

$$\mathcal{E}_\varepsilon(\rho_{\frac{1}{2}}) \leqslant \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^1) + \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^2) - \frac{\lambda}{2}\left(\frac{1}{2}\right)^2 W_2(\rho_*^1, \rho_*^2)^2$$

$$< \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^1) + \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^2) = \varpi$$

which contradicts the definition of ϖ. If $\lambda = 0$, we also know from the Lemma 2.2.5 that $\mathcal{E}_\varepsilon$ is strictly displacement convex, and in this case we also obtain

$$\mathcal{E}_\varepsilon(\rho_{\frac{1}{2}}) < \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^1) + \frac{1}{2}\mathcal{E}_\varepsilon(\rho_*^2) = \varpi.$$

Therefore, there exists a unique ground state for $\mathcal{E}_\varepsilon$. □

2.3 Transport inequalities in dimension 1

We know from Section 2.1 that $\mathcal{E}(\rho_\infty)$ is the minimum value for the energy $\mathcal{E}$ and therefore, we can use the difference $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ as a measure of distance between any $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$ and the ground state ρ_∞. So in this section, we are going to derive several inequalities originated from optimal transportation theory that will be used in the next section to show the exponential convergence of $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ to zero in dimension one.

We also know from the relation (2.1.10) that, once we have the following inequality for a sufficiently large class of functions

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \frac{1}{2\lambda}\mathcal{I}(\rho), \tag{2.3.1}$$

we can prove the exponential convergence of $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ to zero with exponential rate -2λ (but not necessarily the exponential convergence of $\mathcal{I}(\rho)$).

The inequality (2.3.1) is usually called, in the context of optimal transport, Log-Sobolev inequality in the linear diffusion case or generalized Log-Sobolev inequalities otherwise. We will revisit (2.3.1) in the next section by investigating the displacement convexity of the energy $\mathcal{E}(\rho)$. In particular, it becomes the logarithmic Sobolev inequality [46] for linear Fokker-Planck equation [7, 28, 73], and a special family of Gagliardo-Nirenberg inequalities for nonlinear Fokker-Planck equations with porous medium type diffusion [37, 26, 29].

Thus for the rest of this section, we shall prove a generalization of (2.3.1) and use it in the following section to obtain the desired decay for $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$.

Now, let us begin with the following technical lemma about the derivative of the Riesz potential.

Lemma 2.3.1. *Let* $0 < s \leqslant 1$ *and* $\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ *with* $\alpha > \max(1-2s, 0)$. *Then* $(-\partial_{xx})^{-s}\rho \in C^1(\mathbb{R})$ *and for any* $x \in \mathbb{R}$,

$$\partial_x(-\partial_{xx})^{-s}\rho(x) = -c_{1,s}(1-2s)\int_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}}\Big(\rho(y)-\rho(x)\Big)\,dy\ , \quad \textit{if } s \in (0,1/2)\ ,$$

$$\partial_x(-\partial_{xx})^{-s}\rho(x) = -c_{1,\frac{1}{2}}\int_{\mathbb{R}} \frac{x-y}{|x-y|^{2}}\Big(\rho(y)-\rho(x)\Big)\,dy\ , \quad \textit{if } s = \frac{1}{2}\ ,$$

or

$$\partial_x(-\partial_{xx})^{-s}\rho(x) = -c_{1,s}(1-2s)\int_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}}\rho(y)\,dy\ , \quad \textit{if } s \in (1/2,1].$$

Proof. Firstly, let us assume that $s \in (0,1/2)$. To simplify the notation, we write $k_s(x) := c_{1,s}|x|^{2s-1}$. Hence, we note that under the hypothesis on ρ, we have that

$$\mathbf{u}_s(x) := -c_{1,s}(1-2s)\int_{\mathbb{R}} \frac{(x-y)}{|x-y|^{3-2s}}\Big(\rho(y)-\rho(x)\Big)\,dy = k_s' * (\rho - \rho(x))$$

is well defined for all $x \in \mathbb{R}$.

Now, let $\eta \in C^1(\mathbb{R})$ be a radial function such that $0 \leqslant \eta \leqslant 1$, $\eta(x) = 0$ if $|x| \leqslant 1$, $\eta(x) = 1$ if $|x| \geqslant 2$ and $|\eta'| \leqslant 2$. Define $\eta_\varepsilon(x) := \eta(\varepsilon^{-1}x)$ and

$$\begin{aligned} p(x) &:= (-\partial_{xx})^{-s}\rho(x) = k_s * \rho(x)\ , \\ p_\varepsilon(x) &:= (k_s\eta_\varepsilon) * \rho(x). \end{aligned}$$

Since ρ is bounded, we have that $p_\varepsilon \to p$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$ for

$$\begin{aligned} |p(x) - p_\varepsilon(x)| &\leqslant \int_{|x-y|\leqslant 2\varepsilon} k_s(x-y)\Big(1-\eta_\varepsilon(x-y)\Big)\rho(y)\,dy \\ &\leqslant \|\rho\|_\infty \int_{|y|\leqslant 2\varepsilon} \frac{1}{|y|^{1-2s}}\,dy = C\|\rho\|_\infty\varepsilon^{2s} \end{aligned}$$

for all $x \in \mathbb{R}$, where C depends on s.

By the smoothness of $k_s\eta_\varepsilon$ we know that $p_\varepsilon \in C^1$ and $p_\varepsilon'(x) = (k_s\eta_\varepsilon)' * \rho(x)$, and since $k_s\eta_\varepsilon$ is radial, we can write

$$p_\varepsilon'(x) = \int_{\mathbb{R}} (k_s\eta_\varepsilon)'(x-y)\Big(\rho(y)-\rho(x)\Big)\,dy\,.$$

Therefore,

$$
\begin{aligned}
|\mathbf{u}_s(x) - p'_\varepsilon(x)| &= \left| \int_{|x-y|\leqslant 2\varepsilon} (k_s(1-\eta_\varepsilon))'(x-y)\Big(\rho(y)-\rho(x)\Big)\, dy \right| \\
&\leqslant \int_{|x-y|\leqslant 2\varepsilon} \Big(|k'_s(x-y)||1-\eta_\varepsilon(x-y)| + k_s(x-y)|\eta'_\varepsilon(x-y)| \Big) \Big|\rho(y)-\rho(x)\Big|\, dy \\
&\leqslant \int_{|x-y|\leqslant 2\varepsilon} \Big(\frac{c_{1,s}(1-2s)}{|x-y|^{2-2s}} + \frac{2}{\varepsilon}\frac{c_{1,s}}{|x-y|^{1-2s}} \Big) \Big|\rho(y)-\rho(x)\Big|\, dy \qquad (2.3.2) \\
&\leqslant C \int_{|x-y|\leqslant 2\varepsilon} \left(\frac{1}{|x-y|^{2-2s-\alpha}} + \frac{1}{\varepsilon}\frac{1}{|x-y|^{1-2s-\alpha}} \right)\, dy \\
&\leqslant C_1 \varepsilon^{\alpha+2s-1},
\end{aligned}
$$

where the constant C_1 only depends on s, α and on the Hölder constant of ρ. Thus, we also have that p'_ε converges uniformly to $\mathbf{u}_s$ as $\varepsilon \to 0$, and therefore $p' = \mathbf{u}_s$.

Now, if $s \in (1/2, 1]$, we only need to adapt the argument in formula (2.3.2) for the function

$$\mathbf{u}_s(x) := -c_{1,s}(1-2s) \int_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}} \rho(y)\, dy = k'_s * \rho$$

and using that $p'_\varepsilon = (k_s\eta_\varepsilon)' * \rho$ in the following way

$$
\begin{aligned}
|\mathbf{u}_s(x) - p'_\varepsilon(x)| &= C\, \|\rho\|_\infty \int_{|x-y|\leqslant 2\varepsilon} \left(\frac{1}{|x-y|^{2-2s}} + \frac{1}{\varepsilon}\frac{1}{|x-y|^{1-2s}} \right)\, dy \\
&= C_2 \varepsilon^{2s-1},
\end{aligned}
$$

where the constant C_2 only depends on s and on the L^∞ norm of ρ.

Finally, if $s = 1/2$ we have that

$$(-\partial_{xx})^{-\frac{1}{2}} \rho(x) = c_{1,\frac{1}{2}} \int_{\mathbb{R}} \log|x-y| \rho(y)\, dy$$

and

$$\mathbf{u}_{\frac{1}{2}}(x) = -c_{1,\frac{1}{2}} \int_{\mathbb{R}} \frac{(x-y)}{|x-y|^2} \Big(\rho(y)-\rho(x)\Big)\, dy.$$

Arguing as above for $k_{\frac{1}{2}}(x) := c_{1,\frac{1}{2}} \log|x|$ we arrive at the following estimates:

$$|p(x) - p_\varepsilon(x)| \leqslant \|\rho\|_\infty \int_{|y|\leqslant 2\varepsilon} \Big|\log|y|\Big| dy = C\|\rho\|_\infty \varepsilon\Big(\Big|\log 2\varepsilon\Big| + 1\Big)$$

and

$$
\begin{aligned}
|\mathbf{u}_{\frac{1}{2}}(x) - p_\varepsilon'(x)| &\leqslant C \int_{|x-y|\leqslant 2\varepsilon} \left(\frac{1}{|x-y|} + \frac{1}{\varepsilon}\Big| \log |x-y| \Big| \right) \Big| \rho(y) - \rho(x) \Big| \, dy \\
&\leqslant C \int_{|x-y|\leqslant 2\varepsilon} \left(\frac{1}{|x-y|^{1-\alpha}} + \frac{1}{\varepsilon}|x-y|^{\alpha}\Big| \log |x-y| \Big| \right) dy \\
&\leqslant C\varepsilon^{\alpha}\Big(1 + \varepsilon + \varepsilon \Big| \log 2\varepsilon \Big| \Big).
\end{aligned}
$$

Therefore, since all these estimates are uniform in x, we conclude that the lemma is true for all $s \in (0, 1]$. □

Remark 2.3.2. With this expression for the derivative of $(-\partial_{xx})^{-s}\rho$ for $s < \frac{1}{2}$, we obtain the following equality that shall be used in the next proposition:

$$
\begin{aligned}
\frac{\partial_x(-\partial_{xx})^{-s}\rho(x)}{c_{1,s}(2s-1)} &= \lim_{r\to 0} \int_{|x-y|\geqslant r} \frac{x-y}{|x-y|^{3-2s}} \Big(\rho(y) - \rho(x) \Big) \, dy \\
&= \lim_{r\to 0} \int_{|x-y|\geqslant r} \frac{x-y}{|x-y|^{3-2s}} \rho(y) \, dy - \lim_{r\to 0} \rho(x) \int_{|x-y|\geqslant r} \frac{x-y}{|x-y|^{3-2s}} \, dy \\
&= \lim_{r\to 0} \int_{|x-y|\geqslant r} \frac{x-y}{|x-y|^{3-2s}} \rho(y) \, dy,
\end{aligned}
$$

where we only used the fact that k_s is radial and k_s' is integrable at the infinity. For $s > \frac{1}{2}$, the expression is valid without taking the limit, as the kernel is locally integrable.

The generalization of (2.3.1) that we are going to show is the so called HWI inequality which is called so because it was first established in [64] and it relates the relative Kullback information (denoted by H), the Wasserstein distance W_2 and the relative Fisher information (also denoted by I).

The next theorems show that the HWI inequality holds for $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ at least for a class of bounded and Hölder continuous functions on $\mathbb{R}$. The proof follows the arguments given in [54] where the same inequality is proved for the case of the logarithmic interaction ($s = 1/2$) and strongly relies on the fact that the optimal transport map with respect to the Wasserstein distance is a monotone nondecreasing function on $\mathbb{R}$. We point out that the convexity of the confinement due to the drift measured by $\lambda > 0$ appears explicitly in the inequalities as in [25].

Theorem 2.3.3. *Let $s \in (0, 1]$, $\lambda \in \mathbb{R}$ and $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$ such that its density (also denoted* *by ρ) satisfies $\rho \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ with $\alpha > \max(1-2s, 0)$. Then, if ρ_∞ the minimum point of $\mathcal{E}$ on $\mathcal{P}_2(\mathbb{R})$, we have*

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \sqrt{\mathcal{I}(\rho)} W_2(\rho, \rho_\infty) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty).$$

Proof. For $s = 1/2$ this result was proven at [54]. So, let us suppose that $s \in (0, 1/2)$ and, to simplify, let us denote $K\rho(x) = \partial_x(-\partial_{xx})^{-s}\rho(x)$. Since ρ is absolutely continuous with respect to the Lebesgue measure, by Theorem 2.2.1 there exists an nondecreasing transport map $T_\rho^{\rho_\infty}$ such that $T_\rho^{\rho_\infty} \# \rho = \rho_\infty$. In order to simplify the notation let us call $\theta := T_\rho^{\rho_\infty}$.

Then, we can write

$$\sqrt{\mathcal{I}(\rho)} W_2(\rho, \rho_\infty) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty) - \mathcal{E}(\rho) + \mathcal{E}(\rho_\infty) = T_1 + T_2 + T_3 \ ,$$

where

$$\begin{aligned} T_1 := & \left(\int \Big| K\rho(x) + \lambda x \Big|^2 d\rho(x) \right)^{1/2} \left(\int |x - \theta(x)|^2 d\rho(x) \right)^{1/2} \\ & - \int \Big(K\rho(x) + \lambda x \Big) \left(x - \theta(x) \right) d\rho(x) \end{aligned}$$

$$T_2 := \int \left\{ \lambda x (x - \theta(x)) - \frac{\lambda}{2} x^2 + \frac{\lambda}{2} \theta(x)^2 - \frac{\lambda}{2} |x - \theta(x)|^2 \right\} d\rho(x)$$

$$T_3 := \frac{c_{1,s}}{2} \int \frac{d\rho(x) d\rho(y)}{|\theta(x) - \theta(y)|^{1-2s}} - \frac{c_{1,s}}{2} \int \frac{d\rho(x) d\rho(y)}{|x - y|^{1-2s}} - \int K\rho(x) (\theta(x) - x) d\rho(x) \, ,$$

where we added and subtracted several terms. This allows us to show that $T_1 \geqslant 0$ by the Cauchy-Schwarz inequality and $T_2 = 0$ for all $\lambda \in \mathbb{R}$. Now, for T_3 let us call $k_s(x) = c_{1,s} |x|^{2s-1}$. Then, by the Remark 2.3.2

$$K\rho(x) = \lim_{r \to 0} \int_{|y - x| \geqslant r} k_s'(x - y) d\rho(y).$$

And, since $k_s'(x) = -k_s'(-x)$, we can write

$$\begin{aligned} & \int K\rho(x) \Big(\theta(x) - x \Big) d\rho(x) \\ & \qquad = \lim_{r \to 0} \int_{|y - x| \geqslant r} \Big(\theta(x) - x \Big) k_s' \Big(x - y \Big) d\rho(y) d\rho(x) \end{aligned}$$

$$= \frac{1}{2} \lim_{r \to 0} \int_{|y-x| \geqslant r} \Big(\theta(x) - \theta(y) - x + y\Big) k_s'\Big(x - y\Big) d\rho(y) d\rho(x).$$

Furthermore,

$$c_{1,s} \int \frac{d\rho(x)d\rho(y)}{|x-y|^{1-2s}} = \lim_{r\to 0} \int_{|y-x|\geqslant r} k_s(x-y) d\rho(x) d\rho(y)$$

$$c_{1,s} \int \frac{d\rho(x)d\rho(y)}{|\theta(x)-\theta(y)|^{1-2s}} = \lim_{r\to 0} \int_{|y-x|\geqslant r} k_s(\theta(x)-\theta(y)) d\rho(x) d\rho(y)$$

and then,

$$T_3 = \lim_{r\to 0} \frac{1}{2} \int \Big\{ k_s\Big(\theta(x)-\theta(y)\Big) - k_s(x-y) - k_s'\Big(\theta(x)-\theta(y)\Big)\Big(\theta(x)-\theta(y)-x+y\Big)\Big\} d\rho(x)d\rho(y).$$

The integrand is nonnegative by the convexity of k_s on the positive real line and by the monotonicity of θ, so $T_3 \geqslant 0$ as well.

If $s \in (1/2, 1]$, we still have $k_s(x) = c_{1,s}|x|^{2s-1}$ convex because $c_{1,s}$ is negative in this range. Thus, the previous computations still apply. □

Remarks. 1) It is known that, if the HWI inequality holds for some $\lambda > 0$, then the Log-Sobolev inequality also holds. One just needs to maximize the right-hand side for $W_2 \geqslant 0$ or use the Young's inequality for $(\lambda^{-\frac{1}{2}}\sqrt{\mathcal{I}})(\lambda^{\frac{1}{2}}W_2)$. Then we have that

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leqslant \frac{1}{2\lambda}\mathcal{I}(\rho), \tag{2.3.3}$$

for all ρ satisfying the assumptions of the theorem above.

2) Note that in the proof of the Theorem 2.3.3 we did not use the fact that ρ_∞ is the minimum of $\mathcal{E}$, only the fact that $\mathcal{E}(\rho_\infty) < \infty$. In fact, the same inequality holds for any ρ_0 in the place of ρ_∞ as long as $\rho_0 \in D(\mathcal{E})$, and also with ρ_∞ in the place of ρ, since ρ_∞ is absolutely continuous with respect to the Lebesgue measure, which allows the existence of the map θ by the Theorem 2.2.1 from page 37. Therefore, if we exchange ρ and ρ_∞ in the HWI we obtain the fractional version of the so called Talagrand inequality or transportation cost inequality

$$W_2(\rho, \rho_\infty) \leqslant \sqrt{\frac{2}{\lambda}\Big(\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)\Big)}. \tag{2.3.4}$$

We can derive similar results for the ε functionals.

Theorem 2.3.4. *Let $s \in (0, 1]$, $\lambda > 0$, $0 < \varepsilon < \lambda/2\pi$, $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$ such that its density (also* *denoted by ρ) satisfies $\rho \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ with $\alpha > \max(1-2s, 0)$, and ρ_∞^ε the minimum point of $\mathcal{E}_\varepsilon$ on $\mathcal{P}_2(\mathbb{R})$. Then*

$$\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \leqslant \sqrt{\mathcal{I}_\varepsilon(\rho)} W_2(\rho, \rho_\infty^\varepsilon) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty^\varepsilon).$$

Proof. The proof is basically the same, but since we have a new term inside the respective diffusion, we shall include it for completeness.

As in the previous theorem, let $K\rho(x) = \partial_x(-\partial_{xx})^{-s}\rho(x)$ and θ be such that $\theta\#\rho = \rho_\infty^\varepsilon$. Then, we decompose the inequality as

$$\sqrt{\mathcal{I}_\varepsilon(\rho)} W_2(\rho, \rho_\infty^\varepsilon) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty^\varepsilon) - \mathcal{E}_\varepsilon(\rho) + \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) = T_1 + T_2 + T_3; ,$$

where

$$\begin{aligned} T_1 := & \left(\int \left| K\rho(x) + \lambda x + \varepsilon \partial_x \log \rho(x) \right|^2 d\rho(x) \right)^{1/2} \left(\int |x - \theta(x)|^2 d\rho(x) \right)^{1/2} \\ & - \int \Big(K\rho(x) + \lambda x + \varepsilon \partial_x \log \rho(x) \Big) \left(x - \theta(x) \right) d\rho(x); , \end{aligned}$$

$$\begin{aligned} T_2 := & - \int \Big(\varepsilon \partial_x \log \rho(x) + \lambda x \Big) (\theta(x) - x) \, d\rho - \int \left(\frac{\lambda}{2} x^2 + \varepsilon \log \rho \right) d\rho \\ & + \int \left(\frac{\lambda}{2} x^2 + \varepsilon \log \rho_\infty^\varepsilon \right) d\rho_\infty^\varepsilon - \frac{\lambda}{2} \int |x - \theta(x)|^2 d\rho(x) \end{aligned}$$

and

$$T_3 := \frac{c_{1,s}}{2} \int \frac{d\rho(x) d\rho(y)}{|\theta(x) - \theta(y)|^{1-2s}} - \frac{c_{1,s}}{2} \int \frac{d\rho(x) d\rho(y)}{|x - y|^{1-2s}} - \int K\rho(x)(\theta(x) - x) d\rho(x).$$

By the same arguments, we conclude that $T_1, T_3 \geqslant 0$. Now, for T_2, let us define the following functional

$$H(f|g) := \int f(x) \log \left(\frac{f(x)}{g(x)} \right) \, dx$$

for all nonnegative $f, g \in L^1(\mathbb{R})$ with $g > 0$. Then we can re-write T_2 in the following way

$T_2 =$

$$\varepsilon\left(-\int \partial_x \log\left(\frac{\rho(x)}{e^{-\pi x^2}}\right)(\theta(x)-x)\, d\rho(x) - H(\rho|e^{-\pi x^2}) + H(\rho_\infty^\varepsilon|e^{-\pi x^2}) + \pi\int |\theta(x)-x|^2\, d\rho\right)$$
$$+\left(1-\frac{2\pi}{\lambda}\varepsilon\right)\int\left\{-\lambda x(\theta(x)-x) - \frac{\lambda}{2}x^2 + \frac{\lambda}{2}\theta(x)^2 + \frac{\lambda}{2}(\theta(x)-x)^2\right\} d\rho(x).$$

Note that the second line is equal to $(\lambda - 2\pi\varepsilon)\int |\theta(x)-x|^2\, dx$, which is nonnegative for $\varepsilon < \lambda/2\pi$. For the first line, we can use the proof of the HWI inequality made in [64]. Actually, Otto and Villani showed that whenever $\rho, \rho_\infty^\varepsilon \in C_c^\infty(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ and $V \in C^2(\mathbb{R})$ is such that $\int e^{-V} dx = 1$ and $V'' \geqslant \gamma$ for some constant $\gamma \in \mathbb{R}$, then

$$H(\rho_\infty^\varepsilon|e^{-V}) - H(\rho|e^{-V}) - \int \partial_x \log\frac{\rho(x)}{e^{-V(x)}}(\theta(x)-x)\rho(x)\, dx - \frac{\gamma}{2}\int |\theta(x)-x|^2\rho(x)\, dx \ \geqslant 0,$$

and for the density argument given in the proof of the Theorem 9.17 of [78], we have that this inequality holds for all $\rho, \rho_\infty^\varepsilon \in L^1(\mathbb{R}) \cap \mathcal{P}_2(\mathbb{R})$. So, applying this for $V(x) = \pi x^2$ we have that $\gamma = 2\pi$ and we conclude that $T_2 \geqslant 0$. □

Remark 2.3.5. *By the same arguments given for* (2.3.3) *and* (2.3.4)*, we conclude that the following Log-Sobolev and Talagrand inequalities hold for $\mathcal{E}_\varepsilon$, as long as ρ satisfies the assumptions of proposition 2.3.4:*

$$\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \leqslant \frac{1}{2\lambda}\mathcal{I}_\varepsilon(\rho), \tag{2.3.5}$$

$$W_2(\rho, \rho_\infty^\varepsilon) \leqslant \sqrt{\frac{2}{\lambda}\Big(\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)\Big)}.$$

Remark 2.3.6. *These results also work for a general confinement potential $V : \mathbb{R} \to \mathbb{R}$ instead of the quadratic one $\frac{\lambda}{2}x^2$, as long as $V - \frac{\lambda}{2}x^2$ is convex.*

2.4 Exponential Convergence

In this section we shall prove that the energy of the solution decays exponentially fast for the regularized equation with mollified initial data, and then passing the limit on these regularizing parameters we shall be able to prove the same property for the original problem. We conclude the section showing that this exponential decay in the energy implies also a exponential convergence in $L^2(\mathbb{R})$.

Let us begin stating our main result.

Theorem 2.4.1. *Let* $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that*

$$0 \leqslant \rho_0(x) \leqslant Ae^{-a|x|} ,$$

for some constants $a, A > 0$*. Then, for each* $0 < s < 1/2$*, the solution* $\rho(t,\cdot)$ *of* (2.1.1) *with initial data* ρ_0 *satisfies*

$$\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_\infty) \leqslant e^{-2\lambda t}\Big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\Big).$$

We could try to prove this theorem directly since we know from the previous section that the respective Log-Sobolev inequality holds for the energy $\mathcal{E}$ and then, using the relation (2.1.10) the result would follow by the Gronwall's inequality. The only problem of making this argument fully rigorous is that both results demand some smoothness from the solution and a classic regularity theory for the equation (2.1.1) is still an open problem. In spite of the result proved in [19] that the weak solutions are Hölder continuous, no estimate was given for the Hölder exponent, and since the Theorem 2.3.3, and consequently the Log-Sobolev inequality (2.3.3), is valid only for measures whose density belongs to C^α with $\alpha > 1 - 2s$, the argument above cannot be implemented at the moment. One possible way of circumventing this regularity issue would be extending the class of measures for which the HWI inequality holds, but this requires passing limits in the argument of the dissipation term $\mathcal{I}$, which is delicate, specially for two reasons: firstly because it is still not clear for which set of measures ρ the quantity $\mathcal{I}(\rho)$ is finite; and secondly, the dissipation is, in general, only lower semicontinuous, which makes the approximation process hard to implement since this functional appears on the right hand side of both HWI and Log-Sobolev inequalities. Therefore, in order to prove the theorem above we will proceed by an approximation argument as follows: first we will obtain a similar result for an approximate version of the equation (2.1.1) with an small extra linear diffusion for which we have a good regularity result, and then pass the limit on the respective exponential decay as this small diffusion term goes to zero. This approximate equation was also used in [20] to show the existence of weak solutions of the equation (2.1.2).

Proof of Theorem 2.4.1. In order to use the results of Section 2.3, firstly we shall assume that

$$\rho_0 \in C^\infty(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} \rho_0(x)\, dx = 1. \tag{2.4.1}$$

Let $\rho_\infty, \rho_\infty^\varepsilon \in \mathcal{P}(\mathbb{R})$ be the minimizers for $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ respectively. By the assumption on ρ_0 we know from the proofs of Theorems 4.1 and 4.2 in [20] that the solutions ρ and ρ^ε to

$$\begin{cases} \partial_t \rho = \partial_x(\rho \partial_x (-\partial_{xx})^{-s}\rho + \lambda x \rho) & , \text{ in } \mathbb{R}\times(0,\infty) \\ \rho(0) = \rho_0 & , \text{ in } \mathbb{R}, \end{cases} \tag{2.4.2}$$

and

$$\begin{cases} \partial_t \rho^\varepsilon = \partial_x(\rho^\varepsilon \partial_x (-\partial_{xx})^{-s}\rho^\varepsilon + \lambda x \rho^\varepsilon) + \varepsilon \partial_{xx}\rho^\varepsilon & , \text{ in } \mathbb{R}\times(0,\infty) \\ \rho^\varepsilon(0) = \rho_0 & , \text{ in } \mathbb{R} \end{cases} \tag{2.4.3}$$

satisfy $\rho \in C([0,\infty); L^1(\mathbb{R}))$ and $\rho^\varepsilon \in C^1((0,\infty)\times\mathbb{R})$ for all $\varepsilon > 0$ sufficiently small. It is also proved in the above mentioned reference that, because of the ε-regularization in (2.4.3) with a heat term, $\rho^\varepsilon(t,.)$ is in fact in $C^2(\mathbb{R})$ for all fixed time $t \geqslant 0$, and moreover, there exist $C(t), a(t) > 0$, such that

$$0 \leqslant \ \rho(t,x)\ ,\ \rho^\varepsilon(t,x) \ \leqslant \ C(t)e^{-a(t)|x|}. \tag{2.4.4}$$

The energy $\mathcal{E}_\varepsilon$ is related to the equation (2.4.3) in the same way as $\mathcal{E}$ is related to (2.4.2). For example, in both cases the respective energy is nonincreasing with respect to the time and the steady state solutions coincide with the minimums of the energies.

Since $\rho^\varepsilon(t)$ is smooth, we can apply the Log-Sobolev Inequality (2.3.5) for $\mathcal{E}_\varepsilon$ and obtain that for all $t \geqslant 0$,

$$\mathcal{E}_\varepsilon(\rho^\varepsilon(t)) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \leqslant \frac{1}{2\lambda}\mathcal{I}_\varepsilon(\rho^\varepsilon(t)).$$

Furthermore, making use of the fact that

$$\frac{d}{dt}\mathcal{E}_\varepsilon(\rho^\varepsilon(t)) = -\mathcal{I}_\varepsilon(\rho^\varepsilon(t)), \tag{2.4.5}$$

we conclude that

$$\mathcal{E}_\varepsilon(\rho^\varepsilon(t)) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \leqslant e^{-2\lambda t}\Big(\mathcal{E}_\varepsilon(\rho_0) - \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)\Big). \tag{2.4.6}$$

The equality (2.4.5) is easy to show in this case for $\rho^\varepsilon(t)$ is a classical solution to (2.4.3) which decays to zero at infinity and hence, defining $\xi_\varepsilon = \varepsilon \log\rho + \frac{\lambda}{2}x^2 + k_s * \rho$ to simplify the notation, we can write the equation as $\partial_t\rho^\varepsilon = \partial_x(\rho^\varepsilon\partial_x\xi_\varepsilon)$ and we can compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_\varepsilon(\rho^\varepsilon) &= \int\left(\varepsilon\log\rho^\varepsilon\partial_t\rho^\varepsilon + \frac{\lambda}{2}x^2\partial_t\rho^\varepsilon + \partial_t\rho^\varepsilon\frac{k_s}{2}*\rho^\varepsilon + \rho^\varepsilon\frac{k_s}{2}*\partial_t\rho^\varepsilon\right)\,dx\\
&= \int\left(\varepsilon\log\rho^\varepsilon + \frac{\lambda}{2}x^2 + k_s*\rho^\varepsilon\right)\partial_t\rho^\varepsilon\,dx\\
&= \int \xi_\varepsilon(x)\partial_x(\rho^\varepsilon(x)\partial_x\xi_\varepsilon(x))\,dx\\
&= -\int \rho^\varepsilon(x)|\partial_x\xi_\varepsilon(x)|^2\,dx\\
&= -\mathcal{I}(\rho^\varepsilon).
\end{aligned}$$

Now, in order to obtain the desired inequality for the original functional $\mathcal{E}$ we need to take the limits as $\varepsilon\to 0^+$ in (2.4.6). For this, let us analyze each one of the three terms on (2.4.6) separately:

i) The easiest one is the limit $\mathcal{E}_\varepsilon(\rho_0)$, since $\lim\limits_{\varepsilon\to 0^+}\mathcal{E}_\varepsilon(\rho_0) = \mathcal{E}(\rho_0)$ holds as long as $\mathcal{U}(\rho_0) < \infty$ for some, which is true by the assumptions on ρ_0.

ii) For the term $\mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)$, let us first define the following auxiliary functional on $\mathcal{P}_{2,ac}(\mathbb{R})$:

$$\mathcal{H}(\rho) := H(\rho|e^{-\pi x^2}) = \pi\int x^2\rho + \int\rho\log\rho.$$

Since $\int e^{-\pi x^2}dx = 1$, we can write

$$\mathcal{H}(\rho) = \int\frac{\rho}{e^{-\pi x^2}}\log\left(\frac{\rho}{e^{-\pi x^2}}\right)e^{-\pi x^2}\,dx = \int\left[\frac{\rho}{e^{-\pi x^2}}\log\left(\frac{\rho}{e^{-\pi x^2}}\right) - \frac{\rho}{e^{-\pi x^2}} + 1\right]e^{-\pi x^2}\,dx,$$

which is nonnegative since the function $r\mapsto r\log r - r + 1$ is nonnegative as well.

Let us prove that $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)\leqslant\mathcal{E}(\rho_\infty)$. Using the fact that ρ_∞^ε is the minimum for $\mathcal{E}_\varepsilon$, we obtain the following inequality

$$\mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)\leqslant\mathcal{E}_\varepsilon(\rho_\infty) = \mathcal{E}(\rho_\infty) + \varepsilon\int\rho_\infty\log\rho_\infty. \tag{2.4.7}$$

By the characterization of the minimum ρ_∞ in [21, 32], we know that $\rho_\infty\in\mathcal{P}^2_{ac}(\mathbb{R})\cap L^\infty(\mathbb{R})$, and hence the second term on the right hand side of (2.4.7) is finite. Thus, we can take the limit $\varepsilon\to 0$ and obtain that $\limsup_{\varepsilon\to 0^+}\mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)\leqslant\mathcal{E}(\rho_\infty)$.

For the opposite inequality $\liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \geqslant \mathcal{E}(\rho_\infty)$, we can use the fact that ρ_∞ is the minimum for $\mathcal{E}$ and write

$$\begin{aligned}
\mathcal{E}(\rho_\infty) &\leqslant \mathcal{E}(\rho_\infty^\varepsilon)\\
&= \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) + \varepsilon\mathcal{U}(\rho_\infty^\varepsilon) - \varepsilon\left(\mathcal{U}(\rho_\infty^\varepsilon) + \pi\int x^2\rho_\infty^\varepsilon\right) + \varepsilon\pi\int x^2\rho_\infty^\varepsilon\\
&= \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) - \varepsilon\mathcal{H}(\rho_\infty^\varepsilon) + \varepsilon\pi\int x^2\rho_\infty^\varepsilon\\
&\leqslant \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) + \varepsilon\pi\int x^2\rho_\infty^\varepsilon.
\end{aligned} \tag{2.4.8}$$

So, it is sufficient to prove that the second moments of ρ_∞^ε are uniformly bounded for $\varepsilon > 0$ sufficiently small. For this, note that for all $0 < \varepsilon < \min\{\lambda/4\pi, 1\}$ we have

$$\begin{aligned}
0 \leqslant \ & \frac{\lambda}{4}\int x^2\rho_\infty^\varepsilon \ = \frac{\lambda}{2}\mathcal{V}(\rho_\infty^\varepsilon)\\
\leqslant \ & \frac{\lambda}{2}\mathcal{V}(\rho_\infty^\varepsilon) + \varepsilon\mathcal{H}(\rho_\infty^\varepsilon)\\
= \ & \frac{\lambda}{2}\mathcal{V}(\rho_\infty^\varepsilon) + 2\pi\varepsilon\mathcal{V}(\rho_\infty^\varepsilon) + \varepsilon\,\mathcal{U}(\rho_\infty^\varepsilon)\\
\leqslant \ & \lambda\mathcal{V}(\rho_\infty^\varepsilon) + \varepsilon\,\mathcal{U}(\rho_\infty^\varepsilon) + \mathcal{W}(\rho_\infty^\varepsilon)\\
= \ & \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) \ \leqslant \ \mathcal{E}_\varepsilon(\rho_\infty) \ \leqslant \ \mathcal{E}(\rho_\infty) + \left|\int \rho_\infty \log\rho_\infty\right|.
\end{aligned}$$

Therefore, taking the limit as $\varepsilon \to 0^+$ in (2.4.8) we obtain

$$\mathcal{E}(\rho_\infty) \leqslant \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon) + \lim_{\varepsilon\to 0^+} \varepsilon\pi\int x^2\rho_\infty^\varepsilon = \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon).$$

Hence, as ε goes to zero from above, we have that the minimum of $\mathcal{E}_\varepsilon(\rho)$ indeed converge to the minimum of $\mathcal{E}(\rho)$, i.e., $\mathcal{E}(\rho_\infty) = \lim_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho_\infty^\varepsilon)$.

iii) Finally, let us prove that $\mathcal{E}(\rho(t)) \leqslant \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho^\varepsilon(t))$ as a consequence of the convergence of $\rho^\varepsilon(t)$ to $\rho(t)$ in $\mathcal{P}_{2,ac}(\mathbb{R})$ and the lower semi-continuity of the energy $\mathcal{E}_\varepsilon$. For this we can use the bound (2.4.4) to obtain

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|>R} \rho^{\varepsilon_n}(t,x)dx \leqslant \lim_{R\to\infty} C(t)\int_{|x|>R} e^{-a(t)|x|}dx = 0,$$

for every sequence $\varepsilon_n \to 0$, which means that $\rho^{\varepsilon_n}(t)$ is a tight family of probability measures and by Prokhorov's Theorem 2.2.4, there exist a subsequence, still denoted by ε_n, that such that $\rho^{\varepsilon_n}(t) \rightharpoonup \rho(t)$, i.e.,

$$\int_{\mathbb{R}} \varphi(x)\rho^{\varepsilon_n}(t,x)\,dx \to \int_{\mathbb{R}} \varphi(x)\rho(t,x)\,dx \quad , \qquad \text{for all } \varphi \in C_b(\mathbb{R}) \tag{2.4.9}$$

Moreover, due to uniform exponential bound, we also have that

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\geqslant R} x^2 \rho^{\varepsilon_n}(t,x)dx \leqslant \lim_{R\to\infty} C(t) \int_{|x|>R} x^2 e^{-a(t)|x|}dx = 0. \tag{2.4.10}$$

Therefore, by (2.2.6) from the Theorem 2.2.3 we have that (2.4.9) and (2.4.10) imply that $\rho^{\varepsilon_n}(t)$ converges to $\rho(t)$ in $(\mathcal{P}_2(\mathbb{R}), W_2)$. Note that, by (2.4.4), the second moments are uniformly bounded w. r. t. n so, from the following inequality

$$\mathcal{E}(\rho^{\varepsilon_n}(t)) = \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)) - \varepsilon_n \mathcal{H}(\rho^{\varepsilon_n}(t)) + \pi\varepsilon_n \int x^2 \rho^{\varepsilon_n}(t,x) \leqslant \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}) + \pi\varepsilon_n \int x^2 \rho^{\varepsilon_n}(t,x),$$

and by the fact that $\mathcal{E}$ is lower semi-continuous in $(\mathcal{P}_2(\mathbb{R}), W_2)$, we obtain

$$\mathcal{E}(\rho(t)) \leqslant \liminf_{n\to\infty} \mathcal{E}(\rho^{\varepsilon_n}(t)) \leqslant \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)).$$

Putting all the limits together as ε goes to zero, we can conclude the exponential convergence of $\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_\infty)$, that is,

$$\begin{aligned}
\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_\infty) &\leqslant \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)) - \lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}_\infty) \\
&= \liminf_{n\to\infty} \Big(\mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)) - \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon_n}_\infty) \Big) \\
&\leqslant e^{-2\lambda t} \liminf_{n\to\infty} \Big(\mathcal{E}_{\varepsilon_n}(\rho_0) - \mathcal{E}_{\varepsilon_n}(\rho^{\varepsilon}_\infty) \Big) \\
&= e^{-2\lambda t} \Big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty) \Big).
\end{aligned}$$

If the regularity assumption in (2.4.1) is not true, we can proceed the above argument with the mollified initial data $\rho_{0,\delta} = \eta_\delta * \rho_0$, which has the same bound and mass as ρ_0. Since we still have the same exponential bounds for the respective solutions $\rho_\delta(t)$, we can argue as above and conclude that $\mathcal{E}(\rho(t)) \leqslant \liminf_{\delta\to 0} \mathcal{E}(\rho_\delta(t))$ holds for all $t > 0$. For $t = 0$ we can use the exponential bound of the initial data and the Dominated Convergence Theorem to conclude that $\lim_{\delta\to 0} \mathcal{E}(\rho_{\delta,0}) = \mathcal{E}(\rho_0)$. $\square$

As a direct consequence of the Talagrand inequality in (2.3.4), we also obtain the exponential decay in Wasserstein distance.

Corollary 2.4.2. *Assume that ρ_0 satisfies $0 \leqslant \rho_0(x) \leqslant Ae^{-a|x|}$ for all $x \in \mathbb{R}$ and some $a, A > 0$. Then, for each $0 < s < 1/2$, the solution of (2.1.1) with initial data ρ_0 satisfies*

$$W_2(\rho(t), \rho_\infty) \leqslant e^{-\lambda t} \sqrt{\frac{2}{\lambda} \Big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty) \Big)}.$$

For the Fokker-Planck equation or the classic Porous Medium Equations, exponential convergence of the relative entropy $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ implies convergence of ρ to the steady states ρ_∞ in some classical L^p norms, using for this the classical Csiszár-Kullback-Pinsker inequality as in [7, 29]. Here we can show that the convergence in the relative entropy implies the convergence of the norm $\|(-\partial_{xx})^{-\frac{s}{2}}(\rho - \rho_\infty)\|_{L^2}$.

Lemma 2.4.3. *Let ρ_∞ be the unique minimizer of $\mathcal{E}$, then for any $\rho \in \mathcal{P}_2(\mathbb{R})$,*

$$\frac{1}{2} \|(-\partial_{xx})^{-\frac{s}{2}}(\rho - \rho_\infty)\|_{L^2}^2 \leq \mathcal{E}(\rho) - \mathcal{E}(\rho_\infty).$$

Proof. The characterization (2.1.13a) and (2.1.13b) of the global minimizer ρ_∞ and the non-negativity of $\rho - \rho_\infty$ outside of the support of ρ_∞ imply that

$$0 = C_* \int_{\mathbb{R}} (\rho - \rho_\infty) \leq \int_{\mathbb{R}} \left((-\Delta)^{-s} \rho_\infty(x) + \lambda \frac{|x^2|}{2} \right) (\rho - \rho_\infty).$$

Therefore, we deduce

$$\begin{aligned} \mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) &= \frac{1}{2} \int_{\mathbb{R}} \rho (-\partial_{xx})^{-s} \rho - \frac{1}{2} \int_{\mathbb{R}} \rho (-\partial_{xx})^{-s} \rho_\infty + \frac{\lambda}{2} \int_{\mathbb{R}} |x|^2 (\rho - \rho_\infty) \\ &\geq \frac{1}{2} \int_{\mathbb{R}} \rho (-\partial_{xx})^{-s} \rho - \frac{1}{2} \int_{\mathbb{R}} \rho (-\partial_{xx})^{-s} \rho_\infty - \int_{\mathbb{R}} (\rho - \rho_\infty) (-\partial_{xx})^{-s} \rho_\infty \\ &= \frac{1}{2} \int_{\mathbb{R}} (\rho - \rho_\infty) (-\partial_{xx})^{-s} (\rho - \rho_\infty) = \frac{1}{2} \|(-\partial_{xx})^{-\frac{s}{2}} (\rho - \rho_\infty)\|_{L^2}^2. \end{aligned}$$

□

Since $\|(-\partial_{xx})^{-\frac{s}{2}}(\rho - \rho_\infty)\|_{L^2}$ is the $H^{-s/2}$-norm of $\rho - \rho_\infty$, it is unlikely to produce a bound on any stronger L^p norm for the difference $\rho - \rho_\infty$. One way to show the exponential convergence of $\rho(t)$ to ρ_∞ is to assume a uniform bound on a higher order norm of $\rho - \rho_\infty$.

For example, if $\|(-\partial_{xx})^{\frac{s}{2}}(\rho-\rho_\infty)\|_{L^2}$ is uniformly bounded, then we have (easy to establish in Fourier space)

$$\|\rho-\rho_\infty\|_{L^2}^2 \leq \|(-\partial_{xx})^{\frac{s}{2}}(\rho-\rho_\infty)\|_{L^2}\|(-\partial_{xx})^{-\frac{s}{2}}(\rho-\rho_\infty)\|_{L^2}$$

and $\|\rho-\rho_\infty\|_{L^2}$ converges to zero also exponentially fast, but with a smaller rate.

Let us prove that in fact the exponential convergence also holds in L^2 without any additional hypothesis. For this, since $(-\partial_{xx})^{-\frac{s}{2}}u$ usually has more regularity than u, we need to look for an interpolation inequality containing some sort of fractional differentiation, which in our case, it seems natural to be a Hölder semi-norm, i.e., for every $\alpha \in (0,1]$ and $v \in C^\alpha(\mathbb{R})$ we denote the α-Hölder semi-norm of v by

$$[v]_{C^\alpha} := \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha}.$$

Therefore, to obtain the desired decay in L^2 we shall use the following new interpolation inequality, that we will prove for any dimension $d \geqslant 1$. The idea for this result was given by Carrillo and Vázquez.

Theorem 2.4.4. *Let $0<\alpha\leqslant 1$ and $0<s<d/2$ and $0<r<\alpha/2$. There exists a constant $C=C(d,s,\alpha)$ such that*

$$\|u\|_{L^2} \leqslant C\|(-\Delta)^{-\frac{s}{2}}u\|_{L^2}^{\sigma_1}\ [u]_\alpha^{\sigma_2}\ \|u\|_{L^1}^{\sigma_3}\,;, \tag{2.4.11}$$

for all $u \in L^1(\mathbb{R}^d)\cap C^\alpha(\mathbb{R}^d)$ with

$$\sigma_1 = \frac{r}{s+r}\,, \qquad \sigma_2 = \frac{s(d+2r)}{2(d+\alpha)(s+r)}\,, \qquad \sigma_3 = \frac{s(d+2\alpha-2r)}{2(d+\alpha)(s+r)}.$$

In order to prove the Theorem above we are going to need the following result about fractional Sobolev Spaces:

Proposition 2.4.5 (See propositions 3.4 and 3.6 of [39])**.** *Let $\gamma\in(0,1)$ and $H^\gamma(\mathbb{R}^d)$ the Banach space defined by*

$$H^\gamma(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\gamma}}\,dxdy < \infty \right\}$$

with norm given by $\|u\|_{H^\gamma}^2 = \|u\|_{L^2}^2 + [u]_{H^\gamma}^2$ *where* $[.]_{H^\gamma}$ *is a seminorm defined by*

$$[u]_{H^\gamma}^2 := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\gamma}}\, dxdy.$$

Then, there exists a constant $C(d,\gamma)$ *depending only on* d *and* γ *such that*

$$[u]_{H^\gamma}^2 = C(d,\gamma)^{-1}\int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2|\xi|^{2\gamma}\, d\xi = C(d,\gamma)^{-1}\|(-\Delta)^{\frac{\gamma}{2}}u\|_{L^2};, \tag{2.4.12}$$

for all $u \in H^\gamma(\mathbb{R}^d)$*, where* $\widehat{u}$ *is the Fourier transform of* u*.*

Proof of Theorem 2.4.4. For all $0 < s < d/2$, $r > 0$ and $\sigma_1 = r/(s+r)$, we can use Fourier variables, Plancherel's formula, the Hölder's inequality with the conjugate pair $\left(\frac{1}{\sigma_1}, \frac{1}{1-\sigma_1}\right)$ and (2.4.12) to interpolate between $[u]_{H^r}$ and $(-\Delta)^{-\frac{s}{2}}u \in L^2(\mathbb{R}^d)$ obtaining

$$\begin{aligned}\|u\|_{L^2}^2 = \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2\, d\xi &= \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^{2\sigma_1}|\xi|^{-2s\sigma_1}|\widehat{u}(\xi)|^{2(1-\sigma_1)}|\xi|^{2r(1-\sigma_1)}d\xi \\ &\leqslant \left(\int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2|\xi|^{-2s}d\xi\right)^{\sigma_1}\left(\int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2|\xi|^{2r}d\xi\right)^{1-\sigma_1} \\ &= C(d,r)\|(-\Delta)^{-\frac{s}{2}}u\|_{L^2}^{2\sigma_1}[u]_{H^r}^{2(1-\sigma_1)}. \end{aligned} \tag{2.4.13}$$

Our aim now is to bound $[u]_{H^r}$ by $[u]_{C^\alpha}$ and $\|u\|_{L^1}$. Using (2.4.12) again we can split the seminorm $[u]_{H^r}$ for any $R > 0$ as

$$\begin{aligned}[u]_{H^r} &= C(d,r)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2r}}\, dxdy \\ &= C(d,r)^{-1}\left(\iint_{|x-y|\leqslant R} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2r}}\, dxdy + \iint_{|x-y|>R} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2r}}\, dxdy\right) \\ &=: C(d,r)^{-1}\left(\ I_1 + I_2\ \right).\end{aligned}$$

To estimate I_1, we make use of $|u(x)-u(y)| \leq [u]_\alpha\, |x-y|^\alpha$ to get, by the change of variables $z = x-y$, that

$$\begin{aligned} I_1 &= \iint_{|x-y|\leq R} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2r}}\, dxdy \\ &\leqslant [u]_\alpha \iint_{|x-y|\leqslant R} \frac{|u(x)-u(y)|}{|x-y|^{d+2r-\alpha}}\, dxdy \end{aligned}$$

$$
\begin{aligned}
&\leqslant [u]_{C^\alpha} \int_{|z|\leqslant R} \frac{1}{|z|^{\alpha-2r-d}} \int_{\mathbb{R}^d} \left(|u(z+y)| + |u(y)|\right)\, dy\, dz \\
&= 2[u]_{C^\alpha} \|u\|_{L^1} \int_{|z|\leqslant R} |z|^{\alpha-2r-d}\, dz \\
&\leqslant 2[u]_\alpha \|u\|_{L^1} R^{\alpha-2r}\,,
\end{aligned}
$$

where the last step is allowed since $2r < \alpha$. On the other hand, we can similarly estimate the far field term as

$$
I_2 = \iint_{|x-y|\geq R} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dxdy \leqslant 4 \int_{\mathbb{R}^d} |u(y)|^2 dy \int_{|z|\geq R} \frac{dz}{|z|^{d+2r}} \leqslant 4\|u\|_{L^2}^2 R^{-2r}\,.
$$

Joining the two integrals and choosing the optimal value

$$
R = [u]_{C^\alpha}^{\frac{1}{\alpha}} \|u\|_{L^2}^{\frac{2}{\alpha}} \|u\|_{L^1}^{-\frac{1}{\alpha}}\ ,
$$

we infer

$$
\begin{aligned}
[u]_{H^r}^2 &= C(d,r)^{-1} \left(2[u]_{C^\alpha}\|u\|_{L^1} R^{\alpha-2r} + 4\|u\|_{L^2}^2 R^{-2r}\ \right) \\
&= C(d,r)^{-1} \Big(2[u]_\alpha \|u\|_{L^1} [u]_{C^\alpha}^{-(\alpha-2r)/\alpha} \|u\|_{L^2}^{(2\alpha-4r)/\alpha} \|u\|_{L^1}^{-(\alpha-2r)/\alpha} \\
&\qquad\qquad + 4\|u\|_{L^2}^2 [u]_{C^\alpha}^{2r/\alpha} \|u\|_{L^2}^{-4r/\alpha} \|u\|_{L^1}^{2r/\alpha}\Big) \\
&= \widetilde{C}(d,r,\alpha) \|u\|_{L^2}^{2(1-2r/\alpha)} \|u\|_{L^1}^{2r/\alpha} [u]_\alpha^{2r/\alpha}
\end{aligned}
\tag{2.4.14}
$$

We finally use the classical interpolation results between $L^p(\mathbb{R}^d)$ and $C^\alpha(\mathbb{R}^d)$ spaces due to L. Nirenberg in [62], see also [17] for a full statement. This interpolation inequality ensures the existence of a constant depending on α and d such that

$$
\|u\|_{L^2}^2 \leqslant C\, \|u\|_{L^1}^{(d+2\alpha)/(\alpha+d)} [u]_{C^\alpha}^{d/(\alpha+d)}\,.
$$

Putting it together with (2.4.14), it yields

$$
\begin{aligned}
[u]_{H^r}^2 &\leqslant \overline{C}(d,r,\alpha)\, \|u\|_{L^1}^{2r/\alpha} [u]_{C^\alpha}^{2r/\alpha} \left(\|u\|_{L^1}^{(d+2\alpha)/(\alpha+d)} [u]_{C^\alpha}^{d/(\alpha+d)}\right)^{1-2r/\alpha} \\
&\leqslant \overline{C}(d,r,\alpha)\ \|u\|_{L^1}^{(d+2\alpha-2r)/(d+\alpha)} [u]_{C^\alpha}^{(d+2r)/(d+\alpha)}\,.
\end{aligned}
$$

Finally, we plug this into (2.4.13) and we obtain

$$
\begin{aligned}
\|u\|_{L^2}^2 &\leqslant C(d,r)\|(-\Delta)^{-\frac{s}{2}}u\|_{L^2}^{2\sigma_1}[u]_{H^r}^{2(1-\sigma_1)} \\
&\leqslant C\|(-\Delta)^{-\frac{s}{2}}u\|_{L^2}^{2\sigma_1}\left(\|u\|_{L^1}^{(d+2\alpha-2r)/(d+\alpha)}[u]_{C^\alpha}^{(d+2r)/(d+\alpha)}\right)^{1-\sigma_1} \\
&= C\|(-\Delta)^{-\frac{s}{2}}u\|_{L^2}^{2\sigma_1}\|u\|_{L^1}^{s(d+2\alpha-2r)/(d+\alpha)(s+r)}[u]_{C^\alpha}^{s(d+2r)/(d+\alpha)(s+r)},
\end{aligned}
$$

which concludes the proof. □

Therefore, from Theorem 2.4.1 and Theorem 2.4.4, we derive the following decay towards the stationary state under the L^2 norm.

Corollary 2.4.6. *Assume that ρ_0 satisfies $0 \leqslant \rho_0(x) \leqslant Ae^{-a|x|}$ for all $x \in \mathbb{R}$ and some $a, A \geqslant 0$. Then, for each $0 < s < 1/2$, the solution of (2.1.1) with initial data ρ_0 satisfies*

$$
\|\rho(t)-\rho_\infty\|_{L^2} \leqslant C\left(1+[\rho_\infty]_\alpha\right)^{\sigma_2}\left(\mathcal{E}(\rho_0)-\mathcal{E}(\rho_\infty)\right)^{\frac{\sigma_1}{2}} e^{-\lambda\sigma_1 t}.
$$

Proof. Given ρ_0 under the conditions above, we know from Theorem 5.1 of [19] that there exists an $\beta \in (0,1)$ such that the solution ρ of (2.1.1) satisfies $\rho(t) \in C^\beta(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for all $t > 0$ with a uniform bound in time. Since ρ_∞ is bounded and $(1-s)$-Hölder continuous, we have that $u(t) := \rho(t) - \rho_\infty \in C^\alpha(\mathbb{R})$ for all $t > 0$ with $\alpha \in \min\{\beta, 1-s\}$. So we can use inequality (2.4.11) for u and $0 < 2r < \alpha$, the Lemma 2.4.3 and the Theorem 2.4.1 to conclude

$$
\begin{aligned}
\|\rho(t)-\rho_\infty\|_{L^2} &\leqslant C\|(-\Delta)^{-\frac{s}{2}}(\rho(t)-\rho_\infty)\|_{L^2}^{\sigma_1}\|\rho(t)-\rho_\infty\|_{L^1}^{\sigma_3}[\rho(t)-\rho_\infty]_{C^\alpha}^{\sigma_2} \\
&\leqslant C\left(\mathcal{E}(\rho(t))-\mathcal{E}(\rho_\infty)\right)^{\sigma_1/2}\left([\rho(t)]_{C^\alpha}+[\rho_\infty]_{C^\alpha}\right)^{\sigma_2} \\
&\leqslant Ce^{-\lambda\sigma_1 t}\left(\mathcal{E}(\rho_0)-\mathcal{E}(\rho_\infty)\right)^{\sigma_1/2}\left(1+[\rho_\infty]_{C^\alpha}\right)^{\sigma_2}.
\end{aligned}
$$

□

Let us point out that the decay of the entropy in Theorem 2.4.1 implies a uniform in time control of the second moment of the solutions trivially at least for $0 < s < 1/2$. Otherwise, one has to work a bit due to the sign of the constant in the fractional operator. In any case, a uniform in time control of the second moments together with the L^2-decay rates implies L^1-decay rates. For the next result, a similar calculation was performed in [27, Lemma 2.24].

Corollary 2.4.7. *Assume that ρ_0 satisfies $0 \leqslant \rho_0(x) \leqslant Ae^{-a|x|}$ for all $x \in \mathbb{R}$ and some $a, A \geqslant 0$. Then, for each $0 < s < 1/2$, the solution of* (2.1.1) *with initial data ρ_0 satisfies*

$$\|\rho(t) - \rho_\infty\|_{L^1} \leqslant C\left(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\right)^{\frac{1}{5}} \left(1 + [\rho_\infty]_\alpha\right)^{\frac{4\sigma_2}{5}} \left(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\right)^{\frac{4\sigma_1}{10}} e^{-\frac{4\lambda\sigma_1}{5}t}.$$

Proof. For every $R > 0$ we can split the L^1 norm as

$$\begin{aligned}
\|\rho(t) - \rho_\infty\|_{L^1} &\leqslant \int_{|x|<R} |\rho(t,x) - \rho_\infty(x)| dx + \int_{|x|\geq R} |\rho(t,x) - \rho_\infty(x)| dx \\
&\leqslant C\left(R^{1/2}\|\rho(t) - \rho_\infty\|_{L^2} + \int_{\mathbb{R}} \frac{x^2}{R^2}\Big(\rho(t,x) + \rho_\infty(x)\Big) dx\right) \\
&\leqslant C\left(R^{1/2}\|\rho(t) - \rho_\infty\|_{L^2} + \frac{\lambda}{R^2}\left(\mathcal{V}(\rho(t)) + \mathcal{V}(\rho_\infty)\right)\right) \\
&\leqslant C\left(R^{1/2}\|\rho(t) - \rho_\infty\|_{L^2} + \frac{1}{R^2}\left(\mathcal{E}(\rho(t)) + \mathcal{E}(\rho_\infty)\right)\right) \\
&\leqslant C\left(R^{1/2}\|\rho(t) - \rho_\infty\|_{L^2} + \frac{1}{R^2}\left(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\right)\right) \qquad (2.4.15)
\end{aligned}$$

where the last inequality follows from the fact that the energy $\mathcal{E}$ is decreasing. Now, choosing the optimal value

$$R = \left(\frac{\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)}{\|\rho(t) - \rho_\infty\|_{L^2}}\right)^{2/5}$$

we obtain from the (2.4.15)

$$\begin{aligned}
\|\rho(t) - \rho_\infty\|_{L^1} \leqslant C &\left(\left(\frac{\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)}{\|\rho(t) - \rho_\infty\|_{L^2}}\right)^{1/5} \|\rho(t) - \rho_\infty\|_{L^2}\right. \\
&\left. + \left(\frac{\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)}{\|\rho(t) - \rho_\infty\|_{L^2}}\right)^{-4/5} \left(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\right)\right) \\
= C &\left(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\right)^{1/5} \|\rho(t) - \rho_\infty\|_{L^2}^{4/5}.
\end{aligned}$$

Therefore, by the Corollary 2.4.6 we conclude that

$$\|\rho(t) - \rho_\infty\|_{L^1}^5 \leqslant C\left(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\right)\left(1 + [\rho_\infty]_\alpha\right)^{4\sigma_2}\left(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\right)^{\frac{4\sigma_1}{2}} e^{-4\lambda\sigma_1 t}.$$

□

We finally remark that the decay in L^p-norms obtained via Corollary 2.4.6 and 2.4.7 are translated through the change of variables (2.1.6)-(2.1.7) into algebraic decay rates toward self-similar solutions of the original fractional porous medium equation (2.1.2).

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