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Representations of noncommutative Jordan superalgebras

Representações de superálgebras não comutativas de Jordan

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Resumo

Álgebras não comutativas de Jordan foram introduzidas por Albert. Ele observou que as teorias estruturais das álgebras alternativas e de Jordan compartilham tantas propriedades boas que é natural supor que essas álgebras são membros de uma classe mais geral com uma teoria estrutural semelhante. Então, ele introduziu a variedade de álgebras não comutativas de Jordan definidas pela identidade de Jordan e pela identidade da flexibilidade. A classe de (super)álgebras não-comutativas de Jordan se tornou vasta: por exemplo, além das (super)álgebras alternativas e de Jordan, ela contém álgebras quasiassociativas, (super)álgebras quadráticas flexíveis e (super)álgebras anticomutativas. No entanto, a teoria da estrutura dessa classe está longe de ser boa.

No entanto, um certo progresso foi feito no estudo da teoria estrutural de álgebras (e, mais geralmente, superálgebras) não comutativas de Jordan. Particularmente, álgebras simples dessa classe foram estudadas por muitos autores. Superalgebras centrais não-comutativas simples de Jordan de dimensão finita foram descritas por Pozhidaev e Shestakov.

Representações de superalgebras alternativas e de Jordan são um tópico popular atualmente, estudado por muitos autores. Neste trabalho, estudamos representações de álgebras não-comutativas de Jordan. Em particular, classificamos as representações irredutíveis de dimensões finitas de superalgebras simples não-comutativas de Jordan de dimensão finita sobre um corpo algebricamente fechado da característica 0 e mostramos o teorema de fatoração de Kronecker para algumas àlgebras.

Palavras-chave: superálgebra não comutativa de Jordan, teoria de representações de superalgebras não-associativas, representações das superalgebras de Jordan, Teorema da fatoração de Kronecker.

Abstract

Noncommutative Jordan algebras were introduced by Albert. He noted that the structure theories of alternative and Jordan algebras share so many nice properties that it is natural to conjecture that these algebras are members of a more general class with a similar theory. So he introduced the variety of noncommutative Jordan algebras defined by the Jordan identity and the flexibility identity. The class of noncommutative Jordan (super)algebras turned out to be vast: for example, apart from alternative and Jordan (super)algebras it contains quasiassociative (super)algebras, quadratic flexible (super)algebras and (super)anticommutative (super)algebras. However, the structure theory of this class is far from being nice.

Nevertheless, a certain progress was made in the study of structure theory of noncommutative Jordan algebras (and, more generally, superalgebras). Particularly, simple algebras of this class were studied by many authors. Simple finite-dimensional central noncommutative Jordan superalgebras were described by Pozhidaev and Shestakov.

Representations of alternative and Jordan superalgebras is a popular topic nowadays which was studied by many authors. In this work we study representations of noncommutative Jordan algebras. In particular, we classify the irreducible finite-dimensional representations of simple finite-dimensional noncommutative Jordan superalgebras over an algebraically closed field of characteristic 0 and prove Kronecker factorization theorems for some superalgebras.

Keywords: noncommutative Jordan superalgebra, representation theory of nonassociative superalgebras, representations of Jordan superalgebras, Kronecker factorization theorem.

Contents

Introduction

Noncommutative Jordan algebras were introduced by Albert in [Alb48]. He noted that the structure theories of alternative and Jordan algebras share so many nice properties that it is natural to conjecture that these algebras are members of a more general class with a similar theory. So he introduced the variety of noncommutative Jordan algebras defined by the Jordan identity and the flexibility identity. The class of noncommutative Jordan (super)algebras turned out to be vast: for example, apart from alternative and Jordan (super)algebras it contains quasiassociative (super)algebras, quadratic flexible (super)algebras and (super)anticommutative (super)algebras. However, the structure theory of this class is far from being nice.

Nevertheless, a certain progress was made in the study of structure theory of noncommutative Jordan algebras. Particularly, simple algebras of this class were studied by many authors. Schafer proved that a simple finite-dimensional noncommutative Jordan algebra over a field of characteristic 0 is either a Jordan algebra, or a quasiassociative algebra, or a flexible algebra of degree 2 [Sch55]. Oehmke proved an analog of Schafer's classification for simple flexible algebras with strictly associative powers and of characteristic $\neq 2, 3$ [Oeh58], McCrimmon classified simple noncommutative Jordan algebras of degree > 2 and characteristic $\neq 2$ [McC66, McC71], and Smith described such algebras of degree 2 [Smi71]. The case of nodal (degree 1) simple algebras of positive characteristic was considered in the articles of Kokoris [Kok60, Kok58], and the case of infinite-dimensional noncommutative Jordan superalgebras was studied by Shestakov and Skosyrskiy [She71, Sko89].

Simple finite-dimensional Jordan superalgebras over algebraically closed fields of characteristic 0 were classified by Kac [Kac77] and Kantor [Kan92]. The study of superalgebras in positive characteristic was initiated by Kaplansky [Kap80]. Racine and Zelmanov [RZ03] classified finite-dimensional Jordan superalgebras of characteristic $\neq 2$ with semisimple even part, and the case where even part is not semisimple was considered by Martínez and Zelmanov in [MZ10]. Cantarini and Kac described linearly compact simple Jordan and generalized Poisson superalgebras in [CK07].

Simple finite-dimensional central noncommutative Jordan superalgebras were described by Pozhidaev and Shestakov in [PS10a, PS13, PS19]. Structure and derivations of low-dimensional simple noncommutative Jordan algebras were studied in [KLP18]. Generalizations of derivations of simple noncommutative Jordan superalgebras were studied in [Kay10].

We briefly recall the history of the structure theory of other classes of algebras generalizing Jordan algebras. In the paper [Kan72] Kantor generalized the Tits–Koecher–

Kantor construction, extending it to the wide class of algebras, which he called conservative algebras. The Kantor construction puts a graded Lie algebra into correspondence with a conservative algebra. A conservative algebra is said to be of order 2 if its Lie algebra has $(-2, 2)$-grading. In the same paper, he classified the finite dimensional simple conservative algebras of order 2 over an algebraically closed field of characteristic 0.

In [All78], Allison defined a class of nonassociative algebras containing the class of Jordan algebras and allowing the construction of generalizations of the structure algebra and the Tits–Koecher–Kantor construction (Allison's construction). The algebras in this class, called structurable algebras, are unital algebras with involution. The class is defined by an identity of degree 4 and includes associative algebras, Jordan algebras (with the identity map as involution), tensor product of two composition algebras, the 56-dimensional Freudenthal module for E_7 with a natural binary product, and some algebras constructed from hermitian forms in a manner generalizing the usual construction of Jordan algebras from quadratic forms. Central simple finite dimensional structurable algebras over a field of characteristic zero were classified by Allison. The classification of simple structurable algebras over a modular field was obtained by Smirnov [Smi90]. Moreover, he found a new class of simple structurable algebras of characteristic zero, missed by Allison. Simple structurable superalgebras over an algebraically closed field of characteristic 0 were described by Faulkner [Fau10] and by Pozhidaev and Shestakov [PS10b].

Representations of alternative and Jordan superalgebras are considered in various works. In the paper [MZ10] Martínez and Zelmanov used the Tits-Koecher-Kantor construction to describe superbimodules over superalgebras $JP(n), M_{m,n}(\mathbb{F})^{(+)}, Josp(m, 2r)$ and Jordan superalgebras of supersymmetric bilinear superforms over algebraically closed fields of characteristic 0. Also they proved that the universal enveloping superalgebra for unital representations of simple finite-dimensional Jordan superalgebra of degree $\geqslant 3$ is finite-dimensional and semisimple and that every representation over such superalgebra is completely reducible. Some of the Martínez-Zelmanov results were generalized to the case of arbitrary characteristic $\neq 2$. For example, representations of superalgebras $JP(n)$ and $Q(n)^{(+)}, n \geqslant 2$ over fields of characteristic $\neq 2$ were considered by Martínez, Shestakov and Zelmanov in [MSZ10]. Irreducible representations of superalgebras of Poisson-Grassmann bracket were classified by Shestakov and Solarte in [SFS16]. In the papers [Tru05], [Tru08] Trushina described irreducible bimodules over the superalgebras D_t and K_3. In the work [MZ02] the universal envelopes for one-sided representations of simple Jordan superalgebras were constructed, and also irreducible one-sided bimodules over the superalgebras D_t were described. Shtern [Sht87] classified irreducible bimodules over the exceptional Kac superalgebra K_{10}.

Representations of alternative superalgebras were studied by Pisarenko. Particularly, he proved the following: let A be a finite-dimensional semisimple alternative superalgebra over a field of characteristic $\neq 2, 3$. If A contains no ideals isomorphic to the two-dimensional simple associative superalgebra, then every bimodule over A is associative and completely reducible [Pis94]. For the case of two-dimensional simple associative superalgebra he obtained a series of indecomposable alternative superbimodules. López-Díaz and Shestakov described irreducible superbimodules and proved the analogues of Kronecker factorization theorem for exceptional alternative and Jordan superalgebras of characteristic 3 in [LDS02, LDS05]. Infinite-dimensional representations of alternative superalgebras were studied in the paper [ST16].

In the present work we begin the study of representations of central simple finite-dimensional noncommutative Jordan superalgebras.

This work is organized as follows. In Chapter 1 we provide all the preliminary information and prove technical lemmas which will be necessary to work with noncommutative Jordan superalgebras and their representations. Then we reformulate the definitions of a noncommutative Jordan (super)algebra and representation in terms of Jordan multiplication and Poisson brackets. In Chapter 2 we classify finite-dimensional representations of simple noncommutative Jordan superalgebras of degree $\geqslant 3$ and noncommutative Jordan representations of some simple Jordan superalgebras. In Chapter 3 we study representations of low-dimensional noncommutative Jordan superalgebras. Particularly, in Section 3.1 we describe superbimodules over superalgebras $D_t(\alpha, \beta, \gamma)$ and $K_3(\alpha, \beta, \gamma)$, and in Section 3.2 we prove the Kronecker factorization theorem for superalgebras $D_t(\alpha, \beta, \gamma)$ and use it to study representations of the superalgebra $Q(2)$ in Section 3.3. In Chapter 4 we prove a theorem which connects the irreducibility of a noncommutative Jordan module with the irreducibility of the underlying Jordan module and use it to classify irreducible finite-dimensional representations over superalgebras $U(V, f, \star)$ and $K(\Gamma_n, A)$. For a more detailed review of the results of this work, one can see Section 1.13 and the summaries at the beginning of each chapter. The results of this work were published in the paper [Pop20].

1 Preliminaries

In this chapter we briefly recall the definitions, techniques and objects which we work with. Also here we prove some technical lemmas and reproduce the classification of central simple finite-dimensional noncommutative Jordan superalgebras by Pozhidaev and Shestakov. At the end of the chapter we state our main results.

1.1 Notations and defining identities

Since this work deals with representations of nonassociative superalgebras, "(super)algebra" means a not necessarily associative (super)algebra, and "module" and "representation" mean respectively a (super)bimodule and a two-sided representation over a (super)algebra, if not explicitly said otherwise. Also, occasionally we drop the prefix "super-", but it should be always clear from the context in which setting we are working. All algebras and vector spaces in this work are over a field $\mathbb{F}$ of characteristic not 2 (note that there exists an approach to (noncommutative) Jordan algebras which does not assume the characteristic restriction, see, for example, [McC71]). Because this work follows up on and uses the formulas from the papers [PS10a] and [PS13], the operators in this work act on the right.

For a subset S of an $\mathbb{F}$-vector (super)space by $\langle S \rangle$ we denote its $\mathbb{F}$-span.

Let $U = U_{\bar{0}} + U_{\bar{1}}$ be a superalgebra. In what follows, if the parity of an element arises in a formula, this element is assumed to be homogeneous. Idempotents are also assumed to be homogeneous. We assume the following standard notation:

$$(-1)^{xy} = (-1)^{p(x)p(y)},$$

where $p(a) = i$, if $a \in U_{\bar{i}}$ is the parity of a, and

$$(-1)^{x,y,z} = (-1)^{xy+xz+yz}.$$

Definition 1.1.1. Let A and B be superalgebras. By $A \otimes B = C$ we denote their *graded* (*sometimes also called twisted, super or colored*) *tensor product*, which is defined as

$$C_{\bar{0}} = A_{\bar{0}} \otimes B_{\bar{0}} + A_{\bar{1}} \otimes B_{\bar{1}}, \quad C_{\bar{1}} = A_{\bar{0}} \otimes B_{\bar{1}} + A_{\bar{1}} \otimes B_{\bar{0}},$$

and the multiplication is given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{a'b}(aa') \otimes (bb').$$

By L_x, R_x we denote the operators of left and right multiplication by $x \in U$:

$$yL_x = (-1)^{xy}xy, \quad yR_x = yx.$$

The supercommutator and super Jordan product are also denoted in the standard manner:

$$[x, y] = xy - (-1)^{xy}yx, \quad x \circ y = (xy + (-1)^{xy}yx)/2, \quad x \bullet y = xy + (-1)^{xy}yx.$$

Definition 1.1.2. The (super)algebra $(U, \circ)$ is called *the symmetrized (super)algebra of* U and is denoted by $U^{(+)}$.

Definition 1.1.3. A supercommutative superalgebra J is called *Jordan* if it satisfies the following operator identity:

$$R_aR_bR_c+(-1)^{a,b,c}R_cR_bR_a+(-1)^{bc}R_{(ac)b} = R_aR_{bc}+(-1)^{a,b,c}R_cR_{ba}+(-1)^{ab}R_bR_{ac}. \quad (1.1.1)$$

Definition 1.1.4. A superalgebra U is called *noncommutative Jordan* if it satisfies the following operator identities:

$$[R_{a\circ b}, L_c] + (-1)^{a(b+c)}[R_{b\circ c}, L_a] + (-1)^{c(a+b)}[R_{c\circ a}, L_b] = 0, \quad (1.1.2)$$

$$[R_a, L_b] = [L_a, R_b]. \quad (1.1.3)$$

The identity (1.1.3) defines the class of *flexible superalgebras*. If we assume that all elements of U are even we get the notion of a noncommutative Jordan algebra.

The flexibility identity may be written as

$$(-1)^{ab}L_{ab} - L_bL_a = R_{ba} - R_bR_a, \quad (1.1.4)$$

or

$$(x, y, z) = -(-1)^{x,y,z}(z, y, x), \quad (1.1.5)$$

where $(a, b, c) = (ab)c - a(bc)$ is the *associator* of the elements a, b, c.

We would like to clarify the origin of the name "noncommutative Jordan". One can check that a Jordan (super)algebra is noncommutative Jordan. On the other hand, a *commutative* noncommutative Jordan (super)algebra is Jordan — hence the name. In fact, the relation between noncommutative and commutative Jordan (super)algebras can be made precise using the notion of the symmetrized algebra:

Lemma 1.1.5 ([PS10a], Lemma 1.3)**.** *U is a noncommutative Jordan (super)algebra if and only if U is a flexible (super)algebra such that its symmetrized (super)algebra $U^{(+)}$ is a Jordan (super)algebra.*

Using this lemma it is easy to see that (super)algebras from many well-known varieties, such as associative, alternative and anticommutative (super)algebras, are noncommutative Jordan. So while reading this work it is useful to bear in mind associative (or alternative) and Jordan (super)algebras as examples.

From now on, unless otherwise stated, we denote by U a noncommutative Jordan superalgebra over a field $\mathbb{F}$, and by e an even idempotent in U.

1.2 Peirce decomposition

Here we recall some usual facts about Peirce decomposition, which is going to be our main tool during the work. A more detailed exposition of Peirce decomposition can be found in [McC66], [PS10a].

The identity (1.1.2) can be shown to be equivalent to the identity

$$\begin{aligned} R_{a(b\bullet c)} - R_a R_{b\bullet c} + (-1)^{ab}(R_b + L_b)(L_a L_c - (-1)^{ca} L_{ca}) \\ + (-1)^{c(a+b)}(R_c + L_c)(L_a L_b - (-1)^{ab} L_{ba}) = 0. \end{aligned} \tag{1.2.1}$$

Substituting $a = b = c = e$ in (1.2.1) gives us

$$R_e + (R_e + L_e)L_e^2 = (R_e + L_e)L_e + R_e^2.$$

By (1.1.4) we have $L_e - L_e^2 = R_e - R_e^2$. Hence, the last equation is equivalent to

$$(R_e + L_e)(L_e - L_e^2) = (L_e - L_e^2).$$

Put

$$U_i = \{x \in U : ex + xe = ix\} \text{ for } i = 0, 1, 2.$$

Using the standard argument, we get the decomposition

$$U = U_0 \oplus U_1 \oplus U_2. \tag{1.2.2}$$

Definition 1.2.1. The decomposition (1.2.2) is called the *Peirce decomposition of U with respect to e*, and the spaces $U_i = U_i(e)$ are called *Peirce spaces*.

The identities (1.1.2) and (1.1.3) imply that

$$[E_x, F_e] = 0, \text{ if } x \in U_0 + U_2, \ \{E, F\} \subseteq \{R, L\}. \tag{1.2.3}$$

Denote by P_i the associated projections on U_i along the direct sum of two other Peirce spaces. Since P_i are polynomials in $L_e + R_e$, we have

$$[E_x, P_i] = 0, \text{ if } x \in U_0 + U_2, \ \{E, F\} \subseteq \{R, L\}. \tag{1.2.4}$$

The spaces U_0, U_1 and U_2 satisfy the following relations (which we call *the Peirce relations*):

$$U_i^2 \subseteq U_i, \quad U_iU_1 + U_1U_i \subseteq U_1, \ i = 0, 2; \quad U_0U_2 = U_2U_0 = 0, \tag{1.2.5}$$

$$x \in U_i, i = 0, 2 \Rightarrow xe = ex = \frac{1}{2}ix; \quad x, y \in U_1 \Rightarrow x \circ y \in U_0 + U_2. \tag{1.2.6}$$

These relations will be used so frequently throughout our work that referencing them each time would make it messy. Therefore, we only occasionally explicitly reference them (for example, when it is not clear which relation we use), and in other cases when we apply them we say "by Peirce relations", or use them without mentioning.

If $e = \sum_{i=1}^{n} e_i$ is a sum of orthogonal idempotents, then analogously one can obtain the *Peirce decomposition with respect to* $e_1, \dots, e_n$:

$$U = \bigoplus_{i,j=0}^{n} U_{ij}, \tag{1.2.7}$$

where

$$\begin{aligned}
U_{00} &= \{x \in U : e_i x = x e_i = 0 \text{ for all } i\},\\
U_{ii} &= \{x \in U : e_i x = x e_i = x, e_j x = x e_j = 0, j \neq i\},\\
U_{i0} &= \{x \in U : e_i x + x e_i = x, e_j x + x e_j = 0, j \neq i\} = U_{0i},\\
U_{ij} &= \{x \in U : e_i x + x e_j = e_j x + x e_j = x, e_k x + x e_k = 0, k \neq i, j\} = U_{ji}.
\end{aligned}$$

Note that if $i, j \neq 0$ and $x \in U_{ij}$, then $e_i x = x e_j$. As above, there are associated projections P_{ij} on U_{ij} and the following inclusions hold:

$$\begin{gathered}
U_{ii}^2 \subseteq U_{ii}, \quad U_{ii}U_{ij} + U_{ij}U_{ii} \subseteq U_{ij},\\
U_{ij}U_{jk} + U_{jk}U_{ij} \subseteq U_{ik}, \quad U_{ij}^2 \subseteq U_{ii} + U_{ij} + U_{jj}.
\end{gathered}$$

for distinct i, j, k (all other products are zero). Clearly, the decompositions above apply to any subspace $M \subseteq U$ invariant under the multiplication by e, for example, to an ideal of U. For instance, for the usual matrix algebra $U = M_n(\mathbb{F})$ and the set of orthogonal idempotents $\{e_1 = e_{11},\ e_2 = e_{22}, \dots, e_n = e_{nn}\}$ we have

$$U_{ii} = \langle e_{ii} \rangle,\ i = 1, \dots, n, \quad U_{ij} = \langle e_{ij}, e_{ji} \rangle,\ i \neq j.$$

One can see that the Peirce relations for a system of n orthogonal idempotents resemble the multiplication rules for the spaces in the matrix algebra generated by matrix units given above, so one can think of the Peirce decomposition of U as an "approximation of U by an $n \times n$ matrix algebra". In some cases this reasoning can be made rigorous, see, for example, Theorem 1.4.2.

With the aid of the following lemmas one can restore some of the original products in U from the products in $U^{(+)}$ and multiplication by e.

Lemma 1.2.2 ([PS10a], Lemma 1.4)**.** *If* $z, w \in U_1$, *then*

$$P_2(ez \bullet w) = P_2(z \bullet we) = P_2(zw), \quad P_0(w \bullet ez) = P_0(we \bullet z) = P_0(wz). \tag{1.2.8}$$

Lemma 1.2.3. *For $x, y, z \in U_1$ the following relation holds:*

$$x \circ P_1(yz) = P_1(xy) \circ z = (-1)^{x(y+z)} y \circ P_1(zx). \tag{1.2.9}$$

Proof. From [PS10a, Lemma 1.4], it follows that for $i \in \{0, 2\}$ and $x, y, z \in U_1$ the following relation holds:

$$P_i(x \circ P_1(yz)) = P_i(P_1(xy) \circ z) = (-1)^{x(y+z)} P_i(y \circ P_1(zx)).$$

Since $U_1 \circ U_1 \subseteq U_0 + U_2$ by (1.2.6), summing the relations for $i = 0$ and $i = 2$ yields the desired relation. □

The following statement is an obvious yet useful consequence of the last lemma:

Lemma 1.2.4. *Let $x, y \in U_1$ be such that $xy \in U_0 + U_2$. Then*

$$P_1 L_x P_1 (R_y + L_y) = 0, \quad P_1 R_x P_1 (R_y + L_y) = 0. \tag{1.2.10}$$

Note that in contrast to associative or Jordan superalgebras, for arbitrary noncommutative Jordan superalgebra the inclusion $U_1^2 \subseteq U_0 + U_2$ does not hold (see relation (1.2.6)). However, one can find a sufficient condition to ensure that this inclusion holds:

Lemma 1.2.5. *Suppose that there exists a subset $K \subseteq U_1$ such that*

1) $KU_1 \subseteq U_0 + U_2$;

2) $a \in U_1, K \circ a = 0 \Rightarrow a = 0$.

Then $U_1^2 \subseteq U_0 + U_2$.

Proof. Indeed, the relation (1.2.9) implies that for $a, b \in U_1$ we have $K \circ P_1(ab) = P_1(Ka) \circ b = 0$ by the first condition of the lemma. Hence, the second condition of the lemma implies that $P_1(ab) = 0$. □

We will also need a technical lemma by Pozhidaev and Shestakov:

Lemma 1.2.6 ([PS10a], Lemma 1.5)**.** *For $a, b \in U_i$, $i = 0, 2$, the following operator identities hold in U_1:*

$$R_{ab} = R_a R_b + (-1)^{ab} L_b R_a = R_a R_b + (-1)^{ab} R_b L_a;$$
$$L_{ab} = (-1)^{ab} L_b L_a + L_a R_b = (-1)^{ab} L_b L_a + R_a L_b.$$

1.3 Algebras with connected idempotents

For some well-studied classes of (super)algebras, such as associative and Jordan, the Peirce relations are in fact stronger than (1.2.5), (1.2.6), which can be seen a consequence of their defining identities. For example, if J is a Jordan (super)algebra, then its Peirce space J_1 is obviously

$$J_1 = \{x \in J : xe = x/2\},$$

and if A is an associative (super)algebra, then its Peirce space A_1 decomposes as

$$\begin{aligned} A_1 &= A_{10} \oplus A_{01}, \text{ where} \\ A_{10} &= \{x \in A : ex = x,\ xe = 0\}, \\ A_{01} &= \{x \in A : ex = 0,\ xe = x\}. \end{aligned}$$

Therefore, in nice cases the space U_1 decomposes in the direct sum of eigenspaces of L_e. Moreover, if U is associative or Jordan, then $U_1^2 \subseteq U_0 + U_2$.

In this section we consider the general situation, introducing L_e-eigenspaces in U_1 and showing that they satisfy properties analogous to ones described above. Then we introduce related notions of connectedness of idempotents and the degree of an algebra. Finally, we state the results that show that if an algebra U is "sufficiently large" (i.e., has degree $\geqslant 3$) and if "eigenspaces" of U_1 satisfy certain conditions, then U is associative or Jordan.

For $\lambda \in \mathbb{F}$ consider the space $U_1^{[\lambda]} = \{x \in U_1 : xL_e = \lambda x\}$. This set is invariant under the multiplication by $U_i, i = 0, 2$:

Lemma 1.3.1 ([PS10a], Lemma 1.8)**.** *$U_i U_1^{[\lambda]} + U_1^{[\lambda]} U_i \subseteq U_1^{[\lambda]}$ for $i = 0, 2$.*

The following technical lemmas will simplify further computations:

Lemma 1.3.2. *Let $x \in U_1^{[\lambda]}$. Then:*

1) $(1-\lambda)((\mathrm{id} - L_e)L_x + R_e R_x) - \lambda((\mathrm{id} - R_e)R_x + L_e L_x) = 0;$

2) $\lambda(L_x(\mathrm{id} - L_e) + R_x R_e) - (1-\lambda)(L_x L_e + R_x(1 - R_e)) = 0.$

Proof. 1) Let $a = x,\ b = e$ in (1.1.4):

$$\begin{aligned} 0 &= (1-\lambda)L_x - (1-\lambda+\lambda)L_e L_x - \lambda R_x + (1-\lambda+\lambda)R_e R_x \\ &= (1-\lambda)(L_x - L_e L_x + R_e R_x) - \lambda(R_x + L_e L_x - R_e R_x), \end{aligned}$$

which proves the first point of the lemma.

2) Let $a = e,\ b = x$ in (1.1.4):

$$\begin{aligned} 0 &= \lambda L_x - (1-\lambda+\lambda)L_xL_e - (1-\lambda)R_x + (1-\lambda+\lambda)R_xR_e \\ &= \lambda(L_x - L_xL_e + R_xR_e) - (1-\lambda)(L_xL_e + R_x - R_xR_e), \end{aligned}$$

which proves the second point of the lemma. □

Lemma 1.3.3.

1) $U_1^{[0]}U_0 = U_2U_1^{[0]} = 0$;

2) $U_0U_1^{[1]} = U_1^{[1]}U_2 = 0$.

Proof. Let $a \in U_0,\ b = e$ in (1.1.4): $L_eL_a = R_eR_a$. Applying this relation on $z \in U_1^{[0]}$, we get $za = 0$. Now let $a \in U_2,\ b = e$ in (1.1.4): $L_a - L_eL_a = R_a - R_eR_a$. Applying this relation on $z \in U_1^{[0]}$, we get $az = 0$. The second point of the lemma is proved completely analogously. □

Lemma 1.3.4.

1) $U_1^{[0]}U_1 \subseteq U_0 + U_1, \quad U_1U_1^{[0]} \subseteq U_1 + U_2$;

2) $U_1^{[1]}U_1 \subseteq U_1 + U_2, \quad U_1U_1^{[1]} \subseteq U_0 + U_1$;

3) $(U_1^{[0]})^2 \subseteq U_1, \quad (U_1^{[1]})^2 \subseteq U_1$.

Proof. First two items of the lemma follow from (1.2.8), and the third point follows from the first two. □

For any $\lambda \in \mathbb{F}$ the space $S_1^{[\phi]}(e) = U_1^{[\lambda]} + U_1^{[1-\lambda]}$ is completely determined by the value $\phi = \lambda(1-\lambda)$ and can be considered as the "eigenspace" of L_e and R_e corresponding to the "eigenvalue" ϕ. Let (1.2.7) be the Peirce decomposition of U with respect to a system of orthogonal idempotents $e_1, \dots, e_n$. Put

$$S_{ij}^{[\phi]} = S_1^{[\phi]}(e_i) \cap S_1^{[\phi]}(e_j).$$

Definition 1.3.5. We say that e_i and e_j are *evenly connected* if there is a scalar $\phi \in \mathbb{F}$ and even elements $v_{ij}, u_{ij} \in S_{ij}^{[\phi]}$ such that $v_{ij}u_{ij} = u_{ij}v_{ij} = e_i + e_j, i < j$. We say that e_i and e_j are *oddly connected* if there is a scalar $\phi \in \mathbb{F}$ and odd elements $v_{ij}, u_{ij} \in S_{ij}^{[\phi]}, i < j$, such that $v_{ij}u_{ij} = -u_{ij}v_{ij} = e_i - e_j$. Lastly, e_i and e_j are said to be *connected* if they are either evenly or oddly connected. The element ϕ is called *an indicator* of U_{ij}.

Definition 1.3.6. We say that a noncommutative Jordan superalgebra U *is of degree* k if k is the maximal possible number of pairwise orthogonal connected idempotents in $U \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. And say that U *has unity of degree* k if k is the degree of U and the unity of U is a sum of k orthogonal pairwise connected idempotents.

A classical situation in theory of Jordan algebras is that a lot can be said about the structure and representations of a Jordan (super)algebra if it has degree $\geqslant 3$ (see, for example, the coordinatization theorem in [Jac68]). This remains true for the noncommutative case as well: McCrimmon partially described the structure of noncommutative Jordan algebras with unity of degree $\geqslant 3$, and Pozhidaev and Shestakov generalized his results for superalgebras. We state their results:

Lemma 1.3.7 ([PS10a], Lemma 1.10)**.** *If U has unity of degree $\geqslant 3$, then all indicators have the common value ϕ and $U_{ij} = S_{ij}^{[\phi]}, i \neq j$. The element $\phi \in \mathbb{F}$ is then called the indicator of U, and U is said to be of indicator type ϕ.*

Lemma 1.3.8 ([PS10a], Lemma 1.11)**.** *If U has unity of degree at least 3 and is of indicator type $\phi = 1/4$, then U is supercommutative.*

Lemma 1.3.9 ([PS10a], Lemma 1.12)**.** *If U is of degree at least 3 and of indicator type $\phi = 0$, then U is associative.*

It seems that the results stated above only work for specific values of an L_e-eigenvalue λ and indicator ϕ. In fact, we can control the indicator type of U and the eigenvalue λ using the construction called mutation, which we discuss in the next section.

1.4 Mutations

Mutation is a construction which generalizes the symmetrization of an algebra: $A \to A^{(+)}$. Since the class of noncommutative Jordan algebras is large, it is closed under mutations. In fact, the process of mutation is almost always invertible, so it does not really give new interesting examples of algebras. However, using mutations we may greatly simplify the multiplication table of an algebra, and also they allow us to formulate our results in a concise way, so they are still useful.

Definition 1.4.1. Let $A = (A, \cdot)$ be a superalgebra over $\mathbb{F}$ and $\lambda \in \mathbb{F}$. By $A^{(\lambda)}$ we denote the superalgebra $(A, \cdot_\lambda)$, where

$$x \cdot_\lambda y = \lambda x \cdot y + (-1)^{xy}(1-\lambda) y \cdot x.$$

The superalgebra $A^{(\lambda)}$ is called the λ-*mutation* of U.

It is easy to see that $A^{(1/2)}$ is the symmetrized superalgebra $A^{(+)}$, and $A^{(0)}$ is the opposite superalgebra A^{op}. Since $L_x^\lambda = \lambda L_x + (1-\lambda)R_x$, $R_x^\lambda = \lambda R_x + (1-\lambda)L_x$, it is easy to see (by inserting the new operators in relations (1.1.2), (1.1.3)) that a mutation of a noncommutative Jordan superalgebra is again a noncommutative Jordan superalgebra. A mutation $A^{(\lambda)}$ of an associative superalgebra A is called a *split quasiassociative superalgebra*.

A superalgebra U is called a *quasiassociative superalgebra* if there exists an extension Ω of $\mathbb{F}$ such that $U_\Omega = U \otimes_\mathbb{F} \Omega$ is a split quasiassociative superalgebra over Ω.

Consider a double mutation $(A^{(\lambda)})^{(\mu)}$. One can compute that $(A^{(\lambda)})^{(\mu)} = A^{(\lambda \odot \mu)}$, where $\lambda \odot \mu = 2\lambda\mu - \lambda - \mu + 1$. Hence, if $\lambda \neq 1/2$, there exists $\mu \in \mathbb{F}$ such that $\lambda \odot \mu = 1$, and we can recover A from $A^{(\lambda)}$: $A = A^{(1)} = A^{(\lambda\odot\mu)} = (A^{(\lambda)})^{(\mu)}$. However, if $\lambda = 1/2$, it is impossible to immediately recover A from $A^{(1/2)} = A^{(+)}$, since, for example, all mutations of A have the same $A^{(+)}$.

Many results about noncommutative Jordan algebras can be formulated using mutations. For example, in case of degree $\geqslant 3$ we have the noncommutative coordinatization theorem:

Theorem 1.4.2 (Coordinatization theorem, [PS10a])**.** *Let $\mathbb{F}$ be a field which allows square root extraction and U be a noncommutative Jordan superalgebra with unity of degree $n \geqslant 3$ which is not supercommutative. Then $U = (A_n)^{(\lambda)}$ is the λ-mutation of the $n \times n$ matrix algebra over an associative superalgebra A for $\lambda \in \mathbb{F}$.*

Proof. We only give the main idea of the proof. If U is not supercommutative, then its indicator type $\phi = \lambda(1-\lambda) \neq 1/4$ by Lemma 1.3.8. Therefore, $\mathbb{F} \ni \lambda \neq 1/2$ and there exists $\mu \in \mathbb{F}$ such that $\lambda \odot \mu = 1$. Now, one can check that $U^{(\mu)}$ is of indicator type 0. Thus, by Lemma 1.3.9 it is associative and is in fact the $n \times n$ matrix superalgebra over the superalgebra $A = U_{11}$. Hence, by the double mutation rule, $U = (A_n)^{(\lambda)}$. □

Now we can state our results on the classification of simple algebras. Recall that an anticommutative algebra is noncommutative Jordan, and the classification of simple finite-dimensional anticommutative algebras is far from being done. Therefore, we adapt the definition of a simple noncommutative Jordan (super)algebra in the following way:

Definition 1.4.3. A noncommutative Jordan (super)algebra is simple if it has no proper ideals and is not a nilalgebra.

Remark. In the paper [She71] it is shown that a nil noncommutative Jordan algebra with no proper ideals is anticommutative.

It is easy to see that an ideal in A remains an ideal in $A^{(\lambda)}$. Hence, if $\lambda \neq 1/2$, ideals in A and $A^{(\lambda)}$ coincide. In particular, if U is a simple noncommutative Jordan superalgebra, then all its λ-mutations, $\lambda \neq 1/2$, are simple noncommutative Jordan superalgebras.

The list of central simple noncommutative Jordan superalgebras clearly includes all central simple Jordan superalgebras. Also we just understood that it also includes all simple quasiassociative superalgebras. Pozhidaev and Shestakov proved that in the case of finite dimension and degree $\geqslant 3$ there is nothing else:

Theorem 1.4.4 ([PS10a, PS19]). *A finite-dimensional central simple noncommutative Jordan superalgebra U is either*

1) *of degree* 1;

2) *of degree* 2;

3) *quasiassociative*;

4) *supercommutative.*

Therefore, essentially new examples of simple noncommutative Jordan superalgebras must have degree $\leqslant 2$. The next section gives an approach for their classification.

1.5 Poisson brackets

Poisson brackets, Poisson algebras and generic Poisson algebras are important objects in nonassociative algebra. Here we shall see that one can give a definition of a noncommutative Jordan (super)algebra using the notion of generic Poisson bracket and its symmetrized (super)algebra. Also we see that the simplicity of a noncommutative Jordan (super)algebra is equivalent to the simplicity of its symmetrized (super)algebra in the case of degree $\geqslant 2$.

Definition 1.5.1. A superanticommutative binary linear operation $\{\cdot,\cdot\}$ on a superalgebra $(A,\cdot)$ is called a *generic Poisson bracket* [KSU18] if for arbitrary $a, b, c \in A$ we have

$$\{a\cdot b, c\} = (-1)^{bc}\{a,c\}\cdot b + a\cdot\{b,c\}. \tag{1.5.1}$$

In other words, for any homogeneous $c \in A$ the map $\{\cdot, c\}$ is a derivation of degree $p(c)$.

Generic Poisson brackets are important in the study of noncommutative Jordan (super)algebras. We have already seen that the symmetrized (super)algebra of a noncommutative Jordan (super)algebra is a Jordan (super)algebra. So we may ask ourselves: how do we reproduce the original structure of a noncommutative Jordan (super)algebra U having only its symmetrized (super)algebra $U^{(+)}$? Or equivalently, which noncommutative Jordan (super)algebras have a symmetrized (super)algebra isomorphic to a given Jordan (super)algebra J? The answer can be formulated nicely in terms of generic Poisson brackets:

Lemma 1.5.2 ([PS13], Lemma 7). *Let $(J,\circ)$ be a Jordan superalgebra and $\{\cdot,\cdot\}$ be a generic Poisson bracket on J. Then $(J,\cdot)$, where $a\cdot b = \frac{1}{2}(a\circ b + \{a,b\})$ is a noncommutative Jordan superalgebra. Conversely, if U is a noncommutative Jordan superalgebra, then the supercommutator $[\cdot,\cdot]$ is a generic Poisson bracket on a Jordan superalgebra $U^{(+)}$. Moreover, the multiplication in U can be recovered by the Jordan multiplication in $U^{(+)}$ and the Poisson bracket: $ab = \frac{1}{2}(a\circ b + [a,b])$.*

Hence, we can define a noncommutative Jordan superalgebra U as a superalgebra with two multiplications:

Definition 1.5.3. A (super)algebra $U = U(\circ, \{\cdot,\cdot\})$ with two multiplications $\circ$ and $\{\cdot,\cdot\}$, which we call respectively *circle* and *bracket multiplications*, is called *noncommutative Jordan*, if $(U,\circ) = J$ is a Jordan (super)algebra, and $\{\cdot,\cdot\}$ is a generic Poisson bracket on J.

In this setting, an ideal of $(U, \circ, \{\cdot,\cdot\})$ is a subspace invariant with respect to both multiplications. A subspace $I \subseteq U$ such that $I \circ U \subseteq I$ is called a *Jordan ideal* of U. Consider two marginal cases: if $\{\cdot,\cdot\} = 0$, then U is Jordan, and if $\circ = 0$, then U is anticommutative. The passage to the symmetrized algebra in this setting is just forgetting the bracket multiplication: $U^{(+)} = (U, \circ)$.

Now, if A is any algebra, it is obvious that if $A^{(+)}$ is simple, then A is also simple. Pozhidaev and Shestakov proved that the converse holds in the case of noncommutative Jordan superalgebras of degree $\geqslant 2$:

Theorem 1.5.4 ([PS10a, PS13, PS19])**.** *Let U be a central simple finite-dimensional noncommutative Jordan superalgebra of degree $\geqslant 2$. Then $U^{(+)}$ is a simple finite-dimensional Jordan superalgebra.*

In other words, a simple finite-dimensional central noncommutative Jordan superalgebra of degree $\geqslant 2$ is a simple finite-dimensional Jordan superalgebra with a generic Poisson bracket. Therefore, to classify simple superalgebras in degree 2, it suffices to find all generic Poisson brackets on simple finite-dimensional Jordan superalgebras (which are known), and classify the resulting noncommutative superalgebras up to isomorphism. In the next section we consider some examples of this approach.

So the structure theory of noncommutative Jordan (super)algebras can be formulated in a nice manner if we think of them as (super)algebras with two multiplications. In the last section of this section we will see what does the definition of a noncommutative Jordan representation look like in this context.

1.6 Examples and classification in degree $\leqslant 2$

Here we provide some examples of noncommutative Jordan superalgebras of degree $\leqslant 2$ given in [PS10a, PS13]. We also correct a mistake in the classification of the algebras $D_t(\alpha, \beta, \gamma)$ from the paper [PS10a] (the algebra $D_t(1/2, 1/2, 0)$ was omitted there). In the end of the section we state the classification theorem for simple superalgebras of degree $\leqslant 2$.

1.7 The superalgebra $D_t(\alpha,\beta,\gamma)$

Let $t \in \mathbb{F}$. Recall that a Jordan superalgebra D_t is defined in the following way:

$$D_t = (D_t)_{\bar{0}} \oplus (D_t)_{\bar{1}}, \quad (D_t)_{\bar{0}} = \langle e_1, e_2 \rangle, \quad (D_t)_{\bar{1}} = \langle x, y \rangle,$$
$$e_i^2 = e_i, \quad e_1 \circ e_2 = 0,$$
$$e_1 \circ x = e_2 \circ x = x/2, \quad e_1 \circ y = e_2 \circ y = y/2,$$
$$x \circ y = -y \circ x = e_1 + te_2.$$

This superalgebra is simple if $t \neq 0$.

Suppose that U is a noncommutative Jordan superalgebra such that $U^{(+)} = D_t$. Here we describe such algebras and classify them up to isomorphism.

The Peirce decomposition of U with respect to e_1 is as follows: $U_0 = \langle e_2 \rangle$, $U_1 = \langle x, y \rangle$, $U_2 = \langle e_1 \rangle$. be The Peirce relation (1.2.5) implies that

$$e_1 x = \alpha x + \beta y, \quad e_1 y = \gamma x + \delta y, \ \alpha, \beta, \gamma, \delta \in \mathbb{F}.$$

Relation (1.2.8) implies that $P_2(xy) = P_2(e_1 x \bullet y) = \alpha P_2(x \bullet y) = 2\alpha e_1$. Analogously, we obtain $P_2(yx) = -2\delta e_1$. Hence, $\alpha + \delta = 1$. Since $U_1 = U_{\bar{1}}$, we have

$$P_1(x^2) = 0, \quad P_1(y^2) = 0, \quad P_1(xy) = 0, \quad P_1(yx) = 0.$$

Again using the relation (1.2.8) we get

$$\begin{aligned} P_2(x^2) &= -2\beta e_1, & P_0(x^2) &= 2\beta t e_2, \\ P_2(y^2) &= 2\gamma e_1, & P_0(y^2) &= -2\gamma t e_2, \\ P_0(xy) &= 2(1-\alpha) t e_2, & P_0(yx) &= -2\alpha t e_2. \end{aligned}$$

Therefore, multiplication in U is of the following form:

$$\begin{aligned} e_i^2 &= e_i, & e_1 e_2 &= e_2 e_1 = 0, \\ e_1 x &= \alpha x + \beta y = x e_2, & x e_1 &= (1-\alpha)x - \beta y = e_2 x, \\ e_1 y &= \gamma x + (1-\alpha) y = y e_2, & y e_1 &= -\gamma x + \alpha y = e_2 y, \\ xy &= 2(\alpha e_1 + (1-\alpha) t e_2), & yx &= -2((1-\alpha) e_1 + \alpha t e_2), \\ x^2 &= -2\beta(e_1 - te_2), & y^2 &= 2\gamma(e_1 - te_2). \end{aligned}$$

One can check that for all $\alpha, \beta, \gamma \in \mathbb{F}$, U is a flexible superalgebra such that $U^{(+)} = D_t$, thus, by Lemma 1.1.5 it is noncommutative Jordan. Denote this algebra by $D_t(\alpha, \beta, \gamma)$. Putting $t = -1$, we obtain the superalgebra $M_{1,1}(\alpha, \beta, \gamma)$, and putting $t = -2$, we obtain the superalgebra $osp(1,2)(\alpha, \beta, \gamma)$ (see [PS13]). Putting $\alpha = 1/2$, $\beta = \gamma = 0$, we obtain a Jordan superalgebra D_t.

We can classify these algebras up to isomorphism (when this work was being written, the classification also appeared in [PS19]):

Lemma 1.7.1. *If $\mathbb{F}$ is a field which allows square root extraction, then for $\alpha, \beta, \gamma \in \mathbb{F}$, the superalgebra $D_t(\alpha, \beta, \gamma)$ is isomorphic either to $D_t(\lambda, 0, 0) = D_t(\lambda)$ for some $\lambda \in \mathbb{F}$, or to $D_t(1/2, 1/2, 0)$.*

Proof. Let $U = D_t(\alpha, \beta, \gamma)$ and consider the restriction of the operator L_{e_1} to U_1. Since $\mathbb{F}$ allows square root extraction, U_1 has a Jordan basis with respect to L_{e_1}. We denote the elements of this basis by $x' = ax + by,\ y' = cx + dy,\ a, b, c, d \in \mathbb{F}$. Let $\delta = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We consider two cases:

1) $x'L_{e_1} = \lambda x',\ y'L_{e_1} = \mu y'$, where $\lambda, \mu \in \mathbb{F}$. It is easy to see that $x' \circ x' = y' \circ y' = 0$, and
$$x' \circ y' = \delta(e_1 + te_2).$$
Setting $x'' = x/\delta,\ y'' = y'$, we may assume that the multiplication rules in the symmetrized superalgebra for x'', y'' are the same as for x, y. That is, we may assume that $x'' = x$ and $y'' = y$ are eigenvectors with respect to L_{e_1}. Again using relation (1.2.8), we see that $\lambda + \mu = 1$. Repeating the calculations of the structure constants of U as in the general case, we conclude that the original superalgebra is isomorphic to $D_t(\lambda, 0, 0)$.

2) $x'L_{e_1} = \lambda x' + y',\ y'L_{e_1} = \lambda y'$, where $\lambda \in \mathbb{F}$. Setting $x'' = x'/\sqrt{\delta},\ y'' = y'/\sqrt{\delta}$, we see that the multiplication rules for x, y and x'', y'' in the symmetrized algebra D_t coincide. Thus, as in the previous case we may assume $x'' = x,\ y'' = y$. From (1.2.8) it follows that $\lambda = 1 - \lambda = 1/2$. Repeating the calculations of the structure constants of U as in the general case, we see that $D \cong D_t(1/2, 1, 0)$. Repeating these calculations for $\alpha = \beta = 1/2,\ \gamma = 0$, we see that $D_t(1/2, 1, 0) \cong D_t(1/2, 1/2, 0)$ (the last algebra has more symmetrical multiplication rules and will be more convenient to work with). □

1.8 The superalgebra $K_3(\alpha, \beta, \gamma)$

A noncommutative Jordan superalgebra $K_3(\alpha, \beta, \gamma) = U_{\bar{0}} \oplus U_{\bar{1}};\ U_{\bar{0}} = \langle e_1 \rangle,\ U_{\bar{1}} = \langle x, y \rangle$ is defined by the following multiplication table:

	e_1	x	y
e_1	e_1	$\alpha x + \beta y$	$\gamma x + (1-\alpha)y$
x	$(1-\alpha)x - \beta y$	$-2\beta e_1$	$2\alpha e_1$
y	$\alpha y - \gamma x$	$-2(1-\alpha)e_1$	$2\gamma e_1$

The superalgebra $K_3(\alpha, \beta, \gamma)^{(+)}$ is isomorphic to the simple nonunital Jordan superalgebra $K_3 = K_3(1/2, 0, 0)$. In fact, they are characterized by this property:

Lemma 1.8.1 ([PS10a], Lemma 4.6)**.** *Let U be a noncommutative Jordan superalgebra such that $U^{(+)} \cong K_3$. Then $U \cong K_3(\alpha, \beta, \gamma)$ for some $\alpha, \beta, \gamma \in \mathbb{F}$.*

Analogously to Lemma 1.7.1 one can classify these algebras up to isomorphism:

Lemma 1.8.2. *If $\mathbb{F}$ is a field which allows square root extraction, then for $\alpha, \beta, \gamma \in \mathbb{F}$ $K_3(\alpha, \beta, \gamma)$ is isomorphic either to $K_3(\lambda, 0, 0) = K_3(\lambda)$ for some $\lambda \in \mathbb{F}$, or to $K_3(1/2, 1/2, 0)$.*

Note that in fact $K_3(\alpha, \beta, \gamma)$ is of degree 1, but we still list these algebras here because of their similarity to the algebras $D_t(\alpha, \beta, \gamma)$ described above. In fact, the unital hull of $K_3(\alpha, \beta, \gamma)$ is a nonsimple noncommutative Jordan superalgebra $D_0(\alpha, \beta, \gamma)$.

1.9 The superalgebra $U(V, f, \star)$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace over $\mathbb{F}$, and let f be a supersymmetric nondegenerate bilinear form on V. Also let $\star$ be a superanticommutative multiplication on V such that $f(x \star y, z) = f(x, y \star z)$ (that is, f is an invariant form with respect to the product $\star$). Then we can define a multiplication on $U = \mathbb{F} \oplus V$ in the following way:

$$(\alpha + x)(\beta + y) = (\alpha\beta + f(x, y)) + (\alpha y + \beta x + x \star y),$$

and the resulting superalgebra (which is noncommutative Jordan) is denoted by $U(V, f, \star)$.

The superalgebra $U(V, f, \star)^{(+)}$ is isomorphic to a simple Jordan superalgebra of nondegenerate supersymmetric bilinear form which is usually denoted by $J(V, f)$. Again, this property characterizes this family:

Lemma 1.9.1 ([PS10a], Lemma 4.4)**.** *Let $\mathbb{F}$ be a field which allows square root extraction, $J(V, f)$ be a superalgebra of nondegenerate supersymmetric bilinear form, and U be a noncommutative Jordan superalgebra such that $U^{(+)} \cong J(V, f)$. Then there exists a superanticommutative product $\star$ on V such that $U \cong U(V, f, \star)$.*

We remark here that the condition that $\mathbb{F}$ allows square root extraction serves to ensure that algebras $J(V, f)$ and $U(V, f, \star)$ are of degree 2 (see Section 2.2.1, where we describe the idempotents of this algebra). If $\mathbb{F}$ does not allow square root extraction, it is possible that $J(V, f)$ and $U(V, f, \star)$ have degree 1. Note also that $J(V, f) = U(V, f, 0)$. Examples of this type of algebras are generalized Cayley-Dickson algebras of dimension 2^n, see [ZSSS82].

1.10 The classification in degree 2

Analogously to the examples described above, Pozhidaev and Shestakov calculated all possible generic Poisson brackets on simple Jordan superalgebras in the case of characteristic 0, concluding the classification in the case of degree $\geqslant 2$. We provide it here:

Theorem 1.10.1 ([PS13], Theorem 2)**.** *Let U be a simple noncommutative central Jordan superalgebra over a field $\mathbb{F}$ of characteristic zero. Suppose that U is neither supercommutative nor quasiassociative. Then U is isomorphic to one of the following algebras: $K_3(\alpha,\beta,\gamma)$, $D_t(\alpha,\beta,\gamma)$, $K(\Gamma_n, A)$, $\Gamma_n(\mathcal{D})$, or there exists an extension $\mathbb{P}$ of $\mathbb{F}$ of degree $\leqslant 2$ such that $U \otimes_{\mathbb{F}} \mathbb{P}$ is isomorphic as a $\mathbb{P}$-superalgebra to $U(V, f, \star)$.*

The only algebras in the list whose structure we did not describe here are the algebras $K(\Gamma_n, A)$ and $\Gamma_n(\mathcal{D})$. The underlying Jordan superalgebra of $K(\Gamma_n, A)$ is the *Kantor double* of the Grassmann superalgebra Γ_n, its representations will be considered in Section 4.3. The symmetrized superalgebra of $\Gamma_n(\mathcal{D})$ is the Grassmann superalgebra Γ_n (which is not simple and is of degree 1), their descriptions can be found in the paper [PS13]. We will not treat these algebras in this work. Note also that many simple Jordan superalgebras, such as $P(2)$, K_{10}, K_9, do not admit nonzero Poisson brackets and do not give new examples of simple algebras (see [PS13], [PS19]).

In positive characteristic some additional algebras appear, see [PS19].

1.11 Bimodules and representations

In this section we briefly recall the basic notions of representation theory of nonassociative algebras.

Definition 1.11.1. A *superbimodule* over a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a linear superspace $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with two bilinear operations $A \times M \to M, M \times A \to M$ such that $A_{\bar{i}}M_{\bar{j}} + M_{\bar{j}}A_{\bar{i}} + A_{\bar{j}}M_{\bar{i}} + M_{\bar{i}}A_{\bar{j}} \subseteq M_{\bar{i}+\bar{j}}$ for $\bar{i}, \bar{j} \in \mathbb{Z}_2$.

For a subset $S \subset M$ by $\mathrm{Mod}(S)$ we denote the submodule generated by S.

Definition 1.11.2. A set $\{L, R\}$ of two even linear maps $L, R \colon A \to \mathrm{End}(M)$ is called a *representation* of A.

It is clear that the notions of superbimodule and representation are equivalent.

Definition 1.11.3. The *regular superbimodule* $\mathrm{Reg}(A)$ for a superalgebra A is defined on the vector space A with the action of A induced by the multiplication in A.

Definition 1.11.4. For an A-superbimodule M the structure of the *opposite module* on the space M^{op} with $M^{\mathrm{op}}_{\bar 0} = M_{\bar 1}, M^{\mathrm{op}}_{\bar 1} = M_{\bar 0}$ is defined by the action

$$a \cdot m = am, \quad m \cdot a = (-1)^a ma \quad \text{for} \quad a \in A,\ m \in M^{\mathrm{op}}.$$

Note that a module M is irreducible if and only if its opposite M^{op} is irreducible.

Recall the definition of a split null extension:

Definition 1.11.5. The *split null extension of A by a module M* is a superalgebra $E = A \oplus M$ with the multiplication

$$(a_1 + m_1)(a_2 + m_2) = a_1a_2 + a_1 \cdot m_2 + m_1 \cdot a_2 \text{ for } a_1, a_2 \in A,\ m_1, m_2 \in M.$$

Now we give the standard definition of a representation in a variety.

Definition 1.11.6. Suppose that A lies in a homogeneous variety of algebras $\mathfrak{M}$. Then an A-module M is called an $\mathfrak{M}$-*superbimodule* if the split null extension E of A by M also lies in $\mathfrak{M}$. A representation of A is called an $\mathfrak{M}$-*represenation* if the corresponding superbimodule is an $\mathfrak{M}$-superbimodule.

Lemma 1.11.7. *Let M be an $\mathfrak{M}$-superbimodule over A. Then M^{op} is also an $\mathfrak{M}$-superbimodule over A.*

Proof. From the definition it follows that the split null extension $E = A \oplus M$ lies in $\mathfrak{M}$. Consider the superalgebra $P = \langle 1, \bar 1\rangle$ with $P_{\bar 0} = \langle 1\rangle,\ P_{\bar 1} = \langle \bar 1\rangle, 1$ is the unit of P, and $\bar 1^2 = 0$. It is easy to see that P is an associative and supercommutative superalgebra, therefore, $E \otimes P \in \mathfrak{M}$. Note that $E \otimes P$ contains a subalgebra $E' = A \otimes 1 + M \otimes \bar 1$. One can easily check that E' is isomorphic to the split null extension of A by M^{op}. Therefore, M^{op} is a $\mathfrak{M}$-superbimodule over A. □

1.12 Noncommutative Jordan representations

Here we briefly recall the definitions of representation theory specific to noncommutative Jordan algebras. Then we formulate the definition of a noncommutative Jordan representation in the spirit of Definition 1.5.3, and prove some technical results in this setting.

Let M be a noncommutative Jordan bimodule over U. Since $M \subseteq E$ is an ideal, M has the Peirce decomposition with respect to e:

$$M = M_0 \oplus M_1 \oplus M_2,$$

where $M_i = M \cap E_i$ is the ith Peirce component of M. Suppose now that U is unital and $e = 1$. Since $U \subseteq E_2(1)$, Peirce relations (1.2.5) imply that the Peirce components $M_i(1)$ are submodules of M.

The relations (1.2.5) imply that M_0 is a *zero bimodule*, that is, all $R_x, L_x, x \in U$ act as zero operators on it. On M_2 we have $L_1 = R_1 = \mathrm{id}$, so M_2 is called a *unital bimodule*. On M_1 we have $R_1 + L_1 = \mathrm{id}$, such bimodules will be called *one-sided*.

Definition 1.5.3 states that U can be considered as a Jordan superalgebra $(U, \circ)$ with generic Poisson bracket $\{\cdot, \cdot\}$. Here we state the definition of a noncommutative Jordan representation in this framework. The idea is clear: given a module M over $U = (U, \cdot) = (U, \circ, \{\cdot, \cdot\})$, we construct the split null extension $E = U \oplus M$, then extend the circle and the bracket products to E and require that the algebra $(E, \circ, \{\cdot, \cdot\})$ be noncommutative Jordan. Finally, we express the conditions for a representation to be noncommutative Jordan by means of operator identities, similarly to (1.1.2). Here are the details:

Let A be a noncommutative Jordan superalgebra. For $x \in A$, introduce operators $R^+_x, R^-_x \in \mathrm{End}(A)$:

$$R^+_x = \frac{R_x + L_x}{2}, \quad R^-_x = \frac{R_x - L_x}{2}.$$

These operators express circle and bracket multiplications:

$$yR^+_x = y \circ x, \quad yR^-_x = [y, x]/2,$$

where $x, y \in A$ (the denominator of 2 appearing in the formula seems unnatural for now, but it will make our future computations easier).

Let M be a noncommutative Jordan bimodule over U and E be the corresponding split null extension. Then for x in U, M is closed under operators $R^+_x, R^-_x \in \mathrm{End}(E)$. Since $M^2 = 0$ as a subalgebra of E, these operators should be understood as extending the circle and the generic Poisson bracket on U to E. Note that

$$L_x = R^+_x - R^-_x, \quad R_x = R^+_x + R^-_x,$$

so to give a structure of a noncommutative Jordan bimodule on a vector space M it suffices to define the operators R^+_x, R^-_x for every $x \in U$.

We can now state the new definition of a noncommutative Jordan representation:

Definition 1.12.1. Let $U = (U, \circ, \{\cdot, \cdot\})$ be a noncommutative Jordan superalgebra, M be a vector superspace, and $R^+, R^- : U \to \mathrm{End}(M)$ two even maps defined by $R^+ : x \mapsto R^+_x, R^- : x \mapsto R^-_x$. Extend the multiplications on U to the split null extension $E = U \oplus M$ as follows:

$$m \circ a = mR^+_a, \quad \{m, a\} = 2mR^-_a, \quad m \circ n = \{m, n\} = 0,$$

where $a \in U, m, n \in M$. Then M is a noncommutative Jordan superbimodule (equivalently, $\{R^+, R^-\}$ is a noncommutative Jordan representation) iff $(E, \circ, \{\cdot,\cdot\})$ is a noncommutative Jordan superalgebra.

By Lemma 1.5.2 it is obvious that this definition is equivalent to the usual definition of a noncommutative Jordan superbimodule (noncommutative Jordan representation). Now we formulate the explicit conditions on the representation $\{R^+, R^-\}$ for it to be noncommutative Jordan.

Let M be a noncommutative Jordan superbimodule over U with the action given by $m \circ a = mR_a^+, \{m, a\} = 2mR_a^-$. By Lemma 1.5.2, $(E, \circ)$ is a Jordan superalgebra. Thus, M is a Jordan module over $J = U^{(+)}$ with the action given by $m \circ a = mR_a^+$.

The relation (1.5.1) is equivalent to two operator relations:

$$[R_a^+, R_b^-] = \frac{1}{2} R_{[a,b]}^+, \tag{1.12.1}$$

$$R_a^- R_b^+ + (-1)^{ab} R_b^- R_a^+ = R_{a\circ b}^-, \tag{1.12.2}$$

where $a, b \in U$. Thus, if $\{R^+, R^-\}$ is a noncommutative Jordan representation, then R^+ must be a Jordan representation and the two relations above must hold. On the other hand, let M be a module over U. Then from Lemma 1.5.2 and definitions of Jordan and noncommutative Jordan bimodule it follows that if M is a Jordan bimodule over $U^{(+)}$ with the action given by $m \circ a = mR_a^+$ and relations (1.12.1), (1.12.2) hold, then the algebra E is noncommutative Jordan, hence, the representation $R^+, R^- : U \to \mathrm{End}(M)$ is noncommutative Jordan. We state what we have just seen as a definition:

Definition 1.12.2. Let $(U, \circ, \{\cdot,\cdot\})$ be a noncommutative Jordan superalgebra, and M be a vector superspace. A representation $R^+, R^- : U \to \mathrm{End}(M)$ is noncommutative Jordan if and only if $R^+ : U \to \mathrm{End}(M)$ is a Jordan representation of $U^{(+)}$ (that is, the relation (1.1.1) holds for R^+) and relations (1.12.1), (1.12.2) hold.

It seems that the approach which we have just constructed works better with describing representations over superalgebras in which the circle and the bracket products are "more natural" than the usual multiplication (for example, noncommutative Jordan superalgebras that are explicitly built as a Jordan superalgebra with a generic Poisson bracket). For superalgebras in which the usual multiplication is more "natural" than the circle and bracket ones (for example, associative and quasiassociative ones), it appears better to stick with the usual definition of a noncommutative Jordan bimodule which was given in the beginning of the section (that is, to work with relations (1.1.2), (1.1.3), and the relations derived from them).

We finish this section by proving some technical results that will be useful later. A large part of the subsequent calculations of the work will be similar, so we note first that from now on we will use the Peirce relations (1.2.5), (1.2.6) in the operator form. For example, if $x \in U_0$, then we can rewrite the relations (1.2.5) in the following way:

$$\begin{aligned} P_0L_x &= P_0L_xP_0, & P_0R_x &= P_0R_xP_0, \\ P_1L_x &= P_1L_xP_1, & P_1R_x &= P_1R_xP_1, \\ P_2L_x &= 0, & P_2R_x &= 0. \end{aligned}$$

We can transform analogously the relations (1.2.5), (1.2.6) for $x \in U_0, U_1, U_2$ and also with the operators R^+, R^- instead of operators L, R. The relations of this type we also call the Peirce relations.

Lemma 1.12.3. *Let $x \in U_1$. Then*

1) $P_0R^-_x = -P_0R^+_{[e,x]}$;
2) $P_2R^-_x = P_2R^+_{[e,x]}$;
3) $P_1R^-_x(P_0 + P_2) = P_1R^+_{[e,x]}(P_0 - P_2)$.

Proof. Consider the relation (1.12.1) with $a = e,\ b = x$: $\frac{1}{2}R^+_{[e,x]} = [R^+_e, R^-_x]$. Multiply it by P_0 on the left:

$$\frac{1}{2}P_0R^+_{[e,x]} = P_0[R^+_e, R^-_x] = -P_0R^-_xR^+_e = -P_0R^-_xP_1R^+_e = -\frac{1}{2}P_0R^-_x,$$

hence, the first relation follows. The second relation is obtained analogously by multiplying this relation on P_2. Now multiply the same relation by P_1 on the left:

$$\begin{aligned} \frac{1}{2}P_1R^+_{[e,x]} &= P_1[R^+_e, R^-_x] = P_1R^-_x\left(\frac{1}{2}\,\mathrm{id} - R^+_e\right) \\ &= P_1R^-_x(P_0 + P_1 + P_2)\left(\frac{1}{2}\,\mathrm{id} - R^+_e\right) = \frac{1}{2}P_1R^-_x(P_0 - P_2). \end{aligned}$$

Since by the Peirce relations $P_1R^+_{[e,x]} = P_1R^+_{[e,x]}(P_0 + P_2)$, the third relation follows. □

Lemma 1.12.4. *Let $x, y, z \in U_1$ be such that $[x, e] = 0, xy \in U_0 + U_2$. Then*

1) $(P_0 + P_2)R^-_x = 0,\ P_1R^-_x = P_1R^-_xP_1$;
2) $P_1R^+_zR^-_x = 0$;
3) $P_1R^-_xR^+_y = 0,\ P_1R^+_{[x,y]} = 0$;
4) $P_1R^-_{x\circ y} = (-1)^{xy}P_1R^-_yR^+_x = P_1R^+_xR^-_y$. *Particularly, if $[y, e] = 0$, then $P_1R^-_{x\circ y} = 0$.*

Proof. The points 1) and 2) follow directly from the previous lemma.

3) The relation (1.2.10) implies that

$$P_1R_xP_1R_y^+ = 0, \quad P_1L_xP_1R_y^+ = 0.$$

Subtracting the second equation from the first and using 1), we obtain $0 = P_1R_x^-P_1R_y^+ = P_1R_x^-R_y^+$. By 1) and the Peirce relations (1.2.6), we have $P_1R_y^+R_x^- = P_1R_y^+(P_0 + P_2)R_x^- = 0$. Now the identity (1.12.1) implies that $P_1R_{[x,y]}^+ = 0$.

4) The relation (1.12.2) and the previous point imply that

$$P_1R_{x\circ y}^- = P_1(R_x^-R_y^+ + (-1)^{xy}R_y^-R_x^+) = (-1)^{xy}P_1R_y^-R_x^+.$$

Now, the relation $P_1R_{[x,y]}^+ = 0$ implies $P_1(R_x^+R_y^- - (-1)^{xy}R_y^-R_x^+) = 0$, and we have the second equality. The second statement follows from the point 2). □

We also state here the following widely known fact. Let J be a unital Jordan superalgebra with an idempotent e and a Peirce decomposition $J = J_0 + J_1 + J_2$ with respect to e. Then $a \mapsto R_a^+$ is a homomorphism of $J_0 + J_2$ into $\mathrm{End}(J_1)^{(+)}$. Hence, for arbitrary unital noncommutative Jordan superalgebra U with unit of degree $\geqslant 2$ we have

$$P_1R_{x\circ y}^+ = P_1(R_x^+ \circ R_y^+). \tag{1.12.3}$$

1.13 Discussion of main results

The main result of this work, obtained in many theorems dispersed throughout the thesis, is as follows:

Main result: *Irreducible finite-dimensional representations of simple finite-dimensional noncommutative Jordan superalgebras of degree $\geqslant 2$ over an algebraically closed field of characteristic 0 are described.*

In fact, for many algebras we obtain more than just a classification of irreducible modules. Let us describe in an informal manner the results of this work. In Section 2.1 we study representations of noncommutative Jordan superalgebras of degree $\geqslant 3$. Theorem 1.4.4 implies that a finite-dimensional central simple noncommutative Jordan superalgebra of degree $\geqslant 3$ is either Jordan or quasiassociative. In Theorem 2.1.1 we prove that all its representations are also respectively Jordan or quasiassociatve and show that finite-dimensional representations of such algebras are completely reducible.

In Section 2.2 we study noncommutative Jordan representations of simple Jordan superalgebras. In Theorem 2.2.2 we describe irreducible unital noncommutative Jordan modules over the superalgebra $J(V,f)$ of a nondegenerate bilinear superform (in particular, we find one large new family of modules). For low-dimensional Jordan superalgebras $D_t, K_3, P(2), Q(2)^{(+)}, K_{10}$ and K_9 we prove in a uniform way that every noncommutative Jordan module over them is Jordan (Theorems 2.2.4 and 2.2.6 to 2.2.9). Moreover, we extend the Kronecker factorization theorem for K_{10} obtained in [MZ03] to noncommutative Jordan case (Theorems 2.2.10 and 2.2.11).

In Chapter 3 we study modules over superalgebras $D_t(\alpha,\beta,\gamma)$ and $K_3(\alpha,\beta,\gamma)$. By Lemma 1.7.1 a superalgebra $D_t(\alpha,\beta,\gamma)$ is isomorphic either to $D_t(\lambda)$ for $\lambda \in \mathbb{F}$ or to $D_t(1/2,1/2,0)$. In Section 3.1 we study representations of these two algebras separately using different methods, but obtain the same result: except for some values of parameters t,α,β,γ every module over $D_t(\alpha,\beta,\gamma)$ is a direct sum of copies of the regular module and its opposite (Theorems 3.1.2 and 3.1.7, see also Theorems 3.1.3, 3.1.8 and 3.1.9 for results for exceptional values of parameters). As a consequence, in Theorem 3.1.10 we classify representations of $K_3(\alpha,\beta,\gamma)$, which has nontrivial indecomposable modules.

In Section 3.2 we use this result to prove the Kronecker theorem for $D_t(\alpha,\beta,\gamma)$. That is, in Theorems 3.2.8 and 3.2.16 we show that if U is a noncommutative Jordan superalgebra containing $D_t(\alpha,\beta,\gamma)$ as a unital subalgebra, then $U \cong D_t(\alpha,\beta,\gamma) \otimes A$ for an associative-commutative superalgebra A (except for some values of α,β,γ). We use this result in Section 3.3 to prove that every noncommutative Jordan module over the superalgebra $Q(2)$ is associative (Theorem 3.3.2) and show the Kronecker factorization theorem for $Q(2)$ (Theorem 3.3.3).

In Chapter 4 we prove a general result which is useful for classification of irreducible representations. In particular, we show that if M is an irreducible noncommutative module over U, then it is either irreducible as Jordan module over $U^{(+)}$ or equals one of its Peirce components (Theorem 4.1.8). We use this result and classifications of irreducible modules over Jordan superalgebras $J(V,f)$ and $K(\Gamma_n)$ to classify irreducible finite-dimensional modules over superalgebras $U(V,f,\star)$ and $K(\Gamma_n,A)$ in Theorems 4.2.6 and 4.3.2.

2 Representations of simple superalgebras of degree $\geqslant 3$ and Jordan superalgebras

In this chapter we study representations of noncommutative Jordan superalgebras of degree $\geqslant 3$ and Jordan superalgebras. Recall that by Theorem 1.4.4 a finite-dimensional central simple noncommutative Jordan superalgebra of degree $\geqslant 3$ is either Jordan or quasiassociative. In Theorem 2.1.1 we prove that all its representations are also respectively Jordan or quasiassociatve and show that finite-dimensional representations of such algebras are completely reducible. Therefore, to find new examples of noncommutative Jordan representations of simple (algebras) one has to study algebras of degree $\leqslant 2$.

In Section 2.2 we study noncommutative Jordan representations of simple Jordan superalgebras. In Theorem 2.2.2 we describe irreducible unital noncommutative Jordan modules over the superalgebra $J(V, f)$ of a nondegenerate bilinear superform (in particular, we find one large new family of modules which will serve as counterexamples and border cases during the text, see, for example, the remark at the end of Section 3.2.1 and Lemma 4.2.2). Next, for low-dimensional Jordan superalgebras $D_t(t \neq 1)$, K_3, $P(2), Q(2)^{(+)}$, K_{10} and K_9 we prove in a uniform way (by checking that they contain D_t for some appropriate value of t or K_3, see Lemma 2.2.3) that every noncommutative Jordan module over them is Jordan (Theorems 2.2.4 and 2.2.6 to 2.2.9). Moreover, for the case of the base field algebraically closed of characteristic 0, we extend the Kronecker factorization theorem for K_{10} obtained in [MZ03] to noncommutative Jordan case (Theorems 2.2.10 and 2.2.11).

2.1 Superalgebras of degree $\geqslant 3$

We begin with the case of degree $\geqslant 3$: we obtain results analogous to these of McCrimmon [McC66]. Basically, there are no new examples of noncommutative Jordan representations in this case.

Theorem 2.1.1. *Let U be a simple finite-dimensional noncommutative Jordan superalgebra of degree $\geqslant 3$ over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then:*

1) *If U is Jordan, then every unital noncommutative Jordan superbimodule over U is Jordan;*

2) *If U is quasiassociative, then for every unital noncommutative Jordan superbimodule M over U the split null extension $E = U \oplus M$ is quasiassociative;*

3) *Every finite-dimensional noncommutative Jordan superbimodule over U is completely reducible.*

Proof. From Theorem 1.4.4 and the classification of simple finite-dimensional associative and Jordan superalgebras it follows that U has a unit. Let $e_1, \ldots, e_n$ be a system of pairwise orthogonal connected idempotents in U which sum to 1. By Lemma 1.3.7 U has the indicator $\phi = \lambda(1-\lambda)$, and since $\mathbb{F}$ is algebraically closed, $\lambda \in \mathbb{F}$. Let M be a finite-dimensional noncommutative Jordan superbimodule over U. We denote by $E = U \oplus M$ the split null extension of U by M. Let $M = M_0 + M_1 + M_2$ be the Peirce decomposition of M relative to the idempotent $1 \in E$. We have already seen that M_i's are submodules of M. Therefore, we only have to prove the complete reducibility of each M_i.

M_0 is a zero submodule, therefore, it is obviously completely reducible (the submodules are one-dimensional subspaces).

Now consider $M_1 = N$, which is a special U-bimodule. With respect to the system of idempotents $e_1, \ldots, e_n$ it has the Peirce decomposition $N = \sum_{i=0}^{n} N_{i0}$. By connectivity, for distinct i, j not equal to 0, there exist $u_{ij}, v_{ij} \in S_{ij}^{[\phi]}$ such that $u_{ij}v_{ij} = (e_i \pm e_j)$. Then, by Lemmas 1.2.6 and 1.3.1 $N_{i0} = (u_{ij}v_{ij})N_{i0} + N_{i0}(u_{ij}v_{ij}) \subseteq N_{i0}^{[\phi]}$, hence,

$$N = S_1^{[\phi]}(1) = N^{[\lambda]}(1) + N^{[1-\lambda]}(1).$$

Since $U \subseteq E_2(1)$, Lemma 1.3.1 implies that $N^{[\lambda]}(1), N^{[1-\lambda]}(1)$ are submodules. Thus, we only have to show that $N^{[\lambda]}(1)$ and $N^{[1-\lambda]}(1)$ are completely reducible. If the indicator of U is $\phi = 1/4$, then by Lemma 1.3.8 U is commutative and $\lambda = 1 - \lambda = 1/2$, then $L_1 = R_1 = 1/2$ (from now on to until the end of the theorem by $L_x, x \in U$ we denote the restriction of the operator L_x to M) thus, for $x \in U$,

$$L_x - R_x = L_{x \cdot 1} - R_{1 \cdot x} = \text{ (by (1.1.4)) } = L_1 L_x - R_x R_1 = \frac{1}{2}(L_x - R_x),$$

hence, $L_x = R_x$ and N is a one-sided Jordan bimodule over U. From [MZ02, Theorem 1] it follows that N is completely reducible. If $\phi \neq 1/4$, we may perform a mutation of $E' = U \oplus N$ so that ϕ becomes 0 and by Lemma 1.3.9 U becomes associative. Since the ideals of $U \oplus N^{[\lambda]}(1)$ and its μ-mutation ($\mu \neq 1/2$) coincide, it is enough to consider the case where U is associative, $\phi = \lambda = 0$. Then for $x \in U$ we have

$$L_x - R_x = L_{x \cdot 1} - R_{1 \cdot x} = \text{ (by (1.1.4)) } = L_1 L_x - R_x R_1 = -R_x,$$

hence $L_x = 0$ and by (1.1.4) we have $R_{xy} = R_x R_y$. Therefore, $N^{[0]}(1)$ is a finite-dimensional right module (analogously, $N^{[1]}(1)$ is a left module) over a simple associative finite-dimensional superalgebra U, and is completely reducible.

Consider now the bimodule M_2. Since U has unity of degree $n \geqslant 3$, so does $E_2 = U + M$. If the indicator ϕ of U is $1/4$, then E_2 is commutative by Lemma 1.3.8.

Hence, M is a Jordan superbimodule over U and is completely reducible by [MZ10, Theorem 8.1]. If $\phi \neq 1/4$ we can mutate E_2 to reduce to the case $\phi = 0$ as in the proof of the coordinatization theorem. Thus, by Lemma 1.3.9, E_2 is associative, and M is a finite-dimensional associative bimodule which is completely reducible by [Pis94]. □

In the rest of the work we study irreducible **unital** representations over simple noncommutative Jordan superalgebras of degree $\leqslant 2$. From now on, if not explicitly said otherwise, by "representation" we mean a unital noncommutative Jordan representation, and by "bimodule" we mean a unital noncommutative Jordan bimodule.

2.2 Noncommutative Jordan representations of simple Jordan superalgebras

In this section we study unital noncommutative Jordan representations of simple Jordan (super)algebras. For the (super)algebra $J(V, f)$ of nondegenerate vector form we find a class of representations that are not Jordan, and prove that an irreducible noncommutative Jordan representation of such (super)algebra is Jordan or belongs to the described class. For low-dimensional superalgebras D_t, K_3, $P(2)$, $Q(2)^{(+)}$, K_{10} and K_9 we develop a unified approach which allows us to prove that any (not just irreducible) noncommutative Jordan representation over them is Jordan. For the Kac superalgebra K_{10} in the case of $\mathbb{F}$ algebraically closed and characteristic zero it means that any unital K_{10}-module is completely reducible with irreducible summands regular module and its opposite [Sht87], so we check that the Kronecker factorization for K_{10} holds in the class of noncommutative Jordan superalgebras.

First of all, we prove some technical statements that we will need later. Clearly, in our setting (studying noncommutative Jordan representations of Jordan algebras) it is easier to use the approach given by Definitions 1.5.3 and 1.12.2.

Lemma 2.2.1. *Let J be a Jordan superalgebra with an even idempotent e and M a noncommutative Jordan bimodule over J. Then the following statements hold*:

1) *Let $x, y \in J_1$. Then* $R_x^+ R_y^- = R_x^- R_y^+ = 0$;

2) *M_1 is closed under the operators of the form $R_a^{\epsilon_1} R_b^{\epsilon_2}$, where $a, b \in J_1, \epsilon_1, \epsilon_2 \in \{+, -\}$*;

3) $R^-_{J_1 \circ J_1} = 0$;

4) *Let $a \in J_0 + J_2, x \in J_1$. Then* $P_1 R_a^- R_x^+ = 0$, $P_1 R^-_{a \circ x} = (-1)^{ax} P_1 R_x^- R_a^+$;

5) *$M' = M_1 R_J^-$ is a J-submodule of M, and $M' \subseteq M_1$.*

Proof.

1) From Lemma 1.12.4 it follows that $(P_0 + P_2)R_x^- R_y^+ = 0$ and $P_1 R_x^+ R_y^- = 0$.

   Since J is commutative, (1.12.1) simplifies to

$$[R_a^+, R_b^-] = 0. \tag{2.2.1}$$

   Therefore,

$$(P_0 + P_2)R_x^+ R_y^- = (-1)^{xy}(P_0 + P_2)R_y^- R_x^+ = 0, \quad P_1 R_x^- R_y^+ = (-1)^{xy} P_1 R_y^+ R_x^- = 0.$$

2) Easily follows from 1), 1) of Lemma 1.12.4 and Peirce relations.

3) Easily follows from 1) and (1.12.2).

4) From (1.12.2) it follows that

$$P_1(R_{a\circ x}^- - (-1)^{ax} R_x^- R_a^+) = P_1 R_a^- R_x^+.$$

   Using 1) of Lemma 1.12.4 and Peirce relations, it is easy to see that the image of the left part lies in M_1, and the image of the right part lies in $M_0 + M_2$, hence the statement follows.

5) We have to check that M' is closed under all operators $R_x^+, R_x^-, x \in J$. Let $x, y \in J_1,\ z, t \in J_0 + J_2$. Then, using 1), 4), (2.2.1) and 1) of Lemma 1.12.4 it is easy to see that

$$\begin{aligned} &M_1 R_x^- R_y^+ = 0, \quad M_1 R_x^- R_y^- \subseteq M', \quad M_1 R_x^- R_z^+ = M_1 R_z^+ R_x^- \subseteq M', \quad M_1 R_x^- R_z^- \subseteq M', \\ &M_1 R_z^- R_x^+ = 0, \quad M_1 R_z^- R_x^- \subseteq M', \quad M_1 R_z^- R_t^+ = M_1 R_t^+ R_z^- \subseteq M', \quad M_1 R_z^- R_t^- \subseteq M', \end{aligned}$$

   hence, M' is a J-submodule of M, and from 1) of Lemma 1.12.4 it follows that $M' \subseteq M_1$. □

Now we are ready to describe noncommutative Jordan bimodules over simple Jordan superalgebras.

2.2.1 Representations of the superalgebra $J(V, f)$.

Jordan representations of simple superalgebra $J(V, f)$ of symmetric bilinear form over an algebraically closed field of characteristic 0 were considered in [MZ10]. Here we describe irreducible noncommutative Jordan representations of this superalgebra with the restriction that $V_{\bar{0}} \neq 0$ and that $\mathbb{F}$ allows square root extraction (these conditions ensure that the degree of this superalgebra is 2, see below).

Let $J = J(V, f)$ be the superalgebra of nondegenerate supersymmetric bilinear form f on a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $V_{\bar{0}} \neq 0$. If $V_{\bar{0}} \neq 0$, there exists an element

$u \in V_{\bar{0}}$ with $f(w,w) = \alpha \neq 0$. If $\mathbb{F}$ allows square root extraction, then J has a nonzero idempotent $e = 1/2 + v$, where $v = w/2\sqrt{\alpha} \in V_0$ is such that $f(v,v) = 1/4$, and it is easy to see that any idempotent of J is of this form. The superalgebra J has the following Peirce decomposition relative to e:

$$J_0 = \left\langle \frac{1}{2} - v \right\rangle, \quad J_1 = \{u \in V : f(u,v) = 0\}, \quad J_2 = \left\langle \frac{1}{2} + v \right\rangle.$$

Let M be an irreducible noncommutative Jordan bimodule over J. The one-dimensionality of Peirce spaces J_0, J_2 and 1) of Lemma 1.12.4 imply that operators $R_x^-, x \in J$, act nonzero only on M_1. Now, by 5) of Lemma 2.2.1, $M' = M_1 R_J^-$ is a submodule of M. If M' is zero, M is commutative (note that in case of algebraically closed field of characteristic 0, finite-dimensional Jordan irreducible representations of $J(V,f)$ were described in [MZ10, Section 7]). If $M' = M$, then it is easy to see that $R_u^+ = 0$ for any $u \in V$ (for $u \in \langle v\rangle^\perp = J_1$ this follows from Peirce relations, and for v it follows from the fact that $M = M_1$, thus, $R_e^+ = \mathrm{id}/2$). One can check that the action of J given by

$$R_1^+ = \mathrm{id}, \quad R_v^+ = 0, \;\; v \in V,$$

is Jordan. Moreover, if one chooses a basis $\{v_i\}$ of V, then it is easy to see that relations (1.12.1), (1.12.2) hold for any set of operators $R_{v_i}^- \in \mathrm{End}(M)$. Hence, we have the following description of irreducible noncommutative Jordan bimodules over J:

Theorem 2.2.2. *Let $J(V,f)$ be the superalgebra of nondegenerate supersymmetric bilinear form f on vector space $V \neq V_{\bar{1}}$ with a basis $\{v_i\}$ over a field $\mathbb{F}$ which allows square root extraction and M be an irreducible noncommutative Jordan superbimodule over M. Then one of the following holds:*

1) *M is a Jordan superbimodule over J;*

2) *$M = M_1, MR_V^+ = 0$, $R_{v_i}^-$ are linear operators on M such that M has no invariant subspaces with respect to all of $R_{v_i}^-$.*

Remark. One can check that the formulas in Theorem 2.2.2 define a noncommutative Jordan representation not only for $J(V,f)$, but for all algebras $U(V,f,\star)$. In Section 4.2 we shall see that the modules defined in this way are essentially the only nontrivial irreducible modules for the algebras $U(V,f,\star)$.

2.2.2 Representations of superalgebras D_t and K_3.

Finite-dimensional unital Jordan representations of D_t and K_3 were studied in [MZ06] and [Tru05] in the case of characteristic 0, and in [Tru08] in the case of characteristic $p \neq 2$. One-sided representations of D_t were studied in [MZ02]. In this section we study noncommutative irreducible Jordan bimodules over Jordan superalgebras D_t and K_3.

First of all, we prove the following useful technical lemma. It will allow us to consider noncommutative Jordan representations of many Jordan superalgebras in a uniform way.

Lemma 2.2.3. *Let J be a unital Jordan superalgebra containing $D_t = \langle e_1, e_2, x, y\rangle$, $t \neq 1$ as a unital subalgebra (or containing $K_3 = \langle e_1, x, y\rangle$). Then the following statements hold:*

1) *There is no nonzero unital noncommutative Jordan bimodule M over J such that $M = M_1(e_1)$;*
2) *If M is a unital noncommutative Jordan bimodule over J such that $(M_0(e_1) + M_2(e_1))R_a^- = 0$ for all $a \in J_0(e_1) + J_2(e_1)$, then M is commutative;*
3) *If $J_1(e_1)^2 = J_0(e_1) + J_2(e_1)$, then every noncommutative bimodule over J is commutative.*

Proof. 1) Suppose first that $J \supseteq D_t, t \neq 1$ as a unital subalgebra and let M be a bimodule such that $M = M_1(e_1)$. Then D_t also acts unitally on M. Peirce relations (1.2.5), (1.2.6) imply that

$$R_x^+ = 0, \quad R_y^+ = 0, \quad R_{e_1}^+ = R_{e_2}^+ = \mathrm{id}/2.$$

From the definition of a noncommutative Jordan representation it follows that the action of D_t on M defined above should be Jordan. However, substituting in (1.1.1) $a = x,\ b = e_1,\ c = y$ we have $\dfrac{1}{2} = \dfrac{1}{2}\left(\dfrac{1}{2} + \dfrac{t}{2}\right)$, hence, $t = 1$, which contradicts the lemma condition. If a unital Jordan superalgebra J contains K_3, then it contains its unital hull D_0 as a unital subalgebra, and the result follows.

2) From 5) of Lemma 2.2.1 it follows that $M' = M_1(e_1)R_J^-$ is a J-submodule of M, and $M' = M_1'$. Hence, the previous point implies that $M_1(e_1)R_J^- = 0$. From 1) of Lemma 1.12.4 it follows that $(M_0 + M_2)R_x^- = 0$ for all $x \in J_1(e_1)$, therefore, the lemma condition implies that $R_x^- = 0$ for all $x \in J$, and M is commutative.

3) Follows from 3) of Lemma 2.2.1 and the previous point. □

Now we can describe the representations of the superalgebras J. Let M be a noncommutative bimodule over $J = D_t$. Suppose first that $t \neq 1$. Peirce relations and the one-dimensionality of Peirce spaces J_0, J_2 imply that $(P_0 + P_2)R_a^- = 0$ for $a \in J_0 + J_2$. Then 2) of the previous lemma implies that every noncommutative Jordan bimodule over J is Jordan. Consider now the case $t = 1$. In this case it is easy to see that D_1 is a Jordan superalgebra of nondegenerate symmetric form on the space $V = \langle e_1 - e_2, x, y\rangle$ with $V_{\bar{0}} = \langle e_1 - e_2\rangle,\ V_{\bar{1}} = \langle x, y\rangle$, and irreducible bimodules over superalgebras of superforms were classified in the previous subsection. Hence, we have proved the following results:

Theorem 2.2.4. *Every noncommutative Jordan bimodule over $D_t, t \neq 1$, is Jordan.*

Theorem 2.2.5. *Let M be an irreducible noncommutative Jordan bimodule over D_1. Then one of the following holds*:

1) *M is a Jordan bimodule*;
2) *$M = M_1(e_1)$, $MR^+_x = MR^+_y = 0$, $R^-_x, R^-_y, R^-_{e_1} = -R^-_{e_2}$ are linear operators on M such that M has no invariant subspaces with respect to all of them.*

From 3) of Lemma 2.2.3 immediately follows the following theorem:

Theorem 2.2.6. *Every noncommutative Jordan bimodule over K_3 is Jordan.*

2.2.3 Representations of the superalgebra $P(2)$.

Recall that the simple Jordan superalgebra $P(n) \cong \mathrm{H}(M_{n,n}(\mathbb{F}), \mathrm{strp})$ is the Jordan superalgebra of symmetric elements of the simple associative superalgebra $M_{n,n}(\mathbb{F})$ with respect to the transpose superinvolution

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{strp}} = \begin{pmatrix} D^t & -B^t \\ C^t & A^t \end{pmatrix},$$

where $A, B, C, D \in M_n(\mathbb{F})$, and t is the transpose. Jordan representations of $P(n)$ were described in [MZ10, Section 3] in the case of algebraically closed field of characteristic 0 and $n \geqslant 2$, and in [MSZ10] in the case of arbitrary field and $n \geqslant 3$.

In the paper [PS13] it was proved that $P(2)$ does not admit a nonzero generic Poisson bracket. The degree of $P(n)$ is exactly n, so here we will only deal with noncommutative Jordan representations of $P(2)$ (the superalgebra $P(1)$ is not simple). Representations of $P(2)$ are characterized in the following theorem:

Theorem 2.2.7. *All noncommutative Jordan representations of $P(2)$ are Jordan.*

Proof. Let $e_1 = e_{11} + e_{33}$ be an idempotent of $P(2) = J$. We have the following Peirce decomposition relative to e_1:

$$J_0 = \langle e_2 = e_{22} + e_{44},\ f = e_{42}\rangle,$$
$$J_1 = \langle a = e_{12} + e_{43},\ b = e_{21} + e_{34},\ c = e_{14} - e_{23},\ d = e_{32} + e_{41}\rangle,$$
$$J_2 = \langle e_1 = e_{11} + e_{33},\ e = e_{31}\rangle.$$

For the sake of convenience we provide below the multiplication table of $P(2)$ (zero products are omitted).

Note that J has a (unital) subalgebra $J' = \langle e_1, e_2, c, d\rangle$ which is isomorphic to D_{-1}. Also it is easy to see that $J_1(e_1) \circ J_1(e_1) = J_0(e_1) + J_2(e_1)$. Thus, 3) of Lemma 2.2.3 implies that every noncommutative Jordan bimodule over J is Jordan. □

Table 1 – Multiplication table of $P(2)$

$\circ$	e_1	e_2	e	f	a	b	c	d
e_1	e_1		e		$a/2$	$b/2$	$c/2$	$d/2$
e_2		e_2		f	$a/2$	$b/2$	$c/2$	$d/2$
e	e				$d/2$		$b/2$	
f		f				$d/2$	$-a/2$	
a	$a/2$	$a/2$	$d/2$			$(e_1+e_2)/2$		f
b	$b/2$	$b/2$		$d/2$	$(e_1+e_2)/2$			e
c	$c/2$	$c/2$	$-b/2$	$a/2$				$(e_1-e_2)/2$
d	$d/2$	$d/2$			f	e	$(e_2-e_1)/2$	

2.2.4 Representations of the superalgebra $Q(2)^{(+)}$

Finite-dimensional Jordan represenations of a simple Jordan superalgebra $Q(2)^{(+)}$ were studied in [MSZ10]. In particular, irreducible bimodules were described and it was proved that if the characteristic of the field is zero or > 3, then every finite-dimensional representation over $Q(2)^{(+)}$ is completely reducible. Here we describe noncommutative Jordan bimodules over $Q(2)^{(+)}$.

The associative superalgebra $Q(n)$ is a subalgebra of the full matrix superalgebra $M_{n,n}(\mathbb{F})$ with the following grading:

$$Q(n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A \in M_n(\mathbb{F}) \right\}, \quad Q(n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, B \in M_n(\mathbb{F}) \right\}.$$

We can consider $Q(n)$ as the double of the $n \times n$ matrix algebra:

$$Q(n) = M_n(\mathbb{F}) \oplus \overline{M_n(\mathbb{F})},$$

where $\overline{M_n(\mathbb{F})}$ is an isomorphic copy of $M_n(\mathbb{F})$ as a vector space. The grading on $Q(n)$ is then

$$Q(n)_{\bar{0}} = M_n(\mathbb{F}), \quad Q(n)_{\bar{1}} = \overline{M_n(\mathbb{F})}.$$

The multiplication in $Q(n)$ is defined in the following way:

$$a \cdot b = ab, \quad \bar{a} \cdot b = a \cdot \bar{b} = \overline{ab}, \quad \bar{a} \cdot \bar{b} = ab,$$

where $a, b \in M_n(\mathbb{F})$, and ab is the usual matrix product. Therefore, the multiplication is defined in $Q(n)^{(+)}$ in the following way:

$$a \circ b = a \circ b, \quad a \circ \bar{b} = \bar{a} \circ b = \overline{a \circ b}, \quad \bar{a} \circ \bar{b} = [a, b]/2,$$

where $a, b \in M_n(\mathbb{F})$, and $a \circ b$, $[a, b]$ are the matrix Jordan product and commutator.

Regarding to the idempotent e_{11}, we have the following Peirce decomposition of $Q(2)^{(+)} = J$:

$$J_0 = \langle e_{22}, \overline{e_{22}} \rangle, \quad J_1 = \langle e_{12}, e_{21}, \overline{e_{12}}, \overline{e_{21}} \rangle, \quad J_2 = \langle e_{11}, \overline{e_{11}} \rangle.$$

Jordan superalgebras $Q(n)^{(+)}$ are simple and of degree exactly n for all n. Note that J has a unital subsuperalgebra $J' = \langle e_{11}, e_{22}, \overline{e_{12}}, \overline{e_{21}}\rangle$, which is isomorphic to D_{-1}.

Now we are ready to study the representations of J. Let M be a unital noncommutative Jordan bimodule over J, and let $M = M_0 + M_1 + M_2$ be its Peirce decomposition with respect to e_{11}. Substituting $a = e_{12}$, $b = \overline{e_{21}}$ in (1.12.2), by 3) of Lemma 2.2.1 we have $0 = R^-_{\overline{e_{11}}} + R^-_{\overline{e_{22}}}$. Therefore, Peirce relations imply that

$$P_0 R^-_{\overline{e_{22}}} = -P_0 R^-_{\overline{e_{11}}} = 0.$$

Analogously, $P_2 R^-_{\overline{e_{11}}} = 0$. Hence,

$$(P_0 + P_2) R^-_{\overline{e_{11}}} = (P_0 + P_2) R^-_{\overline{e_{22}}} = 0.$$

Combining this with Peirce relations, we have $(M_0 + M_2) R^-_{\overline{J_0 + J_2}} = 0$. Thus, 2) of Lemma 2.2.3 implies that M is Jordan. We state this result as a theorem:

Theorem 2.2.8. *Every noncommutative Jordan bimodule over $Q(2)^{(+)}$ is Jordan.*

2.2.5 Representations of superalgebras K_{10} and K_9.

Representations of the Kac superalgebra K_{10} were studied in the case of algebraically closed field of characteristic 0. In the article [Sht87] Shtern proved that any (Jordan) representation of K_{10} is completely reducible, with irreducible summands being the regular module and its opposite. Later, Martínez and Zelmanov used his results to prove the Kronecker factorization theorem for K_{10}.

The superalgebras K_{10} and K_9 do not admit nonzero generic Poisson brackets (see [PS19, Theorem 4]). In this section we classify noncommutative Jordan representations of K_{10} and K_9. Also we prove the Kronecker factorization theorem in the noncommutative Jordan case for K_{10} if the base field is algebraically closed and of characteristic 0.

Recall the definitions of the simple Jordan superalgebras K_{10} and K_9 over a field $\mathbb{F}$. The even and odd parts of K_{10} are respectively

$$A = A_1 \oplus A_2 = \langle e_1, uz, vz, uw, vw\rangle \oplus \langle e_2\rangle \text{ and } M = \langle u, v, w, z\rangle$$

The even part A is a direct sum of ideals (of A). The unity in A_1 is $e_1, e_2^2 = e_2$, and $e_i \cdot m = 1/2m$ for every $m \in M$. The multiplication table of K_{10} is as follows:

$$\begin{array}{rclrclrclrcl} u \cdot z & = & uz, & u \cdot w & = & uw, & v \cdot z & = & vz, & v \cdot w & = & vw, \\ z \cdot w & = & e_1 - 3e_2, & uz \cdot w & = & -u, & vz \cdot w & = & -v, & uz \cdot vw & = & 2e_1, \end{array}$$

and the remaining nonzero products may be obtained either by applying the skew-symmetries $z \leftrightarrow w$, $u \leftrightarrow v$, or by the substitution $z \leftrightarrow u$, $w \leftrightarrow v$. If the characteristic of $\mathbb{F}$

is not 3, the superalgebra K_{10} is simple, but in the case of characteristic 3 it contains a simple subsuperalgebra $K_9 = A_1 \oplus M$.

Consider first the case of the superalgebra $J = K_{10}$. One can see that it contains a subsuperalgebra $\langle e_1, e_2, z, w\rangle$, which is isomorphic to D_{-3}. The Peirce decomposition of J with respect to e_1 is the following:

$$J_0(e_1) = A_2, \quad J_1(e_1) = M, \quad J_2(e_1) = A_1.$$

Let M be a noncommutative Jordan bimodule over K_{10}. From the multiplication table it is easy to see that

$$uz, vz, uw, vw \in J_1(e_1)^2,$$

thus, 3) of Lemma 2.2.1 implies that

$$R^-_{uz} = 0, \quad R^-_{vz} = 0, \quad R^-_{uw} = 0, \quad R^-_{vw} = 0 \text{ on } M.$$

Moreover, since e_1 and e_2 are orthogonal idempotents that sum to 1, it is obvious that $(M_0(e_1) + M_2(e_1))R^-_{e_1} = (M_0(e_1) + M_2(e_1))R^-_{e_2} = 0$. Hence, 2) of Lemma 2.2.3 implies that M is Jordan.

Now consider the case of $J = K_9$. One can see that it contains a subsuperalgebra $\langle e_1, z, w\rangle \cong K_3$. The Peirce decomposition of J with respect to e_1 is the following:

$$J_0(e_1) = 0, \quad J_1(e_1) = M, \quad J_2(e_1) = A_1.$$

Let M be a noncommutative Jordan bimodule over K_9. Again, 3) of Lemma 2.2.1 implies that

$$R^-_{uz} = 0, \quad R^-_{vz} = 0, \quad R^-_{uw} = 0, \quad R^-_{vw} = 0 \text{ on } M.$$

Also, it is obvious that $(M_0(e_1) + M_2(e_1))R^-_{e_1} = 0$. Hence, 2) of Lemma 2.2.3 implies that M is Jordan. We have proved the following theorem:

Theorem 2.2.9. *Every unital noncommutative Jordan bimodule over K_{10} or K_9 is Jordan.*

2.2.6 Kronecker factorization theorem for K_{10}

A well known theorem, due to Wedderburn, states that if B is an associative algebra and A is a finite dimensional central simple subalgebra of B that contains its unit element, then B is the tensor product of algebras A and Z where Z is the subalgebra of elements of B that commute with every element of A. The statements of this type are usually called *Kronecker factorization theorems*. In the paper [Jac54] Jacobson proved the Kronecker factorization theorem for the split Cayley-Dickson algebra and the exceptional simple Albert algebra. In the case of superalgebras, López-Díaz and Shestakov proved the Kronecker factorization theorem for simple alternative superalgebras $B(1, 2)$ and

$B(4,2)$ [LDS02] and for simple Jordan superalgebras $H_3(B(1,2)), H_3(B(4,2))$ obtained from them [LDS05]. Martínez and Zelmanov used Shtern's classification of irreducible modules over K_{10} to prove the Kronecker factorization theorem this algebra in the case where $\mathbb{F}$ is algebraically closed of characteristic 0. In this subsection we extend their result to noncommutative Jordan case:

Theorem 2.2.10. *Suppose that the base field $\mathbb{F}$ is algebraically closed and is of characteristic 0. Let U be a noncommutative Jordan superalgebra which contains $J = K_{10}$ as a unital subsuperalgebra. Then U is supercommutative and $U \cong Z \otimes J$, where Z is an associative-supercommutative superalgebra.*

Proof. Let $U = U_0 + U_1 + U_2$ be the Peirce decomposition of U with respect to e_1. We need to show that U is supercommutative. Since U is a Jordan bimodule over J, the point 1) of Lemma 1.12.4 implies that $[U_0 + U_2, U_1] = 0,\ [U_1, U_1] \subseteq U_1$.

We prove that we can take $K = \{u, z\}$ in Lemma 1.2.5. Since U is a Jordan superbimodule over J, the Peirce relations (1.2.6) imply that $KU_1 \subseteq U_0 + U_2$. Suppose now that $K \circ a = 0$ for $a \in U_1$. The description of Jordan superbimodules over J ([Sht87]) implies that U is a direct sum of submodules isomorphic either to $\mathrm{Reg}(K_{10})$ or $\mathrm{Reg}(K_{10})^{\mathrm{op}}$. Hence, we can assume that $a \in \mathrm{Reg}(J)$ or $\mathrm{Reg}(J)^{\mathrm{op}}$. Let

$$a = \alpha u' + \beta v' + \gamma w' + \delta z' \in \mathrm{Reg}(J)$$

(we have added primes to distinguish elements of the regular superbimodule from elements of J). Then

$$u \circ a = \beta(e_1' - 3e_2') + \gamma(uw)' + \delta(uz)' = 0,$$

thus, $\beta = \gamma = \delta = 0$. Hence, $z \circ a = -\alpha(uz)' = 0$, and $\alpha = 0$. Therefore, $a = 0$. The case where $a \in \mathrm{Reg}(J)^{\mathrm{op}}$ is considered analogously. We proved that K satisfies the conditions of Lemma 1.2.5, thus, $U_1^2 \subseteq U_0 + U_2$. Hence, $[U_1, U_1] = 0$ and $[U, U_1] = UR^-_{U_1} = 0$.

The structure of U as a bimodule over K_{10} implies that

$$U_0 + U_2 = (J_1 \circ U_1) + (J_1 \circ U_1) \circ (J_1 \circ U_1).$$

Applying (1.12.2) twice, we see that

$$R^-_{J_1 \circ U_1} = 0, \quad R^-_{(J_1 \circ U_1) \circ (J_1 \circ U_1)} = 0.$$

Therefore, $R^-_{U_0 + U_2} = 0$ and U is commutative. By the result of Martínez and Zelmanov, $U \cong Z \otimes J$ for an associative-supercommutative superalgebra Z. □

In fact we can drop the assumption that K_{10} contains the unity of U.

Theorem 2.2.11. *Let U be a noncommutative Jordan superalgebra that contains $J = K_{10}$ as a subsuperalgebra. Then $U \cong (Z \otimes J) \oplus U'$ is a direct sum of ideals, where Z is an associative-supercommutative superalgebra.*

Proof. Let $U = U_0 + U_1 + U_2$ be the Peirce decomposition of U with respect to the unity of J. Then $U^{(+)}$ is a Jordan superalgebra and U_1 is a one-sided Jordan bimodule over J with the action induced by multiplication in U. But since J is an exceptional simple Jordan superalgebra, U_1 must be zero (see, for example, [MZ03, Theorem 2]). Hence, $U = U_0 + U_2$. Applying the previous theorem to $J \subseteq U_2$, we get the desired result. □

3 Representations of low-dimensional algebras

In this chapter we study modules over low-dimensional simple noncommutative Jordan superalgebras $D_t(\alpha,\beta,\gamma)$, $K_3(\alpha,\beta,\gamma)$ and $Q(2)$. By Lemma 1.7.1 for $\alpha,\beta,\gamma\in\mathbb{F}$ a superalgebra $D_t(\alpha,\beta,\gamma)$ is isomorphic either to $D_t(\lambda)$ for $\lambda\in\mathbb{F}$ or to $D_t(1/2,1/2,0)$. In Section 3.1 we study representations of these two algebras separately using different methods described in Section 1.12, but obtain the same result: except for some values of parameters t,α,β,γ every module over $D_t(\alpha,\beta,\gamma)$ is a direct sum of copies of the regular module and its opposite (Theorems 3.1.2 and 3.1.7). For $t=0$ we classify all indecomposable bimodules Theorems 3.1.3 and 3.1.8, and for the algebra $D_{-1}(1/2,1/2,0)$ we only describe its irreducible finite-dimensional modules (Theorem 3.1.9). As a consequence, in Theorem 3.1.10 we classify all indecomposable modules over of $K_3(\alpha,\beta,\gamma)$.

In Section 3.2 we use the results of Section 3.1 to prove the Kronecker theorem for $D_t(\alpha,\beta,\gamma)$. That is, in Theorems 3.2.8 and 3.2.16 we show that if U is a noncommutative Jordan superalgebra containing $D_t(\alpha,\beta,\gamma)$ as a unital subalgebra, then $U\cong D_t(\alpha,\beta,\gamma)\otimes A$ for an associative-commutative superalgebra A (except for some values of α,β,γ). We use this result in Section 3.3 to prove that every noncommutative Jordan module over the superalgebra $Q(2)$ is associative (Theorem 3.3.2) and show the Kronecker factorization theorem for $Q(2)$ (Theorem 3.3.3).

3.1 Representations of $D_t(\alpha,\beta,\gamma)$ and $K_3(\alpha,\beta,\gamma)$

In this section we describe representations of the superalgebras $D_t(\alpha,\beta,\gamma)$ and simple nonunital superalgebras $K_3(\alpha,\beta,\gamma)$, except for two cases:

1) the case $\alpha=1/2,\ \beta=\gamma=0$ is the case of Jordan superalgebra D_t (resp., K_3), which was considered in the previous chapter;

2) the case $t=1$, because in this case D_t is of the type $U(V,f,\star)$. Indeed, its symmetrized superalgebra D_1 is a Jordan superalgebra of nondegenerate symmetric form on the space $V=\langle e_1-e_2,x,y\rangle$ with $V_{\bar 0}=\langle e_1-e_2\rangle$, $V_{\bar 1}=\langle x,y\rangle$. Representations of such superalgebras will be considered in Section 4.2.

In what follows we assume that the base field $\mathbb{F}$ allows square root extraction. Lemma 1.7.1 then tells that for $\alpha,\beta,\gamma\in\mathbb{F}$, a superalgebra $D_t(\alpha,\beta,\gamma)$ belongs to one of the families: $D_t(\lambda,0,0)$ or $D_t(1/2,1/2,0)$, the first family consisting of “almost associative” (up to a mutation), and the second of "almost commutative" superalgebras. We consider the two cases separately, using two different approaches given in the previous section. The results, however, are the same: except for some special values of parameters, every unital bimodule over a superalgebra $D_t(\alpha, \beta, \gamma)$ is completely reducible, with irreducible summands being the regular bimodule and its opposite. Also we classify irreducible representations over non-unital superalgebras $K_3(\alpha, \beta, \gamma)$. Note that in almost any case we make no dimensionality or characteristic restriction.

3.1.1 Representations of $D_t(\lambda),\ \lambda \neq 1/2$

In this subsection we classify all noncommutative Jordan representations over the superalgebra $D_t(\lambda),\ \lambda \neq 1/2,\ t \neq 1$. First of all we describe a certain procedure, which we occasionally refer to as "module mutation", which in our case permits us to consider only the representations of the superalgebra $D_t(1)$, in which case the computations are drastically simplified.

Module mutation. Let U be a noncommutative Jordan (super)algebra and M be a (super)bimodule over U. Then the split null extension $E = U \oplus M$ is a noncommutative Jordan superalgebra. Let $\lambda \neq 1/2$ be an element of the base field, and consider the λ-mutation $E^{(\lambda)}$, which is equal to $U^{(\lambda)} \oplus M$. It is again a noncommutative Jordan superalgebra, and M is an ideal of E such that $M^2 = 0$. Hence, we may consider $E^{(\lambda)}$ as the split null extension of $U^{(\lambda)}$ by the module M. Therefore, M (with the action twisted by mutation) is a noncommutative Jordan bimodule over $U^{(\lambda)}$. This construction is invertible: since $\lambda \neq 1/2$, there exists $\mu \in \mathbb{F}$ such that $\lambda \odot \mu = 1$. Mutating back again by μ, we obtain the original algebra E (that is, we recover the original action of U on M). Therefore, it is equivalent to study representations of a noncommutative Jordan superalgebra U and any its nontrivial mutation $U^{(\lambda)}, \lambda \neq 1/2$. It is also clear that the module mutation preserves irreducibility and direct sum decomposition.

Now we apply this construction to our case: if $1/2 \neq \lambda \in \mathbb{F}$, then one can check that the μ-mutation (where $\lambda \odot \mu = 1$) of $D_t(\lambda)$ is equal to $D_t(1)$. Hence, it suffices to study representations of the superalgebra $D_t(1)$.

For the reference, we provide the multiplication table of the algebra $D_t(1) = D$:

$$D = D_{\bar{0}} \oplus D_{\bar{1}}, \quad D_{\bar{0}} = \langle e_1, e_2 \rangle, \quad D_{\bar{1}} = \langle x, y \rangle,$$
$$e_i^2 = e_i, \quad e_1 e_2 = 0 = e_2 e_1,$$
$$e_1 x = x = x e_2, \quad x e_1 = 0 = e_2 x, \quad e_1 y = 0 = y e_2, \quad e_2 y = y = y e_1,$$
$$xy = 2e_1, \quad yx = -2t e_2, \quad x^2 = 0 = y^2.$$

The Peirce decomposition of D relative to e_1 is as follows:

$$D_0 = \langle e_2 \rangle, \quad D_1 = \langle x, y \rangle, \quad D_2 = \langle e_1 \rangle.$$

Now, let M be a unital bimodule over D and let $M = M_0 + M_1 + M_2$ be its Peirce decomposition with respect to e_1. Our goal is to obtain enough operator relations derived from defining identities (1.1.2), (1.1.3) and Peirce relations in operator form, and then see that they in fact completely define the structure of a module over D.

Apply Lemma 1.3.2 to the split null extension $E = U \oplus M$. Since $x \in D_1^{[1]}$, by 1) of Lemma 1.3.2 we have $(\mathrm{id} - R_{e_1})R_x + L_{e_1}L_x = 0$. Multiplying this relation on P_1 by the left, we get

$$P_1 L_{e_1}(L_x + R_x) = 0. \tag{3.1.1}$$

Multiplying the same relation by P_0 and P_2 on the left, we have

$$P_0 R_x = 0, \tag{3.1.2}$$
$$P_2 L_x = 0. \tag{3.1.3}$$

Now, by 2) of Lemma 1.3.2 we have $L_x(\mathrm{id} - L_{e_1}) + R_x R_{e_1} = 0$. Multiplying this relation by P_0 and P_2 on the left and using relations (3.1.2), (3.1.3), we have

$$P_0 L_x R_{e_1} = 0, \quad P_0 L_x L_{e_1} = P_0 L_x, \tag{3.1.4}$$
$$P_2 R_x R_{e_1} = 0, \quad P_2 R_x L_{e_1} = P_2 R_x. \tag{3.1.5}$$

Analogously, since $y \in U_1^{[0]}$, by Lemma 1.3.2 we obtain the following relations:

$$P_0 L_y = 0, \tag{3.1.6}$$
$$P_1 R_{e_1}(R_y + L_y) = 0,$$
$$P_2 R_y = 0, \tag{3.1.7}$$
$$P_0 R_y L_{e_1} = 0, \quad P_0 R_y R_{e_1} = P_0 R_y, \tag{3.1.8}$$
$$P_2 L_y L_{e_1} = 0, \quad P_2 L_y R_{e_1} = P_2 L_y. \tag{3.1.9}$$

Note that relations (3.1.3), (3.1.6) imply that

$$(P_0 + P_2)(L_{e_1} L_y - L_y) = 0, \quad (P_0 + P_2)(L_{e_1} L_x) = 0. \tag{3.1.10}$$

Combining the relations (3.1.1) and (3.1.4) with Peirce relations, we have

$$\begin{aligned} P_0 L_x(L_x + R_x) &= P_0 L_x P_1(R_{e_1} + L_{e_1})(L_x + R_x) \\ &= P_0 L_x R_{e_1}(L_x + R_x) + P_0 L_x P_1 L_{e_1}(L_x + R_x) = 0. \end{aligned} \tag{3.1.11}$$

Analogously, we have

$$P_2 R_x(L_x + R_x) = 0, \tag{3.1.12}$$
$$P_0 R_y(L_y + R_y) = 0, \quad P_2 L_y(L_y + R_y) = 0.$$

Let $a = b = x$ in (1.1.4): $L_x^2 = R_x^2$. Multiply this relation by P_0 on the left. Then (3.1.2) and (3.1.11) imply that

$$P_0 L_x^2 = 0, \quad P_0 L_x R_x = 0. \tag{3.1.13}$$

Analogously, (3.1.3) and (3.1.12) imply that

$$P_2 R_x^2 = 0, \quad P_2 R_x L_x = 0. \tag{3.1.14}$$

Analogously, substituting $a = b = y$ in (1.1.4), we obtain

$$P_0 R_y^2 = 0, \quad P_0 R_y L_y = 0, \tag{3.1.15}$$
$$P_2 L_y^2 = 0, \quad P_2 L_y R_y = 0. \tag{3.1.16}$$

Let $a = e_1,\ b = x,\ c = y$ in (1.2.1):

$$2R_{e_1} - 2R_{e_1}(R_{e_1} + tR_{e_2}) + (R_x + L_x)(L_{e_1}L_y - L_y) - (R_y + L_y)(L_{e_1}L_x) = 0.$$

Multiply this relation on P_1 on the left:

$$P_1((2-2t)(R_{e_1} - R_{e_1}^2) + (R_x + L_x)(L_{e_1}L_y - L_y) - (R_y + L_y)(L_{e_1}L_x)) = 0.$$

By Peirce relations (1.2.6) $P_1(R_x + L_x)P_1 = P_1(R_y + L_y)P_1 = 0$. Hence, by (3.1.10) the previous relation reduces to

$$(2-2t)P_1(R_{e_1} - R_{e_1}^2) = 0.$$

Since $t \neq 1$, we have $P_1(R_{e_1} - R_{e_1}^2) = 0$. Hence, $P_1 R_{e_1} = (P_1 R_{e_1})^2$ and $P_1 L_{e_1} = (P_1 L_{e_1})^2$ are orthogonal projections that sum to P_1. Thus, $M_1 = M_1^{[0]} + M_1^{[1]}$. Further on we use this fact without mentioning it.

Let $a = y,\ b = x$ in (1.1.4): $2tL_{e_2} - L_x L_y = 2R_{e_1} - R_x R_y$. Multiplying this relation by Peirce projections on the left, using relations (3.1.2) and (3.1.3) and Peirce relations in operator form we have

$$P_0 L_x L_y = 2tP_0, \tag{3.1.17}$$
$$P_1(2(1-t)R_{e_1} - R_x R_y + L_x L_y) = 0, \tag{3.1.18}$$
$$P_2 R_x R_y = 2P_2. \tag{3.1.19}$$

Analogously, substituting $a = x,\ b = y$ in (1.1.4), we have

$$P_0 R_y R_x = -2tP_0, \tag{3.1.20}$$
$$P_1(2(1-t)L_{e_1} - R_y R_x + L_y L_x) = 0, \tag{3.1.21}$$
$$P_2 L_y L_x = -2P_2. \tag{3.1.22}$$

Substituting $e = e_1,\ z \in \{x, y\},\ w \in M_1$, and alternatively, $e = e_1,\ z \in M_1,\ w \in \{x, y\}$ in (1.2.8), we obtain the following operator relations:

$$P_1 R_x P_2 = 0, \quad P_1 L_x P_0 = 0, \tag{3.1.23}$$

$$P_1 R_y P_0 = 0, \quad P_1 L_y P_2 = 0. \tag{3.1.24}$$

The relation (1.2.10) and the multiplication table of $D_t(1)$ imply that for $a, b \in U_1$ we have

$$P_1 L_a P_1 (L_b + R_b) = 0, \quad P_1 R_a P_1 (L_b + R_b) = 0. \tag{3.1.25}$$

Consider the relation (3.1.21):

$$\begin{aligned} 0 &= P_1(2(1-t)L_{e_1} - R_y R_x + L_y L_x) = (\text{by } (3.1.24)) \\ &= 2(1-t)P_1 L_{e_1} + P_1(L_y(P_1 + P_0)L_x - R_y(P_1 + P_2)R_x) \\ &= P_1(2(1-t)L_{e_1} - R_y P_2 R_x + L_y P_0 L_x) + P_1(L_y P_1 L_x - R_y P_1 R_x). \end{aligned}$$

Consider the second summand:

$$\begin{aligned} P_1(L_y P_1 L_x - R_y P_1 R_x) &= (\text{since } P_1(R_y + L_y)P_1 = 0) \\ &= P_1 L_y P_1 (L_x + R_x) = (\text{by } (3.1.25)) = 0. \end{aligned}$$

Therefore, we have

$$2(1-t)P_1 L_{e_1} = P_1(R_y P_2 R_x - L_y P_0 L_x). \tag{3.1.26}$$

Analogously, considering (3.1.18) and using (3.1.23) and (3.1.25), we have

$$2(1-t)P_1 R_{e_1} = P_1(R_x P_0 R_y - L_x P_2 L_y) \tag{3.1.27}$$

Multiply (3.1.21) on the left by $P_0 L_x$:

$$2(1-t)P_0 L_x L_{e_1} = P_0 L_x (R_y R_x - L_y L_x).$$

Combining this relation with (3.1.4) and (3.1.17), we have

$$2P_0 L_x = P_0 L_x R_y R_x. \tag{3.1.28}$$

Multiply this relation by L_y on the right, and combine it with (3.1.17):

$$4tP_0 = 2P_0 L_x L_y = P_0 L_x R_y R_x L_y. \tag{3.1.29}$$

Analogously, multiplying (3.1.18) on the left by $P_0 R_y$ and using (3.1.8) and (3.1.20), we have

$$2P_0 R_y = -P_0 R_y L_x L_y. \tag{3.1.30}$$

The relation (3.1.17) implies that $P_0L_xL_yP_1 = 0$. Hence, Peirce relations imply that

$$P_0L_xR_yP_1 = P_0L_xP_1R_yP_1 = -P_0L_xP_1L_yP_1 = 0.$$

Thus, from (3.1.24) it follows that

$$P_0L_xR_y = P_0L_xR_yP_2. \tag{3.1.31}$$

Now, multiply (3.1.21) on the left by P_2R_x: $2(1-t)P_2R_xL_{e_1} = P_2R_x(R_yR_x - L_yL_x)$. Combining this with (3.1.5) and (3.1.19), we have

$$2tP_2R_x = P_2R_xL_yL_x. \tag{3.1.32}$$

The relation (3.1.19) implies that $P_2R_xR_yP_1 = 0$, hence, Peirce relations imply that $P_2R_xL_yP_1 = 0$. Hence, from (3.1.24) it follows that

$$P_2R_xL_y = P_2R_xL_yP_0. \tag{3.1.33}$$

Analogously, multiplying the relation (3.1.18) on the left by P_2L_y and using (3.1.9) and (3.1.22), we have

$$-2tP_2L_y = P_2L_yR_xR_y. \tag{3.1.34}$$

Consider the identity (1.1.3) for $a = x,\ b = y$: $R_xL_y + L_yR_x = R_yL_x + L_xR_y$. Multiplying it on the left by P_0 and P_2 and using (3.1.2), (3.1.6), (3.1.3) and (3.1.7), we have

$$P_0(R_yL_x + L_xR_y) = 0, \tag{3.1.35}$$

$$P_2(R_xL_y + L_yR_x) = 0. \tag{3.1.36}$$

Multiply (3.1.36) by R_y on the right:

$$P_2R_xL_yR_y = -P_2L_yR_xR_y = (\text{by } (3.1.34)) = 2tP_2L_y. \tag{3.1.37}$$

Using the relations derived above we construct submodules of M isomorphic to the regular module or its opposite. Note that for $t = 0$, the algebra $D_0(1)$ acts irreducibly on the algebra $K_3(1)$ by restricting the regular representation to the submodule generated by e_1.

Lemma 3.1.1.

1) *Let m be a homogeneous nonzero element of M_2. Then*

$$\mathrm{Mod}(m) = \langle m, mR_x, mL_y, mR_xL_y\rangle,$$

the elements $mR_x, mL_y \in M_1$ are linearly independent, $mR_xL_y \in M_0$ and

$$mR_xL_yL_x = 2tmR_x, \quad mR_xL_yR_y = 2tmR_y.$$

(a) *If* $t \neq 0$, *then* $mR_xL_y \neq 0$ *and* $\mathrm{Mod}(m)$ *is isomorphic to* $\mathrm{Reg}(D)$ *or its opposite depending on the parity of* m;

(b) *If* $t = 0$ *and* $mR_xL_y = 0$, *then* $\mathrm{Mod}(m)$ *is isomorphic to* $K_3(1)$ *or the opposite module*;

(c) *If* $t = 0$ *and* $mR_xL_y \neq 0$, *then* $\mathrm{Mod}(m)$ *is isomorphic to the module* $D^0(1) = \langle e_1', x', y', e_2' \rangle$ (*or its opposite*) *with the action given as follows*:

$$x'L_y = -y'R_x = e_2',$$

the only nonzero actions on e_2' *are*

$$e_2'L_{e_2} = e_2'R_{e_2} = e_2'$$

(*that is*, $\mathrm{Mod}(e_2')$ *is isomorphic to* $\mathrm{Reg}(D_0(1))/K_3(1)$), *and all other actions coincide with the actions on* $\mathrm{Reg}(D_0(1))$. *Moreover*, $D^0(1)$ *is indecomposable with the unique nontrivial submodule isomorphic to* $\mathrm{Reg}(D)/K_3(1)$.

2) *Let* m *be a homogeneous nonzero element of* M_0. *If* $t \neq 0$, *then* $\mathrm{Mod}(m)$ *is isomorphic either to* $\mathrm{Reg}(D)$ *or its opposite. If* $t = 0$, *then* $\mathrm{Mod}(M)$ *is isomorphic to* $\mathrm{Reg}(D)$ *or* $\mathrm{Reg}(D_0(1))/K_3(1)$ *or their opposites.*

Proof.

1) Let $0 \neq m$ be a homogeneous element of M_2. The relations (3.1.3) and (3.1.7) imply that

$$mL_x = 0, \quad mR_y = 0,$$

and relations (3.1.19), (3.1.22) imply that the elements $mR_x, mL_y \neq 0$. Relations (3.1.5) and (3.1.9) imply that the elements mR_x and mL_y are linearly independent and that multiplying mR_x and mL_y by e_1 on both sides does not give new elements in $\mathrm{Mod}(m)$. From (3.1.14) and (3.1.16) it follows that

$$mR_x^2 = 0, \quad mR_xL_x = 0, \quad mL_y^2 = 0, \quad mL_yR_y = 0.$$

From relations (3.1.19) and (3.1.22) we get

$$mR_xR_y = 2m = -mL_yL_x.$$

Relations (3.1.36) and (3.1.33) imply that

$$mL_yR_x = -mR_xL_y \in M_0.$$

From relations (3.1.2), (3.1.6) we infer that

$$mR_xL_yR_x = 0, \quad mR_xL_yL_y = 0.$$

Relations (3.1.32) and (3.1.37) show respectively that

$$mR_xL_yL_x = 2tmR_x, \quad mR_xL_yR_y = 2tmL_y.$$

Therefore, $\mathrm{Mod}(m)$ is equal to $\langle m, mL_y, mR_x, mR_xL_y\rangle$, and $mR_xL_y \neq 0$ if $t \neq 0$. Now, one can easily check that if $t \neq 0$, then $\mathrm{Mod}(m)$ is isomorphic to the regular $D_t(1)$-bimodule or its opposite. Indeed, one identifies

$$m \leftrightarrow e_1, \quad mR_x \leftrightarrow x, \quad mL_y \leftrightarrow y, \quad mR_xL_y/2t \leftrightarrow e_2$$

if m is even, and analogously if m is odd.

Now let $t = 0$. If $mR_xL_y = 0$, then the isomorphism between $\mathrm{Mod}(m)$ and $K_3(1)$ (or $K_3(1)^{\mathrm{op}}$) is completely analogous to the one constructed above. If $mR_xL_y \neq 0$, then it is easy to see that $\mathrm{Mod}(m)$ is isomorphic to $D^0(1)$ or its opposite. One can check the relations (1.2.1), (1.1.3) and see that $D^0(1)$ is indeed a noncommutative Jordan module. The space $\langle e_2'\rangle$ is the unique nontrivial submodule of $D^0(1)$ because a Peirce-homogeneous element of degree not zero generates the whole module. Thus, $D^0(1)$ is isomorphic neither to $\mathrm{Reg}(D)$ (they have different socles) nor to $K_3(1) \oplus \mathrm{Reg}(D_0(1))/K_3(1)$ (because $D^0(1)$ has no 3-dimensional submodule).

2) Let $0 \neq m \in M_0$. Analogously to the previous point, using (3.1.2), (3.1.6), (3.1.4), (3.1.8), (3.1.13), (3.1.15), (3.1.17), (3.1.20), (3.1.31), (3.1.30) and (3.1.28) one can see that $\mathrm{Mod}(m) = \langle m, mL_x, mR_y, mL_xR_y\rangle$.

   If $t \neq 0$, then (3.1.29) implies that $\mathrm{Mod}(m) = \mathrm{Mod}(mL_xR_y)$, and (3.1.31) implies that $mL_xR_y \in M_2$. Hence, $\mathrm{Mod}(m)$ is generated by an element of M_2, therefore, by the previous item, it is either the regular $D_t(1)$-bimodule or its opposite.

   If $t = 0$, then from (3.1.28), (3.1.30) and (3.1.35) it follows that the elements mL_x, mR_y and mL_xR_y are either all zero or all nonzero. If they are all nonzero, then $\mathrm{Mod}(m)$ is isomorphic to $\mathrm{Reg}(D_t(1))$. If they are all zero, then $\dim \mathrm{Mod}(m) = 1$ and it is isomorphic to $\mathrm{Reg}(D_0(1))/K_3(1)$. □

Now we are ready to describe the representations of $D_t(\lambda)$. First, we consider the case $t \neq 0$:

Theorem 3.1.2. *Let M be a unital bimodule over $D_t(\lambda)$, $t \neq 0, 1, \lambda \neq 1/2$. Then M is completely reducible and the irreducible summands of M are isomorphic to the regular $D_t(\lambda)$-bimodule or its opposite.*

Proof. As we noted in the beginning of the section, module mutation preserves irreducibility and direct sum decomposition, so we may only consider the case $\lambda = 1$. Since the regular bimodule over $D_t(1) = D$ and its opposite are irreducible, to prove the theorem it suffices to show that the submodule generated by any $m \in M$ is a sum of homomorphic images of the bimodules listed above. It is easy to see that

$$\mathrm{Mod}(m) = \mathrm{Mod}(mP_0, mP_1, mP_2) = \mathrm{Mod}(mP_0) + \mathrm{Mod}(mP_1) + \mathrm{Mod}(mP_2).$$

Hence, we can assume that m is Peirce-homogeneous. Analogously, we can assume that $m \in M_{\bar{0}} \cup M_{\bar{1}}$.

If $m \in M_0 \cup M_2$, the previous lemma implies that $\mathrm{Mod}(m)$ is isomorphic either to $\mathrm{Reg}(D)$ or $\mathrm{Reg}(D)^{\mathrm{op}}$. Suppose that $m \in M_1$. Since $M_1 = M_1^{[0]} \oplus M_1^{[1]}, \mathrm{Mod}(m) = \mathrm{Mod}(mL_{e_1}, mR_{e_1})$. Thus, we can assume that $m \in M_1^{[0]} \cup M_1^{[1]}$. If $m \in M_1^{[1]}$, then (3.1.26) implies that $\mathrm{Mod}(m) = \mathrm{Mod}(mR_yP_2, mL_yP_0)$, therefore, $\mathrm{Mod}(m)$ can be generated by elements of M_0 and M_2, hence, it satisfies the above claim. Analogously, if $m \in M_1^{[0]}$, then (3.1.27) implies that $\mathrm{Mod}(m) = \mathrm{Mod}(mR_xP_0, mL_xP_2)$. $\square$

Now consider the case $t = 0$. Note that $D_0(\lambda)$ is not simple: it contains an ideal $\langle e_1, x, y\rangle$, which is isomorphic to a simple nonunital superalgebra $K_3(\lambda)$. Hence, the regular bimodule over $D_0(\lambda)$ is not irreducible, but indecomposable. Moreover, the module $D^0(\lambda)$, obtained as the λ-mutation of the module $D^0(1)$ from Lemma 3.1.1 is also indecomposable.

Theorem 3.1.3. *Let M be an indecomposable bimodule over $D_0(\lambda), \lambda \neq 1/2$. Then M is isomorphic to one of the following bimodules*:

1) $\mathrm{Reg}(D_0(\lambda))$ *or its opposite*;

2) $K_3(\lambda)$ *with the action restricted from the* $\mathrm{Reg}(D_0(\lambda))$ *or its opposite*;

3) $\mathrm{Reg}(D_0(\lambda))/K_3(\lambda)$ *or its opposite*;

4) $D^0(\lambda)$ *or its opposite.*

Proof. We prove that any D-module M is a direct sum of the above modules. Again we can only consider the case $\lambda = 1$. Consider the subspace $K = \ker L_xR_y \cap M_0$. Let I_0^1 be a complement of a base of K to a base of M_0. By Lemma 3.1.1 $\mathrm{Mod}(m_i)$ is isomorphic to either $\mathrm{Reg}(D_0(1))$ or its opposite for all $i \in I_0^1$. Recall that by (3.1.31) $M_0L_xR_y \subseteq M_2$ and let M_2' be a vector space complement of $M_0L_xR_y$ to M_2. Let I_2^1 be a basis of $\ker R_xL_y \cap M_2'$ and I_2^2 be a basis of a vector space complement of $\ker R_xL_y \cap M_2'$ to M_2'. Then by Lemma 3.1.1 we have

$$\mathrm{Mod}(m_i) \cong \begin{cases} K_3(1) \text{ or } K_3(1)^{\mathrm{op}}, & i \in I_2^1, \\ D^0(1) \text{ or } D^0(1)^{\mathrm{op}}, & i \in I_2^2 \end{cases}.$$

By construction of the module $D^0(1)$, we have $M_2'R_xL_y \subseteq K$. Let M_0' be a vector space complement of $M_2'R_xL_y$ to K, and let I_0^2 be a base of M_0'. By Lemma 3.1.1 we have

$\mathrm{Mod}(m_i) \cong \mathrm{Reg}(D_0(1))/K_3(1)$ for all $i \in I_0^2$. Consider the sum

$$\overline{M} = \sum_{i \in I_0^1} \mathrm{Mod}(m_i) + \sum_{i \in I_2^1} \mathrm{Mod}(m_i) + \sum_{i \in I_2^2} \mathrm{Mod}(m_i) + \sum_{i \in I_0^2} \mathrm{Mod}(m_i)$$

We prove that this sum is direct. By construction, no module in the above sum can lie completely in the sum of others. The modules $\mathrm{Mod}(m_i), i \in I_2^1 \cup I_0^2$, are irreducible, so we only have to check that the modules $\mathrm{Mod}(m_i), i \in I_2^2 \cup I_0^1$, do not intersect with all other ones. Suppose that for $j \in I_2^2$ we have

$$\mathrm{Mod}(m_j) \bigcap \left(\sum_{i \in I_0^1} \mathrm{Mod}(m_i) + \sum_{i \in I_2^1} \mathrm{Mod}(m_i) + \sum_{\substack{i \in I_2^2 \\ j \neq i}} \mathrm{Mod}(m_i) + \sum_{i \in I_0^2} \mathrm{Mod}(m_i) \right) \neq 0.$$

Since the only nontrivial submodule of $\mathrm{Mod}(m_j)$ is $\langle m_j L_x R_y \rangle \subseteq K \subseteq M_0$ and $M_2' L_x R_y \cap M_0' = 0$, we must have

$$\sum_{i \in I_2^2} \mathrm{Mod}(m_i) \bigcap \left(\sum_{i \in I_0^1} \mathrm{Mod}(m_i) + \sum_{i \in I_2^1} \mathrm{Mod}(m_i) + \sum_{i \in I_0^2} \mathrm{Mod}(m_i) \right) = 0,$$

and

$$\mathrm{Mod}(m_j) \bigcap \sum_{\substack{i \in I_2^2 \\ j \neq i}} \mathrm{Mod}(m_i) \neq 0 \Rightarrow m_j L_x R_y = \sum_{k=1}^{n} \alpha_i m_i L_x R_y$$

for some $k = 1, \ldots, n \in I_2^2,\ \alpha_i \in \mathbb{F},\ i_k \neq j$ for all k. But this implies that $m_j - \sum_{i=1}^{k} \alpha_i m_i \in \ker L_x R_y \cap M_2'$, which is a contradiction with the definition of I_2^2. The proof for the modules $\mathrm{Mod}(m_i), i \in I_0^1$ is analogous: the only nontrivial submodule in $\mathrm{Mod}(m_i)$ is isomorphic to $\langle m L_x, m R_y, m L_x R_y \rangle$, and $m L_x R_y \in M_2 \backslash M_2'$, which implies that

$$\sum_{i \in I_0^1} \mathrm{Mod}(m_i) \bigcap \left(\sum_{i \in I_0^2} \mathrm{Mod}(m_i) + \sum_{i \in I_2^1} \mathrm{Mod}(m_i) + \sum_{i \in I_2^2} \mathrm{Mod}(m_i) \right) = 0.$$

The rest of the proof is analogous to the case of $\mathrm{Mod}(m_i), i \in I_2^2$.

The module $M' = M/\overline{M}$ is equal to its Peirce 1-component. Hence, $P_0 = 0,\ P_2 = 0$ in M'. Sum the relations (3.1.26) and (3.1.27):

$$2P_1 = P_1(R_y P_2 R_x - L_y P_0 L_x + R_x P_0 R_y - L_x P_2 L_y).$$

From this relation it follows that $P_1 = 0$, therefore, $M' = 0$ and $M = \overline{M}$. □

3.1.2 Representations of $D_t(1/2, 1/2, 0)$

Let $D = D_t(1/2, 1/2, 0)$. This algebra is very close to being commutative (see the multiplication table below). Therefore, we study its representations using the approach given by Definitions 1.5.3 and 1.12.2, that is, interpreting it as a superalgebra with Jordan and bracket products and using the R^+ and R^- operators.

We start with the multiplication table for D:

$$\begin{array}{rclrcll} e_i^2 &=& e_i,\ \ i = 1,2, & e_1 \circ e_2 &=& [e_1, e_2] & = 0, \\ e_1 \circ x &=& e_2 \circ x = x/2, & [e_1, x] &=& -[e_2, x] & = y, \\ e_1 \circ y &=& e_2 \circ y = y/2, & [e_1, y] &=& [e_2, y] & = 0, \\ [x, x] &=& -2(e_1 - te_2), & [y, y] &=& 0, & \\ x \circ y &=& e_1 + te_2, & [x, y] &=& 0. & \end{array}$$

Fix an idempotent $e = e_1 \in D$. Then $D_0 = D_0(e_1) = \langle e_2 \rangle$, $D_1 = D_1(e_1) = \langle x, y \rangle$, $D_2 = D_2(e_1) = \langle e_1 \rangle$. Let M be a unital bimodule over D, and let $M = M_0 \oplus M_1 \oplus M_2$ be its Peirce decomposition with respect to e_1.

First we prove the following proposition that will allow us to reduce the study of $D_t(1/2, 1/2, 0)$-bimodules to the study of Jordan bimodules over D_t.

Proposition 3.1.4.

1) *The operators $R^-_a, a \in D$ lie in the enveloping associative algebra of the Jordan representation $R^+ \colon D^{(+)} \to \mathrm{End}(M)$. The expressions for R^- operators of basis elements e_1, e_2, x, y do not depend on M;*

2) *A subspace $N \subseteq M$ is a D-submodule if and only if it is a submodule with respect to the representation R^+;*

3) *M is irreducible if and only if it is irreducible with respect to R^+;*

4) *Two bimodules over D are isomorphic if and only if they are isomorphic as Jordan bimodules over $D^{(+)}$ with the symmetrized action.*

Proof. To prove the first point, we need to express the operators $R^-_a, a \in D$ as polynomials in operators $R^+_a, a \in D$. By Lemma 1.12.4 we have

$$(P_0 + P_2)R^-_y = 0, \quad P_1 R^-_y P_1 = P_1 R^-_y. \tag{3.1.38}$$

By Item 2) we have

$$P_0 R^-_x = -P_0 R^+_y, \tag{3.1.39}$$

$$P_2 R^-_x = P_2 R^+_y, \tag{3.1.40}$$

$$P_1 R^-_x (P_0 + P_2) = P_1 R^+_y (P_0 - P_2). \tag{3.1.41}$$

Note that since $[x,y]=[y,y]=0$ and $[x,x]=-2(e_1-te_2)$, from (1.2.10) it follows that

$$P_1R_a^-P_1R_b^+=0,\ \text{where } a,b\in D_1. \tag{3.1.42}$$

Substituting $a=y,\ b=y$ in (1.12.1), we have $[R_y^+,R_y^-]=0$. Multiplying this relation by Peirce projections on the left and using (3.1.38) and Peirce relations, one can see that

$$R_y^+R_y^-=0,\quad R_y^-R_y^+=0. \tag{3.1.43}$$

Substituting $a=y,\ b=x$ in (1.12.1), we have $[R_y^+,R_x^-]=0$. Multiply this relation by P_0+P_2 on the left and by P_1 on the right:

$$0=(P_0+P_2)(R_y^+R_x^-+R_x^-R_y^+)P_1=(P_0+P_2)R_y^+R_x^-P_1. \tag{3.1.44}$$

Substituting $a=y,\ b=x$ in (1.12.2), we have $R_y^-R_x^+-R_x^-R_y^+=-R_{e_1+te_2}^-$. Multiplying this relation by P_0 and P_2 on the left and using (3.1.38), (3.1.39), and (3.1.40), we have

$$\begin{aligned}0&=P_0R_x^-R_y^+=-P_0(R_y^+)^2,\\ 0&=P_2R_x^-R_y^+=P_2(R_y^+)^2,\end{aligned}$$

thus,

$$(P_0+P_2)(R_y^+)^2=0. \tag{3.1.45}$$

Substituting $a=x,\ b=y$ in (1.12.1), we have $[R_x^+,R_y^-]=0$. Multiplying this relation by Peirce projections on the left and using (3.1.38) and (3.1.42), one has

$$R_x^+R_y^-=0,\quad R_y^-R_x^+=0. \tag{3.1.46}$$

Substituting $a=x,\ b=x$ in (1.12.1), we have

$$[R_x^+,R_x^-]=-R_{e_1-te_2}^+. \tag{3.1.47}$$

Multiply this relation by P_0 on the left and by P_0+P_1 on the right:

$$\begin{aligned}tP_0&=P_0(R_x^+R_x^-+R_x^-R_x^+)(P_0+P_1)\\ &=(\text{by } (3.1.39))=P_0(R_x^+P_1R_x^-(P_0+P_1)-R_y^+R_x^+P_0)\\ &=(\text{by } (3.1.41))=P_0((R_x^+R_y^+-R_y^+R_x^+)P_0+R_x^+R_x^-P_1),\end{aligned}$$

hence,

$$P_0(R_x^+R_y^+-R_y^+R_x^+)P_0=tP_0, \tag{3.1.48}$$

$$P_0R_x^+R_x^-P_1=0. \tag{3.1.49}$$

Analogously, multiplying the relation (3.1.47) by P_2 on the left and by $P_1 + P_2$ on the right, we have

$$P_2R_x^+R_x^-P_1 = 0, \tag{3.1.50}$$
$$P_2(R_x^+R_y^+ - R_y^+R_x^+)P_2 = P_2. \tag{3.1.51}$$

Multiplying the relation (3.1.47) by P_1 on the left, we have

$$\begin{aligned}
-\frac{1-t}{2}P_1 &= P_1(R_x^+R_x^- + R_x^-R_x^+)\\
&= (\text{by } (3.1.42)) = P_1(R_x^+(P_0+P_2)R_x^- + R_x^-(P_0+P_2)R_x^+)\\
&= (\text{by } (3.1.39), (3.1.40), (3.1.41)) = P_1(R_x^+(P_2-P_0)R_y^+ + R_y^+(P_0-P_2)R_x^+).
\end{aligned}$$

Therefore (recall that $t \neq 1$),

$$P_1 = \frac{2}{1-t}P_1(R_x^+(P_0-P_2)R_y^+ + R_y^+(P_2-P_0)R_x^+). \tag{3.1.52}$$

Multiply (3.1.52) by R_y^- on the right:

$$\begin{aligned}
P_1R_y^- &= \frac{2}{1-t}P_1(R_x^+(P_0-P_2)R_y^+R_y^- + R_y^+(P_2-P_0)R_x^+R_y^-)\\
&= (\text{by } (3.1.43), (3.1.46)) = 0.
\end{aligned} \tag{3.1.53}$$

Multiply (3.1.52) by $R_{e_1}^-$ on the right:

$$\begin{aligned}
P_1R_{e_1}^- &= \frac{2}{1-t}P_1(R_x^+(P_0-P_2)R_y^+R_{e_1}^- + R_y^+(P_2-P_0)R_x^+R_{e_1}^-) = (\text{by } (1.12.1))\\
&= \frac{2}{1-t}P_1(R_x^+(P_0-P_2)R_{e_1}^-R_y^+ + R_y^+(P_2-P_0)(R_{e_1}^-R_x^+ - \frac{1}{2}R_y^+))\\
&= \frac{1}{1-t}P_1R_y^+(P_0-P_2)R_y^+.
\end{aligned} \tag{3.1.54}$$

Multiply (3.1.52) on $R_x^-P_1$ on the right:

$$\begin{aligned}
P_1R_x^-P_1 &= \frac{2}{1-t}P_1(R_x^+(P_0-P_2)R_y^+R_x^-P_1 + R_y^+(P_2-P_0)R_x^+R_x^-P_1)\\
&= (\text{by } (3.1.44), (3.1.49), (3.1.50)) = 0.
\end{aligned}$$

Therefore,

$$P_1R_x^- = P_1R_x^-(P_0+P_2) = (\text{by } (3.1.41)) = P_1R_y^+(P_0-P_2). \tag{3.1.55}$$

Now, one can see that

$$\begin{aligned}
R_x^- &= (P_0+P_1+P_2)R_x^- = (\text{by } (3.1.39), (3.1.40), (3.1.55))\\
&\qquad = -P_0R_y^+ + P_1R_y^+(P_0-P_2) + P_2R_y^+,\\
R_{e_1}^- &= -R_{e_2}^- = P_1R_{e_1}^- = (\text{by } (3.1.54)) = \frac{1}{1-t}P_1R_y^+(P_0-P_2)R_y^+,\\
R_y^- &= (P_0+P_1+P_2)R_y^- = (\text{by } (3.1.38), (3.1.53)) = 0.
\end{aligned}$$

These relations and the fact that Peirce projections P_i are polynomials in $R^+_{e_1}$ imply that the operators $R^-_a, a \in D$ lie in the enveloping associative algebra of the Jordan representation $R^+ \colon D \to \mathrm{End}(M)$. Also it is clear that the operators $R^-_{e_1}, R^-_{e_2}, R^-_x, R^-_y$ do not depend on the module M. Therefore, the first point is now proved. It follows that the structure of M as a noncommutative Jordan superbimodule is completely determined by its structure as a Jordan superbimodule over $D^{(+)}$. The other points follow immediately from this statement. □

Consider first the case $t \neq -1$. In this case we have the following result:

Lemma 3.1.5 ([Tru05]). *Let $t \neq -1$. Then the operators*

$$E = \frac{2}{1+t}(R^+_x)^2, \quad F = \frac{2}{1+t}(R^+_y)^2, \quad H = \frac{2}{1+t}(R^+_x R^+_y + R^+_y R^+_x)$$

form a basis of the simple Lie algebra $\mathfrak{sl}_2$, that is, $[E,H] = 2E$, $[F,H] = -2F$, $[E,F] = H$.

From Peirce relations it follows that $M_0 + M_2$ is invariant under E, F and H. Hence, (3.1.45) and the multiplication table of $\mathfrak{sl}_2$ imply that

$$(P_0+P_2)(R^+_x)^2 = 0, \quad (P_0+P_2)(R^+_y)^2 = 0, \quad (P_0+P_2)(R^+_x R^+_y + R^+_y R^+_x) = 0. \quad (3.1.56)$$

As in the previous subsection, with the aid of the relations above we can find submodules in M that are isomorphic to the regular one or its opposite. Note that $D_0(1/2,1/2,0)$ acts irreducibly on $K_3(1/2,1/2,0)$ of which it is the unital hull.

Lemma 3.1.6. *Let $t \neq -1, 1$.*

1) *Let m be a nonzero homogeneous element in M_2. Then*

$$\mathrm{Mod}(m) = \langle m, mR^+_x, mR^+_y, mR^+_x R^+_y P_0 \rangle.$$

Moreover, mR^+_x and mR^+_y are linearly independent, and

$$m(R^+_x)^2 = m(R^+_y)^2 = 0, \quad mR^+_x R^+_y P_0 R^+_x = tmR^+_x/2, \quad mR^+_x R^+_y P_0 R^+_y = tmR^+_y/2.$$

   (a) *If $t \neq 0$, $\mathrm{Mod}(m)$ is isomorphic to $\mathrm{Reg}(D)$ or $\mathrm{Reg}(D)^{\mathrm{op}}$;*

   (b) *If $t = 0$ and $mR^+_x R^+_y = 0$, then $\mathrm{Mod}(m)$ is isomorphic to $K_3(1/2,1/2,0)$;*

   (c) *If $t = 0$ and $mR^+_x R^+_y \neq 0$, then $\mathrm{Mod}(m) \cong D^0(1/2,1/2,0) = \langle e_1', x', y', e_2' \rangle$ (or its opposite) with the action given as follows:*

$$x'R^+_y = -y'R^+_x = e_1' + e_2', \quad x'R^-_x = -e_1' + e_2',$$

the only nonzero action on e_2' is $e_2 R^+_{e_2'} = e_2'$ (that is, $\mathrm{Mod}(e_2')$ is isomorphic to $\mathrm{Reg}(D_0(1/2,1/2,0))/K_3(1/2,1/2,0)$), and all other actions coincide with the actions on $\mathrm{Reg}(D)$. Moreover, $D^0(1/2,1/2,0)$ is indecomposable with the unique nontrivial submodule isomorphic to $\mathrm{Reg}(D)/K_3(1/2,1/2,0)$.

2) *Let m be a nonzero homogeneous element in M_0. If $t \neq 0$, then* $\mathrm{Mod}(m)$ *is isomorphic to* $\mathrm{Reg}(D)$ *or* $\mathrm{Reg}(D)^{\mathrm{op}}$. *If $t = 0$, then* $\mathrm{Mod}(m)$ *is isomorphic to* $\mathrm{Reg}(D)$ *or* $\mathrm{Reg}(D)/K_3(1/2, 1/2, 0)$ *or their opposites.*

Proof.

1) Relations (3.1.56) and (3.1.51) imply that

$$m(R_x^+)^2 = m(R_y^+)^2 = 0, \quad mR_x^+R_y^+P_2 = -mR_y^+R_x^+P_2 = m/2.$$

Hence, $mR_x^+, mR_y^+ \neq 0$. Suppose that $mR_x^+ = \alpha mR_y^+$ for some $\alpha \in \mathbb{F}$. Acting by $R_x^+P_2$ on this relation and using (3.1.56) we see that $-\alpha/2\, m = 0$, hence, $\alpha = 0$, a contradiction. Therefore, mR_x^+ and mR_y^+ are linearly independent.

Consider the identity that holds for every Jordan superalgebra:

$$R_{(a,b,c)} = -(-1)^{bc}[[R_a, R_c], R_b]. \tag{3.1.57}$$

Consider this identity for the symmetrized split null extension $E^{(+)} = (D \oplus M)^{(+)}$. Substituting in it $a = x,\ b = y,\ c = x$ we get $\dfrac{1+t}{2}R_x^+ = [(R_x^+)^2, R_y^+]$. Applying this relation on m and using (3.1.56), we get

$$\begin{aligned} m[(R_x^+)^2, R_y^+] &= -mR_y^+(R_x^+)^2 = mR_x^+R_y^+(P_0 + P_2)R_x^+ \\ &= \frac{m}{2}R_x^+ + mR_x^+R_y^+P_0R_x^+ = \frac{1+t}{2}mR_x^+. \end{aligned}$$

Denote $n = mR_x^+R_y^+P_0$. Then the relation above implies that $nR_x^+ = tmR_x^+/2$. Analogously, substituting $a = y,\ b = x,\ c = y$ in (3.1.57) and applying the resulting relation on m, we get $nR_y^+ = tmR_y^+/2$. Thus, the space $\langle m, mR_x^+, mR_y^+, n\rangle$ is closed under all R^+ operators. Proposition 3.1.4 implies that it is also closed under all R^- operators and $\langle m, mR_x^+, mR_y^+, n\rangle = \mathrm{Mod}(m)$.

Suppose that $t \neq 0$. Then, since $nR_x^+ = tmR_x^+/2$, n must be nonzero. Hence, $\mathrm{Mod}(m)$ is isomorphic as a Jordan superbimodule to the regular $D^{(+)}$-bimodule or its opposite. Indeed, one identifies

$$m \leftrightarrow e_1, \quad mR_x^+ \leftrightarrow x/2, \quad mR_y^+ \leftrightarrow y/2, \quad n \leftrightarrow te_2/2$$

if m is even, and analogously if m is odd. Therefore, by Proposition 3.1.4 it is isomorphic to the regular D-bimodule or its opposite.

Now, let $t = 0$. If $n = 0$, then the isomorphism between $\mathrm{Mod}(m)$ and $K_3(1/2, 1/2, 0)$ or its opposite is obtained from the isomorphism above by erasing the last line. If $n \neq 0$, then the mapping

$$m \mapsto e_1, \quad mR_x^+ \mapsto x/2, \quad mR_y^+ \mapsto y/2, \quad n \mapsto e_2'/2$$

gives an isomorphsim between $\mathrm{Mod}(m)$ an $D^0(1/2,1/2,0)$ if m is even, and analogously if m is odd. One can check the relations (3.1.57), (1.12.1) and (1.12.2) to see that $D^0(1/2,1/2,0)$ is indeed noncommutative Jordan. Because Peirce projections are polynomials in $R^+_{e_1}$, it is easy to see that $D^0(1/2,1/2,0)$ is indecomposable with the unique nontrivial submodule equal to

$$\langle e_2' \rangle = \mathrm{Reg}(D_0(1/2,1/2,0))/K_3(1/2,1/2,1/2),$$

hence, it is isomorphic neither to $\mathrm{Reg}(D_0(1/2,1/2,0))$ nor to $K_3(1/2,1/2,1/2) \oplus \mathrm{Reg}(D_0(1/2,1/2,0))/K_3(1/2,1/2,1/2)$.

2) Suppose that $t \neq 0$. From relations (3.1.56) and (3.1.48) it follows that

$$mR_x^+R_y^+P_0 = -mR_y^+R_x^+P_0 = \frac{t}{2}m \Rightarrow mR_x^+, mR_y^+ \neq 0 \text{ if } t \neq 0.$$

Consider the element $n = mR_x^+R_y^+P_2$. Substituting in (3.1.57) $a = x,\ b = y,\ c = x$ and $a = y,\ b = x,\ c = y$ we get

$$nR_x^+ = mR_x^+/2, \quad nR_y^+ = mR_y^+/2, \tag{3.1.58}$$

therefore, $nR_x^+R_y^+P_0 = tm/4$. Hence, if $t \neq 0, \mathrm{Mod}(m) = \mathrm{Mod}(n)$ and the result follows from the previous point. If $t = 0$, then it still holds true that $\mathrm{Mod}(m) = \langle m, mR_x^+, mR_y^+, mR_x^+R_y^+P_2 \rangle$, and from the equations (3.1.58) and (3.1.56) it follows that the elements $mR_x^+, mR_y^+, mR_x^+R_y^+P_2$ are either all zero or all nonzero. If they are all nonzero, then $\mathrm{Mod}(m) \cong \mathrm{Reg}(D_0(1/2,1/2,0))$ or $\mathrm{Reg}(D_0(1/2,1/2,0))^{\mathrm{op}}$. If they are all zero, then $\mathrm{Mod}(m) \cong \mathrm{Reg}(D_0(1/2,1/2,0))/K_3(1/2,1/2,0)$ or its opposite. □

Now we are ready to describe noncommutative Jordan bimodules over D.

Theorem 3.1.7. *Let M be an superbimodule over $D_t(1/2,1/2,0) = D,\ t \neq -1,0,1$. Then M is completely reducible and its irreducible summands are isomorphic either to* $\mathrm{Reg}(D)$ *or* $\mathrm{Reg}(D)^{\mathrm{op}}$.

Proof. It is enough to show that every one-generated bimodule is a sum of homomorphic images of $\mathrm{Reg}(D)$ and $\mathrm{Reg}(D)^{\mathrm{op}}$. Let $m \in M$ and consider $\mathrm{Mod}(m)$. As in the previous subsection, we may suppose that m is homogeneous and Peirce-homogeneous.

If $m \in M_0 \cup M_2$, from the previous lemma it follows that $\mathrm{Mod}(m)$ is isomorphic as a noncommutative Jordan superbimodule to the regular D-bimodule or its opposite. Suppose that $m \in M_1$. Then relation (3.1.52) implies that $\mathrm{Mod}(m)$ can be generated by homogeneous elements of $M_0 + M_2$. Hence, $\mathrm{Mod}(m)$ is a sum of bimodules isomorphic to $\mathrm{Reg}(D)$ or $\mathrm{Reg}(D)^{\mathrm{op}}$. □

Consider now the case $t = 0$. Recall that in Lemma 3.1.6 we found that $D_0(1/2,1/2,0)$ has nontrivial indecomposable modules.

Theorem 3.1.8. *Let M be an indecomposable bimodule over $D_0(1/2,1/2,0) = D$. Then M is isomorphic to one of the following bimodules*:

1) $\mathrm{Reg}(D)$ *or its opposite*;
2) $K_3(1/2,1/2,0)$ *with the action restricted from the* $\mathrm{Reg}(D_0(\lambda))$ *or its opposite*;
3) $\mathrm{Reg}(D)/K_3(1/2,1/2,0)$ *or its opposite*;
4) $D^0(1/2,1/2,0)$ *or its opposite.*

Proof. The proof is completely analogous to the one of Theorem 3.1.3. Consider a D-module M. Consider the subspace $K = \ker R_x^+R_y^+P_2 \cap M_0$. Let I_0^1 be a complement of a base of K to a base of M_0. By Lemma 3.1.6 $\mathrm{Mod}(m_i)$ is isomorphic to either $\mathrm{Reg}(D)$ or its opposite for all $i \in I_0^1$. Let M_2' be a vector space complement of $M_0R_x^+R_y^+P_2$ to M_2. Let I_2^1 be a basis of $\ker R_x^+R_y^+P_0 \cap M_2'$ and I_2^2 be a basis of a vector space complement of $\ker R_x^+R_y^+P_0 \cap M_2'$ to M_2'. Then by Lemma 3.1.6 we have

$$\mathrm{Mod}(m_i) \cong \begin{cases} K_3(1/2,1/2,0) \text{ or } K_3(1/2,1/2,0)^{\mathrm{op}}, & i \in I_2^1, \\ D^0(1/2,1/2,0) \text{ or } D^0(1/2,1/2,0)^{\mathrm{op}}, & i \in I_2^2 \end{cases}.$$

By construction of the module $D^0(1/2,1/2,0)$, we have $M_2'R_x^+R_y^+P_0 \subseteq K$. Let M_0' be a vector space complement of $M_2'R_x^+R_y^+P_0$ to K, and let I_0^2 be a base of M_0'. By Lemma 3.1.6 we have $\mathrm{Mod}(m_i) \cong \mathrm{Reg}(D_0(1))/K_3(1)$ for all $i \in I_0^2$. Consider the sum

$$\overline{M} = \sum_{i\in I_0^1}\mathrm{Mod}(m_i) + \sum_{i\in I_2^1}\mathrm{Mod}(m_i) + \sum_{i\in I_2^2}\mathrm{Mod}(m_i) + \sum_{i\in I_0^2}\mathrm{Mod}(m_i)$$

and note that $M_0 + M_2 \subseteq \overline{M}$. Hence, by Lemma 2.2.3 we have $M/\overline{M} = 0$ and $M = \overline{M}$. The proof that the sum of the submodules constituting $\overline{M}$ is direct is completely analogous to Theorem 3.1.3 and we omit it. □

Consider the case $t = -1$. In this case we only describe finite-dimensional irreducible superbimodules over $D_{-1}(1/2,1/2,0) = D$. Also in this case we assume that $\mathbb{F}$ is algebraically closed.

For $\alpha,\beta,\gamma \in \mathbb{F}$ consider a superbimodule $V(\alpha,\beta,\gamma)$ over $D^{(+)} = J$ with $V_{\bar 0} = \langle v, w\rangle, V_{\bar 1} = \langle z, t\rangle$ and the multiplication table

$$\begin{gathered} v\circ e_1 = v, \quad w\circ e_1 = 0, \quad z\circ e_1 = \frac{z}{2}, \quad t\circ e_1 = \frac{t}{2}, \\ v\circ e_2 = 0, \quad w\circ e_2 = w, \quad z\circ e_2 = \frac{z}{2}, \quad t\circ e_2 = \frac{t}{2}, \\ v\circ x = z, \quad w\circ x = (\gamma-1)z - 2\alpha t, \quad z\circ x = \alpha v, \quad t\circ x = \frac{1}{2}((\gamma-1)v - w), \\ v\circ y = t, \quad w\circ y = 2\beta z - (\gamma+1)t, \quad z\circ y = \frac{1}{2}((\gamma+1)v + w), \quad t\circ y = \beta v. \end{gathered}$$

In the paper [MZ10] it was proved that the modules $V(\alpha,\beta,\gamma)$ are Jordan and every finite-dimensional irreducible Jordan superbimodule over J is isomorphic to one of $V(\alpha,\beta,\gamma)$ (if $\gamma^2-4\alpha\beta-1\neq 0$), or $V_1=\langle w, w\circ J_{\bar{1}}\rangle$, or $V_2=V/V_1$ (if $\gamma^2-4\alpha\beta-1=0$), or its opposite.

Therefore, by Proposition 3.1.4, we have to check whether a module $V(\alpha,\beta,\gamma)$ admits a structure of noncommutative Jordan bimodule over D. Note that with respect to e_1 the Peirce decomposition of V is the following:

$$V_0=\langle w\rangle,\quad V_1=\langle z,t\rangle,\quad V_2=\langle v\rangle.$$

Note also that the operators $(R_x^+)^2,(R_y^+)^2,R_x^+R_y^++R_y^+R_x^+$ act on $V(\alpha,\beta,\gamma)$ as α,β,γ, respectively. Hence, the relation (3.1.45) implies that $\beta=0$.

The proof of Proposition 3.1.4 shows that there is only one way to introduce the noncommutative Jordan action of D in $V(\alpha,0,\gamma)$:

$$\begin{gathered} wR_x^-=-wR_y^+=(\gamma+1)t,\quad vR_x^-=vR_y^+=t,\\ zR_x^-=zR_y^+(P_0-P_2)=\frac{1}{2}(w-(\gamma+1)v),\\ zR_{e_1}^-=\frac{1}{2}zP_1R_y^+(P_0-P_2)R_y^+=-\frac{\gamma+1}{2}t=-zR_{e_2}^-, \end{gathered}$$

and all other R^- operators are zero.

Let $a=x,\ b=e_1$ in (1.12.1): $[R_x^+,R_{e_1}^-]=-\frac{1}{2}R_y^+$. Applying this relation on v, we get $\gamma=0$. Now, one can check that the bimodule $V(\alpha,0,0)$ with the R^- actions introduced above is indeed a noncommutative Jordan D-bimodule. To ensure that we have to check that (1.12.1), (1.12.2) hold for all $a,b\in D$. Some of them have already been checked in the proof of Proposition 3.1.4 and the remaining relations can be easily checked by a direct calculation. Since $\gamma^2-4\alpha\beta-1=-1\neq 0$, this bimodule is irreducible. We denote this noncommutative Jordan bimodule as $V(\alpha)$. The calculation for the opposite module is completely analogous. We have proved the following result:

Theorem 3.1.9. *Let M be an irreducible finite-dimensional noncommutative Jordan bimodule over $D_{-1}(1/2,1/2,0)$ and let the base field $\mathbb{F}$ be algebraically closed. Then M is isomorphic to $V(\alpha),\alpha\in\mathbb{F}$, or its opposite.*

3.1.3 Representations of $K_3(\alpha,\beta,\gamma)$

Here, as a corollary of two previous subsections, we obtain a description of indecomposable bimodules over nonunital simple noncommutative Jordan superalgebra $K_3(\alpha,\beta,\gamma)$.

Note that to study representations of $K_3(\alpha,\beta,\gamma)$ it suffices to study unital representations of $D_0(\alpha,\beta,\gamma)$. Indeed, the superalgebra $D_0(\alpha,\beta,\gamma)=D$ is the unital hull of $K_3(\alpha,\beta,\gamma) = K$, and any noncommutative Jordan K-bimodule M admits a unital noncommutative Jordan action of D by setting $R_{e_2} = \mathrm{id} - R_{e_1}$, $L_{e_2} = \mathrm{id} - L_{e_1}$. Moreover, a structure of a D-module is completely determined by the structure of K-module induced by embedding (in particular, a subspace N of a unital D-bimodule M is a submodule if and only if it is a K-submodule with the action induced by embedding and two D-modules are isomorphic if and only if they are isomorphic as K-modules).

Recall that if $\mathbb{F}$ allows square root extraction, then $K_3(\alpha,\beta,\gamma)$ is isomorphic either to $K_3(\lambda,0,0)$ for $\lambda \in \mathbb{F}$ or to $K_3(1/2,1/2,0)$. Applying module mutation, we can define the modules $D^0(\alpha,\beta,\gamma)$ in the obvious manner. Note that the module $\mathrm{Reg}(D_0(\alpha,\beta,\gamma))/K_3(\alpha,\beta,\gamma)$ is a zero module over $K_3(\alpha,\beta,\gamma)$. Hence, from Theorems 3.1.3 and 3.1.8 we have the following result:

Theorem 3.1.10. *Suppose that the base field $\mathbb{F}$ allows square root extraction. Then every finite-dimensional indecomposable noncommutative Jordan bimodule over $K_3(\alpha,\beta,\gamma)$ is isomorphic either to one of the following modules*:

1) $\mathrm{Reg}(K_3(\alpha,\beta,\gamma))$ *or its opposite*;

2) $\mathrm{Reg}(D_0(\alpha,\beta,\gamma))$ *with the action induced by embedding or its opposite*;

3) $D^0(\alpha,\beta,\gamma)$;

4) *a one-dimensional zero module.*

3.2 Kronecker factorization theorem for $D_t(\alpha,\beta,\gamma)$

Clearly, a necessary condition for the Kronecker factorization over a (super)algebra A to hold is that every A-module be completely reducible and irreducible summands be isomorphic to $\mathrm{Reg}(A)$ (or $\mathrm{Reg}(A)^{\mathrm{op}}$). This is exactly what we proved in the previous section for the superalgberas $D_t(\alpha,\beta,\gamma)$. In this section we investigate if the Kronecker factorization holds for these superalgebras. Again, by different methods for each subclass, we obtain the same result: except for some special values of parameters, any noncommutative algebra U that contains $D_t(\alpha,\beta,\gamma)$ as a unital subalgebra is the graded tensor product of $D_t(\alpha,\beta,\gamma)$ and an associative-commutative superalgebra A. As a consequence, we obtain the classification of noncommutative Jordan representations and the Kronecker factorization for simple associative superalgebra $Q(2)$.

3.2.1 Kronecker factorization theorem for $D_t(\lambda)$

We consider first the case $\lambda = 1$, $t \neq 0,1$ hoping to apply the mutation later. Let U be a noncommutative Jordan superalgebra that contains $D_t(1) = D$ as a unital subsuperalgebra. Then U can be considered as a unital bimodule over D. From Theorem 3.1.2 it follows that U is a direct sum of regular D-bimodules and opposite to them:

$$U = \bigoplus_{i \in I} M_i \oplus \bigoplus_{j \in J} \overline{M}_j,$$

where M_i are isomorphic to $\mathrm{Reg}(D)$, and $\overline{M}_j$ are isomorphic to the $\mathrm{Reg}(D)^{\mathrm{op}}$. For $a \in D$, $i \in I$ ($j \in J$) by a_i ($\overline{a}_j$) we denote the image of a with respect to the module isomorphism $\mathrm{Reg}(D) \to M_i$ ($\mathrm{Reg}(D)^{\mathrm{op}} \to \overline{M}_j$). From now on by $U = U_0 + U_1 + U_2$ we denote the Peirce decomposition of U with respect to $e_1 \in D$.

Consider the space $Z = \{a \in U : [a, D] = 0\}$. It is easy to see that the commutative center of D is equal to $\mathbb{F}$. Therefore, the module structure of U implies that $Z = \langle 1_i, i \in I,\ \overline{1}_j, j \in J\rangle$, thus, $Z \subset U_0 + U_2$.

Lemma 3.2.1. *Z is a subalgebra of U.*

Proof. Let $a, b \in Z,\ c \in D$. Then

$$[a \circ b, c] = 2aR_b^+ R_c^- = (\text{by } (1.12.1)) = (-1)^{bc} 2aR_c^- R_b^+ = 0.$$

Since $Z \subseteq U_0 + U_2$, Z^2 also lies in $U_0 + U_2$ and $[Z^2, e_1] = [Z^2, e_2] = 0$. Therefore, we only have to show that $[[a, b], x] = [[a, b], y] = 0$. This can be showed as follows:

$$\begin{aligned}
[[a, b], x] &= 4aR_b^- R_x^- = 4aR_b^-(P_0 + P_2)R_x^- = (\text{by Lemma } 1.12.3) \\
&= 4aR_b^-(-P_0 + P_2)R_x^+ = (\text{by } (1.2.4)) = 4a(-P_0 + P_2)R_b^- R_x^+ = (\text{by } (1.12.1)) \\
&= (-1)^b 4a(-P_0 + P_2)R_x^+ R_b^- = (-1)^b 4a(P_0 + P_2)R_x^- R_b^- = (-1)^b 4aR_x^- R_b^- = 0.
\end{aligned}$$

Analogously one can show that $[[a, b], y] = 0$. Hence, $[Z^2, D] = 0$ and Z is a subalgebra. □

Note that the module structure of U implies that $U_1 = U_1^{[0]} \oplus U_1^{[1]}$. We will extensively use this property to prove some associativity conditions. In fact, we will show that $U_0 + U_2$ lies in the associative center of U.

Lemma 3.2.2. $(U_1, U_0 + U_2, U_0 + U_2) = 0,\ (U_0 + U_2, U_1, U_0 + U_2) = 0,\ (U_0 + U_2, U_0 + U_2, U_1) = 0.$

Proof. Let $u_0, u_0' \in U_0,\ a \in U_1^{[0]},\ b \in U_1^{[1]},\ u_2, u_2' \in U_2$. Then

$$\begin{aligned}
(a, u_0, u_0') &= a(R_{u_0} R_{u_0'} - R_{u_0 u_0'}) = (\text{by Lemma } 1.2.6) \\
&= -(-1)^{u_0 u_0'} aR_{u_0'} L_{u_0} = (\text{by Lemma } 1.3.3) = 0, \\
(a, u_2, u_2') &= a(R_{u_2} R_{u_2'} - R_{u_2 u_2'}) = (\text{by Lemma } 1.2.6) \\
&= -(-1)^{u_2 u_2'} aL_{u_2'} R_{u_2} = (\text{by Lemma } 1.3.3) = 0.
\end{aligned}$$

Analogously, $(b, u_0, u_0') = 0, (b, u_2, u_2') = 0$.

Also,

$$(a, u_0, u_2) = \ (au_0)u_2 \ = (\text{by Lemma 1.3.3}) = 0,$$

$$(u_0, u_2, a) = -u_0(u_2 a) = (\text{by Lemma 1.3.3}) = 0.$$

Analogously, $(b, u_2, u_0) = 0, (u_2, u_0, b) = 0$.

Also,

$$(u_0, a, u_0') = \ (u_0 a)u_0' \ \in (\text{by Lemma 1.3.1}) \in U_1^{[0]} u_0' = 0,$$

$$(u_2, a, u_2') = -u_2(au_2') \in (\text{by Lemma 1.3.1}) \in u_2 U_1^{[0]} = 0.$$

Analogously, $(u_0, b, u_0') = 0, (u_2, b, u_2') = 0$.

By Lemma 1.3.3 $(u_2, a, u_0) = 0,\ (u_0, b, u_2) = 0$. Finally, the arbitrariness of $u_0, u_0', a, b, u_2, u_2'$ and the flexibility relation (1.1.5) imply the lemma statement. □

Lemma 3.2.3. *$U_0 + U_2$ is an associative subalgebra of U.*

Proof. It suffices to show that U_0 and U_2 are associative. Consider the following identity which is valid in any algebra ([ZSSS82, p. 136]):

$$(ab, c, d) + (a, b, cd) - a(b, c, d) - (a, b, c)d - (a, bc, d) = 0.$$

Substituting in it $a = x;\ b, c, d \in U_0$ by the previous lemma we get $x(b, c, d) = 0$. Then the structure of U as a module over D implies that $(b, c, d) = 0$ and U_0 is associative. Analogously, substituting $a = y;\ b, c, d \in U_2$ we infer that U_2 is associative. □

Lemma 3.2.4. $U_1^2 \subseteq U_0 + U_2,\ (U_1^{[0]})^2 = (U_1^{[1]})^2 = 0,\ U_1^{[0]} U_1^{[1]} \subseteq U_0,\ U_1^{[1]} U_1^{[0]} \subseteq U_2.$

Proof. First we prove that the set $K = \{x, y\}$ satisfies the conditions of Lemma 1.2.5. Indeed, the first condition follows automatically from the bimodule structure of U over D. Suppose now that $a \in U_1$ is such that $K \circ a = 0$. The bimodule structure of U over D implies that

$$a = \sum \alpha_i x_i + \sum \beta_i y_i + \sum \gamma_j \overline{x}_j + \sum \delta_j \overline{y}_j,$$

where $\alpha_i, \beta_i, \gamma_j, \delta_j \in \mathbb{F},\ i \in I,\ j \in J$. Hence,

$$0 = x \circ a = \sum \beta_i (e_{1i} + te_{2i}) + \sum \delta_i (\overline{e_{1i}} + t\overline{e_{2i}}),$$

therefore, $\beta_i = \delta_j = 0, i \in I, j \in J$. Analogously, since $y \circ a = 0,\ \alpha_i = \gamma_j = 0$ for all $i \in I, j \in J$. Therefore, $a = 0$. Thus, Lemma 1.2.5 implies that $U_1^2 \subseteq U_0 + U_2$, and Lemma 1.3.4 implies the lemma statement. □

Lemma 3.2.5. $(U_0 + U_2, U_1, U_1) = 0,\ (U_1, U_0 + U_2, U_1) = 0,\ (U_1, U_1, U_0 + U_2) = 0.$

Proof. Let $u_0 \in U_0,\ a, a' \in U_1^{[0]},\ b, b' \in U_1^{[1]},\ u_2 \in U_2$. Then

$$(u_0, b, a) = (u_0 b)a - u_0(ba) = (\text{by Lemmas 1.3.3 and 3.2.4}) = 0,$$
$$(u_2, a, b) = (u_2 a)b - u_2(ab) = (\text{by Lemmas 1.3.3 and 3.2.4}) = 0.$$

Analogously, $(b, a, u_0) = 0,\ (a, b, u_2) = 0$. Now,

$$(u_0, b, b') = (u_0 b)b' - u_0(bb') = (\text{by Lemmas 1.3.3 and 3.2.4}) = 0,$$
$$(u_2, b, b') = (u_2 b)b' - u_2(bb') = (\text{by Lemmas 1.3.1 and 3.2.4}) = 0.$$

Analogously, $(u_0, a, a') = 0,\ (u_2, a, a') = 0$. Now,

$$\begin{aligned}
(a, u_0, b) &= (au_0)b - a(u_0 b) = (\text{by Lemma 1.3.3}) = 0,\\
(b, u_2, a) &= (bu_2)a - b(u_2 a) = (\text{by Lemma 1.3.3}) = 0,\\
(a, u_0, a') &= (au_0)a' - a(u_0 a') = (\text{by Lemmas 1.3.1 and 3.2.4}) = 0,\\
(b, u_0, b') &= (bu_0)b' - b(u_0 b') = (\text{by Lemmas 1.3.1 and 3.2.4}) = 0.
\end{aligned}$$

Analogously, $(a, u_2, a') = 0,\ (b, u_2, b') = 0$. Finally, arbitrariness of u_0, a, a', b, b', u_2 and the flexibility relation (1.1.5) imply the statement of the lemma. $\square$

Lemma 3.2.6. *U is isomorphic to the graded tensor product of Z and D.*

Proof. Lemmas 3.2.2, 3.2.3 and 3.2.5 imply that $U_0 + U_2$ lies in the associative center of U. Hence, Z also lies in the associative center of U. Let $a, b \in D,\ z, z' \in Z$. Then

$$(za)(z'b) = ((za)z')b = (z(az'))b = (-1)^{az'}(z(z'a))b = (-1)^{az'}((zz')a)b = (-1)^{az'}(zz')(ab).$$

Therefore, U is a homomorphic image of the graded tensor product of Z and D. Since $Z = \langle 1_i, i \in I,\ \overline{1}_j, j \in J\rangle$, it is clear that the equality $z_1 e_1 + z_2 e_2 + z_3 x + z_4 y = 0$ for $z_1, z_2, z_3, z_4 \in Z$ implies $z_1 = z_2 = z_3 = z_4 = 0$. Thus, $U \cong Z \otimes D$. $\square$

By Lemma 3.2.3, Z is an associative superalgebra. Suppose that $t = -1$. Then D is isomorphic to an associative superalgebra $M_{1,1}$, and U is also associative as the graded tensor product of two associative superalgebras. If $t \neq -1$, we can specify the structure of Z further:

Lemma 3.2.7. *Suppose that $t \neq -1$. Then Z is supercommutative.*

Proof. Let $z_1, z_2, z_3 \in Z$. Then by associativity of Z we have

$$(z_1 \otimes x, z_2 \otimes y, z_3 \otimes x) = (-1)^{z_2}(z_1 z_2 z_3) \otimes (x, y, x).$$

The flexibility relation (1.1.5) implies that

$$\begin{aligned}(z_1 \otimes x, z_2 \otimes y, z_3 \otimes x) &= -(-1)^{z_1\otimes x, z_2\otimes y, z_3\otimes x}(z_3 \otimes x, z_2 \otimes y, z_1 \otimes x)\\ &= (-1)^{z_1,z_2,z_3}(z_3 \otimes x, z_2 \otimes y, z_1 \otimes x).\end{aligned}$$

Hence,

$$0 = (z_1z_2z_3 - (-1)^{z_1,z_2,z_3}z_3z_2z_1) \otimes (x, y, x) = 2(1 + t)(z_1z_2z_3 - (-1)^{z_1,z_2,z_3}z_3z_2z_1) \otimes x.$$

Therefore, $z_1z_2z_3 - (-1)^{z_1,z_2,z_3}z_3z_2z_1 = 0$. Taking $z_3 = 1$, we get $z_1z_2 = (-1)^{z_1z_2}z_2z_1$. □

Consider now the general situation, that is, let U be a noncommutative Jordan superalgebra that contains $D_t(\lambda)$ as a unital subsuperalgebra. Suppose that $\lambda \neq 1/2$. Therefore, $U' = U^{(\mu)}$ contains $D_t(1)$ as a unital subsuperalgebra, where $\mu \in \mathbb{F}$ is such that $\lambda \odot \mu = 1$. By what was proved above, $U' = Z \otimes D_t(1)$ for an associative superalgebra Z, and $U = U'^{(\lambda)} = (Z \otimes D_t(1))^{(\lambda)}$. Suppose that Z is supercommutative and let $z, z' \in Z$, $a, b \in D_t(1)$. Then

$$\begin{aligned}(z \otimes a) \cdot_\lambda (z' \otimes b) &= \lambda(z \otimes a)(z' \otimes b) + (-1)^{(z+a)(z'+b)}(1 - \lambda)(z' \otimes b)(z \otimes a)\\ &= (-1)^{az'}\lambda(zz') \otimes (ab) + (-1)^{az'+ab}(1 - \lambda)(zz') \otimes (ba)\\ &= (-1)^{az'}(zz') \otimes (a \cdot_\lambda b).\end{aligned}$$

Therefore, if Z is supercommutative (which holds, for example, when $t \neq -1$), U is isomorphic to $Z \otimes D_t(1)^{(\lambda)} = Z \otimes D_t(\lambda)$. Now we can state our main result:

Theorem 3.2.8. *Let U be a noncommutative Jordan superalgebra that contains $D_t(\lambda)$ as a unital subsuperalgebra, $t \neq 0, 1$, $\lambda \neq 1/2$. Then:*

1) *If $t \neq -1$, then $U \cong Z \otimes D_t(\lambda)$, where Z is an associative-commutative superalgebra;*

2) *If $t = -1$, then $U \cong (Z \otimes M_{1,1})^{(\lambda)}$, where Z is an associative superalgebra. Particularly, any noncommutative Jordan superalgebra containing $D_{-1}(1) = M_{1,1}$ as a unital subsuperalgebra is associative.*

Remark. Note that the condition $\lambda \neq 1/2$ and $t \neq 0, 1$ is necessary for the theorem. Indeed, when $\lambda = 1/2$, the algebra $D_t(1/2)$ is just the Jordan superalgebra D_t which has Jordan bimodules non-isomorphic neither to $\mathrm{Reg}(D_t)$ nor $\mathrm{Reg}(D_t)^{\mathrm{op}}$ [Tru05], [Tru08]. If $t = 0$, then it is easy to see that $\mathrm{Reg}(D_0(\alpha, \beta, \gamma))$ has a 3-dimensional submodule generated by e_1, x, y. Since the superalgebra $D = D_1(\lambda)$ is a superalgebra of the type $U(V, f, \star)$, Theorem 2.2.2 implies that there are bimodules over it which are not isomorphic to $\mathrm{Reg}(D)$ or $\mathrm{Reg}(D)^{\mathrm{op}}$. Hence, Theorem 3.1.2 and Kronecker factorization theorem do not hold for $t = 1$.

3.2.2 Kronecker factorization theorem for $D_t(1/2, 1/2, 0)$

Let $(U, \circ, \{\cdot,\cdot\})$ be a noncommutative Jordan superalgebra containing $D = D_t(1/2, 1/2, 0)$ as a unital subalgebra, $t \neq -1, 0, 1$. Consider U as a unital bimodule over D. As before, Theorem 3.1.7 implies that

$$U = \bigoplus_{i \in I} M_i \oplus \bigoplus_{j \in J} \overline{M}_j, \tag{3.2.1}$$

where $M_i \cong \mathrm{Reg}(D)$, and $\overline{M}_j \cong \mathrm{Reg}(D)^{\mathrm{op}}$. For $a \in D$, $i \in I$ $(j \in J)$ by a_i $(\overline{a}_j)$ we denote the image of a with respect to the module isomorphism $\mathrm{Reg}(D) \to M_i$ $(\mathrm{Reg}(D)^{\mathrm{op}} \to \overline{M}_j)$. By $U = U_0 + U_1 + U_2$ we denote the Peirce decomposition of U with respect to $e_1 \in D$.

As in the previous section, let $Z = \langle 1_i, i \in I,\ \overline{1}_j, j \in J \rangle$. In this section we prove that Z is a supercommutative subalgebra of U and $U \cong Z \otimes D$.

Lemma 3.2.9. *Z is a commutative subalgebra of U.*

Proof. The module structure of U implies that $Z = \{a \in U : [a, D] = 0\} \cap (U_0 + U_2)$. It is obvious that $Z^2 \subseteq U_0 + U_2$, thus, to prove that Z is a subalgebra it suffices to show that $[Z^2, D] = 0$. If $a, b \in Z,\ c \in D$, then

$$[a \circ b, c] = 2aR_b^+R_c^- = (\text{by } (1.12.1)) = (-1)^{bc} 2aR_c^-R_b^+ = 0.$$

To show that $[Z, Z] = 0$ it suffices to prove that $U_0 + U_2$ is a commutative subalgebra of U. In the same way as in Lemma 3.2.4 we can prove that $U_1 U_1 \subseteq U_0 + U_2$. Then, since

$$[e_1, y_i] = 0, \quad [e_1, \overline{y}_j] = 0, \quad [e_1, x_i] = y_i, \quad [e_1, \overline{x}_j] = \overline{y}_j$$

for all $i \in I,\ j \in J$, Lemma 1.12.3 and Lemma 1.12.4 imply that

$$R_{y_i}^- = 0, \quad R_{\overline{y}_j}^- = 0, \quad P_1 R_{x_i}^- = P_1 R_{y_i}^+ (P_0 - P_2), \quad P_1 R_{\overline{x}_j}^- = P_1 R_{\overline{y}_j}^+ (P_0 - P_2).$$

Now,

$$y_i \circ y_j = y_i R_{y_j}^+ (P_0 + P_2) = y_i R_{x_j}^- (P_0 - P_2) = \frac{1}{2}[y_i, x_j](P_0 - P_2) = x_j R_{y_i}^- (P_0 - P_2) = 0.$$

Analogously we can prove that $y_i \circ \overline{y}_j = 0,\ \overline{y}_i \circ \overline{y}_j = 0$.

Consider now the relation (1.12.2) with $a = x,\ b = y_i$: $R_x^- R_{y_i}^+ = R_{(e_1 + te_2)_i}^-$. Apply the resulting relation on the element e_{1_j}:

$$\frac{1}{2}[e_{1_j}, e_{1_i}] = e_{1_j} R_x^- R_{y_i}^+ = \frac{1}{2} y_j \circ y_i = 0.$$

Analogously (using the relation (1.12.2) with $a = x,\ b = \overline{y}_i$) we can show that

$$[e_{1_i}, \overline{e}_{1_j}] = 0, \quad [\overline{e}_{1_i}, \overline{e}_{1_j}] = 0,$$

therefore, U_2 is supercommutative. Since $t \neq 0$, we can apply the same identities to elements $e_{2_i}, \overline{e}_{2_j}, i \in I, j \in J$ to prove that U_0 is supercommutative. Therefore, Z is a supercommutative subalgebra of U. $\square$

With the aid of the following lemma we may consider only the symmetrized Jordan superalgebra:

Lemma 3.2.10. *Suppose that $U^{(+)}$ is the graded tensor product of Jordan superalgebras Z and $D^{(+)} = D_t$. Then U as a noncommutative Jordan superalgebra is the graded tensor product of Z and D.*

Proof. In the proof of the previous lemma we noted that $U_0 + U_2$ is a commutative subsuperalgebra of U, and that $R^-_{y_i} = R^-_{\overline{y}_j} = 0$ for all $i \in I$, $j \in J$. Hence, every nonzero commutator in U is a sum of the commutators of the form $[a, x_i]$, $[b, \overline{x}_j]$, where $a, b \in U_0 + U_2 + \langle x_i, i \in I,\ \overline{x}_j, j \in J\rangle$. Consider, for example, the commutator $[x_i, x_j]$:

$$\begin{aligned}[(1_i \otimes x), (1_j \otimes x)] &= [x_i, x_j] = 2x_i R^-_{x_j} = 2x_i R^+_{y_j}(P_0 - P_2) \\ &= 2(1_i \otimes x) \circ (1_j \otimes y)(P_0 - P_2) = 2(1_i \circ 1_j) \otimes (x \circ y)(P_0 - P_2) \\ &= (\text{by Lemma 3.2.9}) = 2(1_i 1_j) \otimes (-e_1 + te_2) = 1_i 1_j \otimes [x, x].\end{aligned}$$

Thus,

$$\begin{aligned}(1_i \otimes x)(1_j \otimes x) = (1_i \otimes x) \circ (1_j \otimes x) &+ \frac{1}{2}[(1_i \otimes x), (1_j \otimes x)] \\ &= 1_i 1_j \otimes \left(x \circ x + \frac{1}{2}[x, x]\right) = 1_i 1_j \otimes x^2.\end{aligned}$$

Consider also the commutator of elements $\overline{e}_{1_i}$ and x_j:

$$\begin{aligned}[(\overline{1}_i \otimes e_1), (1_j \otimes x)] &= [\overline{e}_{1_i}, x_j] = 2\overline{e}_{1_i} R^-_{x_j} = 2\overline{e}_{1_i} R^+_{y_j} \\ &= 2(\overline{1}_i \otimes e_1) \circ (1_j \otimes y) = 2(\overline{1}_i \circ 1_j) \otimes (e_1 \circ y) \\ &= (\text{by Lemma 3.2.9}) = (\overline{1}_i 1_j) \otimes y = (\overline{1}_i 1_j) \otimes [e_1, x].\end{aligned}$$

Hence,

$$\begin{aligned}(\overline{1}_i \otimes e_1)(1_j \otimes x) = (\overline{1}_i \otimes e_1) \circ (1_j \otimes x) &+ \frac{1}{2}[(\overline{1}_i \otimes e_1), (1_j \otimes x)] \\ &= \overline{1}_i 1_j \otimes \left(e_1 \circ x + \frac{1}{2}[e_1, x]\right) = \overline{1}_i 1_j \otimes e_1 x.\end{aligned}$$

Other cases of a, b can be done completely analogously. For other pairs of elements the product in U coincides with the product in $U^{(+)}$. Therefore, $U \cong Z \otimes D$ as a noncommutative Jordan superalgebra. □

Now it is only left for us to show that $U^{(+)} \cong Z \otimes D_t$. We prove this analogously to the paper [MZ03]. From now on until the end of the section we will be working with a Jordan superalgebra $U^{(+)}$, thus, for convenience we will denote it as U, write Jordan product in U as juxtaposition and denote operators $R^+_x, x \in U$, as R_x. Henceforth we may suppose that $t \neq -3$. Indeed, if char $\mathbb{F} = 3$, then $t = 0$, which we have already excluded, and if char $\mathbb{F} \neq 3$, then $D_{-3} \cong D_{-1/3}$. First of all we need some preliminary data about derivations:

Definition 3.2.11. Let A be a superalgebra, and M be a superbimodule over A. A mapping $d\colon A \to M$ is called a *derivation from A to M* if

$$(ab)d = a(bd) + (-1)^{bd}(ad)b$$

for all $a, b \in A$. The space of derivations from A to M is denoted by $\mathrm{Der}(A, M)$. If A is considered as a module over itself, then an element $d \in \mathrm{Der}(A, A)$ is called a derivation of A. The space $\mathrm{Der}(A) = \mathrm{Der}(A, A)$ with the Lie superalgebra structure is called *the algebra of derivations of A*.

Let J be a Jordan superalgebra. For elements $a, b \in J$ the operator $D(a, b) = [R_a, R_b]$ is a derivation of J. Derivations of the form $\sum D(a_i, b_i),\ a_i, b_i \in J$ are called *inner*. We need to know some facts about the algebra of derivations of D_t.

Lemma 3.2.12. *The superalgebra of derivations of $D_t, t \neq -1$ is a simple 5-dimensional Lie superalgebra, and $D_t/\mathbb{F}$ is an irreducible $\mathrm{Der}(D_t)$-module. Moreover, every derivation of D_t is inner.*

Proof. The computation of this algebra is rather straightforward, so we omit it and only present the base of $\mathrm{Der}(D_t)$ (basis elements of D_t on which the derivations below are not defined map to zero):

$$e\colon x \mapsto y, \quad f\colon y \mapsto x, \quad h\colon \begin{cases} x \mapsto x, \\ y \mapsto -y \end{cases}$$

$$a\colon \begin{cases} e_1 \mapsto x, \\ e_2 \mapsto -x, \\ x \mapsto 0, \\ y \mapsto 2(e_1 - te_2) \end{cases}, \quad b\colon \begin{cases} e_1 \mapsto y, \\ e_2 \mapsto -y, \\ x \mapsto 2(-e_1 + te_2), \\ y \mapsto 0 \end{cases}.$$

Let M be a $\mathrm{Der}(D_t)$-submodule of D_t containing 1. Note that e, f, h span the simple Lie algebra $\mathfrak{sl}_2$ that acts irreducibly on the odd space $\langle x, y\rangle$. Therefore, if M contains an odd element of D_t, it contains all $(D_t)_{\bar{1}}$. Hence (by acting by a), it contains an element $e_1 - te_2$. Since $t \neq -1$, M is equal to the whole D_t. If M contains an even element $\neq 1$, it contains (by acting by a, b) elements x and y and again is equal to the whole D_t. Hence, $D_t/\mathbb{F}$ is an irreducible $\mathrm{Der}(D_t)$-supermodule.

The multiplication table of $\mathrm{Der}(D_t)$ is as follows:

$$\begin{array}{lllllllll}
[e,f] &=& h, & [h,f] &=& -2f, & [h,e] &=& 2e,\\
[a,a] &=& 4(1+t)f, & [a,b] &=& -2(1+t)h, & [b,b] &=& -4(1+t)e,\\
[e,a] &=& -b, & [f,a] &=& 0, & [h,a] &=& -a,\\
[e,b] &=& 0, & [f,b] &=& -a, & [h,b] &=& b.
\end{array}$$

Using this table it is quite easy to see that $\mathrm{Der}(D_t)$ is a simple superalgebra.

One can also check that

$$e=\frac{2}{1+t}[R_y,R_y],\quad f=\frac{-2}{1+t}[R_x,R_x],\quad h=\frac{2}{1+t}[R_x,R_y],$$
$$a=4[R_{e_1},R_x],\quad b=4[R_{e_1},R_y].$$

Therefore, all derivations of D_t are inner. □

Note that the decomposition (3.2.1) also holds for the symmetrized superalgebra: $M_i, i\in I$ as a Jordan bimodule is isomorphic to $\mathrm{Reg}(D_t)$, and $\overline{M}_j, j\in J$ as a Jordan bimodule is isomorphic to $\mathrm{Reg}(D_t)^{\mathrm{op}}$. From this and the previous lemma it follows that for any D_t-submodule $M\subseteq U$ the Lie superalgebra $\mathrm{Der}(D_t)$ acts on M and on $\mathrm{Der}(D_t,M)$. Note also that $\mathrm{Der}(D_t,D_t^{\mathrm{op}})$ as a module over $\mathrm{Der}(D_t)$ is isomorphic to the opposite of the regular bimodule, thus is also irreducible.

Lemma 3.2.9 implies that Z is a subalgebra of $U^{(+)}$. Note also that for $a,b\in D_t,\ i\in I,\ j\in J$ we have

$$(a1_i)b=(ab)1_i,\qquad b(a1_i)=(ba)1_i, \tag{3.2.2}$$
$$(a\overline{1}_j)b=(-1)^b(ab)\overline{1}_j,\quad b(a\overline{1}_j)=(ba)\overline{1}_j. \tag{3.2.3}$$

From this equations it follows easily that $Z\,\mathrm{Der}(D_t)=0$.

We will also need the Jordan identity in the element form:

$$\begin{aligned}((ab)c)d+(-1)^{b,c,d}((ad)c)b+(-1)^{cd}a((bd)c)&\\
=(ab)(cd)+(-1)^{bc}(ac)(bd)+(-1)^{(b+c)d}(ad)(bc)&\\
=a((bc)d)+(-1)^{cd}((ab)d)c+(-1)^{b(c+d)}((ac)d)b.&\end{aligned}$$

Recall that we aim to prove that for $a,b\in D_t,\ z_1,z_2\in Z$ we have $(z_1a)(z_2b)=(-1)^{z_2a}(z_1z_2)(ab)$.

Lemma 3.2.13. *For $a\in D_t,\ z_1,z_2\in Z$ we have $(az_1)z_2=a(z_1z_2)$.*

Proof. We show that the mapping $d\colon D_t\to U$ defined by $a\mapsto(az_1)z_2-a(z_1z_2)$, where $z_1,z_2\in Z$, is a derivation. That is, we show that for $a,b\in D_t$ we have

$$((ab)z_1)z_2-(ab)(z_1z_2)=a((bz_1)z_2-b(z_1z_2))+(-1)^{b(z_1+z_2)}((az_1)z_2-a(z_1z_2))b.$$

From relations (3.2.2), (3.2.3) it follows that $(ab)(z_1z_2) = (-1)^{b(z_1+z_2)}(a(z_1z_2))b$. Thus, we need to prove that

$$((ab)z_1)z_2 = a((bz_1)z_2 - b(z_1z_2)) + (-1)^{b(z_1+z_2)}((az_1)z_2)b. \tag{3.2.4}$$

By the Jordan identity we have

$$(z_2(ab))z_1 + (-1)^{bz_1}(z_2(az_1))b + (-1)^{a(b+z_1)}(z_2(bz_1))a$$
$$= ((z_2a)b)z_1 + (-1)^{a,b,z_1}((z_2z_1)b)a + (-1)^{bz_1}z_2((az_1)b).$$

Since $(z_2(ab))z_1 = ((z_2a)b)z_1$, we have

$$(-1)^{bz_1}(z_2(az_1))b + (-1)^{a(b+z_1)}(z_2(bz_1))a = (-1)^{a,b,z_1}((z_2z_1)b)a + (-1)^{bz_1}z_2((az_1)b),$$

which together with supercommutativity implies (3.2.4). Since $Z\,\mathrm{Der}(D_t) = 0$, the derivation d commutes with $\mathrm{Der}(D_t)$. Thus, the compositions of d with the projections $U \to M_i$, $U \to \overline{M}_j$ belong to $\mathrm{Der}(D_t, M_i)$ and $\mathrm{Der}(D_t, \overline{M}_j)$ respectively and also commute with $\mathrm{Der}(D_t)$. Since the action of $\mathrm{Der}(D_t)$ on $\mathrm{Der}(D_t)$ and $\mathrm{Der}(D_t, D_t^{\mathrm{op}})$ has only zero constants, we conclude that $d = 0$. □

Lemma 3.2.14. *For $a, b \in D_t$, $z_1, z_2 \in Z$ we have $(z_1a)(z_2b) = (-1)^{z_2a}(z_1z_2)(ab)$.*

Proof. For fixed elements $z_1, z_2 \in Z$, $a \in D_t$ consider the mapping

$$d_a\colon D_t \to U, \quad b \mapsto (bz_1)(z_2a) - (-1)^{a(z_1+z_2)}(ba)(z_1z_2).$$

We prove that d_a is a derivation. Indeed, by the Jordan identity for $b, b' \in D_t$ we have

$$((bb')z_1)(z_2a) + (-1)^{b'(z_1+z_2+a)+z_1(z_2+a)}((b(z_2a)z_1)b' + (-1)^{z_1(z_2+a)}(b(b'(z_2a))z_1)$$
$$= (-1)^{b'(z_1+z_2+a)}((bz_1)(z_2a))b' + b((b'z_1)(z_2a)) + (-1)^{z_1(z_2+a)}((bb')(z_2a))z_1.$$

By previous lemma and relations (3.2.2), (3.2.3) we have

$$\begin{aligned}
(-1)^{z_1(z_2+a)}((bb')(z_2a))z_1 &= (-1)^{a(z_2+z_1)}((bb')a)(z_1z_2),\\
(-1)^{z_1(z_2+a)}(b(z_2a))z_1 &= (-1)^{a(z_2+z_1)}(ba)(z_1z_2),\\
(-1)^{z_1(z_2+a)}(b'(z_2a))z_1 &= (-1)^{a(z_2+z_1)}(b'a)(z_1z_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(bb')d_a &= ((bb')z_1)(z_2a) - (-1)^{a(z_2+z_1)}((bb')a)(z_1z_2)\\
&= b((b'z_1)(z_2a) - (-1)^{a(z_2+z_1)}(b'a)(z_1z_2))\\
&+ (-1)^{b'(a+z_1+z_2)}((bz_1)(z_2a) - (-1)^{a(z_2+z_1)}(ba)(z_1z_2))b'\\
&= b \cdot b'd_a + (-1)^{b'd_a}bd_a \cdot b,
\end{aligned}$$

and d_a is a derivation. One can easily check that

$$d_{aDx,y} = [d_a, D(x,y)] \text{ for } a,x,y \in D_t,$$

that is, the map $a \mapsto d_a$ is a $\mathrm{Der}(D_t)$-module homomorphism from D_t to $\mathrm{Der}(D_t, U)$. By the previous lemma, $\mathbb{F}$ lies in the kernel of this homomorphism, therefore, there is a homomorphism of an irreducible $\mathrm{Der}(D_t)$-module $D_t/\mathbb{F}$ into $\mathrm{Der}(D_t, U)$. If this homomorphism is not zero, then one of its compositions with projections to submodules $\mathrm{Der}(D_t, M_i), \mathrm{Der}(D_t, \overline{M}_j)$ is not zero. Hence, $D_t/\mathbb{F}$ is contained in $\mathrm{Der}(D_t, D_t)$ or $\mathrm{Der}(D_t, D_t^{\mathrm{op}})$, which are also irreducible $\mathrm{Der}(D_t, D_t)$-bimodules. Since $\dim \mathrm{Der}(D_t, D_t) = \dim \mathrm{Der}(D_t, D_t^{\mathrm{op}}) = 5$ and $\dim D_t/\mathbb{F} = 3$, we have obtained a contradiction. Therefore, $d_a = 0$ for all $a \in D_t$. □

Lemma 3.2.15. *Z is an associative superalgebra.*

Proof. Consider the Jordan identity for $a = z_1 \otimes x,\ b = z_2 \otimes y,\ c = z_3 \otimes e_1,\ d = 1 \otimes x,\ z_1, z_2, z_3 \in Z$:

$$(-1)^{z_2}\frac{1}{2}(z_1z_2)z_3 \otimes x - (-1)^{z_2}\frac{1}{2}z_1(z_2z_3) \otimes x$$
$$= (-1)^{z_2}\frac{1+t}{4}(z_1z_2)z_3 \otimes x - (-1)^{z_2+z_2z_3}\frac{1+t}{4}(z_1z_3)z_2 \otimes x.$$

Therefore, we have

$$(z_1, z_2, z_3) = (-1)^{z_1z_2}\frac{1+t}{2}(z_2, z_1, z_3) = \frac{(1+t)^2}{4}(z_1, z_2, z_3).$$

Since we have excluded the cases $t = 1, -3$, Z is associative. □

We have now proved the main result of the section:

Theorem 3.2.16. *Let U be a noncommutative Jordan superalgebra containing $D = D_t(1/2, 1/2, 0), t \neq -1, 0, 1$ as a unital subalgebra. Then $U \cong Z \otimes D$, where Z is an associative-supercommutative superalgebra.*

3.3 Representations of $Q(1)$ and $Q(2)$

Recall that $Q(n) = M_n(\mathbb{F}) \oplus \overline{M_n(\mathbb{F})}$, where $\overline{M_n(\mathbb{F})}$ is an isomorphic copy of $M_n(\mathbb{F})$ as a vector space. Also, $Q(n)_{\bar 0} = M_n(\mathbb{F}),\ Q(n)_{\bar 1} = \overline{M_n(\mathbb{F})}$. The multiplication in $Q(n)$ is defined as follows:

$$a \cdot b = ab, \quad \bar a \cdot b = a \cdot \bar b = \overline{ab}, \quad \bar a \cdot \bar b = ab,$$

It is widely known that $Q(n)$ is a simple associative superalgebra for all natural n, and its degree is exactly n [Wal64]. Thus, noncommutative Jordan representations of $Q(n), n \geqslant 3$, are described in Theorem 2.1.1. In this section we describe noncommutative Jordan representations of $Q(1)$ and $Q(2)$.

3.3.1 Representations of $Q(1)$

The superalgebra $Q(1)$ has a basis $1, \overline{1}$, where 1 is the unit of the superalgebra, and $\overline{1}^2 = 1$. Alternative representations of $Q(1)$ were studied by Pisarenko in [Pis94]. In particular, he described all irreducible alternative representations of $Q(1)$ and found a series of indecomposable alternative superbimodules over this algebra.

We note that in fact all unital representations of $Q(1)$ are alternative. Indeed, a superalgebra A is alternative if and only if it satisfies the following operator relations:

$$L_{x\circ y} = L_x \circ L_y, \quad R_{x\circ y} = R_x \circ R_y, \ x, y \in A.$$

Let M be a unital bimodule over $Q(1)$. Then it is easy to see that the above relations trivially hold in the split null extension $Q(1) \oplus M$. Thus, we have proved

Proposition 3.3.1. *Any unital representation of $Q(1)$ is alternative.*

3.3.2 Representations of $Q(2)$

$Q(2)$ is an 8-dimensional simple associative superalgebra. Regarding to an idempotent e_{11}, we have the following Peirce decomposition of $Q(2) = U$:

$$U_0 = \langle e_{22}, \overline{e_{22}} \rangle, \quad U_1 = \langle e_{12}, e_{21}, \overline{e_{12}}, \overline{e_{21}} \rangle, \quad U_2 = \langle e_{11}, \overline{e_{11}} \rangle.$$

Alternative representations of $Q(2)$ were studied by Pisarenko [Pis94]. Particularly, he described irreducible unital bimodules over $Q(2)$ (all of them turned out to be associative and isomorphic either to $\mathrm{Reg}(Q(2))$ or $\mathrm{Reg}(Q(2))^{\mathrm{op}}$) and proved that every bimodule over $Q(2)$ is completely reducible.

In the paper [PS13] it was proved that a noncommutative Jordan superalgebra U such that $U^{(+)} \cong Q(2)^{(+)}$ is necessarily its mutation: $U \cong Q(2)^{(\lambda)}, \ \lambda \in \mathbb{F}$. So, using the module mutation, it suffices to study the representations of $Q(2)$ and $Q(2)^{(+)}$. Description of noncommutative Jordan representations of $Q(2)$ is a consequence of the results of previous section:

Theorem 3.3.2. *Any unital noncommutative Jordan representation of $Q(2)$ is associative.*

Proof. Let M be a noncommutative Jordan bimodule over $Q(2)$ and E be the corresponding split null extension. Note that $Q(2)$ contains a subalgebra $D = \langle e_{11}, \overline{e_{12}}, \overline{e_{21}}, e_{22} \rangle$, which contains the unit of $Q(2)$ and is isomorphic to $D_{-1}(1) \cong M_{1,1}$. Therefore, E contains D as a unital subalgebra, so by Theorem 3.2.8, E is associative and M is an associative bimodule over U. $\square$

Moreover, as a consequence of Theorem 3.2.8, we can prove the Kronecker factorization theorem for $Q(2)$:

Theorem 3.3.3. *Let U be a noncommutative Jordan superalgebra that contains $Q(2)$ as a unital superalgebra. Then U is associative and $U \cong Z \otimes Q(2)$, for an associative superalgebra Z.*

Proof. Since $Q(2)$ contains a noncommutative Jordan subsuperalgebra which is isomorphic to $D_{-1}(1)$, by Theorem 3.2.8, U is an associative algebra. From Pisarenko's classification of unital superbimodules over $Q(2)$ it follows that

$$U = \bigoplus_{i \in I} M_i \oplus \bigoplus_{j \in J} \overline{M}_j,$$

where M_i are isomorphic to $\mathrm{Reg}(Q(2))$, and $\overline{M}_j$ are isomorphic to the $\mathrm{Reg}(Q(2))^{\mathrm{op}}$. For $a \in Q(2)$, $i \in I$ $(j \in J)$ by a_i $(\overline{a}_j)$ we denote the image of a with respect to the module isomorphism $\mathrm{Reg}(Q(2)) \to M_i$ $(\mathrm{Reg}(Q(2))^{\mathrm{op}} \to \overline{M}_j)$.

Consider the set $Z = \langle 1_i,\ i \in I, \overline{1}_j, j \in J \rangle$. Since the commutative center of $Q(2)$ is equal to $\mathbb{F}$, it is clear that $Z = \{a \in U : [a, Q(2)] = 0\}$. Since U is associative, Z is a subalgebra of U. It is clear that

$$(za)(z'b) = (-1)^{az'}(zz')(ab) \text{ for } z, z' \in Z,\ a, b \in Q(2)$$

and the definition of Z implies that every element of U can be uniquely expressed as a sum $\sum_{b \in B} z_b b$, where B is a basis of $Q(2)$, $z_b \in Z, b \in B$. Hence, $U \cong Z \otimes B$. □

4 Representations of superalgebras $U(V, f)$ and $K(\Gamma_n, A)$

So far we have classified finite-dimensional irreducible representations of simple noncommutative Jordan superalgebras of degree $\geqslant 2$, except for the algebras $U(V, f, \star)$ and $K(\Gamma_n, A)$. In this chapter we consider the representations of these algebras. In Section 4.1 we show that if M is an irreducible noncommutative module over U, then it is either irreducible as Jordan module over $U^{(+)}$ or equals one of its Peirce components (Theorem 4.1.8). This result can be used to obtain a generalization of Theorem 1.5.4 without characteristic or dimensionality restrictions. We use this result and classifications of irreducible modules over Jordan superalgebras $J(V, f)$ and $K(\Gamma_n)$ obtained in [MZ10] and [SFS16] to classify irreducible finite-dimensional modules over superalgebras $U(V, f, \star)$ and $K(\Gamma_n, A)$ in Theorems 4.2.6 and 4.3.2. Particularly, we find that there is a large family of irreducible $U(V, f, \star)$-modules that are equal to their Peirce 1-components (previously discovered in Section 2.2.1), and for any irreducible $K(\Gamma_n)$-module there exists a unique structure of a noncommutative Jordan $K(\Gamma_n, A)$-module on it.

4.1 Irreducibility of the symmetrized module

In this section we take a noncommutative Jordan superalgebra U and construct ideals in U out of ideals in $U^{(+)}$. Adapting and simplifying the proof of [McC71] to our new setting, we prove that an irreducible module over a simple noncommutative Jordan superalgebra U of degree $\geqslant 2$ is either an irreducible module over its symmetrized superalgebra $U^{(+)}$ or is equal to one of its Peirce components. As a consequence, we have a new simpler proof of Theorem 1.5.4.

We will need the following lemma:

Lemma 4.1.1 ([PS10a])**.** *If $N_1 \subseteq U_1$ such that $U_i \circ N_1 + [U_i, N_1] \subseteq N_1$ for $i = 0, 2$, then*

$$N_i = P_i(U_1 \circ N_1 + [U_1, N_1]) \trianglelefteq U_i, \;\; i = 0, 2.$$

Lemma 4.1.2. *Let B_1 be a subspace of U_1 such that $[B_1, e] \subseteq B_1$ and $U_i \circ B_1 \subseteq B_1, i = 0, 2$. Then $B_i = P_i(U_1 \circ B_1)$ is an ideal in U_i. The subspace $B = B_0 + B_1 + B_2$ will be an ideal in U if*

1) $U_1 \circ B_i \subseteq B_1$,

2) $P_1([U_1, B_1]) \subseteq B_1$,

in which case B coincides with the ideal in $U^{(+)}$ generated by B_1.

Proof. From 1) and 2) of Lemma 1.12.3 it follows that $[U_i, B_1] \subseteq B_1$. Now 3) of Lemma 1.12.3 implies that for $i = 0, 2$ we have

$$P_i([U_1, B_1]) = U_1 R^-_{B_1} P_i \subseteq U_1 R^+_{[e,B_1]} P_i \subseteq P_i(U_1 \circ B_1).$$

Thus, from Lemma 4.1.1 it follows that B_i is an ideal in U_i. Suppose now that the conditions 1) and 2) hold for B. We have already shown that $B_i, i = 0, 2$ are ideals in U_i, and $U_0 \circ B_2 = [U_0, B_2] = 0 = U_2 \circ B_0 = [B_0, U_2]$ by Peirce relations. Moreover, $P_i(U_1 \circ B_1 + [U_1, B_1]) = B_i$ by definition of P_i, so condition 2) implies that $[U_1, B_1] + U_1 \circ B_1 \subseteq B$. Note also that 1) and 2) of Lemma 1.12.3 imply that $[U_1, B_i] \subseteq B_1$. Therefore, $B \lhd U$. Since $B_i, i = 0, 2$ are generated by B_1 using Jordan products in $U^{(+)}$, B is contained in the ideal of $U^{(+)}$ generated by B_1, and if B is an ideal in U it is also an ideal in $U^{(+)}$. □

Lemma 4.1.3. *For $i = 0, 2$ the space*

$$Z_i = \{z_i \in U_i : U_1 \circ z_i = [U_1, z_i] = 0\} = \{z_i \in U_i : U_1 \circ z_i = 0\}$$

is an ideal of U.

Proof. First of all, Lemma 1.12.3 implies that the two definitions of Z_i in the formula above are equivalent. It suffices to prove the statement for $i = 0$. It is easy to see that $(U_1 + U_2) \circ Z_0 = [U_1 + U_2, Z_0] = 0$, and relations (1.12.3), (1.12.1) imply that $U_0 \circ Z_0 \subseteq Z_0$, $[U_0, Z_0] \subseteq Z_0$, hence Z_0 is an ideal of U. □

Lemma 4.1.4. *Let B be a Jordan ideal of U with $B = B_0 + B_1 + B_2$ relative to an idempotent e. Then there exists a Jordan ideal $C = C_0 + C_1 + C_2$, with $B_0 = C_0$, $B_1 \subseteq C_1$, $B_2 = C_2$ such that*

1) $P_1([U_1, C_1]) \subseteq C_1$,

2) $P_i(U_1 \circ C_1) = P_i(U_1 \circ B_1)$ $(i = 0, 2)$.

Proof. Set $C_1 = \sum_{n=0}^{\infty} B_{1,n}$, where $B_{1,0} = B_1$, $B_{1,n+1} = P_1([U_1, B_{1,n}])$. Is is obvious that $C_1 \supseteq B_1$ satisfies 1) by construction, and to prove 2) it is enough to establish $P_i(U_1 \circ B_{1,n}) \subseteq P_i(U_1 \circ B_1)$ for all n. For $n = 0$ it is trivial, and, if it is true for n, then

$$\begin{aligned} P_i(U_1 \circ B_{1,n+1}) &= U_1 R^+_{P_1([U_1, B_{1,n}])} P_i = U_1 R^+_{[U_1, B_{1,n}]} P_i = \text{ (by (1.12.1))} \\ &= U_1 [R^+_{B_n}, R^-_{U_1}] P_i = [U_1, U_1] R^+_{B_n} P_i \subseteq P_i(U_1 \circ B_{1,n}) \subseteq B_i. \end{aligned}$$

by induction. It remains to verify that C is still a Jordan ideal. From the construction of C, and 1), 2) it suffices to prove that $U_i \circ C_1 \subseteq C_1$. Again, we can show this for $B_{1,n}$. For $n = 0$ it is trivially true, and by induction

$$\begin{aligned} U_i \circ B_{1,n+1} = B_{1,n} R^-_{U_1} P_1 R^+_{U_i} = &\ (\text{by } (1.2.4)) \ = B_{1,n} R^-_{U_1} R^+_{U_i} P_1 \\ \subseteq &\ (\text{by } (1.12.2)) \ \subseteq B_{1,n} R^+_{U_i} R^-_{U_1} P_1 \subseteq B_{1,n} R^-_{U_1} P_1 = B_{1,n+1}. \square \end{aligned}$$

Lemma 4.1.5. *Let B be a Jordan ideal of U, and $x \in U$. Then $D = D(B,x) = \sum_{n=0}^{\infty} B(R^-_x)^n$ is also a Jordan ideal of U, and $DR^-_x \subseteq D$.*

Proof. By construction $DR^-_x \subseteq D$, so we only have to prove that D is still a Jordan ideal. It suffices to prove that for all $n \in \mathbb{N},\ y \in U$ we have $B(R^-_x)^n R^+_y \subseteq \sum_{k=0}^{n} B(R^-_x)^k$. This is trivial for $n = 0$, and by induction

$$\begin{aligned} B(R^-_x)^{n+1} R^+_y = B(R^-_x)^n R^-_x R^+_y \subseteq &\ (\text{by } (1.12.1)) \\ \subseteq B(R^-_x)^n R^+_y R^-_x + B(R^-_x)^n R^+_{[x,y]} \subseteq &\ \sum_{k=0}^{n+1} B(R^-_x)^k. \end{aligned}$$ □

Corollary 4.1.6. *Let B be a Jordan ideal of U, $B = B_0 + B_1 + B_2$ relative to an idempotent e, then there exists a Jordan ideal $E = E_0 + E_1 + E_2$ with $B_0 = E_0,\ B_1 \subseteq E_1,\ B_2 = E_2$ such that $ER^-_e \subseteq E$.*

Proof. Take $E = D(B,e)$, and the inclusions now follow from Peirce relations. □

Lemma 4.1.7. *If B is a Jordan ideal of U, and $B = B_0 + B_1 + B_2$ relative to e, then there exists an ideal $I = I_0 + I_1 + I_2$ of U such that $I_0 \subseteq B_0,\ I_1 \supseteq B_1,\ I_2 \subseteq B_2$.*

Proof. Given a Jordan ideal $J \subseteq U$ let $C(J) \supseteq J$ be the Jordan ideal constructed from J in Lemma 4.1.4, and $E(J) \supseteq J$ be the Jordan ideal constructed in Corollary 4.1.6. Set

$$B_0 = B, \quad B_{2n+1} = E(B_{2n}), \quad B_{2n+2} = C(B_{2n+1}) \quad \text{for } n > 0.$$

Then B_n form an increasing sequence of Jordan ideals, and $K = \sum_{n=0}^{\infty} B_n$ is a Jordan ideal of U which contains B. Since the constructions C and E only increase the Peirce 1-component, it is easy to see that $K_0 = B_0,\ K_1 \supseteq B_1,\ K_2 = B_2$. One can see that $E(K), C(K) \subseteq K$, since

$$E(B_{2n}) = B_{2n+1}, \quad E(B_{2n+1}) = B_{2n+1}, \quad C(B_{2n}) = B_{2n}, \quad C(B_{2n+1}) = B_{2n+2}.$$

Therefore, since K is a Jordan ideal, we have

$$K_1 R^-_e \subseteq K_1, \quad P_1([U_1, K_1]) \subseteq K_1, \quad U_i \circ K_1 \subseteq K_1, \quad U_1 \circ P_i(U_1 \circ K_1) \subseteq U_1 \circ K_i \subseteq K_1.$$

By Lemma 4.1.2 it follows that $I = I_0 + I_1 + I_2$ is an ideal with $I_i = P_i(K_1 \circ U_1) \subseteq K_i = B_i,\ I_1 = C_1 \supseteq B_1$. The lemma is now proved. □

Theorem 4.1.8. *Let U be a noncommutative Jordan algebra with an idempotent $e\neq 0,1$, and M be an irreducible noncommutative Jordan bimodule over U. Suppose that M has a proper Jordan submodule N. Then $M=M_i$, its ith Peirce component with respect to e, $i=0,1,2$.*

Proof. From Lemma 4.1.7 applied to the split null extension E of U by M it follows that there exists a submodule V of M such that $V_i\subseteq N_i$ and $V_1\supseteq N_1$ (from the proofs of the previous lemmas one can easily see that $N\subseteq M$ implies $V\subseteq M$). Suppose now that $N_0+N_2\neq M_0+M_2$. Then $V_0+V_2\neq M_0+M_2$ as well, and $V\neq M$. Hence, $V=0,\ V_1=0$, and $N_1\subseteq V_1=0$. It follows that $(N_0+N_2)\circ(E_1)\subseteq N_1=0$. Hence, $N_i\subseteq Z_i$ in the notation of Lemma 4.1.3, therefore, $Z_i\cap M_i, i=0,2$ are submodules of M. If $M\neq M_0,M_1,M_2$, these modules are proper and the irreducibility of M implies that $N_i=0$ and $N=0$, which is a contradiction.

Consider now the case where N is a Jordan sunmodule of M, and $N_0+N_2=M_0+M_2$. Consider $L_1=(M_0+M_2)\circ U_1$, and $L_i=P_i(L_1\circ U_1),\ i=0,2$. By construction $L=L_0+L_1+L_2\subseteq N$. We prove that L is a submodule of M. For that we check that all conditions of Lemma 4.1.2 hold for L. Since $L\subseteq M$ and $M^2=0$, while checking the conditions we may substitute U_i for E_i. First,

$$L_1R_e^-=(M_0+M_2)R_{U_1}^+R_e^-\subseteq \text{ by (1.12.1) } \subseteq(M_0+M_2)(R_e^-R_{U_1}^++R_{U_1}^+)\subseteq L_1.$$

Second,

$$\begin{aligned}L_1R^+_{U_0+U_2}&=U_1R^+_{M_0+M_2}R^+_{U_0+U_2}\subseteq \text{ by (1.12.3)}\\&\subseteq U_1(R^+_{U_0+U_2}R^+_{M_0+M_2}+R^+_{M_0+M_2})\subseteq U_1R^+_{M_0+M_2}=L_1.\end{aligned}$$

Next, by construction of L_i it is obvious that $U_1\circ L_i\subseteq L_1$. Finally,

$$\begin{aligned}P_1([U_1,L_1])&=U_1R^+_{(M_0+M_2)\circ U_1}P_1\subseteq \text{ (by (1.12.2)) } \subseteq U_1(R^-_{M_0+M_2}R^+_{U_1}+R^-_{U_1}R^+_{M_0+M_2})P_1\\&=U_1R^-_{U_1}R^+_{M_0+M_2}P_1= \text{ (by (1.2.4)) } =U_1R^-_{U_1}P_1R^+_{M_0+M_2}\subseteq U_1R^+_{M_0+M_2}=L_1.\end{aligned}$$

Therefore, L is a submodule of M. Since M is irreducible, $L=N$ or 0. If $L=N$, then $N=M$, a contradiction. If $L=0$, then $L_1=(M_0+M_2)\circ U_1=0$, and by Lemma 1.12.3 M_0,M_2 are submodules of M. Again, if $M\neq M_i,\ i=0,1,2$, they are proper, which contradicts the irreducibility of M. The theorem is now proved. □

As a consequence of the theorem, we obtain an analog of Oehmke's theorem for superalgebras (see [PS10a], [PS19]) which is independent of characteristic and finiteness conditions:

Corollary 4.1.9. *Let U be a simple unital noncommutative Jordan algebra with an idempotent $e\neq 0,1$. Then $U^{(+)}$ is also simple.*

Proof. Since U is simple, its regular bimodule is irreducible. By Theorem 4.1.8, either the regular submodule of $U^{(+)}$ is irreducible (which means that $U^{(+)}$ is simple) or U equals one of its Peirce components with respect to e. It is obvious that the option $U = U_1(e)$ is impossible, and nontriviality of e implies that $U \neq U_0, U_2$. Hence, $U^{(+)}$ is simple. □

In the following sections we use Theorem 4.1.8 and the classification of irreducible finite-dimensional representations over simple Jordan superalgebras obtained in the papers [MZ10, SFS16] to study the representations of noncommutative Jordan superalgebras $U(V, f, \star)$ and $K(\Gamma_n, A)$.

4.2 Irreducible bimodules over $U(V, f, \star)$

In this section we study representations of superalgebras $U(V, f, \star)$. First we recall the definition of the algebra and some its basic properties, and then classify its irreducible finite-dimensional modules. During this section we assume that the base field $\mathbb{F}$ allows square root extraction.

4.2.1 The superalgebra $U(V, f, \star)$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace over $\mathbb{F}$, and let f be a supersymmetric bilinear form on V. Then we can define a multiplication on $J = \mathbb{F} \oplus V$ in the following way:

$$(\alpha + x)(\beta + y) = (\alpha\beta + f(x, y)) + (\alpha y + \beta x),$$

and the resulting superalgebra is denoted $J(V, f)$. This algebra is called a *superalgebra of a supersymmetric bilinear form*. One can check that $J(V, f)$ is simple if f is nondegenerate and $\dim V > 1$. From now on we only consider nondegenerate forms f. Generic Poisson brackets on these algebras were described by Pozhidaev and Shestakov:

Proposition 4.2.1 ([PS10a]). *Let $\star$ be a generic Poisson bracket on the superalgebra $J(V, f)$, where f is nondegenerate. Then $\star$ is a superanticommutative multiplication on V such that $f(x \star y, z) = f(x, y \star z)$.*

The resulting simple noncommutative Jordan superalgebra is denoted by $U(V, f, \star)$. Note that $J(V, f) = U(V, f, 0)$.

We will need the expression of the multiplication $\star$ in the coordinate form. Let $v_1, \ldots, v_n$ be an orthonormal basis of $V_{\bar{0}}$, and let $w_1, \ldots, w_{2m}$ be a basis of $V_{\bar{1}}$ such that $(w_i, w_{2m+1-i}) = 1 = -(w_{2m+1-i}, w_i)$, where all other products are zero. Write the multiplication $\star$ in this basis:

$$\begin{aligned} v_i \star v_j = -v_j \star v_i &= \sum_{k=1}^{n} \alpha_{ijk} v_k, \\ v_i \star w_p = -w_p \star v_i &= \sum_{k=1}^{2m} \beta_{ipk} w_k, \\ w_p \star w_q = w_q \star w_p &= \sum_{k=1}^{n} \gamma_{pqk} v_k. \end{aligned}$$

The superanticommutativity and f-invariance of $\star$ then imply that

$$\begin{array}{r} \alpha_{\sigma(i)\sigma(j)\sigma(k)} = \operatorname{sgn}(\sigma)\alpha_{ijk}, \\ \gamma_{pqi} = \gamma_{qpi}, \quad \gamma_{pqi} = \pm\beta_{ip2m+1-q} \end{array} \quad \text{for all } \sigma \in S_3,\ i,j,k = 1,\dots,n,\ p,q = 1,\dots,2m. \tag{4.2.1}$$

4.2.2 Classification of irreducible representations

By Theorem 4.1.8, if M is an irreducible bimodule over $U = U(V, f, \star)$, then it is either equal to one of Peirce components, or is an irreducible bimodule over $U^{(+)} = J(V, f)$ with the symmetrized action. We start by considering the modules which are Peirce-homogeneous.

Lemma 4.2.2. *Let $J = J(V, f, \star)$ be a superalgebra of a nondegenerate bilinear form, $\dim V > 1$, and let $\star$ be a superanticommutative f-invariant multiplication on V.*

1) *There is no nonzero unital Jordan bimodule M over J such that $M = M_0(e)$ or $M = M_2(e)$ for an even idempotent $e \in J$.*

2) *Let $U = U(V, f, \star)$, and let M be a linear superspace. Define the map $R^+ : U \to \operatorname{End}(M)$ by $R_1^+ = \mathrm{id}$, $R^+|_V = 0$. Then for any linear mapping $R^- : U \to \operatorname{End}(M)$ the pair (R^+, R^-) is a unital noncommutative Jordan representation of U on M, and $M = M_1(e)$ for all idempotents $e \in M$. This module is irreducible if and only if it is irreducible with respect to R^- action (in particular, if M is finite-dimensional, then R^- must be surjective).*

3) *Let M be a unital noncommutative Jordan bimodule over $U = U(V, f, \star)$ such that $M = M_1(e)$ with respect to an even idempotent $e \in U$. Then the action of U on M is of the form above.*

Proof. 1) Recall from Section 2.2.1 that a nontrivial even idempotent e of J must be of the form $e = 1/2 + v$, where $v \in V_{\bar{0}}$ is such that $f(v, v) = 1/4$. The Peirce decomposition of J with respect to e is as follows:

$$J_0 = \left\langle \frac{1}{2} - v \right\rangle, \quad J_1 = \{u \in V : f(u, v) = 0\}, \quad J_2 = \left\langle \frac{1}{2} + v \right\rangle. \tag{4.2.2}$$

Therefore, it suffices to only consider modules M with $M = M_2(e)$. Let M be such module. Peirce relations imply that $MR^+_{J_1} = 0, R^+_v|_M = \mathrm{id}/2$. Note that J_1 is nonzero, otherwise $J \cong \mathbb{F} \oplus \mathbb{F}$. The irreducibility of f implies that there exist vectors $u, w \in J_1$ such that $f(u,w) = 1$. Substituting $a = v,\ b = u,\ c = w$ in (1.1.1) and restricting the relation to M, we obtain $1/2\ \mathrm{id}_M = 0$.

2) This can be checked by direct verification of relations (1.1.1), (1.12.1), (1.12.2) using the description of idempotents in U given in (4.2.2).

3) Let $M = M_1(e)$ for some idempotent $e = 1/2 + v$. Then we have $MR^+_V = 0$ (for elements in J_1 this follows from Peirce relations, and for v it follows from the fact that $R^+_e|_M = \mathrm{id}/2$). The rest follows from the previous point. □

In case of $\mathbb{F}$ algebraically closed of characteristic 0 the classification of irreducible finite-dimensional representations of the simple Jordan superalgebra $J(V,f)$ was obtained in the paper [MZ10]. We provide the classification here.

Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a vector superspace with a nondegenerate supersymmetric form, let $v_1, \ldots, v_n$ be an orthonormal base of $V_{\bar 0}$, and let $w_1, \ldots, w_{2m}$ be a base of $V_{\bar 1}$ such that

$$f(w_i, w_{2m+1-i}) = 1 = -f(w_{2m+1-i}, w_i),$$

while all other products are zero.

Let C be the Clifford algebra of (V,f). The products

$$v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}},\ \text{where } 0 \leqslant i_1, \ldots, i_n \leqslant 1,\ k_1, \ldots, k_{2m} \geqslant 0,$$

form a basis of C. Consider the subspace $C_r = \sum_{i\leqslant r} V\cdots V$ of all basic products of length $\leqslant r$. If $r < 0$, then we set $C_r = \{0\}$. One can check that for an odd r, the space C_r is a J-submodule of C with respect to the regular action. Indeed, for $v, u_1, \ldots, u_r \in V$ we have

$$\begin{aligned} &u_1\cdots u_r v + (-1)^{(u_1\cdots u_r)v} v u_1 \cdots u_r \\ &\quad = \sum_{k=1}^{r} (-1)^{k+1+(u_{k+1}\cdots u_r)v} u_1\cdots u_{k-1}(u_k v + (-1)^{u_k v} v u_k) u_{k+1}\cdots u_r, \end{aligned} \tag{4.2.3}$$

and it is easy to see that the expression on the right side lies in C_r.

Let u be an even vector, $V' = V \oplus \mathbb{F}u$. Let us extend the superform f to V' via $f(u,u) = 1$ and $f(u,V) = 0$. Consider the subspace $C'_r u C_r = \sum_{i\leqslant r} V'\cdots V'$. Analogously, if r is even, then C'_{r+1} is a $J' = \mathbb{F} + V'$-submodule of C'.

Theorem 4.2.3 ([MZ10])**.** *If the ground field $\mathbb{F}$ is algebraically closed and of characteristic 0, then the only finite-dimensional unital irreducible Jordan bimodules over $J(V,f)$ are C_r/C_{r-2} if $r \geqslant 1$ is odd and uC_r/uC_{r-2} if $r \geqslant 0$ is even.*

Note that the module C_1 is isomorphic to the regular J-bimodule. The module $M\mathbb{F}u$ is one-dimensional such that $uR_V^+ = 0$. Thus, $M = M_1(e)$ for any idempotent $e \in J$, and the structures of noncommutative Jordan bimodules on such modules were considered in Lemma 4.2.2.

Now we classify all possible structures of noncommutative Jordan bimodules on the modules above. We begin with the regular module:

Proposition 4.2.4. *Let V be a noncommutative Jordan bimodule over $U = U(V,f,\star)$ such that as a Jordan module over $U^{(+)}$ it is isomorphic to* $\mathrm{Reg}(J(V,f))$. *Then $V \cong$* $\mathrm{Reg}(U(V,f,\star))$ *as a U-module. The same statement holds if $V \cong \mathrm{Reg}(J(V,f))^{\mathrm{op}}$ as a Jordan module.*

Proof. This is done by a direct computation. Let $\{1',\ v_i', i = 1,\dots,n,\ w_j', j = 1,\dots,2m\}$ be a basis of $\mathrm{Reg}\,J(V,f)$ with the obvious Jordan action. Write the minus action in this basis:

$$1'R_{v_i}^- = a_i 1' + \sum_{k=1}^{n} a_{ki} v_k', \qquad 1'R_{w_p}^- = \sum_{q=1}^{2m} b_{qp} w_q',$$

$$v_i'R_{v_j}^- = c_{ij} 1' + \sum_{k=1}^{n} c_{ijk} v_k', \qquad v_i'R_{w_p}^- = \sum_{q=1}^{2m} d_{ipq} w_q',$$

$$w_p'R_{v_i}^- = \sum_{q=1}^{2m} e_{piq} w_q', \qquad w_p'R_{w_q}^- = f_{pq} 1' + \sum_{k=1}^{n} f_{pqk} v_k'.$$

We check the relations (1.12.1), (1.12.2) on the basis elements of V. Note that the right part of the relation (1.12.2) is always zero for $a, b \in V$.

Act by the relation (1.12.2) with $a = b = v_i$ on the element $1'$:

$$0 = 2 \cdot 1'R_{v_i}^- R_{v_i}^+ = 2a_i v_i' + 2a_{ii} 1',$$

so we get $a_i = 0,\ a_{ii} = 0$. Acting by the relation (1.12.1) with $a = v_i,\ b = v_j$ on the element $1'$, we get

$$1'[R_{v_i}^+, R_{v_j}^-] = c_{ij} 1' + \sum_{k=1}^{n} c_{ijk} v_k' - a_{ij} 1' = \frac{1}{2} 1' R_{[v_i,v_j]}^+ = \frac{1}{2} \sum_{k=1}^{n} \alpha_{ijk} v_k',$$

so we get $a_{ij} = c_{ij},\ c_{ijk} = \alpha_{ijk}/2$. Acting by the relation (1.12.1) with $a = v_i,\ b = v_j$ on the element $v_k',\ k \neq i$, we have

$$v_k'[R_{v_i}^+, R_{v_j}^-] = -c_{kj} v_i' - c_{kji} 1' = \frac{1}{2} v_k' R_{[v_i,v_j]}^+ = \frac{1}{2} \alpha_{ijk} 1',$$

which implies $a_{ij} = 0,\ c_{ij} = 0$.

Acting by the relation (1.12.1) with $a = v_i,\ b = w_p$ on the element $1'$, we get

$$1'[R_{v_i}^+, R_{w_p}^-] = \sum_{q=1}^{2m} d_{ipq} w_q' = \frac{1}{2} 1' R_{[v_i,w_p]}^+ = \frac{1}{2} \sum_{q=1}^{2m} \beta_{ipq} w_q',$$

so we get $d_{iqp} = \beta_{iqp}/2$. Analogously, taking $a = w_p,\ b = v_i$ we get $e_{piq} = -\beta_{iqp}/2$.

Act by the relation (1.12.2) with $a = w_p,\ b = w_q$, where $p \neq q$, on the element w'_r:

$$0 = w'_r(R^-_{w_p}R^+_{w_q} - R^-_{w_q}R^+_{w_p}) = f_{rp}w'_q - f_{rq}w'_p,$$

which implies $f_{rp} = 0$. Finally, act by the relation (1.12.1) with $a = w_p,\ b = w_q$ on the element $1'$:

$$1'[R^+_{w_p}, R^-_{w_q}] = \sum_{k=1}^{n} f_{pqk}v'_k \pm b_{2m+1-p,q}1' = \frac{1}{2}1'R^+_{[w_p,w_q]} = \frac{1}{2}\sum_{k=1}^{n}\gamma_{pqk}v'_k,$$

which implies $b_{pq} = 0,\ f_{pqk} = \gamma_{pqk}/2$ for all $p, q = 1, \ldots, 2m,\ k = 1, \ldots, n$. So, all coefficients of the minus action are determined uniquely and are equal to those of the minus action on the regular bimodule. The proof for the opposite of the regular module is completely analogous and we omit it. □

Now we consider the structures of noncommutative Jordan bimodules arising on the modules C_{r+1}/C_{r-1} and uC_r/uC_{r-2}. The main result is that there are no nontrivial such structures:

Proposition 4.2.5. *Let V be a noncommutative Jordan bimodule over $U = U(V, f, \star)$ and suppose that as a Jordan bimodule V is isomorphic to C_r/C_{r-2} or uC_r/uC_{r-2} for $r > 1$. Then $\star = 0$ (that is, $U = J(V, f)$) and V is a Jordan bimodule over U.*

Proof. Note that the second assertion of the theorem follows from the first and the results of the paper [Pop20]. So it suffices to prove that $\star = 0$. We begin with writing down the bases of the modules above and writing down the explicit Jordan action of $J(V, f)$ in those bases.

Let us first consider the module C_r/C_{r-2} for r odd. Recall that the products $v_1^{i_1}\cdots v_n^{i_n}w_1^{k_1}\cdots w_{2m}^{k_{2m}}$, where $0 \leqslant i_1, \ldots, i_n \leqslant 1,\ k_1, \ldots, k_{2m} \geqslant 0$, form a basis of C. We write such element as $v_I w_J$, where I is the subset of the set $I_n = \{1, \ldots, n\}$ defined by $I = \{k \in I : i_k = 1\}$, and J is the multiset of all powers $k_j, j = 1, \ldots, 2m$. Then we can choose a base of C_r/C_{r-2} as $\{v_I w_J, |I| + |J| = r-1, r\}$. Using the formula (4.2.3) one can write down the Jordan action of $J(V, f)$ in this base. Our goal is to obtain the equalities $\alpha_{ijk} = 0 = \beta_{ipq}$ for all possible values of i, j, k, p, q, so we may consider the actions only up to a sign.

Let $|I| + |J| = r - 1$. Then

$$v_I w_J R^+_{v_i} = \begin{cases} 0, & i \in I, \\ \pm v_{I\cup\{i\}}w_J, & i \notin I. \end{cases},$$

$$v_I w_J R^+_{w_j} = v_I w_{J\cup\{j\}}$$

Let $|I|+|J|=r$. Then

$$v_Iw_JR^+_{v_i}=\begin{cases}0, & i\notin I,\\ \pm v_{I\setminus\{i\}}w_J, & i\in I\end{cases},$$

$$v_Iw_JR^+_{w_j}=\begin{cases}0, & 2m+1-j\notin J,\\ \pm k^J_{2m+1-j}v_Iw_{J\setminus\{2m+1-j\}}, & 2m+1-j\in J\end{cases}.$$

Now, for the module uC_r/uC_{r-2}, where $r>0$ is even, we can choose a base of the form $\{uv_Iw_J, |I|+|J|=r-1,r\}$, and using (4.2.3) one can see that the R^+ action of $J(V,f)$ on uC_r/uC_{r-2} is of the same form as above (up to a sign which we can discard). So we will only consider the module C_r/C_{r-2} here.

Note that $C_r/C_{r-2}=S_{r-1}\oplus S_r$, where $S_k,\ k=r-1,r$ is the space of elements of degree r. Denote by π_r the projection onto S_r along S_{r-1}.

Let $J(V,f,\star)$ act on C_r/C_{r-2} in the following way:

$$v_Iw_JR^-_{v_i}=\sum c_{IJiPQ}v_Pw_Q,\quad v_Iw_JR^-_{w_p}=\sum d_{IJpPQ}v_Pw_Q.$$

If this action is noncommutative Jordan, then the relations (1.12.1), (1.12.2) must hold. Act by the relation (1.12.2) with $a=b=v_i$ on the element v_Iw_J:

$$0=v_Iw_JR^-_{v_i}R^+_{v_i}=\sum_{\substack{|P|+|Q|=r-1\\ i\notin P}}\pm c_{IJiPQ}v_{P\cup\{i\}}w_Q+\sum_{\substack{|P|+|Q|=r\\ i\in P}}\pm c_{IJiPQ}v_{P\setminus\{i\}}w_Q.$$

Thus, we have

$$|P|+|Q|=r-1,\ i\notin P\Rightarrow c_{IJiPQ}=0, \tag{4.2.4}$$
$$|P|+|Q|=r,\qquad i\in P\Rightarrow c_{IJiPQ}=0. \tag{4.2.5}$$

Act by (1.12.2) with $a=w_p,\ b=w_q$ for $p\neq q$ on the element v_Iw_J and apply projection π_r:

$$0=v_Iw_J(R^-_{w_p}R^+_{w_q}-R^-_{w_q}R^+_{w_p})\pi_r=\sum_{|P|+|Q|=r-1}\pm d_{IJpPQ}v_Pw_{Q\cup\{q\}}-\sum_{|P|+|Q|=r-1}\pm d_{IJqPQ}v_Pw_{Q\cup\{p\}}.$$

In particular, we get

$$|P|+|Q|=r-1,\ p\notin Q\Rightarrow d_{IJpPQ}=0. \tag{4.2.6}$$

Act by (1.12.2) with $a=v_i,\ b=w_p$ on the element v_Iw_J and apply π_r:

$$\begin{aligned}0&=v_Iw_J(R^-_{v_i}R^+_{w_p}+R^-_{w_p}R^+_{v_i})\pi_r=\ \text{(by (4.2.4), (4.2.6))}\\ &=\sum_{\substack{|P|+|Q|=r-1\\ i\in P}}\pm c_{IJiPQ}v_Pw_{Q\cup\{p\}}+\sum_{\substack{|P|+|Q|=r-1\\ i\notin P,\ p\in Q}}\pm d_{IJpPQ}v_{P\cup\{i\}}w_Q.\end{aligned}$$

Therefore, we have

$$|P|+|Q|=r-1,\ i\in P\Rightarrow c_{IJiPQ}=\pm d_{IJpP\setminus\{i\}Q\cup\{p\}} \text{ for any } p=1,\dots,2m. \tag{4.2.7}$$

Act by (1.12.1) with $a=v_i,\ b=v_j$ on the element v_Iw_J, where $|I|+|J|=r-1,\ i\in I,\ j\neq i$, and apply π_r:

$$v_Iw_J[R^+_{v_i},R^-_{v_j}]\pi_r = \text{ (by (4.2.4)) } = -\sum_{\substack{|P|+|Q|=r-1\\ j\in P,\ i\notin P}} \pm c_{IJjPQ}v_{P\cup\{i\}}w_Q$$
$$=\frac{1}{2}v_Iw_JR^+_{[v_i,v_j]}\pi_r=\frac{1}{2}\sum_{\substack{k\notin I\\ k\neq i,j}}\alpha_{ijk}v_{I\cup\{k\}}w_J.$$

Note that if $n\leqslant 2$, then all $\alpha_{ijk}=0$ by (4.2.1). If $n\geqslant 3$, then taking $I=\{i\}$ and $j,k\neq i$ we see that the vector $v_{\{i,k\}}w_J$ appears only at the right hand side with the coefficient α_{ijk}. Therefore,

$$\alpha_{ijk}=0 \text{ for all } i,j,k=1,\dots,n. \tag{4.2.8}$$

Act by (1.12.1) with $a=v_i,\ b=v_j$ on the element v_Iw_J, where $|I|+|J|=r-1,\ i\notin I,\ j\neq i$:

$$0=v_Iw_J[R^+_{v_i},R^-_{v_j}]\pi_r = \text{ (by (4.2.4), (4.2.5))}$$
$$=\sum_{\substack{|P|+|Q|=r\\ j\notin P}}\pm c_{I\cup\{i\}JjPQ}v_Pw_Q+\sum_{\substack{|P|+|Q|=r-1\\ j\in P,\ i\notin P}}\pm c_{IJjPQ}v_{P\cup\{i\}}w_Q.$$

In particular, we have

$$|I|+|J|=r-1,\ |P|+|Q|=r-1,\ j\in P,\ P\cup I\neq I_n\Rightarrow c_{IJjPQ}=0. \tag{4.2.9}$$

Act by the relation (1.12.1) with $a=v_i,\ b=w_p$ on the element v_Iw_J, where $|I|+|J|=r-1,\ i\in I$, and apply π_r:

$$v_Iw_J[R^+_{v_i},R^-_{w_p}]\pi_r = \text{ (by (4.2.6)) } = -\sum_{\substack{|P|+|Q|=r-1\\ p\in Q,\ i\notin P}}\pm d_{IJpPQ}v_{P\cup\{i\}}w_Q$$
$$=\frac{1}{2}v_Iw_JR^+_{[v_i,w_p]}\pi_r=\frac{1}{2}\sum_{r=1}^{n}\beta_{ipr}v_Iw_{J\cup\{r\}}.$$

For any $p\neq q=1,\dots,2m$ consider $I=\{i\}$ and $J=\{q,\dots,q\}$ ($r-1$ times). Then the vector $v_{\{i\}}w_{\{q,\dots,q\}}$ appears only at the right hand side with the coefficient β_{ipq}. Therefore, if $p\neq q$, then $\beta_{ipq}=0$. We can kill β_{ipp} as well: the same equation yields that for any I,J such that $i\in I,\ |I|+|J|=r-1$ we have

$$\frac{1}{2}\beta_{ipp}=\pm d_{IJpI\setminus\{i\}J\cup\{p\}} = \text{ (by (4.2.7)) } = c_{IJiIJ}.$$

Taking $I=\{i\}$ (and recalling that $n\geqslant 2$), the equation (4.2.9) implies that $\beta_{ijj}=0$. This, together with (4.2.8) and (4.2.1) implies that $\star=0$ and that U is Jordan. $\square$

Therefore, we have proved the following result:

Theorem 4.2.6. *Let M be an irreducible finite-dimensional unital noncommutative Jordan module over $U = U(V, f, \star)$, where f is irreducible, and the ground field $\mathbb{F}$ is algebraically closed and of characteristic* 0. *Then one of the following options holds*:

1) $M \cong \mathrm{Reg}(U)$ *or* $M \cong \mathrm{Reg}(U)^{\mathrm{op}}$;

2) *U is Jordan and M is an irreducible Jordan U-module*;

3) *The restriction of the action R^+ to V is zero, and $R^-|_V$ is an arbitrary surjective linear map onto* $\mathrm{End}(M)$.

4.3 Irreducible bimodules over $K(\Gamma_n, A)$

In this section we study modules over the superalgebra $K(\Gamma_n, A)$. As in the last section, first we recall the definition of the algebra, and then classify its irreducible finite-dimensional representations.

4.3.1 The superalgebra $K(\Gamma_n, A)$

Let Γ be the Grassmann superalgebra in generators $1, x_i,\ i \in \mathbb{N}$, and Γ_n be the Grassmann superalgebra in generators $1, x_1, \dots, x_n$. We define the new operation $\{\cdot, \cdot\}$ on Γ (*the Poisson-Grassmann bracket*) by

$$\{f, g\} = (-1)^f \sum_{j=1}^{\infty} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j}, \tag{4.3.1}$$

where

$$\frac{\partial}{\partial x_j}(x_{i_1} x_{i_2} \dots x_{i_n}) = \begin{cases} (-1)^{k-1} x_{i_1} x_{i_2} \dots x_{i_{k-1}} x_{i_{k+1}} \dots x_{i_n}, & \text{if } j = i_k, \\ 0, & \text{if } j \notin \{i_1, i_2, \dots, i_n\}. \end{cases}$$

Let $\overline{\Gamma}$ be an isomorphic copy of Γ with the isomorphism mapping $x \mapsto \bar{x}$. We define the structure of a Jordan superalgebra on $K(\Gamma) = \Gamma \oplus \overline{\Gamma}$, by setting $K(\Gamma)_{\bar{0}} = \Gamma_{\bar{0}} + \overline{\Gamma}_{\bar{1}},\ K(\Gamma)_{\bar{1}} = \Gamma_{\bar{1}} + \overline{\Gamma}_{\bar{0}}$ and defining multiplication by the rule

$$a \circ b = ab, \quad \bar{a} \circ b = (-1)^b \overline{ab}, \quad a \circ \bar{b} = \overline{ab}, \quad \bar{a} \circ \bar{b} = (-1)^b \{a, b\},$$

where $a, b \in \Gamma_{\bar{0}} \cup \Gamma_{\bar{1}}$ and ab is their product in Γ. By $K(\Gamma_n)$ we will denote the subsuperalgebra $\Gamma_n + \overline{\Gamma}_n$ of $K(\Gamma)$.

If Γ_n is considered as a Poisson superalgebra (with the Poisson bracket $\{\cdot, \cdot\}$), then it is easily seen that $K(\Gamma_n)$ is the *Kantor double* of Γ_n [Kan92].

Further in the paper we will need a basis of this algebra and the multiplication table in this basis, so we provide them here. Recall that Γ_n has a basis $\{1, e_{i_1} \dots e_{i_k},\ 1 \leqslant i_1 < \dots < i_k \leqslant n\}$. For an ordered subset $I = \{i_1, i_2, \dots, i_k\} \subseteq I_n = \{1, 2, \dots, n\}$, we denote

$$e_I = e_{i_1} e_{i_2} \cdots e_{i_k}, \quad \overline{e_I} = \overline{e_{i_1} e_{i_2} \cdots e_{i_k}},$$

in particular, $e_\varnothing = 1$, and $\overline{e_\varnothing} = \bar{1}$. Now, as $e_i e_j = -e_j e_i$, for $i, j \in I_n$, $i \neq j$, if σ is a permutation of the set I, we have $e_I = \operatorname{sgn}(\sigma) e_{\sigma(I)}$, where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

For ordered subsets $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_s\}$, denote by $I \cup J$ the ordered set

$$I \cup J = \{i_1, \dots, i_k, j_1, \dots, j_s\}.$$

Then the multiplication table in the basis $\{e_I, \overline{e_J},\ I, J \subseteq I_n\}$ is as follows:

$$e_I \circ e_J = \begin{cases} e_{I \cup J}, & \text{if } I \cap J = \varnothing \\ 0, & \text{if } I \cap J \neq \varnothing \end{cases},$$

$$e_I \circ \overline{e_J} = \begin{cases} \overline{e_{I \cup J}}, & \text{if } I \cap J = \varnothing \\ 0, & \text{if } I \cap J \neq \varnothing \end{cases},$$

$$\overline{e_I} \circ e_J = \begin{cases} (-1)^s \overline{e_{I \cup J}}, & \text{if } I \cap J = \varnothing \\ 0, & \text{if } I \cap J \neq \varnothing \end{cases},$$

$$\overline{e_I} \circ \overline{e_J} = (-1)^s \{e_I, e_J\} = \begin{cases} (-1)^{s+k+p+q} e_{I' \cup J'}, & \text{if } I \cap J = \{i_p\} = \{j_q\} \\ 0, & \text{otherwise} \end{cases}$$

where $I' = \{i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_k\}$ and $J' = \{j_1, \dots, j_{q-1}, j_{q+1}, \dots, j_s\}$. We will use the notation $\circ$ only in the presence of other multiplications.

Let $A \in (\Gamma_n)_{\bar{0}}$. Define a superanticommutative binary bilinear operation $[\cdot, \cdot]$ on $K(\Gamma_n)$ by the rule

$$[\bar{a}, \bar{b}] = (-1)^b abA,$$

and zero otherwise. One can check that this operation is a generic Poisson bracket on $K(\Gamma_n)$. The resulting superalgebra $(K(\Gamma_n), \cdot, \{\cdot, \cdot\})$ will be denoted by $J(\Gamma_n, A)$. In the paper [PS13] it was proved that any generic Poisson bracket on $K(\Gamma_n)$ is of the form above.

4.3.2 Classification of irreducible representations

In this section we study irreducible representations of superalgebras $K(\Gamma_n, A)$. As Theorem 4.1.8 suggests, if M is an irreducible bimodule over $U = K(\Gamma_n, A)$, then it is either equal to one of its Peirce components, or is an irreducible bimodule over

$U^{(+)} = K(\Gamma_n)$ with the symmetrized action. First we prove that the modules of the first type do not exist:

Lemma 4.3.1. *There is no irreducible U-bimodule M such that M is not irreducible as a Jordan $U^{(+)}$-module and $M = M_i(e),\ i = 0,1,2$ for some idempotent e.*

Proof. Note that $K(\Gamma_n,A)^{(+)}$ contains an even subalgebra $J = \langle 1, \overline{x}_1, \dots, \overline{x}_n\rangle$ which is isomorphic to an algebra of a symmetric nondegenerate bilinear form. Hence, if M is a bimodule over $K(\Gamma_n,A)$ such that $M = M_0$ or M_2, then it has the same property over J. But Lemma 4.2.2 implies that Jordan J-bimodules with the property $M = M_0(e)$ or $M = M_2(e)$ do not exist. Thus Theorem 4.1.8 implies that $M = M_1(e)$ for any idempotent $e \in U$. Consider the family of (non-orthogonal) idempotents $e_k = \dfrac{1+\overline{x}_k}{2} \in U,\ k = 1,\dots,n$. Let us compute the Peirce component $U_1(e_k)$ for each k. It is easy to see that

$$U_i(e_k) = \{x : x \circ e_k = \frac{i}{2}x\} = \{x \circ \overline{x}_k = (i-1)\overline{x}_k\}.$$

Writing any element $x \in U$ in the form $x = a_1 + x_k a_2 + \overline{b_1} + \overline{x_k b_2}$, where $\dfrac{\partial a_1}{\partial x_k} = \dfrac{\partial a_2}{\partial x_k} = \dfrac{\partial b_1}{\partial x_k} = \dfrac{\partial b_2}{\partial x_k} = 0$, one can easily see that $U_1(e_k) = \{a x_k + \overline{b} : \dfrac{\partial a}{\partial x_k} = \dfrac{\partial b}{\partial x_k} = 0\}$. Since $M = M_1(e_k)$ for any k, for the space $U' = \sum_{k=1}^{n} U_1(e_k)$ we must have $MR^+_{U'} = 0$. But it is obvious that U' contains every monomial in letters $x_1,\dots,x_n,\overline{x}_1,\dots,\overline{x}_n$ except 1 and $\overline{x_1 \dots x_n}$. Substituting in the relation (1.1.1) $a = \overline{x_1},\ b = \overline{x}_1,\ c = \overline{1}$ and using the fact that $R^+_a = R^+_b = R^+_c = 0$ on M, we get $R^+_1 = 0$ on M, which is clearly a contradiction. □

Now we may suppose that M is an irreducible Jordan $K(\Gamma_n)$-bimodule. Irreducible finite-dimensional Jordan $K(\Gamma_n)$-bimodules were described in [SFS16]. It was proved that every irreducible module over $K(\Gamma_n)$ is isomorphic to a member of the family $V(\alpha),\ \alpha \in \mathbb{F}$, where each $V(\alpha)$ has the base of the form $v_I,\ \overline{v_I},\ I \subseteq I_n = \{1,\dots,n\}$, and $v_I = \operatorname{sgn}(\sigma) v_{\sigma(I)},\ \overline{v_I} = \operatorname{sgn}(\sigma)\overline{v_{\sigma(I)}}$ for any permutation σ of a set I. Furthermore, let $I, J \subseteq I_n,\ J = \{j_1,\dots,j_{s_1}, j_{s_1+1},\dots,j_{s_1+s_2}\},\ I = \{i_1,\dots,i_{k-s_1}, j_{s_1},\cdots,j_1\}$. Then the action of $K(\Gamma_n)$ on $V(\alpha)$ is defined, up to permutations of the index sets I and J, as follows:

$$v_I e_J = \begin{cases} v(I\backslash J), & s_2 = 0 \\ 0, & \text{otherwise} \end{cases}, \qquad v_I \overline{e_J} = \begin{cases} \overline{v(I\backslash J)}, & s_2 = 0 \\ 0, & \text{otherwise} \end{cases},$$

$$\overline{v_I} e_J = \begin{cases} (-1)^s \overline{v(I\backslash J)}, & s_2 = 0 \\ 0, & \text{otherwise} \end{cases}, \quad \overline{v_I}\,\overline{e_J} = \begin{cases} (-1)^{s_1} v((I\backslash J) \cup \{j_{s_1+1}\}), & s_2 = 1 \\ (-1)^{s-1}\alpha(s-1)\, v(I\backslash J), & s_2 = 0, \\ 0, & \text{otherwise} \end{cases},$$

where $\alpha = R^2_{\overline{1}}$, and $s = s_1 + s_2 = |J|$. Note that the condition $s_2 = 0$ is equivalent to $J \subseteq I$. The module $V(0)$ is isomorphic to the regular $K(\Gamma_n)$-module.

Let $K(\Gamma_n, A)$ act on $V(\alpha)$ in the following way:

$$v_I R^-_{e_J} = \sum_{K\subseteq I_n} c_{IJK} v_K + \sum_{K\subseteq I_n} c_{IJ\overline{K}} \overline{v_K}, \qquad v_I R^-_{\overline{e_J}} = \sum_{K\subseteq I_n} c_{I\overline{J}K} v_K + \sum_{K\subseteq I_n} c_{I\overline{J}\overline{K}} \overline{v_K},$$
$$\overline{v_I} R^-_{e_J} = \sum_{K\subseteq I_n} c_{\overline{I}JK} v_K + \sum_{K\subseteq I_n} c_{\overline{I}J\overline{K}} \overline{v_K}, \qquad \overline{v_I} R^-_{\overline{e_J}} = \sum_{K\subseteq I_n} c_{\overline{I}\overline{J}K} v_K + \sum_{K\subseteq I_n} c_{\overline{I}\overline{J}\overline{K}} \overline{v_K}.$$

For $V(\alpha)$ to be a noncommutative Jordan bimodule, the R^- action has to satisfy the relations (1.12.1), (1.12.2). We check these relations on the basis elements of $V(\alpha)$ and $K(\Gamma_n, A)$. Therefore, we obtain a system of linear equations for the structure constants of the R^- action.

Note that $[e_I, e_J] = [e_I, \overline{e_J}] = [\overline{e_I}, e_J] = 0$ for all $I, J \subseteq I_n$. We first check the relation (1.12.1) for the commutators of this type.

Consider the relation $v_I[R^+_{e_J}, R^-_{e_K}] = 0$.

Case 1: $J \nsubseteq I$.

$$0 = v_I[R^+_{e_J}, R^-_{e_K}] = -(-1)^{|J||K|} v_I R^-_{e_K} R^+_{e_J} = \left(\sum_{L\subseteq I_n} c_{IKL} v_L + c_{IK\overline{L}}\overline{v_L}\right) R^+_{e_J}$$
$$= \sum_{J\subseteq L\subseteq I_n} c_{IKL} v_{L\setminus J} + (-1)^{|J|} c_{IK\overline{L}} \overline{v_{L\setminus J}}.$$

Note that in the equation above we have divided everything by $-(-1)^{|J||K|}$ just for convenience. From now on we will frequently divide the expressions equal to 0 by similar scalars in order to save space. From the above equation we infer that if there exists a set J such that $J \nsubseteq I,\ J \subseteq L$, then we must have $c_{IKL} = c_{IK\overline{L}} = 0$, or in other words

$$L \nsubseteq I \Rightarrow c_{IKL} = c_{IK\overline{L}} = 0. \tag{4.3.2}$$

Case 2: $J \subseteq I$.

$$0 = v_I[R^+_{e_J}, R^-_{e_K}] = \text{ (by (4.3.2)) } = v_{I\setminus J} R^-_{e_K} - (-1)^{|J||K|} \left(\sum_{L\subseteq I} c_{IKL} v_L + c_{IK\overline{L}}\overline{v_L}\right) R^+_{e_J}$$
$$= \sum_{L\subseteq I\setminus J} c_{I\setminus JKL} v_L + c_{I\setminus JK\overline{L}} \overline{v_L} - (-1)^{|J||K|} \left(\sum_{J\subseteq L\subseteq I} c_{IKL} v_{L\setminus J} + (-1)^{|J|} c_{IK\overline{L}} \overline{v_{L\setminus J}}\right).$$

Therefore, if $J \subseteq I,\ J \cap L = \varnothing$, then $c_{I\setminus JKL} = (-1)^{|J||K|} c_{IK(L\cup J)}$, or in other words

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{IKL} = (-1)^{|J||K|} c_{(I\cup J)K(L\cup J)}. \tag{4.3.3}$$

Analogous relation holds for $c_{IK\overline{L}}$:

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{IK\overline{L}} = (-1)^{|J|(|K|+1)} c_{(I\cup J)K\overline{(L\cup J)}}. \tag{4.3.4}$$

Consider the relation $v_I[R^+_{\overline{e_J}}, R^-_{e_K}] = 0$.

Case 1: $J \nsubseteq I$.

$$
\begin{aligned}
0 = v_I[R^+_{\overline{e_J}}, R^-_{e_K}] &= -(-1)^{(|J|+1)|K|} v_I R^-_{e_K} R^+_{\overline{e_J}} = \ \text{(by (4.3.2))} \\
&= \left(\sum_{L\subseteq I} c_{IKL} v_L + c_{IK\overline{L}} \overline{v_L} \right) R^+_{\overline{e_J}} = \text{(since } J \nsubseteq I) \ = (-1)^{|J|-1} \sum_{(*)} c_{IK\overline{L}} v_{L\triangle J},
\end{aligned}
$$

where the condition $(*)$ defines the sets $L \subseteq I$ such that $L = \{l_1, \dots, l_{k-s+1}, j_{s-1}, \dots, j_1\}$ if $J = \{j_1, \dots, j_s\}$, and by $L\triangle J$ we denote the ordered set that is the index of the element $\overline{v_L} R^+_{\overline{e_J}}$ (by definition it is $\{l_1, \dots, l_{k-s+1}, j_s\}$, which is just the symmetric difference of L and J, hence the notation). Note that since $\overline{v_I} = \operatorname{sgn}(\sigma)\overline{v_{\sigma(I)}}$, we can always reorder the sets L to get the the correct action of J, though most of the times it will not matter since the sum will be equal to 0. From now on, we will assume that the sets L are properly ordered (corresponding to the sense above) and denote the condition $(*)$ as "$L \subseteq I,\ |J\backslash I| = 1$".

Note that $L_1\triangle J \neq L_2\triangle J$ for two distinct sets L_1, L_2 satisfying $(*)$. Therefore, if there exists a set J such that $J \nsubseteq I,\ |J\backslash L| = 1$, then $c_{IK\overline{L}} = 0$ for $L \subseteq I$ and any set K. Equivalently, this condition can be written as

$$
I \neq I_n \Rightarrow c_{IK\overline{L}} = 0. \tag{4.3.5}
$$

Case 2: $J \subseteq I$.

$$
\begin{aligned}
0 = v_I[R^+_{\overline{e_J}}, R^-_{e_K}] = \ &\text{(by (4.3.2))} \ \overline{v_{I\backslash J}} R^-_{e_K} - (-1)^{(|J|+1)|K|} \left(\sum_{L\subseteq I} c_{IKL} v_L + c_{IK\overline{L}} \overline{v_L} \right) R^+_{\overline{e_J}} \\
= &\sum_{L\subseteq I\backslash J} c_{\overline{I\backslash J}KL} v_L + c_{\overline{I\backslash J}K\overline{L}} \overline{v_L} \\
&- (-1)^{(|J|+1)|K|} \Bigg(\sum_{J\subseteq L\subseteq I} \big(c_{IKL} \overline{v_{L\backslash J}} + (-1)^{|J|-1} \alpha(|J|-1) c_{IK\overline{L}} v_{L\backslash J} \big) \\
&+ (-1)^{|J|-1} \sum_{\substack{L\subseteq I \\ |J\backslash L|=1}} c_{IK\overline{L}} v_{L\triangle J} \Bigg).
\end{aligned}
$$

Consider the sum involving only the v_L terms:

$$
\sum_{L\subseteq I\backslash J} c_{\overline{I\backslash J}KL} v_L - (-1)^{(|J|+1)|K|+|J|-1} \left(\sum_{J\subseteq L\subseteq I} \alpha(|J|-1) c_{IK\overline{L}} v_{L\backslash J} + \sum_{\substack{L\subseteq I \\ |J\backslash L|=1}} c_{IK\overline{L}} v_{L\triangle J} \right) = 0. \tag{4.3.6}
$$

Taking $J = \varnothing$, we get

$$
\sum_{L\subseteq I} c_{\overline{I}KL} v_L - (-1)^{|K|} \sum_{L\subseteq I} \alpha c_{IK\overline{L}} v_L = 0,
$$

hence,

$$
c_{\overline{I}KL} = (-1)^{|K|} \alpha c_{IK\overline{L}}. \tag{4.3.7}
$$

Suppose now that $J \neq \varnothing$. Note that the sets $L \triangle J$ indexing the basis elements v in the third sum in (4.3.6) contain exactly one element from J, while sets indexing the basis elements in other two sums do not contain elements from J. Thus, if $J, L \subseteq I$ and $|J \backslash L| = 1$, we have $c_{IK\overline{L}} = 0$, or equivalently

$$L \subsetneq I \Rightarrow c_{IK\overline{L}} = 0.$$

The relation above and (4.3.5) imply that the only possible nonzero $c_{IK\overline{L}}$ is $c_{I_nK\overline{I_n}}$. But relations (4.3.4) and (4.3.7) imply that

$$c_{IK\overline{L}} = c_{\overline{I}KL} = 0 \text{ for all } I, K, L \subseteq I_n \tag{4.3.8}$$

and the sum (4.3.6) is 0. Consider now the sum involving only $\overline{v_L}$ terms:

$$\sum_{L \subseteq I \backslash J} c_{\overline{I \backslash J}K\overline{L}} v_{\overline{L}} - (-1)^{(|J|+1)|K|} \sum_{J \subseteq L \subseteq I} c_{IKL} \overline{v_{L \backslash J}} = 0.$$

Taking $J = \varnothing$, we get

$$\sum_{L \subseteq I} c_{\overline{I}K\overline{L}} v_{\overline{L}} - (-1)^{|K|} \sum_{L \subseteq I} c_{IKL} \overline{v_L} = 0,$$

hence,

$$c_{\overline{I}K\overline{L}} = (-1)^{|K|} c_{IKL}. \tag{4.3.9}$$

In particular, by (4.3.2) we have

$$L \nsubseteq I \Rightarrow c_{\overline{I}K\overline{L}} = 0. \tag{4.3.10}$$

The relation for coefficients for general J is a consequence of (4.3.9) and (4.3.3).

Consider the relation $v_I[R^+_{e_J}, R^-_{\overline{e_K}}] = 0$.

Case 1: $J \nsubseteq I$.

In this case, completely analogously to the **Case 1** of the relation $v_I[R^+_{e_J}, R^-_{e_K}] = 0$ we obtain

$$L \nsubseteq I \Rightarrow c_{I\overline{K}L} = c_{I\overline{K}\overline{L}} = 0. \tag{4.3.11}$$

Case 2: $J \subseteq I$.

Analogously, in this case we get

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{I\overline{K}L} = (-1)^{|J|(|K|+1)} c_{(I \cup J)\overline{K}(L \cup J)} \tag{4.3.12}$$

and

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{I\overline{K}\overline{L}} = (-1)^{|J||K|} c_{(I \cup J)\overline{K}\overline{(L \cup J)}}. \tag{4.3.13}$$

Consider the relation $\overline{v_I}[R^+_{e_J}, R^-_{e_K}] = 0$.

Case 1: $J \nsubseteq I$.

In this case, (4.3.8) and (4.3.10) imply that the relation is trivial.
Case 2: $J\subseteq I$.

$$\begin{aligned} 0 = v_I[R^+_{e_J}, R^-_{\overline{e_K}}] &= (-1)^{|J|}\overline{v_{I\setminus J}}R^-_{e_K} - (-1)^{|J||K|}\overline{v_I}R^-_{e_K}R^+_{e_J} = \text{ (by (4.3.8), (4.3.10))} \\ &= (-1)^{|J|}\sum_{L\subseteq I\setminus J} c_{\overline{I\setminus J}K\overline{L}}\overline{v_L} - (-1)^{|J|(|K|+1)}\sum_{J\subseteq L\subseteq I} c_{\overline{I}K\overline{L}}v_{L\setminus J}, \end{aligned}$$

which is zero by (4.3.9) and (4.3.3).

Consider the relation $\overline{v_I}[R^+_{\overline{e_J}}, R^-_{e_K}]=0$.
Case 1: $|J\setminus I|\geqslant 2$.

$$0 = \overline{v_I}[R^+_{\overline{e_J}}, R^-_{e_K}] = \overline{v_I}R^-_{e_K}R^+_{\overline{e_J}} = \text{ (by (4.3.8), (4.3.10)) } = \sum_{L\subseteq I} c_{\overline{I}K\overline{L}}\overline{v_L}R^+_{\overline{e_J}} = 0$$

since $|J\setminus I|=2$.
Case 2: $J\subseteq I$.

$$\begin{aligned} 0 = \overline{v_I}[R^+_{\overline{e_J}}, R^-_{e_K}] &= (-1)^{|J|-1}\alpha(|J|-1)v_{I\setminus J}R^-_{e_K} - (-1)^{(|J|-1)|K|}\overline{v_I}R^-_{e_K}R^+_{\overline{e_J}} \\ &= \text{ (by (4.3.8), (4.3.10)) } = (-1)^{|J|-1}\alpha(|J|-1)\sum_{L\subseteq I\setminus J} c_{I\setminus JKL}v_L \\ &- (-1)^{(|J|-1)|K|+|J|-1}\left(\sum_{J\subseteq L\subseteq I}\alpha(|J|-1)c_{\overline{I}K\overline{L}}v_{L\setminus J} + \sum_{\substack{L\subseteq I\\ |J\setminus L|=1}} c_{\overline{I}K\overline{L}}v_{L\triangle J}\right). \end{aligned}$$

Again, supposing that $J\neq\varnothing$ and noting that the sets $L\triangle J$ contain exactly one element from J, while sets indexing the basis elements in other two sums do not contain elements from J. Thus, using relation (4.3.9) we obtain

$$L\subsetneq I \Rightarrow c_{\overline{I}K\overline{L}} = c_{IKL} = 0. \tag{4.3.14}$$

Moreover, the relations (4.3.9) and (4.3.3) imply that the remaining sum is identically zero.
Case 3: $|J\setminus I|=1$.

$$\begin{aligned} 0 = \overline{v_I}[R^+_{\overline{e_J}}, R^-_{e_K}] &= (-1)^{|J|-1}v_{I\triangle J}R^-_{e_K} - (-1)^{(|J|-1)|K|}\overline{v_I}R^-_{e_K}R^+_{\overline{e_J}} \\ &= \text{ (by (4.3.8), (4.3.10), (4.3.14)) } = c_{I\triangle JKI\triangle J}v_{I\triangle J} - (-1)^{(|J|-1)|K|}c_{\overline{I}K\overline{I}}v_{I\triangle J}, \end{aligned}$$

which is 0 by (4.3.9) and (4.3.3).

Consider the relation $\overline{v_I}[R^+_{e_J}, R^-_{\overline{e_K}}]=0$.
Case 1: $J\nsubseteq I$.
In this case, completely analogously to **Case 1** of the relation $v_I[R^+_{e_J}, R^-_{e_K}]=0$ we obtain

$$L\nsubseteq I \Rightarrow c_{\overline{IK}L} = c_{I\overline{KL}} = 0. \tag{4.3.15}$$

Case 2: $J \nsubseteq I$.

Analogously, in this case we get

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{\overline{IK}L} = (-1)^{|J||K|} c_{\overline{(I\cup J)}\overline{K}(L\cup J)} \tag{4.3.16}$$

and

$$J \cap I = J \cap L = \varnothing \Rightarrow c_{\overline{IKL}} = (-1)^{|J|(|K|+1)} c_{\overline{(I\cup J)}\overline{K}\overline{(L\cup J)}}. \tag{4.3.17}$$

Now we consider relations involving the commutator $[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}]$. Let $A = \sum_{I\subseteq I_n} \alpha_I e_I$. Then

$$[\overline{e_J}, \overline{e_K}] = (-1)^{|K|} e_I A e_J = \begin{cases} 0, & J\cap K \neq \varnothing, \\ \sum\limits_{L\cap J = L\cap K = \varnothing} (-1)^{|K|} \alpha_L e_{J\cup L\cup K}, & J \cap K = \varnothing. \end{cases} \tag{4.3.18}$$

Consider the relation $v_I[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] = \frac{1}{2} v_I R^+_{[\overline{e_J},\overline{e_K}]}$.

Case 1: $J \nsubseteq I$.

In this case the RHS is 0: if $J \cap K \neq \varnothing$, then (4.3.18) implies that $R^+_{[\overline{e_J},\overline{e_K}]} = 0$. If $J \cap K = \varnothing$, then

$$v_I R^+_{[\overline{e_J},\overline{e_K}]} = v_I \sum_{L\cap J = L\cap K=\varnothing} (-1)^{|K|} \alpha_L R^+_{e_{J\cup L\cup K}} = 0$$

since every R^+ operator in this sum is indexed by a set which contains the set $J \nsubseteq I$. Consider the LHS:

$$\begin{aligned} 0 = v_I[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] &= -(-1)^{(|J|+1)(|K|+1)} v_I R^-_{\overline{e_K}} R^+_{\overline{e_J}} = \text{ (by (4.3.11))} \\ &= \left(\sum_{L\subseteq I} c_{\overline{IK}L} v_L + c_{\overline{IKL}} \overline{v_L} \right) R^+_{\overline{e_J}} = \sum_{\substack{L\subseteq I \\ |J\setminus L|=1}} (-1)^{|J|-1} c_{\overline{IKL}} v_{L\triangle J}. \end{aligned}$$

Note that the last sum is nonzero if and only if $|J\setminus I| = 1$. In other words, we have obtained the following relation:

$$I \neq I_n \Rightarrow c_{\overline{IKL}} = 0. \tag{4.3.19}$$

Case 2: $J \subseteq I$.

The RHS is nonzero only if $J \cap K = \varnothing$ and $K \subseteq I$. Then it is equal to

$$\frac{1}{2} v_I R^+_{[\overline{e_J},\overline{e_K}]} = \frac{1}{2} v_I \sum_{L\cap J = L\cap K = \varnothing} (-1)^{|K|} \alpha_L R^+_{e_{J\cup L\cup K}} = \frac{1}{2} \sum_{\substack{L\subseteq I \\ L\cap J = L\cap K=\varnothing}} (-1)^{|K|} \alpha_L v_{I\setminus(J\cup L\cup K)}.$$

Consider the LHS:

$$\begin{aligned} v_I[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] &= \overline{v_{I\setminus J}} R^-_{\overline{e_K}} - (-1)^{(|J|+1)(|K|+1)} \left(\sum_{L\subseteq I} c_{I\overline{K}L} v_L + c_{I\overline{KL}} \overline{v_L} \right) R^+_{\overline{e_J}} \\ &= \text{ (by (4.3.15)) } = \sum_{L\subseteq I\setminus J} c_{\overline{I\setminus J}KL} v_L + c_{\overline{I\setminus J}K\overline{L}} \overline{v_L} \\ &- (-1)^{(|J|+1)(|K|+1)} \left(\sum_{J\subseteq L\subseteq I} \left(c_{I\overline{K}L} \overline{v_{L\setminus J}} + (-1)^{|J|-1} \alpha(|J|-1) c_{I\overline{KL}} \overline{v_{L\setminus J}} \right) \right. \\ &\left. + (-1)^{|J|-1} \sum_{\substack{L\subseteq I \\ |J\setminus L|=1}} c_{I\overline{KL}} v_{L\triangle J} \right). \end{aligned}$$

Consider the last term of the LHS. Note that the sets indexing the basis elements $v_{L\triangle J}$ in this term contain exactly one element from J, while sets indexing the basis elements in other terms in the LHS and the RHS (if it is nonzero) do not contain elements from J. Therefore, if $L, J \subseteq I,\ |J\setminus L| = 1$, we have $c_{I\overline{KL}} = 0$, or in other words

$$L \subsetneq I \Rightarrow c_{I\overline{KL}} = 0.$$

The relation above and (4.3.19) imply that the only possible nonzero $c_{I\overline{KL}}$ is $c_{I_n\overline{KI_n}}$. But relation (4.3.13) then implies that

$$c_{I\overline{KL}} = 0 \text{ for all } I, K, L \subseteq I_n. \tag{4.3.20}$$

Note also that the sum containing the elements $\overline{v_L}$ from the LHS is 0 since the RHS only contains the elements v_L:

$$\sum_{L\subseteq I\setminus J} c_{\overline{I\setminus J}K\overline{L}} \overline{v_L} - (-1)^{(|J|+1)(|K|+1)} \sum_{J\subseteq L\subseteq I} c_{I\overline{K}L} \overline{v_{L\setminus J}} = 0.$$

Taking $J = \varnothing$, we get

$$c_{\overline{I}K\overline{L}} = (-1)^{|K|+1} c_{I\overline{K}L}, \tag{4.3.21}$$

and the relation for the general $J \subseteq I$ follows from (4.3.21) and (4.3.17).
Therefore, the LHS is equal to

$$\sum_{L\subseteq I\setminus J} c_{\overline{I\setminus J}KL} v_L.$$

Case 2.1. $J \cap K \neq \varnothing$. In this case the RHS is 0. Therefore, we have the following relation:

$$L \subseteq I\setminus J,\ J \cap K \neq \varnothing \Rightarrow c_{\overline{I\setminus J}KL} = 0,$$

or equivalently (by (4.3.15)),

$$K \nsubseteq I \Rightarrow c_{\overline{I}KL} = 0. \tag{4.3.22}$$

Case 2.2. $J \cap K = \varnothing,\ K \nsubseteq I$. In this case the RHS is also 0, so we must have

$$K \nsubseteq I,\ J \subseteq I,\ J \cap K = \varnothing \Rightarrow c_{\overline{I\setminus J}KL} = 0,$$

which is equivalent to (4.3.22).

Case 2.3. $J \cap K = \varnothing,\ K \subseteq I$. This is the only case in which the RHS can be nonzero. Particularly, in this case we get

$$\sum_{L \subseteq I \backslash J} c_{\overline{I \backslash J} \overline{K} L} v_L = \frac{1}{2} \sum_{\substack{L \subseteq I \\ L \cap J = L \cap K = \varnothing}} (-1)^{|K|} \alpha_L v_{I \backslash (J \cup L \cup K)}.$$

Note that the sets indexing elements from the RHS do not contain elements from K, which implies that

$$L \subseteq I \backslash J,\ J \cap K = \varnothing,\ L \cap K \neq \varnothing \Rightarrow c_{\overline{I \backslash J} \overline{K} L} = 0,$$

or equivalently (by (4.3.15)),

$$L \cap K \neq \varnothing \Rightarrow c_{\overline{I} \overline{K} L} = 0. \tag{4.3.23}$$

The refined equality then takes the following form:

$$\sum_{M \subseteq I \backslash (J \cup K)} c_{\overline{I \backslash J} \overline{K} M} v_M = \frac{1}{2} \sum_{\substack{L \subseteq I \\ L \cap J = L \cap K = \varnothing}} (-1)^{|K|} \alpha_L v_{I \backslash (J \cup L \cup K)}.$$

Any set M indexing the basis elements from the LHS can be written as $M = I \backslash (J \cup K \cup L)$, where $L \subseteq I,\ L \cap J = \varnothing = L \cap K$. Hence, taking $J = \varnothing$, we get

$$K, L \subseteq I,\ K \cap L = \varnothing \Rightarrow c_{\overline{I} \overline{K} I \backslash (K \cup L)} = \frac{1}{2} (-1)^{|K|} \alpha_L. \tag{4.3.24}$$

Note that the equations (4.3.15), (4.3.22), (4.3.23) and (4.3.24) uniquely determine the coefficients $c_{\overline{I} \overline{K} L}$.

Consider the relation $\overline{v_I} [R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] = \frac{1}{2} \overline{v_I} R^+_{\overline{[e_J, e_K]}}$.

Case 1: $|J \backslash I| \geqslant 2$.

As in the previous relation, if $J \cap K \neq \varnothing$, then $R^+_{\overline{[e_J, e_K]}} = 0$, and if $J \cap K = \varnothing$, then

$$\frac{1}{2} \overline{v_I} R^+_{\overline{[e_J, e_K]}} = \frac{1}{2} v_I \sum_{L \cap J = L \cap K = \varnothing} (-1)^{|K|} \alpha_L R^+_{e_{J \cup L \cup K}} = 0,$$

since every R^+ operator in this sum is indexed by a set which contains the set $J \nsubseteq I$. Consider the LHS:

$$\begin{aligned} \overline{v_I} [R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] = -(-1)^{(|J|+1)(|K|+1)} \overline{v_I} R^-_{\overline{e_K}} R^+_{\overline{e_J}} = &\ \text{(by (4.3.15))} \\ = &\left(\sum_{L \subseteq I} c_{\overline{I} \overline{K} L} v_L + c_{\overline{I} \overline{K} L} \overline{v_L} \right) R^+_{\overline{e_J}} = 0, \end{aligned}$$

since $|J\backslash I| \geqslant 2$. Hence, this case is trivial.
Case 2: $J \subseteq I$. Consider the LHS:

$$\begin{aligned}\overline{v_I}[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] &= (-1)^{|J|-1}\alpha(|J|-1)v_{I\backslash J}R^-_{\overline{e_K}} - (-1)^{(|J|+1)(|K|+1)}\left(\sum_{L\subseteq I} c_{\overline{IKL}}v_L + c_{\overline{IKL}}\overline{v_L}\right)R^+_{\overline{e_J}} \\ &= \text{(by (4.3.20))} = (-1)^{|J|-1}\alpha(|J|-1)\sum_{L\subseteq I\backslash J} c_{I\backslash J\overline{K}L}v_L \\ &- (-1)^{(|J|+1)(|K|+1)}\left(\sum_{J\subseteq L\subseteq I}\left(c_{\overline{IKL}}\overline{v_{L\backslash J}} + (-1)^{|J|-1}\alpha(|J|-1)c_{\overline{IKL}}v_{L\backslash J}\right)\right. \\ &\left. + (-1)^{|J|-1}\sum_{\substack{L\subseteq I\\ |J\backslash L|=1}} c_{\overline{IKL}}v_{L\triangle J}\right).\end{aligned}$$

Note that the RHS is either 0 or a linear combination of elements $\overline{v_L}$. Either way, we must have

$$\alpha(|J|-1)\sum_{L\subseteq I\backslash J} c_{I\backslash J\overline{K}L}v_L - (-1)^{(|J|+1)(|K|+1)}\left(\alpha(|J|-1)\sum_{J\subseteq L\subseteq I} c_{\overline{IKL}}v_{L\backslash J} + \sum_{\substack{L\subseteq I\\ |J\backslash L|=1}} c_{\overline{IKL}}v_{L\triangle J}\right) = 0.$$

Again, the sets indexing the basis elements $v_{L\triangle J}$ in the third sum contain exactly one element from J, while sets indexing the basis elements in other sum do not contain elements from J. Together with (4.3.21) this implies that

$$L \subsetneq I \Rightarrow c_{\overline{IKL}} = c_{I\overline{K}L} = 0. \tag{4.3.25}$$

Moreover, the relations (4.3.25) and (4.3.17) imply that the rest of the sum is zero. Therefore, by (4.3.23) the LHS is equal to

$$-(-1)^{(|J|+1)(|K|+1)}\sum_{\substack{J\subseteq L\subseteq I\\ L\cap K=\varnothing}} c_{\overline{IKL}}\overline{v_{L\backslash J}}.$$

Case 2.1: $J \subseteq I,\ J\cap K \neq \varnothing$. In this case the relations (4.3.23) and (4.3.18) respectively imply that the LHS and the RHS are 0.
Case 2.2: $J \subseteq I,\ J\cap K = \varnothing,\ K \nsubseteq I$. In this case the relation (4.3.22) implies that the LHS is 0. As for the RHS, we get

$$\frac{1}{2}\overline{v_I}R^+_{[\overline{e_J},\overline{e_K}]} = \frac{1}{2}v_I\sum_{L\cap J=L\cap K=\varnothing}(-1)^{|K|}\alpha_L R^+_{e_{J\cup L\cup K}} = 0$$

since every $R^+_{e_{J\cup L\cup K}}$ operator in this sum is indexed by a set which contains the set $K \nsubseteq I$. Thus this case is also trivial.

Case 2.3: $J \subseteq I,\ J \cap K = \varnothing,\ K \subseteq I$. In this case the RHS is equal to

$$\begin{aligned}
\frac{1}{2}\overline{v_I}R^+_{[\overline{e_J},\overline{e_K}]} &= \frac{1}{2}\overline{v_I}\sum_{L\cap J=L\cap K=\varnothing}(-1)^{|K|}\alpha_L R^+_{e_{J\cup L\cup K}} \\
&= \frac{1}{2}(-1)^{|K|}\sum_{\substack{L\subseteq I\\ L\cap J=L\cap K=\varnothing}}\alpha_L(-1)^{|J|+|L|+|K|}\overline{v_{I\setminus(J\cup L\cup K)}} \\
&= (\text{since } A\in(\Gamma_n)_{\bar 0}) \ = \frac{1}{2}(-1)^{|J|}\sum_{\substack{L\subseteq I\\ L\cap J=L\cap K=\varnothing}}\alpha_L\overline{v_{I\setminus(J\cup L\cup K)}}.
\end{aligned}$$

Hence, we have the equation

$$-(-1)^{(|J|+1)(|K|+1)}\sum_{\substack{J\subseteq L\subseteq I\\ L\cap K=\varnothing}}c_{\overline{IKL}}\overline{v_{L\setminus J}} = \frac{1}{2}(-1)^{|J|}\sum_{\substack{L\subseteq I\\ L\cap J=L\cap K=\varnothing}}\alpha_L\overline{v_{I\setminus(J\cup L\cup K)}}.$$

For $J=\varnothing$ the equation is a consequence of relation (4.3.24), and for general $J\subseteq I$ it follows from (4.3.24) and (4.3.16).

Case 3: $|J\setminus I| = 1$. As in the **Case 1**, the RHS is 0. Consider the LHS:

$$\begin{aligned}
0 &= \overline{v_I}[R^+_{\overline{e_J}}, R^-_{\overline{e_K}}] = \ (\text{by } (4.3.15),\ (4.3.25)) \\
&= (-1)^{|J|-1}v_{I\triangle J}R^-_{\overline{e_K}} - (-1)^{(|J|+1)(|K|+1)}\left(\sum_{L\subseteq I}c_{\overline{IKL}}v_L + c_{\overline{IKI}}\overline{v_I}\right)R^+_{\overline{e_J}} \\
&= (\text{since } J\nsubseteq I) \text{ and by } (4.3.25)) \ = c_{I\triangle J\overline{K}I\triangle J}v_{I\triangle J} - (-1)^{(|J|+1)(|K|+1)}c_{\overline{IKI}}v_{I\triangle J},
\end{aligned}$$

thus, we have

$$c_{I\triangle J\overline{K}I\triangle J} = (-1)^{(|J|+1)(|K|+1)}c_{\overline{IKI}},$$

which is a consequence of (4.3.21) and (4.3.17).

Now we check the relation (1.12.2). Consider the relation

$$v_I(R^-_{e_J}R^+_{e_K} + (-1)^{|J||K|}R^-_{e_K}R^+_{e_J}) = v_I R^-_{e_J\circ e_K}.$$

Write the LHS more explicitly:

$$v_I(R^-_{e_J}R^+_{e_K} + (-1)^{|J||K|}R^-_{e_K}R^+_{e_J}) = \ (\text{by } (4.3.2),\ (4.3.14))\ c_{IJI}v_I R^+_{e_K} + (-1)^{|J||K|}c_{IKI}v_I R^+_{e_J}.$$

Suppose that $J,K\subseteq I,\ J\cap K\neq\varnothing$. Then $e_J\circ e_K = 0$ and RHS is 0. The LHS in this case is equal to

$$c_{IJI}v_{I\setminus K} + (-1)|J||K|c_{IKI}v_{I\setminus J} = 0.$$

Thus, if $J\neq K$, we have $c_{IKI}=0$. Clearly I has two different non-trivially intersecting subsets if and only if $|I|\geqslant 2$. Therefore, we have:

$$|I|\geqslant 2,\ \varnothing\neq K\subseteq I \Rightarrow c_{IKI} = 0.$$

Now, let $I, \varnothing \neq K \subseteq I_n$. Recall that $n \geqslant 2$. Then relation (4.3.3) and the relation above imply that $c_{IKI} = 0$. If $K = \varnothing$, then c_{IKI} is automatically 0, since $e_K = 1$ and $R_1^- = 0$. Finally, applying (4.3.2), (4.3.14) and (4.3.9) we conclude that

$$c_{IKL} = c_{\overline{I}K\overline{L}} = 0 \text{ for all } I, K, L \subseteq I_n. \tag{4.3.26}$$

Consider the relation

$$v_I(R^-_{\overline{e_J}} R^+_{e_K} + (-1)^{(|J|+1)|K|} R^-_{e_K} R^+_{\overline{e_J}}) = v_I R^-_{\overline{e_J} \circ e_K}.$$

Write the LHS more explicitly:

$$v_I(R^-_{\overline{e_J}} R^+_{e_K} + (-1)^{(|J|+1)|K|} R^-_{e_K} R^+_{\overline{e_J}}) = \text{ (by (4.3.8), (4.3.11), (4.3.25), (4.3.26)) } = c_{I\overline{J}I} v_I R^+_{e_K}.$$

Case 1: $K \subseteq I,\ J \cap K \neq \varnothing$. In this case $\overline{e_J} \circ e_K = 0$ and the RHS is 0. The LHS in this case is equal to $c_{I\overline{J}I} v_{I\setminus K}$. Hence,

$$J \cap I \neq \varnothing \Rightarrow c_{I\overline{J}I} = 0. \tag{4.3.27}$$

Case 2: $K \subseteq I,\ J \cap K = \varnothing$. In this case the equality takes the form

$$c_{I\overline{J}I} v_{I\setminus K} = (-1)^{|K|} c_{I\overline{(J\cup K)}I} v_I.$$

Hence, taking $K \neq \varnothing,\ J = \varnothing$ we have

$$\varnothing \neq I \Rightarrow c_{I\overline{\varnothing}I} = 0. \tag{4.3.28}$$

Now consider an arbitrary coefficient $c_{I\overline{J}I}$, $I, J \subseteq I_n$. If $J \neq \varnothing$, then (4.3.27) and (4.3.12) imply that $c_{\overline{I}\overline{J}I} = 0$. If $J = \varnothing$, then (4.3.28) and (4.3.12) imply that $c_{\overline{I}\overline{J}I} = 0$. Finally, by (4.3.11), (4.3.25) and (4.3.21) we conclude that

$$c_{I\overline{K}L} = c_{\overline{I}\overline{K}\overline{L}} = 0 \text{ for all } I, K, L \subseteq I_n. \tag{4.3.29}$$

Now, note that there is only one set of variables which satisfies all relations above. Indeed, relations (4.3.8), (4.3.20), (4.3.26) and (4.3.29) imply that

$$c_{IKL} = c_{\overline{I}KL} = c_{I\overline{K}L} = c_{IK\overline{L}} = c_{\overline{I}K\overline{L}} = c_{I\overline{K}\overline{L}} = c_{\overline{I}\overline{K}\overline{K}} = 0 \text{ for all } I, K, L \subseteq I_n,$$

and relations (4.3.15), (4.3.22), (4.3.23), (4.3.24) uniquely determine the coefficients $c_{\overline{I}\overline{K}L}$.

We have to check all the remaining relations of the type (1.12.2). But these relations in fact do not depend on the parameter α (indeed, for a relation to depend on α means to contain a product of the type $\overline{v_I} R^+_{\overline{e_J}}$, where $I, J \subseteq I_n$. But, having killed most of the coefficients c, it is easy to see that the products of this type will not appear in the remaining equations. Therefore, it suffices to check the remainig relations in the case $\alpha = 0$. But in this case for $V(0)$, the regular bimodule over $K(\Gamma_n)$, there exists a structure of a noncommutative Jordan $K(\Gamma_n, A)$-bimodule (the structure of the regular $K(\Gamma_n, A)$-bimodule). Since the coefficients of R^- action are determined uniquely by previous equations, we conclude that they coincide with the coefficients of the R^- action of $K(\Gamma_n, A)$ on the regular bimodule. Hence, all further relations of the type (1.12.2) are satisfied by these coefficients.

We can treat the case of $V(\alpha)^{\mathrm{op}}$ in a completely analogous manner and obtain completely analogous results. Clearly, we will not do it here. We can summarize our result:

Theorem 4.3.2. *Let M be a finite-dimensional irreducible noncommutative Jordan bimodule over $K(\Gamma_n, A)$, where $A = \sum_{I\subseteq I_n} \alpha_I e_I \in (\Gamma_n)_{\bar{0}}$. Then M as a Jordan bimodule over $K(\Gamma_n)$ is isomorphic to $V(\alpha)$ or $V(\alpha)^{\mathrm{op}}$ for some $\alpha \in F$ and the only nonzero R^- action of $K(\Gamma_n, A)$ on M is given by*

$$\overline{v_I} R^-_{\overline{e_K}} = \frac{1}{2}(-1)^{|K|} \sum_{\substack{L\subseteq I \\ L\cap K=\varnothing}} \alpha_L v_{I\setminus(L\cup K)} \text{ if } K \subseteq I,$$

if M is isomorphic to $V(\alpha)$ as a Jordan bimodule over $K(\Gamma_n)$, and analogous action if M is isomorphic to $V(\alpha)^{\mathrm{op}}$ as a Jordan bimodule over $K(\Gamma_n)$.

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