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José Vitor Pena

Attainability of Trudinger-Moser type supremums and related Henón problems

Atingibilidade de supremos do tipo Trudinger-Moser e problemas do tipo Henón relacionados

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Orientador: Djairo Guedes de Figueiredo

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Resumo

Nesta tese apresentar-se-á alguns resultados relativos a não-linearidades do tipo exponencial com pesos apropriados: primeiramente, tratar-se-á a atingibilidade de supremos do tipo Trudinger-Moser e Trudinger-Moser-Hardy na bola unitária e em um domínio com fronteira C^1 simplesmente conexo contendo a origem,respectivamente (ambos os casos em $\mathbb{R}^2$).Chega-se em condições de decaimento para o limite do peso na origem, com condições pouco restritivas para o comportamento do peso numa bola muito pequena no entorno da origem. Tal resultado é atingido utilizando técnicas de Compacidade Concentrada. No terceiro capítulo, estabelecer-se-á um resultado de existência para o caso crítico de um problema de Dirichlet elíptico

$$-\Delta u = f(x,u) \text{ in } B,\ u = 0 \text{ in } \partial B$$

na bola unitária em $\mathbb{R}^2$ com não-linearidade exponencial associada ao primeiro problema de atingibilidade. A condição de criticalidade da função f novamente ignora o comportamento fora de uma pequena vizinhança da origem.

Palavras-chave: Equações diferenciais parciais, Equações diferenciais parciais não-lineares, Análise funcional não-linear.

Abstract

We present some results regarding critical nonlinearities of exponential type with appropriate weights in $\mathbb{R}^2$: first regarding the attainability of supremum of a weighted Trudinger-Moser functional on the unit ball and of a Trudinger-Moser-Hardy type on any simply connected domain containing the origin. We arrive on decay conditions for the limit of the weight at the origin (thus, we do care about the behavior of the weight only on a neighbourhood of zero, requiring less information on the rest of the domain. This is achieved using Concentration-Compactness-type techniques. Moreover, we show an existence result for an elliptical Dirichlet problem

$$-\Delta u = f(x,u) \text{ in } B,\ u = 0 \text{ in } \partial B$$

with exponential nonlinearity on the ball. The criticality condition is associated to the first supremum and again ignores the behaviour of f outside a small ball around the origin.

Key words: Partial differential equations, Nonlinear partial differential equations, Nonlinear functional analysis.

Contents

Introduction

Let $\Omega \subset \mathbb{R}^2$ be a C^1 and bounded domain, and $W_0^{1,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ (compact-supported $C^\infty(\Omega)$ functions) with respect to the norm

$$\|u\|_{W_0^{1,p}} := \left(\int_\Omega |\nabla u|^p \right)^{1/p}.$$

The classical Sobolev inequality states that if $u \in W_0^{1,p}(\Omega)$, then $u \in L^q(\Omega)$, for $1 \leq q \leq p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$ (if $p < 2$), and the embedding is continuous. The borderline case $p = 2$ corresponds to the Trudinger-Moser inequality, first proved by Pohozaev [26] and Trudinger [29] separately, and states that if $u \in H_0^1(\Omega)$, then

$$\int_\Omega e^{\alpha |u|^2}\, dx < \infty, \ \forall \alpha \in \mathbb{R},$$

A sharp result by J. Moser [24] is

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_{L^2} \leq 1} \int_\Omega e^{\alpha u^2}\, dx \begin{cases} \leq c|\Omega|, \ \alpha \leq 4\pi, \\ = \infty, \ \alpha > 4\pi. \end{cases}$$

An interesting question is whether the supremum below is attained, i.e., the existence of a function $u_0 \in H_0^1(B)$ such that

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_{L^2} \leq 1} \int_\Omega e^{\alpha u^2}\, dx = \int_\Omega e^{\alpha u_0^2}\, dx. \tag{1}$$

If $\alpha < 4\pi$ we have compactness of the embedding, so the supremum is achieved. If $\alpha = 4\pi$, this argument fails. However, Carleson and Chang [9] proved that, for $\Omega = B_1(0)$, the supremum is attained, using Concentration Compactness techniques (c.f. [23]). Further improvements on this question were made by Struwe [28] (for small perturbations of $B_1(0)$) and Flucher [21] (for any bounded domain in $\mathbb{R}^2$).

In the present thesis we present some results regarding critical nonlinearities of exponential type with appropriate weights. Embeddings in weighted Sobolev spaces were first considered by Ni [25] and were completely settled by de Figueiredo, Miyagaki and Santos

[16]: if $B \subset \mathbb{R}^N$ is the open ball, for $N > mp$ and $\alpha \geq 0$ one has

$$\sup_{u \in W^{m,p}_{rad}(B), \|u\|_{m,p}=1} \int_B |u|^q |x|^\alpha dx < \infty \Leftrightarrow 1 \leq q \leq \frac{p(N+\alpha)}{N-mp}.$$

The Trudinger-Moser associated case for $N = 2$ was considered first by Calanchi and Terraneo [8]. They established that for any $\beta > 0$

$$\sup_{u \in H^1_{0,rad}(B), \|\nabla u\|_2 \leq 1} \int_B e^{2\pi(2+\beta)u^2} |x|^\beta < \infty, \tag{2}$$

and

$$\sup_{u \in H^1_0(B), \|\nabla u\|_2 \leq 1} \int_B e^{2\pi(2+\beta)u^2} |x|^\beta = \infty. \tag{3}$$

In [15], de Figueiredo, do Ó e dos Santos generalized (2) substituting the weight $|x|^\beta$ for any function h such that

(i) $h(x) = h(|x|) : \bar{B} \to [0, \infty[$ a radial non-decreasing differentiable function such that $h(0) = 0$, and

(ii)
$$0 \leq M := \limsup_{r \to 0^+} \frac{h(r)}{r^\beta} < \infty. \tag{4}$$

Definition. *The case $M = 0$ in (ii) is called* ***subcritical case****, and $M > 0$ is the* ***critical case****.*

More specifically, they established that, under (i) and (ii), the supremum

$$\sup_{u \in H^1_{0,\mathrm{rad}}(B), \|\nabla u\|_2 = 1} \int_B e^{2\pi(2+\beta)u^2} h(x)\ dx. \tag{5}$$

is finite. Regarding the attainability of (5), the subcritical case is solved in [15] under the additional hypothesis

(iii) h is differentiable in $]0, 1[$ and $\limsup_{r \to 0^+} \frac{h'(r)r}{h(r)} < \infty$.

and the critical case is solved for $h(r) = r^\alpha$. Here we will present 3 main results: the first two regarding the attainability of a weighted Trudinger-Moser type and a Hardy-type supremums, and one regarding the solvability of a Henón-type Dirichlet problem.

Attainability of the supremum (5) on the critical case

Let $\beta \geq 0$, $B = B(0, 1) \subset \mathbb{R}^2$ be the open ball. The first goal is to establish decay conditions on the limit (6) to guarantee the attainability of (5) in the critical case where *h does not need to be radial.* It will be assumed (h1) $h(x) : \bar{B} \to [0, \infty[$ a radial continuous function such that $h(0) = 0$,

(h2)

$$0 < M := \lim_{x \to 0} \frac{h(x)}{|x|^\beta} < \infty, \tag{6}$$

Theorem 1. *Let h be a function satisfying (h1) and (h2). Then (5) is finite. Suppose the existence of a $n_0 > 0$ sufficiently large (c.f. remark below, item (ii)) such that*

$$\sup_{|x| \le e^{-\frac{n_0}{2+\beta}}} \left| \frac{h(x)}{|x|^\beta} - M \right| < \frac{M}{n_0}. \tag{7}$$

Then the supremum (5) is achieved.

Remark. *(i) It is important to emphasize that* ***the decay condition on the theorem above does not depend on the value of the weight function*** $h(x)$ ***except on a small neighbourhood of the origin****. This is achieved by using Concentrating Sequences and Concentration-Compactness-type techniques (c.f. [23], [9]).*

(ii) The meaning of sufficiently large will be clarified during the demonstration of the theorem: it depends on the value of M and on the further choice of the test functions used on the proof.

Attainability of the supremum of a Hardy-type embedding

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain containing the origin. Adimurthi and Sandeep [2] established the finiteness of the following singular embedding:

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} < \infty \text{ if } \beta \in [0, 2[,\ \alpha > 0,\ \frac{\alpha}{4\pi} + \frac{\beta}{2} \le 1. \tag{8}$$

Csató and Roy proved the attainability for any domain Ω [11] and for simply connected domain [12] using a much simpler technique relying on the Riemann Mapping Theorem. Here we will replace the weight $\frac{1}{|x|^\beta}$ by $\frac{1}{h(x)}$, where $h : B \to \mathbb{R}$ satisfy hypothesis (h2) (as on Theorem 1) and

(h1)' $h(x) : \bar{\Omega} \to [0, \infty[$ a continuous function such that $h(0) = 0$,

(h3) There exists a constant $C > 0$ such that

$$\frac{h(x)}{|x|^\beta} \ge C \text{ in } B. \tag{9}$$

We arrive again at the necessity of a decay condition for the limit (6):

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a C^1 simply connected bounded domain containing the origin, $\beta \in [0, 2[$, $\alpha > 0$ h be a function satisfying (h1)', (h2) and (h3). Then*

(i) if $\frac{\alpha}{4\pi} + \frac{\beta}{2} < 1$, the supremum

$$\sup_{v \in H_0^1(\Omega), \|\nabla v\|_{L^2} \leq 1} \int_\Omega \frac{e^{\alpha v^2} - 1}{h(x)} < \infty. \tag{10}$$

is attained for any h, while

(ii) if $\frac{\alpha}{4\pi} + \frac{\beta}{2} = 1$, (10) is achieved if there exists $n_0 > 0$ sufficiently large such that

$$\sup_{r \leq e^{-\frac{n_0}{2a}}} \left| \frac{|x|^\beta}{h(x)} - M^{-1} \right| < M^{-1} \frac{1}{n_0} \tag{11}$$

where $a = \frac{\alpha}{4\pi}$.

Here the observations of Remark are still valid.

Existence of solution for a associated Dirichlet Problem

It is known by the Lagrange Multiplier theorem that there exists a constant $\lambda \neq 0$ such that the function $u_0 \in H^1_{0,rad}(B)$ that achieves the supremum (5) is the (classical) solution of a Dirichlet problem

$$\begin{cases} -\Delta u(x) = \lambda u e^{2\pi(2+\beta)u^2} h(x), \ x \in B, \\ u(x) = 0, \ x \in \partial B. \end{cases} \tag{12}$$

The regularity of the solution will be discussed further. Motivated by that, we study the existence problem for a particular class of Henón-type problems, i.e.:

$$\begin{cases} -\Delta u(x) = f(x, u), \ x \in B, \\ u(x) = 0, \ x \in \partial B, \end{cases} \tag{13}$$

where f has exponential critical growth (i.e. $f(x,t) \geq Ce^{Kt^2}$ for t large, $C, K > 0$ constants). Problems of this type have been vastly studied ([1], [3], [14], etc). Here we apply a variational approach in the spirit of Brezis-Nirenberg [7], primarily studied by de Figueiredo, Miyagaki and Ruf (c.f. [14]). Let $B \subset \mathbb{R}^2$ be the open unit ball centered at zero, and f satisfies the following conditions:

(H1)

$$f : \bar{B} \times \mathbb{R} \to \mathbb{R} \text{ continuous and radial on x; } f(x,0) = 0 \ \forall x \in B;$$

(H2)

$$\begin{array}{llll} \exists t_0 > 0,\ M > 0: & 0 & \leq F(x,t) := \int_0^t f(x,s)ds & \leq M f(x,t) \ \forall x \in B,\ \forall t \geq t_0; \\ & 0 & \leq F(x,t) & \leq -M f(x,t) \ \forall x \in B,\ \forall t \leq -t_0 \end{array}$$

(H3)

$$0 \leq F(x,t) \leq \frac{1}{2} t f(x,t) \ \forall x \in B,\ \forall t \in \mathbb{R}.$$

(H4)

$$\limsup_{t \to 0} \frac{2F(x,t)}{t^2} < \mu_1 := \inf\{\frac{\int_B |\nabla u|^2}{\int_B u^2} :\ u \in H^1_{0,rad}(B)\}$$

(H5) There exist some $\alpha > 0$ some $k \geq 0$ and some constant $K > 0$ such that

$$|f(x,t)| \leq K|t|^k e^{2\pi(2+\alpha)t^2} |x|^\alpha \ \forall\ t \in \mathbb{R},\ \forall x \in B,$$

(H6) Given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ and $\delta > 0$ such that for all $t \geq t_\varepsilon$ and for $x \in B_\delta$:

$$f(x,t) t e^{-2\pi(2+\beta)t^2} |x|^{-\beta} > \zeta - \varepsilon,$$

where $\beta > 0$ and ζ is any constant strictly greater than $\frac{2+\beta}{2\pi e}$.

For instance, $f(x,t) = Cte^{2\pi(2+\gamma)t^2} h(x)$ satisfy conditions (H1) to (H6) for any constants $C > 0$, $\gamma > 0$ and any continuous radial function $h : \bar{B} \to \mathbb{R}$ such that $L_1 \leq \frac{h(x)}{|x|^\gamma} \leq L_2 \ \forall\ x \in \bar{B}$ for constants $L_1, L_2 > 0$.

Theorem 3. *Suppose f satisfies conditions (H1) - (H6). Then the problem (13) has a radial nonzero weak solution $u \in H^1_{0,rad}(B) \cap H^2(B)$.*

Remark. *Hypothesis (H1)-(H5) are useful to produce the traditional set-up for appliance of the Mountain Pass Theorem (Th. C.0.3), including the proof that the apropriate functional satisfies the Palais-Smale condition until a certain level $\boldsymbol{d} \in \mathbb{R}$. Condition (H6) is the minimum growth that allows finding a critical value below $\boldsymbol{d}$.*

Chapter 1

Proof of Theorem 1

1.1 Finiteness

First we notice that if a function $u_0 \in H^1_{0,rad}(B)$ achieves the supremum (5), then it also achieves

$$\sup_{u\in H^1_{0,rad}(B), \|\nabla u\|_2 \le 1} \int_B (e^{2\pi(2+\beta)u^2} - 1)h(x)dx, \tag{1.1}$$

since the added part does not depend on u. From now on we will deal with (1.1) instead of (5) whenever it is convenient.

Proof. (Theorem 1.1, finiteness of the supremum) By (h2) one has

$$\int_B e^{2\pi(2+\beta)u^2} h(x)dx \le C \int_B e^{2\pi(2+\beta)u^2} |x|^\beta dx$$

for all $u \in H^1_{0,rad}(B)$, since, given $\eta > 0$, there exists a neighbourhood B_γ (the 0-centered ball with radius γ) of the origin such that

$$\frac{h(x)}{|x|^\beta} \le M - \eta \ \forall \ x \in B_\gamma,$$

and outside B_γ the function $\frac{h(x)}{|x|^\beta}$ is continuous and thus bounded from above. One has

Theorem 1.1.1. *([15], Theorem 1.1(particular case))*

$$\sup_{u\in H^1_{0,rad}(B), \|\nabla u\|_{L^2(B)} \le 1} \int_B e^{2\pi(2+\beta)u^2} |x|^\beta dx \tag{1.2}$$

is finite.

The proof of Theorem 1.1.1 by de Figueiredo, do Ó and Santos goes as follows: we perform the following change of variables

$$z(t) = \left(\frac{1}{2\pi(2+\beta)}\right)^{1/2} u(x), |x| = e^{\frac{-t}{2+\beta}}, \tag{1.3}$$

which is an isometry (c.f. [15]) between $H^1_{0,rad}(B)$ and

$$H := \{w : (0,\infty) \to \mathbb{R} : w \text{ is measurable, has weak derivative, } w(0) = 0,$$
$$\|w\|_H := \int_0^\infty |w'(s)|^2 ds < \infty\},$$

in order to get

$$\begin{aligned} \sup_{u \in H^1_{0,rad}(B), \|\nabla u\|_{L^2} \le 1} \int_B (e^{2\pi(2+\beta)u^2} - 1)|x|^\beta dx &= \sup_{u \in H, \|u\|_H \le 1} \frac{2\pi}{2+\beta} \int_0^\infty (e^{z^2} - 1)e^{-t} dt \\ &= \frac{2\pi}{2+\beta} \sup_{u \in H^1_{0,rad}(B), \|\nabla u\|_{L^2} \le 1} \int_B (e^{4\pi u^2} - 1) dx \end{aligned}$$

which happens to be finite due to a J.Moser result [24]. ■

1.2 Attainability

1.2.1 Concentration-Compactness

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set. We say that $(u_n) \subset H^1_0(\Omega)$ is a normalized concentrating sequence at $x_0 \in \bar{\Omega}$ if*

(C1) $\int_\Omega |\nabla u_n|^2 dt = 1 \ \forall \ n$,

(C2) $\limsup_{n\to\infty} \int_{\Omega - B_\rho(x_0)} |\nabla u_n|^2 dt = 0$ *as* $n \to \infty$ *for all* $\rho > 0$ *fixed.*

The technique used here is based on Concentration-Compactness results, first established by Lions:

Theorem 1.2.2. *(Lions [23], pg. 195-199) Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, $(u_n) \subset W^{1,N}_0(\Omega)$ such that $\|\nabla u_n\|_{L^N} \le 1 \ \forall \ n$. Without loss of generality we assume $u_n \rightharpoonup u$, $|\nabla u|^2 dx \rightharpoonup d\mu$. Set $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Then either*

(i) $d\mu = \delta_{x_0}$, $u = 0$ *and* $\exp(\alpha_N u^{\frac{N}{N-1}}) \rightharpoonup c\delta_{x_0}$ *for some* $c \ge 0$ *and* $x_0 \in \bar{\Omega}$, *or*

(ii) there exists $\alpha > 0$ *such that* $\exp((\alpha_N + \alpha)u_n^{\frac{N}{N-1}})$ *is bounded in* $L^1(\Omega)$ *and thus*

$$\exp(\alpha_N u_n^{\frac{N}{N-1}}) \to \exp(\alpha_N u^{\frac{N}{N-1}}) \text{ in } L^1(\Omega).$$

We will use the above theorem to prove

Theorem 1.2.3. *Define*

$$\mathscr{B}_1(B) := \{u \in H^1_{0,rad}(B) : \ \|u\| \leq 1\}. \tag{1.4}$$

Let B be the unitary open ball in $\mathbb{R}^2$. Let (u_n) in $\mathscr{B}_1(B)$ with $u_n \rightharpoonup u$ and $|\nabla u_n|^2 \rightharpoonup d\mu$. There is a subsequence (still denoted by (u_n)) such that either

(i) (u_n) concentrates at 0 (or equivalently μ is a Dirac measure $c\delta_0$ for some $c \geq 0$) and $u = 0$ or

(ii) compactness holds in the sense that $e^{2\pi(2+\beta)u_n^2}h(x) \to e^{2\pi(2+\beta)u^2}h(x)$ in $L^1(B)$.

Remark 1.2.4. *(i) Since $H^1_0(\Omega)$ is reflexive, if (u_n) is a bounded sequence a theorem by Kakutani (c.f. [6], Theorem 3.17) guarantees the existence of $u \in H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ in $H^1_0(\Omega)$.*

(ii) Since (u_n) is a normalized sequence, i.e, $\int_B |\nabla u_n|^2 dx = 1$ $\forall n$, it is possible to apply Prohorov's Theorem ([5], p. 59) to guarantee the existence of such a measure $d\mu$. More exactly: Prohorov theorem states that a tight subset Π (given $\varepsilon > 0$ there exists $K \subset \mathbb{R}^N$ compact such that $P(K) > 1 - \varepsilon$ for all $P \in \Pi$) of probability measures is relatively compact (there exists a weakly convergent subsequence). The sequence of probability measures $|\nabla u_n|^2 dx$ is tight with respect to $\mathbb{R}^2$ since the functions are supported on the unit ball, which is compact.

Proof. (Theorem 1.2.3) Notice that for any $p > 1$:

$$\begin{aligned}
\int_B e^{2\pi(2+\beta)pu_n^2}h(x)^p dx &\leq C\left(\int_B e^{2\pi(2+\beta)pu_n(|x|)^2}|x|^{p\beta}dx\right) \\
&\leq C\left(\int_B e^{2\pi(2+\beta)pu_n(|x|)^2}|x|^{\beta}dx\right) \\
&\quad \text{(since } |x| \leq 1 \text{ and } \beta p \geq \beta \Rightarrow |x|^{\beta p} \leq |x|^{\beta}) \\
&= C\left(2\pi \int_0^1 e^{2\pi(2+\beta)pu_n(r)^2}r^{\beta+1}dr\right) \\
&\quad \text{(taking } s = r^{\frac{\beta+2}{2}},\ (2+\beta)^{1/2}u_n(r) = 2^{1/2}v_n(s)) \qquad (1.5) \\
&= C\left(\frac{4\pi}{2+\beta}\int_0^1 e^{4\pi p v_n(s)^2}sds\right) \\
&= C\left(\frac{2}{2+\beta}\int_B e^{4\pi p v_n(x)^2}dx\right)
\end{aligned}$$

where $v_n(x) = \left(\frac{2+\beta}{2}\right)^{1/2} u_n(|x|^{\frac{2}{2+\beta}})$. Notice that $\|\nabla v_n\|_{L^2(B)} = \|\nabla u_n\|_{L^2(B)}$ and that, since (u_n) does not concentrate, neither (v_n) concentrates. Thus, applying Holder's inequality for $p > 1$ one has

$$\int_E e^{2\pi(2+\beta)u_n^2}h(x)dx \leq C\left(\int_E e^{4\pi p v_n^2}dx\right)^{1/p}|E|^{1/p^*} \leq C'|E|^{1/p^*},$$

if p is sufficiently close to 1 by Lions' theorem 1.2.2, item (ii). Thus, the sequence of functions $e^{2\pi(2+\beta)u_n^2}h(x)$ is equiintegrable. Since $u_n \rightharpoonup u$ in H_0^1, $u_n \to u$ in L^2 up to a subsequence and finally $u_n \to u$ almost everywhere up to a subsequence. This allows us to conclude the proof by directly applying Vitali's Convergence Theorem. ■

1.2.2 Non-Compactness Level

We now establish the non-compactness level for the functional:

Theorem 1.2.5. *Let $(u_n) \subset \mathscr{B}_{1,rad}(B)$ be a concentrating sequence. Then*

$$\limsup_{n\to\infty} \int_B (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \le \frac{2\pi Me}{2+\beta}. \tag{1.6}$$

In the proof of the theorem above, we use the following lemmas:

Lemma 1.2.6. *(c.f. [18], Proof of Theorem 1.4, page 142, or [9]) If (z_n) is a concentrating sequence just like in Remark 1.2.8, then*

$$\limsup_{n\to\infty} \int_0^\infty (e^{z_n^2} - 1)e^{-t}dt \le e.$$

Lemma 1.2.7. *Let $B \subset \mathbb{R}^2$ and (u_n) be a normalized concentrating sequence around zero, then*

$$\limsup_{n\to\infty} \int_{B-B_\delta} (e^{4\pi u_n^2} - 1)dx \to 0 \text{ as } n \to \infty.$$

Proof. (Lemma 1.2.7) Let $\eta \in C^\infty(\bar{B})$ be a cutoff function such that $\eta \ge 0$ and

$$\eta = 1 \text{ in } B - B_\delta, \;\; \eta = 0 \text{ in } B_{\delta/2}.$$

Therefore

$$\lim_{n\to\infty} \int_{B-B_\delta} \left(e^{4\pi u_n^2} - 1\right) dx \le \limsup_{n\to\infty} \int_{B-B_{\delta/2}} \left(e^{4\pi(\eta u_n)^2} - 1\right) dx \tag{1.7}$$

Notice that $\eta u_n \in H_0^1(B - \overline{B_{\delta/2}})$ and ηu_n does not concentrate at any point, since

$$\int_{B-B_{\delta/2}} |\nabla \eta u_n|^2 dx \le (2\|\eta\|_{C^1}^2 + 2\|\eta\|_{L^2}) \int_{B-B_{\delta/2}} |\nabla u_n|^2 dx \to 0 \text{ as } n \to \infty. \tag{1.8}$$

So the conclusion holds by Lions' Concentration Compactness Alternative (Theorem 1.2.2) second item (ii), which states that the non-concentrating nature of (ηu_n) yields compacity for a subsequence:

$$\int_{B-B_{\delta/2}} \left(e^{4\pi(\eta u_n)^2} - 1\right) dx \to \int_{B-B_{\delta/2}} \left(e^{4\pi(\eta u)^2} - 1\right) dx = 0.$$

■

Proof. (Theorem 1.2.5) We have by (h2), for any $\eta > 0$, that there exists $\rho > 0$ suficiently small such that $M - \eta \leq \frac{h(x)}{|x|^\beta} \leq M + \eta$ if $x \in B_\rho$. So

$$\int_{B_\rho} (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \leq (M + \eta) \int_{B_\rho} (e^{2\pi(2+\beta)u_n^2} - 1)|x|^\beta dx. \tag{1.9}$$

Now we perform again the change of variables (1.3) to conclude that

$$(M + \eta) \int_{B_\rho} (e^{2\pi(2+\beta)u_n^2} - 1)|x|^\beta dx = \frac{2\pi(M + \eta)}{2 + \beta} \int_{(2+\beta)\log\frac{1}{\rho}}^{\infty} (e^{z_n^2} - 1)e^{-t}dt. \tag{1.10}$$

Remark 1.2.8. *Supposing $(u_n) \subset H^1_{0,rad}(\Omega)$ is a normalized concentrationg sequence at $x_0 = 0$ and performing the changes of variables (1.3), we can analogously define a normalized concentrating (at ∞) sequence $(z_n(t))$ by the following conditions:*

(C0) (z_n) is continuous, piecewise differentiable and $z_n(0) = 0$,

(C1) $\int_0^\infty |z_n'|^2 dt = 1 \ \forall \ n$,

(C2) $\int_0^A |z_n'|^2 dt \to 0$ as $n \to \infty$ for any $A > 0$ fixed,

From Lemma 1.2.6 we conclude, putting together (1.9) and (1.10):

$$\limsup_{n\to\infty} \int_{B_\rho} (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \leq \frac{2\pi(M + \eta)}{2 + \beta} e \tag{1.11}$$

Now notice that for any $\delta > 0$ such that $B_\delta \subset B$ one has

$$\lim_{n\to\infty} \int_{B-B_\delta} (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx = 0,$$

by performing the change of variables (1.3) and also by lemma 1.2.7. So instead of (1.11) one actually has

$$\limsup_{n\to\infty} \int_{B} (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \leq \frac{2\pi(M + \eta)}{2 + \beta} e. \tag{1.12}$$

Since (1.12) holds for any $\eta > 0$, we conclude that if (u_n) is a concentrating sequence, then

$$\limsup_{n\to\infty} \int_{B} (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \leq \frac{2\pi M}{2 + \beta} e.$$

This finishes the proof of Theorem 1.2.5. ■

1.2.3 Test functions and decay condition

Given any $\delta > 0$ and again performing the change of variables (1.3) write

$$\begin{aligned}\int_B (e^{2\pi(2+\beta)u^2}-1)h(x)dx &\geq (M-\varepsilon(\delta))\int_{B_\delta}(e^{2\pi(2+\beta)u^2}-1)|x|^\beta dx\\ &= \frac{2\pi(M-\varepsilon(\delta))}{2+\beta}\int_{(2+\beta)\log 1/\delta}^{\infty}(e^{v^2}-1)e^{-t}dt,\end{aligned}$$

where $\varepsilon(\delta) = \sup_{x\in B_\delta}\left|\frac{h(x)}{|x|^\beta} - M\right|$. Consider the sequence

$$y_n(t) = \begin{cases} \frac{t}{n^{1/2}}(1-\delta_n)^{1/2}, \ 0 \leq t \leq n, \\ \frac{1}{(n(1-\delta_n))^{1/2}}\log\frac{A_n+1}{A_n+e^{-(t-n)}} + (n(1-\delta_n))^{1/2}, n \leq t, \end{cases} \tag{1.13}$$

where $\delta_n = 2\frac{\log n}{n}$ and A_n are constants to be chosen. This sequence is contained on the space H and concentrates at ∞. On this context, they will be used as test functions, in order to surpass the non-compactness level given by Lemma 1.2.5. First one needs to establish some properties of (y_n):

Lemma 1.2.9. *The coefficients A_n can be chosen such that $A_n = \frac{1}{n^2 e} + O(\frac{\log^2 n}{n^3})$ $\forall\ n \in \mathbb{N}$ sufficiently large and $\int_0^\infty |y_n'|^2 dt = 1$ $\forall\ n \in \mathbb{Z}_+$.*

Proof. (Lemma 1.2.9) First notice that $\int_0^n |y_n'|^2 dt = 1-\delta_n = 1 - 2\frac{\log n}{n}$ (since $\delta_n = 2\frac{\log n}{n}$ on (1.13)) so we have to show that it is possible to choose A_n such that

$$\int_n^\infty |y_n'|^2 dt = \delta_n = 2\frac{\log n}{n}. \tag{1.14}$$

Indeed,

$$\begin{aligned}\int_n^\infty |y_n'|^2 dt &= \frac{1}{n(1-\frac{2\log n}{n})}\int_n^\infty \left|\frac{d}{dt}[\log(A_n+1)-\log(A_n+e^{-(t-n)})]\right|^2 dt\\ &= \frac{1}{n(1-\delta_n)}\int_n^\infty \left|\frac{e^{-(t-n)}}{A_n+e^{-(t-n)}}\right|^2 dt\\ &= \frac{1}{n(1-\delta_n)}\int_0^\infty \left|\frac{e^{-s}}{A_n+e^{-s}}\right|^2 ds\\ &= \frac{1}{n(1-\delta_n)}\int_1^\infty \left|\frac{1}{A_n r+1}\right|^2 \frac{1}{r}dr \ (r = e^s)\\ &= \frac{1}{n(1-\delta_n)}\int_0^1 \frac{\rho}{(A_n+\rho)^2}dr \ (\rho = 1/r)\\ &= \frac{1}{n(1-\delta_n)}\left(\log\frac{A_n+1}{A_n} - \frac{1}{A_n+1}\right)\end{aligned}$$

By (1.14) this gives the condition

$$\log\frac{A_n+1}{A_n}-\frac{1}{A_n+1}=(1-\delta_n)2\log n \tag{1.15}$$

and thus

$$\frac{A_n+1}{A_n}e^{-\frac{1}{A_n+1}}=n^2e^{-\frac{4\log^2 n}{n}}$$

Since A_n converges to 0 (by (1.15)), $\frac{A_n}{A_n+1}\to 0$ as $n\to\infty$. Since $e^{-\frac{1}{A_n+1}}=e^{-1}e^{\frac{A_n}{A_n+1}}$, we get

$$\frac{A_n+1}{A_n}\left(1+\frac{A_n}{A_n+1}+o(\frac{A_n}{A_n+1})\right)=n^2ee^{-\frac{4\log^2 n}{n}}. \tag{1.16}$$

Thus

$$\frac{A_n+1}{n^2A_n}+\frac{1}{n^2}+\frac{1}{n^2}o(1)=e+e(e^{-\frac{4\log^2 n}{n}}-1),$$

so we conclude that

$$\frac{A_n+1}{n^2A_n}=e-\frac{4\log^2 n}{n}+O(\frac{\log^4 n}{n^2}). \tag{1.17}$$

This implies

$$\frac{1}{A_n}=n^2e-4en\log^2 n+O(\log^4 n),$$

hence

$$A_n=\frac{1}{n^2e}+O(\frac{\log^2 n}{n^3}).$$

■

Lemma 1.2.10.

$$\int_n^\infty e^{y_n^2-t}dt\geq e+\frac{e}{n}+o(\frac{1}{n}). \tag{1.18}$$

Proof. (Lemma 1.2.10)

$$\begin{aligned}
\int_n^\infty e^{y_n^2-t}dt &= \int_0^\infty \exp\left[\left((n(1-\delta_n))^{1/2}+\frac{1}{(n(1-\delta_n))^{1/2}}\log\frac{A_n+1}{A_n+e^{-s}}\right)^2-n-s\right]ds\\
&= \int_0^\infty \exp\left[n(1-\delta_n)+2\log\frac{A_n+1}{A_n+e^{-s}}+\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}-n-s\right]ds\\
&= \frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]\exp\left[\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}\right]ds\\
&\geq \frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]\left(1+\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}\right)\\
&= \frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]ds\ \text{(I)}\\
&+\frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}ds\ \text{(II)}.
\end{aligned} \tag{1.19}$$

Developing (I) from (1.19):

$$\begin{aligned}
\frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]ds &= \frac{1}{n^2}\int_0^\infty\left(\frac{A_n+1}{A_n+e^{-s}}\right)^2 e^{-s}ds\\
&= \frac{(1+A_n)^2}{n^2}\int_0^\infty \frac{e^s}{(A_ne^s+1)^2}ds\\
&= \frac{(1+A_n)^2}{n^2}\int_1^\infty \frac{1}{(A_nr+1)^2}dr\\
&= \frac{(1+A_n)^2}{n^2}\frac{1}{A_n(1+A_n)}\\
&= \frac{1+A_n}{n^2A_n}\\
&= e-\frac{4e\log^2 n}{n}+O(\frac{\log^4 n}{n^2}). \qquad (1.20)
\end{aligned}$$

Developing (II) from (1.19):

$$\begin{aligned}
&\frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}}-s\right]\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}ds\\
&= \frac{1}{n^2}\int_0^\infty\left(\log\frac{A_n+1}{A_ne^s+1}\right)^2\frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}e^s ds\\
&= \frac{1}{n^2n(1-\delta_n)}\int_1^\infty\left(\frac{A_n+1}{A_nr+1}\right)^2\log^2\frac{A_n+1}{A_n+\frac{1}{r}}dr\\
&= \frac{A_n+1}{n^2(n-2\log n)}\int_1^{\frac{A_n+1}{A_n}}\log^2 u\, du\ \left(u=\frac{A_n+1}{A_n+\frac{1}{r}}\right)\\
&= \frac{A_n+1}{n^2(n-2\log n)}\int_0^{\log\frac{A_n+1}{A_n}} z^2e^zdz\ (z=\log u)\\
&= \frac{A_n+1}{n^2(n-2\log n)}\left[z^2e^z-2ze^z+2e^z\right]_0^{\log\frac{A_n+1}{A_n}}\\
&= \frac{A_n+1}{n^2(n-2\log n)}\left[-2+\log^2\left(\frac{A_n+1}{A_n}\right)\frac{A_n+1}{A_n}-2\log\left(\frac{A_n+1}{A_n}\right)\frac{A_n+1}{A_n}+2\frac{A_n+1}{A_n}\right]\\
&= \frac{1}{n-2\log n}\left[e+O(\frac{\log^2 n}{n})\right]\left[4\log^2 n+4\log n+1-4\log n-2+2\right]+O(1/n^2)\\
&\text{(by (1.17) and (1.15))}\\
&= \frac{4e\log^2 n}{n}+\frac{e}{n}+o(\frac{e}{n}).
\end{aligned} \qquad (1.21)$$

■

Proof. (Theorem 1.1, Attainability) By Lemma 1.2.10 one has

$$\frac{2\pi(M-\varepsilon(\delta))}{2+\beta}\int_{(2+\beta)\log 1/\delta}^\infty (e^{y_n^2}-1)e^{-t}dt \geq \frac{2\pi(M-\varepsilon(\delta))}{2+\beta}(e+\frac{e}{n}+o(\frac{1}{n})), \qquad (1.22)$$

where $\varepsilon(\delta) = \sup_{x\in B_\delta}\left|\frac{h(x)}{|x|^\beta} - M\right|$, **provided that** $(2+\beta)\log 1/\delta \geq n$. The goal is to determine a decay condition such that

$$\frac{2\pi(M-\varepsilon(\delta))}{2+\beta}\left(e+\frac{e}{n}+o\left(\frac{1}{n}\right)\right) > \frac{2\pi M}{2+\beta}e \tag{1.23}$$

in order to apply Theorem 1.2.3.

Remark 1.2.11. *(i) Notice that one cannot send n to* ∞ *while* δ *is fixed since it would be impossible to obtain (1.23) and neither send* δ *to* 0 *while n is fixed, otherwise (1.22) would not be true (because* $(2+\beta)\log\frac{1}{\delta} > n$ *and it would be impossible to apply Lemma 1.2.10). The decays of the sequence and of the function h are linked (since the chosen sequence increases the right-hand side of (1.22) and the decay of h decreases it). Therefore a balance between n and* δ *is necessary; here we choose* $n = (2+\beta)\log 1/\delta$ *for sharpness. There is no contradiction on the fact that n is an integer and* $\log 1/\delta$ *is not:* y_n *can be treated as a net or one can take a decreasing sequence of* $\delta_n' s \to 0$ *without loss of generality.*

*(ii) Here it is possible to clarify the meaning of **sufficiently small**: one needs to make n sufficiently big (or* δ *sufficiently small with our choice) in order to compensate the lower order term* $Mo\left(\frac{1}{n}\right)$ *which comes from the choice of test functions and the value of M.*

So it is sufficient that for one δ sufficiently small $\varepsilon(\delta) \leq \frac{M}{(2+\beta)\log\frac{1}{\delta}}$, or, for a n_0 sufficiently big:

$$\sup_{|x|\leq e^{-\frac{n_0}{2+\beta}}}\left|\frac{h(x)}{|x|^\beta} - M\right| \leq \frac{M}{n_0},$$

For this value of n_0, the following is true for any sequence (u_n) concentrating at zero:

$$\int_B (e^{2\pi(2+\beta)y_{n_0}^2} - 1)h(x)dx > \frac{2\pi Me}{2+\beta} \geq \limsup_{n\to\infty}\int_B (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx.$$

Thus a maximizing sequence (u_n) with $\|\nabla u_n\|_{L^2(B)} \leq 1\ \forall\, n$ cannot concentrate. Therefore Theorem 1.2.3, item (ii) implies compacity for a subsequence of (u_n):

$$\begin{aligned}\sup_{v\in H^1_{0,rad}(B), \|\nabla v\|_{L^2}\leq 1}\int_B (e^{2\pi(2+\beta)v^2} - 1)h(x)dx &= \lim_{n\to\infty}\int_B (e^{2\pi(2+\beta)u_n^2} - 1)h(x)dx \\ &= \int_B (e^{2\pi(2+\beta)u^2} - 1)h(x)dx\end{aligned}$$

■

Chapter 2

Proof of Theorem 2

2.1 Finiteness

Proof. (Theorem 1.2, Finiteness) First regarding the finiteness of the supremum

$$\sup_{u\in H_0^1(\Omega),\|\nabla u\|_{L^2}\leq 1}\int_\Omega \frac{e^{\alpha u^2}-1}{h(x)}dx, \tag{2.1}$$

by (h3) one has

$$\int_\Omega \frac{e^{\alpha u^2}-1}{h(x)}dx \leq \int_\Omega C\frac{e^{\alpha u^2}-1}{|x|^\beta}dx$$

and this is finite by the following result by Adimurthi and Sandeep (c.f. [2]):

Theorem 2.1.1. *Let $u \in H_0^1(\Omega)$. Then for every $\alpha > 0$ and $\beta \in [0, 2[$*

$$\int_\Omega \frac{e^{\alpha u^2}-1}{|x|^\beta}dx < \infty.$$

Moreover,

$$\sup_{u\in H_0^1(\Omega),\|\nabla u\|_{L^2}\leq 1}\int_\Omega \frac{e^{\alpha u^2}-1}{|x|^\beta}dx < \infty$$

if and only if $\frac{\alpha}{4\pi}+\frac{\beta}{2}\leq 1$.

■

2.2 Attainability

2.2.1 The case $\frac{\alpha}{4\pi} + \frac{\beta}{2} < 1$

Proof. (Theorem 1.2, Attainability, $\frac{\alpha}{4\pi} + \frac{\beta}{2} < 1$) Let $(u_n) \subset H_0^1(\Omega)$, $\|\nabla u_n\|_{L^2(\Omega)} \leq 1$ be a maximizing sequence, i.e.:

$$\lim_{n\to\infty} \int_\Omega \frac{e^{\alpha u_n^2}}{h(x)} dx = \sup_{u \in H_0^1(\Omega), \|\nabla u\|_{L^2} \leq 1} \int_\Omega \frac{e^{\alpha u^2}}{h(x)} dx. \tag{2.2}$$

Since (u_n) is a bounded sequence and $H_0^1(\Omega)$ is a reflexive space, a theorem by Kakutani (c.f. [6], Theorem 3.17) implies that there exists $u \in H_0^1(\Omega)$ such that

$$\|\nabla u\|_{L^2(\Omega)} \leq 1 \text{ and } u_n \rightharpoonup u.$$

Let $E \subset \Omega$ be an arbitrary measurable set. Let $r := 4\pi/\alpha > 1$ and $\frac{1}{s} = 1 - \frac{1}{r} > \frac{\beta}{2}$ and use Hölder's inequality to obtain

$$\int_E \frac{e^{\alpha u_n^2}}{h(x)} \leq \left(\int_E e^{4\pi u_n^2} \right)^{1/r} \left(\int_E \frac{1}{h(x)^s} \right)^{1/s}.$$

Notice that that $\frac{1}{h(x)^s} \in L^1(\Omega)$, since, by condition (h3), for any sufficiently small $\eta > 0$, $\frac{1}{h(x)^s} \in L^\infty(\bar{\Omega} - B_\eta)$ and by condition (h2) h is very similar to $(1/M)|x|^\beta$, and $\frac{1}{|x|^{\beta s}} \in L^1(\Omega)$ since $\beta s < 2$ and thus $\int_\Omega \frac{1}{|x|^{\beta s}} \leq 2\pi \int_0^R r^{1-\beta s} dr = \frac{2\pi R^{2-\beta s}}{2-\beta s}$, where $R < \infty$ is the radius of a ball containing the bounded domain Ω. Using this fact and the Trudinger-Moser inequality, we conclude then that the sequence $\frac{e^{\alpha u_n^2}}{h(x)}$ is equi-integrable, and apply the Vitali Convergence Theorem to conclude that

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_{L^2} \leq 1} \int_\Omega \frac{e^{\alpha u^2}}{h(x)} dx = \lim_{n\to\infty} \int_\Omega \frac{e^{\alpha u_n^2}}{h(x)} dx = \int_\Omega \frac{e^{\alpha u^2}}{h(x)} dx. \tag{2.3}$$

∎

2.2.2 The case $\frac{\alpha}{4\pi} + \frac{\beta}{2} = 1$

Passing from Ω to B

Let

$$\mathscr{B}_1(E) = \{u \in H_0^1(E) : \|u\| \leq 1\}, \tag{2.4}$$

where $E \subset \mathbb{R}^2$ is any measurable set. Fix

$$F_\Omega^{\alpha,h}(u) = \int_\Omega \frac{(e^{\alpha u^2} - 1)}{h(x)} dx; \; F_\Omega^{\alpha,\beta}(u) = \int_\Omega \frac{(e^{\alpha u^2} - 1)}{|x|^\beta} dx,$$

particularly

$$F_{B_\delta}^{\sup,\alpha,h,B} = \sup\left\{F_{B_\delta}^{\alpha,h}(u);\ u \in \mathscr{B}_1(B)\right\}. \tag{2.5}$$

analogously for $F_{B_\delta}^{\alpha,\beta}$:

$$F_{B_\delta}^{\sup,\alpha,\beta,B} = \sup\left\{F_{B_\delta}^{\alpha,\beta}(u);\ u \in \mathscr{B}_1(B)\right\} \tag{2.6}$$

Let

$$F_{\Omega}^{\mathrm{conc},\alpha,\beta}(x) = \sup\left\{\limsup_{n\to\infty} F_{\Omega}^{\alpha,\beta}(u_n);\ (u_n) \subset \mathscr{B}_1(\Omega) \text{ concentrates at } x \in \bar{\Omega}\right\},$$

and

$$F_{B_\delta}^{\mathrm{conc},\alpha,\beta,B}(x) = \sup\left\{\limsup_{n\to\infty} F_{B_\delta}^{\alpha,\beta}(u_n);\ (u_n) \subset \mathscr{B}_1(B) \text{ concentrates at } x \in \bar{B}_\delta\right\}. \tag{2.7}$$

Remark 2.2.1. *Here one supposes Ω to be simply connected in order to use the Riemann theorem from Complex Analysis (c.f. [10]), guaranteeing the existence of a conformal map $\phi : B \to \Omega$ (which is a C^∞ diffeomorphism in the standard topology) such that $\phi(0) = 0$. The inverse of ϕ is also a conformal map. Thus (by the representation of a holomorphic function by its power series) there exists a conformal map $\varphi : B \to \Omega$ such that $\phi(z) = z\varphi(z)$ and $\varphi(0) = \phi'(0) \neq 0$. This implies also that $\varphi \neq 0$ since ϕ is 1:1.*

Theorem 2.2.2. *Define*

$$\varepsilon(\delta) := \sup_{|x|\le\delta}\left|\frac{|x|^\beta}{h(x)} - M^{-1}\right|. \tag{2.8}$$

For any $v \in H^1_{0,rad}(B) \cap \mathscr{B}_1$ define $u := v \circ \phi^{-1}$. Then $u \in \mathscr{B}_1(\Omega)$ and given any $\delta > 0$ such that $B_\delta(0) \subset \Omega$ we have

$$F_{\Omega}^{\alpha,h}(u) \ge |\phi'(0)|^{2-\beta}(M^{-1} - \varepsilon(\delta))F_{B_\delta}^{\alpha,\beta}(v). \tag{2.9}$$

Proof. (Theorem 2.2.2) Notice that given $\delta > 0$ such that $B_\delta \subset \Omega$ for any $0 < r < \delta$ and $t \in \mathbb{R}$,

$$\frac{(M^{-1} + \varepsilon(\delta))}{r^\beta|\varphi(re^{it})|^\beta} \ge \frac{1}{h(\phi(re^{it}))} \ge \frac{(M^{-1} - \varepsilon(\delta))}{r^\beta|\varphi(re^{it})|^\beta},$$

($\varphi : B \to \Omega$ is the conformal map defined in Remark 2.2.1 such that $\phi(z) = z\varphi(z)$) which implies

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi'(re^{it})|^2}{h(\phi(re^{it}))}dt &\geq \frac{1}{2\pi}\frac{(M^{-1}-\varepsilon(\delta))}{r^\beta}\int_0^{2\pi}\frac{|\phi'(re^{it})|^2}{|\varphi(re^{it})|^\beta}dt && (2.10)\\
&= \frac{(M^{-1}-\varepsilon(\delta))}{r^\beta}|\phi'(0)|^{2-\beta} && (2.11)\\
\Rightarrow 2\pi(M^{-1}-\varepsilon(\delta))|\phi'(0)|^{2-\beta} &\leq r^\beta\int_0^{2\pi}\frac{|\phi'(re^{it})|^2}{h(\phi(re^{it}))}dt\ \forall\ r<\delta. && (2.12)
\end{aligned}$$

The passage from (2.10) to (2.11) holds by the definition of a complex integral of a holomorphic function $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open) over $\Gamma_r := \{z \in \mathbb{C}||z| = r\}$:

$$\begin{aligned}
\int_{\Gamma_r}\frac{f(z)}{z}dz &= \int_0^{2\pi}\frac{f(re^{it})}{re^{it}}ire^{it}dt\\
&= i\int_0^{2\pi}f(re^{it})dt, && (2.13)
\end{aligned}$$

by the fact that $\frac{1}{|\varphi|^\beta}$ is holomorphic since $\varphi \neq 0$ and by a subsequent application of the Cauchy's Integral Formula (c.f. Appendix, Theorem B.0.1).

Since the Jacobian of the change of variables ϕ is $|\phi'(y)|^2$ on each point $y \in B$ by the Cauchy-Riemann equations, we have by Cavalieri's Principle:

$$\begin{aligned}
F_\Omega^{\alpha,h}(u) &\geq \int_0^\delta (e^{\alpha v(r)^2}-1)\left(\int_{\partial B_r}\frac{|\phi'(re^{it})|^2}{h(\phi(re^{it}))}d\sigma(t)\right)dr\\
&\geq 2\pi(M^{-1}-\varepsilon(\delta))|\phi'(0)|^{2-\beta}\int_0^\delta\frac{e^{\alpha v(r)^2}-1}{r^\beta}rdr && (2.14)\\
&= |\phi'(0)|^{2-\beta}(M^{-1}-\varepsilon(\delta))F_{B_\delta}^{\alpha,\beta}(v).
\end{aligned}$$

■

Theorem 2.2.3. *Let $(u_n) \subset \mathscr{B}_1(\Omega)$ be a sequence concentrating at zero. Let $v_n = u_n \circ \phi \in \mathscr{B}_1(B)$. Then (v_n) concentrates at zero and for any $\delta > 0$ sufficiently small one has*

$$\lim_{n\to\infty}F_\Omega^{\alpha,h}(u_n) = M^{-1}|\phi'(0)|^{2-\beta}\lim_{n\to\infty}F_{B_\delta}^{\alpha,\beta}(v_n). \qquad (2.15)$$

In particular, it follows that

$$F_\Omega^{\text{conc},\alpha,h}(0) = M^{-1}|\phi'(0)|^{2-\beta}F_{B_\delta}^{\text{conc},\alpha,\beta}(0). \qquad (2.16)$$

Proof. (Theorem 2.2.3) It has been shown in [12] (Theorem 12) that

Theorem 2.2.4. *Let $(u_n) \subset \mathscr{B}_1(\Omega)$ be a concentrating sequence at zero, let $\phi : B \to \Omega$ be the conformal map as in Remark 2.2.1 and $v_n := u_n \circ \phi$. Then $v_n \in \mathscr{B}_1(B)$ (and thus $F_B^\beta(v_n)$ is well-defined for all i) and (v_n) concentrates at 0.*

From the theorem above follows the first statement of theorem 2.2.3. Now we use the change of variables $x = \phi(y)$ to obtain

$$\lim_{n\to\infty} F_\Omega^{\alpha,h}(u_n) = \lim_{n\to\infty} \int_{\phi(B)} \frac{e^{\alpha u_n^2} - 1}{h(x)} dx = \lim_{n\to\infty} \int_B \frac{e^{\alpha v_n^2} - 1}{h(\phi(y))} |\phi'(y)|^2 dy.$$

Let $\delta > 0$ and split

$$\lim_{n\to\infty} F_\Omega^{\alpha,h}(u_n) = \lim_{n\to\infty} \int_{B_\delta} \frac{e^{\alpha v_n^2} - 1}{h(\phi(y))} |\phi'(y)|^2 dy + \lim_{n\to\infty} \int_{B-B_\delta} \frac{e^{\alpha v_n^2} - 1}{h(\phi(y))} |\phi'(y)|^2 dy. \tag{2.17}$$

Denote the integrals in (2.17) by $A_1^n(\delta)$ and $A_2^n(\delta)$ respectively. First we notice that $\lim_{n\to\infty} A_2^n(\delta) = 0 \ \forall\ \delta > 0$: indeed, since $\phi(z) \neq 0 \ \forall\ z \neq 0$ and $h(x) > 0 \ \forall\ x \neq 0$, so

$$\frac{|\phi'(y)|^2}{h(\phi(y))} \in L^\infty(B - B_\delta).$$

So it suffices to use the fact $\lim_{n\to\infty} \int_{B-B_\delta} (e^{\alpha v_n^2} - 1) dy = 0$, which has been already proven (Lemma 1.2.7). Now define

$$\Xi(y) : \begin{cases} \frac{|y|^\beta |\phi'(y)|^2}{h(\phi(y))} & \text{if } y \neq 0, \\ M^{-1} |\phi'(0)|^{2-\beta} & \text{if } y = 0. \end{cases} \tag{2.18}$$

Ξ is a continuous function on B: indeed, since $\phi(z) = z\varphi(z)$,

$$\frac{|y|^\beta |\phi'(y)|^2}{h(\phi(y))} = \frac{|\phi(y)|^\beta}{h(\phi(y))} \frac{|\phi'(y)|^2}{|\varphi(y)|^\beta},$$

which, together with (h2) and the fact $\varphi(0) = \phi'(0) \neq 0$, yields the desired continuity. If necessary, modify δ so that $|\Xi(y) - \Xi(0)| < \varepsilon$ if $|y| < \delta$. Thus

$$\lim_{n\to\infty} A_1^n(\delta) = \lim_{n\to\infty} \int_{B_\delta} \frac{e^{\alpha v_n^2} - 1}{|y|^\beta} \Xi(y) dy$$

$$\begin{aligned} \Rightarrow \left| \lim_{n\to\infty} F_\Omega^{\alpha,h}(u_n) - M^{-1} |\phi'(0)|^{2-\beta} \lim_{n\to\infty} F_B^{\alpha,\beta}(v_n) \right| &= \left| \lim_{n\to\infty} A_1^n(\delta) - \Xi(0) \lim F_B(v_n) \right| \\ &= \left| \lim_{n\to\infty} \int_{B_\delta} \frac{e^{\alpha v_n^2} - 1}{|y|^\beta} (\Xi(y) - \Xi(0)) dy \right| \\ &\leq \varepsilon F_{B_1}^{\sup,\alpha,\beta}. \end{aligned} \tag{2.19}$$

Since ε is arbitrary, theorem is proved. ■

Concentration-Compactness Result

Definition 2.2.5. *A functional $F_B : \mathscr{B}_1(B) \to \mathbb{R}$ on the form $F_B(u) = \int_\Omega f(x, u(x))dx$ is called* ***compact in the interior of*** $\mathscr{B}_1(B)$ *if given $(u_n) \subset \mathscr{B}_1(B)$ with*

$$\limsup_{n\to\infty} \|\nabla u_n\|_{L^2(B)} < 1 \text{ and } u_n \rightharpoonup u \tag{2.20}$$

implies $f(\cdot, u_n) \to f(\cdot, u)$ in $L^1(B)$ for some subsequence.

The following general result will be useful:

Theorem 2.2.6. *([21], p.473) Let B be the unitary open ball in $\mathbb{R}^2$. Define a general functional $F_\Omega(u) = \int_\Omega f(x, u)dx$. If F_Ω is compact in the interior of $\mathscr{B}_1(B)$, then for every sequence (u_n) in $\mathscr{B}_1(B)$ com $u_n \rightharpoonup u$ and $|\nabla u_n|^2 \rightharpoonup d\mu$ there is a subsequence such that either (u_n) concentrates at 0 and $u = 0$ or compactness holds in the sense that $f(\cdot, u_n) \to f(\cdot, u)$ in $L^1(B)$).*

To prove the attainability of the supremum, we will use the following Lions-type Concentration-Compactness result:

Theorem 2.2.7. *Let $(u_n) \subset \mathscr{B}_1(\Omega)$. Then there exists a subsequence and $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and either*

(i) (u_n) concentrates at a point $x \in \bar{\Omega}$, or

(ii) $\lim_{n\to\infty} F_\Omega(u_n) = F_\Omega(u)$.

Proof. (Theorem 2.2.7) The proof is an application of Theorem 2.2.6 together with the fact that $F_\Omega^{\alpha,h}$ is compact in the interior of $\mathscr{B}_1(\Omega)$ (c.f. Definition (2.2.5)): indeed, the proof is quite similar to the case $\frac{\alpha}{4\pi} + \frac{\beta}{2} < 1$; let $0 < \eta < 1$ and suppose that $(u_n) \subset H_0^1(\Omega)$ is such that

$$\limsup_{n\to\infty} \|\nabla u_n\|_{L^2(\Omega)} \leq \eta \text{ and } u_n \rightharpoonup u \text{ in } H^1(\Omega)$$

for some $u \in H^1(\Omega)$. Assume, up to a subsequence, that $u_n \to u$ almost everywhere on Ω and $\|\nabla u_n\|_{L^2(\Omega)} \leq \theta = \frac{1+\eta}{2} < 1 \ \forall \ n \ \in \mathbb{N}$. Define $v_n := u_n/\theta \in \mathscr{B}_1(\Omega)$, and $\bar{\alpha} = \alpha\theta^2 < \alpha$, so that we have

$$\frac{\bar{\alpha}}{4\pi} + \frac{\beta}{2} < 1.$$

Let $E \subset \Omega$ be an arbitrary measurable set. Let $r := 4\pi/\bar{\alpha} > 1$ and $\frac{1}{s} = 1 - \frac{1}{r} > \frac{\beta}{2}$ and use Hölder's inequality to obtain

$$\int_E \frac{e^{\alpha u_n^2}}{h(x)} = \int_E \frac{e^{\bar{\alpha} v_n^2}}{h(x)} \leq \left(\int_E e^{4\pi v_n^2}\right)^{1/r} \left(\int_E \frac{1}{h(x)^s}\right)^{1/s}.$$

In view of the Trudinger-Moser inequality and using the fact that $\frac{1}{h(x)^s} \in L^1(\Omega)$ (by (h3), for any sufficiently small $\eta > 0$, we have $\frac{1}{h(x)^s} \in L^\infty(\Omega - B_\eta)$ and by (h2) h is very similar to $(1/M)|x|^{\beta}$, and $\frac{1}{|x|^{\beta s}} \in L^1(\Omega)$). We conclude then that the sequence $\frac{e^{\alpha u_n^2}}{h(x)}$ is equi-integrable, and apply the Vitali Convergence Theorem to conclude that the functional is compact at the interior of Ω. ■

Reducing to Radial Functions, Change of Variables

According to Lemma 10 on [11], if $0 < a < \infty$ and u is a radial function defined on B, the map $T_a : H^1_{0,rad}(B) \to H^1_{0,rad}(B)$; $T_a(u) := \sqrt{a}u(|x|^{\frac{1}{a}})$ is invertible, with $(T_a)^{-1} = T_{1/a}$ and satisfies $\|\nabla T_a(u)\|_2 = \|\nabla u\|_2 \; \forall \; u \in H^1_{0,rad}(B)$. Moreover, if $\frac{\alpha}{4\pi} = a = 1 - \frac{\beta}{2}$, we have

$$F^{\alpha,\beta}_{B_\delta}(u) + \int_{B_\delta} \frac{1}{|x|^\beta} dx = \frac{1}{a} \int_{B_{\delta^a}} (e^{4\pi T_a(u)^2} - 1) dx + \frac{|B_{\delta^a}|}{a}, \tag{2.21}$$

This follows by performing the change of variables $r = s^{1/a}$ and using the fact $\frac{\alpha}{4\pi} + \frac{\beta}{2} = 1$:

$$\begin{aligned}
\int_{B_\delta} \frac{e^{\alpha u^2}}{|x|^\beta} &= 2\pi \int_0^\delta e^{\alpha u^2} r^{1-\beta} dr && (2.22)\\
&= \frac{2\pi}{a} \int_0^{\delta^a} e^{4\pi(\sqrt{a}u(s^{1/a}))^2} s^{\frac{1}{a}-1} s^{\frac{1}{a}-2\left(\frac{1}{a}-1\right)} ds \\
&= \frac{1}{a} \int_{B_{\delta^a}} e^{4\pi(T_a(u))^2} dx. && (2.23)
\end{aligned}$$

Given any $u : B \to \mathbb{R}$, denote by u^* its Schwarz rearrangement (c.f. Definition B.0.2). It is shown in Lemma 14 of [11] that if $(u_n) \subset \mathscr{B}_1(B)$ is a sequence concentrating at zero and (u_n^*) concentrates at 0, then $v_n = T_a(u_n^*)$ is a sequence of symmetric functions concentrating at zero.

Lemma 2.2.8. *Let Ω be any bounded domain with C^1 boundary. Let Ω^* be the open ball centered at the origin with $|\Omega| = |\Omega^*|$. Then for any $u \in H^1_0(\Omega)$*

$$F^{\alpha,\beta}_{\Omega}(u) \leq F^{\alpha,\beta}_{\Omega^*}(u^*).$$

Proof. (Lemma 2.2.8) First we need to prove that $u \in H^1_0(\Omega)$ implies $|u| \in H^1_0(\Omega)$, i.e, $|u| \in H^1(\Omega)$ and the trace of $|u|$ on the boundary is identically null. Indeed, $|u| \in H^1(\Omega)$ (c.f. [17], page 308) , with

$$\nabla|u| = \begin{cases} \nabla u \text{ if } u > 0, \\ 0 \text{ if } u = 0, \\ -\nabla u \text{ if } u < 0. \end{cases} \tag{2.24}$$

Now let $(u_n) \subset C_c^\infty(\Omega)$ be a sequence such that $u_n \to u$ in $H^1(\Omega)$, and $T : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace operator. Notice that $T|u_n| = 0$ for all u_n by item (i) of Theorem

B.0.5. Now write

$$\begin{aligned} \|T|u_n| - T|u|\|_{L^2(\partial\Omega)} &= \|T(|u_n| - |u|)\|_{L^2(\partial\Omega)} \\ &\leq C\|\nabla|u_n| - \nabla|u|\|_{L^2(\Omega)} \end{aligned} \tag{2.25}$$

Notice that $\nabla|u_n| \to \nabla|u|$ almost everywhere in Ω, since for almost all $x \in \Omega$,if $u(x) > 0$, there is n sufficiently big such that $u_n(x) > 0$, and thus $\nabla|u_n|(x) = \nabla u_n(n) \to \nabla|u|(x) = \nabla u(x)$; analogously for the case $u < 0$. For the case $u = 0$, just notice that either $\nabla u_n(x) \to 0$ and $-\nabla u_n(x) \to 0$ and $\nabla u_n = 0$ a.e. on $\{u = 0\}$ (c.f. [17], page 308). Now let E be any open subset in Ω. Then $u \in H^1(\Omega)$, and $u_n \to u$ in $H^1(E)$. Thus

$$\begin{aligned} \int_E |\nabla|u_n| - \nabla|u||^2 dx &\leq \int_E 2^2(|\nabla|u_n||^2 + |\nabla|u||^2)dx \\ &= 2^2 \int_E |\nabla u_n|^2 + |\nabla u|^2 dx \\ &\leq C \int_E |\nabla u|^2 dx, \end{aligned}$$

yielding equiintegrability. Applying Vitali's Convergence Theorem one concludes that $T|u| = 0$. Since the boundary $\partial\Omega$ is assumed to be C^1, Theorem B.0.6 implies $|u| \in H_0^1(\Omega)$.

Let $|u|^*$ the Schwarz rearrangement of $|u|$: it is a positive function and belongs to $H^1_{0,rad}(\Omega)$ according to theorem B.0.4. So without loss of generality one may consider u to be positive and radial. Now regarding the functional $F^{\alpha,\beta}(u) = \int_\Omega \frac{e^{\alpha u^2}-1}{|x|^\beta}$: notice that the function $F(t) : [0,\infty[\to [0,\infty[$, $F(t) := e^{\alpha t^2} - 1$ is increasing and thus may be approximated by simple functions of the form

$$\sum_{i=0}^{k-1} \alpha_i \chi_{]a_i,a_{i+1}]} + \alpha_k \chi_{]a_k,\infty[},$$

where $0 = a_0 < a_1 < ... < a_k < \infty$ and $\alpha_{i+1} > \alpha_i$ for all $i = 0, ..., k$. For instance, on ([20], Th. 2.10) the simple functions used to approximate measurable functions have the form

$$\phi = \sum_{k=0}^{2^{2^n}-1} k2^{-n}\chi_{E_n^k} + 2^n\chi_{F_n},$$

where, for $n = 0, 1, 2, ...$ and $k = 0, 1, ..., 2^{2^n} - 1$, one defines

$$E_n^k = f^{-1}(]k2^{-n}, (k+1)2^{-n}]) \text{ and } F_n = f^{-1}(]2^n, \infty])$$

and for a fixed n, the incrising character of F implies that E_n^k is an interval for all k and the superior extremity of E_n^k is equal to the inferior extremity of E_n^{k+1}.

It is worth noticing that $|\{u^* > t\}|$ is a ball centered at the origin, and $\phi(r) = \frac{1}{r^\beta}$ is decreasing, thus assuming greatest values on a ball around the origin rather than on the complement of the same ball; this is valid for any ball centered at the origin. Having this and (i) in mind, one concludes that for any t,

$$\int_{\Omega^*} \frac{\chi_{]t,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\chi_{]t,\infty[}(u)}{|x|^\beta} dx \geq 0. \tag{2.26}$$

To prove it, write

$$\begin{aligned} E &:= \{u > t\} - \{u^* > t\}, \\ E^* &:= \{u^* > t\} - \{u > t\}, \\ E_* &:= \{u^* > t\} \cap \{u > t\}, \end{aligned}$$

and notice that

$$\sup_{x \in E} \frac{1}{|x|^\beta} \leq \inf_{x \in E^*} \frac{1}{|x|^\beta},$$

since, if $x_0 \in E$ and

$$\frac{1}{|x_0|^\beta} > \inf_{x \in E^*} \frac{1}{|x|^\beta},$$

then $\frac{1}{|x_0|^\beta} > \frac{1}{|x_1|^\beta}$ for some $x_1 \in E^*$, implying that $|x_0| \leq |x_1|$, from which one concludes that $x_0 \in \{u^* > t\}$, since $\{u^* > t\}$ is a ball centered at the origin, leading to a contradiction. So one concludes that (by Theorem B.0.3)

$$\int_E \frac{1}{|x|^\beta} dx \leq \left(\sup_{x \in E} \frac{1}{|x|^\beta} \right) |\{u > t\}| \leq \left(\inf_{x \in E^*} \frac{1}{|x|^\beta} \right) |\{u^* > t\}| \leq \int_{E^*} \frac{1}{|x|^\beta} dx. \tag{2.27}$$

Adding $\int_{E_*} \frac{1}{|x|^\beta} dx$ to both sides of (2.27) one arrives at (2.26) . Now fix $i = k$ and consider the following computations for this case:

$$\begin{aligned} &\int_{\Omega^*} \frac{\alpha_k \chi_{]a_k,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_k \chi_{]a_k,\infty[}(u)}{|x|^\beta} dx + \int_{\Omega^*} \frac{\alpha_{k-1} \chi_{]a_{k-1},a_k]}(u^*)}{|x|^\beta} dx \\ &\quad - \int_{\Omega} \frac{\alpha_{k-1} \chi_{]a_{k-1},a_k]}(u)}{|x|^\beta} dx \\ &= \int_{\Omega^*} \frac{\alpha_k \chi_{]a_k,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_k \chi_{]a_k,\infty[}(u)}{|x|^\beta} dx + \int_{\Omega^*} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u^*)}{|x|^\beta} dx \\ &\quad - \int_{\Omega^*} \frac{\alpha_{k-1} \chi_{]a_k,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u)}{|x|^\beta} dx + \int_{\Omega} \frac{\alpha_{k-1} \chi_{]a_k,\infty[}(u)}{|x|^\beta} dx \\ &= (\alpha_k - \alpha_{k-1}) \left(\int_{\Omega^*} \frac{\chi_{]\alpha_k,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\chi_{]\alpha_k,\infty[}(u)}{|x|^\beta} dx \right) \\ &\quad + \int_{\Omega^*} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u)}{|x|^\beta} dx. \\ &\geq \int_{\Omega^*} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_{k-1} \chi_{]a_{k-1},\infty[}(u)}{|x|^\beta} dx. \end{aligned}$$

by (2.26). Performing the analogous computations for $i = k-1, ..., 0$ in the same fashion, one concludes that

$$\int_{\Omega^*} \frac{\sum_{i=0}^{k-1} \alpha_i \chi_{]a_i,a_{i+1}]}(u^*) + \alpha_k \chi_{]a_k,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\sum_{i=0}^{k-1} \alpha_i \chi_{]a_i,a_{i+1}]}(u) + \alpha_k \chi_{]a_k,\infty[}(u)}{|x|^\beta}$$
$$\geq \int_{\Omega^*} \frac{\alpha_0 \chi_{]a_0,\infty[}(u^*)}{|x|^\beta} dx - \int_{\Omega} \frac{\alpha_0 \chi_{]a_0,\infty[}(u)}{|x|^\beta} dx \geq 0.$$

Since it is valid for simple functions, it is also true to $F(t) = e^{\alpha t^2} - 1$ by passing the limit using the Dominated Convergence Theorem. This finishes the proof of Lemma 2.2.8. ■

Thus

$$\begin{aligned} \limsup_{n\to\infty} F_{B_\delta}^{\alpha,\beta}(u_n) &\leq \limsup_{n\to\infty} F_{B_\delta}^{\alpha,\beta}(u_n^*) \\ &= \frac{1}{a} \limsup_{n\to\infty} \int_{B_{\delta^a}} (e^{4\pi v_n^2} - 1) dx + \frac{|B^{\delta^a}|}{a} - \int_{B_\delta} \frac{1}{|x|^\beta} dx \\ &\leq \frac{1}{a} \sup_{(u_n) \text{ concentrates at } 0} \int_{B_{\delta^a}} (e^{4\pi u_n^2} - 1) dx + \frac{|B_{\delta^a}|}{a} - \int_{B_\delta} \frac{1}{|x|^\beta} dx \end{aligned} \tag{2.28}$$

Now we must note that $(u_n) \subset \mathscr{B}_1(B)$ is a sequence concentrating at zero maximizing $F_{B_\delta}^{\text{conc},\alpha,\beta,B}$, then (u_n^*) also concentrates at zero. Indeed (this proof was taken from [12]), since $F_{B_\delta}^{\text{conc},\alpha,\beta,B} = F_B^{\text{conc},\alpha,\beta,B}$ (same argument used in the proof of Theorem 2.2.3), (u_n) also maximizes $F_B^{\text{conc},\alpha,\beta,B}$; since u_n concentrates, then $u_n \to 0$ in $L^2(B)$; if u_n^* do not concentrate we have by the Concentration-Compactness Theorem 2.2.7 that

$$0 < F_B^{\text{conc},\alpha,\beta,B} = \lim_{n\to\infty} F_B^{\alpha,\beta}(u_n) \leq \lim_{n\to\infty} F_B^{\alpha,\beta}(u_n^*) = 0. \tag{2.29}$$

Test functions and decay condition

Lemma 2.2.9. *Define $a = \frac{\alpha}{4\pi}$. If h satisfies the decay condition "There exists a $n_0 > 0$ sufficiently big such that*

$$\sup_{r \leq e^{-\frac{n_0}{2a}}} \left| \frac{|x|^\beta}{h(x)} - M^{-1} \right| < M^{-1} \frac{1}{n_0}\text{''} \tag{2.30}$$

then

$$(M^{-1} - \varepsilon(\delta)) F_{B_\delta}^{\text{sup},\alpha,\beta B} > M^{-1} F_{B_\delta}^{\text{conc},\alpha,\beta,B}. \tag{2.31}$$

Recall that we defined $\varepsilon(\delta) = \sup_{|x|<\delta} \left| \frac{|x|^\beta}{h(x)} - M^{-1} \right|$.

Proof. (Lemma 2.2.9) Using the change of variables by Moser $r = e^{-t/2}$ and supposing that $u \in H_0^1(B)$ is radially symmetric, write $v(t) = (4\pi)^{1/2} u(e^{-t/2})$ and

$$G_{B_{\delta^a}}(u) := \int_{B_{\delta^a}} (e^{4\pi u^2} - 1)dx = \pi \int_{2a \log 1/\delta}^{\infty} (e^{v^2} - 1)e^{-t}dt. \tag{2.32}$$

If (u_n) is any concentrating sequence at zero (on the x-variable), then $\limsup_{n\to\infty} G_{B_\delta}(u_n) \leq \pi e$ (c.f. Lemma 1.2.6). Consider again the sequence (y_n) defined in (1.13). Recalling Remark (1.2.11), one chooses $n = 2\log \frac{1}{\delta^a}$ yielding

$$G_{B_{\delta^a}}(y_n) \geq \pi e + \frac{\pi e}{n} + o\left(\frac{1}{n}\right), \tag{2.33}$$

Performing the change of variables 2.21, Lemma 2.2.9 is implied by

$$(M^{-1} - \varepsilon(\delta))\left[\pi e + \frac{\pi e}{n} + \frac{|B_{\delta^a}|}{a} - \int_{B_\delta} \frac{1}{|x|^\beta}dx\right] > M^{-1}\left[\pi e + \frac{|B_{\delta^a}|}{a} - \int_{B_\delta} \frac{1}{|x|^\beta}dx\right] \tag{2.34}$$

for some $\delta > 0$ or, simplifying and dropping the lower order terms,

$$(M^{-1} - \varepsilon(\delta))\left(\frac{\pi e}{n}\right) > \varepsilon(\delta)\pi e, \tag{2.35}$$

(notice that $\int_{B_\delta} \frac{1}{|x|^\beta}$ and $|B_{\delta^a}|$ tend to zero as $\delta \to 0$, thus $\varepsilon(\delta)\int_{B_\delta} \frac{1}{|x|^\beta}$ and $\varepsilon(\delta)|B_{\delta^a}|$ are low order terms in comparison to the ones in (2.35))or still, for some δ sufficiently small,

$$\sup_{|x|\leq\delta} \left|\frac{|x|^\beta}{h(x)} - M^{-1}\right| \leq \frac{M^{-1}}{2\log\frac{1}{\delta^a}} \tag{2.36}$$

with δ is sufficiently small enough in order to compensate the negative or low order terms we dropped from the expression (2.34). Or, in terms of n_0:

$$\sup_{|x|\leq e^{-\frac{n_0}{2a}}} \left|\frac{|x|^\beta}{h(x)} - M^{-1}\right| \leq \frac{M^{-1}}{n_0} \tag{2.37}$$

This is implied by the decay condition (2.30) with n_0 sufficiently large. So the choice of n_0 depends on the sequence y_n and on the value of M^{-1}. ■

Proof. (Theorem 2, Attainability, $\frac{\alpha}{4\pi} + \frac{\beta}{2} = 1$) Now we put together Theorems 2.2.2, 2.2.3 and Lemma 2.2.9 to conclude that

$$\begin{aligned} F_\Omega^{\sup,\alpha,h} \geq (M^{-1} - \varepsilon(\delta))|\phi'(0)|^{2-\beta} F_{B_\delta}^{\sup,\alpha,\beta,B} > M^{-1}|\phi'(0)|^{2-\beta} F_{B_\delta}^{\mathrm{conc},\alpha,\beta,B} \\ = F_\Omega^{\mathrm{conc},\alpha,h}, \end{aligned} \tag{2.38}$$

which shows that a concentrating sequence cannot be a maximizing sequence if h satisfies the decay condition (D). We conclude that the supremum is achieved by Theorem 2.2.7. ■

Chapter 3

Proof of Theorem 3

3.1 Differentiability of the functional

We will write

$$F(x,t) = \int_0^t f(x,s)ds.$$

Consider the functional $\Phi : H^1_{0,rad}(B) \to \mathbb{R}$

$$\Phi(u) := \frac{1}{2}\int_B |\nabla u|^2 dx - \int_B F(x,u(x))dx. \tag{3.1}$$

Proposition 3.1.1. *Φ is a C^1 functional, with*

$$\langle \Phi'(u), v\rangle = \int_B \nabla u \nabla v - \int_B f(x,u)v \;\forall\; v \in H^1_{0,rad}(B). \tag{3.2}$$

Frechét-differentiability follows from the following lemma together with proposition C.0.1.

Lemma 3.1.2. *Φ is Gateâux-differentiable in $H^1_{0,rad}(B)$ and its Gatêaux-derivative is continuous at $H^1_{0,rad}(B)$.*

Proof. For $t > 0$ we have

$$\frac{\Phi(u+tv)-\Phi(u)}{t} = \frac{1}{t}\left(t\int_B \nabla u\nabla v + \frac{t^2}{2}\int_B |\nabla v|^2 + K\int_B F(x,u+tv) - F(x,u)\right). \tag{3.3}$$

Since F is differentiable, it is possible to write for all t and for almost all $x \in B$:

$$F(x,u+tv) = F(x,u) + tvf(x,u) + r(t), \tag{3.4}$$

where r is a function such that $r(t)/t \to 0$ as $h \to 0$ (thus it can be ignored since it does not depende on $x \in B$). By (H5), we have for all $x \in B$

$$\left|\frac{F(x,u+tv)-F(x,u)}{t}\right| \approx |vf(x,u)| \leq K|v||u|^k e^{2\pi(2+\alpha)u^2}|x|^\alpha,$$

which is integrable due to the Trudinger-Moser embedding ($e^{w^2} \in L^1 \ \forall\ w \in H_0^1$) and the Sobolev embedding $H_0^1(B) \subset L^q(B)\ \forall\ q \in [2,\infty)$. Then we can apply Lebesgue's Dominated Convergence Theorem and conclude that the Gatêaux-drivative exists. For the continuity of the derivative, it suffices to show it is bounded.

Notice that given any $A \subset B$ measurable and $h \in H^1_{0,rad}(B)$, for any $v \in H^1_{0,rad}(B)$ with $\|v\| \leq 1$, we have

$$\begin{aligned}
|\langle \Phi'(u+h), v\rangle_A| &:= \left|\int_A \nabla(u+h)\nabla v dx\right| + \left|\int_A f(x,u+h)v\right| \\
&\leq \|\nabla u\|_{L^2(A)} + \|\nabla h\|_{L^2(A)} + K\int_A |u+h|^k |e^{2\pi(2+\alpha)(u+h)^2}||x|^\alpha |v| dx \\
&\leq \|\nabla u\|_{L^2(A)} + \|\nabla h\|_{L^2(A)} + K\int_A |u+h|^k |e^{8\pi(2+\alpha)(u^2+h^2)}||x|^\alpha |v| dx \\
&\leq \|\nabla u\|_{L^2(A)} + \|\nabla h\|_{L^2(A)} + \qquad (3.5)\\
&\quad K\left(\int_A |u+h|^{8k}\right)^{1/8}\left(\int_A e^{64\pi(2+\alpha)u^2}\right)^{1/8}\left(\int_A e^{32\pi(2+\alpha)h^2}|x|^{4\alpha}\right)^{1/4},
\end{aligned}$$

If $\|h\| \leq \left(\frac{2\pi(2+4\alpha)}{32\pi(2+\alpha)}\right)^{1/2}$ we have

$$\int_A e^{32\pi(2+\alpha)h^2}|x|^{4\alpha} \leq C < \infty,$$

where C is *independent* of h due to theorem 1.1.1. By the Trudinger-Moser embedding, $e^{64\pi(2+\alpha)u^2} \in L^1$, therefore given $\varepsilon > 0$, there exists $\delta > 0$ such that $|A| \leq \delta$ implies $\int_A e^{32\pi(2+\alpha)u^2} \leq \varepsilon$. From these observations and using again the Sobolev Embedding $H^1_{0,rad}(B) \subset L^q(B)\ \forall q \in [2,\infty)$ we conclude that, fixed $u \in H^1_{0,rad}(B)$, the family

$$\mathscr{F} = \left\{ |f(x,u+h)v| : h \in H^1_{0,rad}(B),\ \|h\| \leq \left(\frac{2\pi(2+4\alpha)}{32\pi(2+\alpha)}\right)^{1/2},\ \|v\| \leq 1 \right\}$$

is uniformly integrable in B, allowing us to apply Vitali's Convergence Theorem. Continuity of the derivative follows from taking the limit $h \to 0$. ■

3.2 Palais-Smale condition

Proposition 3.2.1. *Let $(u_n) \subset H^1_{0,rad}(B)$ be a Palais-Smale sequence such that $\Phi(u_n) \to c$. Then if $c < 1/2$, (u_n) admits a strongly convergent subsequence, i.e., Φ satisfies $(PS)_c$ for all $c < 1/2$.*

Proof. (u_n) is a $(\mathrm{PS})_c$ sequence if and only if

$$\begin{aligned} \frac{1}{2}\int_B |\nabla u_n|^2\,dx - \int_B F(x,u_n)\,dx &\to c, & (3.6)\\ \left|\int_B \nabla u_n \nabla v\,dx - \int_B f(x,u_n)v\,dx\right| &\leq \varepsilon_n \|\nabla v\|_{L^2(B)}\ \forall\, v\in H^1_{0,rad}(B),\ \varepsilon_n\to 0. & (3.7)\end{aligned}$$

Now notice that, by (H2) and (H3) given any $\varepsilon>0$ there exists t_ε such that

$$F(x,t)\leq M|f(x,t)|\leq \varepsilon t f(x,t)\forall x\in B,\ \forall |t|\geq t_\varepsilon;$$

just consider t_ε such that $t_\varepsilon>\max\{\frac{M}{\varepsilon},t_0\}$. It follows that, for any $\varepsilon>0$, by condition (H5):

$$\begin{aligned} \frac{1}{2}\|\nabla u_n\|^2_{L^2} &\leq K' + \int_B F(x,u_n)\,dx \leq K' + \int_{\{|u_n|\leq t_\varepsilon\}} F(x,u_n)\,dx\\ &\quad + \varepsilon\int_{\{|u_n|\geq t_\varepsilon\}} f(x,u_n)u_n\,dx\\ &\leq K' + K\frac{\pi}{2}|t_\varepsilon|^{k+1}e^{2\pi(2+\alpha)t_\varepsilon^2} + \varepsilon\int_{\{|u_n|\geq t_\varepsilon\}} f(x,u_n)u_n\,dx\\ &= K_\varepsilon + \varepsilon\int_{\{|u_n|\geq t_\varepsilon\}} f(x,u_n)u_n\,dx, \qquad (3.8)\end{aligned}$$

where $K'>0$ is a constant coming from (3.6), K comes from (H5) and $K_\varepsilon = K' + K\frac{\pi}{2}|t_\varepsilon|^{k+1}e^{2\pi(2+\alpha)t_\varepsilon^2}$. From (3.7) and (3.8) we have

$$\frac{1}{2}\|\nabla u_n\|^2_{L^2(B)} \leq K_\varepsilon + \varepsilon\|\nabla u_n\|^2_{L^2(B)} + \varepsilon_n\|\nabla u_n\|_{L^2(B)},$$

which implies ($C>0$ a constant)

$$\|\nabla u_n\|_{L^2(B)}\leq C,\ \int_B f(x,u_n)u_n\,dx\leq C \text{ by (3.7)},\ \int_B F(x,u_n)\,dx\leq C \text{ by (3.6)}. \qquad (3.9)$$

Now take a subsequence (u_n) such that there exists $u\in H^1_{0,rad}(B)$ such that

$$u_n\rightharpoonup u\in H^1_{0,rad}(B); u_n\to u\in L^q(\Omega)\ \forall q\in[1,\infty); u_n(x)\to u(x) \text{ a.e..} \qquad (3.10)$$

The first convergence follows from Theorem 3.18 in ([6], p.69) and the fact that H^1_0 is reflexive and thus $H^1_{0,rad}$, being a closed linear subspace of H^1_0, is too. The second follows form Rellich-Kondrachov Theorem. Then we have by Lemma C.0.2 and (3.9) $f(x,u_n)\to f(x,u)$ in $L^1(B)$ (note that $f(x,u)\in L^1(B)$ due to the Trudinger-Moser inequality). Therefore it follows from this convergence, condition (H2) and the Generalized Dominated Convergence Theorem (c.f. [19], p.89) that $F(x,u_n)\to F(x,u)$ in $L^1(B)$.

From (3.6) and (3.7) it follows that

$$\lim_{n\to\infty} \|\nabla u_n\|^2_{L^2(B)} = 2(c + \int_B F(x,u)\ dx), \tag{3.11}$$

$$\lim_{n\to\infty} \int_B f(x,u_n)u_n = 2(c + \int_B F(x,u)\ dx). \tag{3.12}$$

Using (3.12) and (H3) we conclude that $c \geq 0$, because (applying Fatou's Lemma)

$$\lim_{n\to\infty} \int_B f(x,u_n)u_n\ dx \leq 2c + \int_B f(x,u)u\ dx \leq 2c + \liminf_{n\to\infty} \int_B f(x,u_n)u_n\ dx.$$

Given any $\psi \in C_c^\infty(B)$, by (3.7) we have

$$\begin{aligned} &\left|\int_B \nabla u_n \nabla\psi\ dx - \int_B f(x,u_n)\psi\ dx\right| \leq \varepsilon_n \|\nabla\psi\|_{L^2(B)} \\ &\Rightarrow \int_B \nabla u \nabla \psi\ dx = \int_B f(x,u)\psi\ dx\ \forall \psi \in C_c^\infty(B). \end{aligned} \tag{3.13}$$

Since $f(x,u) \in L^2(B)$ (again by Trudinger-Moser inequality, Sobolev Embedding and (H5)) then $\forall\ \psi \in C_c^\infty(B)$:

$$\left|\int_B f(x,u)\psi\ dx - \int_B f(x,u)u\ dx\right| \leq \left(\int_B |f(x,u)|^2 dx\right)^{1/2} \|\psi - u\|_{L^2(B)}, \tag{3.14}$$

$$\left|\int_B \nabla u \nabla\psi\ dx - \int_B |\nabla u|^2\ dx\right| \leq \|\nabla u\|_{L^2} \|\psi - u\|_{L^2(B)}. \tag{3.15}$$

Taking a sequence $(\psi_n) \subset C_c^\infty(B)$ such that $\psi_n \to u$ in $H_0^1(B)$, by (3.14), (3.15) and (H3) one has

$$\|\nabla u\|^2_{L^2(B)} = \int_B f(x,u)u\ dx \geq 2\int_B F(x,u)\ dx \Rightarrow \Phi(u) \geq 0.$$

Now we treat 3 different cases:

Case 1: $c = 0$

Then we would have

$$0 \leq \Phi(u) = \frac{1}{2}\int_B |\nabla u|^2 - \int_B F(x,u)\ dx \leq \frac{1}{2}\liminf \int_B |\nabla u_n|^2\ dx - \int_B F(x,u)\ dx = 0$$

(by (3.11)) and thus $\lim \|\nabla u_n\|_{L^2(B)} = \|\nabla u\|_{L^2(B)} \Rightarrow u_n \to u$ in $H_0^1(B)$.

Case 2: $c \neq 0,\ u = 0$

We claim that for some $q > 1$ we have $\int_B |f(x,u_n)|^q\ dx \leq C$ (C independent of n): indeed, since $u = 0$, given $\varepsilon > 0$ $\exists\ n_0$ such that $n \geq n_0$ implies $\|\nabla u_n\|^2_{L^2(B)} \leq 2c + \varepsilon$, which implies

$$\begin{aligned}\int_B |f(x,u_n)|^q\ dx &\le K\int_B |u_n|^{kq} e^{2\pi(2+\alpha)qu_n^2}|x|^{\alpha q}\ dx\\ &\le K\left(\int_B |u_n|^{kql'}\ dx\right)^{\frac{1}{l'}}\left(\int_B e^{2\pi(2+\alpha)qlu_n^2}|x|^{\alpha ql}\ dx\right)^{1/l}\quad (\forall\ l>1).\\ &\le K\left(\int_B e^{2\pi(2+\alpha)ql\|\nabla u_n\|^2_{L^2(B)}\left(\frac{u}{\|\nabla u_n\|_{L^2(B)}}\right)^2}|x|^{\alpha}\ dx\right)^{1/l}\end{aligned} \tag{3.16}$$

Note that K is independent of n, and the last step of (3.16) is true because (u_n) is a convergent sequence in L^r for all $r\in[1,\infty)$. The integral in then last line of (3.16) is bounded independently of n due to theorem 1.1.1 if $ql\|\nabla u_n\|^2_{L^2(B)}<1$, for which is sufficient that $ql(2c+\varepsilon)<1$. Since $c<1/2$, this will happen if we choose l,q sufficiently close to 1 and ε sufficiently close to 0. This proves the claim.

Now from (3.7) and (3.9) we have $\left|\int_B |\nabla u|^2\ dx - \int f(x,u_n)u_n\ dx\right| \le \varepsilon_n\|\nabla u_n\|_{L^2(B)} \le C\varepsilon_n$. We estimate the integral $\int_B f(x,u_n)u_n\ dx$ using Hölder and the claim above; since $u_n\to 0$ in $L^{q'}$ we conclude that $\|\nabla u_n\|_{L^2(B)}\to 0$. This contradicts (3.11), which implies $\|\nabla u_n\|^2_{L^2(B)}\to 2c\neq 0$.

Case 3: $c\neq 0$**,** $u\neq 0$

We claim that $\Phi(u)=c$. $\Phi(u)$ cannot be $>c$ because of Fatou's Lemma. Suppose by contradiction that $\Phi(u)<c$. Then

$$\|\nabla u\|^2_{L^2(B)} < 2(c+\int_B F(x,u)\ dx). \tag{3.17}$$

Let $v_n := \frac{u_n}{\|\nabla u_n\|_{L^2(B)}}$ and $v := \frac{u}{(2(c+\int_B F(x,u)\ dx))^{1/2}}$. Now we use and adaptation of a classical result by Lions [23]:

Lemma 3.2.2. *Let* $(v_n)\subset H^1_{0,rad}(B)$*,* $\|\nabla v_n\|_{L^2(B)}=1\ \forall\ n$*, and* $\|\nabla v\|_{L^2(B)}<1$*. Then*

$$\sup_n \int_B e^{2\pi(2+\alpha)pv_n^2}|x|^\alpha < \infty\ \forall p < \frac{1}{1-\|\nabla v\|^2_{L^2(B)}}. \tag{3.18}$$

Proof. The result by Lions is, under the same conditions,

$$\sup_n \int_B e^{4\pi p v_n^2} < \infty\ \forall\ p < \frac{1}{1-\|\nabla v\|^2_{L^2(B)}}.$$

Given any $z\in H^1_{0,rad}(B)$, we again (just like in the proof of finiteness of the supremum in Theorem 1.1) use the change of variables introduced by Moser [24] $z(r)=w(t)/(4\pi)^{1/2}$, $r=e^{-t/2}$ and, after, the one used in [15] $v(s)=\left(\frac{2+\alpha}{2}\right)^{1/2} w(\frac{2s}{2+\alpha})$, $t=\frac{2+\alpha}{2}s$, together with the fact that $z \mapsto w \mapsto v$ is an isometry from $H^1_{0,rad}(B)$ to

$$H := \{w : [0,\infty) \to \mathbb{R} : w \text{ is measurable, has weak derivative, } w(0) = 0, \int_0^\infty |w'|^2 < \infty\}$$

to compute

$$\begin{aligned}
\sup_{z\in H^1_{0,rad}(B),\|z\|\le 1} \int_B e^{2\pi(2+\alpha)pz(x)^2}|x|^\alpha dx &= \sup_{z\in H^1_{0,rad}(B)\|z\|\le 1} 2\pi \int_0^1 e^{2\pi(2+\alpha)pz(r)^2} r^{\alpha+1} dr \\
&= \sup_{w\in H,\|w\|\le 1} \pi \int_0^\infty e^{\frac{2+\alpha}{2}pw(t)^2} e^{-\frac{\alpha+2}{2}t} dt \\
&= \sup_{v\in H,\|v\|\le 1} \frac{2\pi}{2+\alpha} \int_0^\infty e^{pv(s)^2-s} ds.
\end{aligned}$$

Analogously

$$\sup_{z\in H^1_{0,rad}(B),\|z\|\le 1} \int_B e^{4\pi pz(x)^2} dx = \sup_{v\in H,\|v\|\le 1} \pi \int_0^\infty e^{pv(s)^2-s} ds,$$

and comparing the two equalities the result is proved. ■

Now we proceed just as in case 2 to obtain

$$\int_B |f(x,u_n)|^q \, dx \le K \left(\int_B e^{2\pi(2+\alpha)lq\|\nabla u_n\|^2_{L^2(B)} v_n^2} |x|^\alpha \, dx \right)^{1/l} \tag{3.19}$$

for any $q, l > 1$ (where K depends only on q, l). The integral on the right-hand side of (3.19) is bounded (by Lemma 3.2.2) provided that

$$lq\|\nabla u_n\|^2_{L^2(B)} \le p < \frac{1}{1 - \|\nabla v\|^2_{L^2(B)}} = \frac{1}{1 - \frac{\|\nabla u\|^2_{L^2(B)}}{2(c+\int_B F(x,u)\, dx)}} = \frac{c + \int_B f(x,u)\, dx}{c - \Phi(u)}.$$

Given $\varepsilon > 0$, for n sufficiently big we have $\|\nabla u_n\|^2_{L^2(B)} \le 2(c + \int_B F(x,u)\, dx) + \varepsilon$. Since $\Phi(u) \ge 0$, $\frac{1}{c} \le \frac{1}{c-\Phi(u)}$, so if $lq2(c + \int_B F(x,u)\, dx) + \varepsilon < \frac{c+\int_B F(x,u)\, dx}{c}$ the result follows. Since $c < 1/2$, just take l, q sufficiently close to 1 and ε sufficiently close to zero. ■

3.3 Mountain Pass Theorem set-up

Proposition 3.3.1. *Let Z be a finite dimentional subspace of $H^1_{0,rad}(B)$ spanned by L^∞ functions. Then Φ is bounded from above in Z, and moreover, given $M > 0$ there exists an $R > 0$ such that*

$$\Phi(u) \le -M \ \forall\ u \in Z;\ \|\nabla u\|_{L^2(B)} \ge R.$$

Proof. Notice that by (H2) we have the existence of $t_0 > 0$ and $K > 0$ such that $F(x,t) \geq Ke^{\frac{1}{M}|t|}$ for all $|t| \geq t_0$. We conclude that $F(x,t) \geq (Kt^p - K)\ \forall\ x, t$, where K is a positive constant and $p > 2$ by choice. Let $u_0 \in Z$ with $\|u_0\|_{L^\infty(B)} = 1$; then

$$\begin{aligned}
\Phi(tu_0) &= \frac{t^2}{2}\int_B |\nabla u_0|^2 \ dx - \int_B F(x,u_0)\ dx \\
&\leq \frac{t^2}{2}\|\nabla u_0\|^2_{L^2(B)} - K|t|^p \int_B |u_0|^p\ dx + K \qquad (3.20)\\
&\leq \frac{t^2}{2}\|\nabla u_0\|^2_{L^2(B)} - K|t|^p\|\nabla u_0\|^p_{L^2(B)} + K \to -\infty \text{ as } t \to \infty, \qquad (3.21)
\end{aligned}$$

where (3.21) was achieved from (3.20) using the equivalence of norms in a finite dimensional vector space. ■

Proposition 3.3.2. *There exists $a > 0$ and $\rho > 0$ such that*

$$\Phi(u) \geq a \text{ if } \|\nabla u\|_{L^2(B)} = \rho.$$

Proof. From (H4) there exists $\mu < \mu_1 = \inf\{\frac{\int_B |\nabla u|^2}{\int_B |u^2|} :\ u \in H^1_{0,rad}(B)\}$ and $\delta > 0$ such that $F(x,t) \leq \frac{1}{2}\mu t^2$ if $|t| \leq \delta$. By (H3) and (H5) we have for any $q > \max\{2, k+1\}$ (k comes from the condition (H5)):

$$F(x,t) \leq K|t|^q e^{2\pi(2+\alpha)t^2}|x|^\alpha\ \forall\ |t| \geq 1 \Rightarrow F(x,t) \leq \frac{K}{\delta^q}|t|^q e^{2\pi(2+\alpha)\frac{t^2}{\delta^2}}|x|^\alpha\ \forall\ |t| \geq \delta.$$

Thus

$$F(x,t) \leq \frac{1}{2}\mu t^2 + \frac{K}{\delta^q}|t|^q e^{2\pi(2+\alpha)\frac{t^2}{\delta^2}}|x|^\alpha\ \forall\ x, t. \qquad (3.22)$$

Then for any $p > 1$

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{2}\int_B |\nabla u|^2 \ dx - \frac{1}{2}\mu\int_B u^2\ dx - \frac{K}{\delta^q}\int_B |u|^q e^{2\pi(2+\alpha)\frac{u^2}{\delta^2}}|x|^\alpha\ dx \\
&\geq \frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right)\int_B |\nabla u|^2\ dx - \frac{K}{\delta^q}\left(\int_B e^{2\pi p(2+\alpha)\frac{u^2}{\delta^2}}|x|^{\alpha p}\ dx\right)^{1/p}\left(\int_B |u|^{qp'}\ dx\right)^{1/p'}
\end{aligned}$$

Now if $\|\nabla u\|^2_{L^2(B)} \leq \delta^2/p$ we have

$$\sup_u \int_B e^{2\pi p(2+\alpha)\frac{u^2}{\delta^2}}|x|^{\alpha p}\ dx \leq \sup_u \int_B e^{2\pi p(2+\alpha)\frac{u^2}{\delta^2}}|x|^\alpha\ dx < \infty,$$

(since $|x| \leq 1$ we have $|x|^{\alpha p} \leq |x|^\alpha$ for $p > 1$) and by the Sobolev embedding $H^1_0 \subset L^{qp'}$

$$\left(\int_B |u|^{qp'}\ dx\right)^{1/p'} \leq K\|\nabla u\|^q_{L^2(B)}.$$

So we have for $\|\nabla u\|^2_{L^2(B)} \leq 1/p$

$$\Phi(u) \geq \frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right)\|\nabla u\|^2_{L^2(B)} - \frac{K}{\delta^q}\|\nabla u\|^q_{L^2(B)}. \tag{3.23}$$

Now consider the function $\varphi(s) = \frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right)s^2 - \frac{K}{\delta^q}s^q$. Writing

$$\varphi(s) = s^2\left(\frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right) - \frac{K}{\delta^q}s^{q-2}\right)$$

one can see that there exists a number $\eta > 0$ sufficiently small such that $s \leq \eta$ implies $\phi(s) > 0$. So we take $\rho = \min\{\eta, \delta^2/p\}$ and $a = \varphi(\rho)$. ∎

3.4 Mountain Pass Theorem set-up - test functions

We will apply the Mountain Pass Theorem C.0.3. According to proposition 3.3.1, given any $v \in H^1_{0,rad}(B) \cap L^\infty(B)$, there exists $t_0 > 0$ such that $\Phi(t_0 v) \leq 0$. Recall that $\Phi(0) = 0$ by (H3). By proposition 3.3.2, there exists $a > 0$ and a ball B_ρ such that $\Phi|_{\partial B_\rho} = a$. Suppose there exists a $w_0 \subset H^1_{0,rad}(B)$ with $\|\nabla w_0\|_{L^2} = 1$ such that $\max\{\Phi(tw_0) : t \geq 0\} < \frac{1}{2}$. Set then $U = B_\rho$, $v = t_0 w_0$ (w_0 from 3.24). Let $\mathscr{P}$ denote the class of all paths joining 0 to v. By definition, if such a w_0 exists, we conclude that $c = \inf_{P\in\mathscr{P}} \max_{w\in P} \Phi(w) < 1/2$, which yields, together with theorem C.0.3 and proposition 3.2.1, the existence of the minimizer $u \in H^1_{0,rad}(B)$. Therefore we need to show that there exists an $w_0 \in H^1_0(B)$ with $\|\nabla w_0\|_{L^2} = 1$ such that

$$\max\{\Phi(tw_0) : \ t \geq 0\} < \frac{1}{2}, \tag{3.24}$$

Consider the same sequence used on the proofs of Theorems 1 and 2:

$$y_n(t) = \begin{cases} \frac{t}{n^{1/2}}(1-\delta_n)^{1/2}, \ 0 \leq t \leq n, \\ \frac{1}{(n(1-\delta_n))^{1/2}} \log \frac{A_n+1}{A_n + e^{-(t-n)}} + (n(1-\delta_n))^{1/2}, n \leq t, \end{cases} \tag{3.25}$$

or, in terms of the radio

$$w_n(r) = \left(\frac{1}{2\pi(2+\beta)}\right)^{1/2} \begin{cases} \frac{1}{(n(1-\frac{2\log n}{n}))^{1/2}} \log \frac{A_n+1}{A_n + e^n r^{2+\beta}} + (n(1-\frac{2\log n}{n}))^{1/2}, 0 \leq r \leq e^{\frac{-n}{2+\beta}}, \\ \frac{(2+\beta)\log\frac{1}{r}}{n^{1/2}}(1-\frac{2\log n}{n})^{1/2}, e^{-n/(2+\beta)} \leq r \leq 1, \end{cases} \tag{3.26}$$

after the change of variables

$$y_n(t) = \left(\frac{2}{2+\beta}\right)^{1/2} w_n(r), r = e^{\frac{-t}{2+\beta}}. \tag{3.27}$$

Notice that $(w_n) \subset L^\infty(B)$, allowing us to apply proposition 3.3.1.

Proof. (Theorem 3) Suppose that for all $n \in \mathbb{N}$ we have $\max\{\Phi(tw_n) : t \geq 0\} \geq 1/2$. By Propositions 3.3.1 and 3.3.2, for each $n \in \mathbb{N}$ there exists $t_n \geq 0$ such that

$$\max\{\Phi(tw_n) : t \geq 0\} = \Phi(t_n w_n) \geq \frac{1}{2}. \tag{3.28}$$

Recalling that $\|\nabla w_n\|_{L^2(B)} = 1 \ \forall\ n \in \mathbb{N}$, by (3.28) and (H3) we have

$$t_n^2 \geq 1. \tag{3.29}$$

It follows from (3.28) that $\frac{d}{dt}\Phi(tw_n) = 0$ at $t = t_n$ or

$$t_n - \int_B f(x, t_n w_n) w_n dx = 0. \tag{3.30}$$

Multiplying (3.30) by t_n we have

$$t_n^2 = t_n \int_B w_n f(x, t_n w_n) dx. \tag{3.31}$$

By (H6) there exists some $t_\varepsilon > 0$ and $\delta(\varepsilon)$ such that $t > t_\varepsilon$ and $x \in B_\delta$ implies $f(x,t)t \geq (\zeta - \varepsilon)e^{2\pi(2+\beta)t^2}|x|^\beta$ for some $\varepsilon > 0$ such that $\zeta - \varepsilon > 0$. From (3.26), we see that if $r \leq e^{-n/(2+\beta)}$, $w_n(r) \geq (n(1 - \frac{2\log n}{n}))^{1/2}$, so given any number $L \in \mathbb{N}$ there exists $n_L \in \mathbb{N}$ sufficiently big such that $n \geq n_1$ implies that $w_n(r) \geq L$ if $n \geq n_L$. There exists n_0 sufficiently large such that $n \geq n_0$ implies $B_{e^{-\frac{n}{2+\beta}}} \subset B_\delta$. Then applying (H6) and proceeding just like in the proof of Lemma 1.2.10 to obtain

$$\begin{aligned} t_n^2 &\geq (\zeta - \varepsilon)\int_{B_{e^{-n/(2+\beta)}}} e^{2\pi(2+\beta)t_n^2 w_n^2}|x|^\beta dx \\ &\geq \frac{2\pi(\zeta-\varepsilon)}{2+\beta}\int_n^\infty e^{t_n^2 y_n^2 - t} dt \\ &\geq \frac{2\pi(\zeta-\varepsilon)}{2+\beta}\frac{1}{n^2}\int_0^\infty e^{\left[(t_n^2-1)n + 2\log\frac{A_n+1}{A_n+e^{-s}} - s\right]}\left[1 + \frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}\right] ds, \end{aligned} \tag{3.32}$$

and since, by (1.20) and (1.21),

$$\frac{1}{n^2}\int_0^\infty \exp\left[2\log\frac{A_n+1}{A_n+e^{-s}} - s\right]\left[1 + \frac{1}{n(1-\delta_n)}\log^2\frac{A_n+1}{A_n+e^{-s}}\right] ds \to e,$$

we conclude that t_n is a bounded sequence, otherwise the right side would blow-up faster than the left side. More: from (3.29) we get that t_n^2 converges to 1 (particularly with a decay equal or faster than $1/n$ for the blow-up not to happen). Now recall that the sequence $(w_n) \subset H^1_{0,rad}(B)$ is a concentrating sequence, thus converging a.e. to zero. This implies that there exists $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $t_n w_n(r) < t_\varepsilon$ for all $x \in B$ such that $|r| = \delta$. Since all the functions w_n are radially decreasing, one concludes that if $n \geq n_1$ then $t_n w_n(r) < t_\varepsilon$ for all $x \in B$ such that $|r| \geq \delta$. Thus if n is greater than n_1 one has $\{x \in B : t_n w_n(x) \geq t_\varepsilon\} \subset B_\delta$. So by (H6) and (3.31) we have for $n \geq n_1$:

$$\begin{aligned} t_n^2 &= \int_{\{t_n w_n \geq t_\varepsilon\}} f(x, t_n w_n) t_n w_n + \int_{\{t_n w_n \leq t_\varepsilon\}} f(x, t_n w_n) t_n w_n \\ &\geq (\zeta - \varepsilon) \int_{\{t_n w_n \geq t_\varepsilon\}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx + \int_{\{t_n w_n \leq t_\varepsilon\}} f(x, t_n w_n) t_n w_n \\ &= (\zeta - \varepsilon) \int_B e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx - (\zeta - \varepsilon) \int_{\{t_n w_n \leq t_\varepsilon\}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx \\ &\quad + \int_{\{t_n w_n \leq t_\varepsilon\}} f(x, t_n w_n) t_n w_n \end{aligned} \tag{3.33}$$

Performing the changes of variables (3.27) :

$$\begin{aligned} \int_B e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx &= \int_{B - B_{e^{-n^{1/3}/(2+\beta)}}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx \\ &\quad + \int_{B_{e^{-n^{1/3}/(2+\beta)}} - B_{e^{-n/(2+\beta)}}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx \\ &\quad + \int_{B_{e^{-n/(2+\beta)}}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx \\ &= \int_{B - B_{e^{-n^{1/3}/(2+\beta)}}} e^{2\pi(2+\beta) t_n^2 w_n^2} |x|^\beta dx \ \text{(I)} \\ &\quad + \frac{2\pi}{2+\beta} \int_{n^{1/3}}^n \exp\left[\frac{t^2}{n}\left(1 - \frac{2\log n}{n}\right) - t\right] dt \ \text{(II)} \\ &\quad + \frac{2\pi(M-\eta)}{2+\beta} \int_n^\infty e^{\left[\left(\frac{\log \frac{A_n+1}{A_n + e^{-(t-n)}}}{(n^{1/2}(1-\delta_n))^{1/2}} + (n^{1/2}(1-\delta_n))^{1/2}\right)^2 - t\right]} dt \ \text{(III)} \\ &\geq \int_B |x|^\beta dx + \frac{2\pi}{2+\beta} e + O(1/n) \end{aligned} \tag{3.34}$$

Integral (I) in (3.34) converges to $\int_B |x|^\beta dx$ by the Dominated Convergence Theorem. Integral (II) in (3.34) converges to zero:

$$\begin{aligned} \frac{2\pi}{2+\beta} \int_{n^{1/3}}^n e^{\left[\frac{t^2}{n}\left(1 - \frac{2\log n}{n}\right) - t\right]} dt &= \frac{2\pi}{2+\beta} n \int_{\frac{1}{n^{2/3}}}^1 e^{\left[n\left(s^2\left(1 - \frac{2\log n}{n}\right) - s\right)\right]} ds \\ &\leq \frac{2\pi}{2+\beta} n \int_{\frac{1}{n^{2/3}}}^1 \exp(-2\log n) ds \\ &\leq \frac{2\pi}{2+\beta} \frac{1}{n}, \end{aligned}$$

since for n sufficiently large one has

$$n\left(s^2\left(1-\frac{2\log n}{n}\right)-s\right)\leq -2\log n \text{ in } \left[\frac{1}{n^{2/3}},1\right],$$

because

$$\begin{aligned} n\left(s^2\left(1-\frac{2\log n}{n}\right)-s\right) &= -2\log n \text{ at } s=1,\\ &= \frac{1}{n^{1/3}}\left(1-\frac{2\log n}{n}\right)-n^{1/3} \text{ at } s=\frac{1}{n^{2/3}}. \end{aligned} \tag{3.35}$$

Integral (III) on (3.34) goes to $e+O(1/n)$ by (1.2.10). For the second and third integrals on (3.33) we apply the Dominated Convergence Theorem, and those integrals converge to $\int_B |x|^\beta dx$ and 0 respectively (recall that $w_n \to 0$ almost everywhere on B and f is continuous, thus bounded on $\bar{B}\times[0,t_\varepsilon]$). So taking the limit we conclude that

$$1\geq \frac{2\pi}{2+\beta}e(\zeta-\epsilon)\forall\ \varepsilon>0 \Rightarrow \zeta \leq \frac{1}{\frac{2\pi e}{2+\beta}}, \tag{3.36}$$

a contradiction to (H6). This finishes the proof of Theorem 3. ■

Remark 3.4.1. *Taking the limit $n\to\infty$ on (3.32) one could directly obtain (3.36). In fact, further calculations were made just to show that outside $B_{e^{-\frac{n}{2+\beta}}}$ the integral converges to zero.*

Regarding the regularity of the solution u: Let $w\in H_0^1(B)$ be a weak solution (in $H_0^1(B)$) of

$$\begin{cases} -\Delta w = f(x,u) \text{ in } B,\\ w=0 \ \text{ in } \partial B. \end{cases}$$

then by Elliptic Regularity (c.f. [4]) we have $w\in H^2(B)\cap H_0^1(B)$. Since f and u are radial in x, so is w. Now observe that for any function $\varphi\in C_{0,rad}^\infty(\bar{(B)})$ (C^∞, radial and vanishing on the boundary) we have

$$\int_B w(-\Delta\varphi)dx = \int_B(-\Delta w)\varphi dx \int_B f(x,u)\varphi dx = \int_B \nabla u\nabla\varphi dx \int_B u(-\Delta\varphi)dx.$$

Given any $\psi\in C_{c,rad}^\infty(B)$ let $\varphi\in C_{0,rad}^\infty(\bar{B})$ be the solution of

$$-\Delta\varphi=\psi \ \text{ in } B,\ u=0 \ \text{ in } \partial B.$$

So for any function $\psi\in C_{c,rad}^\infty(B)$ one has

$$\int_B (w-u)\psi dx = 0\ \forall\ \psi\in C_{0,rad}^\infty(B).$$

Thus

$$\int_0^1 (w(r) - u(r))\psi(r) r dr = 0 \;\forall\; \psi \in C_c^\infty(]0,1[).$$

Therefore $u = w$ almost everywhere and regularity is proven.

Chapter 4

Conclusion

On [15], de Figueiredo, do Ó and Santos stated the finiteness of the supremum (5), as well as the attainability for the case where $M = 0$ in (h2). the first theorem states the attainability for the so-called critical case $0 < M < \infty$. We considered the sequence (y_n) (1.13) originally presented on [18] and here we establish an improvement on the estimatives regarding it, namely:

$$\int_n^\infty e^{y_n^2 - t} dt \geq e + \frac{e}{n} + o(\frac{1}{n}). \tag{4.1}$$

Decay conditions on both theorems 1.1 and 1.2 depend explicitely on (4.1), being also a consequence of the Concentration-Compactness technique used. Theorem 1.2 generalizes the result by Csató and Roy [12] changing the weight $|x|^\beta$ for a more general weight $h(x)$). Lemma 2.2.8 is mentioned on literature (for instance, [2], [12]) as an easy consequence of the properties of the Schwartz rearrangement, but since I did not find any proof, I crafted the one we present here. Proof of Theorem 1.3 uses the same approach presented by de Figueiredo, Miyagaki and Ruf, but here we get a radial weak solution (while the solution found in [14]) may not be radial. We use here the same sequence (y_n) used on the proofs of Theorems 1.1 and 1.2, which provides a much better estimative for the constant ζ in (H6) than the Moser sequence originally used in [14]. It is also worth noticing that the so-called *critical growth condition* (H6) does not depend on the values of f outside an arbitrarily small ball around the origin.

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Appendix A

Results used on the Proof of Theorem 1

Definition A.0.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set. We say that $(u_n) \subset H_0^1(\Omega)$ is a normalized concentrating sequence at $x_0 \in \bar{\Omega}$ if*

(C1) $\int_\Omega |\nabla u_n|^2 dt = 1 \ \forall \ n$,

(C2) $\limsup_{n\to\infty} \int_{\Omega - B_\rho(0)} |\nabla u_n|^2 dt = 0$ *as* $n \to \infty$ *for all* $0 < \rho < 1$ *fixed.*

An important property of concentrating sequences is the following:

Proposition A.0.2. *Let $(u_n) \subset H_0^1(B)$ be a normalized concentrating sequence. Then (u_n) converges weakly to 0.*

Proof. Let ϕ be any $C_c^\infty(B)$ (compact support) test function. Then for any $0 < \rho < 1$

$$\begin{aligned} \left|\int_B \nabla\phi\nabla u_n dx\right| &\leq \int_{B-B_\rho} |\nabla\phi||\nabla u_n| dx + \int_{B_\rho} |\nabla\phi||\nabla u_n| dx \\ &\leq \|\phi\|_{L^2(B)}\|\nabla u_n\|_{L^2(B-B_\rho)} + \|\phi\|_{C^1(B)}\sqrt{\pi\rho^2}\|\nabla u_n\|_{L^2(B)} \qquad (A.1) \end{aligned}$$

by Hölder's inequality. Taking $n \to \infty$ and then ρ arbitrarily small one arrives at the desired result. ■

Theorem A.0.3. *(Vitali's Convergence Theorem, c. f. [19]) Let E be a measurable set, $(f_n) \subset L^1(E)$ be a uniformly integrable sequence and $f_n \to f$ almost everywhere. Then $f \in L^1(E)$ and*

$$\int_B |f_n - f| dx \to 0. \qquad (A.2)$$

Appendix B

Results used on the Proof of Theorem 2

Theorem B.0.1. *(Cauchy's Integral Formula) Let U be an open subset of the complex plane, $B_r \subset U$ the open ball of radius r, Γ_r its boundary, counterclockwise oriented and $f : U \to \mathbb{C}$ a holomorphic function. Then for any $a \in B_r$*

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z-a}dz = f(a). \tag{B.1}$$

Definition B.0.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u : \Omega \to \mathbb{R}$ be a measurable function. Then, its Schwarz symmetrization, or the spherically symmetric and decreasing rearrangement is the function $u^* : \Omega^* \to \mathbb{R}$ defined by*

$$\begin{cases} u^*(0) & = \operatorname{ess\,sup} u \\ u^*(x) & = \inf\{t : |\{x : u(x) > t\}| < \alpha_N |x|^N\} \text{ if } x \neq 0, \end{cases}$$

where $|E|$ denotes the N-dimentional Lebesgue Measure of a measurable set $E \subset R^N$ and α_N is such that $|B_{|x|}| = \alpha_N |x|^N$.

Some useful properties of the Schwarz rearragement are:

Theorem B.0.3. *(c.f. [22])*

(i) $|\{u^* > t\}| = |\{u > t\}|$ $\forall$ t *and*

(ii) u^* *is non-negative (if we assume u non-negative), radial and monotonically decreasing (thus, almost everywhere differentiable).*

Theorem B.0.4. *(c.f. [22], Theorem 2.3.1) Let $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded doamin and $u \in W_0^{1,p}(\Omega)$ be such that $u \geq 0$. Then $u^* \in W_0^{1,p}(\Omega^*)$ and*

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq \int_{\Omega} |\nabla u|^p dx. \tag{B.2}$$

Theorem B.0.5. *([17], section 5.5, Theorem 1) Assume Ω is bounded and open and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ *and*

(ii) there exists a constant $C = C(p, \Omega)$ *such that*

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \;\forall\; u \in W^{1,p}(\Omega). \tag{B.3}$$

Theorem B.0.6. *([17], section 5.5, Theorem 2) Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose furthermore that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \Leftrightarrow Tu = 0 \text{ in } \partial\Omega.$$

Appendix C

Results used on the Proof of Theorem 3

Proposition C.0.1 ([13], p.47). *Let X be a Banach space, $\Xi \subset X$ open and $\phi : \Xi \to \mathbb{R}$ such that $\nabla\phi$ (the Gatêaux derivative of ϕ) exists in a neighbourhood of $u \in \Xi$ and is continuous at u. Then $\phi \in C^1$ and $\phi'(u) = \nabla\phi(u)$ (ϕ' denotes the Frechét-derivtive of ϕ).*

Lemma C.0.2. *([14], page 145, Lemma 2.1) Let Ω be a finite measure smooth domain, $(u_n) \subset L^1(\Omega)$ converging to $u \in L^1(\Omega)$. Assume that $g(x, u_n(x))$ and $g(x, u(x))$ are also L^1 functions satisfying (H1). If*

$$\int_\Omega |g(x, u_n)u_n| \leq K, \tag{C.1}$$

then $g(x, u_n) \to g(x, u)$ in $L^1(\Omega)$.

Theorem C.0.3. *(c.f. [7]) Let Φ be a C^1 funcitonal on a Banach Space E. Suppose*

(i) There exists a neighbourhood U of 0 in E and a constant a such that $\Phi(u) \geq a$ for every u in the boundary of U,

(ii) $\Phi(0) < a$ and $\Phi(v) < a$ for some $v \notin U$.

Set

$$c = \inf_{P \in \mathscr{P}} \max_{w \in P} \Phi(w) \geq a,$$

where $\mathscr{P}$ denotes the class of continuous paths joining 0 to v. Then there exists a sequence $(u_j) \subset E$ such that

$$\Phi(u_j) \to c \text{ and } \Phi'(u_j) \to 0 \text{ in } E^*.$$

Thus, if Φ satisfies $(PS)_c$, there exists a subsequence (u_{j_k}) and $u \in E$ such that

$$u_{j_k} \to u \text{ and } \Phi'(u) = \lim \Phi(u_{j_k}) = 0.$$