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Weak and Measure-valued solutions to the Euler and ideal MHD Equations

Soluções fracas e a valores de medidas para as Equações de Euler e da Magnetohidrodinâmica Ideal

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Supervisor: Anne Caroline Bronzi

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Resumo

Nesta dissertação estamos interessados no estudo de noções fracas de solução para algumas equações diferenciais parciais que descrevem fenômenos da Dinâmica de Fluidos, especificamente as equações de Euler para fluidos incompressíveis ideais, e o sistema das equações da Magnetohidrodinâmica ideal, que governam o movimento de fluidos incompressíveis ideais carregados eletricamente, com condutividade perfeita.

Os principais objetos estudados são o método de integração convexa utilizado para obter a não-unicidade de soluções fracas para ambas as EDPs, assim como a abordagem que permite tratar de medidas parametrisadas como soluções muito fracas para estes sistemas. Adaptando um conceito de solução a valor de medidas das equações de Euler para o sistema MHD ideal, pudemos obter um resultado de existência global para o caso completamente tridimensional, assim como um resultado de unicidade fraca-forte para soluções satisfazendo uma condição de simetria planar.

Palavras-chave: Dinâmica dos Fluidos; Soluções Fracas (Matemática); Equações de Euler; Magnetoidrodinâmica; Medidas de Young.

Abstract

In this thesis we are concerned with weaker notions of solutions to some partial differential equations which describe phenomena in the field of fluid dynamics, specifically the incompressible Euler equations for ideal fluids and the ideal Magneto-hydrodynamic system of equations governing the motion of ideal incompressible fluids that are electrically conductive.

The main objects of study are the method of convex integration to obtain non-uniqueness of weak solutions to both PDE systems, as well as the framework for treating parametrised measures as very weak solutions to these equations. By adapting a concept of measure-valued solution from the Euler to the ideal MHD system, we are able to obtain a global existence result for the full $3D$ system, and a weak-strong uniqueness result for solutions satisfying a planar symmetry condition.

Keywords: Fluid dynamics; Weak solutions (Mathematics); Euler Equations; Magnetohydrodynamics; Young Measures.

List of symbols

We denote by $A : B = \sum_{i,j} A_{ij} B_{ij}$ the inner product of two matrices in $\mathbb{R}^{d\times d}$, and the tensor product of two vectors in $\mathbb{R}^d$ by $a \otimes b$, meaning the $d \times d$ matrix whose (i, j)-th entry is $a_i b_j$. Moreover, we denote $v \circ v = v \otimes v - \frac{1}{d}|v|^2$, where v is a vector in $\mathbb{R}^d$. Note that $v \circ v$ is a $d \times d$ symmetric matrix with zero trace. We denote the space of $d \times d$ symmetric matrices by $\mathcal{S}^d$, and the subspace of traceless matrices by $\mathcal{S}^d_0$. If $\phi : X \to \mathbb{R}^{d\times d}$ is a matrix-valued function we denote by div ϕ the vector field defined by $(\text{div } \phi)_i = \sum_j \partial_j \phi_{ij}$.

Let X be a locally compact Hausdorff space. We will typically be interested in cases where X is an open or closed subset of $\mathbb{R}^n$, and in this case it will inherit the subspace topology from it. For a measurable set E, the characteristic function of the set is denoted by χ_E, and is defined as being 1 for points of E and zero elsewhere. For a function $f : X \to \mathbb{R}^d$ its support, which is defined as the closure of the set where it is non-zero, is denoted by supp f. We refer to the space of compactly supported continuous functions on X as $C_c(X)$, and $C_0(X)$ is defined as the closure of $C_c(X)$ in the topology induced by the supremum norm, or, equivalently, the space of functions f for which the set $|f|^{-1}([\varepsilon, \infty))$ is compact for every $\varepsilon > 0$. $BC(X)$ is the set of all bounded continuous functions defined on X.

We define the space $M(X)$ of Radon measures with finite total variation in X, as well as the subsets $M^+(X)$ of non-negative measures and $M^1(X)$ of probability measures, i.e., the non-negative measures with unit mass. Now, given a $\mu \in M^1(X)$ and f μ-integrable, we denote its integral $\int_X f \ d\mu$ by the pairing $\langle \mu, f \rangle$. We can use the Riesz Representation Theorem (Theorem 7.17 on [14]) to identify $M(X)$ with the space of bounded linear functionals on $C_0(X)$. As a dual space it can be endowed with a weak* topology, meaning a sequence (μ_k) of measures converges weakly* to a measure μ whenever the limit

$$\int_X \phi d\mu_k = \langle \phi, \mu_k \rangle \to \langle \phi, \mu \rangle = \int_X \phi d\mu$$

holds for every $\phi \in C_0(X)$. Moreover, if (X, μ) and (Y, ν) are measure spaces we define the product measure on $X \times Y$ by $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$, for measurable sets $A \subset X$ and $B \subset Y$.

Let Y be a normed vector space. For $1 \leq p \leq \infty$ let $L^p(X; Y, \mu)$ be the set of

μ-equivalence classes of maps $f : X \to Y$ for which $\|f\|_Y$ is an $L^p(X, \mu)$ function, and we usually omit the measure and/or Y when it refers to Lebesgue measure on $\mathbb{R}^n$, or when it is well understood from context. One such class of spaces is of special interest to us, which are the spaces $L^\infty_t L^p_x = L^\infty([0, T]; L^p(\mathbb{R}^d))$, and the related $CL^2_w = C([0, T]; L^2_w(\mathbb{R}^d))$ of functions f which are weakly continuous in time and square-integrable in space, meaning that the function

$$t \mapsto \int_{\mathbb{R}^d} \phi(x) f(x, t) dx$$

is continuous for each $\phi \in L^2$. Here the weak topology on L^2 is a particular case of the general identification of $L^q(X)$ with the dual space of $L^p(X)$ for $1 \leq p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. We refer to the weak topology on $L^p(X)$ and the weak* topology on L^∞ as given by these identifications. Finally, for $1 \leq p \leq \infty$ we say a function f is in L^p_{loc} if for every compact set $K \subset X$ the function $f\chi_K$ is in L^p, and (weak) convergence in this space is defined as L^p (weak) convergence for each K.

Contents

Chapter 1

Introduction

1.1 Organisation of the thesis

In the thesis we will be concerned with some relaxed notions of solution to two widely studied partial differential equations in the fields of Fluid Dynamics and Mathematical Analysis, that is, the incompressible Euler equations governing the motion of inviscid incompressible fluids, and the ideal Magneto-hydrodynamic (MHD) equations governing electrically charged fluids. For the ideal MHD equations in $3D$ space, we are also particularly interested in a symmetry-reduced system which bears close resemblance to the $2D$ Euler equations with a tracer, as well as the Navier-Stokes equations for viscous incompressible fluids, where viscosity is embodied by the constant $\nu > 0$ and $\nu = 0$ reduces to the Euler equations; and the viscous, resistive MHD, where similarly, we have the constants $\nu, \mu > 0$ that upon vanishing reduce the system to the ideal MHD. We introduce these equations and some of their properties, and remark on some well-known results regarding classical solutions, their existence and/or uniqueness, and open problems regarding these questions.

In Chapter 2, we are interested in the theory of *weak solutions* to these equations. One can interpret the weak formulation as a way to impose only that functions solve the solutions *on average*. This relaxes the conditions imposed on solution candidates, so as to widen the class of functions to be tested against. Nevertheless, for this weaker notion of solution we are still interested in existence and/or uniqueness (or the lack thereof). One should expect, for these wider spaces of functions, that we could more easily obtain global existence results through methods arising from a typical functional analytic framework.

Alongside this, however, the question of uniqueness will be made significantly harder, and in fact false. This problem can be somewhat remedied by requiring only *weak-strong uniqueness*, that is, that weak solutions must agree with classical solutions whenever the latter exist. We expect to achieve some kind of uniqueness result by adding conditions to the weak solutions, that we know the classical ones to obey. One such class of conditions are energy inequalities, which we will employ to some extent. Nevertheless, we show that requiring simply that a weak solution's energy be bounded, we already gain the property of weak continuity in time, so that we can at least refer to a weak solution as an almost everywhere (in space) well-defined function at each time, which is particularly important in the inclusion of initial data for the weak formulation.

One successfull application of energy inequalities alongside the weak formulation concerns the global existence for a class of solutions to the Navier-Stokes equations, which are called Leray-Hopf solutions. On the other side, we refer the reader to the work of C. De Lellis and L Székelyhidi Jr. in [9], which shows that energy inequalities are generally not sufficient to single out Euler weak solutions. We proceed to show one possible strategy for producing weak solutions to Euler, by taking sequences of Leray-Hopf solutions to the Navier-Stokes equations with the viscosity $\nu \to 0$. We will look at some of the possible consequences of this limit process, and how it motivates the weaker notions of solutions we will later be studying.

After defining weak solutions and looking at some properties they should possess, we proceed in section 2 of Chapter 2 to obtain their non-uniqueness through the method of convex integration. With origin in the geometrical Nash-Kuiper isometric embedding theorem, this method produces somewhat paradoxical weak solutions and was first adapted to the framework of PDE systems of conservation laws by C. De Lellis and L. Székelyhidi in [8], and has since been succesfully employed in many problems of this area, not limited simply to non-uniqueness results (see P. Isett's work [16] for its application on solving the negative direction of the Onsager conjecture).

First we introduce a more general framework to this method, that of a differential inclusion: One separates the linear differential constraints from the pointwise non-linear ones on a PDE, and from certain well-behaved solutions to the linear equations that satisfy a relaxed form of non-linear constraint (subsolutions), one proceeds to show that there exist functions satisfying both the linear differential and strict non-linear constraint simultaneously. This is achieved by showing that the addition of certain highly oscillatory solutions to the linear problem (therefore weakly but not strongly convergent) can approximate toward the strict non-linear constraint. Such constraints can be selectively defined, so as to obtain weak solutions satisfying certain pre-set conditions.

Specifically, we first show how the method can be applied in "directly" obtaining compactly-supported (in space-time) weak solutions to the planar symmetry MHD equations, as done by A. Bronzi, M. Lopes Filho and H. Nussenzveig Lopes in [5]. Since this implies the existence of non-trivial solutions to a stationary initial data, as in a fluid that can spontaneously start and stop moving without any external interference, these solutions are clearly non-physical, and obviously do not conserve energy. This is known as the Scheffer-Shnirelman paradox, named after the two researchers that independently obtained such weak solutions (not through the convex integration method, see [21, 22]).

These sort of solutions are commonly called *wild solutions*. We remark here that, following the recent work by D. Faraco, S. Lindberg and L. Székelyhidi in [13], significant advance has been made in the framework of convex integration for the full $3D$ ideal MHD.

Then, we proceed to use the convex integration framework in the context of the Euler equations, following the work in [9, 25], in which the authors prove that under certain hypotheses there exist infinitely many weak solutions to the Euler arising from given initial data, and presenting a given energy profile. In this case, instead of directly obtaining weak solutions we apply a Baire Category argument to show that the set of such solutions is residual (thus dense) in a purposely defined function space. Any initial data for which this result can be applied (which, in the face of weak-strong uniqueness must exclude smooth functions) is known as *wild* initial data. In [25], the author goes on to prove how the set of wild initial data for which energy conserving weak solutions can be produced is in fact dense in the strong topology of $L^2(\mathbb{R}^d)$, and also shows some cases of functions in this set.

In Chapter 3, we look at another way to treat the limits arising from sequences of approximate weak solutions, such as vanishing viscosity limits. One way to better understand these sequences is to embed them into a larger space, and obtain in the limit a new object, weaker but with more information than a simple weak limit, which ignores oscillatory and other behavior. We introduce the concept of Young measure, meaning a measure parametrised in space-time, which describes the pointwise behaviour of a sequence bounded in L^∞. Oscillatory phenomena in the sequence will appear in the Young measure as any non-Dirac nature.

However, in sequences arising from the equations we are concerned with, boundedness in L^∞ is not the natural thing to ask, but the condition arising from the weak energy inequality, meaning a uniform in time bound of the L^2 norm in space. This condition however allows for more than simply oscillations, and concentration (controlled blow-up) behavior becomes a problem we must also look at in studying the behavior of these sequences. This motivates the notion of generalised Young Measure introduced by R. DiPerna and A. Majda in [11] and further developed by J. Alibert and G. Bouchitté in [2], which we introduce and study some of its properties.

In Chapter 4, we show how the theory of generalised Young Measures can be applied in the definition of measure-valued solutions to systems of partial differential equations. First, we follow along the steps of L. Székelyhidi and E. Wiedemann in [23], which shows how the weak formulation for Euler can be extended to generalised Young measures. Then, we present some of the interesting consequences of requiring that solutions of this kind should arise from bounded energy sequences of weak solutions. Among them, we will see that the energy of such a measure-valued solution is well-defined at each time, and also that weak continuity (in a sense) can also be recovered, so that the inclusion of initial data in the measure-valued formulation can be done accordingly. If we additionally require that the measure-valued solution satisfy a weak energy inequality, we obtain the notion of admissible measure-valued solution, which will imply continuity of the solution at initial time.

We then proceed to show that the notion of measure-valued solution to the Euler equations satisfies some of the conditions for vindicating this (very) relaxed formulation, i.e. we show the global existence (by R. DiPerna and A. Majda in [11]) and weak-strong uniqueness (by Y. Brennier and C. De Lellis in [4]) of admissible measure-valued solutions. To finish, we recall the result of E. Wiedemann in [25], that every bounded energy measure-valued solution in fact does arise as the limit of a sequence of weak solutions.

Finally, we adapt the framework of (admissible) measure-valued solutions to the ideal MHD system, for which the properties of weak continuity and well-defined, bounded energy are promptly guaranteed, as an original result. Furthermore, we obtain for the full $3D$ system the global existence of admissible measure-valued solutions, from the Leray-Hopf theory for the viscous, resistive MHD, and for the planar symmetry-reduced system a weak-strong uniqueness.

1.2 The Euler and Ideal MHD equations

The Incompressible Euler Equations

Let us start by presenting the Euler equations describing the motion of an ideal incompressible fluid in d-dimensional space, for $d \geq 2$. For such a fluid subject to no external forces, its velocity $v : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and scalar pressure $\pi : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ are expected to obey the following nonlinear system of partial differential equations:

$$\begin{aligned} \partial_t v + (v \cdot \nabla) v + \nabla \pi &= 0 \\ \text{div } v &= 0. \end{aligned} \tag{1.1}$$

The non-linear term $(v \cdot \nabla)v$ can be written also as div $(v \otimes v)$, if we use the fact that v is divergence-free. The expression div $(v \otimes v)$ denotes the $d \times d$ matrix with entries $v_i v_j$, and the divergence is taken row-wise. Such fluids are named *ideal*, in contrast with *viscous* fluids, for which the effects of friction arising from the motion of the fluid cannot be disregarded. For a viscous incompressible fluid, viscosity is assumed uniform throughout the domain, and apart from dimensional considerations is embodied by the constant $\nu > 0$, whose inverse is commonly known as the *Reynolds* number $Re = \frac{1}{\nu}$. Such fluids are then governed by the Navier-Stokes equations:

$$\begin{aligned} \partial_t v + (v \cdot \nabla)v + \nabla\pi &= \nu\Delta v \\ \text{div } v &= 0. \end{aligned} \tag{1.2}$$

For fluids inside a domain $\Omega \subset \mathbb{R}^d$, boundary conditions must also be imposed, and these differ depending on which equation is considered. For the Euler equations, we impose the condition that fluid does not flow through the boundary $\partial\Omega$. That is, if $n(x)$ denotes the outward unit normal to $\partial\Omega$ at x, we must have

$$v(x,t) \cdot n(x) = 0 \text{ on } \partial\Omega.$$

Both the Euler and Navier-Stokes equations are obtained through the application of physical principles such as conservation of momentum and continuum conditions, as well as assumptions regarding the action and interaction of forces acting on the fluid's interior. A comprehensive deduction of these equations is available in [6].

The Cauchy problem for the Euler equations consists of finding functions v and p that solve (1.1) and also satisfy, for a given divergence-free vector field v_0, the initial condition

$$v(\cdot, 0) = v_0.$$

A fundamental quantity in the study of the Euler equations is the *kinetic energy*:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} |v(x,t)|^2 dx.$$

In fact, if v, π are a smooth solution to the Euler equations, and v decays sufficiently rapidly at infinity, we can show that kinetic energy is conserved, through the following calculation:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |v|^2 dx &= \int_{\mathbb{R}^d} v \cdot \partial_t v \ dx \\ &= -\int_{\mathbb{R}^d} v \cdot \text{div } (v \otimes v) dx - \int_{\mathbb{R}^d} v \cdot \nabla\pi \ dx \\ &= -\int_{\mathbb{R}^d} \sum_{i,j} v_i v_j \partial_j v_i \ dx + \int_{\mathbb{R}^d} p\text{div } v \ dx = 0. \end{aligned} \tag{1.3}$$

By using integration by parts and the fact that div $v = \sum \partial_j v_j = 0$, we get

$$\int_{\mathbb{R}^d} \sum_{i,j} v_i v_j \partial_j v_i \ dx = -\int_{\mathbb{R}^d} \sum_{i,j} \partial_j v_i v_j v_i \ dx,$$

which assures us that both terms in (1.3) are null. One can similarly obtain for smooth solutions of the Navier-Stokes equations the following identity

$$\frac{1}{2}\int_{\mathbb{R}^d} |v(x,t)|^2 \, dx + \nu \int_0^t \int_{\mathbb{R}^d} |\nabla v(x,t)|^2 \, dxdt = \frac{1}{2}\int_{\mathbb{R}^d} |v_0|^2 \, dx.$$

As can be seen in [19], for sufficiently regular initial data, existence of smooth solutions for both the Euler and Navier-Stokes equations can be guaranteed up to a time $T > 0$ which depends on some Sobolev norm of the initial data, as well as uniqueness for this same time interval. However, for dimension 3 the matter of global in time existence of smooth solutions, or finite time singularity formation, is still an open problem for both equations, regardless of regularity in the initial data. For the Navier-Stokes equations in particular, this is one of the Millenium problems proposed by the Clay institute. However, for dimension 2, we have a positive answer, thanks to a natural L^∞ bound for the *vorticity* ω for local solutions. That is, defining the vorticity as

$$\omega = \text{rot } v = \nabla \times v,$$

we can apply the curl to the Euler equations to see that vorticity is only transported by v, thus obeying a L^∞ form of maximum principle. One can then apply the Beale-Kato-Majda criterion (Theorem 2.2 in [18]) to obtain global in time existence and regularity in 2 dimensions.

The Ideal Magnetohydrodynamic Equations

Another system of PDEs that interests us in this dissertation are the ideal incompressible Magneto-Hydrodynamic equations (MHD). These equations model the motion of electrically charged and conductive fluids, such as plasmas, liquid metals and electrolyte solutions, with applications ranging from chemical processes to studying the dynamics of solar flares and the Earth's magnetic field.

This model can be deduced by coupling the equations governing fluid dynamics to the Maxwell equations of electromagnetism. As with the Euler equations, we are mainly concerned with the case of incompressible ideal fluids with negligible resistivity. Through Ohm's Law, we can obtain the electric field from the velocity and magnetic fields, so that the main quantities characterizing the motion of the fluid at a given point $(x,t) \in \mathbb{R}^d \times [0,T]$ for $d = 2,3$, are the pressure $\pi(x,t) \in \mathbb{R}$, the velocity $v(x,t) \in \mathbb{R}^d$ and the magnetic field $B(x,t) \in \mathbb{R}^d$. Initially considering viscosity ($\nu > 0$) and resistivity ($\mu > 0$), the MHD equations take the form

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - (B \cdot \nabla)B + \nabla\pi = \nu\Delta v \\ \partial_t B + (v \cdot \nabla)B - (B \cdot \nabla)v = \mu\Delta B \\ \text{div } v = \text{div } B = 0. \end{cases} \tag{1.4}$$

If $\nu, \mu = 0$, we get the *ideal MHD* equations:

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - (B \cdot \nabla)B + \nabla\pi = 0 \\ \partial_t B + (v \cdot \nabla)B - (B \cdot \nabla)v = 0 \\ \text{div } v = \text{div } B = 0. \end{cases} \tag{1.5}$$

For these equations, the associated energy is given by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} |v|^2(x,t) + |B|^2(x,t)\ dx.$$

Note that taking $B = 0$ gets us back to the Euler equations.

The MHD Equations with Planar Symmetry

We will also look at specific solutions to (1.5) in $\mathbb{R}^3$, which satisfy the symmetries:

$$\begin{aligned} v = v(x,t) &= (v_1(x,t), v_2(x,t), 0), \\ B = b(x,t) &= (0, 0, b(x,t)) \\ (x,t) &= (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}. \end{aligned}$$

The system is then reduced to

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla(\pi + \frac{|b|^2}{2}) = 0 \\ \partial_t b + (v \cdot \nabla)b = 0 \\ \text{div } v = 0. \end{cases} \tag{1.6}$$

which can be also be interpreted as the $2D$ incompressible Euler equations with a passive tracer (the role of b), if we redefine the pressure $\bar{\pi} = \pi + \frac{|b|^2}{2}$. Despite the apparent $2D$ nature of the equations, solutions to this system will be solutions to the full $3D$ MHD, and not to a $2D$ version, and due to this it has been referred to in the literature a $2.5D$ MHD model.

Chapter 2

Non-uniqueness of weak solutions

2.1 Weak Solutions

A natural way of defining weaker notions of solutions is through integration: One can visualize this as requiring only that the equation be satisfied "on average". That is, integrating the equations multiplied with a smooth test function and formally applying integration by parts and other techniques, we can relax regularity assumptions on the solution candidates, by leaving the "burden" of regularity to fall on the test function.

Specifically for the Euler equations, suppose initially that v, π is a smooth solution of (1.1), and take $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ a divergence-free vector field. Multiplication by ϕ, integration in $\mathbb{R}^d \times [0,T]$, and integration by parts allow us to see that v satisfies

$$\iint (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dxdt = 0. \tag{2.1}$$

Similarly, for a scalar function $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$, the divergence-free condition on v gives us

$$\iint v \cdot \nabla \psi \; dxdt = 0, \tag{2.2}$$

and we say any function satisfying this specific condition is *weakly divergence-free*.

We can see that the integral equations above make sense for a much wider range of functions. In fact, these integrals are finite for any locally p-integrable function v. With this in mind, we say that $v \in L^2_{loc}(\mathbb{R}^d \times [0,T])$ is a *weak solution* of the Euler equations if (2.1) and (2.2) hold for every divergence-free $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T); \mathbb{R}^d)$, and for every $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$.

However, notice that the pressure π is eliminated in (2.1) due to the divergence-free requirement on ϕ. We can in fact recover the pressure, given a weak solution v to

(2.1) and (2.2). That is, we can produce a distribution π such that v and π together satisfy (1.1) in the weak sense, that is, v satisfies (2.2) and also

$$\iint v \cdot \partial_t \phi + v \otimes v : \nabla \phi + \pi \text{ div } \phi \, dxdt = 0$$

holds for every $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T); \mathbb{R}^d)$. In fact, π is obtained as a distributional solution to the equation $-\Delta \pi = \text{div div } (v \otimes v)$. The nature of this non-linear equation, which allows us to express π through a zero-order integral operator applied to $v \otimes v$, also allows us to control the growth of π in terms of v. Specifically, we have the following result:

Theorem 2.1.1 (Lemma 5.1 in [20]). Suppose that for $d = 3$ we have π, v satisfying the equation

$$-\Delta \pi = \text{div div } v \otimes v$$

If $\pi \in L^q$ for some $1 \leq q < \infty$, then the following estimates hold for every $1 < r < \infty$:

$$\|\pi\|_{L^r} \leq C_r \|v\|_{L^{2r}},$$

and

$$\|\nabla \pi\|_{L^r} \leq C_r \|(v \cdot \nabla) v\|_{L^r},$$

for some constant $C^r > 0$.

Since we can look for weak solutions in a much wider class of functions, one expects the question of existence to become much easier, while uniqueness may become a problem. In fact, in the next section we will study some non-uniqueness results for weak solutions to both the Euler and the MHD equations, and in chapter 4 we will see how one can still take advantage of this non-uniqueness to obtain results about an even weaker notion of solution.

We can also repeat this process to define weak solutions to the Navier-Stokes equations. In fact, it is a well-known result that for any divergence-free $v_0 \in L^2$ we can obtain the existence of weak solutions (called Leray-Hopf solutions) $v \in L^\infty(0,T; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d))$ which also satisfy the *strong energy inequality*:

$$\frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 \, dx + \nu \int_s^t \int_{\mathbb{R}^d} |\nabla v(x,\tau)|^2 \, dx d\tau \leq \frac{1}{2} \int_{\mathbb{R}^d} |v(x,s)|^2 \, dx, \tag{2.3}$$

for almost every $s, t \in [0,T]$ for which $s \leq t$. For the Euler equations, we can similarly include energy conditions, for example, in the form of a strong energy inequality of the form

$$\frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v(x,s)|^2 \, dx.$$

for almost every $s, t \in [0, T]$ for which $s \leq t$. The weak form of these energy inequalities arises from setting $s = 0$. More generally, we will be looking for solutions which satisfy the minimal requirement of *bounded energy*, i.e. for $u \in L^\infty_t L^2_x = L^\infty([0, T]; L^2(\mathbb{R}^d))$.

We can similarly define weak solutions for the MHD equations (1.4) and (1.5). Let $v, B \in L^2_{loc}(\mathbb{R}^d \times [0, T))$. We will say that v, B is a *weak solution* to the ideal MHD equations if they are weakly divergence-free and satisfy

$$\int_0^T \int_{\mathbb{R}^d} [v \cdot \partial_t \varphi + (v \otimes v - B \otimes B) : \nabla \varphi] \, dx dt = 0 \tag{2.4}$$

$$\int_0^T \int_{\mathbb{R}^d} [B \cdot \partial_t \varphi + (B \otimes v - v \otimes B) : \nabla \varphi] \, dx dt = 0 \tag{2.5}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^d \times [0, T))$ com div $\varphi = 0$.

With regards to the pressure, an analogous result to Theorem 2.1.1 can be recovered if bounds on some p-norm of both v and B can be secured, since it must obey the equation

$$-\Delta \pi = \operatorname{div} \operatorname{div} (v \otimes v - B \otimes B).$$

Since it is also expected of classical solutions to the ideal MHD equations that they conserve energy, we are also interested, in the very least, in solutions with bounded energy, $v, B \in L^\infty_t L^2_x$. Furthermore, if we extend the analogy to include viscous, resistive fluids, one can also obtain Leray-Hopf-like solutions $v, B \in L^\infty_t L^2_x \cap L^2_t H^1_x$ satisfying the corresponding strong energy inequality:

$$E(t) + \nu \int_s^t \|\nabla v(\cdot, \tau)\|^2_{L^2} d\tau + \mu \int_s^t \|\nabla B(\cdot, \tau)\|^2_{L^2} d\tau \leq E(s). \tag{2.6}$$

Finally, we can define weak solutions for the $2.5D$ MHD system in the following manner: $(v, b) \in L^2_{loc}(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R}^2 \times \mathbb{R})$ is a *weak solution* to (1.6) if v is weakly divergence-free and (v, b)

$$\iint (v \cdot \partial_t \psi + (v \otimes v) : \nabla \psi) dx dt = 0 \tag{2.7}$$

$$\iint (b \cdot \partial_t \varphi + (bv) \cdot \nabla \varphi) dx dt = 0, \tag{2.8}$$

hold for every $\psi \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$ with div $\psi = 0$, and for every $\varphi \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R})$.

2.1.1 Weak continuity in time

Now, if one pays attention to the definitions of weak solutions we have been working with so far, one issue that becomes pressing is the question of how to insert the initial data into the formulation. Say we are looking for a weak solution to Euler in the domain $\mathbb{R}^d \times [0,T)$ and require that $v \in L^2_{loc}(\mathbb{R}^d \times [0,T))$. Inclusion of initial data v_0 into the formal calculations involved in deduction of the weak formulation gives us that v should be weakly divergence-free and satisfy, for any divergence-free $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$, the expression

$$\iint_{\mathbb{R}^d \times [0,T)} (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dxdt + \int_{\mathbb{R}^d} v_0(x)\phi(x,0)dx = 0. \tag{2.9}$$

However, it is not clear, even with this requirement, in what sense the solution v assumes the initial value, or whether if it does at all. Truly, for $v \in L^2(\mathbb{R}^d \times [0,T))$, $v(\cdot,t)$ is only well-defined as a $L^2(\mathbb{R}^d)$ function for *almost every* $t \in [0,T)$. And in fact, many results have been obtained in which weak solutions to this type of equations are produced to be discontinuous at initial time, e.g. solutions for which $\liminf_{t\to 0} \|v(\cdot,t)\|_{L^2} > \|v_0\|_{L^2}$ (see Theorem 2.18 in [25]). Nevertheless, we will show in the following lemma that imposing the condition that a weak solution has essentially bounded energy, it is possible to make sense of $v(\cdot,t)$ as a L^2 function for every time. Furthermore, we obtain that $v(\cdot,t)$ will converge weakly to the initial data in L^2.

Proposition 2.1.1 (Weak continuity in time)**.** Let $v \in L^\infty(0,T;L^2(\mathbb{R}^d))$ be a weak solution to Euler. Then there exists a representative $\tilde{v} \in CL^2_w = C([0,T];L^2_w(\mathbb{R}^d))$, i.e. $\tilde{v}(\cdot,t) = v(\cdot,t)$ as L^2 functions for almost every $t \in [0,T]$.

Proof. Using the Helmholtz Decomposition, density of C_c^∞ in L^2 and separability of L^2, we can produce a sequence $(\phi_i + \nabla p_i)_{i\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$, satisfying:

(*i*) $\phi_i \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ with div $\phi_i = 0$ weakly;

(*ii*) $p_i \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$;

(*iii*) $\{\phi_i + \nabla p_i\}_{i\in\mathbb{N}}$ is dense in $L^2(\mathbb{R}^d;\mathbb{R}^d)$.

Also, take $\chi \in C_c^\infty(0,T)$ and define Φ_i by

$$\Phi_i(t) := \int_{\mathbb{R}^d} (\phi_i(x) + \nabla p_i(x)) \cdot v(x,t)dx = \int_{\mathbb{R}^d} \phi_i(x) \cdot v(x,t)dx,$$

since div $v = 0$ weakly, and take $\chi\phi_i$ as a test function in (2.1) to obtain for every $\chi \in C_c^\infty((0,T))$ the identity:

$$\begin{aligned}\int_0^T \partial_t\chi(t)\Phi_i(t) &= \int_0^T \partial_t\chi(t) \int_{\mathbb{R}^d} \phi_i(x) \cdot v \, dxdt \\ &= -\int_0^T \chi(t) \int_{\mathbb{R}^d} \nabla\phi_i : v \otimes v \, dxdt\end{aligned}$$

which implies that the function defined on $[0,T]$ by

$$t \mapsto \int_{\mathbb{R}^d} (\nabla \phi_i(x) : v(x,t) \otimes v(x,t) dx$$

is a weak derivative for Φ_i. This means we can estimate

$$\begin{aligned} \int_0^T |\Phi_i'|(t) dt &\leq \int_0^T \int_{\mathbb{R}^d} |\nabla \phi_i| \cdot |v \otimes v| \; dxdt \\ &\leq \|v \otimes v\|_{L_t^\infty L_x^1} \|\nabla \phi_i\|_{L_t^1 L_x^\infty} \leq C \|v\|_{L_t^\infty L_x^2} \|\nabla \phi_i\|_{L_t^1 L_x^\infty} < \infty. \end{aligned}$$

This means $\Phi_i' \in L^1([0,T])$, which assures us, by the Lebesgue Differentiation Theorem, that there exists $\tilde{\Phi}_i$ absolutely continuous on $[0,T]$, such that $\tilde{\Phi}_i(t) = \Phi_i(t)$ for almost every $t \in [0,T]$. Since this holds for every $i \in \mathbb{N}$, and the countable union of null sets is a null set, it follows that there exists a null set $\mathcal{N} \subset [0,T]$ for which

$$\tilde{\Phi}_i(t) = \int_{\mathbb{R}^d} (\phi_i(x) + \nabla p_i(x)) \cdot v(x,t) dx$$

for every $i \in \mathbb{N}$, $t \in [0,T] \setminus \mathcal{N}$. Moreover, continuity of $\tilde{\Phi}_i$ gives us

$$|\tilde{\Phi}_i(t)| \leq \|v\|_{L_t^\infty L_x^2} \|\phi_i + \nabla p_i\|_{L_x^2} \text{ for all } t \in [0,T],$$

so that, for every $t \in [0,T]$, the function $\tilde{\Phi}_i(t)$ defines, from the density of $\{\phi_i + \nabla p_i\}$ in L_x^2, a bounded linear functional in L_x^2, which we denote $\mathcal{L}_t$, given by the formula $\mathcal{L}_t(\phi_i + \nabla p_i) = \tilde{\Phi}_i(t)$. The Riesz representation theorem guarantees then that for each $t \in [0,T]$, there exists a function $\tilde{v}(\cdot,t) \in L_x^2$, which coincides with v for all $t \in [0,T] \setminus \mathcal{N}$, and satisfies

$$\|\tilde{v}(\cdot,t)\|_{L_x^2} \leq \|v\|_{L_t^\infty L_x^2} \text{ for all } t \in [0,T],$$

as well as

$$\tilde{\Phi}_i(t) = \int_{\mathbb{R}^d} (\phi_i + \nabla p_i) \cdot v \; dx \text{ for all } t \in [0,T].$$

Lastly, take $\psi \in L_x^2$, and a subsequence $\phi_k + \nabla p_k \xrightarrow{k\to\infty} \psi$ strongly in L^2. Setting $\Psi := \int_{\mathbb{R}^d} \tilde{v} \cdot \psi \; dx$, it follows that

$$|\tilde{\Phi}_k(t) - \Psi(t)| \leq \|v\|_{L_t^\infty L_x^2} \|\phi_k + \nabla p_k - \psi\|_{L_x^2} \to 0 \text{ uniformly in } t \in [0,T],$$

So that Ψ is the uniform limit of continuous functions $\tilde{\Phi}_k$. Since $\psi \in L_x^2$ is arbitrary, it follows that $\tilde{v} \in C([0,T]; L_w^2)$. □

We will see later that this procedure can be applied to other equations, as well as other notions of solution. For now, we just state that it remains valid for weak solutions to the ideal MHD provided we require that both v and B be of bounded energy.

Note that this property assures us that weak solutions defined by conditions such as (2.9) will assume the initial data in the sense of weak convergence.

2.1.2 The vanishing viscosity limit and weaker solutions

It is common in PDE research to try to obtain solutions to a given system by introducing a more well-behaved system of equations, obtaining solutions to the modified equation, and then attempting to recover a solution to the original problem via a limit of modified solutions. This includes a wide variety of methods from the standard mollification arguments, through numerically useful iterative process such as Galerkin approximations used in simulations, and including the convex integration method for obtaining non-physically relevant wild solutions.

One such method arises from the so-called vanishing viscosity approximation, in which one seeks to obtain solutions to the Euler equations arising from solutions of the Navier-Stokes equations, as the viscosity ν tends to 0 (or conversely, as the Reynolds number blows up). For smooth initial data, whence when regular solutions exist for a while, this procedure has been found to produce sequences of solutions which converge *strongly* to a (local in time) solution of the Euler equations. However, numerical simulations have indicated that turbulence plays a large role in increasing the complexity of the flow as time evolves, so that turbulent behavior such as energy cascades leading to high-frequency oscillations may prevent the strong convergence of these sequences after some critical time.

Possible ways around this problem could be to analyze the weak limits of such solutions, which tend to ignore this kind of behavior. To this effect, take initial data $v_0 \in L^2$, a sequence $\nu_k \to 0$ and for each k, a Leray-Hopf solution v_k to the Navier-Stokes equations with viscosity ν_k, and let us look at the sequence (v_k). The energy inequality (2.3) assures us that (v_k) is uniformly bounded in $L^\infty_t L^2_x$, so that the Banach-Alaoglu Theorem gives us (after relabeling a subsequence) the weak* convergence to a limit v, which is also weakly divergence-free. However, the non-linear term $v \otimes v$ prevents us from concluding that the weak limit v is a weak solution to the Euler equations, because only from the weak* convergence we cannot conclude that

$$v_k \otimes v_k \overset{*}{\rightharpoonup} v \otimes v.$$

In fact, oscillations and concentrations can arise in the sequence which destroy the weak convergence of the tensor product. We can, however, obtain the weak* convergence of $v_k \otimes v_k$ in the space $L^\infty(0, T; L^1(\mathbb{R}^d, \mathcal{S}^d_0))$ to a symmetric matrix field u, so that the pair (v, u) satisfies

$$\iint_{\mathbb{R}^d \times [0,T]} (\partial_t \phi \cdot v + \nabla \phi : u) dx dt = 0.$$
$$\iint_{\mathbb{R}^d \times [0,T]} \nabla \psi \cdot v dx = 0$$

for every divergence-free $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T); \mathbb{R}^d)$ and scalar $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$. This is equivalent to saying that, for some distribution q, the triple (v, u, q) is a weak solution to the linear system

$$\begin{cases} \partial_t v + \text{div } u + \nabla q = 0 \\ \text{div } v = 0 \end{cases} \tag{2.10}$$

One can then take at least two approaches to the problem of the failure of weak convergence in the non-linear term. One is to attempt to work with solutions of the linear system above, and from them build a weaker notion of solution, attempting to work with it to gain insight into the structure of the original equation. We will see in the next section how this approach can be useful in obtaining results about proper weak solutions. The other possibility is to weaken even further the form of convergence and obtain another object as a limit, which however has more information on the behavior of the solution. The notion of measure-valued solution developed in chapters 3 and 4 follows along this line.

2.2 Convex Integration

2.2.1 Subsolutions

The notion of subsolution was introduced, for the Euler equations, by C. DeLellis and L. Székelyhidi in [8], inspired by the notion of differential inclusions, and has been successfully utilized in several non-linear PDEs as a powerful tool to obtain results related to weak solutions, for example the non-uniqueness [8], and the unsuitability of energy inequalities as selection principles for weak solutions (see [9]). It is intimately connected with the method of convex integration, which was introduced earlier in connection to the Nash-Kuiper isometric embedding theorem. For an overview of the history of this method from its origins in geometrical problems to applications in fluid dynamics, we recommend [10]. For a very general framework for application of the convex integration method to non-linear partial differential equations, we refer to [7].

The notion of a differential inclusion consists essentially in separating the non-linear nature from the differential structure of the PDE. Specifically, we ask that a non-linear PDE can be written as a system of first order PDEs

$$\sum_{i=1}^{m} \partial A_i \partial_i z = 0 \tag{2.11}$$

coupled with a pointwise non-linear constraint

$$z(y) \in K(y) \subset \mathbb{R}^l,$$

where $z : \mathbb{R}^m \to \mathbb{R}^l$ is the unknown quantity and K is called the *constitutive set.* Then the goal is to look for *plane wave* solutions, that is, solutions of the form

$$z(y) = ah(y \cdot \xi),$$

where $h : \mathbb{R} \to \mathbb{R}$ is called the wave profile, $\xi \in \mathbb{R}^m$ is the direction of the wave, and a lies in the system's *wave cone* Λ, given by

$$\Lambda := \left\{ a \in \mathbb{R}^l : \exists \xi \in \mathbb{R}^m \setminus \{0\} \text{ such that } \sum_{i=1}^{m} \xi_i A_i a = 0 \right\}$$

One is interested then in the relationship between the constitutive set K and the wave cone Λ. However, as is evident, such plane wave solutions are never compactly supported, unless $h \equiv 0$, so in each case we will be looking for ways to localise these plane-wave solutions. The key aspect of the concept of subsolutions is that we relax the condition that solutions to the linear system (2.11) lie necessarily within the constitutive set K, and require only that they be in a set from which one can oscillate along the wave cone Λ, and still be able to access K. Specifically, we define a Λ-convex function as any function $f : \mathbb{R}^l \to \mathbb{R}$ that satisfies the condition

$$a \in \Lambda \qquad \Longrightarrow \qquad t \mapsto f(a_0 + ta) \text{ is convex.}$$

A state $z \in \mathbb{R}^l$ is said to lie in the Λ-convex hull of K, denoted by K^Λ, if for every Λ-convex function f for which $f|_K \leq 0$, it follows that $f(z) \leq 0$. Since all convex functions are Λ-convex, we can see that $K^\Lambda \subset K^{co}$. Typically, subsolutions will be defined as functions $z(y)$ that satisfy (2.11) and lie in the (relative) interior of K^Λ. Then, the convex integration method consists of adding a sequence of highly oscillatory localised plane waves, in such a way as to ensure that weak convergence of the sequence of solutions to the linear system (2.11) is not disturbed (thus generating a solution in the weak limit), while still allowing us to get closer to the constitutive set, and thus producing solutions to the pointwise non-linear constraint. For the remainder of this chapter we will show how this framework can be applied to obtain non-uniqueness of weak solutions to the MHD planar symmetry system (Theorem 2.2.1) and for the Euler equations (Theorem 2.2.2).

2.2.2 Direct Iteration for the planar symmetry MHD

As a demonstration of the method of convex integration we will present the results obtained in [5], which proves the existence of compactly supported weak solutions to the system (1.6). First we describe the system in the context of differential inclusions, as the system of linear PDEs

$$\begin{cases} \partial_t v + \operatorname{div} M + \nabla q = 0 \\ \partial_t b + \operatorname{div} w = 0 \\ \operatorname{div} v = 0, \end{cases} \tag{2.12}$$

coupled with the pointwise non-linear constraints

$$z = (b, w, v, M, q) \in \mathcal{K} = \{(b, bv, v, v \circ v, q\} \quad \text{and } q = p + \frac{|v|^2 + |b|^2}{2} \tag{2.13}$$

We can then find a more succint way to refer to the state variables, which also helps us to describe the wave cone of the linear system:

Proposition 2.2.1 (From section 2 in [5]). *(i)* Let $\mathcal{M}_{3\times 3}$ be the set of 3×3 symmetric matrices, such that $M_{3,3} = 0$, and $\mathcal{M}_{4\times 3}$ the set of 4×3 matrices satisfying $(M_{i,j})_{i,j=1,2,3} \in \mathcal{M}_{3\times 3}$. The following maps are linear isomorphisms:

$$\begin{aligned} \mathbb{R}^2 \times \mathcal{S}_0^2 \times \mathbb{R} &\longrightarrow \mathcal{M}_{3\times 3} \\ (v, M, q) &\longmapsto \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \end{pmatrix} \end{aligned} \tag{2.14}$$

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (b, w) &\longmapsto \begin{pmatrix} w^t & b \end{pmatrix} \end{aligned} \tag{2.15}$$

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}_0^2 \times \mathbb{R} &\longrightarrow \mathcal{M}_{4\times 3} \\ (b, w, v, M, q) &\longmapsto U := \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \\ w^t & b \end{pmatrix} \end{aligned} \tag{2.16}$$

(ii) Introducing the variable $y = (x, t)$, the system (2.12) is equivalent to

$$\operatorname{div}_y U = 0. \tag{2.17}$$

(iii) For every $b \in \mathbb{R}$, $v \in \mathbb{R}^2$, and $M \in \mathcal{S}_0^2$, there exist $q \in \mathbb{R}$ and $w \in \mathbb{R}^2$ for which the kernel of U is non-trivial or, equivalently, (b, w, v, M, q) lies in the wave cone Λ of the system (2.12).

We will be looking for weak solutions (v, b) to equations (1.6) which satisfy the conditions

$$\|v\|_{L^2} = \|b\|_{L^2} = \chi_\Omega,$$

for a given bounded domain $\Omega \subset \mathbb{R}_x^2 \times \mathbb{R}_t$, so that we define the simpler constitutive set $\mathcal{K} = K \times [-1, 1]$, where

$$K = \Big\{(b, w, v, M) \in \{-1, 1\} \times S^1 \times S^1 \times \mathcal{S}_0^2\Big\}$$

It is clear that any solution to (2.12) whose image is contained in $\mathcal{K}$ also solves equations (1.6). We ask then that subsolutions be solutions z to the linear system which satisfy the condition

$$z \in \mathcal{U} = \operatorname{int}(K^{co} \times [-1, 1]).$$

We have the important fact that $0 \in \mathcal{U}$, so that the null function is already a candidate to be a subsolution. Formally, we define subsolutions to (1.6) in the following manner:

Definition 2.2.2. We say (b, w, v, M, q) is a *subsolution* to system (1.6) if $b \in L^2_{loc}(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$, $w, v \in L^2_{loc}(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$, $M \in L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}; \mathcal{S}_0^2)$, q is a distribution, (b, w, v, M, q) solves the linear system (2.12), and the image of (b, w, v, M, q) lies in $\mathcal{U}$.

We will show the following result:

Theorem 2.2.1 (Th. 1.1 in [5])**.** Given a bounded domain $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$, there exists a weak solution $(v, b) \in L^\infty(\mathbb{R}_x^2 \times \mathbb{R}_t; \mathbb{R}^2 \times \mathbb{R})$ to the system (1.6) which also satisfies:

(i) $|v(x, t)| = 1$ and $|b(x, t)| = 1$ for almost every $(x, t) \in \Omega$.

(ii) $v(x, t) = b(x, t) = p(x, t) = 0$ for almost every $(x, t) \in \mathbb{R}^2 \times \mathbb{R} \setminus \Omega$

We will construct a weak solution with the above requirements by constructing a sequence of subsolutions, each obtained from the previous through the addition of highly oscillatory localised plane waves which converge weakly but not strongly, in order to get closer at each step to the constitutive set $\mathcal{K}$. The following lemma provides us with good directions, in which we can oscillate with a sufficiently large amplitude and still solve (2.12) without leaving $\mathcal{U}$:

Lemma 2.2.3 (Lemma 2.1 in [5])**.** There exists a universal constant $C > 0$ such that, for every $(b, w, v, M, q) \in \mathcal{U}$, there exists $(\bar{b}, \bar{w}, \bar{v}, \bar{M}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}_0^2$ satisfying

(i) $(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0) \in \Lambda$.

(ii) The line segment with endpoints $(b, w, v, M, q) \pm (\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)$ is contained in $\mathcal{U}$.

(iii) The following inequality is satisfied:

$$|(\bar{v}, \bar{b})| \geq C(2 - (|v|^2 + |b|^2)) \tag{2.18}$$

Proof. Let $h = (b, w, v, M) \in \operatorname{int} K^{co}$. Carathéodory's theorem for convex hulls tells us that there exist $\lambda_i \in (0, 1)$ with $\sum \lambda_i$, and $h_i = (b_i, w_i, v_i, M_i) \in K$, for $i = 1, ..., N + 1$, where $N = 7$ is the dimension $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}_0^2$, satisfying

$$h = \sum_{i}^{N+1} \lambda_i h_i.$$

Suppose that $\lambda_1 = \max \lambda_i$, and define i^* as the index for which it holds that

$$\lambda_{i^*}^2(|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2) = \max_{i \in 1,...,8} \left\{ \lambda_i^2(|v_i - v_1|^2 + |b_i - b_1|^2) \right\}.$$

Note that, since $h = \sum_{i=1}^8 \lambda_i h_i$ and $h = \sum_{i=2}^8 \lambda_i (h_i - h_1)$, we get

$$\begin{aligned} |(v - v_1, b - b_1)| &= \left| \sum_{i=2}^8 \lambda_i (v_i - v_1, b_i - b_1) \right| \leq \sum_{i=2}^8 \sqrt{\lambda_i^2(|v_i - v_1|^2 + |b_i - b_1|^2)} \\ &\leq 7 \sqrt{\max_{i \in 1,...,8} \{\lambda_i^2(|v_i - v_1|^2 + |b_i - b_1|^2)\}} = 7 \sqrt{\lambda_{i^*}^2(|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2)}. \end{aligned}$$

In this manner, we have that $|v - v_1|^2 + |b - b_1|^2 \leq 49\lambda_{i^*}^2(|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2)$. Therefore, if we define $h := \frac{1}{2}\lambda_{i^*}(h_{i^*} - h_1)$, it follows that

$$\begin{aligned} \frac{1}{28\sqrt{2}}(2 - (|v|^2 + |b|^2)) &\leq \frac{1}{14\sqrt{2}}((1 - |v|) + (1 - |b|)) \leq \frac{1}{14\sqrt{2}} \sqrt{2} \sqrt{((1 - |v|)^2 + (1 - |b|)^2)} \\ &\leq \frac{1}{14} \sqrt{|v - v_1|^2 + |b - b_1|^2} \leq \frac{7}{14} \lambda_{i^*} \sqrt{|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2} = |(\bar{v}, \bar{b})|. \end{aligned}$$

This secures item *iii*. In order to see how $(\bar{h}, 0) \in \Lambda$, write $v_{i^*} = (v_{i^*}^1, v_{i^*}^2)$ e $v_1 = (v_1^1, v_1^2)$, and take $\xi = (-1, 0, v_1^1)$ if $v_{i^*}^1 = v_1^1$, otherwise take $\xi = (-\frac{v_{i^*}^2 - v_1^2}{v_{i^*}^1 - v_1^1}, 1, -\frac{v_{i^*}^1 v_1^2 - v_1^1 v_{i^*}^2}{v_{i^*}^1 - v_1^1})$. Thus, we have $U\xi = 0$, where U is the matrix given by the linear isomorphism in Proposition 2.2.1*i*. $\square$

In order to proceed, we now need to produce a localised plane wave:

Proposition 2.2.4 (Prop. 2.2 in [5])**.** Let $\bar{V} \in \Lambda$ be such that $\bar{V}_{4,3} \neq 0$, $\bar{V}_{3,3} = 0$ and $(\bar{V}_{i,3})_{i=1,2} \neq 0$. Let σ be the line segment with endpoints $\pm\bar{V}$ in $\mathcal{M}_{4\times3}$. Then, for every $\varepsilon > 0$, we can porduce a matrix field V given by

$$V(x,t) = \begin{pmatrix} M(x,t) + q(x,t)I_2 & v(x,t) \\ v^t(x,t) & 0 \\ w^t(x,t) & b(x,t) \end{pmatrix},$$

where $M \in \mathcal{S}_0^2$, $v, w \in \mathbb{R}^2$, $b \in \mathbb{R}$, $x \in \mathbb{R}^2$, satisfy the following:

(i) $\operatorname{div}_{x,t} V = 0$;

(ii) supp $V \subset B_1(0)$;

(iii) Im $V \subset \sigma_{\bar{V},\varepsilon} = \{A \in \mathcal{M}_{4\times 3} | \text{dist } (A, \sigma) < \varepsilon\}$;

(iv) $\fint |v(y)|dy \geq \alpha |\bar{v}|$ e $\fint |b(y)|dy \geq \alpha|\bar{b}|$, where $\bar{v} = (\bar{V}_{i,3})_{i=1,2}$, $\bar{b} = \bar{V}_{4,3}$, and $\alpha > 0$ is a universal constant.

Proof. Let us write $\bar{V} = \begin{pmatrix} \bar{U} \\ \bar{W}^t \end{pmatrix}$, where $\bar{U} = \begin{pmatrix} M + qI_2 & \bar{v} \\ \bar{v}^t & 0 \end{pmatrix}$ and $\bar{W} = (\bar{w}, \bar{b})$. We will see later that $\bar{U}$ is the matrix arising in the differential inclusion associated in the incompressible Euler equations, and will produce localised plane waves more directly for them later. For now, we refer to Proposition 3.2 in [8], to obtain a matrix field $U : \mathbb{R}^2 \times \mathbb{R} \to \mathcal{M}_{3\times 3}$ such that div $_{x,t}U = 0$, supp $U \subset B_1(0)$, Im $U \subset \sigma_{\bar{U},\varepsilon}$, where $\sigma_{\bar{U},\varepsilon}$ is the ε-neighborhood of the line segment with endpoints $\pm\bar{U}$ in $\mathcal{M}_{3\times 3}$, and $\fint |Ue_3(y)|dy \geq \alpha|\bar{v}|$.

Now, we construct $\bar{W} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ such that div $_{x,t}W = 0$, supp $W \subset B_1(0)$, Im $W \subset \sigma_{\bar{W},\varepsilon}$, where $\sigma_{\bar{W},\varepsilon}$ is the ε-neighborhood of the line segment with endpoints $\pm\bar{W}$ in $\mathbb{R}^3$, and $\fint |W \cdot e_3(y)|dy \geq \alpha|\bar{b}|$. Once we have done this we define $V = \begin{pmatrix} U \\ W^t \end{pmatrix}$ so that it is clear that V satisfies the given conditions.

In order to do so, we divide the construction in two parts. First, supose that $\bar{W} = (0, \bar{w}_2, \bar{b})$ with $\bar{b} \neq 0$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth cutoff function with $|\phi| \leq 1$,$\phi \equiv 1$ on $B_{1/2}(0)$ and supp $(\phi) \subset B_1(0)$. Define

$$W(y) = \frac{1}{N^2} \begin{pmatrix} \partial_{12}^2(\bar{w}_2\phi \sin(Ny_1)) + \partial_{13}^2(\bar{b}\phi \sin(Ny_1)) \\ -\partial_{11}^2(\bar{w}_2\phi \sin(Ny_1)) \\ -\partial_{11}^2(\bar{b}\phi \sin(Ny_1)) \end{pmatrix}.$$

Note that W is a smooth divergence-free vector field with support contained in $B_1(0)$. Moreover, for $y \in B_{1/2}(0)$ we have $W(y) = \bar{W}\sin(Ny_1)$, thus

$$\int |W(y) \cdot e_3|dy \geq \int_{B_{1/2}(0)} |W(y) \cdot e_3|dy = |\bar{W} \cdot e_3| \int_{B_{1/2}} |\sin(Ny_1)|dy \geq \alpha|\bar{b}|,$$

for some $\alpha > 0$.

Define $\tilde{W} = \begin{pmatrix} 0 \\ (\bar{w}_2 \sin(Ny_1)) \\ (\bar{b} \sin(Ny_1)) \end{pmatrix}$, taking values in the line segment with endpoints $\pm\bar{W}$, and observe that $\|W - \phi\tilde{W}\|_\infty \leq \frac{C}{N^2}\|\phi\|_{C^2}$. Therefore, for sufficiently large N we have $\|W - \phi\tilde{W}\|_\infty < \varepsilon$. Since $|\phi| \leq 1$, this means that the image of W is contained in $\sigma_{\bar{W},\varepsilon}$.

Consider the general case now. By hypothesis, $\bar{W} \cdot e_3 \neq 0$ and $\bar{W} \cdot \xi = 0$ for some $\xi \in \mathbb{R}^3 \setminus \{0\}$. Since e_3 and ξ must be linearly independent, choose a $\eta \in \mathbb{R}^3 \setminus \{0\}$ so as to complete a basis for $\mathbb{R}^3$, and let A be the 3×3 matrix given by the requirements $Ae_1 = \xi$, $Ae_2 = \eta$ and $Ae_3 = e_3$. We can in fact choose ξ, η so that A is an isometry.

If we define $\bar{B} = A^t\bar{W}$, it is clear that $\bar{B} \in \mathbb{R}^3$, $\bar{B}_1 = 0$, and $\bar{B}_3 \neq 0$. Then we use the above construction to construct a smooth divergence-free map $B : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ with compact support in $B_1(0)$ and image lying in the ε-neighborhood of the line segment $\sigma_{\bar{B}}$ with endpoints $\pm\bar{B}$.

Set $W(y) = (A^{-1})^t B(A^t y)$. Observe that W is supported in $(A^{-1})^t B_1(0)$, and that, since the isomorphism $T : X \mapsto (A^{-1})^t X$ maps $\sigma_{\bar{B}}$ into $\sigma_{\bar{W}}$, the image of W is contained in $\sigma_{\bar{W},\varepsilon}$. It is a straightforward calculation to see that W is divergence-free, and that $\int |W(y) \cdot e_3| dy \geq \alpha|\bar{b}|$, so that we have all the desired properties.

$\square$

There remains only one step before we begin the iteration. We need to define a function space whose topological and analytic properties we can take advantage of. Let X_0 be the set of functions $(b, w, v, M, q) \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$ satisfying the following conditions:

(*i*) supp $(b, w, v, M, q) \subset \Omega$;

(*ii*) (b, w, v, M, q) solves the system (2.12) in $\mathbb{R}^2 \times \mathbb{R}$;

(*iii*) $(b, w, v, M, q) \in \mathcal{U}$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$.

We endow X_0 with the weak* topology of L^∞ and denote by X its closure in this topology. By direct calculation we can see that for any $(b, w, v, M, q) \in X$ such that $|v(x,t)| = 1$, $|b(x,t)| = 1$ for almost every $(x,t) \in \Omega$, then v, b and $p := q - \frac{1}{2}(|v|^2 + |b|^2)$ define a weak solution of (1.5) with $v(x,t) = b(x,t) = p(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^2 \times \mathbb{R} \setminus \Omega$. We can now begin with the iteration scheme. Given a subsolution, we will produce one closer to the constitutive set in the following manner:

Lemma 2.2.5 (Lemma 2.2 in [5])**.** There exists a constant $\beta > 0$ such that, for each $(b_0, w_0, v_0, M_0, q_0) \in X_0$, one can produce a sequence $(b_k, w_k, v_k, M_k, q_k) \in X_0$ satisfying

$$\|v_k\|^2_{L^2(\Omega)} + \|b_k\|^2_{L^2(\Omega)} \geq \|v_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)} + \beta(2|\Omega| - (\|v_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)}))^2,$$

while also

$$(b_k, w_k, v_k, M_k, q_k) \overset{*}{\rightharpoonup} (b_0, w_0, v_0, M_0, q_0) \text{ in } L^\infty.$$

Proof. Define $z_0 := (b_0, w_0, v_0, M_0, q_0) \in X_0$. Applying Lemma 2.2.3 to each element of the compact set $\mathrm{Im}(z_0) \subset \mathcal{U}$, we obtain for every $(x,t) \in \Omega$ a direction

$$\bar{z}(x,t) := (\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0) \in \Lambda$$

such that the line segment with endpoints $z_0(x,t) \pm \bar{z}(x,t)$ is contained in $\mathcal{U}$, and it also holds that

$$|\bar{v}(x,t)| + |\bar{b}(x,t)| \geq \sqrt{|\bar{v}(x,t)|^2 + |\bar{b}(x,t)|^2} \geq C(2 - (|v_0|^2(x,t) + |b_0|^2(x,t))).$$

Since $z_0 \in X_0$, observe that $|\bar{v}(x,t)| + |\bar{b}(x,t)| > 0$ for every $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Moreover, it is clear from the construction of $(\bar{b}, \bar{w}, \bar{v}, \bar{M})$ and from the uniform continuity of z_0 that there exists $\varepsilon > 0$ such that, for every $(x,t), (x_0, t_0) \in \Omega$ with $|x - x_0| + |t - t_0| < \varepsilon$, the ε-neighborhoodof the line segment with endpoints $z_0(x,t) \pm \bar{z}(x_0, t_0)$ also lies inside $\mathcal{U}$. Now, since $\bar{v} \neq 0$ and $\bar{b} \neq 0$, we can use Proposition 2.2.4 with $(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)(x_0, t_0) \in \Lambda$ and $\varepsilon > 0$ to obtain a smooth solution (b, w, v, M, q) of (2.12) which also satisfies the properties of the Proposition. For every $r < \varepsilon$, Let

$$(b_r, w_r, v_r, M_r, q_r)(x,t) = (b, w, v, M, q)\left(\frac{x - x_0}{r}, \frac{t - t_0}{r}\right).$$

Claim 2.2.6. Defined as above, $(b_r, w_r, v_r, M_r, q_r)$ solves (2.12) and has the following properties:

(i) $\mathrm{supp}((b_r, w_r, v_r, M_r, q_r)) \subset B_r(x_0, t_0)$;

(ii) The image $\mathrm{Im}((b_r, w_r, v_r, M_r, q_r))$ is contained in the ε-neighborhood of the line segment with endpoints $\pm(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)(x_0, t_0)$;

(iii) $\int |v_r| dxdt \geq \alpha |\bar{v}(x_0, t_0)||B_r(x_0, t_0)|$ and $\int |b_r| dxdt \geq \alpha |\bar{b}(x_0, t_0)||B_r(x_0, t_0)|$.

Therefore, combining the properties above to the fact that the line segment with endpoints $z_0(x,t) \pm \bar{z}(x_0, t_0)$ lies within the open set $\mathcal{U}$, it's easy to see that $(b_0, w_0, v_0, M_0, q_0) + (b_r, w_r, v_r, M_r, q_r) \in X_0$. We now have all the tools required to proceed in the construction. Very well, since z_0 is uniformly continuous, we can find a radius $r_0 > 0$ for which, whenever $r < r_0$, there exists a finite family of pairwise disjoint balls, $B_{r_j}(x_j, t_j)$, with $r_j < r$, satisfying

$$\begin{aligned} &\int_\Omega (2 - (|v_0(x,t)|^2 + |b_0(x,t)|^2)) dxdt \\ &\leq 2 \sum_j (2 - (|v_0(x_j, t_j)|^2 + |b_0(x_j, t_j)|^2)) |B_{r_j}(x_j, t_j)|. \end{aligned} \tag{2.19}$$

In what follows, we fix $k \in \mathbb{N}$ for which $\frac{1}{k} < \min\{r_0, \varepsilon\}$, and choose a finite family of pairwise disjoint balls $B_{r_{k,j}}(x_{k,j}, t_{k,j}) \subset \Omega$ satisfying $r_{k,j} < \frac{1}{k}$, for which (2.19) holds. In each ball we apply the above construction to obtain a sequence of smooth solutions to (2.12), denoted $(b_{k,j}, w_{k,j}, v_{k,j}, M_{k,j}, q_{k,j})$, which satisfy corresponding versions of the properties in Claim 2.2.6. Particularly, we have:

$$(b_k, w_k, v_k, M_k, q_k) := (b_0, w_0, v_0, M_0, q_0) + \sum_j (b_{k,j}, w_{k,j}, v_{k,j}, M_{k,j}, q_{k,j}) \in X_0,$$

and by (iii), we also have

$$\int (|v_k - v_0| + |b_k - b_0|) dxdt \geq \frac{C\alpha}{2} \int_\Omega (2 - (|v_0|^2 + |b_0|^2)) dxdt. \tag{2.20}$$

Finally, observe that $(b_k, w_k, v_k, M_k, q_k) \overset{*}{\rightharpoonup} (b_0, w_0, v_0, M_0, q_0)$ in L^∞. Consequently, we have

$$\begin{aligned} &\liminf_{k\to\infty} \|v_k\|^2_{L^2(\Omega)} + \|b_k\|^2_{L^2(\Omega)} \\ &\quad = \|v_0\|^2_{L^2(\Omega)} + \liminf_{k\to\infty} (2\langle v_0, v_k - v_0\rangle + \|v_k - v_0\|_{L^2(\Omega)}) \\ &\qquad\qquad + \|b_0\|^2_{L^2(\Omega)} + \liminf_{k\to\infty} (2\langle b_0, b_k - b_0\rangle + \|b_k - b_0\|_{L^2(\Omega)}) \\ &\quad \geq \|v_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)} + \frac{1}{|\Omega|} \liminf_{k\to\infty} (\|v_k - v_0\|_{L^1(\Omega)} + \|b_k - b_0\|_{L^1(\Omega)})^2 \end{aligned} \tag{2.21}$$

In view of (2.20) and (2.21), we are left with

$$\liminf \|v_k\|^2_{L^2(\Omega)} + \|b_k\|^2_{L^2(\Omega)} \geq \|v_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)} + \frac{C^2\alpha^2}{4|\Omega|} (2|\Omega| - (\|v_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)}))^2,$$

proving the Lemma for $\beta = \frac{C^2\alpha^2}{4|\Omega|}$. $\square$

Proof of Theorem 2.2.1. The main idea is to use Lemma 2.2.5 to inductively define a sequence $(z_k) \in X_0$ satisfying the following conditions:

(i) There exists $z = (b, w, v, M, q) \in X$ for which z_k converges strongly em $L^2(\mathbb{R}^2 \times \mathbb{R})$;

(ii) The inequality

$$\|v_{k+1}\|^2_{L^2} + \|b_{k+1}\|^2_{L^2} \geq \|v_k\|^2_{L^2} + \|b_k\|^2_{L^2} + \beta(2|\Omega| - (\|v_k\|^2_{L^2} + \|b_k\|^2_{L^2}))^2$$

holds for every step $k \in \mathbb{N}$.

We can then use $i)$ to take the limit in $ii)$ and obtain

$$\|v\|^2_{L^2(\Omega)} + \|b\|^2_{L^2(\Omega)} \geq \|v\|^2_{L^2(\Omega)} + \|b\|^2_{L^2(\Omega)} + \frac{C^2\alpha^2}{4|\Omega|} (2|\Omega| - (\|v\|^2_{L^2(\Omega)} + \|b\|^2_{L^2(\Omega)}))^2,$$

which can only happen if $\|v\|^2_{L^2(\Omega)} + \|b\|^2_{L^2(\Omega)} = 2|\Omega|$. Since $|v|, |b| \leq 1$ in Ω, and they're supported in Ω, we conclude that $|v| = |b| = \chi_\Omega$ almost everywhere. Clearly, $(b, v, w, M) \in K^{co}$ for almost every $(x,t) \in \Omega$, since $(b, w, v, M, q) \in X$. This implies $(b, w, v, M)(x,t) \in K$ for almost every $(x,t) \in \Omega$, as desired. There remains only to construct the sequence $(z_k) \in X_0$ satisfying $i)$ and $ii)$. For such, we define $(b_1, w_1, v_1, M_1, q_1) \equiv 0$ in $\mathbb{R}^2 \times \mathbb{R}$. This is possible since $0 \in \mathcal{U}$. Let ρ_ε be a standard mollifier in $\mathbb{R}^2 \times \mathbb{R}$. Alongside the sequence $(b_k, w_k, v_k, M_k, q_k) \in X_0$ we will inductively obtain an auxilliary sequence of numbers $\eta_k > 0$, as follows. Supposing we have $z_j := (b_j, w_j, v_j, M_j, q_j)$, for $j \leq k$, and $\eta_1, ..., \eta_{k-1}$, we choose

$$\eta_k < 2^{-k} \tag{2.22}$$

for which

$$\|z_k - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} < 2^{-k}. \tag{2.23}$$

We now apply Lemma 2.2.5 to construct $z_{k+1} = (b_{k+1}, w_{k+1}, v_{k+1}, M_{k+1}, q_{k+1}) \in X_0$ satisfying

$$\begin{aligned} &\|v_{k+1}\|^2_{L^2\Omega} + \|b_{k+1}\|^2_{L^2\Omega} \\ &\qquad \geq \|v_k\|^2_{L^2\Omega} + \|b_k\|^2_{L^2\Omega} + \beta(2|\Omega| - (\|v_k\|^2_{L^2\Omega} + \|b_k\|^2_{L^2\Omega}))^2 \end{aligned} \tag{2.24}$$

and

$$\|(z_{k+1} - z_k) * \rho_{\eta_j}\|_{L^2(\Omega)} < 2^{-k} \text{ for all } j \leq k \qquad , \tag{2.25}$$

where weak* convergence of the sequence provided in 2.2.5 guarantees the existence of z_{k+1} satisfying (2.25). Since this sequence (z_k) is bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{R})$, there exists a subsequence, which we still label (z_k), and a function $z = (b, w, v, M, q) \in X$, such that $z_k \overset{*}{\rightharpoonup} z$ in $L^\infty(\mathbb{R}^2 \times \mathbb{R})$. Moreover, the sequences (z_k) and (η_k) satisfy properties (2.22), (2.23), (2.24) and (2.25). Therefore, for every $k \in \mathbb{N}$, it holds that

$$\|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} \leq \sum_{j=0}^{\infty} \|z_{k+j} * \rho_{\eta_k} - z_{k+j+1} * \rho_{\eta_k}\|_{L^2(\Omega)} \leq \sum_{j=0}^{\infty} 2^{-(k+j)} = 2^{-k+1},$$

and, since we also have

$$\|z_k - z\|_{L^2(\Omega)} \leq \|z - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} + \|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} + \|z * \rho_{\eta_k} - z\|_{L^2(\Omega)},$$

it follows that $z_k \to z$ strongly in $L^2(\Omega)$. This concludes the proof. $\square$

2.2.3 Wild solutions to the Euler Equations

In this section, we present some of the results of [9], but follow also along their presentation in [25]. These results are a further development of the work started in [8], which introduced the framework of differential inclusion to the Euler equations and proceeded similarly as we have shown above for the planar symmetry MHD equations, to obtain weak solutions that are compactly supported in space-time. Since these solutions are in clear violation of energy principles, because they neither conserve nor dissipate energy, the authors in [9] were interested in seeing if one could employ the convex integration method to obtain weak solutions that could satisfy energy inequalities. The answer was somewhat surprising: One could obtain infinitely many weak solutions obeying nearly *every* reasonable form of energy inequality, as long as a single corresponding subsolution is known. This result is very important in the overall consideration of weak solutions: it shows that most energy considerations (admissibility criteria) may not be sufficient to single out a single weak solution. First, let's recall the incompressible Euler equations (1.1):

$$\begin{aligned} \partial_t v + \operatorname{div}\,(v \otimes v) + \nabla p = 0 \\ \operatorname{div}\, v = 0. \end{aligned} \tag{2.26}$$

We can define the traceless symmetric matrix $u := v \otimes v - \frac{|v|^2}{n} I_d := v \circ v$ and the scalar $q = p + \frac{|v|^2}{n}$, and then consider $z = (v, u, q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} \simeq \mathbb{R}^{\frac{d(d+3)}{2}}$. At the same time, we define

$$U = \begin{pmatrix} u + qI_d & v \\ v^t & 0 \end{pmatrix},$$

and write $(x,t) = y \in \mathbb{R}^{d+1}$. We will see later that z and U are linearly equivalent ways to refer to the unknown, so that the Euler equations can then be rewritten as the coupling of the linear PDE

$$\operatorname{div}_{\,y} U = 0$$

with the non-linear pointwise constraint

$$z(y) \in K = \left\{(v,u,q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} : \ u = v \circ v\right\}.$$

Formally, we have the following results:

Lemma 2.2.7. Let $v \in L^\infty([0,T]; L^2(\mathbb{R}^d;\mathbb{R}^d))$, $u \in L^\infty([0,T]; L^1(\mathbb{R}^d;\mathcal{S}_0^d))$ and q be a distribution so that (v, u, q) satisfy

$$\begin{aligned} \partial_t v + \operatorname{div} u + \nabla q &= 0, \\ \operatorname{div} v &= 0, \end{aligned} \tag{2.27}$$

in the weak sense. If it also holds that

$$u = v \circ v = v \otimes v - \frac{1}{d}|v|^2 I_d, \tag{2.28}$$

for almost every $(x, t) \in \mathbb{R}^d \times [0, T]$, then v and $p = q - \frac{1}{d}|v|^2$ define a weak solution to the Euler equations. Conversely, if (v, p) is a bounded energy weak solution to the Euler equations, then the triple (v, u, q), with $u := v \circ v$, $q := p + \frac{1}{d}|v|^2$ solves (2.27) and (2.28) in the sense of distributions.

Proposition 2.2.8 (Prop 2.3 in [25]). *(i)* Let $\mathcal{M} \subset \mathcal{S}^{d+1}$ be the linear subspace of symmetric matrices U satisfying $[U]_{(d+1),(d+1)} = 0$. The mapping defined by

$$\mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} \longrightarrow \mathcal{M} \tag{2.29}$$

$$(v, u, q) \mapsto U = \begin{pmatrix} u + qI_d & v \\ v^t & 0 \end{pmatrix} \tag{2.30}$$

is a linear isomorphism.

(ii) If we denote $y = (x, t) \in \mathbb{R}^d \times [0, T]$, the equation (2.27) is equivalent to

$$\operatorname{div}_y U = 0. \tag{2.31}$$

(iii) For every $v \in \mathbb{R}^d$ and $u \in \mathcal{S}_0^d$, there exists $q \in \mathbb{R}$ such that the corresponding matrix U has null determinant.

Proof. i) and ii) are direct calculations. To see that iii) also holds, let $V^\perp$ be the orthogonal complement of $V = \{v\}$ in $\mathbb{R}^d$, and define $P_{V^\perp}$ as the orthogonal projection of $\mathbb{R}^d$ onto $V^\perp$. Since u is symmetric, i.e. self-adjoint, so will be its restriction to $V^\perp$, $P_{V^\perp}u$. Therefore, there exists at least one eigenvalue to this operator, which we denote by $-q$, and also and eigenvector $\xi \in V^\perp$, so that $P_{V^\perp}(u + qI_d)\xi = 0$. Therefore, there exists $\lambda \in \mathbb{R}$ satisfying

$$(u + qI_d)\xi = \lambda v.$$

It follows that the non-zero vector $(\xi, -\lambda)$ lies in the kernel of U, and therefore U has null determinant. $\square$

The importance of item iii) in the above proposition is that it, together with the independence of the constitutive set K on q, tells us we don't need to worry too much about what happens to the "pressure". That is, given only $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, there exist $q \in \mathbb{R}$ and a direction $\eta = (\xi, \lambda) \in (\mathbb{R}^d \times \mathbb{R}) \setminus \{0\}$ for which, given any wave profile $h : \mathbb{R} \to \mathbb{R}$, the function

$$h(y \cdot \eta)(v, u, q),$$

is a solution to (2.27). The following Lemma tells us how to localise these plane waves.

Lemma 2.2.9 (Lemma 3.4 in [8])**.** Suppose that for $i, j, k, l = 1, ..., d+1$, we have functions $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{d+1})$, such that E is an antisymmetric tensor in ij and kl, and such that $E_{(d+1)i}^{(d+1)j} = 0$ for all i, j. Then, the matrix U defined by the formula

$$U_{ij} = \mathcal{L}(E)_{ij} = \frac{1}{2} \sum_{k,l} \partial_{k,l}^2 (E_{kj}^{il} + E_{ki}^{jl})$$

takes values in $\mathcal{M}$ and is divergence-free, that is, it satisfies (2.31).

Now, since we will be looking for weak solutions that fall below a given energy bound, we must also be concerned with particular sections of the constitutive set:

$$K_r = \left\{ (v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d \mid u = v \circ v \text{ e } |v| = r \right\} \tag{2.32}$$

Also, it will be useful to have an analogous energy associated to the pairs (v, u):

Definition 2.2.10. For $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, we define the generalised energy as

$$e(v,u) \; := \; \frac{d}{2} \lambda_{\max}(v \otimes v - u),$$

where $\lambda_{\max}$ denotes the largest eigenvalue function.

Lemma 2.2.11. The generalised energy defined above has the following properties:

(i) $e : \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ is convex.

(ii) For all $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, we have

$$\frac{1}{2}|v|^2 \leq e(v,u),$$

and equality holds if, and only if, $u = v \circ v$.

(iii) For all $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, we have that

$$|u|_\infty \leq 2\frac{d-1}{d} e(v,u),$$

where $|u|_\infty$ is the operator norm of the matrix u.

(iv) Given $r \geq 0$, the convex hull of K_r is given by

$$K_r^{co} = \left\{ (v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d \,\middle|\, e(v,u) \leq \frac{r^2}{2} \right\}.$$

(v) If $(u,v) \in \mathbb{R}^d \times \mathcal{S}_0^d$, then $\sqrt{2e(v,u)}$ is the minimal $\rho \geq 0$ for which it holds that $(u,v) \in K_\rho^{co}$.

We can finally define subsolutions:

Definition 2.2.12. Let

$$\bar{e} \in C(\mathbb{R}^d \times (0,T)) \cap L^\infty([0,T]; L^1(\mathbb{R}^d)) \cap C([0,T]; L^1(\mathbb{R}^d)) \tag{2.33}$$

and suppose we have

$$v \in C^\infty(\mathbb{R}^d \times (0,T)) \cap L^\infty([0,T]; L^2(\mathbb{R}^d)) \cap C([0,T]; L^2_w(\mathbb{R}^d))$$

with $v(\cdot,0) = v_0$, and

$$u \in C^\infty(\mathbb{R}^d \times (0,T)) \cap L^\infty([0,T]; L^1(\mathbb{R}^d)).$$

If there exists a scalar function $q \in C^\infty(\mathbb{R}^d \times (0,T))$, such that (v,u,q) satisfies (2.27), and also if

$$e(v(x,t), u(x,t)) < \bar{e}(x,t) \tag{2.34}$$

for all $x \in \mathbb{R}^d$ and $t > 0$, we say that (v,u) is a (smooth) *subsolution* with respect to the energy profile $\bar{e}$ and initial data v_0

Note that we did not require that the solution be smooth up to time zero.

We are now prepared to announce the result we wish to show:

Theorem 2.2.2 (Proposition 2 in [9])**.** Let $v_0 \in L^2(\mathbb{R}^d)$ be a weakly divergence-free vector field, and $\bar{e}$ be an energy profile obeying (2.33). If a smooth subsolution w.r.t. the initial data v_0 and energy profile $\bar{e}$ exists, then there exist infinitely many weak solutions $v \in C([0,T]; L^2_w(\mathbb{R}^d))$ to the Euler equations that satisfy

$$v(\cdot,0) = v_0 \quad \text{and} \quad \tfrac{1}{2}|v(x,t)|^2 = \bar{e}(x,t) \quad \text{for every } t > 0 \text{ and a.e. } x \in \mathbb{R}^d.$$

The first step in proving the theorem is to define a function space of weak solution candidates, whose functional analytic properties we can exploit in order to find our desired solutions:

Definition 2.2.13. Given v_0 and $\bar{e}$ as above in definition 2.2.12, We denote by X_0 the set of $v \in C([0,T]; L^2_w(\mathbb{R}^d))$ for which a corresponding matrix field u exists, such that (v,u) is a subsolution with respect to v_0 and $\bar{e}$. The space X is defined as the closure of X_0 in the topology of $C([0,T]; L^2_w(\mathbb{R}^d))$.

The assumptions on $\bar{e}$ guarantee that there exist a bounded set $B \subset L^2_x$, such that $v(\cdot,t) \in B$ for all t and $v \in X$. We can therefore safely assume that B is weakly compact, which makes the weak topology on B metrisable by a metric d_B. The space $C([0,T]; B) \subset C([0,T]; L^2_w)$ is consequently also metrisable, by defining

$$d(v_1, v_2) := \sup_{t\in[0,T]} d_B(v_1(\cdot,t), v_2(\cdot,t)).$$

Such a metric makes $C([0,T]; B)$ complete. Being a closed subset of $C([0,T]; B)$, it follows that (X,d) is also a complete metric space. We can see also that it consists of "weak" subsolutions:

Proposition 2.2.14. Every $v \in X$ satisfies $v(\cdot,0) = v_0$, and there exist $u \in L^\infty([0,T]; L^1_x)$ and a distribution q, such that (2.27) holds weakly. Furthermore, we have that $e(v,u) \leq \bar{e}$ almost everywhere.

Proof. Let $v \in X$ with $v_k \xrightarrow{d} v$, and $v_k \in X_0$. Take also $u_k \in L^\infty_t L^1_x$ the corresponding matrix fields, which are pointwise bounded by $(2(d-1)/d)\bar{e}$ (by item (iii) in Lemma 2.2.11). Therefore, (u_k) is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^d \times (0,T))$. It follows, after passing to a subsequence, that $u_k \overset{*}{\rightharpoonup} u$ in L^∞_{loc}, and that u is a.e. bounded by $(2(d-1)/d)\bar{e}$, so that $u \in L^\infty_t L^1_x$. Weak convergence preserves the equations (2.27), and the convexity of e guarantees that $e(v,u) \leq \bar{e}$ almost everywhere. $\square$

We define now an *error functional*, which tells us how far $v \in X$ is from being a proper weak solution to the Euler equations:

Definition 2.2.15. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and $[t_1,t_2] \subset (0,T)$ be a proper interval. We define an error functional in X by:

$$I_{\Omega,t_1,t_2}(v) = \inf_{t\in[t_1,t_2]} \int_\Omega \left(\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right) dx.$$

Since the functional as defined above, is in essence the L^2 norm of v, it is easy to see that it is lower semi-continuous w.r.t the metric d, which induces the topology of $X \subset CL^2_w$. A direct proof is available in [9]. The hypotheses on $\bar{e}$ tell us also that it is bounded by below.

Combining the definition of X, lower-semicontinuity of I_{Ω,t_1,t_2} and the condition that $e(v,u) \leq \bar{e}$ for every $v \in X$ from proposition 2.2.14, it becomes clear that

$I_{\Omega,t_1,t_2}(v) \le 0$ with equality for every Ω, t_1 and t_2 if, an only if, $u = v \circ v$ almost everywhere, i.e. if v is a weak solution to the Euler equations with initial data v_0 and energy density $\bar{e}$ for almost every $x \in \mathbb{R}^d$ and for every $t > 0$.

The next essential step to prove 2.2.2, is the following "perturbation property":

Proposition 2.2.16. Fix Ω, t_1 and t_2 as above. For every $\alpha > 0$, there exists $\beta > 0$ such that, if $v \in X_0$ satisfies $I_{\Omega,t_1,t_2}(v) < -\alpha$, then there exists a sequence $(v_k) \subset X_0$, with $v_k \xrightarrow{d} v$ (that is, in the topology of CL^2_w), but such that

$$\liminf_{k\to\infty} I_{\Omega,t_1,t_2}(v_k) \ge I_{\Omega,t_1,t_2}(v) + \beta.$$

To prove this proposition, we will add highly oscillating localised plane wave solutions (v_k, u_k) of (2.27) to (v, u), so that $(v + v_k, u + u_k)$ converges weakly but not strongly to (v, u), and the sequence stays in X_0. For that, we need to guarantee that there is "room" for these oscillations to be large enough to prevent strong convergence, without breaking the energy condition, which is the purpose of the following result, whose proof follows along the same line as 2.2.3.

Lemma 2.2.17. There exists a universal constant C, such that for every $(v, u) \in \mathbb{R}^d \times \mathcal{S}^d_0$ with $e(v, u) < \frac{r^2}{2}$, that is $(v, u) \in \operatorname{int}K^{co}_r$, there exists a $(\bar{v}, \bar{u}, \bar{q}) \in \Lambda$ such that the line segment with endpoints $(v, u) \pm (\bar{v}, \bar{u})$ is contained in $\operatorname{int}K^{co}_r$, and such that

$$|\bar{v}| \ge \frac{C}{r}(r^2 - |v|^2).$$

Let us proceed with the proof. We decompose the domain $\Omega \times [t_1, t_2]$ in tiny cubes and discretize the solution and energy density to be constant inside each cube. For each cube we will produce a localised plane wave, oscillating in the direction provided by Lemma 2.2.17. Now, since the definition of the error functional has a uniform estimate in time, we need to guarantee that at each $t \in [t_1, t_2]$ there are enough oscillations. We do so by using a "shifted grid".

Specifically, for $\zeta \in \mathbb{Z}^d$ and sidelength $h > 0$, define the families of cubes (Q_ζ) and $(\tilde{Q}_\zeta)$ in $\mathbb{R}^d{}_x$ by

$$Q_\zeta = h\zeta + \left[-\tfrac{h}{2}, \tfrac{h}{2}\right) \quad \text{and} \quad \tilde{Q}_\zeta = h\zeta + \left[-\tfrac{3h}{8}, \tfrac{3h}{8}\right),$$

so that $\tilde{Q}_\zeta \subset Q_\zeta$. Moreover, for $(\zeta, i) \in \mathbb{Z}^{d+1}$, define cubes in $\mathbb{R}^d{}_x \times \mathbb{R}_t$ by

$$C_{\zeta,i} = \begin{cases} Q_\zeta \times [ih, (i+1)h) & \text{for } \sum_{j=1}^d \zeta_j \text{ even,} \\ Q_\zeta \times [(i - \frac{1}{2})h, (i + \frac{1}{2})h) & \text{for } \sum_{j=1}^d \zeta_j \text{ odd} \end{cases}$$

so as to obtain a shifted grid in space-time. Similarly, we define $\tilde{C}_{\zeta,i} \subset C_{\zeta,i}$ of sidelength

$\frac{3}{4}h$ by

$$\tilde{C}_{\zeta,i} = \begin{cases} \tilde{Q}_\zeta \times [(i+\frac{1}{8})h, (i+\frac{7}{8})h) & \text{for } \sum_{j=1}^d \zeta_j \text{ even,} \\ \tilde{Q}_\zeta \times [(i-\frac{3}{8})h, (i+\frac{3}{8})h) & \text{for } \sum_{j=1}^d \zeta_j \text{ odd.} \end{cases}$$

Furthermore, define the *cutoff* function $0 \le \phi^h \le 1$ as a smooth function in $\mathbb{R}^{d+1}$ which equals 1 in the "smaller cubes", that is, in $\bigcup_{\mathbb{Z}^{d+1}} \tilde{C}_{\zeta,i}$, and is zero near the boundaries of the larger cubes, i.e. on

$$\left\{ (x,t) \in \mathbb{R}^{d+1} \middle| \operatorname{dist}\left((x,t), \bigcup_{\mathbb{Z}^{d+1}} \partial C_{\zeta,i} \right) \le \frac{h}{16} \right\}.$$

Next, define

$$\Omega_1^h = \bigcup \left\{ \tilde{Q}_\zeta \middle| \sum_{j=1}^d \zeta_j \text{ even, } \mathbb{Q}_\zeta \subset \Omega \right\}$$

and

$$\Omega_2^h = \bigcup \left\{ \tilde{Q}_\zeta \middle| \sum_{j=1}^d \zeta_j \text{ odd, } \mathbb{Q}_\zeta \subset \Omega \right\},$$

and note that

$$\lim_{h\to 0} \mathcal{L}^d(\Omega_l^h) = \frac{1}{2}\left(\frac{3}{4}\right)^d \mathcal{L}^d(\Omega)$$

for $l = 1, 2$, adn that, thanks to the shift, for every time $t \in [t_1, t_2]$ the set $\{x \in \Omega | \phi^h(x,t) = 1\}$ contains at least one of the sets Ω_l^h.

Take now the hypothesis that the smooth subsolution (v, u) satisfies $I_{\Omega,t_1,t_2}(v) < -\alpha$ for some $\alpha > 0$, and let E_h be the cube-wise constant approximation of the error integrand in $\Omega \times [t_1, t_2]$, defined by

$$E_h(x,t) = \frac{1}{2}|v(h\zeta, hi)|^2 - \bar{e}(h\zeta, hi), \text{ for}(x,t) \in C_{\zeta,i}.$$

Uniform continuity of v and $\bar{e}$ in $\Omega \times [t_1, t_2]$, gives us the following, for $l = 1, 2$:

$$\lim_{h\to 0} \int_{\Omega_l^h} E_h(x,t)dx = \frac{1}{2}\left(\frac{3}{4}\right)^d \int_\Omega \left(\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right) dx$$

uniformly in $t \in [t_1, t_2]$. Consequently, there exists a constant $c > 0$ for which $\int_\Omega (\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t))dx \le -\frac{\alpha}{2}$ implies

$$\int_{\Omega_l^h} |E_h(x,t)|dx \ge c\alpha \tag{2.35}$$

if h is small enough. Define $z_{\zeta,i} = (v(h\zeta, hi), u(h\zeta, hi))$. Now, if for sufficiently small $\delta > 0$, we have $C_{\zeta,i} \subset \Omega \times [t_1 - \delta, t_2 + \delta]$, Lemma 2.2.17 assures us that there exists

$\bar{z}_{\zeta,i} = (\bar{v}_{\zeta.i}, \bar{u}_{\zeta,i}) \in \mathbb{R}^d \times \mathcal{S}_0^d$, such that every point in the line segment

$$\sigma_{\zeta,i} = [z_{\zeta,i} - \bar{z}_{\zeta,i}, z_{\zeta,i} + \bar{z}_{\zeta,i}]$$

has generalised energy less than $\bar{e}(h\zeta, hi)$, and such that

$$|\bar{v}_{\zeta,i}|^2 \geq \frac{C}{\bar{e}(h\zeta, hi)} |E_h(h\zeta, hi)|^2 \geq \frac{C}{M} |E_h(h\zeta, hi)|^2, \tag{2.36}$$

where we take $r = \sqrt{2\bar{e}(h\zeta, hi)}$ and fix $M = \sup\{\bar{e}(x,t) | (x,t) \in \Omega \times [t_1 - \delta, t_2 + \delta]\}$. Finally, uniform continuity of $z := (v, u)$ and $\bar{e}$ allows us to choose h small enough to make

$$e(z(x,t) + \lambda \bar{z}_{\zeta,i}) < \bar{e}(x,t) \tag{2.37}$$

for every $\lambda \in [-1, 1]$ and $(x,t) \in C_{\zeta,i}$. We fix h small enough to make all estimates so far valid.

Now to define the perturbations. Consider a fixed (ζ, i). Let $\bar{U}_{\zeta,i}$ be the matrix corresponding to $(\bar{v}_{\zeta,i}, \bar{u}_{\zeta,i}, \bar{q}_{\zeta,i})$, meaning there exists $\eta_{\zeta,i} \in \mathbb{R}^{d+1}$, such that $h(y \cdot \eta_{\zeta,i})\bar{U}_{\zeta,i}$ solves (2.31) for any wave profile h. Moreover, since $|\bar{v}_{\zeta,i}| > 0$, we have that $\bar{\eta}_{\zeta,i}$ is not parallel to the time direction e_{d+1}. Let's assume for the moment that $\bar{\eta}_{\zeta,i} = e_1$.

Then we define a tensor field E_{jk}^{lm}, $j, k, l, m = 1, ..., d+1$, by

$$E_{j1}^{k1} = -E_{1j}^{k1} = -E_{j1}^{1k} = E_{1j}^{1k} = (\bar{U}_{\zeta,i})_{jk} \frac{\sin(Ny_1)}{N^2},$$

with all other entries zero. This tensor satisfies all conditions required in Lemma 2.2.9, and it holds that

$$\mathcal{L}(E) = \bar{U}_{\zeta,i} \sin(Ny_1),$$

where $\mathcal{L}$ is the differential operator defined there 2.2.9. Now let $\chi_{\zeta,i}$ be the characteristic function of $C_{\zeta,i}$, and consider the *cutoff* function $\phi_{\zeta,i} := \phi^h \chi_{\zeta,i}$. Since $\mathcal{L}$ is a homogeneous second order differential operator, we have that

$$\begin{aligned} \|\mathcal{L}(\phi_{\zeta,i}E) - \phi_{\zeta,i}\mathcal{L}(E)\|_\infty &\leq C\|\phi_{\zeta,i}\|_{C^2}\|E\|_{C^1} \\ &\leq C\|\phi_{\zeta,i}\|_{C^2}\frac{1}{N}, \end{aligned} \tag{2.38}$$

where C is independent of N. The case $\eta_{\zeta,i} \neq e_1$ can be reduced to the previous case through a simple linear algebra exercise, by utilising the Galilean invariance of (2.27). This can be found in [8]. The perturbation is then defined as

$$\tilde{U}_N := \sum_{(\zeta,i):C_{\zeta,i}\in\Omega\times[t_1-\delta,t_2+\delta]} \mathcal{L}(\phi_{\zeta,i}E)$$

with the perturbed subsolution given by

$$(v_N, u_N) = (v, u) + (\tilde{v}_N, \tilde{u}_N),$$

where $(\tilde{v}_N, \tilde{u}_N)$ is obtained from $\tilde{U}_N$ by the linear isomorphism in Proposition 2.2.8(i). Together with (2.37) and (2.38), we get that $\tilde{v}_N \in X_0$ if N is large enough. We recall now that, at each time $t \in [t_1, t_2]$, there exists $l \in \{1, 2\}$ for which $\phi^h(x,t) \equiv 1$ if $x \in \Omega_l^h$. If $\tilde{Q}_\zeta \subset \Omega_l^h$, then, we have

$$\begin{aligned}\lim_{N\to\infty} \int_{\tilde{Q}_\zeta} |\tilde{v}_N(x,t)|^2 dx &= \lim_{N\to\infty} \int_{\tilde{Q}_\zeta} |\bar{v}_{\zeta,i}|^2 \sin^2(N\eta_{\zeta,i}\cdot(x,t))dx \\ &= \frac{1}{2}\int_{\tilde{Q}_\zeta} |\bar{v}_{\zeta,i}|^2 dx\end{aligned}$$

uniformly in t, since $\eta_{\zeta,i}$ is not parallel to e_{d+1}. In this step, i is determined from t. From (2.36) we can then arrive at

$$\lim_{N\to\infty} \int_{\Omega_l^h} \frac{1}{2}|\tilde{v}_N(x,t)|^2 dx \geq \frac{C}{M}\int_{\Omega_l^h} |E_h(x,t)|^2 dx \tag{2.39}$$

uniformly in t, for an appropriate $l = l(t)$.

Finally, if $t \in [t_1, t_2]$, then by definition on v_N, it holds that

$$\int_\Omega \left(\frac{1}{2}|v_N(x,t)|^2 - \bar{e}(x,t)\right) dx$$

$$= \int_\Omega \left(\frac{1}{2}|v|^2(x,t) - \bar{e}(x,t)\right) dx + \int_\Omega \frac{1}{2}|\tilde{v}_N(x,t)|^2 dx + \int_\Omega \tilde{v}_N(x,t)\cdot v(x,t)dx.$$

Since $\tilde{v}_N$ converges weakly to 0, uniformly in t, the last integral can be made arbitrarily small. Therefore, thanks to (2.39), we can estimate

$$\begin{aligned}\liminf_{N\to\infty} I_{\Omega,t_1,t_2}(v_N) &\geq \inf_{t\in[t_1,t_2]} \left(\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e}\right) dx + \int_\Omega \frac{1}{2}|\tilde{v}_N|^2 dx\right) \\ &\geq \inf_{t\in[t_1,t_2]} \left[\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e}\right) dx + \frac{C}{M}\min_{l\in\{1,2\}} \int_{\Omega_l^h} |E_h|^2 dx\right] \\ &\geq \inf_{t\in[t_1,t_2]} \left[\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e}\right) dx + \frac{C}{\mathcal{L}^d(\Omega)M}\min_{l\in\{1,2\}} \left(\int_{\Omega_l^h} |E_h| dx\right)^2\right].\end{aligned}$$

Taking (2.35) into account, we conclude that

$$\liminf_{N\to\infty} I_{\Omega,t_1,t_2}(v_N) \geq \min\left\{-\frac{\alpha}{2}, I_{\Omega,t_1,t_2}(v) + \frac{C}{\mathcal{L}^d(\Omega)M}\alpha^2\right\}$$
$$I_{\Omega,t_1,t_2}(v) + \min\left\{\frac{\alpha}{2}, \frac{C}{\mathcal{L}^d(\Omega)M}\alpha^2\right\},$$

since, by hypothesis, $I_{\Omega,t_1,t_2}(v) > -\alpha$. this proves Proposition 2.2.16, with

$$\beta = \min\left\{\frac{\alpha}{2}, \frac{C}{\mathcal{L}^d(\Omega)M}\alpha^2\right\}$$

Now we can conclude the proof of Theorem 2.2.2. The lower bound and lower semi-continuity of I_{Ω,t_1,t_2} imply, [see Proposition 7.11 in [14]] that I_{Ω,t_1,t_2} is a Baire-1 map, so that its points of continuity form a residual set Ξ_{Ω,t_1,t_2} in (X,d). However, the perturbation property shows us that any v for which $I_{\Omega,t_1,t_2}(v) < 0$ cannot be a point of continuity, so that any point of continuity v of I_{Ω,t_1,t_2} must satisfy $I_{\Omega,t_1,t_2}(v) = 0$. At last, we can use an exhaustion argument for $\Omega_k \nearrow \mathbb{R}^d$ e $[t_1^k, t_2^k] \nearrow [0,T]$, to obtain a residual set $\Xi = \bigcap_k \Xi_{\Omega_k,t_1^k,t_2^k}$, for which it holds that, if $x \in \Xi$, then we must have $\liminf_{k\to\infty} I_{\Omega,t_1,t_2}(v) = 0$ for every Ω, t_1 and t_2. It follows that v is a weak solution to Euler, as desired. Note that, by the hypotheses in Theorem 2.2.2, X_0, and therefore X, is non-empty. Moreover, one can see by adding sufficiently small amplitude plane waves, that X_0 actually has infinite cardinality, so that Ξ, as a residual set of an infinite metric space, must also be infinite (and dense in X). This proves Theorem 2.2.2.

Chapter 3

Young Measures

3.1 Classical Young Measure theory

In this chapter we look at another tool developed in order to deal with the failure of strong convergence in sequences of solutions to a PDE system. It arose through the method of compensated compactness developed by L. Tartar in [24]), which takes advantage of the interplay between properties of weaker limits and differential constraints on a sequence of solutions, to obtain better convergence results. A key instrument utilised by Tartar relies on embedding the function space where one could not obtain the desired convergence into a larger space, whose topological and analytical properties allow the existence of a limit. The theory of Young measures is one application of the compensated compactness method, in which we embed a set of functions $f : \Omega \to \mathbb{R}^d$ into a space of parametrised measures, through the mapping:

$$f(x) \mapsto \delta_{f(x)} \in M(\mathbb{R}^d).$$

In this context, typical behavior which is not perceivable to the weak topology, such as rapid oscillations or concentration, can be recorded in the limit as the collapse of the atomic measure δ into more structurally complex measures. One is interested in finding ways to represent and manipulate these limits, in order to obtain results for which the failure of convergence was an impediment. Among some of the first applications of this theory to evolution equations is the study of L^∞-bounded sequences of solutions to scalar conservation laws. We will now introduce the theory as exposed in [11].

Let $\Omega \subset \mathbb{R}^n$ be a given domain, and take an arbitrary sequence (v_k) of vector fields $v_k : \Omega \to \mathbb{R}^d$, obeying an uniform L^∞ bound, i.e., $|v_k(y)| \leq C$ for a.e. $y \in \Omega$ and every $k \in \mathbb{N}$, and suppose it converges weakly to a function v in some space $L^p(\Omega; \mathbb{R}^d)$, for $1 \leq p < \infty$. We can best describe the behaviour of the sequence in the following manner:

Theorem 3.1.1 (Fundamental theorem of Young Measures)**.** Let (v_k), v be as above and assume further that $v_k(y) \in K$ a.e., where $K \subset \mathbb{R}^d$ is bounded. Then, there exists a subsequence (v_m) and a Lebesgue-measurable mapping

$$y \to \nu_y \in M^1(\mathbb{R}^d),$$

such that supp $\nu_y \subset \bar{K}$, and also the limit

$$f(v_m(y))dm \overset{*}{\rightharpoonup} \langle \nu_y, f\rangle dm \text{ in } M(\Omega) \tag{3.1}$$

holds for every continuous function $f : \mathbb{R}^d \to \mathbb{R}$, that is, for every $\phi \in C_0(\Omega)$, we have:

$$\lim_{m\to\infty} \int \phi(y) f(v_m(y))\, dy = \int \phi(y)\langle \nu_y, f\rangle\, dy.$$

Moreover, for $1 \leq p < \infty$ we have:

$$v_m \to v \text{ in } L^p \iff \nu_y = \delta_{v(y)} \text{ for a.e. } y \in \Omega.$$

Sketch of the proof. The theorem relies on embedding the sequence (v_k) into $M(\mathbb{R}^d \times \Omega)$ and taking advantage of the topological properties it possesses as the dual space of $C_0(\mathbb{R}^d \times \Omega)$. Specifically, we define the sequence μ_k of measures by

$$d\mu_k(x, y) := d\delta_{v_k(y)}(x) \otimes dm(y).$$

One can then obtain from the L^∞ bounds on v_k the boundedness of this sequence in $M(\Omega \times \mathbb{R}^d)$, whence the Banach-Alaoglu theorem guarantees the existence of corresponding subsequences (v_m), (μ_m), and of a weak* limit $\mu \in M(\mathbb{R}^d \times \Omega)$ for which we have

$$\iint_{\mathbb{R}^d\times\Omega} \phi(x, y) d\mu_m(x, y) \to \iint_{\mathbb{R}^d\times\Omega} \phi(x, y) d\mu(x, y),$$

for every $\phi \in C_0(\mathbb{R}^d \times \Omega)$. If we take test functions ϕ depending only on $y \in \Omega$, which is equivalent to testing against the projection of μ to Ω, that is, the measure defined by $\sigma(E) = \mu(\mathbb{R}^d \times E)$, we readily see that σ is equal to the Lebesgue measure on Ω. We can then use the Disintegration Theorem (see the Appendix) to obtain a measurable paramerisation of probability measures ν_y on $\mathbb{R}^d$, for which it holds that $d\mu(x, y) = d\nu_y(x) \otimes dm(y)$.

Next, taking test functions of the form $\phi(x)$ supported in $\mathbb{R}^d \setminus \bar{K}$ shows that the support of ν_y is contained in $\bar{K}$, so that, lastly, for any continuous $f : \mathbb{R}^d \to \mathbb{R}$ and

$\phi \in C_0(\Omega)$ we can evaluate the limit

$$\begin{aligned}\lim \int_\Omega f(v_k(y))\phi(y)dy = \lim \iint_{\mathbb{R}^d\times\Omega} f(x)\phi(y)d\mu_m(x,y) &= \iint_{\mathbb{R}^d\times\Omega} f(x)\phi(y)d\mu(x,y)\\ &= \iint_{\mathbb{R}^d\times\Omega} f(x)\phi(y)d\nu_y(x)dm(y)\\ &= \int_\Omega \langle \nu_y, f\rangle \phi(y)dy\end{aligned}$$

□

We follow this very technical result with an example:

Example 3.1.1 (Oscillations in the Young Measure)**.** Consider on $[0,1]$ the function $w_0(x) = \chi_{(0,1/2)}(x) - \chi_{(1/2,1)}(x)$, extend it periodically to $\mathbb{R}$, and define $w_n(x) = w_0(nx)$. The Young Measure theorem then gives us $\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. In fact, take a test function $\phi \in C_c(\mathbb{R})$ and f bounded and continuous, and write:

$$\begin{aligned}\int \phi(x)f(w_n(x))dx &= f(1)\int_{E_n^1}\phi(x)dx + f(-1)\int_{E_n^2}\phi(x)dx,\\ &= f(1)\int \phi(x)d\mu_n^1(x) + f(-1)\int \phi(x)d\mu_n^2(x),\end{aligned}$$

where $E_n^1 = \bigcup_{k\in\mathbb{Z}}[\frac{k}{n}, \frac{k}{n}+\frac{1}{2n})$ and $E_n^2 = (E_n^1)^c$. Here we define the measures $\mu_n^i(A) = m(A\cap E_n^i)$, for $i = 1,2$, where m denotes Lebesgue measure in $\mathbb{R}$. We have that $\mu_n^1 + \mu_n^2 = m$, and it also holds that $\mu_n^i \to \frac{1}{2}m$ strongly in $M(\mathbb{R})$ for $i = 1,2$. Strong convergence implies weak convergence, and we have

$$\begin{aligned}\lim \int \phi(x)f(w_n(x))dx &= f(1)\lim \int_{\mathbb{R}} \phi\ d\mu_n^1 + f(-1)\lim\int_{\mathbb{R}} \phi\ d\mu_n^2\\ &= \left(\frac{f(1)+f(-1)}{2}\right)\int \phi(x)dx\\ &= \int_{\mathbb{R}} \left\langle \frac{\delta_1+\delta_{-1}}{2}, f\right\rangle \phi(x)dx.\end{aligned}$$

In fact, we have the more general result: For every continuous 1-periodic function w on $\mathbb{R}$, defining $w_n(x) = w(nx)$ we have that if $f:\mathbb{R}\to\mathbb{R}$ is continuous, the limit is given by:

$$f(w_n(x))dx \overset{*}{\rightharpoonup} \alpha dx, \text{ where } \alpha = \int_0^1 f(w(x))dx.$$

As we have seen above, the Young Measure Theorem gives us a form to represent through the measure ν_y, all composite weak limits of the L^∞-bounded sequence (w_k). Non-Dirac structure of the measure can be understood as the persistence of oscillations in the sequence, whereas simple weak convergence in L^p is unable to capture this kind of behaviour.

3.2 Generalised Young Measures

Let us return to the incompressible Euler equations. We are interested in using Young measures to understand the behavior vanishing viscosity sequences of Leray-Hopf solutions to the Navier-Stokes equations, in order to look for solutions in some sense to the incompressible Euler equations. One notes, however, that for these sequences the natural uniform bound, arising from the various energy inequalities, is not in L^∞ but a uniform in time estimate for the L^2 norm. In attempting to apply the methods above, one stumbles upon one type of behavior expected from sequences which satisfy (locally) uniform L^2 bounds, which is not contemplated by the original Young Measure Theorem: concentration of energy. For instance, we have the following very simple example:

Example 3.2.1. Take the sequence (w_k) defined by

$$w_k(x) = k\chi_{(0,\frac{1}{k^2})}(x), \text{ for } x \in [0,1].$$

Such a sequence can be easily seen to satisfy $\|w_k\|_{L^2} = 1$ for every k. However, all energy is increasingly confined to the interval $(0, 1/k^2)$, so that as k grows it becomes concentrated near the origin. We can see that for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$, we have in $M([0,1])$ the weak* limit $f(w_k(x))dx \overset{*}{\rightharpoonup} f(0)dx$, as expected from the Young measure theorem, but the energy satisfies $|w_k(x)|^2 dx \overset{*}{\rightharpoonup} d\delta_0(x)$. This shows that the behavior of the composite sequence $f(w_k)$ is distinct between bounded and quadratic f, and the Young measure theorem can no longer provide a representation valid for all continuous f.

Since the weak formulation of the incompressible Euler equations includes the non-linear unbounded term $v \otimes v$, and it is desirable also to be able to understand what happens to the energy density $\frac{1}{2}|v|^2$, this is a problem we cannot simply ignore, if we intend to develop the framework of Young measures applied to the Euler equations. In that regard, we will develop a theory of generalised Young Measures which is equipped to deal with such cases, following the results developed by DiPerna and Majda, Alibert and Bouchitté in [11, 2]. We will see that it is still possible to represent the behaviour of $f(w_k)$ similarly to (3.1) for a wide class of functions, but we will need to restrict their growth.

In fact, considering the L^2 bound of the sequences considered, one cannot expect to be able to know the behaviour of these composite weak limits if they present higher than quadratic growth. Conversely, we need to include quadratic functions like $w \otimes w$ and $|w_k|^2$, and expect their asymptotic behaviour to be the determinant factor to interfere in the weak limits (since wherever the sequence (w_k) is L^∞ bounded the original Young measure is sufficient to explain its behaviour). Contemplating other applications, we generalise these considerations and aim at developing a theory of Young measures arising from bounded sequences in L^p, for $p \geq 1$, and expect to obtain representations of the composite weak limits if f has growth of order at most p. In that regard, the largest class of functions we hope to consider is of the form

$$f(w) = \tilde{f}(w)(1 + |w|^p),$$

assuming merely $\tilde{f} \in BC(\mathbb{R}^d)$. For this we have the most general result

Theorem 3.2.1 (Theorem 4.1 in [11])**.** Let Ω be a bounded domain in $\mathbb{R}^n$ and (w_k) a bounded sequence in $L^p(\Omega; \mathbb{R}^d)$. There exists a subsequence (w_m), a non-negative measure $\sigma \in M^+(\Omega)$ and a bounded linear transformation

$$T : BC(\mathbb{R}^d) \to L^\infty(\sigma),$$

such that

$$(1 + |w_m|^p)\, dy \overset{*}{\rightharpoonup} \sigma, \quad \text{and}$$

$$f(w_m(y))dy \overset{*}{\rightharpoonup} T(\tilde{f})d\sigma.$$

Note, however, that the generality of this result does not provide any information about what happens to the composite weak limit, since this behavior is hidden within the all-encompassing linear transformation T. We can understand what happens under this linear transformation and find concrete representations of it, by specifying restrictions on the space of admissible f. Note also that direct dependency of the function f on $y \in \Omega$ can be admitted with only the slightest conditions of boundedness and continuity on $\bar{\Omega}$, without altering the results. Besides, the results also hold for unbounded Ω and $f \in C_c^\infty$, by an exhaustion argument. With this intent, a very general representation can still be obtained if we consider the concept of the L^p-recession function:

Definition 3.2.2. For $f \in C(\Omega \times \mathbb{R}^d)$, define the L^p *recession function* f^∞ as

$$f^\infty(y, z) = \lim_{\substack{y' \to y \\ z' \to z \\ s \to \infty}} \frac{f(y', sz')}{s^p},$$

whenever the limit exists. Moreover, define the class of functions

$$\mathcal{F}^p(\Omega) = \left\{ f \in C(\Omega \times \mathbb{R}^d) : f^\infty \text{ exists and is continuous on } S^{d-1} \right\}.$$

For f which does not depend on $y \in \Omega$, we adopt the simplified notation $\mathcal{F}^p$.

For this class of functions we have the following result:

Theorem 3.2.2 (Adapted from Theorem 2.5 in [2])**.** Let $1 \leq p < \infty$, and take (w_k) a (locally) uniformly bounded sequence in $L^p(\Omega, \mathbb{R}^d)$. There exists a subsequence (w_m), a measure $\lambda \in M^+(\Omega)$, and parametrisations

$$y \in \Omega \mapsto \nu_y \in M^1(\mathbb{R}^d) \text{ Lebesgue measurable, and}$$

$$y \in \Omega \mapsto \nu_y^\infty \in M^1(S^{d-1}), \ \lambda\text{-measurable,}$$

which satisfy in $M(\Omega)$, for every $f \in \mathcal{F}^p(\Omega)$, the weak* limit

$$f(y, w_m(y))dy \overset{*}{\rightharpoonup} \langle \nu_y, f(y, \cdot)\rangle dy + \langle \nu_y^\infty, f^\infty(y, \cdot)\rangle d\lambda(y).$$

Moreover, it holds that

$$\int_\Omega \langle \nu_y, |\cdot|^p\rangle < \infty.$$

The triplet $(\nu, \lambda, \nu^\infty)$ is called the *(generalised) Young measure generated* by the subsequence (w_m), and this is denoted by

$$w_m \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty).$$

When specification is necessary, we say $(\nu, \lambda, \nu^\infty)$ is of type p to indicate the space of functions in which the composite limit can be evaluated.

We will also use the notation

$$\langle\!\langle \nu, \lambda, \nu^\infty; f\rangle\!\rangle = \int_\Omega \langle \nu_y, f\rangle dy + \int_\Omega \langle \nu_y^\infty, f^\infty\rangle d\lambda(y),$$

so that the main result of the theorem can be rewritten as:

$$\int_\Omega f(y, w_n(y))\, dy \to \langle\!\langle \nu, \lambda, \nu^\infty; f\rangle\!\rangle \text{ for all } f \in \mathcal{F}^p(\Omega).$$

Similarly, we define a notion of weak convergence on the set of Young measures: we say that $(\nu^k, \lambda^k, \nu^{\infty,k}) \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty)$ if it holds that

$$\langle\!\langle \nu^k, \lambda^k, \nu^{\infty,k}; f\rangle\!\rangle \to \langle\!\langle \nu, \lambda, \nu^\infty; f\rangle\!\rangle \text{ for all } f \in \mathcal{F}^p(\Omega).$$

In fact, identifying the functions w_k with Dirac measures δ_{w_k}, the result of the theorem becomes a specific case of this notion of convergence. Let us see some properties of these generalised Young Measures:

Proposition 3.2.3 (Prop. 5 in [23])**.** The following properties hold:

(*i*) There exists a countable set of functions $f_k = \phi_k \otimes h_k$, where $k \in \mathbb{N}$, $\phi_k \in C_c(\Omega)$,

$h_k \in \mathcal{F}^p$, such that

$$\langle\!\langle \nu, \lambda, \nu^\infty; f_k \rangle\!\rangle = \langle\!\langle \tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty; f_k \rangle\!\rangle \text{ for all } k \implies (\nu, \lambda, \nu^\infty) = (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty).$$

(*ii*) If $w_n \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty)$, $\tilde{w}_n \xrightarrow{\mathbf{Y}_p} (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$, and $w_n - \tilde{w}_n \to 0$ locally in measure, then $\nu = \tilde{\nu}$.

(*iii*) If $w_n \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty)$ and $w_n - \tilde{w}_n \to 0$ in L^p_{loc}, then $\tilde{w}_n \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty)$.

(*iv*) $w_n \to w$ strongly in L^p_{loc} if and only if $w_n \xrightarrow{\mathbf{Y}_p} \delta_w$.

(*v*) (*Translation*) Suppose $w_n \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty)$, and let $w \in L^p(\Omega)$. Then, $w_n + w \xrightarrow{\mathbf{Y}_p} (T_w\nu, \lambda, \nu^\infty)$, where $T_w\nu$ is the parametrised measure defined by the identity

$$\langle (T_w\nu)_y, f \rangle = \int_{\mathbb{R}^l} f(z + w(y)) d\nu_y(z) \text{ for every } f \in C_0(\mathbb{R}^l), \text{ and a.e. } y \in \Omega.$$

An immediate consequence of property (*i*), is that the notion of convergence introduced is metrisable on bounded sets. This allows us to extract diagonal sequences as follows:

Proposition 3.2.4. Supposes that, for all $k \in \mathbb{N}$, it holds that

$$(\nu^{k,n}, \lambda^{k,n}, \nu^{\infty,k,n}) \xrightarrow{\mathbf{Y}_p} (\nu^k, \lambda^k, \nu^{\infty,k}),$$

and also that

$$(\nu^k, \lambda^k, \nu^{\infty,k}) \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty).$$

Then, there exists a sequence $n(k) \xrightarrow{k\to\infty} \infty$ satisfying

$$(\nu^{k,n(k)}, \lambda^{k,n(k)}, \nu^{\infty,k,n(k)}) \xrightarrow{\mathbf{Y}_p} (\nu, \lambda, \nu^\infty).$$

Chapter 4

Measure-valued solutions

4.1 Measure-valued solutions to the Euler Equations

Following the work done in [23], we apply the theory of generalised Young measures to obtain a definition of measure-valued solutions to the Euler equations. Remember the weak formulation of the Euler equations: Given an initial data $v_0 \in L^2(\mathbb{R}^d)$ such that div $v_0 = 0$ weakly, and a positive time $0 < T \leq \infty$, a vector field $v \in L^2_{loc}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ is a *weak solution* if it is weakly divergence-free and also satisfies

$$\int_0^T \int_{\mathbb{R}^d} (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dxdt + \int_{\mathbb{R}^d} v_0(x)\phi(x,0)dx = 0, \tag{4.1}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T); \mathbb{R}^d)$ com div $\phi = 0$.

Suppose now that we have a sequence (v_k) of weak solutions, (locally) uniformly bounded in $L^2(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$. Such a sequence will generate a generalised Young measure $(\nu, \lambda, \nu^\infty)$ of type 2 in $\mathbb{R}^d$, parametrised over $\mathbb{R}^d \times [0,T]$, and we can evaluate according to Theorem 3.2.2 the integrals in (4.1) to arrive at

$$\int_0^T \int_{\mathbb{R}^d} (\langle \nu, \xi \rangle \cdot \partial_t \phi + \langle \nu, \xi \otimes \xi \rangle : \nabla \phi) dxdt + \int_{\mathbb{R}^d \times (0,T)} \nabla \phi : \langle \nu^\infty, \theta \otimes \theta \rangle d\lambda = 0. \tag{4.2}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T); \mathbb{R}^d)$ such that div $\phi = 0$.

Similarly, we obtain that

$$\int \nabla \psi \cdot \langle \nu_{x,t}, \xi \rangle dx = 0, \tag{4.3}$$

for all $\psi \in C_c^\infty(\mathbb{R}^d)$ and almost every $t \in [0,T]$. By defining $\bar{v}(x,t) = \langle \nu_{x,t}, \xi \rangle$ as the *barycenter* of $\nu_{x,t}$, (4.3) is equivalent to saying that $\bar{v}$ is weakly divergence-free.

Moreover, remembering that we are looking for solutions arising from the vanishing viscosity limit of Leray-Hopf solutions, we can assume further that the sequence (v_k) in fact satisfies the stronger uniform bound $v_k \in L^\infty(0,T;L^2(\mathbb{R}^d)) = L^\infty_t L^2_x$. If we require that measure-valued solutions also satisfy this bound, we can also gain further knowledge of the structure of the measure. That is, we have the following:

Proposition 4.1.1. Let (v_k) be a sequence of vector fields bounded in $L^\infty_t L^2_x$, generating the generalised Youn measure $(\nu, \lambda, \nu^\infty)$ in $L^2(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$. We have the following:

$$\mathrm{esssup}_t \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle dx \right) < \infty, \tag{4.4}$$

and the measure λ admits a disintegration of the form

$$d\lambda(x,t) = d\lambda_t(x) \otimes dt, \tag{4.5}$$

where $t \mapsto \lambda_t$ is a measurable parametrisation which is bounded in the total variation norm:

$$\lambda_{(\cdot)} : [0,T] \to M^+(\mathbb{R}^d).$$

The first property guarantees, by Jensen's Inequality, that $\bar{v} \in L^\infty_t L^2_x$, which combined to (4.2), let us obtain, as in Theorem 2.1.1, the following:

Proposition 4.1.2. Given a generalised Young measure $(\nu, \lambda, \nu^\infty)$ in L^2, generated by a sequence in $L^\infty_t L^2_x$, which satisfies (4.2), the barycenter $\bar{v}(x,t) \in L^\infty_t L^2_x$ has a representative $\tilde{v} \in CL^2_w = C([0,T];L^2_w(\mathbb{R}^d))$, i.e. $\bar{v}(\cdot,t) = \tilde{v}(\cdot,t)$ for a.e. $t \in [0,T]$ if viewed as L^2 functions.

Now, knowing that $\bar{v} \in CL^2_w$, the initial average $\bar{v}(x,0)$ is well defined as a L^2 function, and is attained in the sense that $\bar{v}(\cdot,t) \overset{t\to 0}{\rightharpoonup} \bar{v}(\cdot,0)$ weakly in L^2. Then, we can expand (4.2) to obtain by the Young measure theorem:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (\langle \nu, \xi \rangle \cdot \partial_t \phi + \langle \nu, \xi \otimes \xi \rangle : \nabla \phi) dx dt + \int_0^T \int_{\mathbb{R}^d} \nabla \phi : \langle \nu^\infty, \theta \otimes \theta \rangle \lambda_t(dx) dt \\ = - \int_{\mathbb{R}^d} \phi(x,0) \bar{v}(x,0) dx,\end{aligned} \tag{4.6}$$

for all $\phi \in C^\infty_c(\mathbb{R}^d \times [0,T), \mathbb{R}^d)$ with div $\phi = 0$.

Lastly, we can define for almost all time the *energy* of a Young measure satisfying all properties above, by

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle dx + \frac{1}{2} \lambda_t(\mathbb{R}^d). \tag{4.7}$$

Having these considerations in mind, we can properly define measure-valued solutions:

Definition 4.1.3 (Measure valued solutions to the incompressible Euler equations)**.** Let $(\nu, \lambda, \nu^\infty)$ be a generalised Young measure of type 2 in $\mathbb{R}^d$, with parameters in $\mathbb{R}^d \times [0, T]$ and barycenter $\bar{v} = \langle \nu, \xi \rangle$. Then:

(*i*) we say $(\nu, \lambda, \nu^\infty)$ is a *measure-valued solution* to the incompressible Euler equations if it satisfies (4.2) and (4.3).

(*ii*) we say $(\nu, \lambda, \nu^\infty)$ is an *admissible measure-valued solution* to the incompressible Euler equations with initial data $v_0 \in L^2(\mathbb{R}^d)$ if it satisfies (4.3)-(4.6), $\bar{v}(\cdot, 0) = v_0$, and

$$E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx \text{ for a.e. } t > 0. \tag{4.8}$$

Proposition 4.1.4. Let $(\nu, \lambda, \nu^\infty)$ be an admissible measure-valued solution to the Euler equations and $\bar{v}$ its barycenter. Then it holds that

$$\bar{v}(\cdot, t) \to \bar{v}(\cdot, 0) = v_0$$

strongly in $L^2(\mathbb{R}^d)$, as $t \to 0$.

Armed with this very weak notion of solution, we show a very simple criterion from [11], for a sequence of functions to generate a measure-valued solution:

Proposition 4.1.5. Let (v_k) be a sequence of weakly divergence-free vector fields defined on $\Omega \subset \mathbb{R}^d \times [0, T]$ for $0 < T \leq \infty$, satisfying:

(*i*) *Weak Stability:* For some constant $C(t) > 0$, we have for all $k \in \mathbb{N}$ and for every finite time t with $0 < t \leq T$

$$\iint_{\Omega \cap (\mathbb{R}^d \times [0,t])} |v_k(x, \tau)|^2 dx d\tau \leq C(t).$$

(*ii*) *Weak Consistency:* For every divergence-free test function $\phi \in C_c^\infty(\Omega)$,

$$\lim_k \iint_\Omega \phi_t \cdot v_k + \nabla\phi : v_k \otimes v_k \, dxdt = 0. \tag{4.9}$$

Then any generalised Young measure $(\nu, \lambda, \nu^\infty)$ generated by a subsequence of (v_k) defines a measure-valued solution of the ideal incompressible Euler equations on Ω.

In particular, we can generate admissible measure-valued solutions of Euler from the vanishing viscosity limit of the Navier-Stokes equations:

Theorem 4.1.1. Let v_0 be a weakly divergence-free vector field in $L^2(\mathbb{R}^d)$ and for $\varepsilon > 0$ let v_ε be a Leray-Hopf solution of the Navier-Stokes equations with initial data v_0 and viscosity $\varepsilon > 0$, and suppose they are all defined up to a time $0 < T \leq \infty$. Then, any generalised Young measure generated by a sequence $(v_{\varepsilon_k})_{\varepsilon_k \to 0}$ defines an admissible measure-valued solution to the incompressible Euler equations on $\mathbb{R}^d \times [0, T]$, with initial data v_0.

Proof. Let $(\nu, \lambda, \nu^\infty)$ be the generalised Young measure generated by a sequence $(v_{\varepsilon_k})_{\varepsilon_k \to 0}$. First we note that for all $\varepsilon > 0$ the solution v_ε will satisfy, from the strong energy inequality (2.3), the uniform energy bound

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} |v_\varepsilon(x,t)|^2 dx \leq \int_{\mathbb{R}^d} |v_0(x)|^2 dx,$$

so that weak stability and boundedness of (v_{ε_k}) in $L^\infty_t L^2_x$ is guaranteed, as well as admissibility of $(\nu, \lambda, \nu^\infty)$, as long as it defines a solution.

Also, the weak formulation of the Navier-Stokes equations with initial data v_0 reads as

$$\iint_{\mathbb{R}^d \times [0,T]} (\phi_t \cdot v_\varepsilon + \nabla \phi : v_\varepsilon \otimes v_\varepsilon) dx dt + \int_{\mathbb{R}^d} \phi(x, 0) \cdot v_0(x) dx = \varepsilon \iint \Delta \phi \cdot v_\varepsilon dx dt,$$

which, since $|\iint \Delta\phi \cdot v_\varepsilon| \leq \|\Delta \phi\|_{L^1_t L^2_x} \|v_\varepsilon\|_{L^\infty_t L^2_x} \leq C \|v_0\|_{L^2}$, implies weak consistency as $\varepsilon \to 0$, as well as attainment of the initial data, so that by the proposition above, $(\nu, \lambda, \nu^\infty)$ defines an admissible measure-valued solution to the incompressible Euler equations with initial data v_0. $\square$

Since for $d = 3$ global existence of Leray-Hopf solutions is guaranteed, we immediately obtain as a consequence the global existence of admissible measure-valued solutions to the incompressible Euler equations for any initial data $v_0 \in L^2(\mathbb{R}^3)$. Having guaranteed existence, another essential property in validating the notion of measure-valued solutions we have developed is the weak-strong uniqueness property. This assures us that in weakening the notion of solution we have not gone too far and allowed wild solutions to exist even when existence of a classical solution can be obtained. We will adapt the proof of this result in the case of planar symmetry MHD, so we choose to omit it in this instance. Specifically, we have for measure-valued solutions of the incompressible Euler equations the following result:

Theorem 4.1.2 (Weak-Strong Uniqueness, Theorem 2 in [4])**.** Let $v_0 \in L^2(\mathbb{R}^d)$ with div $v_0 = 0$. Suppose that we have $v \in C([0, T], L^2(\mathbb{R}^d))$ a classical solution to the Euler equations with initial data v_0, and suppose further that we also have

$$\int_0^T \|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^d)} dt < \infty, \tag{4.10}$$

and let $(\nu, \lambda, \nu^\infty)$ be an admissible measure-valued solution. Then $\lambda = 0$ and $\nu_{x,t} = \delta_{v(x,t)}$ for almost every (x, t).

Finally, we have the following result, which was obtained by connecting the non-uniqueness result of Theorem 2.2.2 and the notion of measure-valued solutions:

Theorem 4.1.3 (Theorems 4.1−2 in [25])**.** A type 2 generalised Young measure $(\nu, \lambda, \nu^\infty)$ on $\mathbb{R}^d$ with parameters on $\mathbb{R}^d \times [0, T]$ is a measure-valued solution of the Euler equations with bounded energy if and only if it can be generated by a sequence $(v_k)_{k\in\mathbb{N}}$ of weak solutions to the Euler equations bounded in $C([0, T]; L^2_w(\mathbb{R}^d, \mathbb{R}^d))$. Moreover, if it is an admissible measure-valued solution with initial data $v_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$, the generating sequence can be chosen to satisfy also

$$\|v_k(\cdot, 0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{k}$$

and

$$\sup_{t\in[0,T]} \frac{1}{2} \int_{\mathbb{R}^d} |v_k(x, t)|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_k(x, 0)|^2 dx$$

4.2 Measure-valued solutions to the MHD Equations

We now proceed similarly to define measure-valued solutions to the ideal incompressible MHD equations. First, recall the weak formulation of these equations: Given initial velocity and magnetic fields, respectively $v_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ and $B_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d)$, both weakly divergence-free, and a positive time $0 < T \leq \infty$, a pair $(v, B) \in L^2_{loc}(\mathbb{R}^d \times [0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ is a *weak solution* to the ideal MHD if v, B are weakly divergence-free and satisfy:

$$\int_0^T \int_{\mathbb{R}^d} [v \cdot \partial_t\varphi + (v \otimes v - B \otimes B) : \nabla\varphi]\, dxdt + \int_{\mathbb{R}^d} v_0(x) \cdot \varphi(x, 0)dx = 0$$
$$\int_0^T \int_{\mathbb{R}^d} [B \cdot \partial_t\varphi + (B \otimes v - v \otimes B) : \nabla\varphi]\, dxdt + \int_{\mathbb{R}^d} B_0(x) \cdot \varphi(x, 0)dx = 0$$

for all $\phi \in C^\infty_c(\mathbb{R}^d \times [0, T); \mathbb{R}^d)$ with div $\varphi = 0$.

Suppose now that we have a sequence (v_k, B_k) of weak solutions. Note that the natural energy bounds for the MHD equations allow us to require that both v and B obey the same L^2 bound, so that the sequence (v_k, B_k) is uniformly bounded in $L^2(\mathbb{R}^d \times [0, T]; \mathbb{R}^d \times \mathbb{R}^d)$. For a discussion of Young measures generated by sequences satisfying nonhomogeneous integrability conditions, see [15]. The sequence (v_k, B_k) will then generate a generalised Young measure $(\nu, \lambda, \nu^\infty)$ of type 2 on $\mathbb{R}^d \times \mathbb{R}^d = \{(\xi, \zeta)\}$, with parameters in $\mathbb{R}^d \times [0, T]$, with Theorem 3.2.2 allowing us to obtain in the limit:

$$\int_0^T \int_{\mathbb{R}^d} [\langle \nu, \xi \rangle \cdot \partial_t \varphi + \langle \nu, (\xi \otimes \xi - \zeta \otimes \zeta) \rangle : \nabla \varphi] \, dxdt \\ + \int_{\mathbb{R}^d \times (0,T)} \langle \nu^\infty, (\xi \otimes \xi - \zeta \otimes \zeta)^\infty \rangle : \nabla \varphi \; d\lambda(x,t) = 0 \tag{4.11}$$

$$\int_0^T \int_{\mathbb{R}^d} [\langle \nu, \zeta \rangle \cdot \partial_t \varphi + \langle \nu, (\zeta \otimes \xi - \xi \otimes \zeta) \rangle : \nabla \varphi] \, dxdt \\ + \int_{\mathbb{R}^d \times (0,T)} \langle \nu^\infty, (\zeta \otimes \xi - \xi \otimes \zeta)^\infty \rangle : \nabla \varphi \; d\lambda(x,t) = 0 \tag{4.12}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d \times (0,T); \mathbb{R}^d)$ with div $\varphi = 0$, where $(\xi, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d$ represents respectively the fields v and B, so that in the case that $\nu_{x,t} = \delta_{(v,B)(x,t)}$ and $\lambda = 0$, we recover $\langle \nu_{x,t}, \xi \rangle = v(x,t)$ and $\langle \nu_{x,t}, \zeta \rangle = B(x,t)$.

Alongside this, we obtain also

$$\begin{aligned} \int \nabla \psi \cdot \langle \nu_{x,t}, \xi \rangle dx = 0 \\ \int \nabla \psi \cdot \langle \nu_{x,t}, \zeta \rangle dx = 0, \end{aligned} \tag{4.13}$$

for all $\psi \in C_c^\infty(\mathbb{R}^d)$ and almost every $t \in [0,T]$.

Defining $\bar{v}(x,t) = \langle \nu_{x,t}, \xi \rangle$ and $\bar{B}(x,t) = \langle \nu_{x,t}, \zeta \rangle$ as the *barycenters* of $\nu_{x,t}$, (4.13) is equivalent to the requirement that $\bar{v}$ and $\bar{B}$ are weakly divergence-free.

It is convenient to remark that, while the definition presented above is the most general, and allows us to manage composite weak limits $f(x,t;v,b)$ for all $f \in \mathcal{F}_2(\mathbb{R}^d \times [0,t], \mathbb{R}^d \times \mathbb{R}^d)$, this class of functions presents a difficulty in interpretation, when it comes possible concentration behaviour. This comes because the space where the concentration angle measure ν^∞ is defined, which is S^{2d-1}, does not allow us to understand where the concentration effects may arise whether only on the velocity v, the magnetic field B, or both.

If we consider, however, for a sequence (v_k, B_k) bounded in L^2, the component sequences $(v_k), (B_k)$ and, after passing through a subsequence common to all limits taken, the respective generalised Young measures generated by each, denoted $(\nu_v, \lambda_v, \nu_v^\infty)$ for (v_k) and $(\nu_B, \lambda_B, \nu_B^\infty)$ for (B_k), we can gain some insight.

Specifically, since we know λ is the singular part of the weak* limit $|v_k|^2 + |B_k|^2 dxdt$, and likewise, λ_v, λ_B the respective singular parts of the limits $|v_k|^2 dxdt$, $|B_k|^2 dxdt$, we can assert that $\lambda_v + \lambda_B = \lambda$. Therefore, we will have $\Theta_v, \Theta_B \in L^1(\mathbb{R}^d \times [0,T], \lambda)$, satisfying $|\Theta_v|, |\Theta_B| \leq 1$ λ-a.e. and $\lambda_v = \Theta_v \lambda$, $\lambda_B = \Theta_B \lambda$, by the Lebesgue Differentiation Theorem on regular measures.

Moreover, for functions $f = f(x,t;\xi,\zeta) \in \mathcal{F}_2(\mathbb{R}^d \times [0,T]; \mathbb{R}^d \times \mathbb{R}^d)$, independent of ζ (or ξ) (as is the case of $\xi \otimes \xi$ in (4.11)), it still holds that $f = f(x,t;\xi) \in \mathcal{F}_2(\mathbb{R}^d \times$

$[0, T]; \mathbb{R}^d)$, so that we may apply Theorem 3.2.2 and obtain

$$\langle \nu, f \rangle dxdt + \langle \nu^\infty, f^\infty \rangle d\lambda(x,t) = \langle \nu_v, f \rangle dxdt + \langle \nu_v^\infty, f_1^\infty \rangle d\lambda_v(x,t),$$

where $f^\infty : \mathbb{R}^d \times [0, T] \times S^{2d-1} \to Y$ refers to the recession function as defined for $f \in \mathcal{F}_2(\mathbb{R}^d \times [0, T]; \mathbb{R}^d \times \mathbb{R}^d)$, and $f_1^\infty : \mathbb{R}^d \times [0, T] \times S^{d-1} \to Y$ refers to its definition for $f \in \mathcal{F}_2(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)$. Defining the notation f_2^∞ for f independent of ξ, the same holds, exchanging $(\nu_v, \lambda_v, \nu_v^\infty)$ by $(\nu_B, \lambda_B, \nu_B^\infty)$.

Let us now continue. Recall the Leray-Hopf type theory of solutions to the viscous and resistive MHD equations, for which existence in time can also be obtained, and in analogy to what was done for the Euler equations let $(v_{\nu,\mu}, B_{\nu,\mu})_{\nu,\mu>0}$ be a sequence of solutions in this fashion. The energy inequality (2.6) guarantees a uniform bound to $v_{\nu,\mu}, B_{\nu,\mu} \in L^\infty(0, T; L^2(\mathbb{R}^d)) = L_t^\infty L_x^2$, so that if we include this condition into our formulation, we get the following:

Proposition 4.2.1. Let (v_k, B_k) be a bounded sequence of vector fields in $L_t^\infty L_x^2$, generating a generalised Young measure $(\nu, \lambda, \nu^\infty)$ in $L^2(\mathbb{R}^d \times [0, T]; \mathbb{R}^d \times \mathbb{R}^d)$. It holds that:

$$\operatorname{esssup}_t \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |(\xi, \zeta)|^2 \rangle dx \right) < \infty, \tag{4.14}$$

and the concentration measures $\lambda, \lambda_v, \lambda_B$ admit disintegrations

$$d\lambda(x,t) = d\lambda_t(x) \otimes dt \tag{4.15}$$

$$d\lambda_v(x,t) = d\lambda_{v,t}(x) \otimes dt \tag{4.16}$$

$$d\lambda_B(x,t) = d\lambda_{B,t}(x) \otimes dt, \tag{4.17}$$

where $t \mapsto \lambda_{\cdot,t}$ is a measurable parametrisation, bounded in the total variation norm

$$\lambda_{(\cdot)} : [0, T] \to M^+(\mathbb{R}^d).$$

Proof. Regarding the disintegration, it suffices to demonstrate that λ admits such a characterisation, and absolute continuity of λ_v, λ_B in relation to λ assures us of the result for them. Therefore, let σ be the measure defined on $[0, T]$ by the expression $\sigma(A) = \lambda(\mathbb{R}^d \times A)$, where A is a Borel subset of $[0, T]$. By the Disintegration Theorem, there exists a measurable mapping $t \mapsto \tilde{\lambda}_t$, with $\tilde{\lambda}_t \in M^1(\mathbb{R}^d)$, such that

$$\lambda(dx, dt) = \tilde{\lambda}_t(dx) \otimes \sigma(dt).$$

Applying the Fundamental Theorem on generalised Young measures for $f =$

$|(\xi,\zeta)|^2$ (so that $f^\infty \equiv 1$), and integrating over $x \in \mathbb{R}^d$, we have

$$\|(v_k(\cdot,t), B_k(\cdot,t))\|^2_{L^2(\mathbb{R}^d)}dt \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |(\xi,\zeta)|^2\rangle dx\right) dt + \sigma(dt),$$

which implies, for every $\phi \in C([0,T])$, that we have

$$\int_0^T \phi(t)(\|(v_k(\cdot,t), B_k(\cdot,t))\|^2_{L^2(\mathbb{R}^d)})\ dt \to \int_0^T \phi(t) \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |(\xi,\zeta)|^2\rangle dx\right)\ dt + \int_0^T \phi(t)\sigma(dt).$$

From this we deduce

$$\begin{aligned}
\underbrace{\left|\int_0^T \phi(t) \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |(\xi,\zeta)|^2\rangle dx\right)\ dt\right|}_{|I(\phi)|} + \underbrace{\left|\int_0^T \phi(t)\sigma(dt)\right|}_{|J(\phi)|}& \\
\le \sup_{k\in\mathbb{N}} \left|\int_0^T |\phi(t)| \|(v_k(\cdot,t), B_k(\cdot,t))\|^2_{L^2(\mathbb{R}^d)}\ dt\right|& \\
\le \|\phi\|_{L^1([0,T])} \sup_k \|(v_k(\cdot,t), B_k(\cdot,t))\|^2_{L^\infty_t L^2_x}& \\
\le M\|\phi\|_{L^1([0,T])},&
\end{aligned}$$

where $I(\phi)$ and $J(\phi)$ denote the linear functionals defined on $C([0,T])$ by the corresponding formulas. The bounds above mean that these functionals can then be extended to bounded linear functionals in $L^1([0,T])$. Therefore, the Riesz representation theorem on L^p secures the existence of $g, h \in L^\infty([0,T])$, satisfying:

$$\int_{\mathbb{R}^d} \langle \nu_{x,t}, |(\xi,\zeta)|^2\rangle dx = g(t)$$

for almost every $t \in [0,T]$, and

$$\sigma(dt) = h(t)dt.$$

The first identity gives (4.14), and definining $\lambda_t(dx) = h(t)\tilde{\lambda}_t(dx)$, we get (4.15), which concludes the proof. □

The first property assures us that, by Jensen's Inequality, $\bar{v}, \bar{B} \in L^\infty_t L^2_x$, and similarly to the Euler equations, together with (4.11), we arrive at the following:

Proposition 4.2.2 (Weak continuity in time)**.** Given a generalised Young measure $(\nu, \lambda, \nu^\infty)$ in L^2, generated by a sequence in $L^\infty_t L^2_x$, that satisfies (4.11), the barycenters $\bar{v}(x,t)$, $\bar{B}(x,t)$ admit representatives $\tilde{v}, \tilde{B} \in CL^2_w = C([0,T]; L^2_w(\mathbb{R}^d))$, i.e. $\bar{v}(\cdot,t) = \tilde{v}(\cdot,t)$ and $\bar{B}(\cdot,t) = \tilde{B}(\cdot,t)$, as L^2 functions for almost every $t \in [0,T]$.

Proof. Defining the matrix fields

$$\begin{aligned} M &= \langle \nu, \xi \otimes \xi - \zeta \otimes \zeta \rangle \\ R &= \langle \nu^\infty, (\xi \otimes \xi - \zeta \otimes \zeta)^\infty \rangle \\ N &= \langle \nu, \zeta \otimes \xi - \xi \otimes \zeta \rangle \\ S &= \langle \nu^\infty, (\zeta \otimes \xi - \xi \otimes \zeta)^\infty \rangle, \end{aligned}$$

we have from (4.14) that $M, N \in L_t^\infty L_x^1$, and that $\|R(\cdot,t)\|_{L^1(\mathbb{R}^d,\lambda_t)}$, $\|S(\cdot,t)\|_{L^1(\mathbb{R}^d,\lambda_t)}$ are $L^\infty([0,T])$ functions, and by hypothesis, $\bar{v}$ and $\bar{B}$ are weakly divergence-free and satisfy

$$\int_0^T \left[\int_{\mathbb{R}^d} \bar{v} \cdot \partial_t \varphi \, dx + \int_{\mathbb{R}^d} M : \nabla \varphi \, dx + \int_{\mathbb{R}^d} R : \nabla \varphi \, d\lambda_t(x) \right] dt = 0 \tag{4.18}$$

$$\int_0^T \left[\int_{\mathbb{R}^d} \bar{B} \cdot \partial_t \varphi \, dx + \int_{\mathbb{R}^d} N : \nabla \varphi \, dx + \int_{\mathbb{R}^d} S : \nabla \varphi \, d\lambda_t(x) \right] dt = 0, \tag{4.19}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ with div $\varphi = 0$. Using the Helmholtz Decomposition in L^2, let $(\phi_i + \nabla p_i)_{i \in \mathbb{N}}$ be a sequence $L^2(\mathbb{R}^d)$, satisying (see the Appendix in [25]):

(*i*) $\phi_i \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ with div $\phi_i = 0$ weakly;

(*ii*) $p_i \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$;

(*iii*) $\{\phi_i + \nabla p_i\}_{i \in \mathbb{N}}$ is dense in $L^2(\mathbb{R}^d; \mathbb{R}^d)$,

and take $\chi \in C_c^\infty(0,T)$.

Define Φ_i by

$$\Phi_i(t) := \int_{\mathbb{R}^d} (\phi_i(x) + \nabla p_i(x)) \cdot \bar{v}(x,t) dx = \int_{\mathbb{R}^d} \phi_i(x) \cdot \bar{v}(x,t) dx$$

where we have used that div $\bar{v} = 0$ weakly, and take $\chi \phi_i$ as the test function in (4.18) to get for every $\chi \in C_c^\infty(0,T)$ the following identity:

$$\begin{aligned} \int_0^T \partial_t \chi(t) \Phi_i(t) &= \int_0^T \partial_t \chi(t) \int_{\mathbb{R}^d} \phi_i(x) \cdot \bar{v} \, dx dt \\ &= -\int_0^T \chi(t) \left[\int_{\mathbb{R}^d} \nabla \phi_i : M \, dx + \int_{\mathbb{R}^d} \nabla \phi_i : R \, \lambda_t(dx) \right] dt \end{aligned}$$

This means that the function

$$\int_{\mathbb{R}^d} (\nabla \phi_i : M) dx + \int_{\mathbb{R}^d} (\nabla \phi_i : R) \lambda_t(dx)$$

is a weak derivative for Φ_i. Therefore, we can estimate

$$\begin{aligned}\int_0^T |\Phi_i'|(t)dt &\leq \int_0^T \int_{\mathbb{R}^d} |\nabla\phi_i| \cdot |M| \; dxdt + \int_0^T \int_{\mathbb{R}^d} |\nabla\phi_i| \cdot |R| \lambda_t(dx)dt \\ &\leq \left(\|M\|_{L_t^\infty L_x^1} + \text{esssup}_{\; t\in[0,T]} \|R(\cdot,t)\|_{L^1(\mathbb{R}^d,\lambda_t)} \right) \|\nabla\phi_i\|_{L^1{}_t L^\infty{}_x} < \infty.\end{aligned}$$

Therefore, $\Phi_i' \in L^1([0,T])$, which guarantees, by Lebesgue's Differentiation Theorem, the existence of $\tilde{\Phi}_i$ absolutely continuous on $[0,T]$, with $\tilde{\Phi}_i(t) = \Phi_i(t)$ for almost every $t \in [0,T]$. Since this holds for every $i \in \mathbb{N}$, and the countable union of null sets is null, there exists a null set $\mathcal{N} \subset [0,T]$ such that

$$\tilde{\Phi}_i(t) = \int_{\mathbb{R}^d} (\phi_i(x) + \nabla p_i(x)) \cdot \bar{v}(x,t)dx$$

for all $i \in \mathbb{N}$, $t \in [0,T] \setminus \mathcal{N}$. Therefore, by the continuity of $\tilde{\Phi}_i$, it holds that

$$|\tilde{\Phi}_i(t)| \leq \|\bar{v}\|_{L_t^\infty L_x^2} \|\phi_i + \nabla p_i\|_{L_x^2} \text{ for all } t \in [0,T],$$

so that, for every $t \in [0,T]$, the functions $\tilde{\Phi}_i(t)$ define, by density of $\{\phi_i + \nabla p_i\}$ in L_x^2, a bounded linear functional on L_x^2, which we denote by L_t, through the formula $L_t(\phi_i + \nabla p_i) = \tilde{\Phi}_i(t)$. The Riesz Representation Theorem assures us, for every $t \in [0,T]$, of the existence of a function $\tilde{v}(\cdot,t) \in L_x^2$ which coincides with $\bar{v}(\cdot,t)$ for every $t \in [0,T] \setminus \mathcal{N}$, and also satisfies

$$\|\tilde{v}(\cdot,t)\|_{L_x^2} \leq \|\bar{v}\|_{L_t^\infty L_x^2} \text{ for all } t \in [0,T],$$

as well as

$$\tilde{\Phi}_i(t) = \int_{\mathbb{R}^d} (\phi_i + \nabla p_i) \cdot \tilde{v} \; dx \text{ for all } t \in [0,T].$$

Lastly, take $\psi \in L_x^2$, and a sequence $\phi_k + \nabla p_k \overset{k\to\infty}{\longrightarrow} \psi$ strongly in L^2. if $\Psi := \int_{\mathbb{R}^d} \tilde{v} \cdot \psi \; dx$, it follows that

$$|\tilde{\Phi}_k(t) - \Psi(t)| \leq \|\bar{v}\|_{L_t^\infty L_x^2} \|\phi_k + \nabla p_k - \psi\|_{L_x^2} \to 0 \text{ uniformly on } t \in [0,T],$$

so that Ψ is the uniform limit of conituous $\tilde{\Phi}_k$. Being $\psi \in L_x^2$ arbitrary, it follows that $\tilde{v} \in C([0,T]; L_w^2)$. Repeating the same argument with $\bar{B}$, N and S, one obtains $\tilde{B} \in CL_w^2$ coinciding with $\bar{B}$ for almost every $(x,t) \in \mathbb{R}^d \times [0,T]$.

□

Now, knowing that $\bar{v}(x,0)$, $\bar{B}(x,0)$ are well defined as L^2 functions with $\bar{v}(\cdot,t) \overset{t\to 0}{\rightharpoonup} \bar{v}(\cdot,0)$ and $\bar{B}(\cdot,t) \overset{t\to 0}{\rightharpoonup} \bar{B}(\cdot,0)$ in L^2, we can use the Young measure convergence to expand (4.11) to:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} [\langle \nu, \xi \rangle \cdot \partial_t \varphi + \langle \nu, (\xi \otimes \xi - \zeta \otimes \zeta) \rangle : \nabla \varphi]\, dxdt \\ + \int_{\mathbb{R}^d \times (0,T)} \langle \nu^\infty, (\xi \otimes \xi - \zeta \otimes \zeta)^\infty \rangle : \nabla \varphi \; d\lambda(x,t) \\ = - \int_{\mathbb{R}^d} \varphi(x,0) \cdot \bar{v}(x,0) dx\end{aligned} \tag{4.20}$$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} [\langle \nu, \zeta \rangle \cdot \partial_t \varphi + \langle \nu, (\zeta \otimes \xi - \xi \otimes \zeta) \rangle : \nabla \varphi]\, dxdt \\ + \int_{\mathbb{R}^d \times (0,T)} \langle \nu^\infty, (\zeta \otimes \xi - \xi \otimes \zeta)^\infty \rangle : \nabla \varphi \; d\lambda(x,t) \\ = - \int_{\mathbb{R}^d} \varphi(x,0) \cdot \bar{B}(x,0) dx\end{aligned} \tag{4.21}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T), \mathbb{R}^d)$ with div $\varphi = 0$.

We define also the *energy*:

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle dx + \frac{1}{2} \lambda_t(\mathbb{R}^d). \tag{4.22}$$

Definition 4.2.3 (Measure-valued solutions to the ideal incompressible MHD equations)**.** Let $(\nu, \lambda, \nu^\infty)$ be a type 2 generalised Young measure on $\mathbb{R}^d \times \mathbb{R}^d$, with parameters on $\mathbb{R}^d \times [0,T]$ and barycenters $\bar{v} = \langle \nu, \xi \rangle$ and $\bar{B} = \langle \nu, \zeta \rangle$.

(*i*) We say $(\nu, \lambda, \nu^\infty)$ is a *measure-valued solution* to the ideal incompressible MHD equations if it satisfies (4.11) and (4.13).

(*ii*) We say $(\nu, \lambda, \nu^\infty)$ is an *admissible measure-valued solution* with initial data $v_0 \in L^2(\mathbb{R}^d)$ and $B_0 \in L^2(\mathbb{R}^d)$, if it satisfies (4.13), (4.14), (4.15), (4.20),(4.21), $\bar{v}(\cdot,0) = v_0$, $\bar{B}(\cdot,0) = B_0$ and

$$E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 + |B_0(x)|^2 dx \text{ for almost every } t > 0. \tag{4.23}$$

Proposition 4.2.4. Let $(\nu, \lambda, \nu^\infty)$ be an admissible measure-valued solution to ideal incompressible MHD equations, and $\bar{v}$, $\bar{B}$ be its barycenters. Then it holds that

$$(\bar{v}, \bar{B})(\cdot, t) \to (\bar{v}, \bar{B})(\cdot, 0) = (v_0, B_0)$$

strongly in $L^2(\mathbb{R}^d)$, as $t \to 0$.

Proof. We have already seen that $(\bar{v}, \bar{B}) \in CL^2_w$, so that

$$\liminf_{t\to 0} \|(\bar{v}, \bar{B})(t)\|_{L^2} \geq \|(\bar{v}, \bar{B})(0)\|_{L^2},$$

by lower semi-continuity of the norm in the weak topology. On the other hand,

$$\begin{aligned}\int |(\bar{v}, \bar{B})(t)|^2 dx &= \int |\langle \nu_{x,t}, |\xi|^2 + |\zeta|^2 \rangle dx \\ &\leq \int |\langle \nu_{x,t}, |\xi|^2 + |\zeta|^2 \rangle dx + \lambda_t(\mathbb{R}^d) \\ &= 2E(t) \leq \int |(\bar{v}, \bar{B})(0)|^2 dx,\end{aligned}$$

where we have used the energy inequality (4.23). Combining both inequalities, we arrive at $\|(\bar{v}, \bar{B})(t)\|_{L^2} \to \|(\bar{v}, \bar{B})(0)\|_{L^2}$ as $t \to 0$. Weak convergence together with convergence of norms gives strong convergence. □

Now, we can easily adapt the arguments from Theorem 4.1.1, to obtain admissible measure-valued solutions to the ideal incompressible MHD system from Leray-Hopf solutions to the viscous, resistive MHD.

Theorem 4.2.1. Let v_0, B_0 be weakly divergence-free vector fields in $L^2(\mathbb{R}^d)$ and for $\varepsilon > 0$ let $v_\varepsilon B_\varepsilon$ be a Leray-Hopf solution to the viscous, resistive MHD system with initial data v_0, B_0, with viscosity and resistivity equal to ε, and suppose they are all defined up to a time $0 < T \leq \infty$. Then, any generalised Young measure generated by a sequence $(v_{\varepsilon_k}, B_{\varepsilon_k})_{\varepsilon_k \to 0}$ defines an admissible measure-valued solution to the ideal incompressible MHD equations on $\mathbb{R}^d \times [0, T]$, with initial data v_0, B_0.

Moreover, if $d = 3$, global existence of Leray-Hopf solutions [12] implies the global existence of admissible measure-valued solutions to the ideal incompressible MHD equations for any initial data $v_0, B_0 \in L^2(\mathbb{R}^3)$.

4.3 The planar symmetry MHD system

We consider measure-valued solutions also in the planar symmetry MHD system. After the desired simplifications, we have:

Definition 4.3.1 (Measure-valued solutions to the planar symmetry MHD system)**.** Let $(\nu, \lambda, \nu^\infty)$ be a type 2 generalised Young measure 2 on $\mathbb{R}^2 \times \mathbb{R}$, with parameters in $\mathbb{R}^2 \times [0, T]$ and barycenters $\bar{v}(x, t) = \langle \nu_{x,t}, \xi \rangle \in \mathbb{R}^2$ and $\bar{b} = \langle \nu_{x,t}, \zeta \rangle \in \mathbb{R}$.

(i) We say $(\nu, \lambda, \nu^\infty)$ is a *measure-valued solution* to the system (1.6) if $\bar{v}$ is weakly divergence-free and $(\nu, \lambda, \nu^\infty)$ satisfies

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} \left[\bar{v} \cdot \partial_t \varphi + \langle \nu, \xi \otimes \xi \rangle : \nabla \varphi\right] dxdt + \int_0^T \int_{\mathbb{R}^2} \langle \nu^\infty, (\xi \otimes \xi)^\infty \rangle : \nabla \varphi \lambda_t(dx)dt = 0 \\ \int_0^T \int_{\mathbb{R}^2} \left[\bar{b} \partial_t \psi + \langle \nu, \xi\zeta \rangle \cdot \nabla \psi\right] dxdt + \int_0^T \int_{\mathbb{R}^2} \langle \nu^\infty, (\xi\zeta)^\infty \rangle \cdot \nabla \psi \lambda_t(dx)dt = 0\end{aligned} \tag{4.24}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times (0, T); \mathbb{R}^2)$ with div $\varphi = 0$ and all $\psi \in C_c^\infty(\mathbb{R}^2 \times (0, T))$.

(ii) We say $(\nu, \lambda, \nu^\infty)$ is an *admissible measure-valued solution* for initial data $v_0 \in L^2(\mathbb{R}^2)$, div $v_0 = 0$ and $b_0 \in L^2(\mathbb{R}^2)$, if it satisfies the following conditions:

a) div $\bar{v} = 0$;

b) $(\nu, \lambda, \nu^\infty)$ satisfies

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} \left[\bar{v} \cdot \partial_t \varphi + \langle \nu, \xi \otimes \xi \rangle : \nabla \varphi\right] dxdt + \int_0^T \int_{\mathbb{R}^2} \langle \nu^\infty, (\xi \otimes \xi)^\infty \rangle : \nabla \varphi \lambda_t(dx)dt \\ = - \int_{\mathbb{R}^2} v_0 \cdot \varphi(x, 0)\ dx \\ \int_0^T \int_{\mathbb{R}^2} \left[\bar{b} \partial_t \psi + \langle \nu, \xi\zeta \rangle \cdot \nabla \psi\right] dxdt + \int_0^T \int_{\mathbb{R}^2} \langle \nu^\infty, (\xi\zeta)^\infty \rangle \cdot \nabla \psi \lambda_t(dx)dt \\ = - \int_{\mathbb{R}^2} b_0 \psi(x, 0)\ dx\end{aligned} \tag{4.25}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0, T); \mathbb{R}^2)$ with div $\varphi = 0$ and for all $\psi \in C_c^\infty(\mathbb{R}^2 \times [0, T))$;

c) It holds that

$$\text{esssup}_{t \in [0,T]} \left\{ \int_{\mathbb{R}^2} \langle \nu_{x,t}, |\xi|^2 + |\zeta|^2 \rangle dx \right\} < \infty; \tag{4.26}$$

d) The concentration measure λ admits the disintegration

$$\lambda(dx, dt) = \lambda_t(dx) \otimes dt, \tag{4.27}$$

where $\lambda_t \in L^\infty([0, T]; M^+(\mathbb{R}^2))$;

e) $\bar{v}(\cdot, 0) = v_0$ e $\bar{b}(\cdot, 0) = b_0$; and

f) the energy inequality

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^2} \langle \nu_{x,t}, |\xi|^2 + |\zeta|^2 \rangle dx + \frac{1}{2} \lambda_t(\mathbb{R}^2) \leq \frac{1}{2} \int_{\mathbb{R}^2} |v_0(x)|^2 + |b_0(x)|^2 dx \tag{4.28}$$

holds for almost every $t \in [0, T]$.

Note that since the symmetry assumption on the system precludes integrability of any non-zero solution in $\mathbb{R}^3$, the existence result from Theorem 4.2.1, which relies on Leray-Hopf solutions for the full-$3D$ MHD, cannot be employed. Nevertheless, we can adapt the methods used by [4] to obtain a weak-strong uniqueness result:

Theorem 4.3.1 (Weak-strong uniqueness)**.** Let $v_0,\ b_0 \in L^2(\mathbb{R}^2)$, satisfying div $v_0 = 0$ be given. Suppose that $v, b \in C([0,T], L^2(\mathbb{R}^2))$ is a classical solution to the planar symmetry MHD equations 1.6 with the given initial data v_0, b_0 e assume further that

$$\int_0^T \left(\|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^2)} + \|\nabla b\|_{L^\infty(\mathbb{R}^2)}\right) dt < \infty \text{ holds,} \tag{4.29}$$

and let $(\nu, \lambda, \nu^\infty)$ be an admissible measure-valued solution with the same initial data. Then we have $\lambda = 0$ and $\nu_{x,t} = \delta_{v(x,t),b(x,t)}$ for almost every $(x,t) \in \mathbb{R}^2 \times [0,T]$.

Proof. Define

$$F(t) = \frac{1}{2}\int_{\mathbb{R}^2} \langle \nu_{x,t}, |(\xi - v, \zeta - b)|^2\rangle dx + \frac{1}{2}\lambda_t(\mathbb{R}^2)$$

and also, for $\phi \in C_c^\infty(\mathbb{R}^2)$, the approximation

$$F^\phi(t) = \frac{1}{2}\int_{\mathbb{R}^2} \phi(x)\langle \nu_{x,t}, |(\xi - v, \zeta - b)|^2\rangle dx + \frac{1}{2}\int_{\mathbb{R}^2} \phi(x)\lambda_t(dx).$$

Note that $F^\phi \in L^\infty(0,T)$, by the hypotheses on $(\nu, \lambda, \nu^\infty)$. Take $\chi \in C_c^\infty(0,T)$ and consider the following:

$$\begin{aligned}
\int_0^T \chi'(t)F^\phi(t)\ dt = &\frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \chi'(t)\phi(x)\langle \nu_{x,t}, |\xi|^2\rangle dxdt + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \chi'(t)\phi(x)\langle \nu_{x,t}, |\zeta|^2\rangle dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \chi'(t)\phi(x)|v|^2 dxdt + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \chi'(t)\phi(x)|b|^2 dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \chi'(t)\phi(x)\lambda_t(dx)dt \underbrace{- \int_0^T \int_{\mathbb{R}^2} \chi'\phi\bar{v}\cdot v\ dxdt}_{(I)} \\
&\underbrace{- \int_0^T \int_{\mathbb{R}^2} \chi'\phi\bar{b}b\ dxdt}_{(II)}.
\end{aligned}$$

Looking closely at (I), we can use the fact that

$$\chi'\phi v = \partial_t(\chi\phi v) + \chi\phi(\text{div }(v\otimes v) + \nabla\left(\pi + \frac{|b|^2}{2}\right)$$

from (1.6), to arrive at

$$
\begin{aligned}
(I) &= \iint -\partial_t(\chi\phi v)\cdot\bar{v} - \chi\phi\bar{v}\cdot\left(\text{div }(v\otimes v) + \nabla\left(\pi + \frac{|b|^2}{2}\right)\right) dxdt\\
&= \iint \chi\nabla(\phi v) : \langle \nu, \xi\otimes\xi\rangle - \chi\phi\bar{v}\cdot\text{div }(v\otimes v)\ dxdt\\
&\quad + \iint \chi\nabla(\phi v) : \langle \nu^\infty, (\xi\otimes\xi)^\infty\rangle\lambda_t(dx)dt\\
&\quad - \iint \chi\phi\bar{v}\cdot\nabla\left(\pi + \frac{|b|^2}{2}\right)\ dxdt,
\end{aligned}
$$

by using $\varphi = \chi\phi v$ as a test function in (4.25). Now, by using the vector calculus identities

$$
\begin{aligned}
\nabla(\phi v) : (\bar{v}\otimes v) &= \phi\bar{v}\cdot\text{div }v\otimes v + (\nabla\phi\cdot v)(v\cdot\bar{v}),\\
\nabla(\phi v) : (v\otimes\bar{v}) &= \nabla\phi\cdot\bar{v}\frac{|v|^2}{2} + \nabla\left(\phi\frac{|v|^2}{2}\right)\cdot\bar{v}\\
\nabla(\phi v) : (v\otimes v) &= \nabla\phi\cdot v\frac{|v|^2}{2} + \nabla\left(\phi\frac{|v|^2}{2}\right)\cdot v,
\end{aligned}
$$

together with div $v = 0$ and div $\bar{v} = 0$ weakly, we can rearrange (I) as

$$
\begin{aligned}
(I) &= \iint \chi\nabla(\phi v) : \langle \nu, (\xi - v)\otimes(\xi - v)\rangle dxdt\\
&\quad + \iint \chi\nabla(\phi v) : \langle \nu^\infty, (\xi\otimes\xi)^\infty\rangle\lambda_t(dx)dt\\
&\quad + \iint \chi\nabla\phi\cdot\left((\bar{v} - v)\frac{|v|^2}{2} + \bar{v}\left(\pi + \frac{|b|^2}{2}\right)\right) + \chi(\nabla\phi\cdot v)(v\cdot\bar{v})dxdt. \qquad (4.30)
\end{aligned}
$$

We can do similarly with (II), by using the fact that

$$\chi'\phi b = \partial_t(\chi\phi b) + \chi\phi\text{div }(bv), \text{ by (1.6), and obtain:}$$

$$
\begin{aligned}
(I) &= \iint -\partial_t(\chi\phi b)\cdot\bar{b} - \chi\phi\bar{b}\text{div }(bv)dxdt\\
&= \iint \chi\nabla(\phi b) : \langle \nu, \zeta\xi\rangle - \chi\phi\bar{b}\text{div }(bv)\ dxdt\\
&\quad + \iint \chi\nabla(\phi b) : \langle \nu^\infty, (\zeta\xi)^\infty\rangle\lambda_t(dx)dt,
\end{aligned}
$$

where we used $\varphi = \chi\phi b$ as a test function in (4.25). This, together with the identities

$$
\begin{aligned}
\nabla(\phi b)\cdot(\bar{b}v) &= \phi\bar{b}\text{div }(bv) + (\nabla\phi\cdot v)b\bar{b}\\
\nabla(\phi b)\cdot(b\bar{v}) &= \nabla\phi\cdot\bar{v}\frac{|b|^2}{2} + \nabla(\phi\frac{|b|^2}{2})\cdot\bar{v}\\
\nabla(\phi b)\cdot(bv) &= \nabla\phi\cdot v\frac{|b|^2}{2} + \nabla(\phi\frac{|b|^2}{2})\cdot v,
\end{aligned}
$$

and div $v = 0$ e div $\bar{v} = 0$ weakly, allow us to rearrange (II) as

$$\begin{aligned}(II) = &\iint \chi \nabla(\phi v) : \langle \nu, (\zeta - b)(\xi - v)\rangle dxdt \\ &+ \iint \chi \nabla(\phi b) : \langle \nu^\infty, (\zeta\xi)^\infty \rangle \lambda_t(dx)dt \\ &+ \iint \chi \nabla \phi \cdot (\bar{v} - v)\frac{|b|^2}{2} + \chi(\nabla\phi \cdot v)(b\bar{b})dxdt. \end{aligned} \tag{4.31}$$

Now, observe that $\|v\|_{L^p(\mathbb{R}^2)}$ can be bounded in terms of $\|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}$, for every p satisfying $2 \leq p \leq \infty$. Truly, for every ball $B_1(x_0)$, Korn's inequality (see [1]) in $W^{1,2}$ gives us:

$$\begin{aligned}\|v\|_{W^{1,2}(B_1(x_0))} &\leq C(\|\nabla v + (\nabla v)^t\|_{L^2(B_1(x_0))} + \|v\|_{L^2(B_1(x_0))}) \\ &\leq C(\|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)})\end{aligned}$$

Therefore, through the Sobolev embedding $W^{1,2}(B_1(x_0)) \subset L^6(B_1(x_0))$, and a new application of Korn's inequality in $W^{1,6}(B_1(x_0))$, we get

$$\|v\|_{W^{1,6}(B_1(x_0))} \leq C(\|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}).$$

Lastly, we apply also the Sobolev embedding $W^{1,6}(B_1(x_0)) \subset L^\infty(B_1(x_0))$, which gives us, since $x_0 \in \mathbb{R}^2$ is arbitrary,

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq C(\|\nabla v + (\nabla v)^t\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}).$$

So that we have $v \in L^2 \cap L^\infty \subset L^p$ for $2 \leq p \leq \infty$. In particular, $v \in L^4(\mathbb{R}^2)$, and thus, $\pi \in L^2$. Also, using only the Sobolev embeddings,we have $\|b\|_{L^p(\mathbb{R}^2)} \leq C(\|\nabla b\|_{L^\infty(\mathbb{R}^2)} + \|b\|_{L^2(\mathbb{R}^2)})$ for $2 \leq p \leq \infty$. Therefore, we can take a sequence (φ_k) of test functions such that $0 \leq \varphi_k \leq 1$, $\varphi_k \equiv 1$ in $B_k(0)$, and $\|\nabla\varphi_k\|_{C_0}$ is uniformly bounded. Appropriately combining (4.30) and (4.31) and the bounds above with the generalised Hölder Inequality, we can use the Dominated Convergence Theorem to obtain, as $k \to \infty$, the following:

$$\iint \chi \nabla \phi_k \cdot \left((\bar{v} - v)\frac{|v|^2}{2} + \bar{v}\left(\pi + \frac{|b|^2}{2}\right)\right) + \chi(\nabla\phi_k \cdot v)(v \cdot \bar{v})dxdt \overset{k\to\infty}{\longrightarrow} 0,$$

and

$$\iint \chi \nabla \phi_k \cdot (\bar{v} - v)\frac{|b|^2}{2} + \chi(\nabla\phi_k \cdot v)b\bar{b}\; dxdt \overset{k\to\infty}{\longrightarrow} 0,$$

so that we end up with

$$\begin{aligned}-\iint \chi' \phi_k \bar{v} \cdot v\; dxdt \overset{k\to\infty}{\longrightarrow} &\iint \chi \nabla v : \langle \nu, (\xi - v) \otimes (\xi - v)\rangle dxdt \\ &+ \iint \chi \nabla v : \langle \nu^\infty, (\xi \otimes \xi)^\infty \rangle \lambda_t(dx)dt\end{aligned} \tag{4.32}$$

and

$$-\iint \chi'\phi_k \bar{b}\cdot b\ dxdt \overset{k\to\infty}{\longrightarrow} \iint \chi\nabla b\cdot\langle\nu,(\zeta-b)\cdot(\xi-v)\rangle dxdt \\ + \iint \chi\nabla b : \langle\nu^\infty,(\zeta\xi)^\infty\rangle\lambda_t(dx)dt \tag{4.33}$$

Since we can also see that $F^{\phi_k}\to F$, we have after symmetrizing the ∇v terms the following:

$$\begin{aligned}\int_0^T \chi'(t)F(t)dt = &\int_0^T \chi'(t)E(t)dt + \frac{1}{2}\int_0^T \chi'(t)\left(\int_{\mathbb{R}^2}|v|^2+|b|^2dx\right)dt\\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\chi(t)(\nabla v+(\nabla v)^t):\langle\nu,(\xi-v)\otimes(\xi-v)\rangle dxdt\\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\chi(t)(\nabla v+(\nabla v)^t):\langle\nu^\infty,(\xi\otimes\xi)^\infty\rangle\lambda_t(dx)dt\\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\chi(t)\nabla b\cdot\langle\nu,(\zeta-b)\cdot(\xi-v)\rangle dxdt\\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\chi(t)\nabla b\cdot\langle\nu^\infty,(\zeta\xi)^\infty\rangle\lambda_t(dx)dt.\end{aligned} \tag{4.34}$$

Finally, we can use the Cauchy-Schwarz inequality and the fact that (v,b) conserve energy to obtain

$$-\int_0^T \chi'(t)F(t)dt \le -\int_0^T\chi'(t)E(t) + C\int_0^T(\|\nabla v+(\nabla v)^t\|_{L^\infty}+\|\nabla b\|_{L^\infty})F(t)dt.$$

That is, for almost every $s,t\in(0,T)$ it holds that

$$F(t)-F(s)\le E(t)-E(s)+C\int_s^t(\|\nabla v+(\nabla v)^t\|_{L^\infty}+\|\nabla b\|_{L^\infty})F(\tau)d\tau. \tag{4.35}$$

Now, note that

$$F(s)=E(s)-\int_{\mathbb{R}^2}v(s)\cdot\bar{v}(s)+b(s)\bar{b}(s)\ dx+\int|v(s)|^2+|b(s)|^2\ dx,$$

so that (4.35) becomes

$$F(t)\le E(t)-\int_{\mathbb{R}^2}v(s)\cdot\bar{v}(s)+b(s)\bar{b}(s)\ dx+\int|v(s)|^2+|b(s)|^2\ dx \\ +C\int_s^t(\|\nabla v+(\nabla v)^t\|_{L^\infty}+\|\nabla b\|_{L^\infty})F(\tau)d\tau. \tag{4.36}$$

Since $\bar{v},v,\bar{b}$ and b are all in CL^2_w, and also that $\bar{v}(s),\ v(s)\to v_0$ and $\bar{b}(s),\ b(s)\to b_0$

strongly in L^2 as s vanishes, we can pass to the limit to get

$$\begin{aligned} F(t) &\leq E(t) - E(0) + C \int_0^t (\|\nabla v + (\nabla v)^t\|_{L^\infty} + \|\nabla b\|_{L^\infty}) F(\tau) d\tau \\ &\leq C \int_0^t (\|\nabla v + (\nabla v)^t\|_{L^\infty} + \|\nabla b\|_{L^\infty}) F(\tau) d\tau, \end{aligned}$$

where we used that $E(t) - E(0) \leq 0$ by admissibility of the measure-valued solution. Grönwall's inequality then assures us that $F = 0$ almost everywhere. It follows that $\lambda_t(\mathbb{R}^2) = 0$ for almost every $t \in [0, T]$, and that

$$\langle \nu_{x,t}, |(\xi - v(x,t), \zeta - b(x,t))|^2 \rangle = 0, \text{for a.e.}(x,t) \in \mathbb{R}^2 \times [0, T].$$

Now, since ν is a non-negative measure and the function $(\xi, \zeta) \mapsto |(\xi - v(x,t), \zeta - b(x,t))|^2$ is strictly positive in every open set not containing the point $(v(x,t), b(x,t))$, this means $\nu_{x,t}$ is supported in $\{(v(x,t), b(x,t))\}$. By definition, $\nu_{x,t}$ is a probability measure, so that this must mean

$$\nu_{x,t} = \delta_{(v(x,t), b(x,t))},$$

as desired. $\square$

Future directions

Finally, we would like to remark that, throughout the period of study relating to this thesis, we were interested in achieving a result similar to Theorem 2.2.2 for the planar symmetry MHD equations. In this problem the main obstacles seemed to be concerning how to express the constraint set K and its convex hull, for a given energy profile, and therefore how to properly define subsolutions. Note that, if K is taken generally as the set of states $z = (v, b, u, w)$ satisfying

$$u = v \circ v \quad \text{and} \quad w = bv,$$

then it holds that for every $z_1, z_2 \in K$, their difference $z_1 - z_2$ lies in Λ, the wave-cone for the system (2.12), (see Remark 2.1 in [5]), so that the $\Lambda-$convex hull of K should coincide with the traditional convex hull. However, we were unable to define a similar generalised energy density e for these states, satisfying properties (in analogy to Lemma 2.2.11) like:

(i) e is convex;

(ii) $e(v, b, u, w) \geq \frac{|v|^2+|b|^2}{2}$ with equality if and only if $z \in K$;

(iii) $\left\{z : e(z) \leq \frac{r^2}{2}\right\} = K_r^{co}$, for some definition of K_r (with possibly more parameters).

Assuming that we have achieved a description of K_r^{co}, and with it a functioning definition of subsolution, we can still apply the results shown in section 2.2.2 regarding the existence of good wave directions (Lemma 2.2.3) and localised plane waves (Proposition 2.2.4) to proceed similarly as was done in section 2.2.3 with the Euler equations.

We remark that, in contrast to the planar symmetry case, for the full $3D$ ideal MHD case the set K^{Λ} does not coincide with the traditional convex hull, and has empty interior, and both these properties make the problem significantly harder. In this case, D. Faraco, S. Lindberg and L. Székelyhidi in [13] were able to describe the relative interior of certain convex-hulls of the sections $K_{r,s}^{\Lambda}$ of the constitutive set, where r, s are parameters on the norms of the *Elsässer variables* $z^{\pm} = v \pm B$. This change of variables is introduced in order to prescribe both the total energy density $\frac{|v|^2+|B|^2}{2}$ and the *cross-helicity* density $u \cdot B$, because both define conserved quantities for regular solutions. However, for the planar symmetry system the cross-helicity necessarily vanishes, and the two parameters reduce to a single one, so that the same process does not seem to work.

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APPENDIX A: Disintegration of measures defined on a product space

Here we present a result which is used extensively in chapters 3 and 4, and concerns the representation of a measure defined on a product of measure spaces. We adapt from the statement of Theorem 10.4.14 in [3], where the result can be found in greater generality.

Theorem. Let X and Y be secound countable, locally compact Hausdorff spaces, with the respective Borel σ-algebras $\mathcal{B}_X$ and $\mathcal{B}_Y$, and take the product space $X \times Y$ with the product σ-algebra $\mathcal{A} = \mathcal{B}_X \otimes \mathcal{B}_Y$. Let also μ be a non-negative Radon measure on the product, and define μ_X as the natural projection of μ to X, defined on Borel sets E of X by

$$\mu_X(E) = \mu(E \times Y).$$

Define also for $x \in X$ and $F \in \mathcal{A}$ the set

$$F_x = \{y : (x, y) \in F\},$$

Then, for every $x \in X$ there exists a probability measure ν_x defined on $\mathcal{B}_Y$, so that the map $x \mapsto \nu_x(F_x)$ is μ_X-measurable for every $F \in \mathcal{A}$, and it holds that for $B \in \mathcal{B}_X$

$$\mu(F \cap (B \times Y)) = \int_B \nu_x(F_x) d\mu_X(x).$$

In addition, for every $\mathcal{A}$-measurable and μ-integrable function f we have

$$\iint_{X \times Y} f(x, y) d\mu(x, y) = \int_X \int_Y f(x, y) d\nu_x(y) d\mu_X(x).$$

Moreover, if $\{\nu_x'\}_{x \in X}$ is a family of measures with the stated properties, then it holds that $\nu_x' = \nu_x$ for μ_X-a.e. $x \in X$. We write informally that $d\mu(x, y) = d\nu_x(y) \otimes d\mu(x)$.