UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e
Computação Científica

KENNERSON NASCIMENTO DE SOUSA LIMA

Bifurcation and Local Rigidity of Homogeneous Solutions to the Yamabe Problem on Maximal Flag Manifolds

Bifurcação e Rigidez Local de Soluções Homogêneas do Problema de Yamabe Sobre Variedades Flag Maximais

Campinas

2019

Kennerson Nascimento de Sousa Lima

Bifurcation and Local Rigidity of Homogeneous Solutions to the Yamabe Problem on Maximal Flag Manifolds

Bifurcação e Rigidez Local de Soluções Homogêneas do Problema de Yamabe Sobre Variedades Flag Maximais

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Advisor: Lino Anderson da Silva Grama

Co-advisor: Caio José Colletti Negreiros

Este exemplar corresponde à versão final da Tese defendida pelo aluno Kennerson Nascimento de Sousa Lima, orientada pelo Prof. Dr. Lino Anderson da Silva Grama.

Campinas

2019

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

	Lima, Kennerson Nascimento de Sousa, 1984-
L628b	Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on maximal flag manifolds / Kennerson Nascimento de Sousa Lima. – Campinas, SP : [s.n.], 2019.
	Orientador: Lino Anderson da Silva Grama. Coorientador: Caio José Colletti Negreiros. Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.
	1. Teoria da bifurcação. 2. Variedades bandeira. 3. Yamabe, Problema de. I. Grama, Lino Anderson da Silva, 1981-. II. Negreiros, Caio José Colletti, 1955-. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações para Biblioteca Digital

Título em outro idioma: Bifurcação e rigidez local de soluções homogêneas do problema de Yamabe sobre variedades flag maximais
Palavras-chave em inglês:
Bifurcation theory
Flag manifolds
Yamabe problem
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Lino Anderson da Silva Grama
Pedro Jose Catuogno
Ricardo Miranda Martins
Llohann Dallagnol Sperança
Andrew James Clarke
Data de defesa: 09-08-2019
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0002-9761-525X
- Currículo Lattes do autor: http://lattes.cnpq.br/3918727202912115

Tese de Doutorado defendida em 09 de agosto de 2019 e aprovada pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). LINO ANDERSON DA SILVA GRAMA

Prof(a). Dr(a). PEDRO JOSE CATUOGNO

Prof(a). Dr(a). RICARDO MIRANDA MARTINS

Prof(a). Dr(a). LLOHANN DALLAGNOL SPERANÇA

Prof(a). Dr(a). ANDREW JAMES CLARKE

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Este trabalho é dedicado aos meus pais,
Damião de Sousa Lima e Ivânia Nascimento de Sousa Lima.

Acknowledgements

- I thank my family for the unconditional support, especially my parents Damião de Sousa Lima and Ivânia Nascimento de Sousa Lima, my sister Karolina, and my brother Kelson.
- I thank my wife Andressa Gomes for her support, understanding and motivation at all times.
- I thank the Universidade Estadual de Campinas and the IMECC for the opportunity they gave me.
- I thank all my teachers that I had the pleasure of knowing them, especially my advisor Prof. Lino Anderson da Silva Grama for the patience, support and guidance throughout the doctorate.
- I thank my friends and colleagues at UNICAMP.
- This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.
- I thank the Universidade Federal de Campina Grande and specially its Unidade Acadêmica de Matemática for giving me the opportunity to do my doctorate at UNICAMP.

”As long as algebra and geometry have been separated,
their progress have been slow and their uses limited;
but when these two sciences have been united,
they have lent each mutual forces,
and have marched together towards perfection.”
(Joseph-Louis Lagrange)

Resumo

Neste trabalho, construimos famílias de 1-parâmetro de soluções do problema Yamabe a partir de submersões Riemannianas com fibras totalmente geodésicas. Consideramos como espaços totais dessas submersões variedades flag maximais munidas de uma métrica normal. Em seguida, determinamos os instantes de bifurcação e rigidez local para essas famílias de soluções olhando para as mudanças do índice de Morse dessas métricas quando o parâmetro varia no intervalo $(0, 1]$. Um ponto de bifurcação para tais famílias é um ponto de acumulação de outras soluções para o problema de Yamabe conformes a soluções homogêneas. Já um ponto de rigidez local é uma solução isolada para este problema em sua classe conforme, ou seja, não é um instante de bifurcação.

Também calculamos o índice Morse das variações canônicas sobre as variedades flag maximais $SU(n+1)/T^n$ e $SO(2n+1)/T^n$, com o parâmetro variando no intervalo $(0, 1]$.

Finalmente, obtemos resultados sobre a multiplicidade de soluções do problema Yamabe em nossa situação utilizando resultados de R. G. Bettiol e P. Piccione que garantem um número mínimo de soluções em uma determinada classe conforme se o índice Morse da métrica correspondente for positivo.

Palavras-chave: Instante de bifurcação; Instante de rigidez local; Variedade flag; Variação canônica; Problema de Yamabe.

Abstract

In this work, we construct 1-parameter families of well known solutions to the Yamabe problem from Riemannian submersions with totally geodesic fibers. We consider as total spaces maximal flag manifolds equipped with a normal metric. Thereafter, we determine bifurcation and local rigidity instants for these families looking for changes of the Morse index of these metrics when the parameter varies on the interval $(0, 1]$. A bifurcation point for such families is an accumulation point of others solutions to the Yamabe problem conformal to homogeneous solutions. Already a local rigidity point is an isolated solution to this problem in its conformal class, i.e., it is not a bifurcation instant.

We also compute the Morse index of the canonical variations defined on the maximal flag manifolds $SU(n+1)/T^n$ and $SO(2n+1)/T^n$, for the parameter varying on the interval $(0, 1]$.

Finally, we obtain results about multiplicity of solutions of the Yamabe problem in our situation by using results of R. G. Bettiol and P. Piccione that guarantee a minimum number of solutions in a given conformal class if the Morse index of the corresponding metric is positive.

Keywords: Bifurcation instant; Local rigidity instant; Flag manifold; Yamabe problem.

Contents

Introduction

Given a compact, orientable Riemannian manifold (M, g), with dimension $m \geqslant 3$, the *Yamabe problem* concerns the existence of constant scalar curvature metrics on M conformal to g. Solutions to this problem, called *Yamabe metrics*, can be characterized variationally as critical points of the Hilbert-Einstein functional restricted to the set $[g]$ of metrics conformal to g. The existence of such solutions are consequence of the successive works of Yamabe [1960], Trudinger [1968], Aubin [1976] and Schoen [1984].

It is said that $g_t, t \in [a, b]$, is a 1-parameter family of solutions to the Yamabe problem if

$$d(\mathcal{A}|_{[g_0]_1})(\hat{g}_t) = 0 \ \ \forall \ \ t \in [a, b],$$

where $\hat{g}_t$ is the unit volume metric homothetic to g_t, $g_1 = g$ and $\mathcal{A}$ is the Hilbert-Einstein functional. The families that we study in this thesis are formed by homogeneous metrics, which are trivial solutions to the Yamabe problem.

A classic method to obtain new solutions of a PDE is to use *bifurcation theory*. In our main result, we found *bifurcation and local rigidity points* among certain homogeneous solutions to the Yamabe problem, namely homogeneous metrics defined on *maximal flag manifolds*. Bifurcation point of a family of such solutions is an accumulation point of other solutions to the Yamabe problem conformal to homogeneous solutions. Already a local rigidity point is an isolated solution to this problem in its conformal class, i.e., not a bifurcation point.

The bifurcation theory applied here is based on finding bifurcation instants for a given 1-parameter family of homogeneous metrics $g_t, 0 < t \leqslant 1$. For this, we analyze the occurrence of *jump* in the Morse index of g_t by using strongly the expressions of the Laplacian defined on (M, g_t) and the scalar curvature of g_t. The Morse index is equal to the number (counting multiplicity) of positive eigenvalues that are less than $\dfrac{\mathrm{scal}(g_t)}{m-1}$, being $\mathrm{scal}(g_t)$ scalar curvature of g_t. These tools allow to find new examples of positively curved manifolds.

Given a *Riemannian submersion with totally geodesic fibers*, a 1-parameter family of other such submersions can be constructed by scaling the original metric of the total space in the direction of the fibers. This family is called *canonical variation*, and a generalization of this is, for example, the *Cheeger deformations*. Particular cases of Riemannian submersion with totally geodesic fibers are *Homogeneous fibrations*, which we use in order to construct canonical variations on maximal flag manifolds.

Applying ideas of bifurcation theory, Paolo Piccione and Renato G. Bettiol obtained in their work [11] [2013] bifurcation and local rigidity instants for canonical variations of round metrics on spheres. These families were constructed from homogeneous metrics on total spaces of Hopf fibrations (the total spaces equipped with such deformed metrics are also often referred to as *Berger spheres*). More precisely, they defined on the total space of each Hopf fibration

$$S^1 \cdots S^{2n+1} \to \mathbb{CP}^n,\ S^3 \cdots S^{4n+3} \to \mathbb{HP}^n,\ S^7 \cdots S^{15} \to S^8(\frac{1}{2})$$

canonical variations depending on one parameter $t > 0$, obtained by scaling the round metric by a factor t^2 in the subbundle tangent to the Hopf fibration. It is also proved that there are classes of the Berger spheres above whose conformal classes contain at least 3 solutions to the Yamabe problem.

Bettiol and Piccione treated in [10]the homogeneous fibrations

$$H/K \cdots G/K \to G/H$$

with $K \subsetneq H \subsetneq G$ compact connected Lie groups, $\dim H/K \geqslant 2$, either H is normal in G or K is normal in H and H/K has positive scalar curvature. From such homogeneous fibration they construct a canonical variation g_t of an initial G-invariant metric $g = g_1$ on the total space G/K. It was proved that in this situation there exists a subset $\mathcal{G} \subset (0, 1[$, accumulating at 0, such that for each $t \in \mathcal{G}$ there are at least 2 solutions to the Yamabe problem in the conformal class $[g_t]_1$, other than $\hat{g}_t$, where $\hat{g}_t$ is the unit volume metric homothetic to g_t.

For the case of canonical variations on nonhomogeneous fibrations, Bettiol and Piccione also obtained similar bifurcations results in their work [10]. Indeed, they proved existence of bifurcations instants accumulating at 0 for the canonical variation of a Riemannian submersion

$$F \cdots M \to B$$

with totally geodesic fibers isometrics to F, under assumption of upper boundaries for Ricci and scalar curvatures.

In this thesis, we deform homogeneous metrics on maximal flag manifolds, total spaces of the following homogeneous fibrations

(i) $SU(n)/T^{n-1} \cdots SU(n+1)/T^n \to SU(n+1)/S(U(1) \times U(n)) = \mathbb{CP}^n, n \geqslant 2;$

(ii) $SO(2n)/T^n \cdots SO(2n+1)/T^n \to SO(2n+1)/SO(2n) = S^{2n}, n \geqslant 2, n \neq 3;$

(iii) $SU(n)/T^{n-1} \cdots Sp(n)/T^n \to Sp(n)/U(n), n \geqslant 3;$

(iv) $SU(n)/T^{n-1} \cdots SO(2n)/T^n \to SO(2n)/U(n), n \geqslant 4;$

(v) $SO(4)/T \cong S^2 \times S^2 \cdots G_2/T \to G_2/SO(4);$

More specifically, we equip each total space above with a *normal homogeneous metric* and deform these metrics by shrinking the fibers by a factor t^2, $0 < t \leqslant 1$, getting the canonical variations $(SU(n+1)/T^n, \mathbf{g}_t), (SO(2n+1)/T^n, \mathbf{h}_t)$, $(Sp(n)/T^n, \mathbf{k}_t), (SO(2n)/T^n, \mathbf{m}_t), (G_2/T, \mathbf{n}_t)$ in Section 2.2.2. Recall that a normal homogeneous metric on G/K is obtained from the restriction to the tangent space (isotropy representation) of a bi-invariant inner product on the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. The maximal flag manifolds are homogeneous spaces such that K is compact, hence they admit bi-invariant metrics and hence normal homogeneous metrics.

A *degeneracy point* for a given canonical variation $g_t, t > 0$, is a degenerate critical point g_{t_*} for the Hilbert-Einstein functional, at some $t_* > 0$. It is established in Section 1.1 that every bifurcation point is a degeneracy point for this canonical variation, however not all degeneracy point is a bifurcation point. Since the Morse index of each $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, changes as t crosses *degeneracy instants* $t_* \in (0,1[$ (Proposition 1.1.9, Section 1.1), we prove existence of new solutions to the Yamabe problem accumulating at $\mathbf{g}_{t_*}, \mathbf{h}_{t_*}, \mathbf{k}_{t_*}, \mathbf{m}_{t_*}, \mathbf{n}_{t_*}$, respectively. Such instants t_* are called *bifurcation instants* and $\mathbf{g}_{t_*}, \mathbf{h}_{t_*}, \mathbf{k}_{t_*}, \mathbf{m}_{t_*}, \mathbf{n}_{t_*}$ are bifurcation points for the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, respectively.

From Proposition 1.1.6 we also determine the local rigidity instants for each family $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$ in the interval $(0,1]$. Indeed, we obtain in Section 3.2 the following results.

Theorem A *For the canonical variations* $(SU(n+1)/T^{n+1}, \mathbf{g}_t)$ *and* $(SO(2n+1)/T^n, \mathbf{h}_t)$*, there are sequences* $\{t_q^{\mathbf{g}}\}, \{t_q^{\mathbf{h}}\} \subset (0,1]$*, of bifurcation instants for* $\mathbf{g}_t$ *and* $\mathbf{h}_t$*, given in* (3.14) *an* (3.16)*, such that* $t_q^{\mathbf{g}}, t_q^{\mathbf{h}} \to 0$ *as* $q \to 0$*. Moreover,* $\mathbf{g}_t$ *and* $\mathbf{h}_t$ *are locally rigid for all* $t \in (0,1] \setminus \{t_q^{\mathbf{g}}\}$ *(respectively* $t \in (0,1] \setminus \{t_q^{\mathbf{h}}\}$*).*

Theorem B *For the canonical variations* $(Sp(n)/T^n, \mathbf{k}_t), n \geqslant 5$*, and* $(SO(2n)/T^n, \mathbf{m}_t), n \geqslant 4$*, introduced in 3.2.2, the bifurcation instants are discrete sets* $\{t^g_{x_1 x_2 \dots x_l}\}_{x_1, x_2, \dots, x_l \in \mathbb{Z}_+} \subset (0,1]$*,* $1 \leqslant l \leqslant n$*, with infinite elements accumulating close to zero as* $x_1, x_2, \dots, x_l$ *vary over* $\mathbb{Z}_+$*. Moreover,* $\mathbf{k}_t$ *and* $\mathbf{m}_t$ *are locally rigid for all* $t \in (0,1] \setminus \{t^g_{x_1 x_2 \dots x_l}\}_{x_1, x_2, \dots, x_l \in \mathbb{Z}_+}$.

Theorem C *The elements of the set* $\{t_{rs}^{\mathbf{n}}\} \subset (0,1]$ *given by*

$$t_{rs}^{\mathbf{n}} = \frac{\sqrt{\sqrt{\left(-66r^2 - 33rs - 99r - 22s^2 - 55s + 24\right)^2 + 64} - 66r^2 - 33rs - 99r - 22s^2 - 55s + 24}}{2\sqrt{2}},$$

$\mathbb{Z} \ni r, s \geqslant 0$*, are bifurcation instants for* $(G_2/T, \mathbf{n}_t)$*. Moreover,* $\mathbf{n}_t$ *is locally rigid at all* $0 < t \leqslant 1$ *such that* $t \notin \{t_{rs}^{\mathbf{n}}\}$*.*

In other words, we proved that for $\{\mathbf{g}_t; 0 < t < 1\}$ and $\{\mathbf{h}_t; 0 < t < 1\}$ there are infinitely many other solutions to the Yamabe problem that accumulate close to the homogeneous solutions $\mathbf{g}_{t_q^{\mathbf{g}}}$ and $\mathbf{h}_{t_q^{\mathbf{h}}}$. For $\mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ the set of bifurcation instants are also (infinite) discrete, however such bifurcation instants depend on several (integer) indexes. In addition, we also determine the sets of local rigidity instants for each canonical variation $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$ in the interval $(0, 1]$.

Our results also can be understood from the view point of dynamical systems, where *bifurcation* means a topological or qualitative change in the structure of the set of fixed points of a 1-parameter family of systems when we vary this paremeter. Critical points of the Hilbert-Einstein functional in a conformal class $[g]$ are fixed points of the so-called *Yamabe flow*, the corresponding L^2-gradient flow of the Hilbert-Einstein functional, which gives a dynamical system in this conformal class. Hence, the bifurcation results above mentioned can be interpreted as a local change in the set of fixed points of the Yamabe flow near homogeneous metrics (which are always fixed points) when varying the conformal class $[g]$ with g in one of the families $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$. An interesting question, proposed by Bettiol and Piccione in [11], would be to study the dynamics near these new fixed points.

In Section 3.1 we calculate expressions for the scalar curvatures $\mathrm{scal}(\mathbf{g}_t), \mathrm{scal}(\mathbf{h}_t), \mathrm{scal}(\mathbf{k}_t), \mathrm{scal}(\mathbf{m}_t), \mathrm{scal}(\mathbf{n}_t)$ applying the formula for scalar curvature of reductive homogeneous spaces due W. Ziller and M. Wang in their work [31]. We also apply the Lie theoretic description of the flag manifolds to determine these formulae in our situation. The coefficients of the invariant homogeneous metrics $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ and the structure constants of the Lie algebras which define each total space in (i)-(v) above were determined for this purpose.

In order to prove the bifurcation results claimed, we analyze the second variation of the Hilbert-Einstein functional at every homogeneous metrics $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, defined on the total spaces in (i)-(v). Given the second variation formula (1.2), Section 1.1, for this functional, this analysis amounts comparing the eigenvalues $\lambda^{kj}(t)$ (see expression (2.3), Corollary 2.2.5) of the Laplacian Δ_t of $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, respectively, with the scalar curvatures $\mathrm{scal}(\mathbf{g}_t), \mathrm{scal}(\mathbf{h}_t), \mathrm{scal}(\mathbf{k}_t), \mathrm{scal}(\mathbf{m}_t), \mathrm{scal}(\mathbf{n}_t)$. More precisely, a critical point $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ or $\mathbf{n}_t$ is degenerate if and only if $\mathrm{scal}(t) \neq 0$ and $\dfrac{\mathrm{scal}(t)}{m-1}$ is an eigenvalue of Δ_t.

The spectrum of the Laplacian Δ_t of the canonical variations of Riemannian submersions with totally geodesic fibers is well-understood. Roughly, it consists of linear combinations (that depends on t) of eigenvalues of the original metric on the total space with eigenvalues of the fibers. In particular, we compute formulae for the first positive eigenvalues of $\Delta_{\mathbf{g}_t}$ and $\Delta_{\mathbf{h}_t}$, in addition to lower and upper bounds for the first positive eigenvalues of $\Delta_{\mathbf{k}_t}$, $\Delta_{\mathbf{m}_t}$ and $\Delta_{\mathbf{n}_t}$, respectively (Proposition 2.2.22). Combining this knowledge of the spectra of $\Delta_{\mathbf{g}_t}$, $\Delta_{\mathbf{h}_t}$, $\Delta_{\mathbf{k}_t}$, $\Delta_{\mathbf{m}_t}$ and $\Delta_{\mathbf{n}_t}$ with the formulae for the scalar curvatures $\mathrm{scal}(\mathbf{g}_t), \mathrm{scal}(\mathbf{h}_t), \mathrm{scal}(\mathbf{k}_t), \mathrm{scal}(\mathbf{m}_t), \mathrm{scal}(\mathbf{n}_t)$ we are able to identify all degeneracy instants for $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$ and prove existence of bifurcation at all degeneracy instants in the interval $(0, 1[$. The local rigidity instants for these canonical variations are also determined for all other $0 < t < 1$ (by applying our results, namely Proposition 3.2.2, Lemma 3.2.2 and

Lemma 3.2.3). We also compute the Morse index of each $\mathbf{g}_t$ and $\mathbf{h}_t$, $0 < t \leqslant 1$ (Proposition 3.2.5 and Proposition 3.2.7).

We remark that the induced metrics on the fibers and on the basis of the homogeneous fibrations in (i)-(v) are such that the fibers become some type of maximal flag manifold endowed with a normal homogeneous metric and the basis spaces are compact isotropy irreducible Hermitian symmetric spaces. The spectra of the Laplacian on these spaces are described in terms of weights of some irreducible representations. In respect to spectra of maximal flag manifolds, we use the description of Yamaguchi [1979] [33] where the spectra of the Laplacian on maximal flag manifolds endowed with a normal homogeneous metric are specified in terms of the representation theory of the related compact Lie groups. Already in the case of the spectrum of the Laplacian on compact isotropy irreducible Hermitian symmetric space G/H, we used in this thesis the description presented in [30] and [29]. In this case, the spectrum is given in terms of integer coefficients of highest weights of spherical representations of the symmetric pair (G, H).

In addition by exploring the existence of infinitely many bifurcations of $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, we obtain in Section 3.3 the following multiplicity result.
Theorem D *Let $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ be the families of homogeneous metrics obtained as described above. Then, there exists, for each of such families, a subset $\mathcal{G} \subset (0, 1[$, accumulating at 0, such that for each $t \in \mathcal{G}$, there are at least 3 solutions to the Yamabe problem in each conformal class $[\mathbf{g}_t], [\mathbf{h}_t], [\mathbf{k}_t], [\mathbf{m}_t], [\mathbf{n}_t]$.*

Renato G. Bettiol and Paolo Piccione proved in [10] a result that guarantees existence of other solutions to the Yamabe problem in the conformal class $[g_t]$ of a canonical variation g_t if its Morse index is positive over some subset of $(0, 1)$. Theorem D is a consequence of this result.

We have restricted our study to the interval $(0, 1)$ since for $t > 1$ would be necessary more refinements for the expressions of the eigenvalues $\lambda^{kj}(t)$ of the Laplacian Δ_t on the total spaces of the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$, respectively. For the above mentioned canonical variations of the Hopf fibrations, Bettiol and Piccione in [11] obtained refinements for expressions of the eigenvalues $\lambda^{kj}(t)$ applying standard theory of spherical harmonics on spheres. This enabled them obtaining bifurcation results on spheres for all $t > 0$.

An extension of the results presented in this thesis consists to find all bifurcation and local rigidity instants for the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ in $(0, +\infty)$. To do so, it is necessary, in principle, a more detailed description of the spectra of the Laplacians of such canonical variations. In order to study cononical variations defined on total spaces more general than maximal flag manifolds, one can consider initially homogeneous fibrations on *partial flag manifolds*, as long as the spectra of the Laplacians on the basis and on the fiber have well-known descriptions. Furthermore, another approach is to study homogeneous fibrations without hypothesis of normality of subgroups considered by Bettiol and Piccione in [10].

This thesis is organized as follows. In Chapter 1 we introduce the variational characterization of the Yamabe problem and the basic notions of bifurcation and local rigidity, in addition to establish the definitions of generalized flag manifold, invariant metrics and isotropy representation. The study of the Laplacian of the Riemannian submersions with totally geodesic fibers, the definition of the canonical variations and the description of their spectra, as well as the construction of the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ are studied in Chapter 2. In Chapter 3 we obtain formulae for scalar curvature of $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$. We prove also the main results of the thesis, namely Theorems A, B, C and D, on bifurcation and local rigidity of solutions to the Yamabe problem, for the families $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ respectively. We also determine the Morse index of $\mathbf{g}_t$ and $\mathbf{h}_t$. The last section of this chapter contains multiplicity of solutions to the Yamabe problem for all the families previously introduced.

1 Preliminaries

In this chapter, we present some necessary prerequisites for the development of this work.

1.1 Variational Setup For the Yamabe Problem

Before we present the notions of bifurcation and local rigidity instants, we will describe the set $\mathcal{R}^k(M)$ of all C^k *Riemannian metrics* on M, $m = \dim M \geqslant 3$, and some properties of the Hilbert-Einstein functional. For further references and details of the concepts and properties presented here see, e.g, [11], [14] and [28].

Let g_R be a fixed auxiliary Riemannian metric on M; g_R and its Levi-Civita connection ∇^R induce naturally Riemannian metrics and connections on all tensor bundles over M, respectively. For each $k \geqslant 0$, denote by $\Gamma^k(TM^* \vee TM^*)$ the space of symmetric C^k sections of $TM^* \otimes TM^*$, i.e, symmetric $(0,2)$-tensors of class C^k on M. This becomes a Banach space when equipped with the C^k norm

$$\|\tau\|_{C^k} = \max_{j=1,\dots,k} \left(\max_{x \in M} \left\| (\nabla^R)^j \tau(x) \right\|_R \right),$$

where $\|\cdot\|_R$ denote the norms induced by g_R on each appropriate space.

Note that the set $\mathcal{R}^k(M)$ of all C^k *Riemannian metrics* on M is a open convex cone inside $(\Gamma^k(TM^* \vee TM^*), \|\cdot\|_{C^k})$ and, therefore, contractible. Thus, $\mathcal{R}^k(M)$ inheriting a natural differential structure. We remark also that $\mathcal{R}^k(M)$ is a open set of a Banach space and, thus, we can identify its tangent space with the vector space $\Gamma^k(TM^* \vee TM^*)$. From now on, we assume that $k \geqslant 3$.

For each $g \in \mathcal{R}^k(M)$, denote by vol_g the volume form on M (we assume that M is orientable); in this case, $L^2(M, \mathrm{vol}_g)$ will denote the usual Hilbert space of real square integrable functions on M. Consider over $\mathcal{R}^k(M)$ the maps

$$\mathrm{scal} : \mathcal{R}^k(M) \longrightarrow C^{k-2}(M) \quad \text{and} \quad \mathrm{Vol} : \mathcal{R}^k(M) \longrightarrow \mathbb{R},$$

the *scalar curvature* and the *volume*, that for each Riemannian metric $g \in \mathcal{R}^k(M)$ associate, respectively, its scalar curvature $\mathrm{scal}(g) : M \longrightarrow \mathbb{R}$ and its volume $\mathrm{Vol}(g) = \displaystyle\int_M \mathrm{vol}_g$. Define the *Hilbert-Einstein functional* as the function $A : \mathcal{R}^k(M) \longrightarrow \mathbb{R}$ given by

$$\mathcal{A}(g) = \frac{1}{\mathrm{Vol}(g)} \int_M \mathrm{scal}(g) \mathrm{vol}_g. \tag{1.1}$$

In order to enunciate some properties of the Hilbert-Einstein functional, we will establish the more appropriate regularity for our manifold of metrics and maps. Denote by $\mathcal{R}^k_1(M) =$

$\mathrm{Vol}^{-1}(1)$ the suset of $\mathcal{R}^k(M)$ consisting of unit volume metrics volume. Note that $\mathcal{R}_1^k(M)$ is a smooth embedded codimension 1 submanifold of $\mathcal{R}^k(M)$.

For each $g \in \mathcal{R}^k(M)$, the conformal class of g is the set $[g] \subset \mathcal{R}^k(M)$ formed by conformal metrics to g. The space $\Gamma^k(TM^* \vee TM^*)$ defined above induces a differential structure on each conformal class.

Later, some Fredholm's conditions must be satisfied. For this, it is enough we introduce the notion of $C^{k,\alpha}$ *conformal class* and, therefore, define on the conformal class $g \in \mathcal{R}^k(M)$ a Hölder $C^{k,\alpha}(M)$ regularity. Let

$$[g]_{k,\alpha} = \left\{\phi g; \phi \in C^{k,\alpha}(M), \quad \phi > 0\right\}$$

be the $C^{k,\alpha}(M)$ *conformal class* of g, which can be identified with the open subset of $C^{k,\alpha}(M)$ formed by the positive functions, which allows $[g]_{k,\alpha}$ to have a natural differential structure. Working with this regularity, the necessary Fredholm's conditions of the second variation of the Hilbert-Einstein functional are satisfied. The set

$$\mathcal{R}_1^{k,\alpha}(M,g) = \mathcal{R}_1^k(M) \cap [g]_{k,\alpha}$$

is a smooth embedded codimension 1 submanifold of $[g]_{k,\alpha}$. Proposition 1.1.1 contains well-known facts about $\mathcal{A}$ and its critical points, see, e.g. [14].

Proposition 1.1.1. *The following hold for the functional $\mathcal{A}$:*

(i) *The functional $\mathcal{A}$ is smooth on $\mathcal{R}^k(M)$ and on the $C^{k,\alpha}$ conformal class of any metric $g \in \mathcal{R}^k(M)$. In particular, $\mathcal{A}$ is smooth on the submanifolds $\mathcal{R}_1^{k,\alpha}(M,g)$ and $\mathcal{R}_1^k(M)$*

(ii) *The metric $g_0 \in \mathcal{R}_1^k(M)$ is a critical point of $\mathcal{A}$ on $\mathcal{R}_1^k(M)$ if and only if it is a Einstein metric;*

(iii) *A metric $g_0 \in \mathcal{R}_1^{k,\alpha}(M,g)$ is a critical point of $\mathcal{A}$ on $\mathcal{R}_1^{k,\alpha}(M,g)$ if and only if it has constant scalar curvature;*

(iv) *At a critical point $g_0 \in \mathcal{R}_1^{k,\alpha}(M,g)$ of $\mathcal{A}$, the second variation $\mathrm{d}^2\mathcal{A}(g_0)$ can be identified with the quadratic form*

$$\mathrm{d}^2\mathcal{A}(g_0)(\psi,\psi) = \frac{m-2}{2}\int_M ((m-1)\Delta_{g_0}\psi - \mathrm{scal}(g_0)\psi)\psi \mathrm{vol}_{g_0}, \qquad (1.2)$$

defined on the tangent space at $g_0 = \phi g$, given by

$$T_{g_0}\mathcal{R}_1^{k,\alpha}(M,g) \cong \left\{\psi \in C^{k,\alpha}(M); \int_M \frac{\psi}{\phi}\mathrm{vol}_{g_0} = 0\right\}.$$

Remark 1.1.2. We are considering $\Delta_g = -\mathrm{div}_g \circ \mathrm{grad}_g$, the Laplacian operator of (M,g), acting on $C^\infty(M)$. We observe that, given $\lambda \in \mathbb{R}^+$, one has $\Delta_{\lambda g} = \dfrac{1}{\lambda}\Delta_g$ e $\mathrm{scal}(\lambda g) =$

$\frac{1}{\lambda}\mathrm{scal}(g)$. Therefore, the negative eigenvalues (respect. positive) of Δ_g remain negative (respect. positive), that is, we can normalize metrics in order to have unit volume, without changing the spectral theory of the operator $\Delta_g - \dfrac{\mathrm{scal}(g)}{m-1}\cdot \mathrm{Id}$, $m = \dim M$.

Now, we will introduce the concepts of *bifurcation* and *local rigidity* for a 1-parameter family of solutions to the Yamabe problem. Let

$$[a,b] \ni t \mapsto g_t \in \mathcal{R}^k(M), \quad k \geqslant 3,$$

or simply g_t, denote a smooth 1-parameter family (smooth path) of Riemannian metrics on M, such that each g_t has constant scalar curvature.

Definition 1.1.3. *An instant $t_* \in [a,b]$ is a* bifurcation instant *for the family g_t if there exists a sequence $\{t_q\}$ in $[a,b]$ that converges to t_* and a sequence $\{g_q\}$ in $\mathcal{R}^k(M)$ of Riemannian metrics that converges to g_{t_*} satisfying*

(i) $g_{t_q} \in [g_q]$, *but* $g_q \neq g_{t_q}$;

(ii) $\mathrm{Vol}(g_q) = \mathrm{Vol}(g_{t_q})$;

(iii) $\mathrm{scal}(g_q)$ *is constant.*

If $t_* \in [a,b]$ is not a bifurcation instant, it is said that the family g_t is *locally rigid* at t_*. More precisely, the family is locally rigid at $t_* \in [a,b]$ if there exists an open set $U \subset \mathcal{R}^k(M)$ containing g_{t_*} such that if $g \in U$ is another metric with constant scalar curvature and there exists $t \in [a,b]$ with $g_t \in U$ and

(a) $g \in [g_t]$;

(b) $\mathrm{Vol}(g) = \mathrm{Vol}(g_t)$,

then $g = g_t$.

In order to identify the occurrence of the above two situations, we must define *degeneracy instant* and Morse index of g_t.

Definition 1.1.4. *An instant $t_* \in [a,b]$ is a* degeneracy instant *for the family g_t if* $\mathrm{scal}(g_{t_*}) \neq 0$ *and* $\dfrac{\mathrm{scal}(g_{t_*})}{m-1}$ *is an eigenvalue of the Laplacian operator Δ_{t_*} of g_{t_*}.*

Remark 1.1.5. In face of the second variation expression of the Hilbert-Einstein functional, given in Proposition 1.1.1, item (iv), the above definition is equivalent to the fact that g_{t_*} being a degenerate critical point (in the usual sense of Morse theory) of the Hilbert-Einstein functional on $\mathcal{R}_1^{k,\alpha}(M,g)$.

The *Morse index* $N(g_t)$ of g_t is given by the number of positive eigenvalues of Δ_t counted with multiplicity that are less than $\dfrac{\mathrm{scal}(g_{t_*})}{m-1}$. For each $t > 0$, $N(g_t)$ is a non-negative integer number. In other words, $N(g_t)$ counts the number of directions which the functional decreases, since the second variation is negative definite.

Next, we will present a general criterion for classifying local rigidity instants for a 1-parameter family g_t of solutions to the Yamabe problem.

Proposition 1.1.6 ([11]). *Let g_t be a smooth path of metrics of class C^k, $k \geqslant 3$, such that $\mathrm{scal}(g_t)$ is constant for all $t \in [a, b]$, and let Δ_t be the Laplacian operator of g_t. If t_* is not a degeneracy instant of g_t, then g_t is locally rigid at t_*.*

Corollary 1.1.7 ([11]). *Suppose that, in addition to the hypotheses of Proposition 2, there exists an instant t_* when $\dfrac{\mathrm{scal}(g_{t_*})}{m-1}$ is less than the first positive eigenvalue $\lambda_1(t_*)$ of Δ_{t_*}. Then g_{t_*} is a local minimum for the Hilbert-Einstein functional in its conformal class. In particular, g_t locally rigid at t_*.*

Remark 1.1.8. Note that, in fact, the Corollary 1.1.7 is an immediate consequence from Proposition 1.1.6, since Morse index of the non-degenerate critical point g_{t_*} is $N(g_{t_*}) = 0$. In addition, we remark that, if t_* is a bifurcation instant for g_t, then t_* is necessarily a degeneracy instant for g_t. However, the reciprocal is not true in general.

The next result provides a sufficient condition to determinate if a degeneracy instant is a bifurcation instant, given in terms of a jump in the Morse index when passing a degeneracy instant.

Proposition 1.1.9 ([11]). *Let g_t a smooth path of metrics of class C^k, $k \geqslant 3$, such that $\mathrm{scal}(g_t)$ is constant for all $t \in [a, b]$, Δ_t the Laplacian g_t and $N(g_t)$ the Morse index of g_t. Assume that a and b are not degeneracy instants for g_t and $N(g_a) \neq N(g_b)$. Then, there exists a bifurcation instant $t_* \in]a, b[$ for the family g_t.*

1.2 Generalized Flag Manifolds

Let G be a compact semissimple Lie group and let $x \in \mathfrak{g}$ be a regular element, i.e, its centralizer is equal to a maximal toral subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$, Lie algebra of G. The *adjoint orbit* of x is the set $M = G \cdot x = \mathrm{Ad}(G)x \subset \mathfrak{g}$. Let

$$K = G_x = \{g \in G : \mathrm{Ad}(g)x = x\} \subset G$$

be isotrpy subgroup of x. Then M is diffeomorphic to the left coset G/K.

Let x_0 be the point which corresponds to the class $1 \cdot K \in G/K$ ($1 \in G$ is the identity element of G). If we take $T_{x_0} = \overline{\exp \mathbb{R} x_0}$, it is well known that T_{x_0} is a toral subgroup of G (diffeomorphic to the Lie group $S^1 \times S^1 \times \ldots \times S^1$). Furthermore, one has $K = K_x = C(T_x)$, where $C(T_x)$ is the centralizer of T_x in G. If T_x is maximal, $C(T_x) = T_x$.

Using the above notations, we can introduce the concept of *generalized flag manifold* which we will consider in this work.

Definition 1.2.1. *Let G be a compact semissimple Lie group with Lie algebra $\mathfrak{g}$. A* generalized flag manifold *is the adjoint orbit of a regular element in the Lie algebra $\mathfrak{g}$. Equivalently, a generalized flag manifold is a homogeneous space of the form $G/C(T)$, where T is a torus in G. When T is maximal, it is said that $G/C(T) = G/T$ is a* maximal flag manifold*.*

Example 1.2.2 (Maximal flags in $\mathbb{C}^n$)**.** Let $G = U(n)$ be unitary Lie group, with $\mathfrak{g} = \mathfrak{u}(n)$. Taking $x = \mathrm{diag}(i\lambda_1, \ldots, i\lambda_n)$, with $\lambda_1, \ldots, \lambda_n$ distinct real numbers, we have $K = K_x = T_n = S^1 \times \ldots \times S^1$ (n times), a n-torus in $U(n)$. Hence, $\mathrm{Ad}(U(n))x \cong U(n)/T_n$. If F_n is the set of all increasing sequences $\{V_1 \subset V_2 \subset \ldots \subset V_{n-1}\}$ of complex vector spaces of $\mathbb{C}^n$ (flags of subspaces), one has $U(n)$ acts on F_n by $\{gV_1 \subset gV_2 \ldots \subset gV_{n-1}\}$, $\forall g \in U(n)$, and the isotropy subgroup on the flag $e = \{[e_1, \ldots, e_{r_1}] \subset [e_1, \ldots, e_{r_2}] \subset \ldots \subset [e_1, \ldots e_{r_{n-1}}]\}$ is T_n. Therefore, the adjoint orbit of x can be identified with the homogeneous space $U(n)/U(1) \times \ldots \times U(1)$ (n times). This space is also diffeomorphic to the homogeneous space $SU(n)/S(U(1) \times \ldots \times U(1))$, where $SU(n)$ is the special unitary Lie group given by the unitary matrices A with $\det A = 1$.

Example 1.2.3 (Partial Flags in $\mathbb{C}^n$)**.** Consider $G = SU(n)$ with $\mathfrak{g} = \mathfrak{su}(n)$. Take $x = \mathrm{diag}(i\lambda_1 I_{n_1}, \ldots, i\lambda_s I_{n_s})$ with $\lambda_1 n_1 + \ldots + \lambda_s n_s = 0$ and I_{n_i} the $n_i \times n_i$ identity matrix. It Follows that $\mathrm{Ad}(SU(n))x \cong SU(n)/S(U(n_1) \times \ldots \times U(n_s))$, with $n = n_1 + \ldots + n_s$. This adjoint orbit also can be identified with the set $F(n_1, \ldots n_s)$ of all partial flags $\{V_1 \subset V_2 \subset \ldots \subset V_s\}$ in $\mathbb{C}^n$ with $\mathrm{dim}V_i = n_1 + \ldots + n_i$, $1 \leqslant i \leqslant s$.

Example 1.2.4. Flag manifolds of a Classical Lie group:

A: $SU(n)/S(U(n_1) \times \ldots \times U(n_s) \times U(1)^m)$,

$n = n_1 + \ldots + n_s + m, n_1 \geqslant n_2 \geqslant \ldots \geqslant n_s > 1, s, m \geqslant 0$

B: $SO(2n+1)/U(n_1) \times \ldots \times U(n_s) \times SO(2t+1) \times U(1)^m$

C: $Sp(n)/U(n_1) \times \ldots \times U(n_s) \times Sp(t) \times U(1)^m$

D: $SO(2n)/U(n_1) \times \ldots \times U(n_s) \times Sp(2t) \times U(1)^m$

$n = n_1 + \ldots + n_s + m + t, n_1 \geqslant n_2 \geqslant \ldots \geqslant n_s > 1, s, m, t \geqslant 0, t \neq 1$

We will define generalized flag manifold from a complex Lie algebra.

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semissimple Lie algebra. Given a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, denote by R the set of *roots* of $\mathfrak{g}^{\mathbb{C}}$ with relation to $\mathfrak{h}^{\mathbb{C}}$. Consider the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where $\mathfrak{g}^{\mathbb{C}}_{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}}; \forall H \in \mathfrak{h}^{\mathbb{C}}, [X, H] = \alpha(H)X\}$. Let $R^{+} \subset R$ be a choice of positive roots, Σ the correspondent system of simple roots and Θ a subset of Σ. Will be denoted by $\langle \Theta \rangle$ the span of Θ, $R_M = R \backslash \langle \Theta \rangle$ the set of the *complementary roots* and by R_{M^+} the set of positive complementary roots.

A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$ is called *parabolic* if it contains a Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$ (i.e., a maximal soluble Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$). Take

$$\mathfrak{p}_{\Theta} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \langle \Theta \rangle^{+}} \mathfrak{g}^{\mathbb{C}}_{\alpha} \oplus \sum_{\alpha \in \langle \Theta \rangle^{+}} \mathfrak{g}^{\mathbb{C}}_{-\alpha} \oplus \sum_{\beta \in \Pi^{+}_{M}} \mathfrak{g}^{\mathbb{C}}_{\beta}$$

the canonical parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ determined by Θ which contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum\limits_{\beta \in \Pi_{+}} \mathfrak{g}^{\mathbb{C}}_{\beta}$.

The generalized flag manifold $\mathbb{F}_{\Theta}$ associated with $\mathfrak{g}^{\mathbb{C}}$ is defined as the homogeneous space

$$\mathbb{F}_{\Theta} = G^{\mathbb{C}}/P_{\Theta},$$

where $G^{\mathbb{C}}$ is the complex compact connected simple Lie group with Lie algebra $\mathfrak{g}$ and $P_{\Theta} = \{g \in G^{\mathbb{C}}; \mathrm{Ad}(g)\mathfrak{p}_{\Theta} = \mathfrak{p}_{\Theta}\}$ is the normalizer of $\mathfrak{p}_{\Theta}$ in $G^{\mathbb{C}}$.

Let G be the real compact form of $G^{\mathbb{C}}$ corresponding to $\mathfrak{g}$, i.e, G is the connected Lie Group with Lie algebra $\mathfrak{g}$, the compact real form of $\mathfrak{g}^{\mathbb{C}}$. The subgroup $K_{\Theta} = G \cap P_{\Theta}$ is the centralizer of a torus. Furthermore, G acts transitively on $\mathbb{F}_{\Theta}$. Since G is compact, we have that $\mathbb{F}_{\Theta}$ is a compact homogeneous space, that is,

$$\mathbb{F}_{\Theta} = G^{\mathbb{C}}/P_{\Theta} = G/G \cap P_{\Theta} = G/K_{\Theta},$$

accordingly characterization given above from adjoint orbits of a regular element of $\mathfrak{g}$.

There are two classes of generalized flag manifolds. The first occurs when $\Theta = \varnothing$. Therefore, the parabolic subalgebra is given by $\mathfrak{p}_{\Theta} = \mathfrak{h}^{\mathbb{C}} \oplus \sum\limits_{\beta \in \Pi_{+}} \mathfrak{g}^{\mathbb{C}}_{\beta}$, that is, is equal to the Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and $T = P_{\Theta} \cap G$ is a maximal torus. In this case, $\mathbb{F}_{\Theta} = G^{\mathbb{C}}/P_{\Theta} = G/G \cap P_{\Theta} = G/T$ is called *maximal* flag manifold. When $\Theta \neq \varnothing$, $\mathbb{F}_{\Theta}$ is called *partial* flag manifold.

We consider now a *Weyl basis* for $\mathfrak{g}^{\mathbb{C}}$ given by $\{X_{\alpha}\}_{\alpha \in R} \cup \{H_{\alpha}\}_{\alpha \in \Sigma} \subset \mathfrak{g}^{\mathbb{C}}$, $X_{\alpha} \in \mathfrak{g}^{\mathbb{C}}_{\alpha}$. From this basis we determine a basis for $\mathfrak{g}$, the compact real form of $\mathfrak{g}^{\mathbb{C}}$, putting

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}H_{\alpha}, A_{\alpha}, S_{\alpha}\},$$

with $A_\alpha = X_\alpha - X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha + X_{-\alpha})$ ($A_\alpha = S_\alpha = 0$ if $\alpha \notin R$) and $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ is such that $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$, $\alpha \in R$. The *structure constants* of this basis are determined by the following relations

$$\begin{cases} [A_\alpha, S_\beta] &= m_{\alpha,\beta}A_{\alpha+\beta} + m_{-\alpha,\beta}A_{\alpha-\beta} \\ [S_\alpha, S_\beta] &= -m_{\alpha,\beta}A_{\alpha+\beta} - m_{\alpha,-\beta}A_{\alpha-\beta} \\ [A_\alpha, S_\beta] &= m_{\alpha,\beta}S_{\alpha+\beta} + m_{\alpha,-\beta}S_{\alpha-\beta} \end{cases},$$

$$\begin{cases} [\sqrt{-1}H_\alpha, A_\beta] &= \beta(H_\alpha)S_\beta \\ [\sqrt{-1}H_\alpha, S_\beta] &= -\beta(H_\alpha)A_\beta \\ [A_\alpha, S_\alpha] &= 2\sqrt{-1}H_\alpha \end{cases},$$

where $m_{\alpha,\beta}$ is such that $[X_\alpha, X_\beta] = m_{\alpha,\beta}X_{\alpha+\beta}$, with $m_{\alpha,\beta} = 0$ if $\alpha + \beta \notin R$ and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$. We remark that this basis is $-B$-orthogonal and $-B(A_\alpha, A_\alpha) = -B(S_\alpha, S_\alpha) = 2$, where B Cartan-Killing form of $\mathfrak{g}^{\mathbb{C}}$ (the Cartan-Killing form of $\mathfrak{g}^{\mathbb{C}}$ restricted to $\mathfrak{g}$ coincides with the Cartan-Killing form of its compact form $\mathfrak{g}$ and $\mathfrak{h}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}H_\alpha\}_{\alpha \in R}$ is a Cartan subalgebra of $\mathfrak{g}$). Moreover, if $\mathfrak{q}$ and $\mathfrak{s}$ are the subspaces spanned by $\{H_\alpha, A_\alpha\}$ and $\{S_\alpha\}$ respectively, one has

$$[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{s}, \quad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{q}.$$

The above construction of the Weyl basis can be found, e.g., in [26], p. 334. This will be the basis used when dealing with flag manifolds in this work.

Remark 1.2.5 ([1]). An invariant order R_M^+ in R_M is a choice of a subset $\mathbb{R}_M^+$ such that

(i) $R = R_K \cup R_M^+ \cup R_M^-$, where $R_M^- = \{-\alpha; \alpha \in R_M^+\}$,

(ii) If $\alpha \in R_K \cup R_M^+$, $\beta \in R_K \subset R_M^+$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R_M^+$,

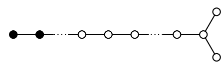
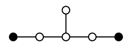
where $\mathbb{R}_K = \langle \Theta \rangle \cap R$. One has $\alpha > \beta$ if and only if $\alpha - \beta \in R_M^+$. Note that the choice $R_M^+ = R^+ \backslash \langle \Theta \rangle^+$ determines an invariant order in R, called natural invariant order.

Generalized flag manifolds $M = G/K$ can be classified by mean the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$, the Lie algebra of $G^{\mathbb{C}}$. Indeed, let $\Gamma = \Gamma(\Sigma)$ be the Dynkin diagram of the set of simple roots of R. Painting black the nodes of Γ corresponding to simple roots in R_M we obtain the *Dynkin diagram of* $M = G/K$, denotaded here by $\Gamma(\Sigma \backslash \Theta)$. The subdiagram formed by white nodes connected by lines determines the semissimple part of the Lie algebra of K, and each black node determines a $\mathfrak{u}(1)$-summand of this Lie algebra of K. Therefore, this painted Dynkin diagram determines the reducible decomposition of $\mathfrak{g}$, in addition to determine the flag manifold M. The case where all nodes of $\Gamma(\Sigma)$ have been painted black corresponds to the manifold G/T of maximal flags, where T is a maximal torus in G.

Proposition 1.2.6. ([3]) *If G is not $SO(2n)$, and it is not a excepitional Lie group, then different painted connected Dynkin diagrams Γ and Γ_1 define equivalent flag manifolds G/K and G'/K', i.e, there exists a isomorphism $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(K) = K'$, if the subdiagrams Γ' and Γ'_1 of white roots corresponding to R_K and $R_{K'}$ are isomorphic.*

From the above proposition, it is possible to give a complete list of all flag manifolds G/K, where G is either a classical or an exceptional Lie group (up to isomorphism).

Example 1.2.7. Painted Dynkin diagrams of some generalized flag manifolds are given in the next table.

G/K	$\Sigma\backslash\Theta$	$\Gamma(\Sigma\backslash\Theta)$
$SU(6)/S(U(2) \times U(2) \times U(2))$	$\{\alpha_2, \alpha_4\}$	
$SO(2n+1)/T^n$	$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$	
$SO(2n)/SO(2(n-2)) \times U(1) \times U(1)$	$\{\alpha_1, \alpha_2\}$	
$E_6/SO(8) \times U(1) \times U(1)$	$\{\alpha_1, \alpha_5\}$	

1.3 Invariant Metrics and Isotropy Representation

Let $M = G/K$ be a homogeneous space, provided with a transitive and differentiable action $G \times M \longrightarrow M$ of the Lie group G over M, where K is the isotropy subgroup of G on some point of M. A Riemannian metric $m(\cdot, \cdot)$ on M is said to be *G-invariant* by the the action of G or if the elements of this group are isometry of the metric m, i.e., if for all $h \in G$ and $x \in M$

$$m_{hx}(dh_x u, dh_x v) = m_x(u, v), \forall u, v \in T_x M.$$

Note that the G-invariant metric over a homogeneous space $M = G/K$ is completely determined by its value m_{x_0} at the origin $x_0 = 1.K$, where $1 \in G$ is the identity of G. In this case, m_{x_0} is a inner product over $T_{x_0}M$. In fact, we can define for each $x = gK \in G/K$,

$$m_x(u, v) = m_{gK}(u, v) = m_{x_0}(d(g^{-1})_{gK}u, d(g^{-1})_{gK}v),$$

$\forall u, v \in T_{gK}M$ e $g \in G$.

1.3.1 Isotropy Representation

From the notion of *isotropy representation* we can determine invariant metrics on certain homogeneous spaces as follows.

Let $G \times M \longrightarrow M$ be a defferentiable and transitive action of a Lie group G on the homogeneous space (M, m) endowed with a G-invariant metric m. Given $x \in G$, let $K = G_x$ be the isotropy subgroup of x. The *isotropy representation* of K is the homomorphism $g \in K \mapsto dg_x \in \mathrm{Gl}(T_x M)$. Note that m_{x_0} is a inner product on $T_{x_0}(G/K)$, invariant by such representation, where $x_0 = 1 \cdot K$.

A homogeneous space G/K is *reducible* if G has a Lie algebra $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

with $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. If K is compact, this decomposition always exists, namely, if we take $\mathfrak{m} = \mathfrak{k}^{\perp}$, $-B$-orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$, where B is the Cartan-Killing form of $\mathfrak{g}$.

If G/K is reducible, then the isotropy representation of K is equivalent to $\mathrm{Ad}|_K$, restriction to K of the adjoint representation of G in $\mathfrak{m}$:

$$j(k) = \mathrm{Ad}(k)|_{\mathfrak{m}}, \forall k \in K.$$

In particular, the same assertion holds if K is compact. Indeed, for each $k \in K$, if

- $\mathcal{P} : \mathfrak{m} \longrightarrow T_{x_0}(G/K), X \mapsto \mathcal{P}(X) = \widetilde{X}(x_0) = \dfrac{d}{dt}(\mathrm{Ad}(\exp(tX))x_0|_{t=0},$
- $\mathrm{Ad}(k) : \mathfrak{m} \longrightarrow \mathfrak{m}, \mathrm{Ad}(k)X = d(C_k)_1 X, (C_k : K \longrightarrow K, C_k(h) = khk^{-1})$
- $j(k) : T_{x_0}(G/K) \longrightarrow T_{x_0}(G/K), j(k)\widetilde{X} = dk_{x_0}\widetilde{X},$

the following diagram commutes

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\mathrm{Ad}(k)"] \arrow[d, "\mathcal{P}"'] & \mathfrak{m} \arrow[d, "\mathcal{P}"] \\ T_{x_0}(G/K) \arrow[r, "j(k)"'] & T_{x_0}(G/K) \end{tikzcd}$$

being $\mathcal{P}$ a linear isomorphism.

A representation of a compact Lie group K is always orthogonal (preserves inner product) on the representation space. We can conclude that every reductive homogeneous space G/K has a G-invariant metric, since such a metric is completely determined by an inner product on the tangent space at the origin $T_{x_0}(G/K)$.

We remark that the set of all G-invariant metrics on G/K is in 1-1 correspondence the set of inner products $\langle , \rangle$ on $\mathfrak{m}$, invariant by $\mathrm{Ad}(k)$ on$\mathfrak{m}$, for each $k \in K$, by virtue the equivalence between the adjoint and isotropy representations on K cited above:

$$\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle = \langle X, Y \rangle, \forall X, Y \in \mathfrak{m}, k \in K.$$

The isotropy representation of K leaves $\mathfrak{m}$ invariant, i.e, $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ and decomposes it into irreducible submodules

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \ldots \oplus \mathfrak{m}_n.$$

These submodules are inequivalent each other, which happens when G/K is a generalized flag manifold. The submodules $\mathfrak{m}_i$ are called *isotropy summands.*

It follows that a G-invariant metric g on G/K is represented by a inner product

$$g_{1\cdot K} = t_1 Q|_{\mathfrak{m}_1} + t_2 Q|_{\mathfrak{m}_2} + \ldots + t_n Q|_{\mathfrak{m}_n}$$

on $\mathfrak{m}$, with t_i positive constants and Q is (the extension of) a inner product on $\mathfrak{m}$, $\mathrm{Ad}(K)$-invariant.

In particular, if $Q = (-B)$, with B Cartan-Killing form of G and $t_i = 1$ for all $1 \leqslant i \leqslant n$ above, the G-invariant metric g on G/K represented by the inner product

$$g_{1\cdot K} = (-B)|_{\mathfrak{m}_1} + (-B)|_{\mathfrak{m}_2} + \ldots + (-B)|_{\mathfrak{m}_n}$$

on $\mathfrak{m}$ is called *normal metric.*

Example 1.3.1. Consider $K = T^2 = S(U(1) \times U(1) \times U(1))$, $H = S(U(2) \times U(1))$, $G = SU(3)$. By using the above notations, we have the $(-B)$-orthogonal decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{k} \oplus \mathfrak{m},$$

with $B(X,Y) = 6\mathrm{Tr}(XY), \quad X, Y \in \mathfrak{g}$, being the Cartan-Killing form of $\mathfrak{g}$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, with

$$\left\{ \begin{array}{rcl} \mathfrak{k} & = & \{d = \mathrm{diag}(ia, ib, -i(a+b)) \in \mathfrak{su}(3); a, b \in \mathbb{R}\} \\ \mathfrak{m}_1 & = & \mathrm{spam}_{\mathbb{C}}\,(E_{12} - E_{21}) \subset \mathfrak{su}(3) \\ \mathfrak{m}_2 & = & \mathfrak{m}_2 = \mathrm{spam}_{\mathbb{C}}\,(E_{13} - E_{31}) \subset \mathfrak{su}(3) \\ \mathfrak{m}_3 & = & \mathfrak{m}_3 = \mathrm{spam}_{\mathbb{C}}\,(E_{23} - E_{32}) \subset \mathfrak{su}(3) \end{array} \right. , \tag{1.3}$$

where E_{ij} is the matrix having 1 in the (i,j) position and 0 elsewhere.

The decomposition of the isotropy representation $\mathfrak{m}$ of T^2 into pairwise inequivalent irreducible $\mathrm{Ad}(T^2)$-modules $\mathfrak{m}_i$ is given by

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

and an invariant metric on $SU(3)/T^2$ is determined by an inner product on $\mathfrak{m}$ of the form

$$g_{eT^2} = (-t_1 B)|_{\mathfrak{m}_1} + (-t_2 B)|_{\mathfrak{m}_2} + (-t_3 B)|_{\mathfrak{m}_3}, \quad t_1, t_2, t_3 > 0.$$

The normal metric on $SU(3)/T^2$ corresponds to the above inner product with $t_1 = t_2 = t_3 = 1$. Any homothetic metric to this is also called normal metric.

On the other hand, if we take $t_1 = 2, t_2 = t_3 = 1$ we have in particular the Kähler-Einstein metric on $SU(3)/T^2$.

We will now examine the properties of the isotropy representation of a maximal flag manifold G/K, associated with a complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ whose the real compact form is $\mathfrak{g}$. The toral maximal Lie subalgebra of $\mathfrak{g}$ is denoted by $\mathfrak{t}$.

Consider the reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ of $\mathfrak{g}$ with respect to the negative of the Cartan-Killing form $-B(\, , \,)$ of $\mathfrak{g}$, that is, $\mathfrak{m} = \mathfrak{t}^{\perp}$ and $\mathrm{Ad}(T)\mathfrak{m} \subset \mathfrak{m}$.

Take a Weyl basis $\{H_{\alpha_1}, \ldots, H_{\alpha_l}\} \cup \{X_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha\}$ with $B(X_\alpha, X_{-\alpha}) = -1$, $[X_\alpha, X_{-\alpha}] = -H_\alpha \in \mathfrak{k}$ and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$, with $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not root and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$. The numbers are called structure constants of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Then, the real subalgebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \sum_{j=1}^{l} \mathbb{R}\sqrt{-1}H_{\alpha_j} \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}S_\alpha) = \mathfrak{t} \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}S_\alpha),$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = \sqrt{-1}(X_\alpha + X_{-\alpha})$.

Since $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{\sqrt{-1}H_{\alpha_j;1 \leqslant j \leqslant l}\}$, then the reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ implies that

$$\mathfrak{m} = \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}S_\alpha). \tag{1.4}$$

Set $\mathfrak{m}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}S_\alpha$ for any $\alpha \in R^+$. The linear space $\mathfrak{m}_\alpha$ is a irreducible $\mathrm{Ad}(T)-$module which does not depends on the choice of an ordering in R. Furthermore, since the roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ are distinct, and the roots spaces are one-dimensional, it is obvious that $\mathfrak{m}_\alpha \ncong \mathfrak{m}_\beta$ as $\mathrm{Ad}(T)-$representations, for any two roots $\alpha, \beta \in R^+$. Thus, by using (1.4), one has the following

Proposition 1.3.2 ([5])**.** *Let $M = G/T$ be a maximal flag manifold of a compact simple Lie group G. Then, the isotropy representation of M decomposes into a discrete sum of 2-dimensional pairwise inequivalent irreducible T-submodules $\mathfrak{m}_\alpha$ as follows:*

$$\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha.$$

The number of these submodules is equal to the cardinality $|R^+|$.

Table 1 – The number of isotropy summands for maximal flag manifolds G/T

Maximal flag manifold G/T	Number of roots $\|R\|$	$\mathfrak{m} = \bigoplus_{j=1}^{l}$
$SU(n+1)/T^n, n \geqslant 1$	$n(n+1)$	$l = n(n+1)/2$
$SO(2n+1)/T^n, n \geqslant 2$	$2n^2$	$l = n^2$
$Sp(n)/T^n, n \geqslant 3$	$2n^2$	$l = n^2$
$SO(2n)/T^n, n \geqslant 4$	$2n(n-1)$	$l = n(n-1)$
G_2/T	12	$l = 6$
F_4/T	48	$l = 24$
E_6/T	72	$l = 36$
E_7/T	126	$l = 63$
E_8/T	240	$l = 120$

For $n = 1$ the maximal flag $SU(n+1)/T^n$ is $SU(2)/S(U(1) \times U(1)) \cong \mathbb{CP}^1$, which is an isotropy irreducible Hermitian symmetric space.

2 Laplacian and Canonical Variation of Riemannian Submersions With Totally Geodesic Fibers

Some useful properties of the spectrum of the Laplacian acting on functions defined on the total space of a Riemannian submersion will be present here. In particular, we will consider the family of submersions with totally geodesic fibers. This family includes many classical examples, among other, the *homogeneous fibrations*, with the Hopf fibrations as particular cases.

The Lie theoretic description of flag manifolds is applied in order to obtain the horizontal and vertical distributions for each homogeneous fibration whose the total space represents a class of maximal flag manifold G/T provided with a *normal* metric g, where G is a compact simple Lie group and $T \subset G$ is a maximal torus in G.

The maximal flag manifolds dealt with in this work are associated with one of the classical complex simple Lie algebras.

2.1 Horizontal and Vertical Laplacians

Let (M, g) and (B, g') be two completes Riemannian manifolds of dimensions m and n, respectively. Let $\pi : M \longrightarrow B$ be a submersion. The map π is called *Riemannian Submersion* from (M, g) to (B, g') if at each $p \in M$, $\mathrm{d}\pi_p|_{\mathcal{H}} : (\mathcal{H}, g_p|_{\mathcal{H}}) \longrightarrow (T_{\pi(p)}B, g'_{\pi(p)})$ is an isometry, where $\mathcal{V} = \ker(\mathrm{d}\pi_p)$ is the *vertical space* and $\mathcal{H} = \mathcal{V}^{\perp}$ is the *horizontal space*.

A fiber $F_x = \pi^{-1}(x)$, $x \in B$, is a closed submanifold of M with dimension $r = m - n$. The submersion $\pi : M \longrightarrow B$ has *totally geodesic fibers* if, for all $x \in B$, the submanifold $F_x \subset M$ is totally geodesic in M, i.e., if $p \in F_x$, there exists a neighborhood $\Omega \subset T_pF_x$ of $0 \in T_pF_x$ such that $\exp : \Omega \longrightarrow F_x$ is injective. This means that geodesics over a fiber F_x are geodesics over M too and that, reciprocally, any geodesic over M tangent to F_x at some point $p \in F_x$ is contained in F_x for a time interval sufficiently small.

We denote by $\Delta_g = -\mathrm{div}_g \circ \mathrm{grad}_g$ the Laplacian operator of (M, g) acting on $C^{\infty}(M)$. The operator Δ_g, densely defined on $L^2(M, \mathrm{vol}_g)$, is symmetric (hence closable), non-negative has compact resolvent. Furthermore, it is well-known that Δ_g is essentially self-adjoint with this domain. We denote its unique self-adjoint extension also by Δ_g. Analogously, let Δ_k be the unique self-adjoint extension of the Laplacian of the fiber (F, k), where k is the metric induced by (M, g) on F.

Definition 2.1.1 ([7]). *The* vertical Laplacian Δ_v *acting on* $L^2(M, \mathrm{vol}_g)$ *is the operator defined at* $p \in M$ *by*

$$(\Delta_v \psi)(p) = (\Delta_k \psi|_{F_p})(p),$$

and the horizontal Laplacian Δ_h*, acting on the same space, is defined by the difference*

$$\Delta_h = \Delta_g - \Delta_v.$$

Both Δ_h and Δ_v are non-negative self-adjoint unbounded operators on $L^2(M, \mathrm{vol}_g)$, but in general, are not elliptic (unless π is a covering). We now consider the spectrum of such operators.

As remarked above, Δ_g is non-negative and has compact resolvent, that is, its spectrum is non-negative and discrete. Since the fibers are isometrics, Δ_v also has discrete spectrum equal to the fibers. However, the spectrum of Δ_h need not be discrete.

One of the main observation on which this work is based is given by the following theorem.

Theorem 2.1.2 ([7]). *If the fibers of the Riemannian submersion* $\pi : M \longrightarrow B$ *are totally geodesics the operators* Δ_g*,* Δ_v *e* $\Delta_{g'}$ *commute with each other.*

When Δ_g and Δ_v commute, we have the following decomposition of $L^2(M, g)$.

Theorem 2.1.3 ([7]). *The Hilbert space* $L^2(M, g)$ *admits a Hilbert basis consisting of simultaneous eigenfunctions of* Δ_g *e* Δ_v*.*

2.2 Canonical Variation

From a Riemannian submersion with totally geodesic fibers, it is possible to define a 1-parameter family g_t, $t > 0$, of other such submersions by scaling the original metric of the total space in the direction of the fibers.

This construction, called *the canonical variation*, will be fundamental in the determination of bifurcation and local rigid instants. Indeed, the criterion to be applied to calculate these instants, according to Corollary 1.1.7 enunciated in the previous section, makes use of the first eigenvalue of the Laplacian Δ_t of the family g_t. Furthermore, the spectrum of Δ_t has description in terms of the vertical and horizontal Laplacians.

Definition 2.2.1 ([7]). *Let* $F \cdots (M, g) \xrightarrow{\pi} B$ *be a Riemannian submersion with totally geodesic fibers. Consider the 1-parameter family of Riemannian submersions given by* $\{F \cdots (M, g_t) \xrightarrow{\pi} B, t > 0\}$*, where* $g_t \in \mathcal{R}^k(M)$ *is defined by*

$$g_t(v, w) = \begin{cases} t^2 g(v, w), & \textit{se} \quad v, w \quad \textit{are verticals} \\ 0, & \textit{se} \quad v \quad \textit{is vertical and} \quad w \quad \textit{is horizontal} \\ g(v, w), & \textit{se} \quad v, w \quad \textit{are horizontals.} \end{cases}$$

Such family of Riemannian submersions is called the canonical variation *of* $F \cdots (M,g) \xrightarrow{\pi} B$ *or, for simplicity, we may also refer to the family of* total spaces *of these submersions, i.e., the Riemannian manifolds* (M, g_t)*, as the canonical variation of* (M, g)*.*

Proposition 2.2.2 ([7])**.** *The family* $\left\{F \cdots (M, g_t) \xrightarrow{\pi} B, t > 0\right\}$ *of Riemannian submersions has totally geodesic fibers, for each* $t > 0$*. Furthermore, its fibers are isometrics to* $(F, t^2 k)$*, where* (F, k) *is the original fiber of* $\pi : M \longrightarrow B$*.*

Remark 2.2.3. Note that, for $a \neq b$, the metrics g_a e g_b are not conformal. Furthermore, for each $t > 0$, g_t is the unique Riemannian metric that satisfy the conditions of Definition 2.2.1.

The following shows how to decompose Δ_t in terms of the vertical and horizontal Laplacians.

Proposition 2.2.4 ([11])**.** *Let* Δ_t *the Laplacian of* (M, g_t)*. Then*

$$\Delta_t = \Delta_h + \frac{1}{t^2}\Delta_v = \Delta_g + (\frac{1}{t^2} - 1)\Delta_v. \tag{2.1}$$

Corollary 2.2.5 ([11])**.** *For each* $t > 0$*, the following inclusion holds*

$$\sigma(\Delta_t) \subset \sigma(\Delta_g) + (\frac{1}{t^2} - 1)\sigma(\Delta_v), \tag{2.2}$$

where $\sigma(\Delta_t)$*,* $\sigma(\Delta_g)$ *and* $\sigma(\Delta_v)$ *are the respective spectrum of* Δ_t*,* Δ_g *e* Δ_v*. Since the above spectra are discrete, this means that every eigenvalue* $\lambda(t)$ *of* Δ_t *is of the form*

$$\lambda^{k,j}(t) = \mu_k + (\frac{1}{t^2} - 1)\phi_j, \tag{2.3}$$

for some $\mu_k \in \sigma(\Delta_g)$ *and some* $\phi_j \in \sigma(\Delta_v)$*.*

Proof: By Theorem 2.1.2, Δ_g and Δ_v commute. Hence, by the Spectral Theorem, such operators are simultaneously diagonalizable, in the sense that there exists a unitary operator U of $L^2(M, \mathrm{vol}_g)$ such that $U\Delta_g U^{-1} = f_M$ and $U\Delta_v U^{-1} = f_v$ are multiplication operators by functions f_M and f_v respectively. For such multiplication operators, the spectrum $\sigma(T_f)$ is the essential range, ess.Im$(f)(\subset \overline{\mathrm{Im}(f)})$, of f. Then, by the expression of Δ_t in (2.2.5),

$$\sigma(\Delta_t) = \mathrm{ess.Im}(f_M + (\frac{1}{t^2} - 1)f_v) \subset \overline{\mathrm{ess.Im}(f_M) + \mathrm{ess.Im}((\frac{1}{t^2} - 1)f_v)} = \overline{\sigma(\Delta_g) + (\frac{1}{t^2} - 1)\sigma(\Delta_v)}.$$

Since both spectra are discrete, we may remove the closure and the inclusion (2.2) is proved.

□

Corollary 2.2.6. *If $\lambda_1(t)$ is the first positive eigenvalue of Δ_t, then*

$$\lambda_1(t) \geqslant \mu_1, \forall \ \ 0 < t \leqslant 1,$$

where μ_1 is the first positive eigenvalue of Δ_g.

Proof: Since $\lambda_1(t) = \mu_k + (\frac{1}{t^2} - 1)\phi_j$, for some $\mu_k \in \sigma(\Delta_g)$ and $\phi_j \in \sigma(\Delta_v)$, if $0 < t \leqslant 1$, $(\frac{1}{t^2} - 1)\phi_j \geqslant 0$ and $\lambda_1(t) = \mu_k + (\frac{1}{t^2} - 1)\phi_j \geqslant \mu_k \geqslant \mu_1$, where $\mu_1 \in \sigma(\Delta_g)$ is the first positive eigenvalue of the operator Δ_g.

□

We remark also that not all possible combinations of μ_k and ϕ_j in the expression (2.3), Corollary 2.2.5, give rise to an eigenvalue of Δ_t. In fact, this only happens when the total space (M, g) of the submersion is a Riemannian product. Determining which combinations are allowed is complicated in general and depends on the global geometry of the submersion. Moreover,the ordering of the eigenvalues of Δ_t may change with t.

We have the following important property of Δ_t.

Proposition 2.2.7 ([10])**.** *Using the same notations above, it has that*

$$\sigma(\Delta_B) \subset \sigma(\Delta_t),$$

for all $t > 0$.

Proof: For each $\psi : B \longrightarrow \mathbb{R}$ and its lift $\widetilde{\psi} := \psi \circ \pi$,

$$\Delta_g \widetilde{\psi} = (\Delta_B \psi) \circ \pi + g(\operatorname{grad}_g \widetilde{\psi}, \vec{H}), \tag{2.4}$$

where $\vec{H}$ is the *mean curvature* vector field of the fibers. Since we assumed the fibers of π are totally geodesics, $\vec{H} \equiv 0$ over the fibers. It follows that, if ψ is a eigenfunction of Δ_B, then its lift $\widetilde{\psi}$ is an eigenfunction of Δ_g with the same eigenvalue (and constant along the fibers) and therefore

$$\sigma(\Delta_B) \subset \sigma(\Delta_g).$$

Since the fibers of π are totally geodesics with respect to g_t, the above inclusion above holds when Δ_g is replaced with Δ_t, i.e,

$$\sigma(\Delta_B) \subset \sigma(\Delta_t), \quad t > 0.$$

□

It follows from (2.4) other property of the spectrum of Δ_h; $\sigma(\Delta_h)$ contains but not coincides with the spectrum of the basis B. In fact, if $\overline{f}$ is a C^∞ function on the basis B, then

$$(\Delta_{g'}\overline{f}) \circ \pi = \Delta_g(\overline{f} \circ \pi) = \Delta_h(\overline{f} \circ \pi),$$

where $\Delta_{g'}$ is the Laplacian operator acting on functions in $C^\infty(B, g')$.

Corollary 2.2.8. *Denoting by β_1 the first positive eigenvalue of Δ_B and by $\lambda_1(t)$ the first positive eigenvalue of Δ_t, the following inequality holds*

$$\lambda_1(t) \leqslant \beta_1, \quad \forall\, t > 0.$$

As consequence of the Corollaries 3 and 4, we have that $\mu_1 \leqslant \lambda_1(t) \leqslant \beta_1$, where μ_1 the first positive eigenvalue of the Laplacian Δ_g on the total space M.

When $j = 0$ in the expression (2.3), if $\lambda^{k,0}(t) = \mu_k \in \sigma(\Delta_g)$ remains an eigenvalue of Δ_t for $t \neq 1$, such eigenvalues will be called *constant eigenvalue of* Δ_t, since they are independent of t. We stress that $\lambda^{k,0}(t)$ is not necessarily a constant eigenvalue of Δ_t for all k. A simple criterion to determinate when $\lambda^{k,0}(t) \in \sigma(\Delta_t)$ is consequence of the following Proposition.

Proposition 2.2.9 ([10])**.** *If $\pi : M \longrightarrow B$ is a Riemannian submersion with totally geodesic fibers, then the eigenfunction $\widetilde{\psi}$ of Δ_g is constant along the fibers if and only if ψ is the lift of some eigenfunction of Δ_B, that is, $\widetilde{\psi} = \psi \circ \pi$, for some eigenfunction $\psi : B \longrightarrow \mathbb{R}$ of $\Delta_B$on the basis.*

Corollary 2.2.10. *$\mu_k = \lambda^{k,0}(t) \in \sigma(\Delta_t)$ for $t \neq 1 \Leftrightarrow \mu_k \in \sigma(\Delta_B)$.*

Proof: Fixed $t \neq 1$, since

$$\lambda^{k,j}(t) = \mu_k + (\frac{1}{t^2} - 1)\phi_j \quad \text{and} \quad \Delta_t = \Delta_g + (\frac{1}{t^2} - 1)\Delta_v,$$

we have that $\lambda^{k,j}(t) \in \sigma(\Delta_t)$ if and only if there exists $\widetilde{\psi} \in C^\infty(M)$ such that

$$\Delta_g\widetilde{\psi} = \mu_k\widetilde{\psi} \quad \text{and} \quad \Delta_v\widetilde{\psi} = \phi_j\widetilde{\psi},$$

with $\mu_k \in \sigma(\Delta_g)$ and $\phi_j \in \sigma(\Delta_v)$. In particular, if we have $\lambda^{k,0}(t) = \mu_k + (\frac{1}{t^2} - 1)0 \in \sigma(\Delta_t)$, there exists $\widetilde{\psi}$ such that

$$\Delta_g\widetilde{\psi} = \mu_k\widetilde{\psi} \quad \text{and} \quad \Delta_v\widetilde{\psi} = 0 \cdot \widetilde{\psi} = 0.$$

However, $\Delta_v\widetilde{\psi} = 0$ implies that $\widetilde{\psi}$ is constant along the fibers. Therefore, by Proposition 2.2.9, there exists a eigenfunction $\psi : B \longrightarrow \mathbb{R}$ in $C^\infty(B)$ such that $\Delta_B\psi = \mu_k\psi$ and $\widetilde{\psi} = \psi \circ \pi$. It follows that $\mu_k \in \sigma(\Delta_B)$. The reciprocal results from the fact that $\sigma(\Delta_B) \subset \sigma(\Delta_t)$.

□

2.2.1 Homogeneous Fibration

In our main result, the canonical variations g_t, obtained from a Riemannian submersion with totally geodesic fibers, are *homogeneous metrics*, which are trivial solutions to the Yamabe problem, since every homogeneous metric has constant scalar curvature. This makes these metrics good candidates for admitting other solutions in their conformal class.

The homogeneous fibrations are obtained from the following construction. Let $K \subsetneq H \subsetneq G$ be compact connected Lie groups, such that $\dim K/H \geqslant 2$. Consider the natural fibration

$$\begin{array}{rccc} \pi: & G/K & \longrightarrow & G/H \\ & \alpha K & \mapsto & \alpha H, \end{array}$$

with fibers H/K (em eK) and structural group H. More precisely, π is the associated bundle with fiber H/K to the H-principal bundle $p : G \longrightarrow G/H$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h} \supset \mathfrak{k}$ the Lie algebras of H and K, respectively. Given a inner product on $\mathfrak{g}$, determined by the Cartan-Killing form B of $\mathfrak{g}$, since K, H and G are compacts, we can consider a $\mathrm{Ad}_G(H)$-invariant orthogonal complement $\mathfrak{q}$ to $\mathfrak{h}$ in $\mathfrak{g}$, i.e., $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, and a $\mathrm{Ad}_G(K)$-invariant orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$, i.e., $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. It follows that $\mathfrak{p} \oplus \mathfrak{q}$ is a $\mathrm{Ad}_G(K)$-invariant orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$.

The $\mathrm{Ad}_G(H)$-invariant inner product $(-B)|_{\mathfrak{q}}$ on $\mathfrak{q}$ define a G-invariant Riemannian metric $\breve{g}$ on G/H, and the inner product $(-B)|_{\mathfrak{p}}$, $\mathrm{Ad}_G(K)$-invariant on $\mathfrak{p}$, define on H/K a H-invariant Riemannian metric $\hat{g}$ on H/K. Finally, the orthogonal direct sum of these inner products on $\mathfrak{p} \oplus \mathfrak{q}$ define a G-invariant metric g on G/K, determined by

$$g(X + V, Y + W)_{eK} = (-B)|_{\mathfrak{q}}(X, Y) + (-B)|_{\mathfrak{p}}(V, W), \tag{2.5}$$

for all $X, Y \in \mathfrak{q}$ and $V, W \in \mathfrak{p}$; g is a normal homogeneous metric on G/K and the $(-B)$-orthogonal direct sum $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ is the isotropy representation of K.

Theorem 2.2.11 (([8], p. 257))**.** *The map $\pi : (G/K, g) \longrightarrow (G/H, \breve{g})$ is a Riemannian submersion with totally geodesic fibers and isometric to $(H/K, \hat{g})$.*

If we take for each $t > 0$ the metric g_t that corresponds to the inner product on $\mathfrak{p} \oplus \mathfrak{q}$

$$\langle \cdot, \cdot \rangle_t = (-B)|_{\mathfrak{q}} + (-t^2 B)|_{\mathfrak{p}}, \tag{2.6}$$

where B is the Cartan-Killing form of $\mathfrak{g}$, we have that the map $\pi_t : (G/K, g_t) \longrightarrow (G/H, \breve{g})$ is a Riemannian submersion with totally geodesic fibers and isometric to H/K provided with the induced metric $-t^2 B|_{\mathfrak{p}}$. Hence, we obtain the canonical variation of the original homogeneous fibration $\pi : (G/K, g) \longrightarrow (G/H, \breve{g})$.

Example 2.2.12. Consider $K = T^2 = S(U(1) \times U(1) \times U(1))$, $H = S(U(2) \times U(1))$, $G = SU(3)$ and

$$\pi : SU(3)/T^2 \longrightarrow SU(3)/S(U(2) \times U(1)) = \mathbb{CP}^2$$

the canonical map defined by $\pi(xK) = xH$, $x \in G$. According 1.3.1, we have the $(-B)$-orthogonal decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q},$$

with $B(X,Y) = 6\mathrm{Tr}(XY)$, $\quad X, Y \in \mathfrak{g}$, being the Cartan-Killing form of $\mathfrak{g}$ and

$$\left\{ \begin{array}{lcl} \mathfrak{k} & = & \left\{d = \sqrt{-1}\mathrm{diag}(a, b, -(a+b)) \in \mathfrak{su}(3); a, b \in \mathbb{R}\right\} \\ \mathfrak{p} & = & \mathrm{spam}_{\mathbb{C}}\left(E_{12} - E_{21}\right) \subset \mathfrak{su}(3) \\ \mathfrak{q} & = & \mathrm{spam}_{\mathbb{C}}\left(E_{13} - E_{31}, E_{23} - E_{32}\right) \subset \mathfrak{su}(3) \\ \mathfrak{h} & = & \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1)) = \mathfrak{k} \oplus \mathfrak{p} \end{array} \right. , \tag{2.7}$$

where E_{ij} is the matrix having 1 in the (i,j) position and 0 elsewhere.

It follows that $\pi : SU(3)/T^2 \longrightarrow SU(3)/S(U(2) \times U(1)) = \mathbb{CP}^2$ is a Riemannian submersion with totally geodesic fibers isometric to $(H/K, \hat{g})$, $H/K = SU(2)/S(U(1) \times U(1)) = \mathbb{CP}^1$ and $\hat{g}$ the metric induced by the inner product $(-B)|_{\mathfrak{p}}$. The metric g on $SU(3)/T^2$ obtained from the above decomposition, according 2.5, is exactly the normal metric. In fact, we have that the decomposition of the isotropy representation $\mathfrak{m}$ of T^2 into pairwise inequivalent irreducible $\mathrm{Ad}(T^2)$-modules $\mathfrak{m}_i$ is given by

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

with $\mathfrak{m}_1 = \mathfrak{p}$, $\mathfrak{m}_2 = spam_{\mathbb{C}}\left(E_{13} - E_{31}\right)$ and $\mathfrak{m}_3 = spam_{\mathbb{C}}\left(E_{23} - E_{32}\right)$. An invariant metric on $SU(3)/T^2$ is determined by an inner product on $\mathfrak{m}$ of the form

$$g_{eT^2} = (-t_1 B)|_{\mathfrak{m}_1} + (-t_2 B)|_{\mathfrak{m}_2} + (-t_3 B)|_{\mathfrak{m}_3}, \quad t_1, t_2, t_3 > 0.$$

The normal metric on $SU(3)/T^2$ corresponds to the above inner product with $t_1 = t_2 = t_3 = 1$. Any homothetic metric to this is also called normal metric.

Moreover, note that the metric $\breve{g}$ on the basis $G/H = SU(3)/S(U(1) \times U(2))$ is determined by the inner product on $\mathfrak{q}$ given by $(-B)|_{\mathfrak{q}}$ and

$$(\mathbb{CP}^2 = SU(3)/S(U(1) \times U(2)), \breve{g})$$

is a compact isotropy irreducible symmetric space, as well as the fiber $(\mathbb{CP}^1 = SU(2)/S(U(1) \times U(1)), \hat{g})$.

Therefore, the one-parameter family of Riemannian submersions

$$\mathbb{CP}^1 \cdots (SU(3)/T^2, g_t) \xrightarrow{\pi} \mathbb{CP}^2),$$

is the canonical variation of the original submersion

$$(\mathbb{CP}^1\hat{g})\cdots(SU(3)/T^2, g) \xrightarrow{\pi} (\mathbb{CP}^2, \breve{g}),$$

with totally geodesic fibers isometric to $(\mathbb{CP}^1, t^2\hat{g})$, where g_t according (2.6), can be determined by the following inner product on $\mathfrak{m}$,

$$(g_t)_{eT^2} = (-t^2B)|_{\mathfrak{m}_1} + (-B)|_{\mathfrak{m}_2} + (-B)|_{\mathfrak{m}_3}, \quad t > 0. \tag{2.8}$$

We remark that at $t = 1$, g_1 is the normal metric on $SU(3)/T^2$.

Example 2.2.13. Let $SO(5)$ be the special orthogonal group and $T^2 = U(1) \times U(1) \subset SO(5)$ maximal torus in $SO(5)$. The space $SO(5)/U(1) \times U(1)$ is a maximal flag manifold, according definition. The Lie algebra $\mathfrak{so}(5)$ (classical Lie algebra of the type D_2) of $G = SO(5)$ decomposes into the $(-B)$-orthogonal direct sum,

$$\mathfrak{so}(5) = \mathfrak{k} \oplus \mathfrak{m},$$

where B is the Cartan-Killing form of $\mathfrak{so}(5)$, defined by $B(X, Y) = 3\mathrm{Tr}(XY), X, Y \in \mathfrak{so}(5)$, $\mathfrak{k}$ is the Lie algebra of T^2, and $\mathfrak{m}$ is the isotropy representation of $K = T^2$. The Lie subalgebra $\mathfrak{k} \subset \mathfrak{so}(5)$ is a maximal abelian Lie subalgebra of $\mathfrak{so}(5)$ formed by the diagonal null trace matrices belonging to $\mathfrak{so}(5)$. With respect the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$, the set of positive roots of $\mathfrak{so}(5)^{\mathbb{C}}$ is $\{\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2, \alpha_3 = \lambda_1, \alpha_4 = \lambda_1 + \lambda_2\}$, where the functional λ_i is given by $\mathrm{diag}(a_1, \ldots, a_4) \mapsto \lambda_i(\mathrm{diag}(a_1, \ldots, a_4)) = a_i$, for each $1 \leqslant i \leqslant 4$. The space $\mathfrak{m}$ decomposes into four pairwise inequivalent irreducible $\mathrm{Ad}(T^2)$-modules $\mathfrak{m}_{\alpha_i}$, $1 \leqslant i \leqslant 4$, as follows

$$\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_3} \oplus \mathfrak{m}_{\alpha_4},$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, with $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{so}(5)$.

Setting $G = SO(5), H = SO(4) \cong SO(3) \times SO(3)$ and $K = T^2 = U(1) \times U(1) \cong SO(2) \times SO(2)$, we have that $K \subsetneq H \subsetneq G$, with G, H and K compact connected Lie groups. Consider the canonical map $\pi : SO(5)/T^2 \longrightarrow SO(5)/SO(4) = S^4$. Let $\mathfrak{g} = \mathfrak{so}(5)$, $\mathfrak{h} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{k} = \{d = \mathrm{diag}(0, ia, ib, -ia, -ib)) \in \mathfrak{so}(5); a, b \in \mathbb{R}\}$ be the Lie algebras of G, H and K, respectively. As we saw previously, since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in $\mathfrak{so}(5)$, and a $\mathrm{Ad}(K)$-invariant $(-B)$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(5)$ decomposes as

$$\mathfrak{so}(5) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and therefore, as previously stated, $\pi : (SO(5)/T^2, g) \longrightarrow (SO(5)/SO(4) = S^4, \breve{g})$ is a Riemannian submersion with totally geodesic fibers isometric to $(H/K, \hat{g})$, $H/K =$

$SO(4)/T^2 \cong SO(4)/SO(2) \times SO(2) \cong S^2 \times S^2$, g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$.

In order to determine the vertical distribution $\mathfrak{p}$ (tangent space to the fiber) and the horizontal distribution $\mathfrak{q}$ (tangent space to the basis), we note that, since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim SO(5)/SO(4) = S^4 = 4$, by the fact that $\dim \mathfrak{m}_{\alpha_i} = 2$, for each $i = 1, 2, 3, 4$, we have that $\mathfrak{q}$ is equal to the direct sum of two irreducible submodules of $\mathfrak{m}$, as well as $\mathfrak{p}$. We will determinate the submodules that form each of these spaces, by using the relations

$$[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$$

and the property of Weyl basis

$$\begin{cases} [A_\alpha, S_\beta] & = & m_{\alpha,\beta} A_{\alpha+\beta} + m_{-\alpha,\beta} A_{\alpha-\beta} \\ [S_\alpha, S_\beta] & = & -m_{\alpha,\beta} A_{\alpha+\beta} - m_{\alpha,-\beta} A_{\alpha-\beta} \\ [A_\alpha, S_\beta] & = & m_{\alpha,\beta} S_{\alpha+\beta} + m_{\alpha,-\beta} S_{\alpha-\beta} \end{cases},$$

where $m_{\alpha,\beta}$ is such that $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$, with $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not root and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$. We remark that this basis is $(-B)$-orthogonal and $-B(A_\alpha, A_\alpha) = -B(S_\alpha, S_\alpha) = 2$, B Cartan-Killing form of $\mathfrak{g}$. This relations allow us conclude that $\mathfrak{q} = \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_3}$ and $\mathfrak{p} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_4}$, since $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. Hence, by scaling the normal metric $g_{eT^2} = (-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_3}} + (-B)|_{\mathfrak{m}_{\alpha_4}}$ of the total space $SO(5)/T^2$ in the direction of the fibers, i.e., multiplying by $t^2, t > 0$, $(-B)|_{\mathfrak{m}_{\alpha_1}}$ and $(-B)|_{\mathfrak{m}_{\alpha_4}}$ in the expression of g we obtain the following canonical variation of the metric g

$$(g_t)_{eT^2} = t^2(-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_3}} + t^2(-B)|_{\mathfrak{m}_{\alpha_4}},$$

according (2.6).

Example 2.2.14. Let $\mathfrak{g}_2$ be the exceptional complex simple Lie algebra of type G_2. We will denote by G_2 the compact simple Lie group whose Lie algebra is $\mathfrak{g}_2$ and by $T \subset G_2$ a maximal torus there, where $T = U(1) \times U(1)$. The full flag manifold associated with $\mathfrak{g}_2$ is G_2/T. The Lie algebra $\mathfrak{g}_2$ decomposes into the $(-B)$-orthogonal direct sum

$$\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{m},$$

where B is the Cartan-Killing form of G_2, defined by $B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$, $X, Y \in \mathfrak{g}_2$, $\mathfrak{k}$ is the Lie algebra of T, maximal abelian Lie subalgebra of $\mathfrak{g}_2$ formed by the diagonal null trace matrices belonging to $\mathfrak{g}_2$ and $\mathfrak{m}$ its isotropy representation of T. With respect the Cartan Lie subalgebra $\mathfrak{k}$, the root system of $\mathfrak{g}_2$ can be chosen as by

$$\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2), \pm(\alpha_1 + 3\alpha_2), \pm(2\alpha_1 + 3\alpha_2)\},$$

and we fix a system of simple roots to be $\Pi = \{\alpha_1, \alpha_2\}$. With respect to Π the positive roots are given by

$$\{\alpha_1, \alpha_2, (\alpha_1 + \alpha_2), (\alpha_1 + 2\alpha_2), (\alpha_1 + 3\alpha_2), (2\alpha_1 + 3\alpha_2)\}.$$

The maximal root is $\alpha = 2\alpha_1 + 3\alpha_2$. The angle between α_1 and α_2 is $5\pi/6$ and we have $\|\alpha_1\| = \sqrt{3}\,\|\alpha_2\|$. Moreover, the roots of $\mathfrak{g}_2$ form successive angles of $\pi/6$. Since in this case it has six positive roots, the space $\mathfrak{m}$ decomposes into six pairwise inequivalent irreducible $\mathrm{Ad}(T)$-modules as follows

$$\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_1+\alpha_2} \oplus \mathfrak{m}_{\alpha_1+2\alpha_2} \oplus \mathfrak{m}_{\alpha_1+3\alpha_2} \oplus \mathfrak{m}_{2\alpha_1+3\alpha_2},$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, where $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{g}_2$.

Setting $G = G_2, H = SO(4) \cong SO(3) \times SO(3)$ and $K = T = U(1) \times U(1) \cong SO(2) \times SO(2)$, we have that $K \subsetneq H \subsetneq G$, with G, H and K compact connected Lie groups. Consider the canonical map $\pi : G_2/T \longrightarrow G_2/SO(4)$. Let $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{h} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{k} = \{d = \mathrm{diag}(ia, ib, -i(a+b)) \in \mathfrak{sl}(3) \subset \mathfrak{g}_2; a, b \in \mathbb{R}\}$ be the Lie algebras of G, H and K, respectively. As we saw since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in $\mathfrak{g}_2$, and a $\mathrm{Ad}(K)$-invariant $(-B)$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}_2$ decomposes into the sum

$$\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and follows that

$$\pi : (G_2/T, g) \longrightarrow (G_2/SO(4), \breve{g})$$

is a Riemannian submersion with totally geodesic fibers isometric to $(H/K, \hat{g})$, $H/K = SO(4)/T \cong SO(4)/SO(2) \times SO(2) \cong S^2 \times S^2$, g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$.

Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim SO(4)/T \cong S^2 \times S^2 = 4$, by the fact that $\dim \mathfrak{m}_\alpha = 2$, for each positive root α, we have that $\mathfrak{q}$ is equal to the direct sum of four irreducible submodules of $\mathfrak{m}$ while $\mathfrak{p}$ is the direct sum of two irreducible submodules. In order to find the submodules that composes each of these distributions we apply the relations $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and the same property of the Weyl basis in the previous example, obtaining $\mathfrak{p} = \mathfrak{m}_{\alpha_1+\alpha_2} \oplus \mathfrak{m}_{\alpha_1+3\alpha_2}$ and $\mathfrak{q} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_1+2\alpha_2} \oplus \mathfrak{m}_{2\alpha_1+3\alpha_2}$, since $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. Hence, by scaling the normal homogeneous metric

$$\begin{aligned}(\mathbf{n}_t)_{eT} &= (-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_1+2\alpha_2}} \\ &\quad + t^2(-B)|_{\mathfrak{m}_{\alpha_1+3\alpha_2}} + (-B)|_{\mathfrak{m}_{2\alpha_1+3\alpha_2}},\end{aligned}$$

defined on the total space G_2/T in the direction of the fibers, i.e., multiplying by $t^2, t > 0$, the parcel $(-B)|_{\mathfrak{m}_{\alpha_1+\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_1+3\alpha_2}}$ in the expression of g we obtain the following canonical variation $\mathbf{n}_t$ of the metric g

$$\begin{aligned}(\mathbf{n}_t)_{eT} &= (-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_1+2\alpha_2}} \\ &\quad + t^2(-B)|_{\mathfrak{m}_{\alpha_1+3\alpha_2}} + (-B)|_{\mathfrak{m}_{2\alpha_1+3\alpha_2}},\end{aligned}$$

according (2.6).

2.2.2 Canonical Variations of Normal Metrics on Maximal Flag Manifolds

In this section we will present in detail the construction of the 1-parameter families of homogeneous metrics from normal homogeneous metrics on maximal flag manifolds. Such homogeneous spaces are total spaces of a homogeneous fibration with fibers and basis spaces being well known homogeneous spaces, namely symmetric spaces or a maximal flag manifold of another type.

Remark 2.2.15. At first, we will consider the following

1. From now on, the total spaces of the original homogeneous fibrations are endowed with its normal homogeneous metric induced by the Cartan-Killing form of the respective complex classical simple Lie algebras that determine it;

2. Given $K \subsetneq H \subsetneq G$, compact connected Lie groups, with Lie algebras $\mathfrak{k}, \mathfrak{h}$ and $\mathfrak{g}$ respectively, the homogeneous fibrations $\pi : G/K \longrightarrow G/H$ are such that the basis space G/H is a Hermitian symmetric space with irredudcible isotropy representation and a metric induced by the restriction of the Cartan-Killing form of $\mathfrak{g}$ to its isotropy representation (horizontal distribution) and the fiber H/K (with $\dim H/K \geqslant 2$) is some maximal flag manifold, i.e, K is a maximal torus contained in H, and H/K is provided with the normal metric induced by the Cartan-Killing form of $\mathfrak{g}$ restricted to its isotropy representation (vertical distribution).

Proposition 2.2.16. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be complex classic simple Lie algebras such that $\mathfrak{h} \subset \mathfrak{g}$, and let $\mathfrak{t}$ be a Cartan subalgebra for $\mathfrak{g}$ with Cartan-Killing form B. If R is the root system of $\mathfrak{t}$ associated with $\mathfrak{g}$, then there exists a subset R' of R such that*

(a) *The subspace $\mathfrak{t}(R')$ of $\mathfrak{h}$ spanned by the dual vectors $H_\alpha, \alpha \in R'$, is a Cartan subalgebra of $\mathfrak{h}$ and R' is a system of roots of $\mathfrak{t}(R')$.*

(b) *The Lie algebra $\mathfrak{h}$ has a $(-B)$-orthogonal decomposition into direct sum given by*

$$\mathfrak{h} = \mathfrak{t}(R') + \sum_{\alpha \in R'} \mathfrak{g}_\alpha,$$

where the $\mathfrak{g}_\alpha$ are root spaces with $\alpha \in R'$.

Proof: (a) Since $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, the Cartan subalgebra of $\mathfrak{h}$ must be contained in $\mathfrak{t}$. Then, there are dual vectors $H_{\alpha_1}, \ldots, H_{\alpha_l} \in \mathfrak{t}, \alpha_i \in R$, that span the Cartan subalgebra of $\mathfrak{h}$, with l equal to its dimension. Take $R' = \operatorname{span}(\alpha_1, \ldots, \alpha_l) \subset R$. Thus, we can consider $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ as a simple root system and follows that R' is a system of roots of the Cartan subalgebra $\mathfrak{t}(R')$.

(b) It is a direct consequence of (a).

$\square$

Corollary 2.2.17. *Let $\pi : G/K \longrightarrow G/H$ be a homogeneous fibration as above, with totally geodesic fiber H/K ($\dim H/K \geqslant 2$) and let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H. Accordingly the current notations, if $\mathfrak{p}$ and $\mathfrak{q}$ are the vertical and horizontal distribution, i.e, $\mathfrak{p}$ is the tangent space to the fiber H/K and $\mathfrak{q}$ is the tangent space to the basis space G/H, then $\mathfrak{p}$ is equal to the $(-B)$-orthogonal direct sum of the root spaces $\mathfrak{g}_\alpha$ which $\alpha \in R'$ and hence $\mathfrak{q}$ is equal to the $(-B)$-orthogonal direct sum of the root spaces $\mathfrak{g}_\beta$ which $\beta \in R \backslash R'$.*

The next lemma will be applied to find vertical and horizontal distributions.

Lemma 2.2.18 ([26])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and Σ a system of simple roots of the Cartan subalgebra $\mathfrak{h}$. Take a subset $\Theta \subset \Sigma$ and denote by $\langle\Theta\rangle$ the set of roots spanned by Θ, i.e., $\langle\Theta\rangle$ is the set of roots which are linear combinations of Θ only. Consider the space*

$$\mathfrak{g}(\Theta) = \mathfrak{h}(\Theta) + \sum_{\alpha \in \langle\Theta\rangle} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}(\Theta)$ is the subspace of $\mathfrak{h}$ spanned by the dual vectors $H_\alpha, \alpha \in \Theta$. Then, $\mathfrak{g}(\Theta)$ is a semisimple Lie subalgebra of $\mathfrak{g}$ spanned by $\{\mathfrak{g}_\alpha; \pm\alpha \in \Theta\}$. Its Dynkin diagram coincides with the diagram associated with Θ, seen as a subset of Σ.

We can now introduce the construction of the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t, \mathbf{n}_t$, of the normal metrics on the maximal flag manifolds $SU(n+1)/T^n, SO(2n+1)/T^n, Sp(n)/T^n$, $SO(2n)/T^n, G_2/T$, respectively, as in the following Sections.

2.2.2.1 $(SU(n+1)/T^n, \mathbf{g}_t), n \geqslant 2$:

In first, we observe that for $n = 1$ the maximal flag $SU(n+1)/T^n$ is $SU(2)/S(U(1) \times U(1)) \cong \mathbb{CP}^1$, which is an isotropy irreducible Hermitian symmetric space.

We will denote by $G = SU(n+1)$ the compact simple Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{su}(n+1)$ and by $T^n \subset G$ a maximal torus there, given by $T^n = S(U(1) \times \ldots \times U(1)) = S(U(1)^{n+1})$. The maximal flag manifold associated with $\mathfrak{su}(n+1)$ is $SU(n+1)/T^n$. The Lie algebra $\mathfrak{su}(n+1)$ decomposes into the $(-B)$-orthogonal direct sum

$$\mathfrak{su}(n+1) = \mathfrak{k} \oplus \mathfrak{m},$$

where B is the Cartan-Killing form of $\mathfrak{su}(n+1)$, defined by $B(X,Y) = 2(n+1)\mathrm{Tr}(XY)$, $X, Y \in \mathfrak{su}(n+1)$, $\mathfrak{k}$ is the Lie algebra of T^n, maximal abelian Lie subalgebra of $\mathfrak{su}(n+1)$ formed by the diagonal matrices of the form

$$\mathfrak{k} = \left\{ \sqrt{-1} \cdot \mathrm{diag}(a_1, \ldots, a_{n+1}); \sum_{i=1}^{n} a_i = 0 \right\}$$

and $\mathfrak{m}$ the $\mathrm{Ad}(T^n)$-invariant isotropy representation of T^n. With respect the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$, the root system of $\mathfrak{su}(n+1)^{\mathbb{C}}$ can be chosen as by

$$R = \{\alpha_{ij} = \pm(\lambda_i - \lambda_j); i \neq j\},$$

where λ_i is given by $\mathrm{diag}(a_1, \ldots, a_{n+1}) \mapsto \lambda_i(\mathrm{diag}(a_1, \ldots, a_{n+1})) = a_i$, for each $1 \leqslant i \leqslant n+1$. We fix a system of simple roots to be $\Sigma = \{\alpha_{ii+1}; 1 \leqslant i \leqslant n\}$ and with respect to Σ the positive roots are given by

$$R^+ = \{\alpha_{ij} = \lambda_i - \lambda_j; i < j\}.$$

In this case it has $\dfrac{n(n+1)}{2}$ positive roots, hence $\mathfrak{m}$ decomposes into $\dfrac{n(n+1)}{2}$ pairwise inequivalent irreducible $\mathrm{Ad}(T^n)$-modules as follows

$$\mathfrak{m} = \bigoplus_{i<j\leqslant n+1} \mathfrak{m}_{\alpha_{ij}}, \tag{2.9}$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, where $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{su}(n+1)$. Remembering that the root vectors $\{H_{\alpha_{12}}, \ldots, H_{\alpha_{nn+1}}\} \cup \{X_\alpha \in \mathfrak{su}(n+1)^{\mathbb{C}}_\alpha\}$ of this basis satisfy $B(X_\alpha, X_{-\alpha}) = -1$ and $[X_\alpha, X_{-\alpha}] = -H_\alpha \in \mathfrak{k}$.

Setting $H = S(U(1) \times U(n))$ and $K = T^n$, we have that $K \subsetneq H \subsetneq G$, with G, H and K compact connected Lie groups. Consider the canonical map

$$\pi : SU(n+1)/T^n \longrightarrow SU(n+1)/S(U(1) \times U(n)).$$

Let $\mathfrak{g} = \mathfrak{su}(n+1)$, $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$ and $\mathfrak{k}$ be the Lie algebras of G, H and K, respectively. As we saw previously, since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and a $\mathrm{Ad}(K)$-invariant $(-B)$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ decomposes into the sum

$$\mathfrak{g} = \mathfrak{su}(n+1) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and follows that

$$\pi : (SU(n+1)/T^n, g) \longrightarrow (SU(n+1)/S(U(1) \times U(n)), \breve{g}) \tag{2.10}$$

is a Riemannian submersion with totally geodesic fibers isometric to $(H/K, \hat{g})$,

$$H/K = S(U(1) \times U(n))/T^n = S(U(1) \times U(n))/S(U(1)^{n+1}) = SU(n)/T^{n-1},$$

$T^{n-1} = S(U(1)^n)$, g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$. Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim SU(n)/T^{n-1} = n(n-1)$, by the fact that $\dim \mathfrak{m}_\alpha = 2$, for each positive root α, we have that the vertical space $\mathfrak{p}$ (tangent to the fiber $SU(n)/T^{n-1}$) is equal to the direct sum of $\dfrac{n(n-1)}{2}$ irreducible isotropy summands of $\mathfrak{m}$ while $\mathfrak{q}$ is the direct sum of $\dfrac{n(n+1)}{2} - \dfrac{n(n-1)}{2} = n$ isotropy summands, once dimension of the total space is equal to $n(n+1)$.

In order to find the submodules that compose these distributions, observe that the fiber $SU(n)/T^{n-1}$ of the original fibration is a maximal flag manifold and the space $\mathfrak{p}$ is exactly the isotropy representation of T^{n-1}, which in turn decomposes into $\dfrac{n(n-1)}{2}$ pairwise inequivalent irreducible $\mathrm{Ad}(T^{n-1})$-modules equal to the root spaces relative to the positive roots in $R' \subset R$,

$$R' = \{\alpha_{ij} = \lambda_i - \lambda_j; i < j \leqslant n\}.$$

The Riemannian metric $\hat{g}$ on the fiber $SU(n)/T^{n-1}$, represented by $(-B)|_{\mathfrak{p}}$, is the normal metric represented by the inner product

$$g_{eT^{n-1}} = \sum_{i<j\leqslant n} (-B)|_{\mathfrak{m}_{\alpha_{ij}}}$$

and the homogeneous metric $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$ makes the basis $(SU(n+1)/S(U(1) \times U(n)), \breve{g})$ an isotropy irreducible compact symmetric space.

Hence, by scaling the normal metric

$$g_{eT^n} = \sum_{i<j\leqslant n+1} (-B)|_{\mathfrak{m}_{\alpha_{ij}}}$$

on the total space $SU(n+1)/T^n$ in the direction of the fibers by t^2, i.e., multiplying by $t^2, t > 0$, the parcels $(-B)|_{\mathfrak{m}_{\alpha_{ij}}}, 1 \leqslant i < j \leqslant n$ in the expression of g we obtain the canonical variation $\mathbf{g}_t$ of the metric g, according (2.6).

2.2.2.2 $(SO(2n+1)/T^n, \mathbf{h}_t), n \geqslant 4$:

For $n = 2$ we have the maximal flag $SO(5)/T^2 = SO(5)/U(1) \times U(1)$, for which the canonical fibration already been obtained in Example 2.2.13, 2.2.1.

We will denote by $G = SO(2n+1)$ the compact simple Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n+1)$ and by $T^n \subset G$ a maximal torus there, given by $T^n = U(1) \times \ldots \times U(1) = U(1)^n$. The maximal flag manifold associated with $\mathfrak{so}(2n+1)$ is $SO(2n+1)/T^n$. The Lie algebra $\mathfrak{so}(2n+1)$ decomposes into the $(-B)$-orthogonal direct sum

$$\mathfrak{so}(2n+1) = \mathfrak{k} \oplus \mathfrak{m},$$

where B is the Cartan-Killing form of $\mathfrak{so}(2n+1)$, defined by $B(X,Y) = (2n-1)\mathrm{Tr}(XY)$, $X, Y \in \mathfrak{so}(2n+1)$, $\mathfrak{k}$ is the Lie algebra of T^n, maximal abelian Lie subalgebra of $\mathfrak{so}(2n+1)$ formed by the diagonal matrices of the form

$$\mathfrak{k} = \left\{\sqrt{-1} \cdot \mathrm{diag}(0, a_1, \ldots, a_n, -a_1, \ldots, -a_n); a_i \in \mathbb{R}\right\}$$

and $\mathfrak{m}$ the $\mathrm{Ad}(T^n)$-invariant isotropy representation of T^n. With respect the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$, the root system of $\mathfrak{so}(2n+1)^{\mathbb{C}}$ can be chosen as by

$$R = \{\pm\lambda_i \pm \lambda_j, \pm\lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\},$$

where λ_i is given by $\mathrm{diag}(a_1, \ldots, a_n) \mapsto \lambda_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$, for each $1 \leqslant i \leqslant n$. The system of simple roots is $\Sigma = \{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n\}$ and with respect to Σ the positive roots are given by $R^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j, \lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\}$. In this case it has n^2 positive roots, hence $\mathfrak{m}$ decomposes into n^2 pairwise inequivalent irreducible $\mathrm{Ad}(T^n)$-modules as follows

$$\mathfrak{m} = \bigoplus_{\alpha \in R^+} \mathfrak{m}_\alpha, \tag{2.11}$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, where $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{so}(2n+1)^{\mathbb{C}}$. Remembering that the root vectors $\{H_{\alpha_1}, \ldots, H_{\alpha_n}\} \cup \{X_\alpha \in \mathfrak{so}(2n+1)^{\mathbb{C}}_\alpha\}$ of this basis satisfy $B(X_\alpha, X_{-\alpha}) = -1$ and $[X_\alpha, X_{-\alpha}] = -H_\alpha \in \mathfrak{k}$.

Setting $H = SO(2n)$ and $K = T^n$, we have that $K \subsetneq H \subsetneq G$, with G, H and K compact connected Lie groups. Consider the canonical map

$$\pi : SO(2n+1)/T^n \longrightarrow SO(2n+1)/SO(2n).$$

Let $\mathfrak{g} = \mathfrak{so}(2n+1)$, $\mathfrak{h} = \mathfrak{so}(2n)$ and $\mathfrak{k}$ be the Lie algebras of G, H and K, respectively. As we saw previously, since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and a $\mathrm{Ad}(K)$-invariant $(-B)$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ decomposes into the sum

$$\mathfrak{g} = \mathfrak{so}(2n+1) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and follows that

$$\pi : (SO(2n+1)/T^n, g) \longrightarrow (SO(2n+1)/SO(2n), \breve{g}) \tag{2.12}$$

is a Riemannian submersion with totally geodesic fibers isometric to the maximal flag manifold $(H/K, \hat{g})$,

$$H/K = SO(2n)/T^n,$$

g the normal metric determined by the inner product $-B|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$. Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim SO(2n)/T^n = 2n(n-1)$, by the fact that $\dim \mathfrak{m}_\alpha = 2$, for each positive root α, we have that the vertical space $\mathfrak{p}$ (tangent to the fiber $SO(2n)/T^n$) is equal to the direct sum of $n(n-1)$ irreducible isotropy summands of $\mathfrak{m}$ while $\mathfrak{q}$ is the direct sum of $n^2 - n(n-1) = n$ isotropy summands, once dimension of the total space is $2n^2$.

Since the fiber $SO(2n)/T^n$ of the original fibration is a maximal flag manifold, the space $\mathfrak{p}$ is exactly the isotropy representation of T^n, which in turn decomposes into $n(n-1)$ pairwise inequivalent irreducible $\mathrm{Ad}(T^n)$-modules equal to the root spaces relative to the positive roots in $R' \subset R$,

$$R' = \{\pm\lambda_i \pm \lambda_j; 1 \leqslant i < j \leqslant n\}.$$

The Riemannian metric $\hat{g}$ on the fiber $SO(2n)/T^n$, represented by $(-B)|_{\mathfrak{p}}$, is the normal metric determined by the inner product

$$g_{eT^n} = \sum_{\alpha \in R' \cap R^+} (-B)|_{\mathfrak{m}_\alpha}$$

and the homogeneous metric $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$ makes the basis $(SU(2n+1)/SO(2n), \breve{g})$ a irreducible Hermitian symmetric space, isometric to the round sphere.

Hence, by scaling the normal metric

$$g_{eT^n} = \sum_{\alpha \in R^+} (-B)|_{\mathfrak{m}_\alpha}$$

on the total space $SO(2n+1)/T^n$ in the direction of the fibers by t^2, i.e., multiplying by $t^2, t > 0$, all the $(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+$ in the expression of g, we obtain the canonical variation $\mathbf{h}_t$ of g.

2.2.2.3 $(Sp(n)/T^n, \mathbf{k}_t), n \geqslant 3$:

Let $G = Sp(n)$ be the compact simple Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{sp}(n)$ and by $T^n \subset G$ a maximal torus there, given by $T^n = U(1) \times \ldots \times U(1) = U(1)^n$. The maximal flag manifold associated is $Sp(n)/T^n$ and the Lie algebra $\mathfrak{sp}(n)$ decomposes into the $(-B)$-orthogonal direct sum

$$\mathfrak{sp}(n) = \mathfrak{k} \oplus \mathfrak{m},$$

where B is its the Cartan-Killing form of defined by $B(X, Y) = 2(n+1)\mathrm{Tr}(XY)$, $X, Y \in \mathfrak{sp}(n)$, $\mathfrak{k}$ is the Lie algebra of T^n, maximal abelian Lie subalgebra of $\mathfrak{sp}(n)$ formed by the diagonal matrices of the form

$$\mathfrak{k} = \left\{\sqrt{-1} \cdot \mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n); a_i \in \mathbb{R}\right\}$$

and $\mathfrak{m}$ the $\mathrm{Ad}(T^n)$-invariant isotropy representation of T^n. With respect the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$, the root system of $\mathfrak{sp}(n)^{\mathbb{C}}$ can be chosen as by

$$R = \{\pm\lambda_i \pm \lambda_j, \pm 2\lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\},$$

where λ_i is given by $\mathrm{diag}(a_1, \ldots, a_n) \mapsto \lambda_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$, for each $1 \leqslant i \leqslant n$. The system of simple roots is $\Sigma = \{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_n\}$ and with respect to Σ the positive roots are given by $R^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j, 2\lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\}$. In this case it has n^2 positive roots, hence $\mathfrak{m}$ decomposes into n^2 pairwise inequivalent irreducible $\mathrm{Ad}(T^n)$-modules as follows

$$\mathfrak{m} = \bigoplus_{\alpha \in R^+} \mathfrak{m}_\alpha, \tag{2.13}$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, where $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{sp}(n)^{\mathbb{C}}$.

Setting $H = U(n)$ and $K = T^n$, we have that $K \subsetneq H \subsetneq G$, with G, H and K compact connected Lie groups. Consider the canonical map

$$\pi : Sp(n)/T^n \longrightarrow Sp(n)/U(n).$$

Let $\mathfrak{h} = \mathfrak{u}(n)$ and $\mathfrak{k}$ be the Lie algebra H. Since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and a $\mathrm{Ad}(K)$-invariant $-B$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ decomposes into the sum

$$\mathfrak{g} = \mathfrak{sp}(n) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and follows that

$$\pi : (Sp(n)/T^n, g) \longrightarrow (Sp(n)/U(n), \breve{g}) \tag{2.14}$$

is a Riemannian submersion with totally geodesic fibers isometric to the maximal flag manifold $(H/K, \hat{g})$,

$$H/K = U(n)/T^n \cong SU(n)/S(U(1) \times \ldots \times U(1)) = SU(n)/S(U(1)^n),$$

g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$. Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim H/K = U(n)/T^n \cong SU(n)/S(U(1)^n) = n(n-1)$, by the fact that $\dim \mathfrak{m}_\alpha = 2$, for each positive root α, we have that the vertical space $\mathfrak{p}$ (tangent to the fiber $SU(n)/S(U(1)^n)$) is equal to the direct sum of $\dfrac{n(n-1)}{2}$ irreducible isotropy summands of $\mathfrak{m}$ while $\mathfrak{q}$ is the direct sum of $n^2 - \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2}$ isotropy summands, once dimension of the total space is $2n^2$.

Since the fiber $SU(n)/S(U(1)^n)$ of the original fibration is a maximal flag manifold, the space $\mathfrak{p}$ is exactly the isotropy representation of the maximal torus $T^{n-1}_{SU(n)} = S(U(1)^n)$ of $SU(n)$, which in turn decomposes into $n(n-1)$ pairwise inequivalent irreducible $\mathrm{Ad}(T^{n-1}_{SU(n)})$-modules equal to the root spaces relative to the positive roots in $R' \subset R$,

$$R' = \{\pm(\lambda_i - \lambda_j); 1 \leqslant i < j \leqslant n\}.$$

The Riemannian metric $\hat{g}$ on the fiber $SU(n)/S(U(1)^n)$, represented by $(-B)|_{\mathfrak{p}}$, is the normal metric rerpesented by the inner product

$$g_{eT^{n-1}_{SU(n)}} = \sum_{\alpha \in R' \cap R^+} (-B)|_{\mathfrak{m}_\alpha}$$

and the homogeneous metric $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$ makes the basis $(Sp(n)/U(n), \breve{g})$ a irreducible Hermitian symmetric space.

By scaling the normal metric

$$g_{eT^n} = \sum_{\alpha \in R^+} (-B)|_{\mathfrak{m}_\alpha}$$

on the total space $Sp(n)/T^n$ in the direction of the fibers by t^2, i.e., multiplying by $t^2, t > 0$, all the $(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+$ in the expression of g, we obtain the canonical variation $\mathbf{k}_t$ of g.

2.2.2.4 $(SO(2n)/T^n, \mathbf{m}_t), n \geqslant 4$:

Let $G = SO(2n)$ be the compact simple Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n)$ and by $T^n \subset G$ a maximal torus with $T^n = U(1) \times \ldots \times U(1) = U(1)^n$. The maximal flag manifold associated is $SO(2n)/T^n$ and the Lie algebra $\mathfrak{so}(2n)$ decomposes into the $(-B)$-orthogonal direct sum

$$\mathfrak{so}(2n) = \mathfrak{k} \oplus \mathfrak{m},$$

where B is its the Cartan-Killing form of defined by $B(X,Y) = 2(n-1)\mathrm{Tr}(XY)$, $X, Y \in \mathfrak{sp}(n)$, $\mathfrak{k}$ is the Lie algebra of T^n, maximal abelian Lie subalgebra of $\mathfrak{so}(2n)$ formed by the diagonal matrices of the form

$$\mathfrak{k} = \left\{\sqrt{-1}\cdot \mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n); a_i \in \mathbb{R}\right\}$$

and $\mathfrak{m}$ the $\mathrm{Ad}(T^n)$-invariant isotropy representation of T^n. With respect the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$, the root system of $\mathfrak{so}(2n)^{\mathbb{C}}$ can be chosen as by

$$R = \{\pm\lambda_i \pm \lambda_j; 1 \leqslant i < j \leqslant n, \},$$

according to the above notation. The system of simple roots is

$$\Sigma = \{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}$$

and with respect to Σ the positive roots belong to

$$R^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j; 1 \leqslant i < j \leqslant n\}.$$

It has $n(n-1)$ positive roots, hence $\mathfrak{m}$ decomposes into $n(n-1)$ pairwise inequivalent irreducible $\mathrm{Ad}(T^n)$-modules as follows

$$\mathfrak{m} = \bigoplus_{\alpha \in R^+} \mathfrak{m}_\alpha, \tag{2.15}$$

where each two dimensional irreducible submodule $\mathfrak{m}_\alpha$ above is generated by $\{A_\alpha, S_\alpha\}$, where $A_\alpha = X_\alpha + X_{-\alpha}$, $S_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})$ and X_α belongs to the Weyl basis of $\mathfrak{so}(2n)^{\mathbb{C}}$.

Take the canonical map

$$\pi : SO(2n)/T^n \longrightarrow SO(2n)/U(n).$$

Let $\mathfrak{h} = \mathfrak{u}(n)$ and $\mathfrak{k}$ be the Lie algebras H and $K = T^n$, respectively. Since K, H and G are compacts, we can consider a $\mathrm{Ad}(H)$-invariant $(-B)$-orthogonal complement $\mathfrak{q}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and a $\mathrm{Ad}(K)$-invariant $(-B)$-orthogonal complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ decomposes into the sum

$$\mathfrak{g} = \mathfrak{so}(2n) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{m},$$

and follows that

$$\pi : (SO(2n)/T^n, g) \longrightarrow (SO(n)/U(n), \breve{g}) \tag{2.16}$$

is a Riemannian submersion with totally geodesic fibers isometric to the maximal flag manifold $(H/K, \hat{g})$,

$$H/K = U(n)/T^n \cong SU(n)/S(U(1) \times \ldots \times U(1)) = SU(n)/S(U(1)^n),$$

g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$. Since $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and $\dim H/K = U(n)/T^n \cong SU(n)/S(U(1)^n) = n(n-1)$, by the fact that $\dim \mathfrak{m}_\alpha = 2$, for each positive root α, we have that the vertical space $\mathfrak{p}$ (tangent to the fiber $SU(n)/S(U(1)^n)$) is equal to the direct sum of $\dfrac{n(n-1)}{2}$ irreducible isotropy summands of $\mathfrak{m}$ while $\mathfrak{q}$, from the fact that $2n(n-1)$ is the dimension of the total space, is equal to the direct sum of $n(n-1) - \dfrac{n(n-1)}{2} = \dfrac{n(n-1)}{2}$ isotropy summands.

Since the fiber $SU(n)/S(U(1)^n)$ of the original fibration is a maximal flag manifold, the space $\mathfrak{p}$ is exactly the isotropy representation of $T^{n-1}_{SU(n)} = S(U(1)^n)$, which in turn decomposes into $n(n-1)$ pairwise inequivalent irreducible $\mathrm{Ad}(T^{n-1}_{SU(n)})$-modules equal to the root spaces relative to the positive roots in $R' \subset R$,

$$R' = \{\pm(\lambda_i - \lambda_j); 1 \leqslant i < j \leqslant n\}.$$

The Riemannian metric $\hat{g}$ on the fiber $SU(n)/S(U(1)^n)$, represented by $(-B)|_{\mathfrak{p}}$, is the normal metric given by the inner product

$$g_{eT^{n-1}_{SU(n)}} = \sum_{\alpha \in R' \cap R^+} (-B)|_{\mathfrak{m}_\alpha}$$

and the homogeneous metric $\breve{g}$ on the basis, defined by the inner product $-B|_{\mathfrak{q}}$, makes $(Sp(n)/U(n), \breve{g})$ a irreducible Hermitian symmetric space.

By scaling the normal metric

$$g_{eT^n} = \sum_{\alpha \in R^+} (-B)|_{\mathfrak{m}_\alpha}$$

on the total space $SO(2n)/T^n$ in the direction of the fibers by t^2, i.e., multiplying by $t^2, t > 0$, all the $(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+$ in the expression of g, we obtain the canonical variation $\mathbf{m}_t$ of g.

2.2.3 Spectra of Maximal Flag Manifolds and Symmetric spaces

In [33], S. Yamaguchi describes the spectrum of the Laplacian defined on $C^\infty(G/T, g)$, where G/T is a maximal flag manifold equipped with a normal metric g. In this case, we have the following description of $\sigma(\Delta_g)$.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}$ complex and simple Lie algebra. Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, R the root system of $(\mathfrak{g}, \mathfrak{h})$ and by $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ the associated system of simple roots. Let $(\cdot,\cdot) = -B$ the inner product on $\mathfrak{g}$ induced by the Cartan-Killing form B $\mathfrak{g}$. Consider the *system of fundamental weights* $\{\omega_1, \cdots, \omega_l\}$ of $\mathfrak{h}$ and denote by $\mathcal{P}$ the set of all integral dominant weights,

$$\mathcal{P} = \{\Lambda = \sum_{i=1}^{l} s_i\omega_i \in \mathfrak{h}^*; s_i \geqslant 0, s_i \in \mathbb{Z}\}.$$

Let $\widehat{\mathcal{P}}$ the set of all elements of $\mathcal{P}$ which are of class one relative to $\mathfrak{h}$, i.e., the irreducible representation (ξ_Λ, V_Λ) of $\mathfrak{g}$ in V_Λ, with the highest weight equal to $\Lambda \in \mathcal{P}$, has a non zero $\xi_\Lambda(\mathfrak{h})$-invariant vector in the representation space V_Λ.

Theorem 2.2.19 (H. Freudenthal [33]). *Let (ξ_Λ, V_Λ) be the irreducible representation with highest weight $\Lambda \in \mathcal{P}$ $(\Lambda \neq 0)$. Then, $\Lambda \in \widehat{\mathcal{P}}$ if and only if $\Lambda = \sum_{i=1}^{l} p_i\alpha_i, p_i \geqslant 1,\ p_i \in \mathbb{Z}$, $1 \leqslant i \leqslant l$.*

$$\text{We identify } \Lambda = \sum_{i=1}^{l} p_i\alpha_i \in \widehat{\mathcal{P}} \text{ with } P = (p_1, \ldots, p_l).$$

Theorem 2.2.20 (S. Yamaguchi [33]). *If G/T is a maximal flag manifold associated with a complex classical simple Lie algebra $\mathfrak{g}$ of the type A_n, B_n, C_n, D_n or with the exceptional Lie algebra of the type G_2, the spectrum of the Laplacian Δ_g, g normal metric induced by the Cartan-Killing form B of $\mathfrak{g}$, is determined by the irreducible representations with highest weight $\Lambda \in \widehat{\mathcal{P}}$, in such a way that each eigenvalue $\mu \in \sigma(\Delta_g)$ is given by $\mu(\Lambda)$, for some irreducible representation (ξ_Λ, V_Λ) with highest weight $\Lambda = \sum_{i=1}^{l} p_i\alpha_i$, as follows:*

Type $A_n, n \geqslant 1$:

(1) $$\mu(\Lambda) = \frac{1}{n+1}\left\{\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} p_ip_{i+1} + \sum_{i=1}^{n} p_i\right\},$$

(2) $$\left\{\begin{array}{rcl} 2p_1 - p_2 & \geqslant & 0 \\ -p_1 + 2p_2 - p_3 & \geqslant & 0 \\ \vdots & & \\ -p_{n-2} + 2p_{n-1} - p_n & \geqslant & 0 \\ -p_{n-1} + 2p_n & \geqslant & 0 \end{array}\right. ,$$

(3) First positive eigenvalue $\mu_1 = \mu(1, \ldots, 1) = 1$,

$$P_0 = (1, \ldots, 1) \longleftrightarrow \Lambda_0 = \alpha_1 + \ldots + \alpha_n \quad \textbf{\textit{highest root}},$$

ξ_{Λ_0}=adjoint representation.

Type $B_n, n \geqslant 2$:

(1) $$\mu(\Lambda) = \frac{1}{4n-2}\left\{\sum_{i=1}^{n-1} 2p_i^2 + p_n^2 - 2\sum_{i=1}^{n-1} p_ip_{i+1} + 2\sum_{i=1}^{n-1} p_i + p_n\right\},$$

(2) $$\begin{cases} 2p_1 - p_2 & \geqslant 0 \\ -p_1 + 2p_2 - p_3 & \geqslant 0 \\ \vdots & \\ p_{n-2} + 2p_{n-1} - p_n & \geqslant 0 \\ -p_{n-1} + p_n & \geqslant 0 \end{cases},$$

(3) First positive eigenvalue $\mu_1 = \mu(1, \dots, 1) = \dfrac{n}{2n-1}$,

$$P_0 = (1, \dots, 1) \longleftrightarrow \Lambda_0 = \omega_1 = \alpha_1 + \dots + \alpha_n,$$

ξ_{Λ_0}=*representation with highest weight* ω_1.

Type $C_n, n \geqslant 3$:

(1) $$\mu(\Lambda) = \frac{1}{2(n+1)} \left\{ \sum_{i=1}^{n-1} p_i^2 + 2p_n^2 - \sum_{i=1}^{n-2} p_i p_{i+1} - p_{n-1}p_n + \sum_{i=1}^{n-1} p_i + 2p_n \right\},$$

(2) $$\begin{cases} 2p_1 - p_2 & \geqslant 0 \\ -p_1 + 2p_2 - p_3 & \geqslant 0 \\ \vdots & \\ -p_{n-3} + 2p_{n-2} - p_{n-1} & \geqslant 0 \\ -p_{n-2} + 2p_{n-1} - 2p_n & \geqslant 0 \\ -p_{n-1} + 2p_n & \geqslant 0 \end{cases},$$

(3) First positive eigenvalue $\mu_1 = \mu(1, 2, \dots, 2, 1) = \dfrac{4n-1}{4(n+1)}$,

$$P_0 = (1, \dots, 1) \longleftrightarrow \Lambda_0 = \omega_2,$$

ξ_{Λ_0}=*representation with highest weight* ω_2.

Type $D_n, n \geqslant 4$:

(1) $$\mu(\Lambda) = \frac{1}{2n-1} \left\{ \sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-2} p_i p_{i+1} - p_{n-2}p_n + \sum_{i=1}^{n} p_i \right\},$$

(2) $$\begin{cases} 2p_1 - p_2 & \geqslant 0 \\ -p_1 + 2p_2 - p_3 & \geqslant 0 \\ \vdots & \\ -p_{n-4} + 2p_{n-3} - p_{n-2} & \geqslant 0 \\ -p_{n-3} + 2p_{n-2} - p_{n-1} - p_n & \geqslant 0 \\ -p_{n-2} + 2p_{n-1} & \geqslant 0 \\ -p_{n-2} + 2p_n & \geqslant 0 \end{cases},$$

(3) *First positive eigenvalue* $\mu_1 = \mu(1, 2, \ldots, 2, 1, 1) = 1$,

$$P_0 = (1, 2, \ldots, 2, 1, 1) \longleftrightarrow \Lambda_0 \quad \textbf{\textit{highest root}},$$

ξ_{Λ_0}=*adjoint representation.*

Type G_2:

(1) $\mu(\Lambda) = \frac{1}{12}\left\{p_1^2 + 3p_2^2 - 3p_1p_2 + p_1 + 3p_2\right\}$,

(2) $\begin{cases} 2p_1 - 3p_2 & \geqslant & 0 \\ -p_1 + 2p_2 & \geqslant & 0 \end{cases}$,

(3) *First positive eigenvalue* $\mu_1 = \mu(2, 1) = \frac{1}{2}$, $P_0 = (2, 1) \longleftrightarrow \Lambda_0 = \omega_1$,
ξ_{Λ_0}=*representation with highest weight* ω_1.

We note that the basis spaces $(G/H, \breve{g})$ of the homogeneous fibrations that have been studied so far in this work are irreducible Hermitian symmetric space of compact type. The spectrum of such spaces are well known, and can be determined as follows (see [30], page 64).

Let G be a compact simply connected simple Lie group, H closed subgroup of G. Let $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H, respectively, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, the Cartan decomposition. The inner product on $\mathfrak{q}$ is $(-B)|_{\mathfrak{q}}$, where B is the Cartan-Killing form of $\mathfrak{g}$. Let $\breve{g}$ the G-invariant Riemannian metric on G/H induced by B. Then, it is known that the spectrum of the Laplacian of $(G/H, \breve{g})$ is given by

$$\mu(\Lambda) = -B(\Lambda + 2\delta, \Lambda), \tag{2.17}$$

with multiplicities

$$d_\Lambda = \prod_{\alpha \in R^+} \frac{-B(\Lambda + \delta, \alpha)}{-B(\delta, \alpha)}. \tag{2.18}$$

Here, Λ varies over the set $D(G, H)$ of the highest weights of all spherical representations of (G, H), δ is equal to the sum of the positive roots $\alpha \in R^+$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ relative to the maximal abelian subalgebra $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ and d_Λ is the dimension of the irreducible spherical representation of (G, H) with highest weight Λ.

For simple compact connected G, Krämer [22] provides a classification of possible subgroups H along with a set of dominant weights whose integral non-negative combinations give all spherical representations.

In particular, in Krämer's classification the basis of the representations in $D(G_2, SO(4))$ is given by

$$\mathcal{B} = \{2\pi_1, 2\pi_2\},$$

with $\{\pi_1, \pi_2\}$ being the set of fundamental weights of the maximal abelian Lie algebra $\mathrm{Lie}(T) = \mathfrak{k}$. Therefore, by applying (2.17) and the fact that all $\Lambda \in D(G_2, SO(4))$ is linear combination of the weights in $\mathcal{B}$, follows the case of the spectrum of the Laplacian on the isotropy irreducible symmetric space $G_2/SO(4)$ equipped with the metric determined by the Cartan-Killing form of G_2.

Proposition 2.2.21. *The spectrum of the Laplacian of the isotropy irreducible Hermitian space $(G_2/SO(4), \breve{g})$ is given by* $\sigma(\Delta_{\breve{g}}) = \left\{\frac{1}{6}(9r + 6r^2 + 5s + 6rs + 2s^2); \mathbb{Z} \ni r, s \geqslant 0\right\}$, *with first positive eigenvalue* $\beta_1 = \frac{7}{6}$.

Assume that B is the Cartan-Killing form of G, G/T total space of the homogeneous fibration

$$\pi : (G/T, g) \longrightarrow (G/H, \breve{g})$$

and $\breve{g}$ is the homogeneous metric given by the inner product $(-B)|_{\mathfrak{q}}$, $\mathfrak{q}$ $(-B)$-orthogonal complement to $\mathrm{Lie}(H) = \mathfrak{h}$ in $\mathrm{Lie}(G) = \mathfrak{g}$, i.e, $\mathfrak{q}$ is the horizontal distribution relative to the submersion π. Moreover, each $(G/H, \breve{g})$ below is an isotropy irreducible Hermitian symmetric space.

In the follow, we will describe the spectrum of the Laplacian on each basis space $(G/H, \breve{g})$ of the homogeneous fribations $\pi : (G/T, g) \longrightarrow (G/H, \breve{g})$ constructed in the previous Section on the maximal flag manifolds $SU(n+1)/T^n, SO(2n+1)/T^n, Sp(n)/T^n$, $SO(2n)/T^n$ and G_2/T, respectively.

Spectrum of the Complex Projective Space, $\sigma(\Delta_{\mathbb{CP}^n})$

The basis space $(G/H, \breve{g}) = (SU(n+1)/S(U(1) \times U(n)), \breve{g})$ of the homogeneous fibration

$$\pi : (SU(n+1)/T^n, g) \longrightarrow (SU(n+1)/S(U(1) \times U(n)), \breve{g}), n \geqslant 2,$$

is the homogeneous realization of the complex projective space $\mathbb{CP}^n$ endowded with a homothetical metric to the Fubini-Study metric. In fact, the induced metric $(-B)|_{\mathfrak{q}}$ on $\mathbb{CP}^n$, B Cartan-Killing form given by $B(X, Y) = 2(n+1)\mathrm{Tr}(XY), X, Y \in \mathfrak{su}(n+1)$, is $n+1$ times the Fubini-Study metric. According [19], the spectra of the Laplacian acting on functions on the complex projective space $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$ with the Fubini-Study metric is

$$\sigma(\Delta_{FS}) = \{\xi_k = k(k+n); k \in \mathbb{N}\}.$$

Hence, the spectrum of the Laplacian on $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$ provided with the metric $\breve{g}$ represented by the inner product $-B|_{\mathfrak{q}}$ is

$$\sigma(\Delta_{\breve{g}}) = \{\beta_k = \frac{\xi_k}{n+1} = \frac{k(k+n)}{n+1}; k \in \mathbb{N}\}. \tag{2.19}$$

Note that the first positive eigenvalue in this case is $\beta_1 = 1$.

Spectrum of the Round Sphere, $\sigma(\Delta_{S^{2n}})$

The basis space $(G/H, \breve{g}) = (SO(2n+1)/SO(2n)), \breve{g})$ of the homogeneous fibration

$$\pi : (SO(2n+1)/T^n, g) \longrightarrow (SO(2n+1)/SO(2n), \breve{g}), n \geqslant 2,$$

is the homogeneous realization of the round sphere S^{2n} provided with a homothetical metric to the canonical metric. In fact, the metric induced from the Cartan-Killing form given by

$$B(X,Y) = (2n-1)\mathrm{Tr}(XY), X, Y \in \mathfrak{so}(2n+1),$$

is $2(2n-1)$ times the usual one. According [19], the spectra of the Laplacian acting on functions on sphere $S^{2n} = SO(2n+1)/SO(2n)$ with the usual metric h is

$$\sigma(\Delta_h) = \{\xi_k = k(k+n-1); k \in \mathbb{N}\}.$$

Hence, the spectrum of the Laplacian on $S^{2n} = SO(2n+1)/SO(2n)$ provided with the metric $\breve{g}$ represented by the inner product $-B|_{\mathfrak{q}}$ is

$$\sigma(\Delta_{\breve{g}}) = \{\beta_k = \frac{\xi_k}{2(2n-1)} = \frac{k(k+2n-1)}{2(2n-1)}; k \in \mathbb{N}\}. \tag{2.20}$$

Note that the first positive eigenvalue in this case is $\beta_1 = \dfrac{n}{2n-1}$.

Spectrum of the Symmetric Space $Sp(n)/U(n)$, $\sigma(\Delta_{Sp(n)/U(n)})$

In the case of the basis space $(G/H, \breve{g}) = (Sp(n)/U(n), \breve{g})$ of the homogeneous fibration $\pi : (Sp(n)/T^n, g) \longrightarrow (Sp(n)/U(n), \breve{g}), n \geqslant 3$, Krämer's classification [22] give us the basis of the representations in the set $D(Sp(n), U(n))$ of the highest weights of all spherical representations of the pair $(Sp(n), U(n))$, namely

$$\mathcal{B} = \{2\pi_l; 1 \leqslant l \leqslant n\},$$

with $\{\pi_l; 1 \leqslant l \leqslant n\}$ being the fundamental weights of the maximal abelian Lie algebra $\mathrm{Lie}(T^n) = \mathfrak{k}^{\mathbb{C}}$. Therefore, by applying (2.17) and the fact that all $\Lambda \in D(Sp(n), U(n))$ is linear combination of the weights in $\mathcal{B}$, we have the spectrum of the Laplacian on the isotropy irreducible symmetric space $Sp(n)/U(n)$ provided with the metric determined by the Cartan-Killing form B of $Sp(n)$, given by

$$B(X,Y) = 2(n+1)\mathrm{Tr}(XY), X, Y \in \mathfrak{sp}(n).$$

The first positive eigenvalue in this case, according [30], is $\beta_1 = 1$.

Spectrum of the Symmetric Space $SO(2n)/U(n)$, $\sigma(\Delta_{SO(2n)/U(n)})$

In the case of the basis space $(G/H, \breve{g}) = (SO(2n)/U(n), \breve{g})$ of the homogeneous fibration $\pi : (SO(2n)/T^n, g) \longrightarrow (SO(2n)/U(n), \breve{g}), n \geqslant 4$, Krämer's classification [22] give us the basis of the representations in the set $D(SO(2n), U(n))$ of the highest weights of all spherical representations of the pair $(SO(2n), U(n))$, namely

$$\mathcal{B}_1 = \{\pi_2, \pi_4, \ldots, \pi_{n-2}, 2\pi_n\}, \quad \text{if} \quad n \quad \text{is even}$$

and

$$\mathcal{B}_2 = \{\pi_2, \pi_4, \ldots, \pi_{n-3}, \pi_{n-1} + \pi_n\}, \quad \text{if} \quad n \quad \text{is odd}$$

with $\{\pi_l; 1 \leqslant l \leqslant n\}$ being the fundamental weights of the maximal abelian Lie algebra $\mathrm{Lie}(T^n) = \mathfrak{k}^{\mathbb{C}}$. Therefore, by applying (2.17) and the fact that all $\Lambda \in D(SO(2n), U(n))$ is linear combination of the weights in $\mathcal{B}_1$ or $\mathcal{B}_2$, according to n is even or odd, we have the spectrum of the Laplacian on the isotropy irreducible symmetric space $SO(2n)/U(n)$ provided with the metric determined by the Cartan-Killing form B of $SO(2n)$, given by

$$B(X, Y) = 2(n-1)\mathrm{Tr}(XY), X, Y \in \mathfrak{so}(2n).$$

The first positive eigenvalue in both of the cases, i.e, when n is even or odd, according [30], is $\beta_1 = 1$.

From the above, we obtain the following useful properties of the first positive eigenvalue $\lambda_1(t)$ of the Laplacian $\Delta_t = \Delta_{g_t}$, g_t canonical variation of the normal metrics on the flag manifolds in Theorem 2.2.20.

Proposition 2.2.22. *Considering the canonical variations* $(SU(n+1)/T^n, g_t)$, $(SO(2n+1)/T^n, g_t)$, $(Sp(n)/T^n, g_t)$, $(SO(2n)/T^n, g_t)$ *and* $(G_2/T, g_t)$, *one has the following estimates for the first positive eigenvalue* $\lambda_1(t)$ *of the Laplacian* Δ_t *on the canonical variations* $(G/T, g_t)$*:*

$(G/T, g_t)$	$\lambda_1(t), 0 < t \leqslant 1$
$(SU(n+1)/T^n, \mathbf{g}_t)$	$\lambda_1(t) = 1$
$(SO(2n+1)/T^n, \mathbf{h}_t)$	$\lambda_1(t) = \dfrac{n}{2n-1}$
$(Sp(n)/T^n, \mathbf{k}_t)$	$\dfrac{4n-1}{4(n+1)} \leqslant \lambda_1(t) \leqslant 1$
$(SO(2n)/T^n, \mathbf{m}_t)$	$\lambda_1(t) = 1$
$(G_2/T, \mathbf{n}_t)$	$\dfrac{1}{2} \leqslant \lambda_1(t) \leqslant \dfrac{7}{6}$

Proof: By Corollaries 2.2.6 and 2.2.8 in Section 2.2,

$$\mu_1 \leqslant \lambda_1(t) \leqslant \beta_1,$$

for all $0 < t \leqslant 1$, where $\lambda_1(t), \mu_1$ and β_1 are the first positive eigenvalues of Δ_t, Δ_g and Δ_h, which are the Laplacians on the total space of the canonical variation, on the original total space and on the basis, respectively. From the above description of the spectra of the Laplacians on maximal flag manifolds and isotropy irreducible Hermitian symmetric spaces, we have all the first positive eigenvalues of the Laplacians both on the original total spaces and on the basis, i.e, we have the values of β_1 and μ_1, which allow us applying the inequality $\mu_1 \leqslant \lambda_1(t) \leqslant \beta_1$ in order to obtain the respective values and estimates for $\lambda_1(t)$ given in the table above.

□

3 Bifurcation and Local Rigidity Instants for Canonical Variations on Maximal Flag Manifolds

In this chapter, we prove the main results of this thesis, determining the bifurcation and local rigidity instants for the canonical variations which were constructed in the previous chapter. The criterion used to find such instants is based on comparison between an expression of the eigenvalues of the Laplacian relative to the respective canonical variation and a multiple of its scalar curvature. This method is related to the notion of Morse index.

Firstly, will be determined an expression for the scalar curvature of each canonical variation described above by mean a general formula obtained by W. Ziller in [31] that can be applied to calculate the scalar curvature of homogeneous metrics on reductive homogeneous spaces. From this formula, we can see clearly that these kind of metrics have constant scalar curvature and are trivial solutions to the Yamabe problem.

Necessary conditions for classifications of bifurcation and local rigidity instants in the interval $(0,1)$ can be deduced by using the expressions of the scalar curvature in all cases.

3.1 Scalar Curvature

As stated, it is necessary an expression for $\mathrm{scal}(g_t)$ of given 1-parameter family g_t of homogeneous metrics on a maximal flag manifold, defined by (2.6) in Section 2.2.1. The general formula of the scalar curvature for reductive homogeneous spaces which we use here is obtained in [31].

Let G be a compact connected Lie group and $K \subset G$ a closed subgroup of G. Assume that K is connected, which corresponds to the case that G/K is simply connected. Let $\mathfrak{m}$ be the $(-B)$-orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively, and B is the Cartan-Killing form of $\mathfrak{g}$. It is known that the isotropy representation $\mathfrak{m}$ of K decomposes into a direct sum of inequivalent irreducible submodules,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \ldots \oplus \mathfrak{m}_r,$$

with $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$ such that $\mathrm{Ad}(K)\mathfrak{m}_i \subset \mathfrak{m}_i$, for all $i = 1, \ldots, r$. Thus, each G-invariant metric on G/K can be represented by a inner product on $\mathfrak{m}$ given by

$$t_1(-B)|_{\mathfrak{m}_1} + t_2(-B)|_{\mathfrak{m}_2} + \ldots + t_r(-B)|_{\mathfrak{m}_r}, \quad t_i > 0, \quad i = 1, \ldots, r.$$

Let X_α be a $(-B)$-orthonormal basis adapted to the $\mathrm{Ad}(K)$-invariant decomposition of $\mathfrak{m}$, i.e., $X_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ with $X_\alpha \in \mathfrak{m}_i$ and $X_\beta \in \mathfrak{m}_j$. Set $A^\gamma_{\alpha\beta} = -B([X_\alpha, X_\beta], X_\gamma)$, so that $[X_\alpha, X_\beta]_{\mathfrak{m}} = \sum_\gamma A^\gamma_{\alpha\beta} X_\gamma$, and define

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2,$$

where the sum is taken over all indices α, β, γ with $X_\alpha \in \mathfrak{m}_i, X_\beta \in \mathfrak{m}_j$ and $X_\gamma \in \mathfrak{m}_k$. Note that $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the $(-B)$-orthogonal basis chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, but it depends on the choice of the decomposition of $\mathfrak{m}$. In addition, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is continuous function on the space of all $(-B)$-orthogonal ordered decomposition of $\mathfrak{m}$ into $\mathrm{Ad}(K)-$irreducible summands and also is non-negative and symmetric in all 3 indices. The set $\{X_\alpha/\sqrt{t_i}; X_\alpha \in \mathfrak{m}_i\}$ is a orthonormal basis of $\mathfrak{m}$ with respect to

$$\langle \cdot, \cdot \rangle = t_1(-B)|_{\mathfrak{m}_1} + t_2(-B)|_{\mathfrak{m}_2} + \ldots + t_r(-B)|_{\mathfrak{m}_r}.$$

Then (see [31]), the scalar curvature of g determined by $\langle \cdot, \cdot \rangle$ is

$$\mathrm{scal}(g) = \frac{1}{2} \sum_{l=1}^{r} \frac{d_l}{t_l} - \frac{1}{4} \sum_{i,j,k} \begin{bmatrix} k \\ ij \end{bmatrix} \frac{t_k}{t_i t_j}, \tag{3.1}$$

for all $x \in G/K$, $d_i = \dim \mathfrak{m}_i$, $i = 1, \ldots, r$.

Example 3.1.1. In [20], page 303, was obtained formula (3.1) for a G-invariant metric g on a homogeneous space G/K for which $\mathfrak{m}$ has 3 isotropy summands. In this case, g is determined by a inner product $t_1(-B)|_{\mathfrak{m}_1} + t_2(-B)|_{\mathfrak{m}_2} + t_3(-B)|_{\mathfrak{m}_3}$ and

$$scal(g) = \frac{1}{2} \sum_{i=1}^{3} \frac{d_i}{t_i} - \frac{1}{2} \begin{bmatrix} 3 \\ 12 \end{bmatrix} \left(\frac{t_3}{t_1 t_2} + \frac{t_2}{t_1 t_3} + \frac{t_1}{t_2 t_3} \right). \tag{3.2}$$

In particular, considering the 1-parameter family g_t on $SU(3)/T^2$, given by the inner product

$$(g_t)_{eT^2} = (-t^2 B)|_{\mathfrak{m}_1} + (-B)|_{\mathfrak{m}_2} + (-B)|_{\mathfrak{m}_3}, \quad t > 0,$$

from the decomposition of the isotropy representation of $K = T^2$,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

com $\mathfrak{m}_1 = \mathfrak{p}$, $\mathfrak{m}_2 = \mathrm{spam}_{\mathbb{C}}(E_{13} - E_{31})$ and $\mathrm{spam}_{\mathbb{C}}(E_{23} - E_{32})$, we take the $(-B)$-orthornormal basis of $\mathfrak{m}$, $\mathcal{W} = \{A_{12}, A_{13}, A_{23}, S_{12}, S_{13}, S_{23}\}$,

$$A_{ij} = \frac{1}{\sqrt{12}}(E_{ij} - E_{ji}), S_{ij} = \frac{\sqrt{-1}}{\sqrt{12}}(E_{ij} + E_{ji}).$$

We have that $\mathcal{W}$ is a adapted basis to this decomposition of $\mathfrak{m}$, with $A_{12}, S_{12} \in \mathfrak{m}_1$, $A_{23}, S_{23} \in \mathfrak{m}_2$ and $A_{13}, S_{13} \in \mathfrak{m}_3$. Since $d_1 = d_2 = d_3 = 2$, $\begin{bmatrix} 3 \\ 12 \end{bmatrix} = \frac{1}{3}$ and $t_1 = t^2, t_2 = t_3 = 1$, by (3.2), follows

$$scal(g_t) = \frac{2}{3t^2} - \frac{t^2}{6} + 2. \tag{3.3}$$

After application of the formula (3.1) in the previous example, we will use it to exhibit a formula of the scalar curvature scal(t) for each 1-parameter family of homogeneous metrics g_t defined on some type of maximal flag manifold associated with one of the complex classical simple Lie algebras.

First, some observations about the numbers $\begin{bmatrix} k \\ ij \end{bmatrix}$ are necessary. For a maximal flag manifold G/K, we have the $(-B)$-orthogonal decomposition $\sum\limits_{\alpha \in R^+} \mathfrak{m}_\alpha$ of its isotropy representation $\mathfrak{m}$, where B is the Cartan-Killing form of $\mathfrak{g}$ and $\mathfrak{m}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}S_\alpha$, with the vectors A_α and S_α defined by the elements of a Weyl basis of $\mathfrak{g} = \mathrm{Lie}(G)$. This allows us rewrite the above splitting of $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s$, where $s = \left|R^+\right|$. Since $B(A_\alpha, A_\alpha) = B(S_\alpha, S_\alpha) = -2$ and $B(A_\alpha, S_\alpha) = 0$, the set

$$\left\{ X_\alpha = \frac{A_\alpha}{\sqrt{2}}, Y_\alpha = \frac{S_\alpha}{\sqrt{2}}; \alpha \in R^+ \right\}, \tag{3.4}$$

is a $(-B)$-orthonormal basis of $\mathfrak{m}$. If we denote for simplicity such a basis by $e_\alpha = \{A_\alpha, S_\alpha\}$, then the notation $\begin{bmatrix} k \\ ij \end{bmatrix}$ can be rewritten as $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ where e_α, e_β and e_γ are the $(-B)$-orthogonal bases of the modules $\mathfrak{m}_\alpha$, $\mathfrak{m}_\beta$ and $\mathfrak{m}\gamma$, respectively.

Remark 3.1.2. Recall that, if $\alpha, \beta \in R$ such that $\alpha + \beta \neq 0$, then root spaces satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ and $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$. Since $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \neq 0$ if and only if $B([\mathfrak{m}_\alpha, \mathfrak{m}_\beta], \mathfrak{m}\gamma) \neq 0$, we can conclude that $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \neq 0$ if and only if the positive roots α, β and γ are such that $\alpha + \beta - \gamma = 0$. Thus, we have $\begin{bmatrix} \alpha+\beta \\ \alpha\beta \end{bmatrix} \neq 0$, for any α, β such that $\alpha + \beta \in R$. Moreover, for $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we will have in the second summation of (3.1) the following:

(a) $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \times \frac{1}{t^2}$ when $t^2(-B)|_{\mathfrak{m}_\alpha}, t^2(-B)|_{\mathfrak{m}_\beta}, t^2(-B)|_{\mathfrak{m}_\gamma}$ are vertical components and $\alpha + \beta$ is a positive root,

(b) $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \times \dfrac{1}{t^2}$ when $t^2(-B)|_{\mathfrak{m}_\alpha}$ is vertical, $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components and $\alpha+\beta$ is a positive root,

(c) $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} \times t^2$ when $(-B)|_{\mathfrak{m}_\alpha}$ is vertical, $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components and $\alpha+\beta$ is a positive root

(d) $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = 0 \times t^2$ otherwise.

Proposition 3.1.3 ([5]). *For a maximal flag manifold G/K the triples $\begin{bmatrix} \alpha+\beta \\ \alpha\beta \end{bmatrix} \neq 0$ are given by*

$$\begin{bmatrix} \alpha+\beta \\ \alpha\beta \end{bmatrix} = 2m^2_{\alpha,\beta},$$

where $m^2_{\alpha,\beta}$ are the structure constants of the Weyl basis of $\mathfrak{g} = \mathrm{Lie}(G)$.

Remark 3.1.4 ([5]). If $\alpha, \beta \in R$ are such that $\alpha - \beta \in R$, similarly one obtains

$$\begin{bmatrix} \alpha-\beta \\ \alpha\beta \end{bmatrix} = 2m^2_{\alpha,-\beta}.$$

Lemma 3.1.5 ([25]). *For $SU(n+1)/T^n$, considering the decomposition (2.9) of the isotropy representation $\mathfrak{m}$ of T^n, one has*

$$\begin{bmatrix} \alpha+\beta \\ \alpha\beta \end{bmatrix} = \begin{cases} \dfrac{1}{n+1}, & if \quad k \neq i,j \\ 0, & otherwise \end{cases}, \tag{3.5}$$

where $\alpha = \lambda_i - \lambda_k$ and $\beta = \lambda_k - \lambda_j$ are positive roots of the root system

$$R = \{\alpha_{ij} = \pm(\lambda_i - \lambda_j); i \neq j\}$$

of the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$ relative to $\mathfrak{su}(n+1)^{\mathbb{C}}$, λ_i given by $\lambda_i(diag(a_1, \ldots, a_{n+1})) = a_i$, for each $1 \leqslant i \leqslant n+1$ and

$$\mathfrak{k} = \left\{ \sqrt{-1} \cdot diag(a_1, \ldots, a_{n+1}); \sum_{i=1}^{n} a_i = 0 \right\}.$$

Proposition 3.1.6. $[(\mathbf{SU(n+1)/T^n, g_t}), \mathbf{n} \geqslant \mathbf{2}]$ *Considering the hypothesis of Lemma 3.1.5, let $(SU(n+1)/T^n, \mathbf{g}_t)$ be the canonical variation of $(SU(n+1)/T^n, g)$, where g is the normal metric on $SU(n+1)/T^n$. Then, the function* $\mathrm{scal}(t)$, *that for each $t > 0$ gives the scalar curvature of $\mathbf{g}_t$, is given by*

$$\mathrm{scal}(t) = \frac{-2n + n^2(n+1) + 4n(n+1)t^2 + n(1-n)t^4}{4(n+1)t^2}. \tag{3.6}$$

Proof: The 1-parameter family $\mathbf{g}_t$ is obtained by multiplying the components

$$(-B)|_{\mathfrak{m}_{\alpha ij}}, 1 \leqslant i < j \leqslant n,$$

by t^2, that is, by scaling in the direction of the fibers by t^2 the normal metric g given by the inner product

$$g_{eT^n} = \sum_{i<j\leqslant n+1} (-B)|_{\mathfrak{m}_{\alpha_{ij}}}.$$

It is known that one has $\dfrac{n(n-1)}{2}$ vertical components, i.e,

$$\left|\{(-B)|_{\mathfrak{m}_{\alpha_{ij}}}, 1 \leqslant i < j \leqslant n\}\right| = \frac{n(n-1)}{2},$$

and $\left|\{(-B)|_{\mathfrak{m}_{\alpha ij}}, 1 \leqslant i < j = n+1\}\right| = n$ horizontal components of the normal metric g. In addition, the dimension of each submodule $\mathfrak{m}_\alpha$ is equal to 2. Therefore, the first summation in (3.1) is

$$\frac{1}{2}\sum_{\alpha\in R^+} \frac{d_\alpha}{t_\alpha} = \frac{(n-1)n}{2t^2} + n.$$

In order to obtain the second summation $\displaystyle\sum_{\alpha,\beta,\gamma\in R^+} \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \frac{t_\gamma}{t_\alpha t_\beta}$ in (3.1), with $t_\gamma, t_\alpha, t_\beta > 0$ coefficients of $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}$, respectively, it is sufficient to know the number of triples $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ multiplying t^2 and the number of these constants multiplying $\dfrac{1}{t^2}$. In fact, $\dfrac{t_\gamma}{t_\alpha t_\beta} = t^2$ or $\dfrac{1}{t^2}$, since the coefficients of the canonical variation g_t are $t_\alpha = t^2$ or $t_\alpha = 1$, depending on whether the component $(-B)|_{\mathfrak{m}_\alpha}$ is vertical or horizontal. Furthermore, the nonzero triples are equal to the same value, namely $0 \neq \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \dfrac{1}{n+1}$ according Lemma 3.1.5.

Let us now determine the number of triples for each of the cases (a), (b) and (c) in Remark 3.1.2 above.

CASE (a): Let N_1 the total number of triples in (a). Using the above notations, $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k - \lambda_j$ and $\gamma = \alpha + \beta = \lambda_i - \lambda_j$, $i < k < j \leqslant n$, since $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are vertical components. Fixed $i = 1$, we will have $\dfrac{(n-1)(n-2)}{2}$ symbols of the type $\begin{bmatrix} \lambda_1 - \lambda_j \\ \lambda_1 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $\dfrac{(n-2)(n-3)}{2}$ symbols of the type $\begin{bmatrix} \lambda_2 - \lambda_j \\ \lambda_2 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$, and so on, getting $\dfrac{(n-i)(n-i-1)}{2}$ symbols of the type $\begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-2$. It follows that the number of symbols in (a), not counting the permutations, is

$$\sum_{i=1}^{n-2}\frac{1}{2}(n-i)(n-i-1)=\frac{1}{6}(2n-3n^2+n^3).$$

Recall that in (a), $\begin{bmatrix} \lambda_i-\lambda_j \\ \lambda_i-\lambda_k\lambda_k-\lambda_j \end{bmatrix}=\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$,with $i<k<j\leqslant n$. Then, $t_\alpha, t_\beta, t_\gamma$ are equal to t^2, which implies that $\dfrac{t_\gamma}{t_\alpha t_\beta}=\dfrac{1}{t^2}$, and by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering six times the number of symbols above to obtain the total number of these symbols in (a), hence

$$N_1=6\cdot\frac{1}{6}(2n-3n^2+n^3)=(2n-3n^2+n^3).$$

CASE (b): Let N_2 the total number of triples in (b); $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}=\begin{bmatrix} \lambda_i-\lambda_j \\ \lambda_i-\lambda_k\lambda_k-\lambda_j \end{bmatrix}$, with $\alpha=\lambda_i-\lambda_k$, $\beta=\lambda_k-\lambda_j$ and $\gamma=\alpha+\beta=\lambda_i-\lambda_j$, $i<k<j=n+1$, since $(-B)|_{\mathfrak{m}_\alpha}$ is vertical and $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components. Fixed $i=1$, we will have $(n-1)$ triples of the type $\begin{bmatrix} \lambda_1-\lambda_j \\ \lambda_1-\lambda_k\lambda_k-\lambda_j \end{bmatrix}\neq 0$; if we fix $i=2$, one has $(n-2)$ triples of the type $\begin{bmatrix} \lambda_2-\lambda_j \\ \lambda_2-\lambda_k\lambda_k-\lambda_j \end{bmatrix}\neq 0$, and so on, getting $(n-i)$ triples of the type $\begin{bmatrix} \lambda_i-\lambda_j \\ \lambda_i-\lambda_k\lambda_k-\lambda_j \end{bmatrix}\neq 0$ for each $1\leqslant i\leqslant n-1$. It follows that the number of triples in (b), not counting the permutations, is

$$\sum_{i=1}^{n-1} i=\frac{n(n-1)}{2}.$$

In (b), $\begin{bmatrix} \lambda_i-\lambda_j \\ \lambda_i-\lambda_k\lambda_k-\lambda_j \end{bmatrix}=\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ then, $t_\alpha=t^2, t_\beta=t_\gamma=1$, which implies $\dfrac{t_\gamma}{t_\alpha t_\beta}=\dfrac{t_\gamma}{t_\beta t_\alpha}=\dfrac{t_\beta}{t_\gamma t_\alpha}=\dfrac{t_\beta}{t_\alpha t_\gamma}=\dfrac{1}{t^2}$. By symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering four times the number of triples above to obtain the total number N_2 of these triples in (b), hence

$$N_2=4\cdot\frac{n(n-1)}{2}=2n(n-1).$$

CASE (c): Similarly to the previous case, not counting the permutations, we have $\dfrac{n(n-1)}{2}$ symbols $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix}$ multiplying $\dfrac{t_\alpha}{t_\gamma t_\beta}=\dfrac{t_\alpha}{t_\beta t_\gamma}=t^2$. By symmetry, in this case, we must consider two times the number above in order to obtain the total number N_3 of triples $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix}$, which multiply in (c), i.e, $N_3=2\cdot\dfrac{n(n-1)}{2}=n(n-1)$.

Therefore, since $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} = \dfrac{1}{n+1}$ whenever $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} \neq 0$,

$$\begin{aligned} \mathrm{scal}(t) &= \frac{1}{2}\sum_{\alpha\in R^+}\frac{d_\alpha}{t_\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma\in R^+}\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}\frac{t_\gamma}{t_\alpha t_\beta} \\ &= \frac{(n-1)n}{2t^2} + n - \frac{1}{4}\left(\frac{N_1}{(n+1)t^2} + \frac{N_2}{t^2(n+1)} + \frac{N_3 t^2}{n+1}\right) \\ &= \frac{-2n + n^2(n+1) + 4n(n+1)t^2 + n(1-n)t^4}{4(n+1)t^2}. \end{aligned}$$

□

Let us now to consider the case of the canonical variation $(SO(2n+1)/T^n, \mathbf{h}_t)$ of the normal metric on $SO(2n+1)/T^n$.

Lemma 3.1.7 ([25])**.** *For $SO(2n+1)/T^n$, considering the decomposition (2.11) of the isotropy representation $\mathfrak{m}$ of T^n, one has*

$$\begin{bmatrix} \lambda_i - \lambda_k \\ \lambda_i - \lambda_j \lambda_j - \lambda_k \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i - \lambda_j \lambda_j \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i \lambda_j \end{bmatrix} = \frac{1}{2n-1}, \tag{3.7}$$

and $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = 0$ *otherwise, where α, β and γ are positive roots of the root system*

$$R = \{\pm\lambda_i \pm \lambda_j, \pm\lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\},$$

of the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$ relative to $\mathfrak{so}(2n+1)^{\mathbb{C}}$, λ_i given by $\lambda_i(diag(a_1, \ldots, a_{n+1})) = a_i$, for each $1 \leqslant i \leqslant n$ and

$$\mathfrak{k} = \left\{\sqrt{-1} \cdot diag(0, a_1, \ldots, a_n, -a_1, \ldots, -a_n); a_i \in \mathbb{R}\right\}.$$

Proposition 3.1.8. $[(\mathbf{SO(2n+1)/T^n, h_t}), \mathbf{n} \geqslant \mathbf{4}]$ *Considering the hypothesis of Lemma 3.1.7, let $SO(2n+1)/T^n, \mathbf{h}_t)$ be the canonical variation of $(SO(2n+1)/T^n, g)$, where g is the normal metric on $SO(2n+1)/T^n$. Then, the function* scal(t)*, that for each $t > 0$ gives the scalar curvature of $\mathbf{h}_t$, is given by*

$$\mathrm{scal}(t) = \frac{5n^3 - 2n^2t^4 + 8n^2t^2 - 7n^2 + 2nt^4 - 4nt^2 + 2n}{4(2n-1)t^2}. \tag{3.8}$$

Proof: The 1-parameter family $\mathbf{h}_t$ was obtained multiplying by $t^2, t > 0$, all the $(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+$ in the expression of the normal metric g, which on the isotropy representation $\mathfrak{m}$ of T^n, is represented by the inner product

$$g_{eT^n} = \sum_{\alpha\in R^+}(-B)|_{\mathfrak{m}_\alpha}.$$

It is known that there are $n(n-1)$ vertical components in the above expression of g, i.e, $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+\}\right| = n(n-1)$, and $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R^+ \backslash R'\}\right| = n$ horizontal components of the normal metric. Moreover, the dimension of each submodule $\mathfrak{m}_\alpha$ is equal to 2. Therefore, the first summation in (3.1) is

$$\frac{1}{2}\sum_{\alpha\in R^+}\frac{d_\alpha}{t_\alpha} = \frac{(n-1)n}{t^2} + n.$$

To obtain the second summation $\displaystyle\sum_{\alpha,\beta,\gamma\in R^+}\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix}\frac{t_\gamma}{t_\alpha t_\beta}$ in (3.1), with $t_\gamma, t_\alpha, t_\beta > 0$ coefficients of $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}$, respectively, we will determine, as well as in the previous example, the number of factors $\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix}$ multiplying t^2 and the number of these constants multiplying $\dfrac{1}{t^2}$.

Let us now determine the number of triples for each of the cases (a), (b) and (c) in Remark 3.1.2 above.

CASE (a): Let N_1 the total number of triples in (a). For $\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix} = \begin{bmatrix}\lambda_i-\lambda_j\\ \lambda_i-\lambda_k\lambda_k-\lambda_j\end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k - \lambda_j$ and $\gamma = \alpha+\beta = \lambda_i - \lambda_j$, $i < k < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are vertical components. Fixed $i = 1$, we will have $\dfrac{(n-1)(n-2)}{2}$ symbols of the type $\begin{bmatrix}\lambda_1-\lambda_j\\ \lambda_1-\lambda_k\lambda_k-\lambda_j\end{bmatrix} \neq 0$; if we fix $i = 2$, one has $\dfrac{(n-2)(n-3)}{2}$ symbols of the type $\begin{bmatrix}\lambda_2-\lambda_j\\ \lambda_2-\lambda_k\lambda_k-\lambda_j\end{bmatrix} \neq 0$, and so on, getting $\dfrac{(n-i)(n-i-1)}{2}$ symbols of the type $\begin{bmatrix}\lambda_i-\lambda_j\\ \lambda_i-\lambda_k\lambda_k-\lambda_j\end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-2$. It follows that the number such triples, not counting the permutations, is

$$\sum_{i=1}^{n-2}\frac{1}{2}(n-i)(n-i-1) = \frac{1}{6}(2n-3n^2+n^3).$$

Since $\begin{bmatrix}\lambda_i-\lambda_j\\ \lambda_i-\lambda_k\lambda_k-\lambda_j\end{bmatrix} = \begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix}$ the coefficients $t_\alpha, t_\beta, t_\gamma$ are equal to t^2 and, by symmetry of $\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix}$, we must considering six times the number of triples to obtain the total number N_1 multiplying $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{1}{t^2}$ in this case, hence

$$N_1 = 6\cdot\frac{1}{6}(2n-3n^2+n^3) = (2n-3n^2+n^3)$$

.

CASE (b): For $\begin{bmatrix}\gamma\\ \alpha\beta\end{bmatrix} = \begin{bmatrix}\lambda_i+\lambda_j\\ \lambda_i-\lambda_k\lambda_k+\lambda_j\end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k + \lambda_j$ and

$\gamma = \alpha - \beta = \lambda_i + \lambda_j$, $i < k \leqslant n, k \neq j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are vertical components of the normal metric g. Fixed $i = 1$, one has $(n-2)(n-1)$ triples of the type $\begin{bmatrix} \lambda_1 + \lambda_j \\ \lambda_1 - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $(n-2)(n-3)$ symbols of the type $\begin{bmatrix} \lambda_2 + \lambda_j \\ \lambda_2 - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)(n-i-1)$ triples of the type $\begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-1$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-2} (n-i)(n-i-1) = \frac{1}{3}(2n - 3n^2 + n^3).$$

When $\begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ the coefficients $t_\alpha, t_\beta, t_\gamma$ in the expression of the canonical fibration are equal to t^2. It follows that $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{1}{t^2}$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering six times the number of triples above to obtain the total number N_2 of triples multiplying $\dfrac{1}{t^2}$ in this case, hence

$$N_2 = 6\frac{1}{3}(2n - 3n^2 + n^3) = 2N_1.$$

CASE (c): For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i - \lambda_j \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_j$, $\beta = \lambda_j$ and $\gamma = \alpha + \beta = \lambda_i$, $i < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}$ corresponds to a vertical component and $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components of the normal metric g. Fixed $i = 1$, one has $(n-1)$ triples of the type $\begin{bmatrix} \lambda_1 \\ \lambda_1 - \lambda_j \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $(n-2)$ triples of the type $\begin{bmatrix} \lambda_2 \\ \lambda_2 - \lambda_j \lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)$ triples of the type $\begin{bmatrix} \lambda_i \\ \lambda_i - \lambda_j \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-1$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-1} (n-i) = \frac{1}{2} n(n-1).$$

When $\begin{bmatrix} \lambda_i \\ \lambda_i - \lambda_j \lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ one has $t_\alpha = t^2, t_\beta = t_\gamma = 1$ in the expression of the canonical fibration. It follows that $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{t_\gamma}{t_\beta t_\alpha} = \dfrac{t_\beta}{t_\alpha t_\gamma} = \dfrac{t_\beta}{t_\gamma t_\alpha} = \dfrac{1}{t^2}$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering four times the number of triples above to obtain the total number N_3 of triples multiplying $\dfrac{1}{t^2}$ in this case, hence

$$N_3 = 4\frac{1}{2} n(n-1) = 2n(n-1),$$

and considering two times this same number in order to obtain the total number of triples multiplying $\dfrac{t_\alpha}{t_\gamma t_\beta} = \dfrac{t_\alpha}{t_\beta t_\gamma} = t^2$, hence

$$N_4 = 2\frac{1}{2}n(n-1) = n(n-1).$$

If $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i\lambda_j \end{bmatrix}$, with $\alpha = \lambda_i$, $\beta = \lambda_j$ and $\gamma = \alpha + \beta = \lambda_i + \lambda_j$, $1 \leqslant i < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}$ and $(-B)|_{\mathfrak{m}_\beta}$ are horizontal components and $(-B)|_{\mathfrak{m}_\gamma}$ is a vertical component of the normal metric g. Fixed $i = 1$, one has $(n-1)$ triples of the type $\begin{bmatrix} \lambda_1 + \lambda_j \\ \lambda_1\lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $(n-2)$ triples of the type $\begin{bmatrix} \lambda_2 + \lambda_j \\ \lambda_2\lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)$ triples of this type, for each $1 \leqslant i < j \leqslant n$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-1}(n-i) = \frac{1}{2}n(n-1).$$

When $\begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i\lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ one has $t_\alpha = t_\beta = 1, t_\gamma = t^2$ in the expression of the canonical fibration. It follows that $\dfrac{t_\gamma}{t_\alpha t_\beta} = t^2$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering two times the number of triples that was been found above to obtain the total number N_4 of triples multiplying $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{t_\gamma}{t_\beta t_\alpha} = t^2$ in this case, hence

$$N_5 = 2\frac{1}{2}n(n-1) = n(n-1),$$

and considering four times this same number in order to obtain the total number of triples multiplying $\dfrac{t_\alpha}{t_\gamma t_\beta} = \dfrac{t_\alpha}{t_\beta t_\gamma} = \dfrac{t_\beta}{t_\gamma t_\alpha} = \dfrac{t_\beta}{t_\alpha t_\gamma} = \dfrac{1}{t^2}$ in this case, hence $N_6 = 4\dfrac{1}{2}n(n-1) = 2n(n-1)$.

Therefore, since $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} = \dfrac{1}{2n-1}$ whenever $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} \neq 0$, according Lemma 3.1.7, follows that

$$\begin{aligned} \mathrm{scal}(t) &= \frac{1}{2}\sum_{\alpha\in R^+}\frac{d_\alpha}{t_\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma\in R^+}\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}\frac{t_\gamma}{t_\alpha t_\beta} \\ &= \frac{(n-1)n}{t^2} + n - \frac{1}{4}\left(\frac{N_1}{(2n-1)t^2} + \frac{N_2}{t^2(2n-1)} + \frac{N_3}{t^2(2n-1)}\right) - \\ &\quad \frac{1}{4}\left(\frac{N_4 t^2}{(2n-1)} + \frac{N_5 t^2}{(2n-1)} + \frac{N_6}{t^2(2n-1)}\right) \\ &= \frac{5n^3 - 2n^2t^4 + 8n^2t^2 - 7n^2 + 2nt^4 - 4nt^2 + 2n}{4(2n-1)t^2}. \end{aligned}$$

□

The next case is the scalar curvature $\mathrm{scal}(t)$ of the canonical variation of the maximal flag manifold associated with the complex simple Lie algebra $\mathfrak{sp}(n)^{\mathbb{C}}$, provided with the normal metric.

Lemma 3.1.9 ([25]). *For $Sp(n)/T^n$, considering the decomposition (2.13) of the isotropy representation $\mathfrak{m}$ of T^n, one has*

$$\begin{bmatrix} \lambda_i - \lambda_k \\ \lambda_i - \lambda_j \lambda_j - \lambda_k \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix} = \frac{1}{2(n+1)}, \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_j 2\lambda_j \end{bmatrix} = \frac{1}{n+1}, \tag{3.9}$$

and $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = 0$ *otherwise, where α, β and γ are positive roots of the root system*

$$R = \{\pm\lambda_i \pm \lambda_j, \pm 2\lambda_k; 1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n\},$$

of the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$ relative to $\mathfrak{sp}(n)^{\mathbb{C}}$, λ_i given by $\lambda_i(diag(a_1, \ldots, a_{n+1})) = a_i$, for each $1 \leqslant i \leqslant n$ and

$$\mathfrak{k} = \left\{\sqrt{-1} \cdot diag(a_1, \ldots, a_n, -a_1, \ldots, -a_n); a_i \in \mathbb{R}\right\}.$$

Proposition 3.1.10. $[(\mathbf{Sp(n)/T^n, k_t}), \mathbf{n \geqslant 3}]$ *Considering the hypothesis of the previous Lemma, let $(Sp(n)/T^n, \mathbf{k}_t)$ be the canonical variation of $(Sp(n)/T^n, g)$, where g is the normal metric on $Sp(n)/T^n$. Then, the function* $\mathrm{scal}(t)$, *that for each $t > 0$ gives the scalar curvature of $\mathbf{k}_t$, is given by*

$$\mathrm{scal}(t) = \frac{-2n^3t^4 + 24n^3t^2 + 5n^3 + 48n^2t^2 + 9n^2 + 2nt^4 + 24nt^2 - 14n}{24(n+1)t^2}. \tag{3.10}$$

Proof: The 1-parameter family $\mathbf{k}_t$ was obtained multiplying by $t^2, t > 0$, all the $(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+$ in the expression of the normal metric g, which on the isotropy representation $\mathfrak{m}$ of T^n, is represented by the inner product

$$g_{eT^n} = \sum_{\alpha \in R^+} (-B)|_{\mathfrak{m}_\alpha}.$$

It is known that there are $\dfrac{n(n-1)}{2}$ vertical components in the above expression of g, i.e, $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+\}\right| = \dfrac{n(n-1)}{2}$, and $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R^+ \backslash R'\}\right| = \dfrac{n(n+1)}{2}$ horizontal components of the normal metric. Moreover, the dimension of each submodule $\mathfrak{m}_\alpha$ is equal to 2. It follows that the first summation in (3.1) is

$$\frac{1}{2} \sum_{\alpha \in R^+} \frac{d_\alpha}{t_\alpha} = \frac{(n-1)n}{2t^2} + (n+1)n.$$

The second summation $\displaystyle\sum_{\alpha,\beta,\gamma \in R^+} \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \frac{t_\gamma}{t_\alpha t_\beta}$ in (3.1), with $t_\gamma, t_\alpha, t_\beta > 0$ coefficients of $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}$, respectively, we will determine, as well as in the previous examples, the number of factors $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ multiplying t^2 and the number of these constants multiplying $\dfrac{1}{t^2}$.

Let us now determine the number of triples for each of the cases (a), (b) and (c) in Remark 3.1.2 above.

CASE (a): Let N_1 the total number of triples in (a). For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k - \lambda_j$ and $\gamma = \alpha + \beta = \lambda_i - \lambda_j$, $i < k < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are vertical components. Fixed $i = 1$, we will have $\dfrac{(n-1)(n-2)}{2}$ triples $\begin{bmatrix} \lambda_1 - \lambda_j \\ \lambda_1 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $\dfrac{(n-2)(n-3)}{2}$ triples $\begin{bmatrix} \lambda_2 - \lambda_j \\ \lambda_2 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$, and so on, getting $\dfrac{(n-i)(n-i-1)}{2}$ triples $\begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-2$. It follows that the number of triples that multiply $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{1}{t^2}$ in this case, not counting permutations, is

$$\sum_{i=1}^{n-2} \frac{1}{2}(n-i)(n-i-1) = \frac{1}{6}(2n - 3n^2 + n^3).$$

Since $\begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ the coefficients $t_\alpha, t_\beta, t_\gamma$ are equal to t^2 and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering six times the number of such triples to obtain the total number N_1 of triples that multiply $\dfrac{1}{t^2}$ in this case, hence

$$N_1 = 6 \cdot \frac{1}{6}(2n - 3n^2 + n^3) = (2n - 3n^2 + n^3).$$

CASE (b): For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k + \lambda_j$ and $\gamma = \alpha - \beta = \lambda_i + \lambda_j$, $i < k \leqslant n, k \neq j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}$ is vertical component and $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components of the normal metric g. Fixed $i = 1$, one has $(n-2)(n-1)$ triples of the type $\begin{bmatrix} \lambda_1 + \lambda_j \\ \lambda_1 - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $(n-2)(n-3)$ symbols of the type $\begin{bmatrix} \lambda_2 + \lambda_j \\ \lambda_2 - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)(n-i-1)$ triples of the type $\begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-1$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-2}(n-i)(n-i-1) = \frac{1}{3}(2n-3n^2+n^3).$$

When $\begin{bmatrix} \lambda_i+\lambda_j \\ \lambda_i-\lambda_k\lambda_k+\lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ one has $t_\alpha = t^2, t_\beta = t_\gamma = 1$ in the expression of the canonical fibration. It follows that $\dfrac{t_\alpha}{t_\gamma t_\beta} = t^2$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering two times the number of triples above to obtain the total number N_2 of triples multiplying $\dfrac{t_\alpha}{t_\gamma t_\beta} = \dfrac{t_\alpha}{t_\beta t_\gamma} = t^2$ in this case, hence

$$N_2 = 2\frac{1}{3}(2n-3n^2+n^3) = \frac{2}{3}N_1,$$

and in addition, four times the value above to obtain the total number N_3 of triples multiplying $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{t_\gamma}{t_\beta t_\alpha} = \dfrac{t_\beta}{t_\gamma t_\alpha} = \dfrac{t_\beta}{t_\alpha t_\gamma} = \dfrac{1}{t^2}$ in this case, hence

$$N_3 = 4\frac{1}{3}(2n-3n^2+n^3) = \frac{4}{3}N_1.$$

CASE (c) For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i-\lambda_j 2\lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_j$, $\beta = \lambda_j$ and $\gamma = \alpha+\beta = 2\lambda_i$, $i < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}$ corresponds to a vertical component and $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components of the normal metric g. Fixed $i = 1$, one has $(n-1)$ triples of the type $\begin{bmatrix} \lambda_1 \\ \lambda_1-\lambda_j 2\lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $(n-2)$ triples of the type $\begin{bmatrix} \lambda_2 \\ \lambda_2-\lambda_j 2\lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)$ triples of the type $\begin{bmatrix} \lambda_i \\ \lambda_i-\lambda_j 2\lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-1}(n-i) = \frac{1}{2}n(n-1).$$

When $\begin{bmatrix} \lambda_i \\ \lambda_i-\lambda_j 2\lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ one has $t_\alpha = t^2, t_\beta = t_\gamma = 1$ in the expression of the canonical fibration. It follows that $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{1}{t^2}$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering two times the number of triples above to obtain the total number N_4 of triples multiplying $\dfrac{t_\alpha}{t_\gamma t_\beta} = \dfrac{t_\alpha}{t_\beta t_\gamma} = t^2$ in this case, hence

$$N_4 = 2\frac{1}{2}n(n-1) = n(n-1),$$

and considering four times the same number in order to obtain the total number N_5 of triples multiplying $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{t_\gamma}{t_\beta t_\alpha} = \dfrac{t_\beta}{t_\gamma t_\alpha} = \dfrac{t_\beta}{t_\alpha t_\gamma} = \dfrac{1}{t^2}$, hence

$$N_5 = 4\frac{1}{2}n(n-1) = 2n(n-1).$$

Thus, once $\left[\begin{array}{c}\alpha\\ \gamma\beta\end{array}\right]=\dfrac{1}{2(n+1)}$ or $\dfrac{1}{n+1}$ whenever $\left[\begin{array}{c}\alpha\\ \gamma\beta\end{array}\right]\neq 0$, according Lemma 3.1.9, follows that

$$\begin{aligned}
\operatorname{scal}(t) &= \frac{1}{2}\sum_{\alpha\in R^+}\frac{d_\alpha}{t_\alpha}-\frac{1}{4}\sum_{\alpha,\beta,\gamma\in R^+}\left[\begin{array}{c}\gamma\\ \alpha\beta\end{array}\right]\frac{t_\gamma}{t_\alpha t_\beta}\\
&= \frac{(n-1)n}{2t^2}+(n+1)n-\frac{1}{4}\left(\frac{N_1}{2(n+1)t^2}+\frac{N_2t^2}{2(n+1)}+\frac{N_3}{2(n+1)t^2}\right)-\\
&\quad \frac{1}{4}\left(\frac{N_4t^2}{(n+1)}+\frac{N_5}{(n+1)t^2}\right)\\
&= \frac{-2n^3t^4+24n^3t^2+5n^3+48n^2t^2+9n^2+2nt^4+24nt^2-14n}{24(n+1)t^2}.
\end{aligned}$$

□

The last expression for $\operatorname{scal}(t)$ of the canonical variations for the maximal flag manifolds associated with a complex classical simple Lie algebra is the one associated with $\mathfrak{so}(2n)^{\mathbb{C}}$ provided with its normal metric.

Lemma 3.1.11 ([25]). *For $SO(2n)/T^n$, considering the decomposition (2.15) of the isotropy representation $\mathfrak{m}$ of T^n, one has*

$$\left[\begin{array}{c}\lambda_i-\lambda_k\\ \lambda_i-\lambda_j\lambda_j-\lambda_k\end{array}\right]=\left[\begin{array}{c}\lambda_i+\lambda_j\\ \lambda_i-\lambda_k\lambda_k+\lambda_j\end{array}\right]=\frac{1}{2(n-1)},\ (k\neq i,j) \tag{3.11}$$

and $\left[\begin{array}{c}\gamma\\ \alpha\beta\end{array}\right]=0$ *otherwise, where α, β and γ are positive roots of the root system*

$$R=\{\pm\lambda_i\pm\lambda_j;1\leqslant i<j\leqslant n,\},$$

of the Cartan Lie subalgebra $\mathfrak{k}^{\mathbb{C}}$ relative to $\mathfrak{so}(2n)^{\mathbb{C}}$, λ_i given by $\lambda_i(diag(a_1,\ldots,a_{n+1}))=a_i$, for each $1\leqslant i\leqslant n$ and

$$\mathfrak{k}=\left\{\sqrt{-1}\cdot diag(a_1,\ldots,a_n,-a_1,\ldots,-a_n);a_i\in\mathbb{R}\right\}.$$

Proposition 3.1.12. $[(\mathbf{SO(2n)/T^n},\mathbf{m_t}),\mathbf{n}\geqslant\mathbf{4}]$ *Considering the hypothesis of the previous Lemma, let $(SO(2n)/T^n,\mathbf{m}_t)$ be the canonical variation of $(SO(2n)/T^n,g)$, where g is the normal metric on $SO(2n)/T^n$. Then, the function* $\operatorname{scal}(t)$, *that for each $t>0$ gives the scalar curvature of $\mathbf{m}_t$, is given by*

$$\operatorname{scal}(t)=\frac{-2n^2t^4+24n^2t^2+5n^2+4nt^4-24nt^2+2n}{24t^2}. \tag{3.12}$$

Proof: The 1-parameter family $\mathbf{m}_t$ was obtained multiplying by $t^2, t>0$, all the $(-B)|_{\mathfrak{m}_\alpha},\alpha\in R'\cap R^+$ in the expression of the normal metric g, which on the isotropy representation $\mathfrak{m}$ of T^n, is represented by the inner product

$$g_{eT^n}=\sum_{\alpha\in R^+}(-B)|_{\mathfrak{m}_\alpha}.$$

It is known that there are $\dfrac{n(n-1)}{2}$ vertical components in the above expression of g, i.e, $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R' \cap R^+\}\right| = \dfrac{n(n-1)}{2}$, and $\left|\{(-B)|_{\mathfrak{m}_\alpha}, \alpha \in R^+\backslash R'\}\right| = \dfrac{n(n-1)}{2}$ horizontal components of the normal metric g. Moreover, the dimension of each submodule $\mathfrak{m}_\alpha$ is equal to 2. It follows that the first summation in (3.1) is

$$\frac{1}{2}\sum_{\alpha \in R^+} \frac{d_\alpha}{t_\alpha} = \frac{(n-1)n}{2t^2} + (n-1)n.$$

The second summation $\displaystyle\sum_{\alpha,\beta,\gamma \in R^+} \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} \frac{t_\gamma}{t_\alpha t_\beta}$ in (3.1), with $t_\gamma, t_\alpha, t_\beta > 0$ coefficients of $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\alpha}$, respectively, we will determine, as well as in the previous examples, the number of factors $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ multiplying t^2 and the number of these constants multiplying $\dfrac{1}{t^2}$.

CASE (a): For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k - \lambda_j$ and $\gamma = \alpha + \beta = \lambda_i - \lambda_j$, $i < k < j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}, (-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are vertical components. Fixed $i = 1$, we will have $\dfrac{(n-1)(n-2)}{2}$ triples $\begin{bmatrix} \lambda_1 - \lambda_j \\ \lambda_1 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$; if we fix $i = 2$, one has $\dfrac{(n-2)(n-3)}{2}$ triples $\begin{bmatrix} \lambda_2 - \lambda_j \\ \lambda_2 - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$, and so on, getting $\dfrac{(n-i)(n-i-1)}{2}$ triples $\begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-2$. It follows that the number of that multiply $\dfrac{1}{t^2}$ in this case, not counting the permutations, is

$$\sum_{i=1}^{n-2} \frac{1}{2}(n-i)(n-i-1) = \frac{1}{6}(2n - 3n^2 + n^3).$$

Since $\begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i - \lambda_k \lambda_k - \lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ the coefficients $t_\alpha, t_\beta, t_\gamma$ are equal to t^2 and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering six times the number of such triples to obtain the total number N_1 of triples that multiply $\dfrac{1}{t^2}$ in this case, hence

$$N_1 = 6 \cdot \frac{1}{6}(2n - 3n^2 + n^3) = (2n - 3n^2 + n^3).$$

CASE (b): For $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix} = \begin{bmatrix} \lambda_i + \lambda_j \\ \lambda_i - \lambda_k \lambda_k + \lambda_j \end{bmatrix}$, with $\alpha = \lambda_i - \lambda_k$, $\beta = \lambda_k + \lambda_j$ and $\gamma = \alpha - \beta = \lambda_i + \lambda_j$, $i < k \leqslant n, k \neq j \leqslant n$, we have that $(-B)|_{\mathfrak{m}_\alpha}$ is vertical component and $(-B)|_{\mathfrak{m}_\beta}, (-B)|_{\mathfrak{m}_\gamma}$ are horizontal components of the normal metric g. Fixed $i = 1$, one has $(n-2)(n-1)$ triples of the type $\begin{bmatrix} \lambda_1+\lambda_j \\ \lambda_1-\lambda_k\lambda_k+\lambda_j \end{bmatrix} \neq 0$; if we fix $i=2$, one has $(n-2)(n-3)$ symbols of the type $\begin{bmatrix} \lambda_2+\lambda_j \\ \lambda_2-\lambda_k\lambda_k+\lambda_j \end{bmatrix} \neq 0$, and so on, getting $(n-i)(n-i-1)$ triples of the type $\begin{bmatrix} \lambda_i+\lambda_j \\ \lambda_i-\lambda_k\lambda_k+\lambda_j \end{bmatrix} \neq 0$ for each $1 \leqslant i \leqslant n-1$. It follows that the number of such triples, not counting the permutations, is

$$\sum_{i=1}^{n-2}(n-i)(n-i-1) = \frac{1}{3}(2n-3n^2+n^3).$$

When $\begin{bmatrix} \lambda_i+\lambda_j \\ \lambda_i-\lambda_k\lambda_k+\lambda_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$ one has $t_\alpha = t^2, t_\beta = t_\gamma = 1$ in the expression of the canonical fibration. It follows that $\dfrac{t_\alpha}{t_\gamma t_\beta} = \dfrac{t_\alpha}{t_\beta t_\gamma} = t^2$ and, by symmetry of $\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}$, we must considering two times the number of triples above to obtain the total number N_2 of triples multiplying t^2 in this case, hence

$$N_2 = 2\frac{1}{3}(2n-3n^2+n^3) = \frac{2}{3}N_1,$$

in addition considering four times the value above to obtain the total number N_3 of triples multiplying $\dfrac{t_\gamma}{t_\alpha t_\beta} = \dfrac{t_\gamma}{t_\beta t_\alpha} = \dfrac{t_\beta}{t_\alpha t_\gamma} = \dfrac{t_\beta}{t_\gamma t_\alpha} = \dfrac{1}{t^2}$ in this case, hence

$$N_3 = 4\frac{1}{3}(2n-3n^2+n^3) = \frac{4}{3}N_1.$$

From (3.11), $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} = \dfrac{1}{2(n-1)}$ whenever $\begin{bmatrix} \alpha \\ \gamma\beta \end{bmatrix} \neq 0$, and it follows that

$$\begin{aligned} \mathrm{scal}(t) &= \frac{1}{2}\sum_{\alpha\in R^+}\frac{d_\alpha}{t_\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma\in R^+}\begin{bmatrix} \gamma \\ \alpha\beta \end{bmatrix}\frac{t_\gamma}{t_\alpha t_\beta} \\ &= \frac{(n-1)n}{2t^2} + (n+1)n - \frac{1}{4}\left(\frac{N_1}{2(n-1)t^2} + \frac{N_2 t^2}{2(n-1)} + \frac{N_3}{2t^2(n-1)}\right) \\ &= \frac{-2n^2t^4+24n^2t^2+5n^2+4nt^4-24nt^2+2n}{24t^2}. \end{aligned}$$

□

Consider now the formula for $\mathrm{scal}(t)$ of the canonical variation described in Example 2.2.14, section 2.2.1, of the normal metric on the maximal flag manifold associated with the exceptional simple Lie algebra of the type G_2. Before, we will enunciate the following useful lemma.

Lemma 3.1.13 ([5]). *According Example 2.2.14, Section 2.2, for the isotropy representation* $\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_1+\alpha_2} \oplus \mathfrak{m}_{\alpha_1+2\alpha_2} \oplus \mathfrak{m}_{\alpha_1+3\alpha_2} \oplus \mathfrak{m}_{2\alpha_1+3\alpha_2}$ *of* T^2*, put* $\mathfrak{m}_1 = \mathfrak{m}_{\alpha_1}$, $\mathfrak{m}_2 = \mathfrak{m}_{\alpha_2}$, $\mathfrak{m}_3 = \mathfrak{m}_{\alpha_1+\alpha_2}$, $\mathfrak{m}_4 = \mathfrak{m}_{\alpha_1+2\alpha_2}$, $\mathfrak{m}_5 = \mathfrak{m}_{\alpha_1+3\alpha_2}$ *and* $\mathfrak{m}_6 = \mathfrak{m}_{2\alpha_1+3\alpha_2}$*. Using the current notation, the non zero triples* $\begin{bmatrix} k \\ ij \end{bmatrix}$ *of the maximal flag* G_2/T^2 *are*

$$\begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix} = \begin{bmatrix} 6 \\ 34 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix} = \frac{1}{4}, \quad \text{and} \quad \begin{bmatrix} k \\ ij \end{bmatrix} = \frac{1}{3}.$$

Proposition 3.1.14. $[(\mathbf{G_2/T}, \mathbf{n_t})]$ *Let* $(G_2/T^2, \mathbf{n}_t)$ *be the canonical variation of* $(G_2/T^2, g)$*, where* g *is the normal metric on* G_2/T*. Then, the function* $\mathrm{scal}(t)$*, that for each* $t > 0$ *gives the scalar curvature of* $\mathbf{n}_t$*, is given by*

$$\mathrm{scal}(t) = \frac{2 + 12t^2 - 2t^4}{3t^2}. \tag{3.13}$$

Proof: We had see that the inner product

$$(\mathbf{n}_t)_{eT^2} = (-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_1+2\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+3\alpha_2}} + (-B)|_{\mathfrak{m}_{2\alpha_1+3\alpha_2}},$$

on the isotropy representation

$$\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_1+\alpha_2} \oplus \mathfrak{m}_{\alpha_1+2\alpha_2} \oplus \mathfrak{m}_{\alpha_1+3\alpha_2} \oplus \mathfrak{m}_{2\alpha_1+3\alpha_2}$$

of T, is the canonical variation g_t of the normal metric g on G_2/T^2, where we took, with respect $\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{m}$, the root system of the Cartan Lie subalgebra $\mathfrak{k}$ as being

$$\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2), \pm(\alpha_1+3\alpha_2), \pm(2\alpha_1+3\alpha_2)\},$$

and we fix a system of simple roots to be $\Sigma = \{\alpha_1, \alpha_2\}$. With respect to Σ the positive roots are given by

$$\{\alpha_1, \alpha_2, (\alpha_1+\alpha_2), (\alpha_1+2\alpha_2), (\alpha_1+3\alpha_2), (2\alpha_1+3\alpha_2)\}.$$

By Lemma 3.1.13 and applying the formula (3.1) in this case, since $d_i = \dim \mathfrak{m}_i = 2$, follows that

$$\begin{aligned} \mathrm{scal}(t) &= \sum_{l=1}^{6} \frac{d_l}{t_l} - \frac{1}{4} \sum_{i,j,k} \begin{bmatrix} k \\ ij \end{bmatrix} \frac{t_k}{t_i t_j} \\ &= \frac{1}{2}\left(8 + \frac{4}{t^2}\right) - \frac{1}{4}\left(2t^2 + \frac{4}{t^2}\right) - \frac{1}{4} \cdot \frac{1}{3}\left(2t^2 + \frac{4}{t^2}\right) \\ &= \frac{2 + 12t^2 - 2t^4}{3t^2}. \end{aligned}$$

$\square$

3.2 Bifurcation and Local Rigidity Instants

It is known from Proposition 1.1.9, Section 1.1, that the Morse index changes when passing a degeneracy instant and thus this is a bifurcation instant. We will determine these changes in the Morse index in order to find bifurcation instants for the canonical variations $g_t, 0 < t < 1$.

It was defined previously that a *degeneracy instant* $t_* > 0$ for g_t in $\mathcal{R}^k(M)$, with $g_1 = g$, is an instant such that $\dfrac{\operatorname{scal}(g_{t_*})}{m-1} \in \sigma(\Delta_{t_*})$.

Denoting by

$$\{0 < \lambda_1^g \leqslant \lambda_2^g \leqslant \ldots \leqslant \lambda_j^g \leqslant \ldots\}$$

the sequence of positive eigenvalues of Δ_g, the Morse index of a Riemannian metric g is

$$N(g) = \max\left\{ j \in \mathbb{N}; \lambda_j^g < \frac{\operatorname{scal}(g)}{m-1} \right\},$$

where is the Laplacian Δ_g acting on $C^\infty(M)$, M provide with the Riemannian metric g and $m = \dim M$.

The following result is a sufficient condition for a degeneracy instant $0 < t_* < 1$ to be a bifurcation instant when $\dfrac{\operatorname{scal}(t_*)}{m-1}$ is a constant eigenvalue of the Laplacian Δ_{t_*}. We will apply this in order to obtain the main theorems of this current work, namely the determination of all bifurcation and local rigidity instants in the interval $(0,1)$, for each canonical variation constructed on the maximal flag manifolds in the previous Chapter.

Proposition 3.2.1 ([10])**.** *Let (M, g) be a closed Riemannian manifold with $\dim M \geqslant 3$ and $\pi : (M, g) \longrightarrow (B, h)$ a Riemannian submersion with totally geodesic fibers isometric to (F, κ), where $\dim F \geqslant 2$ and scal$(F) > 0$. Denote by $\lambda \in \sigma(\Delta_h) \subset \sigma(\Delta_g)$ a constant eigenvalue of Δ_{t_*} such that $\dfrac{scal(g_{t_*})}{m-1} = \lambda$, g_{t_*} canonical variation of g at $0 < t_* < 1$ and Δ_{t_*} the Laplacian on (M, g_{t_*}). If*

$$\frac{\operatorname{scal}(g_{t_*})}{m-1} < \lambda^{1,1}(t_*) = \mu_1 + (\frac{1}{t_*^2} - 1)\phi_1,$$

$\mu_1 \in \sigma(\Delta_g)$ the first positive eigenvalue of Δ_g and $\phi_1 \in \sigma(\Delta_v)$ the first positive eigenvalue of the vertical Laplacian Δ_v, then t_* is a bifurcation instant for g_t.

Proof: It is sufficient to show that the Morse index changes when passing the degeneracy value $0 < t_* < 1$, i.e, for $\epsilon > 0$ sufficiently small, $N(g_{t_*-\epsilon}) \neq N(g_{t_*+\epsilon})$ and, by Proposition 1.1.9, Section 1.1, t_* is a bifurcation instant.

Observe that, if the Morse index does not change, there must be a compensation of eigenvalues. Namely, there must exist nonconstant eigenvalues $\lambda^{k_1,j_1}(t), \ldots, \lambda^{k_n,j_n}(t)$ of Δ_t, whose combined multiplicity equals the multiplicity of λ, such that,

$$\lambda < \frac{\mathrm{scal}(g_t)}{m-1} < \lambda^{k_i, j_i}(t), \quad \forall t < t_* \text{ (close to } t_*\text{) and } 1 \leqslant i \leqslant n,$$

$$\lambda > \frac{\mathrm{scal}(g_t)}{m-1} > \lambda^{k_i, j_i}(t), \quad \forall t > t_* \text{ (close to } t_*\text{) and } 1 \leqslant i \leqslant n.$$

If $\dfrac{\mathrm{scal}(g_{t*})}{m-1} < \mu_1 + (\dfrac{1}{t_*^2} - 1)\phi_1$, $\mu_1 \in \sigma(\Delta_g)$ first positive eigenvalue of Δ_g and $\phi_1 \in \sigma(\Delta_v)$ first positive eigenvalue of the vertical Laplacian Δ_v, then

$$\lambda = \frac{\mathrm{scal}(g_{t*})}{m-1} < \mu_1 + (\frac{1}{t_*^2} - 1)\phi_1 < \lambda^{k,j}(t_*), \quad \forall \mathbb{Z} \ni k, j > 0,$$

since $\lambda^{k,j}(t_*) = \mu_k + (\dfrac{1}{t_*^2} - 1)\phi_j$, $\mu_1 < \mu_k \in \sigma(\Delta_g)$, $\phi_1 < \phi_k \in \sigma(\Delta_v)$ and $(\dfrac{1}{t_*^2} - 1) > 0$ when $0 < t_* < 1$. It follows that every nonconstant eigenvalue $\lambda^{k,j}(t)$ is strictly greater than $\lambda = \dfrac{\mathrm{scal}(g_{t*})}{m-1}$ for t sufficiently close to t_*, so that there is no compensation of eigenvalue and the Morse index must change when passing t_*. Since $\dfrac{\mathrm{scal}(g_{t*})}{m-1} = \lambda \in \sigma(\Delta_{t*})$, t_* is a degeneracy instant for g_t, and by Proposition 1.1.9, Section 1.1, t_* is bifurcation instant for g_t.

□

Later on, g is the normal homogeneous metric on the maximal flag manifold G/T, represented by $(-B)|_{\mathfrak{m}}$, B Cartan Killing form of the Lie group G, $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ the isotropy representation of the maximal torus $T \subset G$, H/T represents the fibers provided with the induced homogeneous metric $\hat{g}$ represented by the inner product $(-B)|_{\mathfrak{p}}$ and the basis space G/H is provided with the symmetric metric $\breve{g}$ induced by $(-B)|_{\mathfrak{q}}$, $\mathfrak{p}$ and $\mathfrak{q}$ vertical and horizontal distributions, respectively.

For each canonical variation introduced in Section 2.2.2 defined on the maximal flag manifolds associated with a complex classical simple Lie algebra, we obtain the following properties

Lemma 3.2.2. *Consider the canonical variations* $(SU(n+1)/T^n, \mathbf{g}_t)$, $(SO(2n+1)/T^n, \mathbf{h}_t)$, $(Sp(n)/T^n, \mathbf{k}_t)$, $(SO(2n)/T^n, \mathbf{m}_t)$ *and* $(G_2/T, \mathbf{n}_t)$ *constructed in 2.2.2 and in 2.2.14, respectively, from the homogeneous fibrations* $\pi : G/T \longrightarrow G/H$ *described below,*

(a) $\pi : (SU(n+1)/T^n, g) \longrightarrow (SU(n+1)/S(U(1) \times U(n)), \breve{g})$, $n \geqslant 2$, $(H/T = SU(n)/T^{n-1}, \hat{g})$;

(b) $\pi : (SO(2n+1)/T^n, g) \longrightarrow (SO(2n+1)/SO(2n), \breve{g})$, $n \geqslant 2, n \neq 3$, $(H/T = SO(2n)/T^n, \hat{g})$;

(c) $\pi : (Sp(n)/T^n, g) \longrightarrow (Sp(n)/U(n), \breve{g})$, $n \geqslant 5$, $(H/T = SU(n)/T^{n-1}, \hat{g})$;

(d) $\pi : (SO(2n)/T^n, g) \longrightarrow (SO(n)/U(n), \breve{g})$, $n \geqslant 4$, $(H/T = SU(n)/T^{n-1}, \hat{g})$;

(e) $\pi : (G_2/T, g) \longrightarrow (G_2/SO(4), \breve{g})$, $(H/T = SO(4)/T \cong S^2 \times S^2, \hat{g})$.

Therefore, for each of the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ *and* $\mathbf{n}_t$ *of the homogeneous fibrations* (a)-(e), *with their respective fibers* $(H/T, \hat{g})$,

$$\frac{scal(t)}{m-1} < \lambda^{1,1}(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1, \forall\, 0 < t \leqslant 1,$$

with $m = \dim G/T$, $scal(t)$ *the respective scalar curvatures of* $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ *and* $\mathbf{n}_t$, μ_1 *the first positive eigenvalue of the Laplacian* Δ_g *and* ϕ_1 *the first positive eigenvalue of the Laplacian* $\Delta_{\hat{g}}$.

Proof: For each of the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ of the homogeneous fibrations (a)-(e) we have:

(a) $m = \dim SU(n+1)/T^n = n(n+1)$,

$$\mathrm{scal}(t) = \frac{-2n + n^2(n+1) + 4n(n+1)t^2 + n(1-n)t^4}{4(n+1)t^2},$$

$\mu_1 = 1$, $\phi_1 = 1$ and $\lambda_1(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1 = \frac{1}{t^2}$;

(b) $m = \dim SO(2n+1)/T^n = 2n^2$,

$$\mathrm{scal}(t) = \frac{5n^3 - 2n^2t^4 + 8n^2t^2 - 7n^2 + 2nt^4 - 4nt^2 + 2n}{4(2n-1)t^2},$$

$\mu_1 = \frac{n}{2n-1}$, $\phi_1 = 1$ and $\lambda_1(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1 = \frac{n}{2n-1} + \left(\frac{1}{t^2} - 1\right)$;

(c) $m = \dim Sp(n)/T^n = 2n^2$,

$$\mathrm{scal}(t) = \frac{-2n^3t^4 + 24n^3t^2 + 5n^3 + 48n^2t^2 + 9n^2 + 2nt^4 + 24nt^2 - 14n}{24(n+1)t^2},$$

$\mu_1 = \frac{4n-1}{4(n+1)}$, $\phi_1 = 1$ and $\lambda_1(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1 = \frac{4n-1}{4(n+1)} + \left(\frac{1}{t^2} - 1\right)$;

(d) $m = \dim SO(2n)/T^n = 2n(n-1)$,

$$\mathrm{scal}(t) = \frac{-2n^2t^4 + 24n^2t^2 + 5n^2 + 4nt^4 - 24nt^2 + 2n}{24t^2},$$

$\mu_1 = 1$, $\phi_1 = 1$ and $\lambda_1(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1 = \frac{1}{t^2}$.

(e) $m = \dim G_2/T = 12$,

$$\mathrm{scal}(t) = \frac{2 + 12t^2 - 2t^4}{3t^2},$$

$$\mu_1 = \frac{1}{2},\ \phi_1 = 1 \text{ and } \lambda_1(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1 = \frac{1}{2} + \left(\frac{1}{t^2} - 1\right).$$

In order to prove the inequality

$$\frac{\mathrm{scal}(t)}{m-1} < \lambda^{1,1}(t) = \mu_1 + \left(\frac{1}{t^2} - 1\right)\phi_1, \forall\, 0 < t \leqslant 1,$$

for each of the above (a)-(e) cases we define a function f given by

$$f(t) = \frac{\mathrm{scal}(t)}{m-1} - \lambda^{1,1}(t) = \frac{\mathrm{scal}(t)}{m-1} - \mu_1 - \left(\frac{1}{t^2} - 1\right)\phi_1, \forall\, 0 < t \leqslant 1,$$

and we verify that f is strictly negative for $0 < t \leqslant 1$. Indeed, in the case of the canonical variation $\mathbf{g}_t$ on $SU(n+1)/T^n$ we obtain

$$f(t) = \frac{n^3 - n^2t^4 + 4n^2t^2 + n^2 + nt^4 + 4nt^2 - 2n}{4(n+1)(n(n+1)-1)t^2} - \frac{1}{t^2},$$

where $f(1) = \dfrac{n^3 + 4n^2 + 3n}{4(n+1)(n(n+1)-1)} - 1 < 0$, for all $n \geqslant 2$ and

$$\frac{df}{dt}(t) = \frac{-4n^2t^3 + 8n^2t + 4nt^3 + 8nt}{4(n+1)(n(n+1)-1)t^2} - \frac{n^3 - n^2t^4 + 4n^2t^2 + n^2 + nt^4 + 4nt^2 - 2n}{2(n+1)(n(n+1)-1)t^3} + \frac{2}{t^3} > 0$$

for $0 < t \leqslant 1$. In the other cases, the proof is analogous.

□

Lemma 3.2.3. *Every degeneracy instant t_* for $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ introduced in the previous lemma is such that*

$$\frac{scal(t_*)}{m-1} \in \sigma(\Delta_{\breve{g}}) \subset \sigma(\Delta_{t_*}),$$

with $\sigma(\Delta_{t_*})$ spectrum of the Laplacians on the total spaces $SU(n+1)/T^n$, $SO(2n+1)/T^n, Sp(n)/T^n, SO(2n)/T^n$, G_2/T and $\sigma(\Delta_{\breve{g}})$ the spectrum on the basis spaces $(G/H, \breve{g})$, respectively, $m = \dim G/T$. In other words, $\dfrac{scal(t_*)}{m-1}$ is eigenvalue of Δ_{t_*} if and only if $\dfrac{scal(t_*)}{m-1}$ is a constant eigenvalue $\lambda^{k,0}(t) \in \sigma(\Delta_{t_*})$, for some $1 \leqslant k \in \mathbb{Z}$.

Proof: It is known that an eigenvalue of Δ_t can be written as

$$\lambda^{k,j}(t) = \mu_k + \left(\frac{1}{t^2} - 1\right)\phi_j,$$

for some $\mu_k \in \sigma(\Delta_g)$ and $\phi_j \in \sigma(\Delta_{\hat{g}})$. From Lemma 3.2.2,

$$\frac{\operatorname{scal}(t)}{m-1} < \lambda^{1,1}(t) < \lambda^{k,j}(t), \forall k, j > 0,$$

and $\dfrac{\operatorname{scal}(t)}{m-1}$ can not be equal to an eigenvalue of the type $\lambda^{k,j}(t), \forall k, j > 0$.

Furthermore, it can be shown, by applying elementary differential calculus, that there is no $0 < t < 1$ such that $\dfrac{\operatorname{scal}(t)}{m-1} = \lambda^{0,j}(t)$, for all integer $j \geqslant 1$. We will present the proof of this property for the canonical variation $\mathbf{g}_t$. The proof of the another cases are entirely analogous.

From the description of the spectra of maximal flag manifolds and symmetric spaces in Section 2.2.3, the first positive eigenvalue of the Laplacian Δ_t acting on functions on the total space $(SU(n+1)/T^n, \mathbf{g}_t)$ is $\lambda_1(t) = 1$, for all $0 < t \leqslant 1$. Since

$$\operatorname{scal}(t) = \frac{-2n + n^2(n+1) + 4n(n+1)t^2 + n(1-n)t^4}{4(n+1)t^2}, n \geqslant 2,$$

for all $t > 0$, and $m = \dim SU(n+1)/T^n = n(n+1)$, we have

$$\begin{aligned} \frac{\operatorname{scal}(g_t)}{m-1} &= \frac{n^3 - n^2t^4 + 4n^2t^2 + n^2 + nt^4 + 4nt^2 - 2n}{4(n+1)(n(n+1)-1)t^2} \\ &< \lambda_1(t) = 1 \\ &\Leftrightarrow \sqrt{\sqrt{\frac{4n^4 + 17n^3 + 26n^2 + 16n + 4}{n^2}} - \frac{2\left(n^2 + 2n + 1\right)}{n}} < t \leqslant 1, \end{aligned}$$

Thus, we have that $\mathbf{g}_t$ is locally rigidity at all instant

$$t_* \in \left]\sqrt{\sqrt{\frac{4n^4 + 17n^3 + 26n^2 + 16n + 4}{n^2}} - \frac{2\left(n^2 + 2n + 1\right)}{n}}, 1\right],$$

accordingly Corolário 1.1.7, Section 1.1. Hence, there are no degeneracy instants for $\mathbf{g}_t$ in the interval $(b, 1]$, with

$$b = \sqrt{\sqrt{\frac{4n^4 + 17n^3 + 26n^2 + 16n + 4}{n^2}} - \frac{2\left(n^2 + 2n + 1\right)}{n}}.$$

The fiber of the canonical fibration $(SU(n+1)/T^n, \mathbf{g}_t)$ is the maximal flag manifold $(SU(n)/T^{n-1}, \hat{g})$, with the induced metric $\hat{g}$, represented by the inner product $(-B)|_{\mathfrak{p}}$. With this homogeneous metric, according Theorem 2.2.20, Section 2.2.3, the first positive eigenvalue ϕ_1 of the Laplacian $\Delta_{\hat{g}}$ is equal to 1, i.e., if $\phi \in \sigma(\Delta_{\hat{g}})$, then $\phi \geqslant 1$.

Define $\varphi_r : (0,1) \longrightarrow \mathbb{R}$ by

$$\varphi_r(t) = \frac{\operatorname{scal}(g_t)}{m-1} - \left(\frac{1}{t^2} - 1\right) \cdot r,$$

for some fixed $r \geqslant 1$. We have that

$$\frac{d}{dt}(\varphi_r(t)) = \frac{n^3(4r-1) + n^2\left(8r - t^4 - 1\right) + n\left(t^4 + 2\right) - 4r}{2(n+1)\left(n^2 + n - 1\right)t^3} > 0,$$

and

$$\varphi_r(b) = \frac{n((C_n - 5)r + C_n - 4) - 2n^2(r+1) - 2(r+1)}{(C_n - 4)n - 2n^2 - 2} < 0,$$

$\forall\, 0 < t < 1, r \geqslant 1, n \geqslant 2$, where $C_n = \sqrt{4n^2 + \frac{4}{n^2} + 17n + \frac{16}{n} + 26}$. It follows that $\varphi_r(t)$ is negative for all $0 < t < b$ and

$$\frac{\mathrm{scal}(g_t)}{m-1} < \left(\frac{1}{t^2} - 1\right) \cdot r,$$

for any $r \geqslant 1$, that is, there is no $0 < t < 1$ such that $\frac{\mathrm{scal}(t)}{m-1} = \lambda^{0,j}(t) = \left(\frac{1}{t^2} - 1\right) \cdot \phi_j$, for all $\phi_j \in \sigma(\Delta_{\hat{g}})$ and we complete the proof for the case of the canonical variation $\mathbf{g}_t$.

As well as for the case of $\mathbf{g}_t$, define for $\mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ the function $\varphi_r(t)$ as above with $r \geqslant 1$, since the first positive eigenvalues of the Laplacians on their respective fibers are also equal to 1. The statement follows by checking that $\frac{d}{dt}(\varphi_r(t)) > 0, \forall\, 0 < t < 1$ and $\varphi_r(b) < 0$, where $0 < b < 1$ is such that

$$\frac{\mathrm{scal}(g_t)}{m-1} < \lambda_1(t), \forall\, b < t < 1,$$

with $\lambda_1(t)$ first positive eigenvalues of the Laplacians, given in Proposition 2.2.22. We found the following values for b, in addition to the expressions for $\varphi_r(t)$ for the other cases:

$(SO(2n+1)/T^n, \mathbf{h}_t))$

$$b = \sqrt{\frac{\sqrt{8n^2 + 5n - 2}}{\sqrt{2}} - 2n},$$

$$\begin{aligned}\varphi_r(t) \;=\; & \frac{\left(2n - 2n^2\right)t^2}{4(2n-1)\left(2n^2-1\right)} + \frac{-16n^3r + 5n^3 + 8n^2r - 7n^2 + 8nr + 2n - 4r}{4(2n-1)\left(2n^2-1\right)t^2} \\ & + \frac{16n^3r - 8n^2r + 8n^2 - 8nr - 4n + 4r}{4(2n-1)\left(2n^2-1\right)}.\end{aligned}$$

$(Sp(n)/T^n, \mathbf{k}_t))$

$$b = \sqrt{\sqrt{\frac{2n\left(77n^5 - 351n^4 + 143n^3 + 747n^2 + 212n + 72\right) + 9}{4n^2(n^2-1)^2}} - \frac{3\left(4n^3 - 10n^2 - 8n + 1\right)}{2n(n^2-1)}},$$

$$\begin{aligned}\varphi_r(t) &= \frac{-48n^3r + 5n^3 - 48n^2r + 9n^2 + 24nr - 14n + 24r}{24(n+1)\left(2n^2-1\right)t^2} \\ &+ \frac{48n^3r + 24n^3 + 48n^2r + 48n^2 - 24nr + 24n - 24r}{24(n+1)\left(2n^2-1\right)} \\ &+ \frac{\left(2n - 2n^3\right)t^2}{24(n+1)\left(2n^2-1\right)}.\end{aligned}$$

$(SO(2n)/T^n, \mathbf{m}_t)$

$$b = \sqrt{\frac{\sqrt{77n^4 - 152n^3 - 76n^2 + 144n + 72}}{\sqrt{2}(n-2)n} - \frac{6\left(n^2 - n - 1\right)}{(n-2)n}},$$

$$\begin{aligned}\varphi_r(t) &= \frac{-48n^2r + 5n^2 + 48nr + 2n + 24r}{24\left(2n^2 - 2n - 1\right)t^2} + \frac{48n^2r + 24n^2 - 48nr - 24n - 24r}{24\left(2n^2 - 2n - 1\right)} \\ &+ \frac{\left(4n - 2n^2\right)t^2}{24\left(2n^2 - 2n - 1\right)}.\end{aligned}$$

$(G_2/T, \mathbf{n}_t)$

$$b = \frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{145} - 9\right)},\ \varphi_r(t) = \frac{33rt^2 - 33r - 2t^4 + 12t^2 + 2}{33t^2}.$$

□

Observe that the proof of the last Lemma does not guarantees that the set of all instants $0 < t < 1$ such that $\dfrac{\mathrm{scal}(g_t)}{m-1} < \lambda_1(t)$ is equal to the interval $]b, 1]$, for the canonical variations $\mathbf{k}_t$ and $\mathbf{n}_t$. In fact, we don't have expressions of $\lambda_1(t)$ in such cases, therefore we can not determine the interval of rigidity instants, only subintervals of them. We used the lower bounds of $\lambda_1(t)$ in order to determine a subset $]b, 1]$ of local rigidity instants for the canonical variations $\mathbf{k}_t$ and $\mathbf{n}_t$.

For the canonical variations $\mathbf{g}_t$, $\mathbf{h}_t$ and $\mathbf{m}_t$ on $SU(n+1)/T^n$, $SO(2n+1)/T^n$ and $SO(2n)/T^n$ we determined all $t \in (0, 1]$ such that $\dfrac{\mathrm{scal}(g_t)}{m-1} < \lambda_1(t)$, since $\lambda_1(t)$ is constant on $(0, 1]$ in these cases.

The degeneracy instants for the canonical variation $\mathbf{g}_t$, $0 < t < 1$, on $SU(n+1)/T^n$ are given in our following theorem.

Theorem 3.2.4. *Let $\mathbf{g}_t$ be the above canonical variation on $SU(n+1)/T^n$ and take $b = \sqrt{\sqrt{\frac{4n^4+17n^3+26n^2+16n+4}{n^2}} - \frac{2(n^2+2n+1)}{n}}$. Thus, the degeneracy instants for $\mathbf{g}_t$ in $(0,1)$ form a decreasing sequence $\{t_q^{\mathbf{g}}\} \subset (0,b]$ such that $t_q^{\mathbf{g}} \to 0$ as $q \to 0$, with $t_1^{\mathbf{g}} = b$ and for $q > 1$,*

$$t_q^{\mathbf{g}} = \sqrt{\sqrt{\xi(q)} - \eta(q)}. \tag{3.14}$$

where

$$\begin{aligned} \xi(q) \quad = \quad & \frac{1}{n^2(n-1)^2} \cdot \big(4n^6q^2 + n^5\left(8q^3+8q^2-8q+1\right) + \\ & 4n^4\left(q^4+4q^3-3q^2-4q+1\right) + n^3\left(8q^4-8q^3-24q^2+5\right) + \\ & n^2\left(-4q^4-16q^3+4q^2+8q+6\right) + 8nq^2\left(-q^2+q+1\right) + 4q^4\big) \end{aligned}$$

and $\eta(q) = \dfrac{2\left(n^3q + n^2\left(q^2+q-1\right) + n\left(q^2-q-1\right) - q^2\right)}{(n-1)n}$.

Proof: We must determine all $t \in \ (0,b]$ such that $\frac{\mathrm{scal}(t)}{m-1} \in \sigma(\Delta_t)$, since by the previous theorem g_t is locally rigidity for $b < t < 1$. From Lemma 3.2.3, if $\frac{\mathrm{scal}(t)}{m-1}$ is a eigenvalue of Δ_t, then $\frac{\mathrm{scal}(t)}{m-1} \in \Delta_{\breve{g}}$.

Thus, it remains verify for which instants $0 < t < b < 1$ one has $\frac{\mathrm{scal}(t)}{m-1} = \lambda^{k,0}(t)$.

From Corollary 2.2.10, Section 2.2, $\lambda^{k,0}(t)$ is eigenvalue of Δ_t if and only if $\lambda^{k,0}(t)$ belongs to the spectrum of the Laplacian on the basis $\mathbb{CP}^n = SU(n+1)/S(U(1)\times U(n))$, provide with the symmetric metric $\breve{g}$ represented by the inner product $(\cdot,\cdot) = -B|_{\mathfrak{q}}$, $\mathfrak{q}$ horizontal space of the original fibration. According the spectrum of the complex projective space given in 2.19, the spectrum of the Laplacian $\Delta_{\breve{g}}$ on $(\mathbb{CP}^n, \breve{g})$ is

$$\sigma(\Delta_{\breve{g}}) = \{\beta_q = \frac{q(q+n)}{n+1}; q \in \mathbb{N}\}.$$

The degeneracy instants for $\mathbf{g}_t$ in $(0,b]$ are the real values t_q^g, solutions of the equation

$$\frac{\mathrm{scal}(t)}{m-1} = \beta_q = \frac{q(q+n)}{n+1}.$$

The explicit formula for $t_q^{\mathbf{g}}$, which represents the solution of the above equation in t, is presented in (3.14). Note that the constant eigenvalues $\beta_q = \frac{q(q+n)}{n+1}$ tend to $+\infty$ as $q \to \infty$ and, since $\frac{\mathrm{scal}(t)}{m-1}$ is continuous and tends to $+\infty$ as $t \to 0$, $t_q^{\mathbf{g}} \to 0$.

□

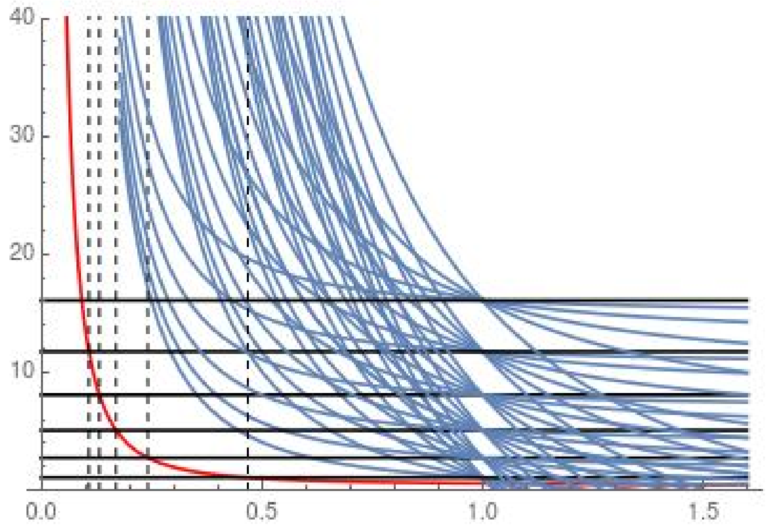


Figure 1 – For n=2, we have the maximal flag manifold $SU(3)/T^2$. The graph of the function $\dfrac{\mathrm{scal}(t)}{m-1}$ is given in red and of the eigenvalues $\lambda^{k,j}(t)$ of the Laplacian Δ_t are given in blue; the constants eigenvalues have their graphics in black and correspond to $(k,0), 1 \leqslant k \leqslant 6$, and non-constants ones to (k,j) where $1 \leqslant j \leqslant k \leqslant 6$. The dashed vertical lines mark the first five degeneracy instants (which are all bifurcation instants) starting at $b = t_1^{\mathbf{g}}$.

We also compute the Morse index of each $\mathbf{g}_t$ as a critical point of the Hilbert-Einstein functional (1.1).

Proposition 3.2.5. *The Morse index of* $\mathbf{g}_t$ *is given by*

$$N(t) = \begin{cases} \Sigma_{q=1}^{r} m_q(\mathbb{CP}^n), & \text{if } \ t_{r+1}^{\mathbf{g}} \leqslant t < t_r^{\mathbf{g}} \\ 0, & \text{if } \ t_1^{\mathbf{g}} \leqslant t \leqslant 1 \end{cases}, \tag{3.15}$$

where $m_q(\mathbb{CP}^n)$ *is the multiplicity of the qth eigenvalue of the basis* $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$, *see* (2.18).

Proof: We established in Lemma 3.2.3 that $\dfrac{\mathrm{scal}(t)}{m-1} \leqslant \lambda_1(t)$ for all $t \in [b,1]$, with $b = t_1^{\mathbf{g}}$, so that there are no eigenvalues of Δ_t that are less than $\dfrac{\mathrm{scal}(t)}{m-1}$. Hence, $N(t) = 0$ for $t \in [b,1]$. When $t \to 0$, whenever t crosses a degeneracy instant $t_q^{\mathbf{g}}$, the constant eigenvalue $\lambda^{q,0}(t)$ becomes smaller than $\dfrac{\mathrm{scal}(t)}{m-1}$.

It follows that the Morse index increases by the multiplicity of $\lambda^{q,0}(t)$, which is the dimension of the corresponding eigenspace E_q^0. This dimension is given in (2.18) and is positive. Thus, $\dim E_q^0$ also coincides with the multiplicity of the complex projective space $\mathbb{CP}^n$, concluding the proof.

□

The degeneracy instants for the canonical variation $\mathbf{h}_t$, $0 < t \leqslant 1$, on $SO(2n+1)/T^n$ are given in our following theorem.

Theorem 3.2.6. *Let $\mathbf{h}_t$ be the above canonical variation on $SO(2n+1)/T^n$ and take $b = \sqrt{\frac{\sqrt{8n^2+5n-2}}{\sqrt{2}}} - 2n$. Thus, the degeneracy instants for $\mathbf{h}_t$ in $(0, b]$ form a decreasing sequence $\{t_q^{\mathbf{h}}\} \subset (0, b]$ such that $t_q^{\mathbf{h}} \to 0$ as $q \to 0$, with $t_1^{\mathbf{h}} = b$ and for $q > 1$,*

$$t_q^{\mathbf{h}} = \sqrt{\sqrt{f(q)} + g(q)}. \tag{3.16}$$

where

$$\begin{aligned} f(q) \;\; = \;\; & \frac{1}{(n-1)^2n^2}(10n^5 - 8n^4 + 2n^3 + \left(4n^4 - 4n^2 + 1\right) q^4 + \\ & \left(16n^5 - 8n^4 - 16n^3 + 8n^2 + 4n - 2\right) q^3 + \left(-32n^5 + 32n^4 + 8n^3 - 16n^2 + 4n\right) q + \\ & \left(16n^6 - 16n^5 - 28n^4 + 24n^3 + 8n^2 - 8n + 1\right) q^2) \end{aligned}$$

and $g(q) = \dfrac{-4n^3q - 2n^2q^2 + 2n^2q + 4n^2 + 2nq - 2n + q^2 - q}{2(n-1)n}$.

Proof: We must determine all $t \in \;(0, b]$ such that $\dfrac{\mathrm{scal}(t)}{m-1} \in \sigma(\Delta_t)$, since by the previous theorem $\mathbf{h}_t$ is locally rigidity for $b < t < 1$. From Lemma 3.2.3, if $\dfrac{\mathrm{scal}(t)}{m-1}$ is a eigenvalue of Δ_t, then $\dfrac{\mathrm{scal}(t)}{m-1} \in \Delta_{\breve{g}}$.

Thus, it remains verify for which instants $0 < t < b < 1$ one has $\dfrac{\mathrm{scal}(t)}{m-1} = \lambda^{k,0}(t)$.

From Corollary 2.2.10, Section 2.2, $\lambda^{k,0}(t)$ is eigenvalue of Δ_t if and only if $\lambda^{k,0}(t)$ belongs to the spectrum of the Laplacian on the basis $S^{2n} = SO(2n+1)/SO(2n)$, provide with the symmetric metric $\breve{g}$ represented by the inner product $(\cdot,\cdot) = -B|_{\mathfrak{q}}$, $\mathfrak{q}$ horizontal space of the original fibration. According 2.20, the spectrum of the Laplacian $\Delta_{\breve{g}}$ on $(S^{2n}, \breve{g})$ is

$$\sigma(\Delta_{\breve{g}}) = \{\beta_q = \frac{q(q+2n-1)}{2(2n-1)}; q \in \mathbb{N}\}.$$

The degeneracy instants for $\mathbf{h}_t$ in $(0, b]$ are the real values $t_q^{\mathbf{h}}$, solutions of the equation

$$\frac{\mathrm{scal}(t)}{m-1} = \beta_q = \frac{q(q+2n-1)}{2(2n-1)}.$$

The explicit formula for t_q^g, which represents the solution of the above equation in t, is presented in (3.16). Note that the constant eigenvalues $\beta_q = \dfrac{q(q+2n-1)}{2(2n-1)}$ tend to $+\infty$ as $q \to \infty$ and, since $\dfrac{\mathrm{scal}(t)}{m-1}$ is continuous and tends to $+\infty$ as $t \to 0$, $t_q^{\mathbf{h}} \to 0$.

□

By a totally analogous argument, we also have the Morse index $N(t)$ for each $\mathbf{h}_t$, as we obtained for $\mathbf{g}_t$ above.

Proposition 3.2.7. *The Morse index of* $\mathbf{h}_t$ *is given by*

$$N(t) = \begin{cases} \Sigma_{q=1}^{r} m_q(S^{2n}), & \text{if } \; t^{\mathbf{g}}_{r+1} \leqslant t < t^{\mathbf{g}}_{r} \\ 0, & \text{if } \; t^{\mathbf{g}}_{1} \leqslant t \leqslant 1 \end{cases}, \tag{3.17}$$

where $m_q(S^{2n})$ *is the multiplicity of the qth eigenvalue of the basis* $S^{2n} = SO(2n+1)/SO(2n)$, *see* (2.18).

As consequence of the above results, we have obtained the bifurcation and rigidity instants for $(SU(n+1)/T^n, \mathbf{g}_t)$ and $(SO(2n+1)/T^n, \mathbf{h}_t)$, $0 < t \leqslant 1$, in the following theorem.

Theorem 3.2.8. *For the canonical variations* $(SU(n+1)/T^n, \mathbf{g}_t)$ *and* $(SO(2n+1)/T^n, \mathbf{h}_t)$, *constructed above, the elements of the sequences* $\{t^{\mathbf{g}}_q\}, \{t^{\mathbf{h}}_q\} \subset (0, b]$, *of degeneracy instants for* $\mathbf{g}_t$ *and* $\mathbf{h}_t$, *given in* (3.14) *an* (3.16), *are bifurcation instants for* $\mathbf{g}_t$ *and* $\mathbf{h}_t$, *respectively. Moreover,* $\mathbf{g}_t$ *and* $\mathbf{h}_t$ *are locally rigid for all* $t \in (0,1] \setminus \{t^{\mathbf{g}}_q\}$ *(respectively* $t \in (0,1] \setminus \{t^{\mathbf{h}}_q\}$*).*

Proof: From Proposition 1.1.6, Section 1.1, we know that, if $t_* > 0$ is not a degeneracy instant for $\mathbf{g}_t$ and $\mathbf{h}_t$, then $\mathbf{g}_t$ and $\mathbf{h}_t$ are locally rigidity at t_*. In the previous theorems, we proved that the degeneracy instants for $\mathbf{g}_t$ and $\mathbf{h}_t$ form the sequences $t^{\mathbf{g}}_q, t^{\mathbf{h}}_q$, and thus, if $t \notin \{t^{\mathbf{g}}_q\}$ or $t \notin \{t^{\mathbf{h}}_q\}$, t must be a local rigidity instant for $\mathbf{g}_t$ and $\mathbf{h}_t$, respectively.

The fact that each $t^{\mathbf{g}}_q$ or $t^{\mathbf{h}}_q$ are a bifurcation instants follows from Proposition 3.2.1 and Lemma 3.2.2.

□

Remark 3.2.9. The spectra of the Laplacian on the basis spaces G/H of the canonical variations

$$\pi : (Sp(n)/T^n, \mathbf{k}_t) \longrightarrow (Sp(n)/U(n), \breve{g}), n \geqslant 5 \;\text{ and}$$

$$\pi : (SO(2n)/T^n, \mathbf{m}_t) \longrightarrow (SO(2n)/U(n), \breve{g}), n \geqslant 4,$$

as we introduce in 2.17, Section 2.2.3, are given by

$$\sigma(\Delta_{\breve{g}}) = \{\mu(\Lambda) = -B(\Lambda + 2\delta, \Lambda); \Lambda \in D(G, H)\},$$

where Λ varies over the set $D(G, H)$ of the highest weights of all spherical representations of (G, H) and δ is equal to the sum of the positive roots of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ relative to the maximal abelian subalgebra $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$.

The set $D(Sp(n), U(n))$ is spanned by $\mathcal{B} = \{2\pi_l; 1 \leqslant l \leqslant n\}$, while for $D(SO(2n), U(n))$ one has the generators

$$\mathcal{B}_1 = \{\pi_2, \pi_4, \ldots, \pi_{n-2}, 2\pi_n\}, \quad \text{if } \ n \ \text{ is even}$$

and

$$\mathcal{B}_2 = \{\pi_2, \pi_4, \ldots, \pi_{n-3}, \pi_{n-1} + \pi_n\}, \quad \text{if } \ n \ \text{ is odd},$$

with $\{\pi_l; 1 \leqslant l \leqslant n\}$ being the fundamental weights of the maximal abelian Lie algebra $\mathrm{Lie}(T^n) = \mathfrak{k}^{\mathbb{C}}$. Therefore, an element Λ of $D(Sp(n), U(n))$ or $D(SO(2n), U(n))$ is linear combination of elements in some of these basis, with coefficients belonging to the set of non-negative integers. It is well known that 2δ can also be written as linear combination of fundamental weights (the coefficients of this linear combination are called *Koszul numbers*.)

Hence, any eigenvalue of the Laplacian on the basis spaces $Sp(n)/U(n)$ and $SO(2n)/U(n)$ is equal to a polynomial depending on several (integers) variables, so that we can not to obtain general formulae for bifurcation instants in these cases, as in 3.14 and 3.16, for the canonical variations $\mathbf{g}_t$ and $\mathbf{h}_t$ defined on $SU(n+1)/T^n$ and $SO(2n+1)/T^n$, respectively.

Theorem 3.2.10. *For the canonical variations* $(Sp(n)/T^n, \mathbf{k}_t), n \geqslant 5$, *and* $(SO(2n)/T^n, \mathbf{m}_t)$, $n \geqslant 4$, *introduced in Lemma 3.2.2, the bifurcation instants form discrete sets* $\{t^g_{x_1 x_2 \ldots x_l}\}_{x_1, x_2, \ldots, x_l \in \mathbb{Z}_+} \subset (0, 1]$, $1 \leqslant l \leqslant n$, *with infinite elements accumulating close to zero as* $x_1, x_2, \ldots, x_l$ *vary over* $\mathbb{Z}_+$. *Moreover,* $\mathbf{k}_t$ *and* $\mathbf{m}_t$ *are locally rigid for all* $t \in (0, 1] \setminus \{t^g_{x_1 x_2 \ldots x_l}\}_{x_1, x_2, \ldots, x_l \in \mathbb{Z}_+}$.

Proof: We must find the instants $0 < t < 1$ such that $\dfrac{\mathrm{scal}(t)}{m-1}$ is a constant eigenvalue, where m represents the dimension of each spaces $Sp(n)/T^n$ and $SO(2n)/T^n$, while $\mathrm{scal}(t)$ denotes the scalar curvature of $\mathbf{k}_t$ and $\mathbf{m}_t$.

From Corollary 2.2.10, Section 2.2, $\lambda^{k,0}(t)$ is eigenvalue of Δ_t if and only if $\lambda^{k,0}(t)$ belongs to the spectrum of the Laplacian on the basis space provide with the symmetric metric $\breve{g}$ represented by the inner product $(\cdot, \cdot) = -B|_{\mathfrak{q}}$, $\mathfrak{q}$ horizontal space of the original fibration. According 2.17, the spectrum of the Laplacian $\Delta_{\breve{g}}$ on the basis is given by

$$\sigma(\Delta_{\breve{g}}) = \{\mu(\Lambda) = -B(\Lambda + 2\delta, \Lambda); \Lambda \in D(G, H)\},$$

implying that any eigenvalue of the Laplacian on basis spaces $Sp(n)/U(n)$ and $SO(2n)/U(n)$ is equal to a polynomial expression depending on several (integers) variables, as has been seen in the above remark. Let $\mu(\Lambda) = \mu(x_1, \ldots, x_l) \in \sigma(\Delta_{\breve{g}})$, for some $1 \leqslant l \leqslant n$, a constant eigenvalue of the Laplacian Δ_t acting on $(Sp(n)/T^n, \mathbf{k}_t), n \geqslant 5$, or $(SO(2n)/T^n, \mathbf{m}_t)$, $n \geqslant 4$.

The degeneracy instants for $\mathbf{k}_t$ and $\mathbf{m}_t$ in $(0,1)$ are the real values $t^g_{x_1 x_2 \dots x_l}$ which are solutions of the equation

$$\frac{\mathrm{scal}(t)}{m-1} = \mu(x_1, \dots, x_l).$$

The above equations in t, both for $\mathbf{k}_t$ and $\mathbf{m}_t$, have infinite solutions, since the functions $(0,1) \ni t \mapsto \dfrac{\mathrm{scal}(t)}{m-1}$ are continuous and assume any positive value greater than or equal to the first positive eigenvalue of the Laplacian $\Delta_{\breve{g}}$ on the respective basis spaces $Sp(n)/U(n)$ and $SO(2n)/U(n)$, namely, greater than 1, see Section 2.2.3.

The fact that the degeneracy instants $t^g_{x_1 x_2 \dots x_l}$ are bifurcation instants follows from Proposition 3.2.1 and Lemma 3.2.2.

By Proposition 1.1.6, Section 1.1, we know that, if $t_* > 0$ is not a degeneracy instant for $\mathbf{k}_t$ and $\mathbf{m}_t$, then $t_* > 0$ is a local rigidity instant for them. Therefore, $\mathbf{k}_t$ and $\mathbf{m}_t$ are locally rigid at all

$$t \in (0,1] \setminus \{t^g_{x_1 x_2 \dots x_l}\}_{x_1, x_2, \dots, x_l \in \mathbb{Z}_+}.$$

Note that in fact $t^g_{x_1 x_2 \dots x_l} \in (0,1)$, since we obtain positive numbers $0 < b < 1$ in Lemma 3.2.3 such that $\mathbf{k}_t$ and $\mathbf{m}_t$ are locally rigid at all $t \in]b, 1]$. Thus,

$$t^g_{x_1 x_2 \dots x_l} \in (0, b[\subset (0,1).$$

□

Using the notations in Example 2.2.14, 2.2.1, it was established that the homogeneous fibration

$$\pi : (G_2/T, g) \longrightarrow (G_2/SO(4), \breve{g})$$

is a Riemannian submersion with totally geodesic fibers isometric to $(H/K, \hat{g})$, $H/K = SO(4)/T \cong SO(4)/SO(2) \times SO(2) \cong S^2 \times S^2$, g the normal metric determined by the inner product $(-B)|_{\mathfrak{m}}$, $\hat{g}$ the metric given by $(-B)|_{\mathfrak{p}}$ and $\breve{g}$ defined by the inner product $(-B)|_{\mathfrak{q}}$. Furthermore, the canonical variation g_t of this submersion is represented by the inner product

$$(\mathbf{n}_t)_{eT} = (-B)|_{\mathfrak{m}_{\alpha_1}} + (-B)|_{\mathfrak{m}_{\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+\alpha_2}} + (-B)|_{\mathfrak{m}_{\alpha_1+2\alpha_2}} + t^2(-B)|_{\mathfrak{m}_{\alpha_1+3\alpha_2}} + (-B)|_{\mathfrak{m}_{2\alpha_1+3\alpha_2}},$$

according (2.6).

The first positive eigenvalue of the Laplacian $\Delta_{\hat{g}}$ on the fiber is, according to [30], equal to $\phi_1 = 1$ and the first positive eigenvalue of the Laplacian Δ_g on the total space is equal to $\mu_1 = \dfrac{1}{2}$. Hence,

$$\lambda^{1,1}(t) = \mu_1 + (\frac{1}{t^2} - 1)\phi_1 = \frac{1}{2} + (\frac{1}{t^2} - 1).$$

In Proposition 3.1.14, Section 3.1, we obtained the expression (3.13) of $\mathrm{scal}(t)$, scalar curvature of the canonical variation $\mathbf{n}_t$, namely

$$\mathrm{scal}(t) = \frac{2 + 12t^2 - 2t^4}{3t^2}, \quad t > 0.$$

Since $m = \dim G_2/T = 12$, follows that

$$\frac{\mathrm{scal}(t)}{m-1} = \frac{2 + 12t^2 - 2t^4}{33t^2}.$$

Furthermore, by Proposition 2.2.21, we have that the spectrum of the Laplacian on the basis space $(G_2/SO(4), \breve{g})$ is

$$\sigma(\Delta_{\breve{g}}) = \left\{\frac{1}{6}(9r + 6r^2 + 5s + 6rs + 2s^2); \mathbb{Z} \ni r, s \geqslant 0\right\}.$$

Now, we can determine the bifurcation and local rigidity instants for $\mathbf{n}_t$ in the interval $(0, 1]$.

Theorem 3.2.11. *The elements of the set $\{t^{\mathbf{n}}_{rs}\} \subset (0, 1]$ given by*

$$t^{\mathbf{n}}_{rs} = \frac{\sqrt{\sqrt{(-66r^2 - 33rs - 99r - 22s^2 - 55s + 24)^2 + 64} - 66r^2 - 33rs - 99r - 22s^2 - 55s + 24}}{2\sqrt{2}},$$

$\mathbb{Z} \ni r, s \geqslant 0$, are bifurcation instants for $(G_2/T, \mathbf{n}_t)$. Moreover, $\mathbf{n}_t$ is locally rigid at all $0 < t \leqslant 1$ such that $t \notin \{t^{\mathbf{n}}_{rs}\}$.

Proof: By Proposition 3.2.1, if $0 < t_* < 1$ such that $\dfrac{\mathrm{scal}(t_*)}{m-1} \in \sigma(\Delta_{\breve{g}})$ and $\dfrac{\mathrm{scal}(t_*)}{m-1} < \lambda^{1,1}(t_*)$, then t_* is a bifurcation instant.

Recall that

$$\frac{\mathrm{scal}(t)}{m-1} = \frac{\mathrm{scal}(t)}{11} = \frac{2 + 12t^2 - 2t^4}{33t^2} < \lambda^{1,1}(t) = \mu_1 + (\frac{1}{t^2} - 1)\phi_1 = \frac{1}{2} + (\frac{1}{t^2} - 1),$$

for all $0 < t \leqslant 1$, according 3.2.2. The instants that satisfy $\dfrac{\mathrm{scal}(t_*)}{11} \in \sigma(\Delta_{\breve{g}}) \subset \sigma(\Delta_t)$ are the solutions of

$$\frac{\mathrm{scal}(t)}{11} = \frac{2 + 12t^2 - 2t^4}{33t^2} = \frac{1}{6}(9r + 6r^2 + 5s + 6rs + 2s^2), 0 < t \leqslant 1$$

which are exactly the $t^{\mathbf{n}}_{rs}$ given above. The elements of the set $\{t^{\mathbf{n}}_{rs}\}$ are such that

$$\frac{\mathrm{scal}(t^{\mathbf{n}}_{rs})}{11} < \lambda^{1,1}(t^{\mathbf{n}}_{rs})$$

and $t^{\mathbf{n}}_{rs}$ is a bifurcation instant for all $0 \leqslant r, s \in \mathbb{Z}$. In order to prove that $\mathbf{n}_t$ is locally rigid at all $0 < t < 1$ such that $t \notin \{t^{\mathbf{n}}_{rs}\}$ we apply Proposition 1.1.6, 1.1.

□

Remark 3.2.12. By continuity of $\dfrac{\mathrm{scal}(t)}{11}$ and by the fact that $\lim\limits_{t\to 0}\dfrac{\mathrm{scal}(t)}{11} = +\infty$, since the eigenvalues of $\Delta_{\breve{g}}$ goes to $+\infty$ when $r, s \to +\infty$, we have that the sequence t^{g}_{rs} obtained above is such that $t^{\mathbf{n}}_{rs} \to 0$ when $r, s \to +\infty$ and, then, we determine a sequence of bifurcation instants accumulating close to zero.

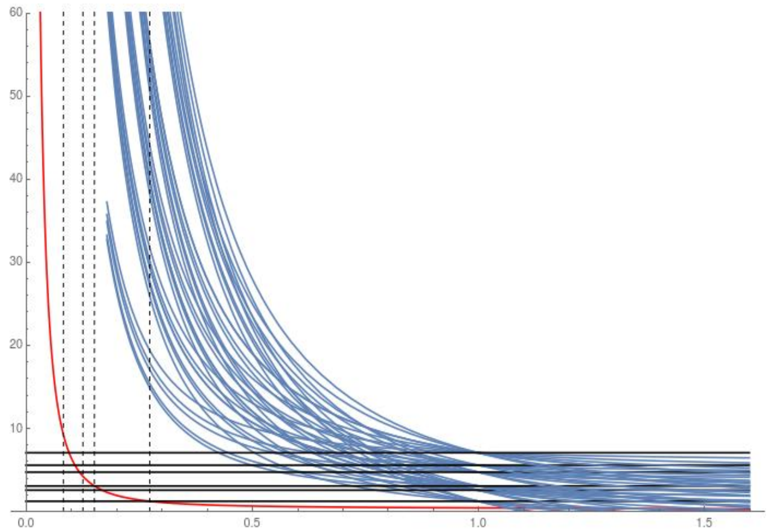


Figure 2 – For G_2/T, graphs of the functions $\dfrac{\mathrm{scal}(t)}{m-1}$ in red and $\lambda^{k,j}(t)$ in black, the constants corresponding to $(k,0), 1 \leqslant k \leqslant 6$, and non-constants to (k,j) where $1 \leqslant j \leqslant k \leqslant 6$. The dashed vertical lines mark five degeneracy instants (which are all bifurcation instants) starting at $t^{\mathbf{n}}_{1,0}$.

3.3 Multiplicity of Solutions to the Yamabe Problem

We now explain how to obtain multiplicity results for the canonical variations $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ applying the next proposition due R. G. Bettiol and P. Piccione which is in their work [10]. We have been interested in determine which conformal classes carry multiple unit volume metrics with constant scalar curvature.

Proposition 3.3.1 ([10])**.** *Let g_t, with $t \in (0, b[$, be a family of metrics on M with $N(t) = N(g_t) > 0$, $N(t)$ the Morse index of g_t, and suppose there exists a sequence $\{t_q\}$ in $(0, b[$, that converges to 0, of bifurcation values for g_t. Then, there is an infinite subset $\mathcal{G} \subset (0, b[$ accumulating at 0, such that for each $t \in \mathcal{G}$, there are at least 3 solutions to the Yamabe problem in the conformal class $[g_t]$.*

Applying the last Proposition and our bifurcation results, we can determine a lower bound for the number of unit volume metrics with constant scalar curvature in each conformal class $[\mathbf{g}_t], [\mathbf{h}_t], [\mathbf{k}_t], [\mathbf{m}_t], [\mathbf{n}_t]$, respectively, for instants in a given subset $\mathcal{G} \subset (0, 1)$.

Theorem 3.3.2. *Let $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ be the families of homogeneous metrics obtained as described above. Then, there exists, for each of such families, a subset $\mathcal{G} \subset (0, 1)$, accumulating at 0, such that for each $t \in \mathcal{G}$, there are at least 3 solutions to the Yamabe problem in each conformal class $[\mathbf{g}_t], [\mathbf{h}_t], [\mathbf{k}_t], [\mathbf{m}_t], [\mathbf{n}_t]$.*

Proof: It is only necessary to verify that there exists $0 < b < 1$ such that $N(t) > 0$ in $(0, b[$. For each canonical variation $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$ take a positive number $0 < b < 1$ such that $\frac{\mathrm{scal(t)}}{m-1} < \lambda_1(t)$ for all $t \in]b, 1)$, with m denoting the dimension of the respective total space. This implies that $N(t) = 0$, for all $t \in]b, 1)$, since there are no eigenvalues of Δ_t less than $\frac{\mathrm{scal(t)}}{m-1}$. If $t = b$, one has $\frac{\mathrm{scal(t)}}{m-1} = \beta_1$, where β_1 is the first positive eigenvalue of the Laplacian on the basis. We proved that $t = b$ is a bifurcation instant and then the Morse index changes from 0 to a positive integer. Hence, for $t \in (0, b[\subset (0, 1)$, we have $N(t) \geqslant N(b - \epsilon) > 0$, since by definition, the Morse index is equal to the number (counting multiplicity) of positive eigenvalues that are lees than $\frac{\mathrm{scal}(t)}{m-1}$, which is strictly decreasing for $0 < t < 1$ and $\frac{\mathrm{scal}(t)}{m-1} \to \infty$ as $t \to 0.$, for $\mathbf{g}_t, \mathbf{h}_t, \mathbf{k}_t, \mathbf{m}_t$ and $\mathbf{n}_t$.

□

Bibliography

[1] ALEKSEEVSKY, D. V. *Flag Manifolds.* Yugoslav Geometrical Seminar, Divcibare, 10-17 October 1996, 3-35.

[2] ALEKSEEVSKY, D. V.; ARVANITOYEORGOS, A. *Riemannian flag manifolds with homogeneous geodesics* Trans. Am Math. Soc. 359 (8), 3769-3789, 2007.

[3] ALEKSEEVSKY, D. V.; PERELOMOV, A. M. *Invariant Khäler–Einstein metrics on compact homogeneous spaces,* Funct. Anal. Appl. 20, 3769-3789, 2007.

[4] ARVANITOYEORGOS, A.; CHRYSIKOS, I.; *Invariant Einstein Metrics on Flag Manifolds With Four Isotropy Summands* Ann Glob Anal Geometry,(2010) 37 pp 185-219.

[5] ARVANITOYEORGOS, A.; CHRYSIKOS, I.; SAKANE, Y. *Homogeneous Einstein Metrics on* G_2/T Proceedings of the American Mathematical Society, 141, (2013) pp 2485-2499.

[6] BERGER, M. *Sur Le Prèmiere Valeurs Propre des Variété Riemannienne* Lecture Notes in Math., Vol. 194, Springer (1971).

[7] BERGERY, L. B.; BOURGUIGNON, J. P. *Laplacians and Riemannian Submersions With Totally Geodesic Fibres.* Illinois Journal of Mathematics Vol. 26, No. 2, Summer (1982).

[8] BESSE, A. L. *Einstein Manifolds.* Classics in Mathematics, Springer-Verlag Berlin Heidelberg, 1987.

[9] BESSON, G.; BORDONI, M. *On the Spectrum of Riemannian Submersions With Totally Geodesic Fibers.* Rendiconti Lincei Matematica e Applicazioni, Vol. 1, 1990, n. 4, p. 335-340.

[10] BETTIOL, R. G.; PICCIONE, P. *Multiplicity of Solutions to the Yamabe Problem On Collapsing Riemannian Submersions.* Pacific Journal of Math. Vol. 266, No. 1, (2013).

[11] BETTIOL, R. G.; PICCIONE, P. *Bifurcation and Local Rigidity of Homogeneous Solutions to the Yamabe Problem On Spheres.* Calculus of Variations Vol. 47, Springer-Verlag (2013), pag. 789-807.

[12] BURSTALL, F. E.; RAWNSLEY, J. H. *Twistor Theory for Riemannian Symmetric Spaces.* Lectures Notes in Mathematics, Springer-Verlag, 1980.

[13] COHEN, N.; GRAMA, L.; NEGREIROS, C. J. C. *Equigeodesics on flag manifolds.* Houston Journal of Mathematics. 37 (2011) 113-125.

[14] DE LIMA, L.L.; PICCIONE, P.; ZEDDA, M. *On bifurcation of solutions of the Yamabe problem in product manifolds.* Annales de l'I.H.P. Analyse non linéaire, Volume 29 (2012) no. 2, pp. 261-277.

[15] FALCITELLI, M.; IANUS, S.; PASTORE, A. M.; *Riemannian Submersions and Related Topics.* World Scientific Publishing Co. Pte. Ltd., 2004.

[16] HELGASON, S. *Groups and Geometric Analysis.* Amarican Mathematical Society, Mathematical Surveys and Monographs, Vol. 83, 1984.

[17] HELGASON, S. *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, Inc., 1978.

[18] HUMPHREYS, J. E. *Introdution to Lie Algebras and Representation Theory.* Springer-Verlag New York, Thirdy Edition, 1972.

[19] IKEDA, A.; TANIGUCHI, Y., *Spectra and Eigenforms of the Laplacian on S^n and $\mathbb{CP}^n$.* Osaka J. Math. 15 (1978), 515-546.

[20] KIMURA, M. *Homogeneous Einstein metrics on certain Kähler C-spaces* Recent topics in differential and analytic geometry, 303-320, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990.

[21] KOBAYASHI, M. *Transformation Groups in Differential Geometry* Classics in Math., Band 70, Srpinger, Berlin (1995).

[22] KRAMER, M; *Sphärische Untergruppen In Kompakten Zusammenhängenden Liegruppen.* Compositio Mathematica, tome 38, Number 2, 1979, p. 129-153.

[23] NAGANO, T. *On the Minimum Eigenvalues of the Laplacians in Riemannian Manifolds.* University of Tokyo, 1961.

[24] PANELLI, F.; PODESTÀ, F. *On the First Eigenvalue of Invariant Kähler Metrics.* Mathematische Zeitschrift (2014).

[25] SAKANE, Y. *Homogeneous Einstein Metrics on Flag manifolds* Lobachevskii J. Math., Vol. 4, 1999, 71-87.

[26] SAN MARTIN, L. A. B. *Álgebras de Lie* 2.ed. Editora Unicamp, 2010.

[27] SAN MARTIN, L. A. B. *Grupos de Lie*, 2015.

[28] SCHOEN, R. *Variational Theory for the Total Scalar Curvature Functional for Riemannian Manifolds and Related Topics* Topics in Calculus of Variations, Lecture Notes in Mathematics, 1365 (1989), 120-154.

[29] SUGIURA, M. *Representation of compact groups realized by spherical functions on symmetric spaces*. Proc. Japan Acad., 38 (1962) 111-113.

[30] URAKAWA, H. *The First Eigenvalue of the Laplacian for a Positively Curved Homogeneous Riemannian Manifold*. Compositio Mathematica, tome 59, Number 1, 1986, p. 57-71.

[31] WANG, M.; ZILLER, W. *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math., 84 (1986), p. 171-194.

[32] WARNER, F. W. *Foundations Of Differentiable Manifolds And Lie Groups*. Sccot, Foresman and Company, Glenview, Illinois, 1971.

[33] YAMAGUCHI, S. *Spectra of Flag Manifolds*. Memois of the Faculty of Science, Kyushu University, Ser. A, Vol. 33, Number 1, 1979.