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Intrinsically harmonic forms and characterization of flat circle bundles

Formas intrinsecamente harmônicas e caracterização de fibrados de círculos flat

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Advisor: Francesco Mercuri

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Topology has the peculiarity that questions belonging in its domain may under certain circumstances be decidable even though the continua to which they are addressed may not be given exactly, but only vaguely, as is always the case in reality.
(H. Weyl, Philosophy of Mathematics and Natural Science, 1949)

Transversality unlocks the secrets of the manifold.
(H.E. Winkelnkemper)

Resumo

Nesta tese, abordamos o problema de caracterização de formas intrinsecamente harmônicas e relacionados. Dentre os resultados obtidos, os mais relevantes são: caracterização de fibrados de círculos *flat*, em termos da classe de cohomologia do *pullback* da forma volume; caracterização do n-torus como sendo a única variedade fechada, admitindo uma coleção linearmente independente em todos os pontos de $(n-1)$ formas fechadas de grau 1, de tal modo que o produto entre elas determina uma classe de cohomologia não trivial; caracterização de $(n-1)$-formas sem singularidades que são intrinsecamente harmônicas como aquelas que induzem um fluxo geodesicável; e, por fim, duas condições suficientes para que um fluxo de classe C^1 sobre uma variedade fechada seja periódico.

Palavras-chave: Fibrado de círculos, fluxos periódicos, formas harmônicas, teoria de correntes, fibrados folheados.

Abstract

In this thesis, we address the problem of characterizing intrinsically harmonic forms and related ones. Among the results obtained, the most relevant are: characterization of flat circles bundles, in terms of the cohomological class determined by the pullback of the volume form on the base space; characterization of n-torus as the only closed manifold, admitting an everywhere linearly independent set, of $(n-1)$ closed 1-forms, in such a way that the product between them determines nontrivial cohomological class; characterization of nowhere-vanishing intrinsically harmonic $(n-1)$-forms, such as those that induce a geodesible flow; and, finally, two sufficient conditions for a flow of class C^1 over a closed manifold to be periodic.

Keywords: Circle bundles, periodic flows, harmonic forms, currents theory, foliated bundles.

Contents

Introduction

A classical theorem, due to W.V.D. Hodge, states that in each De Rham cohomology class of a compact Riemannian manifold, there is one, and only one, harmonic form (Section 1.7). A natural question is the following.

Given a closed form ω on a compact manifold M, is there a Riemannian metric g on M such that ω is harmonic concerning g?

If such a metric exists, ω is called an *intrinsically harmonic* form. The problem of obtaining an intrinsic characterization of harmonic forms was first placed by E. Calabi. He showed in his beautiful paper [7] that under suitable conditions on its zero-set, an one form ω is intrinsically harmonic if and only if it is transitive. Transitivity means that for every point x that is not a zero for ω, there is an embedded circle containing x, such that the restriction of ω to the it never vanishes. In Section 2.4 we sketch a proof of Calabi's theorem as well as provide a proof of the following assertion of E. Calabi (Theorem 2.4.2)

Theorem A nowhere-vanishing closed 1-form on a closed manifold is intrinsically harmonic.

E. Volkov made the more notable advance in this subject. He generalized Calabi's characterization of intrinsically harmonic one forms by showing that a closed 1-form is intrinsically harmonic if, and only if, it is harmonic in a neighborhood of its zero set and transitive. The condition of transitivity can be generalized for higher degree forms. Roughly speaking, a p-form ω is transitive if for any "regular point" there is a closed submanifold, containing the point, such that the ω restricts to a volume form (see Section 2.2).

In [37] Ko Honda proved a "dual" version of Calabi-Volkov's result: roughly speaking, he showed that a closed $(n-1)$-form is intrinsically harmonic, under suitable conditions on its zero-set, provided that it is transitive. Naturally comes up the following issue. Is a nowhere-vanishing closed $(n-1)$-form intrinsically harmonic? The main objective of this thesis was to try to answer this question. We want to point out a difference between this and the degree one case. From Hodge's Theorem, it follows that the pullback of the volume form in the Hopf's fibration $S^3 \longrightarrow S^2$ is a nowhere-vanishing closed 2-form which is *not* intrinsically harmonic since it represents the trivial cohomological class (in general, given a map $f : M \to N$, N orientable, Ω a volume form on N, the pullback of the volume form is $f^*\Omega$). On the other hand, on a closed manifold, a nowhere-vanishing closed 1-form always determines a nontrivial cohomological class.

It is known that given a volume form Ω on an orientable manifold, every closed

$(n-1)$-form corresponds to a unique vector field inducing a flow preserving Ω. Conversely, any volume-preserving flow induces a closed $(n-1)$-form. The fixed points of the flow are exactly the singular points of the associate form. We obtain the following criteria (Theorem **??** and Theorem 2.4.4)

Theorem Let M be a closed orientable manifold. A nowhere-vanishing volume-preserving flow defined on M admits global cross-section if and only if the induce closed nowhere-vanishing $(n-1)$-form is intrinsically harmonic.

Theorem Let M be a closed orientable manifold. A nowhere-vanishing C^r-volume-preserving flow on M admits a C^r-global cross-section if and only if it admits a C^r-transversal foliation $(r \geqslant 2)$.

So one objective of this work is to study conditions for a volume-preserving flow admits global cross-sections (or transversal foliations).

We search examples of $(n-1)$-forms that are *not* intrinsically harmonic among nowhere-vanishing *closed-decomposable* forms; these forms are products of $(n-1)$ linearly independent closed 1-forms. It is known that a closed manifold admitting such a form becomes the total space of a fiber bundle with base an (n-1)-torus (Tischler's argument 1.8). We prove the following (Theorem 5.0.4)

Theorem The n-torus is the unique closed n-dimensional manifold admitting a set of $(n-1)$ everywhere linearly independent nowhere-vanishing closed 1-forms $\{\omega_i\}$, such that the product $\omega = \omega_1 \wedge \cdots \wedge \omega_{n-1}$ determines a nonzero cohomological class.

This fact led us to the study of circle bundles. By Theorem 2.4.4, a closed nowhere-vanishing $(n-1)$-form is intrinsically harmonic if, and only if the induced flow admits a complementary foliation. So the pullback of the volume form on a circle bundle is an intrinsically harmonic form if and only if the bundle is smoothly foliated.

As already said, we have characterized nowhere-vanishing intrinsically harmonic $(n-1)$-forms as being the ones that induce a flow admitting a global cross-section. The first criterion to decide if a flow admits a global cross-section is due to S. Schwartzmann [74]. He introduced the notion of asymptotic cycles. Such a cycle is a real homology class defined for each invariant measure. A global cross-section is determined by an integral 1-dimensional cohomology class that is positive on all asymptotic cycle. Using this criterion, we give examples of intrinsically harmonic forms and characterize flat circle bundles. The main results are:

Theorem Let M be a closed smooth manifold with a nowhere-vanishing closed $(n-1)$-form ω inducing a pointwise periodic flow. If each orbit of the induced flow is homologous to each other and $[\omega] \neq 0$ in $H^{n-1}_{DR}(M)$, then ω is intrinsically harmonic. If ω is intrinsically harmonic, then there exists a smooth $\mathbb{S}^1$-action on M with the same orbits as the one from the flow induced by ω.

Theorem Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a differentiable principal circle bundle with M closed and orientable. Then $\mathcal{B}$ admits a flat connection if and only if the form $p^*(\Omega_M)$ is intrinsically harmonic.

For fiber bundles over orientable manifolds we obtain the following result (Lemma 5.2.7).

Lemma Let $\mathcal{B} = \{B, p, M, F\}$ be a differentiable fiber bundle with compact total space and base be an orientable manifold. Let Ω be a volume form on M with $\int_M \Omega = 1$. Then, the Poincaré dual to the fiber of $\mathcal{B}$ can be represented by $p^*\Omega$. In particular, $[F] = 0$ in $H_{\dim F}(B; \mathbb{R})$ if and only if $p^*\Omega$ is an exact form.

From this, using the Gysin sequence, we obtain a known result, but by a different approach, about smooth foliated circle bundles. It is contained in Theorem 5.2.11 highlighted below.

Theorem A differentiable principal circle bundle $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ with closed orientable base space admits a flat connection if and only if one (and consequently all) of the four conditions below holds:

(1) $p^*(\Omega_M)$ is intrinsically harmonic;

(2) $p^*(\Omega_M)$ determines a nonzero cohomological class;

(3) the fiber represents a nonzero class in $H_1(B; \mathbb{R})$;

(4) $\chi(\mathcal{B})$ is a torsion element.

Similar results for circle bundles over non-orientable manifolds are presented in Section 5.2.1. For forms of a degree different from $0, 1, n-1, n$, the question of obtaining an intrinsic characterization of harmonic forms is quite open. A notable aspect of the difficulty to obtain a general characterization is that Calabi's argument does not apply to intermediate degrees. In fact, a concrete example, due to J. Latshev, of a transitive 2-form on a 4-manifold not intrinsically harmonic was presented by E. Volkov in [85]. We generalize this example in Section 3.1.

For a p-form ω, the kernel space at a point x is the space of all $X \in T_xM$ such that $\imath_X\omega$ is the zero $(p-1)$-form. The dimension of the kernel space at x, denoted by $\nu(x)$, is called the *nullity* of ω at x. The number $r(x) = \dim M - \nu(x)$ is called *rank* of ω at x. It is well known that if the form has a constant rank, the kernel spaces form a smooth distribution, denoted by $\ker\omega$, which is integrable when, for example, the form is closed. In Section 3.3, we provided the following result (Theorem 3.3.5).

Theorem Let M be a closed Riemannian manifold. If ω is a harmonic p-form of constant rank p, the kernel distributions of ω and $*\omega$ are orthogonal, and the leaves of the two foliations induced by theses distributions are minimal concerning a suitable modification of the metric.

We also give characterizations of intrinsically harmonic p-forms ω, of nullity $(n-p)$, in terms of the existence of foliations complementary to the nullity foliation of ω (theorems 2.1.2 and 3.3.9)

Theorem Let ω be a closed p-form of rank p. Then ω is intrinsically harmonic if and only if there exists a closed form η of rank $(n-p)$ such that $\ker\omega \cap \ker\eta = \{0\}$.

Theorem Let M be a manifold and ω be a closed p-form of rank p on M. Suppose that there exists a Riemannian metric g on M such that every leaf of the foliation induced by ω is a minimal submanifold concerning g. If $(\ker\omega)^{\perp}$ is an integrable distribution, then ω is intrinsically harmonic.

We do not know if we can weaken the hypothesis in Theorem 2.1.2 by merely supposing that the foliation induced by the differential form ω admits complementary foliation. This problem can be studied in the context of foliated fiber bundles. In this direction, we obtain the following result (Theorem 3.3.10).

Theorem Let $\mathcal{B} = \{B, \pi, M, F, \mathfrak{F}\}$ be a foliated bundle with total space and base orientated manifolds. Let Ω a volume for on M. If the fundamental group of M is finite then $\pi^*\Omega$ is a transitive intrinsically harmonic form.

S. Shwartzamnn [74] showed that a recurrent flow has only one asymptotic cycle. By a recurrent flow Φ on a metric space X we mean a flow with some sequence $t_n \longrightarrow \infty$ such that $\Phi_{t_n} \longrightarrow 1_X$ uniformly. We can show that the class of a closed $(n-1)$-form induces an asymptotic cycle to the induced flow (Lemma 5.1.1). Hence, a nowhere-vanishing closed $(n-1)$-form inducing a recurrent flow is intrinsically harmonic if, and only if it determines a nontrivial cohomological class. It is well-known from dynamical systems' theory that a volume-preserving flow is pointwise recurrent. Hence, it is natural to ask if we can obtain a uniform recurrence for that kind of flow. This question directs the last part of this thesis. We noted that the equicontinuity of a flow implies recurrence of it. So, based on a criterion of Y. Carriere [11] and H. Rummler [70] to a flow be geodesible, we give another characterization of a nowhere-vanishing intrinsically harmonic $(n-1)$-form, the ones inducing a geodesible flow. More precisely, we provide the following result (Theorem 6.3.8).

Theorem Let ω be a nowhere-vanishing closed $(n-1)$-form on a closed manifold M and X be a vector field that generates the foliation induced by ω. Suppose that $[\omega] \neq 0$ in $H^{n-1}_{DR}(M)$. Then are equivalents: :

(1) there exists a 1-form η on M satisfying $\eta \wedge \omega > 0$ and $i_X d\eta = 0$;

(2) X is geodesible;

(3) X is Killing for some Riemannian metric on M;

(4) ω is intrinsically harmonic.

We give attention to the problem of obtaining a uniform recurrence to pointwise periodic flows. We want to obtain a global cross-section for a given flow. Since this is a topological condition, independent of flow reparametrization, it makes sense to ask if the orbits of this flow are the orbits of a (continuous) recurrent flow. Let Φ be a pointwise periodic flow with period function λ. The flow $\Psi(x,t) = \Phi(x, \lambda(x)t)$ (no necessarily continuous) has the same orbits of Φ and is periodic (hence recurrent). This fact led us to study the set of continuity of λ. D.B. Epstein proved in [20] that this set is invariant, open, and dense for flows defined on manifolds. On the other hand, if a flow is recurrent on a dense subset, it is easy to show that this flow is recurrent. These observations directed us to the study of pointwise periodic flows with continuous period function. We showed that Epstein's remark holds for pointwise periodic flows defined on any locally compact metric space (Theorem 6.5.3). For this reason, we deal with flows in these spaces.

An old problem in the theory of pointwise periodic flows is to obtain a condition so that there exists a continuous action of $\mathbb{S}^1$ possessing the same orbits as the one of the flow. For instance, this is trivially true when the period function is continuous. By a very delicate argument, D.B. Epstein showed in [20] that this condition always holds for flows on compact three-dimensional manifolds. Hence, in dimension 3, a pointwise periodic nowhere-vanishing volume-preserving flow admits a global cross-section if and only if the induced closed 2-form determines a nontrivial cohomological class. A necessary condition for the orbits of a C^1-flow be the same as one given by a $\mathbb{S}^1$-action is the following: the period function must be locally bounded. In particular, on compact manifolds, the period function must be bounded. We obtain the following result (Theorem 6.8.10).

Theorem Let Φ be a pointwise periodic C^1-flow on a manifold M, compact or not. Then the period function λ is locally bounded if, and only if, Φ is locally weakly almost periodic.

As showed by W. H. Gottschalk in [29], on compact metric space, locally weakly almost periodic flows are characterized as one where the orbit space satisfies the Hausdorff property. Hence, this property on the orbit space and the boundedness (local) of the period function are equivalent. The latter theorem is probably a particular case of a more general result present in [21]. However, in any case, we give proof of this fact. Then there are two significant problems in the theory of pointwise periodic flows: one is characterizing when the period function is locally bounded and another when the orbits of the flow are given by a $\mathbb{S}^1$-action with the same regularity of the flow. The former was known as *pointwise orbit conjecture*. In [86], A. W. Wadsley showed that the latter occurs when the flow (on a smooth manifold) is of class C^3 and geodesible. We improve this result to flows on closed manifolds by showing the following result 6.8.1:

Theorem Let M be a closed manifold. Let Φ be a pointwise periodic flow without singularity of class C^1 on M. Let N the continuity set of the period function λ. Suppose that Φ is locally weakly almost periodic. If Φ is equicontinuous in N for some metric inducing its topology, then Φ is periodic.

As we will see, it follows from this theorem that if d is a metric inducing the topology of M and if Φ is a pointwise periodic C^1-flow on M preserving d, then Φ is periodic.

A problem similar to these regarding flows is determining conditions for a pointwise periodic homomorphism to be periodic. D. Montgomery showed in [58] that any pointwise periodic homomorphism of a connected and locally Euclidean space is periodic. This result is essential to study pointwise periodic flows. It is noteworthy that in dimension 5, D. Sullivan [79] obtained an example of a pointwise periodic flow on a closed manifold where the period is unbounded and gave a rich discussion about this problem, relating it with one in compact foliation theory. In dimension 4, a counterexample for the pointwise orbit conjecture was given by D.B. Epstein and E. Vogt [22]. At the end of the work, we conclude that pointwise periodic flows are not necessarily recurrent. Actually, this is a very restrictive condition, as shown in the following result (Theorem 6.9.5).

Theorem Let Φ be a pointwise periodic C^1-flow without singularity with bounded period function on a manifold M. Then Φ is recurrent if and only if Φ it is locally periodic. If, also, M is compact, then Φ is recurrent if, and only if, it is periodic.

Some questions for which we did not get an answer appear at the end of this work. We have made an effort to present this thesis as self-contained as possible. In no case does the absence of a reference imply any claim to originality on our part. This not omits our results since we stated them clearly. We believe that many pieces of arguments in mathematics, not all, can already be considered in the "public domain". However, the most relevant ones are presented here with the author's mention or reference where it appears. We encourage the reader to contact us by the email *elizeufranca@ufam.edu.br* if you want to give any criticism, suggestion, or commentary of any order.

1 Background

In this chapter, we collect the basic definitions and some important results that will be used throughout the thesis.

1.1 Conventions

(1) By a *neighborhood* of a subset Y of a topological space X, we mean a subset of X that contains each element of Y as an interior point.

(2) Throughout this work, by manifold we mean a smooth (class C^∞), finite-dimensional, connected (hence path connected), Hausdorff, second enumerable space, whithout boundary, unless it is said explicitely otherwise. These manifolds are known to be paracompact. Also, for every covering of such a manifold by open subsets, there exists a partition of unity subordinate to it [33, 45, 87]. We reserve the symbol M to denote a smooth n-dimensional manifold and the letters p and q represent nonnegative integer numbers satisfying $p+q=n$ in this context. The tangent bundle of M will be denoted by τ_M and the total space this bundle will be denoted by TM

(3) A *coordinate neighborhood (chart)* around a point $x \in M$ means a homeomorphism

$$\phi : U \subset M \longrightarrow \phi(U) \subset \mathbb{R}^n$$

where U is an open connected set containing x and ϕ belongs to a maximal atlas defining the differentiable structure of M [45].

(4) Given a smooth map $f : M \longrightarrow N$, M and N smooth manifolds, the *derivative* of f will be denoted by f_*. It is defined by $f_*(X) = \partial_t|_0 f(\alpha(t))$, where $X \in T_xM$ and $\alpha : (-\epsilon, \epsilon) \longrightarrow M$ is a smooth curve with $\alpha(0) = x$ and $\partial_t|_0\alpha(t) = X$. A smooth map is said to be a *submersion* if $f_* : T_xM \longrightarrow T_{f(x)}N$ is surjective for all $x \in M$. The *pullback* of f in the space of differential forms will be denoted by f^*. It is defined by $f^*\omega(X_1, \ldots, X_p) = \omega(f_*X_1, \ldots, f_*X_p)$ [45].

(5) The domain and range of a map are occasionally omitted. For example, if $\phi : U \subset M \longrightarrow \phi(U) \subset \mathbb{R}^n$ and $\psi : V \subset M \longrightarrow \phi(V) \subset \mathbb{R}^n$ are coordinate neighborhoods, we consider the transition function $\psi \circ \phi^{-1}$ (if exists) without mention to its domain.

(6) Given an *equivalence relation* $\sim$ on a set X we always denote by $[x]$ the class determined by $x \in X$, that is, the set $[x] = \{y \in X / y \sim x\}$. The set of equivalence classes determined by $\sim$ is denoted $X/\sim$. We have a natural map $\pi : X \longrightarrow X/\sim$,

given by $\pi(x) = [x]$. When X is a topological space, the topology considered on $X/\sim$ is the smallest (coarser) topology that makes the projection π continuous. In particular, a subset U of $X/\sim$ is open if and only if $\pi^{-1}(U)$ is open [41].

(7) Let X be a nonempty set. The identity map of X is denoted by the symbol "1", that is, the map $1 : X \longrightarrow X$ given by $1(x) = x$. In some contexts, to emphasize the space in consideration, we denote the identity map of X by 1_X.

(8) Let $p : X \longrightarrow Y$ be a continuous map. By a *local cross-section for* p trhough $y \in Y$ we mean a continuous function $s : V \longrightarrow X$ defined on a neighborhood of x such that $f \circ s = 1$. If $V = Y$, then s is called a *global cross-section*.

1.2 Fiber bundles

In some references in foliation theory, the notion of the fiber bundle has been given as a triple (B, p, X) where $p : B \longrightarrow X$ is a smooth submersion and, around each point of B, there exists a local trivialization such that the map p behaves like a projection on a certain factor (for example, [9, 8]). This notion is a particular case of an *Ehresmann-Feldbau bundle*, in which the group of diffeomorphisms of the fiber is the structural group of the fiber bundle. This definition of fiber bundle (given in [19]) is very general, because the structural group is not necessarily topological and the natural action on the fiber is not required to be continuous. In a later work, C. Ehresmann [17] considered a notion of fiber bundle where the topology of the structural group plays a rule. This notion is equivalent to the given in Steenrod's book [78], where the structural group is a topological group that acts effectively and continuously on the fiber. This is meaningful to us, because the topology of the structural group plays an important role in classification theorems, and some of them are used in this thesis. Suppose that the fiber is a Lie group that acts on itself by translations, that the total space and base space are smooth manifolds and that the coordinate functions are diffeomorphisms. Then the definition of fiber bundle according to N. E. Steenrod is a particular case of the a *differentiable principal fiber bundle* defined in Kobayashi-Nomizu's book [42].

In this section, these notions of fiber bundles are discussed. We study when an Ehresmann-Feldbau bundle is associated to a Steenrod bundle and we show that the existence of a continuous morphism between two differentiable principal bundles implies the existence of a differentiable morphism (in particular, two differentiable principal bundles that are isomorphic by a continuous bundle map are shown to be isomorphic in the sense of Kobayashi-Nomizu). We finish the section by showing that two differentiable principal circle bundles that are $\mathrm{Diff}^+(\mathbb{S}^1)$-equivalents are equivalents as differentiable principal bundles. The most important references for this section are Kobayashi-Nomizu's and Steenrod's books [42, 78].

1.2.1 Eheresmann-Feldbau bundle

Definition 1.2.1 (Ehresmann-Feldbau [19])**.** Let E be a connected topological space with an equivalence relation $\sim$ on E, $B = E/\sim$ the quotient space (*base space*), $p : E \longrightarrow B$ the canonic projection, F a topological space and G a group of homeomorphisms of F. Suppose that for each $x \in B$ there exists a family H_x of homeomorphisms from $p^{-1}(x)$ to F such that

(1) If $h, k \in H_x$ then $hk^{-1} \in G$;

(2) To each point $x \in B$ exists a neighborhood U_x of x and a homeomorphism $\varphi : p^{-1}(U_x) \longrightarrow U_x \times F$ such that for each $y \in U_x$, $\varphi_y : p^{-1}(y) \longrightarrow \{y\} \times F$ is a homeomorphims and the canonic projection $\pi_2 \circ \varphi_y$ is an element of H_y.

In this situation, the family $\mathcal{H} = \{H_x\}$ defines on E a *structure of fiber space associated with group* G denoted by $E(B, F, G, \mathcal{H})$.

Remark 1.2.2. If G is the group of all homeomorphisms of F then H_x is the family of all homeomorphims of $p^{-1}(x)$ to F, the conditions (1) and (2) reduces the following: to each $x \in B$ there exists a neighborhood U_x of x and a homeomorphims $\varphi : p^{-1}(U) \longrightarrow U_x \times F$ such that $p = \pi_1 \circ \varphi$. In the smooth category, that is, when E, B and F are smooth manifolds and G is the group of all diffeomorphisms of the fiber, we have a fiber bundle as defined in [9, 8].

Lemma 1.2.3 (Ehresmann's lemma[1], [18])**.** Let $p : E \longrightarrow B$ a submersion such that p is a proper map[2]. Then E is the total space of a fiber space over B with projection p. When $\partial E \neq \varnothing$, the result is still valid since $p|_{\partial E} : \partial E \longrightarrow B$ be a submersion.

Proof. Fixe $x \in B$ and let W be a tubular neighborhood of $p^{-1}(x)$ in E with smooth retraction $r : W \longrightarrow p^{-1}(x)$ (see [33], page 109. The hypothesis that $p|_{\partial E}$ is a submersion ensures that, for each $x \in B$, $p^{-1}(x)$ is a *neat* submanifold, hence admits a tubular neighborhood). The differential of the map

$$p \times r : W \longrightarrow B \times p^{-1}(x)$$

is nonsingular at each point of $p^{-1}(x) \subset W$. Since $p^{-1}(x)$ is compact, we can obtain an open neighborhoord W' of $p^{-1}(x)$ such that $p \times r : W' \longrightarrow B \times p^{-1}(x)$ is an embendding. Since p is a proper map, we can obtain an open set U of B such that $p^{-1}(x) \subset p^{-1}(U)$ and $p^{-1}(U) \subset W'$. Thus,

$$p \times r : p^{-1}(U) \longrightarrow B \times p^{-1}(x)$$

[1] For an extention of this lemma see [15].

[2] A map $f : X \longrightarrow Y$ is said *proper* if the inverse image of any compact subset of Y under it is a compact subset of X.

is a diffeomorphism satisfying $\pi_1 \circ (r \times p) = p$. It remains only to show that given $x, y \in B$, then $p^{-1}(y)$ is diffeomorphic to $p^{-1}(x)$, that is, there exists a well-defined fiber. First, the condition "$p^{-1}(y)$ is diffeomorphic to $p^{-1}(x)$" is an open condition, since for $y \in p^{-1}(U)$, the restriction $p \times r : p^{-1}(y) \longrightarrow \{y\} \times p^{-1}(x)$ is a diffeomorphism. Let $y \in B$ such that there exists a sequence $y_n \longrightarrow y$ and, furthemore, $p^{-1}(y_n)$ and $p^{-1}(x)$ are diffeomorphic for all $n \in \mathbb{N}$. By the bove construction, for sufficiently large n, $p^{-1}(y_n)$ is diffeomorphic to $p^{-1}(y)$, concluding that the condition "$p^{-1}(y)$ is diffeomorphic to $p^{-1}(x)$" is a closed condition. Since B is a connected space, this proves the claim. □

1.2.2 Fiber bundles according Steenrod

N. E. Steenrod defined a fiber bundle as a maximal coordinate bundle. The definition of a coordinate bundle is given below.

Definition 1.2.4 (Coordinate bundle)**.** A *coodinate bundle* $\mathcal{B}$ is a collection as follows:

(1) A space B called *bundle space*;

(2) a space X called *base space*;

(3) a surjective map $p : B \longrightarrow X$ called *projection*;

(4) a space Y called *fiber*;

(5) an effective topological transformation group G of F called the *group of the bundle*;

(6) a family $\{V_j\}$ of open sets covering X indexed by a set J, the V_j's are called *coordinate neighborhoods*, and

(7) for each $j \in J$, a homeomorphism

$$\phi_j : V_j \times Y \longrightarrow p^{-1}(V_j)$$

called *coordinate function (local trivialization).* The coordinate functions are required to satisfy the following conditions:

(8) $p\phi_j(x, y) = x$, for all $x \in V_j$, $y \in Y$,

(9) if the map

$$\phi_{j,x} : Y \longrightarrow p^{-1}(x)$$

is defined by setting

$$\phi_{x,j}(y) = \phi_j(x, y),$$

then, for each pair $i, j \in J$, and each $x \in V_i \cap V_j$, the homeomorphism

$$\phi_{j,x}^{-1}\phi_{i,x} : Y \longrightarrow Y$$

coincides with the operation of an element of G (it is unique since G is effective), and

(10) For each pair i, j the map

$$g_{ji} : V_i \cap V_j \longrightarrow G$$

defined by $g_{ji}(x) = \phi_{j,x}^{-1}\phi_{i,x}$ is continuous.

Definition 1.2.5 (Fiber bundle)**.** Two coordinate bundle $\mathcal{B}$ and $\mathcal{B}'$ are said to be *equivalent in the strict sense* if they have the same bundle space, base space, projection, fiber, and group, and the union $\mathcal{B} \cup \mathcal{B}'$ is a coordinate bundle. This notion of equivalence between coodinate bundles defines an equivalence relation on the collection of all coordinate bundles with the same bundle space, base space, projection, fiber, and group. A *fiber bundle* is defined to be an equivalence class of coordinate bundles. We eventually call such a fiber bundle *Steenrod bundles.*

This definition is analogous to the Ehresmann-Feldbau's definition of fiber space but here the topology of G plays a rule. The definition given by N.E. Steenrod is equivalent to the definition given by C. Ehresmann in [17].

Definition 1.2.6. Let $\mathcal{B}$ and $\mathcal{B}'$ be two coordinate bundles having the same fiber, group. By a *bundle map* (*fiber preserving map*) $h : \mathcal{B} \longrightarrow \mathcal{B}'$, we mean a continuous function $h : B \longrightarrow B$ having the following properties:

(1) h carries each fiber Y_x homeomorphically onto a fiber $Y_{x'}$ of $\mathcal{B}'$, thus inducing a continuous function $\overline{h} : X \longrightarrow X'$, such that

$$p'h = \overline{h}p$$

(2) If $x \in V_j \cap \overline{h}^{-1}(V_k')$ and $h_x : Y_x \longrightarrow Y_{x'}$ is the function induced by h ($x' = h(x)$), then the map

$$\overline{g}_{kj} = {\phi'_{k,x'}}^{-1} h_x \phi_{j,x} : Y \longrightarrow Y$$

coincides with the operation of an element of G, and

(3) the map

$$\overline{g}_{kj} : V_i \cap \overline{h}^{-1}(V_k') \longrightarrow G$$

so obtained is continuous.

Definition 1.2.7. Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ with the same base space, fiber, and group are said to be *equivalent* if there exists a bundle map $\mathcal{B} \longrightarrow \mathcal{B}'$ that induces the identity map of the common base space. Two fiber bundles having the same base space, fiber and group are said to be equivalent if they have a representative coordinate bundles that are equivalent.

Definition 1.2.8. Let G be a topological group, and X a space. By a *system of coordinate transformations* in X with values in G is meant an indexed covering $\{V_j\}$ of X by open sets and a collection of continuous maps

(1)
$$g_{ji} : V_i \cap V_j \longrightarrow G, i, j \in J$$
such that

(2) $g_{kj}(x)g_{ji}(x) = g_{ki}(x)$, for all $x \in V_i \cap V_j \cap V_k$.

Theorem 1.2.9 (Construction of fiber bundles)**.** Let G a topological group acting efectively on a topological space Y, $\{V_j\}$ and $\{g_{ij}\}$ a system of coordinate transformation on a topological space X. There exists a unique fiber bundle $\mathcal{B}$ with base space X, fiber Y and group G with coordinate transformations $\{g_{ij}\}$.

Proof's sketch. The construction is given by consider on the disjoint union

$$T = \bigsqcup V_i \times Y \times \{j\}$$

the equivalence relation

$$(x, y, j) \sim (x^{'}, y^{'}, k)$$

if

$$x = x^{'} \text{ e } g_{kj}(x)y = y^{'}.$$

Defining $q : T \longrightarrow B$ the natural projection and $p[(x, y, j)] = x$, the coordinate functions are given by

$$\phi_j(x, y) = [(x, y, j)].$$

(Compare with [17], § 3). □

Definition 1.2.10. A fiber bundle $\mathcal{B} = \{B, p, X, Y, G\}$ is called *principal fiber bundle* if $Y = G$ and the action is the left translation. Principal bundles are denoted briefly by $\mathcal{B} = \{B, p, X, G\}$.

Definition 1.2.11. Let $\mathcal{B} = \{B, p, X, Y, G\}$ be an arbitrary bundle. The *associated principal bundle* $P(\mathcal{B})$ of $\mathcal{B}$ is the bundle given by the construction in Theorem 1.2.9 using the same base space X, the same $\{V_j\}$ the same $\{g_{ij}\}$ and the same group G as for $\mathcal{B}$ but replacing F by G and allowing G to operate on itself by left translations.

Definition 1.2.12. Let $\mathcal{B}$ be a fiber bundle. We call $\mathcal{B}$ a *fiber bundle of class C^r* and write C^r-fiber bundle for short if it satisfies the following two condition:

(1) Its total space and base space are manifolds of class C^r;

(2) the representative coordinate functions of $\mathcal{B}$ are diffeomorphisms of class C^r.

Let $h : \mathcal{B} \longrightarrow \mathcal{B}'$ a bundle map between two C^r-fiber bundles. We say that h is a C^r*-bundle map* if h is a map of class C^r that carries the fibers diffeomorphically. Two differentiable fiber bundles are said to be *differentiable equivalent* if they have representative coordinate bundle equivalent by a C^r-bundle map.

Theorem 1.2.13. Let $\mathcal{B}$, $\mathcal{B}'$ be C^r-fiber bundles with the same group and same base M. Let $h : \mathcal{B} \longrightarrow \mathcal{B}'$ be a C^r-bundle map that induces a diffeomorphism $\overline{h} : M \longrightarrow M$. Then h is a diffeomorphism.

Proof. Since $\dim B = \dim B'$, it is enough to show that h_* is injective. Let $\phi : U \times F \longrightarrow p^{-1}(U)$ be a coordinate function for $\mathcal{B}$ and

$$\phi_x : F \longrightarrow B, \phi^f : U \longrightarrow B$$

defined by $\phi_x(f) = \phi(x, f)$ and $\phi^f(x) = \phi(x, f)$. Given $b \in \pi^{-1}(U)$, any $v \in T_bB$ has unique representation

$$v = (\phi_x)_*(v_1) \oplus (\phi^f)_*(v_2).$$

Suppose that $h_*v = 0$. Denoting the projections of $\mathcal{B}$ and $\mathcal{B}'$ by p and q, respectively, the map $\overline{h}$ satisfies $\overline{h}p = qh$. Since $\overline{h}_*p_*(v) = v_2$, it follows that $v_2 = 0$. Thus, $v = (\phi_x)(v_1)$ is tangent to fiber over x. Since h restricts to a diffeomorphism in the fibers and $h_*v = 0$, it follows that $v = 0$. □

1.2.3 Associating a coordinate bundle to an Ehresmann-Feldbau bundle

We next discuss when an Eheresmann-Feldbau bundle is actually a Steenrod bundle. Recall that the latter one is required to possess a topologized structural group that act continuously on the fiber. But the structural group in the former one is not required to satisfy any topological condition.

In some cases, a group G of homeomorphisms (C^r- diffeomorphisms) of a topological space F (a manifold) has a natural topology τ such that G is a topological group under τ and the natural action $(g, y) \in G \times Y \longrightarrow g(y)$ is continuous. In some of these cases, an Ehresmann-Feldbau bundle $E(B, F, G, \mathcal{H})$ determines a coordinate bundle. Now we give some examples.

Definition 1.2.14 (Compact-open topology, R.H. Fox [23])**.** Let X, Y be topological spaces and F a set of continuous functions from X to Y. Given A and B subsets of X and Y, respectively, denote by $M(A, B)$ the set of all elements $f \in F$ satisfying $f(A) \subset B$. The *compact-open topology (co.o.-topology)* on G is the topology that has as a subbasis for open sets of F the sets $M(K, U)$, where K ranges over all compact subsets of X and U ranges over all open subsets of Y.

Definition 1.2.15 (Admissible topology, S. Myers [60])**.** Let X, Y be topological spaces and F a set of continuous functions from X to Y. A topology on F is said *admissible* if the natural map (evaluation map) $F \times X \longrightarrow Y$ is continuous.

Theorem 1.2.16 (R.H. Fox, [23])**.** Let X be a regular and locally compact space and Y be an arbitrary topological space. The co.o.-topology for a collection of continuous maps from X to Y is admissible.

Theorem 1.2.17 (S. Myers, [60])**.** Let X and Y be arbitrary topological spaces. The co.o.-topology on a set of continuous maps from X to Y is smaller than any admissible topology.

Theorem 1.2.18 (S. Myers, [60])**.** Let X be a metric space and G be a group of homeomorphisms of X. Suppose that G is an equicontinuous set of functions. Then, under the co. o.-topology, G is a topological group.

Theorem 1.2.19 (R. Arens, [2])**.** The group of all homeomorphisms of a compact Hausdorff space under the co.o.-topology is a topological group.

Theorem 1.2.20 (R. Arens,[2])**.** Let X be a locally connected, locally compact, and Hausdorff space. The group of all homeomorphisms of X under the co.o.-topology is a topological group.

Let $E = E(B, F, G, \mathcal{H})$ be an Ehresmann-Feldbau bundle such that its structural group G is a subgroup of the group of all homeomorphisms of the fiber (equipped with the co.o-topology, see Remark 1.2.21). It follows from theorems 1.2.16-1.2.20 that if F satisfies one of the conditions (1)-(3) below, then E is a Stenrood bundle.

(1) F is compact and Hausdorff.

(2) F is locally connected, locally compact and Hausdorff.

(3) F is a metric space and G is equicontinuous.

Remark 1.2.21. Let H be the group of all homeomorphisms of a topological space Y and let $E = E(B, F, G, \mathcal{H})$ be an Ehresmann-Feldbau bundle with fiber Y and structural group G. Suppose that H is a topological group under the co.o-topology and that G is a subgroup of H such that the natural action $G \times Y \longrightarrow Y$ is continuous (under this topology). Then E determines a coordinate bundle:

(1) the natural action of G on Y is continuous, since it is the restriction of the natural action $H \times Y \longrightarrow Y$;

(2) the group G is a topological group. Indeed, let V an open set of G. We want to show that $p_H^{-1}(U \cap H)$ and $i_H^{-1}(U \cap H)$ are open in $H \times H$ and H, respectively, where p_H denotes the product of H and i_H the inversion of H. We have the equalities

$$p_H^{-1}(U \cap H) = p_G^{-1}(U) \cap (H \times H) \text{ and } i_H^{-1}(U \cap H) = i_G^{-1}(U) \cap H.$$

Since the product topology of $H \times H$ is the same as the subspace topology, we conclude that p_H and i_H are continuous.

(3) Given homeomorphisms $\varphi_i : p^{-1}(V_i) \longrightarrow V_i \times F$ as in 1.2.1, putting $\phi_i = \varphi_i^{-1}$, we claim that the function $x \in V_j \cap V_i \longrightarrow g_{ji}(x) = \phi_{j,x}^{-1}\phi_{i,x} \in G$ is continuous. R.H. Fox [23], observes that for any tpoological spaces X, Y, T, if a function $h : X \times T \longrightarrow Y$ is continuous and the space $C^0(X,Y)$ of all continuous functions of X to Y has the co.o.-topology, then the function $h^* : T \longrightarrow C^0(X,Y)$ defined by $h^*(t)(y) = h(y,t)$ is continuous. Taking $h = p_2\phi_j\phi_i^{-1} : (V_i \cap V_j) \times Y \longrightarrow Y$, we have that $g_{ji}(x) = h^*(x)$ is continuous, concluding the claim by Fox's observation (note that $h^* : V_j \cap V_i \longrightarrow G$ is continuous, since the image of h^* is contained in $G \subset C^0(Y,Y)$ and $h^* : V_j \cap V_i \longrightarrow C^0(Y,Y)$ is continuous).

Remark 1.2.22. Given an effective continuous action of a topological group G on a topological space Y, we can see the group G as a group of homeomorphisms of Y. In this case, G has an admissible topology since acts continuously on Y. Now, suppose that G is compact and Y is Hausdorff. Then the topology of G is the co.o.-topology. Indeed, denote by τ the original topology of G and τ_c the co.o.-topology on G. We have:

(1) τ_c is a Hausdorff topology. Let $g, h \in G$, $g \neq h$. Given $y \in Y$ with $g(y) \neq h(y)$, since Y is Hausdorff, there exists open disjoint sets U and V with $g(y) \in U$ and $h(y) \in V$. So $M(\{x\}, U)$ and $M(\{y\}, V)$ are disjoint open sets for the co.o.-topology on G with $g \in M(\{x\}, U)$ and $h \in M(\{y\}, V)$. Therefore G under the co.o.-topology is Hausdorff.

(2) Now, any co.o.-open set is open in the original topology of G, since the original topology of G is admissible and any admissible topology is finer than co.o.-topology by 1.2.17. It follows that the identity map $I : (G,\tau) \longrightarrow (G,\tau_c)$ is continuous. Since (G,τ) is compact and (G,τ_c) Hausdorff, this map is closed. Therefore I^{-1} is continuous concluding that $\tau = \tau_c$.

Thus, given a fiber bundle $\mathcal{B} = \{B, p, X, Y, G\}$ with G compact and Y Hausdorff, the topology of G is necessarily the co.o.-topology. More generally, the same proof shows that any compact admissible topology on a set G of homeomorphisms of Y coincides with the co.o.-topology.

We will now deal with the differentiable case. The notation $\mathcal{B} = \{B, p, M, F\}$ will be fixed to denote a C^r fiber space, $1 \leqslant r \leqslant \infty$, with the group of all C^r-diffeomorphisms of the fiber (see 1.2.2). Next, we describe a topology for this group so that $\mathcal{B}$ is a fiber bundle and present a characterization of bundle map. Let M, N be C^r-manifolds (without boundary) and denote $C^r(M, N)$ the space of all C^r maps $f : M \longrightarrow N$. Let $J^r(M, N)$ the space of all r-jets of functions $f \in C^r(M, N)$. The spaces $J^r(M, N)$ are finite dimensional manifolds for all $1 \leqslant r < \infty$ and for $r = \infty$ the space $J^\infty(M, N)$ is an infinite dimensional manifold modeled on $\mathbb{R}^\infty$. The natural projections

$$J^r(M,N) \xrightarrow{\pi_r} M$$

$$J^r(M,N) \xrightarrow{\pi_{r,0}} M$$

$$J^r(M,N) \xrightarrow{\pi_{r,k}} J^s(M,N)$$

$1 \leqslant s < r \leqslant r \leqslant \infty$, given by $\pi_r j^r_x f = x$, $\pi_{r,0} j^r_x f = f(x)$ and $\pi_{r,k} j^r_x f = j^s_x f$ are surjective submersions (see [72] for details). We have an injective inclusion

$$J^r : C^r(M,N) \longrightarrow C^0(M, J^r(M,N))$$

given by $J^r(f)(x) = j^r_x f$. Considering on $C^0(M, J^r(M, N))$ the co.o.-topology, we can topologize the space $C^r(M, N)$ so that J^r be an embbeding. This topology is called *C^r-topology* (or *weak topology*). When M is noncompact, it is convenient consider the strong topology on $C^r(M, N)$, induced by the graph-topology on $C^0(M, J^r(M, N))$. Those topologies coincide when M is compact. We will restrict our attention to the compact case. Many properties about this topology can be found in Hirsch's book [33], chapter 2.

Theorem 1.2.23. The C^r-topology is admissible, $1 \leqslant r \leqslant \infty$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
C^r(M,N) \times M \arrow[r, "J^r \times 1"] \arrow[d, "e_1"] & C^0(M, J^r(M,N)) \times M \arrow[d, "e_2"] \\
N & J^r(M,N) \arrow[l, "\pi_{r,0}"]
\end{tikzcd}$$

where e_i, $i = 1, 2$, denote the evaluation maps. Since e_2 is continuous by 1.2.16, it follows that the C^r-topology is admissible. $\square$

Theorem 1.2.24 ([33], chapter 2)**.** Let M be a C^r-closed manifold. The group $\mathrm{Diff}^r(M)$ under the C^r-topology is a topological group.

Theorem 1.2.25. A C^r fiber space with group $G = \mathrm{Diff}^r(F)$ equipped with the C^r-topology is a fiber bundle. Let $\mathcal{B} = \{B, p, M, F\}$ and $\mathcal{B}' = \{B', p', M', F\}$ be C^r-fiber bundles with group $G = \mathrm{Diff}^r(F)$. A C^r-map $h : \mathcal{B} \longrightarrow \mathcal{B}'$ that carries each fiber of $\mathcal{B}$ diffeomorphically to a fiber of $\mathcal{B}'$ is a bundle map.

Proof. In general, given C^r-function $f : X \times Y \longrightarrow Z$, denoting f_x the function $f_x(y) = f(x, y)$ we have a continuous map

$$\begin{aligned} k : X \times Y &\longrightarrow J^r(Y, Z) \\ (x, y) &\longrightarrow j^r_y f_x. \end{aligned}$$

Since k is continuous, the function $k^* : X \longrightarrow C^0(Y, J^r(Y, Z))$ given by $k^*(x)(y) = k(x, y)$ is continuous (see 1.2.21 item 3).

Now, let $\mathcal{B} = \{B, p, M, F\}$ a C^r-fiber space and $\varphi_i : p^{-1}(V_i) \longrightarrow V_i \times F$ be a C^r-diffeomorphism as in Definition 1.2.1. Denoting $\phi_i = \varphi_i^{-1}$, we want to show that the function

$$g_{ji} : V_j \cap V_i \longrightarrow \mathrm{Diff}^r(F)$$

given by $g_{ji}(x) = \phi_{j,x}\phi^{-1}_{i,x}$ is continuous. Via the identification of $C^r(F, F)$ as a subspace of $C^0(F, J^r(F, F))$, this function is given by $g_{ji}(x)(y) = j^r_y(\phi_{j,x}\phi^{-1}_{i,x})$. With the notation of the preceeding paragraph, taking $f(x, y) = p_2\phi_j\phi_i(x, y)$, then $k^*(x) = g_{ji}(x)$ is continuous, since f is a C^r-function. Therefore $\mathcal{B}$ is a fiber bundle.

Now, let (ϕ, U), (ϕ', V) local trivializations for $\mathcal{B}$ and $\mathcal{B}'$, respectively, $x \in M$ and $x' = \overline{h(x)}$. As in the preceeding paragraph, we can show that the map

$$\begin{aligned} U \cap \overline{h}^{-1}(V) &\longrightarrow \mathrm{Diff}^r(F) \\ x &\longrightarrow {\phi'^{-1}}_{x'} h_x \phi_x \end{aligned}$$

is continuous. Thus, since h carries the fibers of $\mathcal{B}$ homeomorphically into the fibers of $\mathcal{B}'$, it follows that h is a bundle map. □

Corollary 1.2.26. Two C^r-fiber bundles $\mathcal{B} = \{B, p, M, F\}$ and $\mathcal{B}' = \{B', p', M, F\}$ are equivalent if there exists a C^r-diffeomorphism $h : \mathcal{B} \longrightarrow \mathcal{B}'$ satisfying $p'h = p$.

In what follows, we will consider only smooth fiber bundles (C^∞-fiber bundles). The group of all C^∞-diffeomorphisms of a manifold F equipped with the C^∞-topology will be denoted by $\mathrm{Diff}(F)$.

1.2.4 Differentiable principal fiber bundles

In the classical book [42], S. Kobayashi and K. Nomizu define the notion of principal fiber bundle as follows. Let M be a smooth manifold and G a Lie group. A manifold P is called a *differentiable principal fiber bundle* provided the following conditions are satisfied:

(1) G acts differentiably on P to the right without fixed point: $(b,g) \in P \times G \longrightarrow bg = R_g \cdot b \in P$;

(2) M is the quotient space of P by the equivalence relation induced by G and the canonical projection $\pi : P \longrightarrow M$ is differentiable;

(3) P is *locally trivial*, that is, every point $x \in M$ has a neighborhood U such $\pi^{-1}(U)$ is isomorphic with $U \times G$ in sense that $b \in \pi^{-1}(U) \longrightarrow (\pi(b), \varphi(b)) \in U \times G$ is differentible isomorphism satisfying $\varphi(bg) = \varphi(b)g$ for all $g \in G$. In this case, M is called the base of P and G the structural group of P. We denote by $P(M,G)$ a principal bundle with base M and group G.

Remark 1.2.27. There exists a 1-1 correspondence between local trivializations and local cross-sections in a differentiable principal bundle. Indeed, given a local trivialization $\psi : \pi^{-1}(U) \longrightarrow U \times G$, then $s_\psi(x) = \psi^{-1}(x,e)$ is a smooth local cross-section. Conversely, given a local cross-section $s : U \longrightarrow \pi^{-1}(U)$, the function $\phi : U \times G \longrightarrow \pi^{-1}(U)$ given by $\phi(x,g) = s(x)g$ is a diffeomorphism and $\psi = \phi^{-1}$ become a local trivialization satisfying $s = s_\psi$. Note that the map $\phi_x : G \longrightarrow \pi^{-1}(x)$ satisfies $\phi_x^{-1}(bg) = \phi_x^{-1}(b)g$.

Definition 1.2.28. Let $P(M,G)$ and $P'(M,G')$ be differentiable principal fiber bundles. A differentiable mapping f of P into P' is called a *homomorphism* if there is a homomorphism, denoted by the same letter f, of G into G' such that $f(bg) = f(b)f(g)$ for all $b \in P$ and $g \in G$. If f is a diffeomorphism, it is called an *isomorphism*.

Theorem 1.2.29. The following statements hold:

(1) Let B and X be smooth manifolds and let G be a Lie group. Suppose that $\mathcal{B} = \{B, p, X, G\}$ is a principal bundle and that there exists a representative coordinate bundle for $\mathcal{B}$ such that the coordinate functions concerning this coordinate bundle are diffeomorphisms. Then $\mathcal{B}$ is a differentiable principal bundle;

(2) a differentiable principal bundle is a Steenrod bundle;

(3) Let h be a bundle map between two differentiable principal bundles. Then h is an homomorphism between differentiable principal bundles;

(4) let h be a homomorphism between two differentiable principal bundles $P(M,G)$ and $P'(M,G)$. Suppose that that the induced map $G \longrightarrow G$ is an homeomorphism. Then h is a differentiable bundle map.

Proof. The proof requires only routine definitions and computations.

(1) - Let $\mathcal{B} = \{B, p, X, G\}$ be a principal bundle with B, X being smooth manifolds, G being a Lie group, and a family $\{V_j\}$ of open sets covering X with coordinate

functions $\phi_j : V_j \times G \longrightarrow p^{-1}(V_j)$ being smooth diffeomorphisms. We will construct a right action of G on B such that $B/G = X$. Let $b \in B$ with $\pi(b) = x$. Set

$$b \cdot g = \phi_{j,x}(g^{-1}p_2\phi_j^{-1}(b)),$$

where $\phi_{x,j}(g) = \phi_j(x, g)$. Note the following equalities

$$\phi_{j,x}(p_2\phi_j^{-1}(b)) = b \text{ and } p_2\phi_j^{-1}\phi_{i,x}(g) = g.$$

Given $x \in V_i \cap V_j$, we have

$$\phi_{j,x}^{-1}\phi_{i,x}(p_2\phi_i^{-1}(b)) = p_2\phi_j^{-1}(b).$$

Since $\phi_{j,x}^{-1}\phi_{i,x}$ conicides with an operation of an element of G, then

$$\phi_{j,x}^{-1}\phi_{i,x}(g^{-1}p_2\phi_i^{-1}(b)) = g^{-1}p_2\phi_j^{-1}(b)$$

for all $g \in G$. Thus, the rule $(b, g) \longrightarrow b \cdot g$ is well-defined. This rule satisfies the axioms of an action:

$$b \cdot e = \phi_{j,x}(e^{-1}p_2\phi_j^{-1}(b)) = b;$$

$$(b \cdot g) \cdot h = \phi_{j,x}(h^{-1}p_2\phi_j^{-1}(\phi_{j,x}(g^{-1}p_2\phi_j^{-1}(b)))) = \phi_{j,x}(h^{-1}g^{-1}p_2\phi_j^{-1}(b)) = b \cdot (gh).$$

This action is smooth, since it is given in terms of smooth maps. Note that the orbit of a point $b \in B$ is equal the fiber containing b. It follows that the function $p : B \longrightarrow B/G$ is equal to $p : B \longrightarrow X$. Now, we will give the local trivializations. For each j, let $\varphi_j(b) = ({\phi_{j,\pi(b)}}^{-1}(b))^{-1}$ and $\psi_j(b) = (\pi(b), \varphi_j(b))$. Denoting $x = \pi(b)$, we have

$$\varphi_j(b \cdot g) = (\phi_{j,x}^{-1}(b \cdot g))^{-1} = (\phi_{j,x}^{-1}(\phi_{j,x}(g^{-1}p_2\phi_j^{-1}(b)))^{-1} = (g^{-1}p_2\phi_j^{-1}(b))^{-1} = \varphi(b)g.$$

It follows that $\mathcal{B}$ determines a differentiable principal fiber bundle.

(2) - Let $P(M, G)$ be a differentiable principal bundle. We will associate to $P(M, G)$ a coordinate bundle as in Definition 1.2.4. The bundle space, base space, projection, and fiber are, respectively, P, M, π and G. Since $P(M, G)$ is a differentiable principal bundle, we can consider a family of open sets $\{V_j\}$ covering M with local cross-section $s_j : V_j \longrightarrow \pi^{-1}(V_j)$ such that $\phi_j(x, g) = s_j(x)g$ is a diffemorphism for all j (see 1.2.27). We have

$$\phi_{j,x}^{-1}\phi_{i,x}(g) = \phi_{j,x}^{-1}(s_i(x)g) = \phi_{j,x}^{-1}(s_i(x)) \cdot g,$$

hence $\phi_{j,x}^{-1}\phi_{i,x}$ conicides with the operation by an element of G. Note that $\phi_{j,x}^{-1}\phi_{i,x} = \phi_{j,x}^{-1}(s_j(x))$ is a smooth function. It follows that the collection $P, \pi, M, G, V_j, \phi_j$ determine a (smooth) coordinate bundle.

(3) - Now, let $h : P(M, G) \longrightarrow P^{'}(M, G)$ be a bundle map between differentiable principal bundles. Fixed $b \in P$, let $s, s^{'}$ local cross-sections with $s(x) = b$ and $s(x^{'}) = h(b)$ ($x^{'} = \overline{h}(x)$) and ϕ, $\phi^{'}$ given by $\phi(y, g) = s(y)g$, $\phi^{'}(y, g) = s^{'}(y)g$. By the item 1 above, ϕ and $\phi^{'}$ are coordinate functions for P and $P^{'}$, respectively. Then

$$\phi^{'\,-1}_{x^{'}} h(bg) = \phi^{'\,-1}_{x^{'}} h_x \phi_x(g) = (\phi^{'\,-1}_{x^{'}} h_x \phi_x) \cdot g,$$

since $\phi^{'\,-1}_{x^{'}} h_x \phi_x$ coincides with an operation of an element of G, and

$$\phi^{'\,-1}_{x^{'}} h(bg) = (\phi^{'\,-1}_{x^{'}} h_x \phi_x(e)) \cdot g = \phi^{'\,-1}_{x^{'}} (h(b)) \cdot g = \phi^{'\,-1}_{x^{'}} (h(b)g),$$

concluding that $h(bg) = h(b)g$.

(4) - Now, let $f : P(M, G) \longrightarrow P^{'}(M, G)$ be a homomorphism with the induced map of G onto itself being a homeomorphism. Since $f(bg) = f(b)f(g)$ and a continuous homomorphism with continuous inverse between Lie groups is a diffeomorphism (see [48]), then f carries the fibers of P diffeomorphically onto the fibers of $P^{'}$. With the notations of the item 3, we have

$$\phi^{'\,-1}_{x^{'}} f_x \phi_x(g) = \phi^{'\,-1}_{x^{'}} f_x(s(x)g) = \phi^{'\,-1}_{x^{'}} (f_x(s(x))f(g)) = (\phi^{'\,-1}_{x^{'}} f_x(s(x))) \cdot f(g).$$

Thus, $\phi^{'\,-1}_{x^{'}} f_x \phi_x = \phi^{'\,-1}_{x^{'}} f_x(s(x))$ is a smooth map that conincides with an operation of an element of G. Hence f is a bundle map.

□

Definition 1.2.30. Let P be a differentiable principal fiber bundle with base M and structural Lie group G. A *connection* in P is a smooth distribution Γ in P satisfying

(1) $T_x P = \Gamma_x \oplus G_x$, where G_x denotes the tangent space of the fiber at x.

(2) Given $x \in P$ and $g \in G$, then $\Gamma_{xg} = (R_g)_* \Gamma_x$, where R_g denote the right translation of G by g.

The subspaces Γ_x are called *horizontal subspaces* of the connection Γ. A connection in a principal fiber bundle is said *flat* if it is given by an integrable distribution[3].

Theorem 1.2.31 ([42], page 79)**.** A homomorphism $f : P(B, G) \longrightarrow P^{'}(B, G^{'})$ such that the induced map in the base spaces is a diffeomorphism, maps a connection Γ into a connection $\Gamma^{'}$ in such way that f maps the horizontal subspaces of Γ into the horizontal subspaces of $\Gamma^{'}$.

[3] This is an equivalent condition to flatness. A connection is said flat if the associated curvature two form vanishes.

Remark 1.2.32. Let $P(M, G)$ be a differentiable principal fiber bundle and F be a differentiable manifold on which G acts to the left. Consider the action of G on $P \times F$ given by $(x, f)g = (xg, g^{-1}f)$. Let $B = (P \times F)/G$ be the quotient space. The canonical projection $\pi_P : P \longrightarrow M$ induces a projection π_B of B onto M in the following way: the class determined by class (x, f) is mapped to $\pi_P(x)$. Space B is called a fiber bundle with base space M that is associated with P. This definition of associated bundle appear in Kobayashi-Nomizu [42]. It is easy to show that if $\mathcal{B} = \{B, p, X, Y, G\}$ is a smooth Stenrod fiber bundle and if G is a Lie group that acts smoothly on the fiber, then the associated principal bundle $P = P(\mathcal{B})$ is a differentiable principal bundle and the bundle that is associated to P is equivalent to $\mathcal{B}$ (equivalence as defined by Steenrod of course).

1.2.5 Circle bundles

By a *circle bundle* we means a fiber bundle $\mathcal{B} = \{B, p, X, Y, G\}$ with fiber $Y = \mathbb{S}^1$. Next, we will show that two differentiable principal circle bundles are isomorphic Kobayashi-Nomizu's sense (differentiable isomorphic) if, and only if, are equivalents as $\mathrm{Diff}^+(\mathbb{S}^1)$-bundles[4].

Theorem 1.2.33 ([78], §6.7)**.** Let $\mathcal{B} = \{B, p, X, F, G\}$ be a smooth fiber bundle and ρ a metric that generates the topology of B. If there exists a continuous cross-section[5] s for $\mathcal{B}$, given $\epsilon > 0$, there exists a smooth cross-section s' for $\mathcal{B}$ such that $\rho(s(x), s'(x)) < \epsilon$ for all $x \in M$.

Let $\mathcal{B}_1, \mathcal{B}_2$ be differentiable principal fiber bundles with the same base space M and the same group G. Consider the action of G on $B_1 \times B_2$ given by $(b_1, b_2)g = (b_1 g, b_2 g)$. Then, $B = (B_1 \times B_2)/G$ is the total space of a smooth fiber bundle $\mathcal{B}$ with base space M, group G, fiber B_2 and projection given by $p[(b_1, b_2)] = p_1(b_1)$ (Defining a left action on B_2 by $g \cdot b = b \cdot g^{-1}$, this bundle is the associated to $\mathcal{B}_1$ with fiber B_2, see 1.2.32). Considering $B_1 \times B_2 \longrightarrow B_1$ as a trivial bundle, we have that π is a bundle map and the diagram

$$\begin{tikzcd}
B_1 \times B_2 \arrow[r, "\pi"] \arrow[d, "\pi_1"] & B \arrow[d, "p"] \\
B_1 \arrow[r, "p_1"] & M
\end{tikzcd}$$

is commutative. To see this, we will describe the coordinate functions for the bundle $\mathcal{B}$. Let (ψ, U) a local trivialization of $\mathcal{B}_1$ and s the associated cross-section given by 1.2.27. The function $\phi : U \times B_2 \longrightarrow p^{-1}(U)$ given by $\phi(x, b_2) = [(s(x), b_2)]$ is a coordinate transformation as in the item (7) of 1.2.4. Since the map $(x, g) \longrightarrow s(x)g$ is a diffeomorphism,

[4] The symbol $\mathrm{Diff}^+(M)$ denote the subgroup of $\mathrm{Diff}(M)$ of all orientation preserving diffeomorphisms of M.

[5] A *cross-section* on a fiber bundle $\mathcal{B} = \{B, p, X, F, G\}$ is a function $s : X \longrightarrow B$ satisfying $ps = 1$.

the map $b \longrightarrow g_b$ such that $s(x)g_b = b$ is a smooth map. Then,

$$(\phi_x)^{-1}([b_1, b_2]) = \phi_x^{-1}[(s(x)g_{b_1}, b_2)] = \phi_x^{-1}[(s(x), b_2 g_{b_1}^{-1}]) = b_2 g_{b_1}^{-1}$$

and

$$\phi_{p_1(b_1)}^{-1} \pi_{b_1}(b_2) = b_2 g_{b_1}^{-1} = g_{b_1} b_2, \tag{1.1}$$

concluding that $\phi_{p_1(b_1)}^{-1} \pi_{b_1}$ coincides with the operation of an element of G and the function $b_1 \longrightarrow \phi_{p_1(b_1)}^{-1} \pi_{b_1}$ is smooth. Now, given $b_1 \in B_1$ and s be a local cross-section with $s(x) = b_1$, we have that

$$\pi_{b_1}(b_2) = \phi(x, b_2)$$

is a diffeomorphism. It follows that π is a bundle map.

Theorem 1.2.34. Under the above hypothesis, there is a 1-1 correspondence between the smooth (continuous) cross-sections of $\mathcal{B}$ and the smooth (continuous) bundle maps of $\mathcal{B}_1$ to $\mathcal{B}_2$.

Proof. To proof this theorem we will use the following general fact about submersions. Let $f : M \longrightarrow N$ be a submersion. Then, a map $g : N \longrightarrow Z$ is smooth (continuous) if, and only if, gf is smooth (continuous). Indeed, there exists smooth local cross-section for f trough every point $y \in N$. Thus, if gf is smooth (continuous), then given $y \in N$ and smooth local cross-section $s : U \longrightarrow M$, the map $(gfs)|_U$ is smooth (continuous), but $(gfs)|_U = g|_U$. It follows that g is smooth (continuous).

Given a smooth (continuous) bundle map $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$, then $s : M \longrightarrow \mathcal{B}$ given by $s(x) = [(b, h(b))]$, where b is any point in the fiber over x, is well-defined cross-section. Indeed, given $g \in G$, we have

$$[(bg, h(bg))] = [(bg, h(b)g)] = [(b, h(b))].$$

By the commutativity of the diagram

$$\begin{tikzcd} B_1 \arrow[r, "1\times h"] \arrow[d, "p_1"] & B_1 \times B_2 \arrow[d, "\pi"] \\ M \arrow[r, "s"] & B, \end{tikzcd}$$

it follows that $sp_1 = \pi(1 \times h)$ is smooth (continuous). Since p_1 is a submserion, we conclude that s is smooth (continuous).

Conversely, let $s : M \longrightarrow B$ be a smooth (continuous) cross-section. Since π is a bundle map, for each $b_1 \in B_1$, the map π_{b_1} is a diffeomorphism from B_2 to the fiber over $p_1(b_1)$. Thus, the map $h(b_1) = \pi_{b_1}^{-1}(s(p_1(b_1)))$ is well-defined and satisfies:

(1) $\pi(1 \times h) = sp_1$, hence $h(b_1 g) = h(b_1)g$ for all $b_1 \in B_1$; in particular, h carries fiber in fiber diffeomorphically.

(2) Given a trivialization $\phi(x, b_2) = [(s(x), b_2)]$ as above, by Equation 1.1 we have that

$$\phi^{-1}_{p_1(b_1)} h(b_1) = \phi^{-1}_{p_1(b_1)} \pi^{-1}_{b_1}(sp_1(b_1)) = (sp_1(b_1))g^{-1}_{b_1}$$

is a smooth (continuous) map. It follows that h is a smooth (continuous) map.

□

Let $\mathcal{B}_1, \mathcal{B}_2$ bundle with the same base space and group and $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ a bundle map that covers the identity. Let $s : M \longrightarrow B$, $s(x) = [(b, h(b))]$ be the local cross-section given by Theorem 1.2.34. Since h covers the identity, we have $p_1(b) = p_2 h(b)$. Conversely, if $s : M \longrightarrow B$ is a cross-section, say $s(x) = [(b, f(x))]$, satisfying $p_1(b) = p_2(f(x))$, given the bundle map $h(b) = \pi^{-1}_b(s(p_1(b)))$ associeated with s, it is easy to see that h covers the identity. Thus, we have characterized the cross-sections that correspond to bundle maps that covers the identity. It is not difficult to see that $B^{'} = \{[(b_1, b_2)] \in B / p_1(b_1) = p_2(b_2)\}$ is an embendding submanifold of B and the restricition $\pi : B^{'} \longrightarrow M$ determines a fiber bundle with fiber G. Thus, a cross-section $s : M \longrightarrow B$ determines a bundle map that covers the identity if and only if take values in $\mathcal{B}^{'}$, that is, if and only if it is a cross-section $s : M \longrightarrow B^{'}$. Thus, we have:

Theorem 1.2.35. Two differentiable principal fiber bundles equivalents by a continuous bundle map are differentiable isomorphic.

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ two differentiable principal bundle with same base space M and group G. Suppose that there exists a continuous bundle map $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ that covers the identity. Then, by 1.2.34 there exists a cross-section $s : M \longrightarrow B$ determined by h. Since h covers the identity, s is a cross-section $s : M \longrightarrow B^{'}$. By Theorem 1.2.33, there exists a smooth cross-section $s^{'} : M \longrightarrow B^{'}$ that apporoximates s. This cross-section corresponds to a smooth bundle map $h^{'} : B_1 \longrightarrow B_2$ that covers the identity. By 1.2.13 and 1.2.35, this bundle map determines an isomorphism of differentiable principal bundles. □

Lemma 1.2.36 ([78], theorem 29.2.)**.** Let $\mathcal{B}$ be a fiber bundle with fiber F and base space a finite complex K of dimension n. If Y is arcwise connected and $\pi_i(Y) = 0$ for $i = 1, \ldots, n-1$, then $\mathcal{B}$ admits a cross-section.

Theorem 1.2.37. Let K be a finite complex of dimension n, G be a topological group and H a closed subgroup of G such that G/H is arcwise connected. Suppose that $\pi_i(G/H) = 0$ for all $i = 1, \ldots, n-1$. If H has a local cross-section in G[6], then any two H-bundles over K that are G-equivalents are also H-equivalents. In particular, two H-bundles over a smooth manifold M that are G-equivalents are also H-equivalent.

[6] This means that there exists a neighborhood U of $H \in G/H$ and a continuous cross-section $s : U \longrightarrow G$.

Proof's sketch. This theorem follows as in [78], § 12.5. The hypothesis about K and G/K enable us to obtain cross-section of any bundle with fiber G/H over K by 1.2.36. In particular, a G-bundle over a n-complex K is equivalent to a H-bundle over K (See [78], § 9.2 for details). To finish, every smooth manifold M admit a smooth triangulation $t : |K| \longrightarrow X$ [7]. Thus, a fiber bundle $\mathcal{B} = \{B, p, M, Y, G\}$ is completely determined by the fiber bundle $\mathcal{B} = \{B, t^{-1}p, K, Y, G\}$ over the n-complex K. □

Theorem 1.2.38. The space $\mathrm{Diff}^+(\mathbb{S}^1)$ has as a deformation retract the rotation group $SO(2)$.

Proof's sketch. Denote by $\mathrm{Diff}^+{}_0(\mathbb{S}^1)$ the subgroup of $\mathrm{Diff}^+(\mathbb{S}^1)$ that keep fixed $1 \in \mathbb{S}^1$.

(1) $\mathrm{Diff}^+(\mathbb{S}^1) = \mathrm{Diff}^+{}_0(\mathbb{S}^1)SO(2)$. Indeed, given $f \in \mathrm{Diff}^+(\mathbb{S}^1)$, and $\gamma \in \mathbb{S}^1$ with $f(\gamma) = 1$, there exists $g \in SO(2)$ with $g^{-1}(\gamma) = 1$. Thus, $f \circ g(1) = \theta$, and $f \circ g \in \mathrm{Diff}^+{}_0(\mathbb{S}^1)$. It follows that $f \in \mathrm{Diff}_0^+(\mathbb{S}^1)SO(2)$ and, therefore, $\mathrm{Diff}^+(\mathbb{S}^1) = \mathrm{Diff}^+{}_0(\mathbb{S}^1)SO(2)$.

(2) Consider the usual coverging $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ with $\pi(0) = 1$. Any function $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$ extend to a function $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ safisfying $\tilde{f}(x+1) = \tilde{f}(x) + d$. Since f is an orientation-preserving diffeomorphism, we have $d = 1$. It follows that $\mathrm{Diff}^+{}_0(\mathbb{S}^1)$ can be idendified with

$$G = \{\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}/\tilde{f}(0) = 0 \text{ and } \tilde{f}(x+1) = \tilde{f}(x) + 1\} \subset \mathrm{Diff}^+(\mathbb{R}).$$

Now, the function $F : G \times I \longrightarrow G$ given by $F(\tilde{f}, t) = (1-t)\tilde{f} + tx$ is well-defined, continuous, hence give us a homotopy between the identity map 1_G and the point $1_{\mathbb{R}} \in G$.

(3) To finish, we have that the function $r : \mathrm{Diff}^+(\mathbb{S}^1) \times I \longrightarrow \mathrm{Diff}^+(\mathbb{S}^1)$ given by $r(fg, t) = \pi(F(\tilde{f}, t))g$, $f \in \mathrm{Diff}_0^+(\mathbb{S}^1)$, $g \in SO(2)$, satisfies $r_0 = 1$ and $r_f(g) = g$ for all $g \in SO(2)$. It follows that r_t is a deformation retraction of $\mathrm{Diff}^+(\mathbb{S}^1)$ onto $SO(2)$.

□

Theorem 1.2.39. Two differentiable principal circle bundles are differentiable isomorphic if and only if are equivalents as $\mathrm{Diff}^+(\mathbb{S}^1)$-bundles.

Proof. In Theorem 1.2.38 we have showed that $SO(2)$ has a global cross-section in $\mathrm{Diff}^+(\mathbb{S}^1)$ and $\mathrm{Diff}^+(\mathbb{S}^1)/SO(2)$ is arcwise connected and contractible. In particular

$$\pi_i(\mathrm{Diff}^+(\mathbb{S}^1)/SO(2)) = 0$$

[7] The map $t : |K| \longrightarrow X$ is homeomorphism such that for each simplex σ of K, the restriction $t : \sigma \longrightarrow t(\sigma)$ is a smooth submersion. The existence of smooth triangulations was established, for example, in [6].

for all $i \geqslant 1$. It follows by 1.2.37 that two $SO(2)$-principal bundles over a finite complex K that are $\mathrm{Diff}^{+}(\mathbb{S}^1)$-equivalents, are $SO(2)$-equivalentes. If those bundles are differentiable principal bundles over a manifold M, it follows by 1.2.35 that we can obtain a differentiable isomorphism. $\square$

1.3 Foliations

Intuitively, a manifold M is "foliated" if it is expressed as the union of l-dimensional submanifolds that fit alongside each other, locally, like parallel l-planes in Euclidian n-space. Following Deahna-Clebsch-Frobenius' theorem[8], a (smooth) foliation $\mathfrak{F}$ can be defined as an involutive subbundle E of the tangent bundle of M. If the fibers of E are $l-$dimensional, the maximal integral manifolds of E are immersed l-dimensional submanifolds of M called *leaves* of the foliation $\mathfrak{F}$. This result allows constructing an atlas for M as in the next definition.

Definition 1.3.1. Let $\mathfrak{F}$ be a foliation on M. A *foliated chart* for M is a coordinate neighborhood (U, φ) for it, $\varphi = (x, y)$, such that the leaves of $\mathfrak{F}$ are determined locally by $y = constant$. Each set $y = constant \subset U$ is called *plaque* of $\mathfrak{F}$. A *foliated atlas* $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}_{\alpha\in J}$ for M is an atlas formed only by foliated charts. A foliated atlas is said to be *regular* provided:

(1) for each $\alpha \in J$, $\overline{U_\alpha}$ is a compact subset of a foliated chart (W, φ) and $\phi_\alpha = \varphi\mid_{U_\alpha}$;

(2) the cover $\{U_\alpha\}_{\alpha\in J}$ is locally finite[9];

(3) if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are elements of $\mathcal{U}$, $\alpha, \beta \in J$, then the interior of each closed plaque $P \subset \overline{U}_\alpha$ meets at most one plaque in $\overline{U}_\beta$.

Given a foliated atlas $\mathcal{U}$ concerning to a foliation $\mathfrak{F}$, the coordinate maps

$$x_\alpha : U_\alpha \longrightarrow \mathbb{R}^p \text{ and } y_\alpha : U_\alpha \longrightarrow \mathbb{R}^q$$

are submersions on its image and the local coordinate change on $U_\alpha \cap U_\beta$ has, therefore, the form

$$\begin{aligned} x_\alpha &= \phi_\alpha(x_\beta, y_\beta), \\ y_\alpha &= \psi_\alpha(y_\beta). \end{aligned}$$

Fixing for each foliated chart $(U_\alpha, \varphi_\alpha)$ a submanifold $S_\alpha = \{x_\alpha = \text{constant}\}$, we have that

$$S_{\mathcal{U}} = \coprod_{\alpha\in J} S_\alpha$$

[8] See historical note of [53].
[9] This means that given $I \subset J$, if $\bigcap_{\alpha\in I} U_\alpha \neq \varnothing$, then I is finite.

is an embedded submanifold of M transverse to $\mathfrak{F}$, that is, transverse to the leaves of $\mathfrak{F}$ (see [30] chapter 2). The equation $y_\alpha = \psi_\alpha(y_\beta)$ can be viewed as local diffeomorphism $\gamma_{\alpha\beta}$ of S_α to S_β. These diffeomorphisms generate a pseudogroup $\Gamma_{\mathcal{U}}$ called *holonomy pseugroup* of $\mathfrak{F}$ defined on $S_{\mathcal{U}}$; the definition of pseudogroup is given in the sequence. Geometrically theses pseudogroups are obtained by sliding along the leaves. For a discussion about the equivalence between these pseudogroups see [31]. In what follows, we will refer to holonomy pseudogroup of $\mathfrak{F}$ as the pseudogroup $\Gamma_{\mathcal{U}}$ for a regular foliated atlas $\mathcal{U}$.

Definition 1.3.2. A *pseudgroup of transformation* on a topological space S is a set Γ of transformations satisfying the following conditions:

(1) Each element $f \in \Gamma$ is a homeomorphism of an open set of S (called the domain of f) onto another open set of S (called the range of f);

(2) If $f \in \Gamma$ then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

(3) If $f_i : U_i \longrightarrow V_i$, $i \in J$, are elements in Γ, then $f : \bigcup_{i\in J} U_i \longrightarrow \bigcup_{i\in J} V_i$ defined by $f(x) = f_i(x)$ if $x \in U_i$, is an element of Γ;

(4) For every open set U of S the identy transformation of U is in Γ;

(5) If $f \in \Gamma$, then $f^{-1} \in \Gamma$;

(6) The composition of elements of Γ where have sense is in Γ.

Remark 1.3.3. There are other definitions of regularity of a foliation. For example, foliations where the individual leaves are C^k-submanifolds but $T\mathfrak{F}$, when exists, have class C^r, $r \leqslant k$. Foliations also can be defined in other topological spaces (see [21]).

Let $\mathfrak{F}$ be a foliation defined on M and L be a leaf of $\mathfrak{F}$. Given $x \in L$ and Σ a transversal embedding submanifold containing x, to each loop α with base at $x \in M$, we can associate a germ of a local diffeomorphism $\varphi_\alpha : U \subset \Sigma \longrightarrow \Sigma$ by sliding along the leaves "covering" α. The definition of this map depends only on the homotopy class of α. It is well-comported concerning the concatenation of loops. Therefore, induces a homomorphism

$$\varphi : \pi_1(M) \longrightarrow \mathrm{Germ}(\Sigma)$$

called *holonomy* of L at x. Given $x, y \in L$, the holonomy of L at x is equivalent the holonomy of L at y. For a precise definition see [8, 9].

Example 1.3.4. Let ω be a closed p-form of constant rank defined on manifold M. Then, the distribution $x \longrightarrow \ker \omega_x$ is integrable and determines a foliation denoted by $\mathfrak{F}_\omega$. Indeed, if X, Y are vector fields tangent to $\ker\omega$, then

$$d\omega(X_1,\ldots,X_{p+1}) = \sum_{1\leqslant i\leqslant k+1}(-1)^{i-1}X_i(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})) + \sum_{1\leqslant i<j\leqslant p+1}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1}),$$

for any smooth vector fields $X_1,\ldots,X_{p+1}$, where $\hat{X}$ mens that X not is considered. Given $X, Y \in \ker\omega$, putting $X = X_1, Y = X_2$ and X_i arbitrary for $i > 2$, we have

$$d\omega(X_1,\ldots,X_{p+1}) = -\omega([X,Y],X_3,\ldots,X_{p+1}) = i_{[X,Y]}\omega(X_3,\ldots,X_{p+1}).$$

Thus, if $d\omega = 0$ then $i_{[X,Y]}\omega = 0$. Therefore, when ω is a closed form of constant rank, the distribution $\ker\omega$ gives a foliation on M. When $p = 1$, the holonomy of each leaf of $\mathfrak{F}_\omega$ is trivial. Conversely, if $\mathfrak{F}$ is a C^2-codimension one foliation such that every leaf has trivial holonomy group, then $\mathfrak{F}$ is topologically equivalent to a foliation given by the kernel of a closed 1-form ([8], page 82).

1.4 Foliated bundles

A foliated bundle $\mathcal{B} = \{B, p, M, F, \mathfrak{F}\}$ is a smooth fiber bundle (group $G = \mathrm{Diff}(F)$) with a foliation $\mathfrak{F}$ satisfying the following condition: for each $x \in X$ there exists a local trivialization of the bundle $\mathcal{B}$, $\psi : p^{-1}(U) \longrightarrow U \times F$, such that $\mathfrak{F}|_{\pi^{-1}(U)}$ is the trivial foliation $\{U \times \{y\}\}_{y\in F}$. In a foliated bundle, each leaf is a covering of the base space and the holonomy pseudogroup of $\mathfrak{F}$ have a simpler description by the existence of the global sections for $\mathfrak{F}$, the fibers of $\mathcal{B}$ (in fact we have a group). For each $x, y \in M$, and continuous path $\alpha : [0,1] \longrightarrow M$ conecting x to y in M, we have a diffeomorphism

$$\varphi(\alpha) : p^{-1}(x) \longrightarrow p^{-1}(y)$$

defined by

$$\varphi^{-1}(\alpha)(z) = \lambda_y(z),$$

where $\lambda : [0,1] \longrightarrow p^{-1}(z)$ is the unique curve that covers α concerning to the covering map $p : L_z \longrightarrow M$ (where L_z is the leaf of $\mathfrak{F}$ containing x). The covering proprerties implies that $\varphi(\alpha)$ depend only of the homotopy class of α. For $x = y$, the function

$$\varphi : \pi_1(M,x) \longrightarrow \mathrm{Diff}(p^{-1}(x))$$

is a homomorphism called *holonomy homomorphism* of the $\mathcal{B}$ with image Γ called *holonomy group* of $\mathcal{B}$. This homomorphism independs of the choice of base point and we can see it as a homomorphism from $\pi_1(M)$ to $\mathrm{Diff}(F)$. Conversely, given a homomorphism

$$\varphi : \pi_1(M) \longrightarrow \mathrm{Diff}(F)$$

we can construct a smooth fiber bundle with base space M, fiber F and discret structural group Γ =image of φ. We know that $\pi_1(M)$ act on $\tilde{M}$ by *Deck transformations* and on F by φ. Then, $\pi_1(M)$ act on $\tilde{M} \times F$ by $\alpha(\tilde{x}, f) = (\alpha \cdot \tilde{x}, \varphi(\alpha)^{-1}(f))$. Let

$$B = (\tilde{X} \times F)/\pi_1(M)$$

with natural projection

$$p : (\tilde{M} \times F)/\pi_1(M) \longrightarrow \tilde{X}/\pi_1(M) = M.$$

We have that $\mathcal{B}_\varphi = \{B, p, M, F, \mathfrak{F}\}$ is a foliated bundle where the leaves of $\mathcal{F}$ are given by the projection of the $\tilde{M} \times y$, $y \in Y$. If $\varphi : \pi_1(M) \longrightarrow \Gamma \subset \text{Diff}(F)$ is the holonomy homomorphism of the foliated bundle $\mathcal{B} = \{B, p, M, F, \mathfrak{F}\}$, then $\mathcal{B}_\varphi$ is equivalent to $\mathcal{B}$ with the natural equivalence preserving both foliations. To refer, we will highlight those results.

Theorem 1.4.1 ([8], chapter V). Let M and F be connected manifolds and $\varphi : \pi_1(M) \longrightarrow \text{Diff}(F)$ be a homomorphism. There exists a foliated fiber bundle $\mathcal{B}_\varphi = \{B, p, M, F, \mathfrak{F}(\varphi)\}$ whose the holonomy homomorphism is φ. The structural group of $\mathcal{B}_\varphi$ can be taken as the image of φ with the discrete topology.

Theorem 1.4.2 ([8], chapter V). Let M and F be connected manifolds, $\varphi, \varphi' : \pi_1(M) \longrightarrow \text{Diff}(F)$ be homomorphisms and $f : F \longrightarrow F$ be a diffemorphism satisfying $\varphi(\alpha) = f^{-1}\varphi'(\alpha)f$. There exists a smooth bundle map $h : \mathcal{B}_\varphi \longrightarrow \mathcal{B}_{\varphi'}$ that covers the identity and take the leaves of $\mathfrak{F}(\varphi)$ on the leaves of $\mathfrak{F}(\varphi')$.

Theorem 1.4.3 ([8], chapter V). Let $\mathcal{B} = \{B, p, M, F, \mathcal{F}\}$ be a foliated bundle with holonomy homomorphism $\varphi : \pi_1(M) \longrightarrow \text{Diff}(F)$. There exists a bundle map $\mathcal{B} \longrightarrow \mathcal{B}_\varphi$ that covers the identity and take the leaves of $\mathfrak{F}$ on the leaves of $\mathfrak{F}(\varphi)$.

Remark 1.4.4. Those theorems hold when the fiber has a boundary. Let F any topological space and X a topological space possessing universal covering. Let $\varphi : \pi_1(X) \longrightarrow \text{Homeo}(F)$ be a homomorphism. With this data, we can construct a "foliated bundle" as already described in this section (see [34]).

Remark 1.4.5. In Theorem 1.4.1, the topology of $G = \varphi(\pi_1(M))$ as a subspace of $\text{Diff}(F)$ can be a no discrete topology. However, we can get a coordinate bundle equivalent to $\mathcal{B}_\varphi$ with the system of coordinate transformation being constant (taking values on $\text{Image}(\varphi)$). Thus, we can see $\mathcal{B}_\varphi$ as a fiber bundle with structural group G equipped with the discrete topology. As a subspace of $\text{Diff}(F)$, we can show that the topology of G is totally disconnected. Indeed, we have that $\pi_1(M)$ is a finitely presented group, hence an enumerable set[10]. We claim that every enumerable subspace A of $\text{Diff}(F)$ is totally

[10] Since M is compact, it has a simplicial decomposition with a finite number of cells. It follows by Van Kampen's theorem that the fundamental group of M is finitely generated, hence enumerable.

disconnected. More generally, let X be a topological space that separates points[11] and A be an enumerable subset of X. Given $x \neq y$ elements of A, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ satisfying $f(x) = 0$ and $f(y) = 1$. Since A is an enumerable set, there exists $\alpha \in (0,1)$ with $f(z) \neq \alpha$ for all $z \in A$. Then, $x \in f^{-1}(-\infty, \alpha) \cap A$, $y \in f^{-1}(\alpha, \infty) \cap A$ and, therefore, we may write $A = (f^{-1}(-\infty, \alpha) \cap A) \bigcup (f^{-1}(\alpha, \infty) \cap A)$ (note that this union is disjoint). Since $x, y \in A$ were arbitrarily chosen, it follows that A is totally disconnected. Now, since the space $C^\infty(F, F)$ is metrizable ([33], chapter 2 theorem 4.4), we conclude that $\varphi(\pi_1(M))$ is a totally disconnected subgroup of $\mathrm{Diff}(F)$. This proves the claim. Recall that any subgroup of a topological group is, itself, a topological group (see 1.2.21). Hence, it follows from Theorem 1.4.4 that the structural group $\mathrm{Diff}(F)$ of a foliated bundle can be reduced to a totally disconnected group, namely, the image of the holonomy homomorphism φ. In general, fiber bundles where the structural group is totally disconnected have similar characterizations as the one given above (see [78], §13).

A smooth fiber bundle can admit transversal foliations that are not compatible. That is, there may exist leaves that do not cover the base space. C. Ehresmann showed that in this case, the fiber is necessarily noncompact. More precisely, we have the following theorem.

Theorem 1.4.6 (C. Ehresmann, [17])**.** Let $\mathcal{B} = \{B, p, M, F\}$ be a fiber bundle. If $\mathcal{B}$ has compact fiber and admits a transversal foliation, then $\mathcal{B}$ is a foliated fiber bundle.

Proof's sketch. Let $\mathcal{B} = \{B, p, M, F\}$ be a fiber bundle with compact fiber and $\mathfrak{F}$ be a foliation transversal to the fibers of $\mathcal{B}$. Let L be a leaf of $\mathfrak{F}$. Since L is transversal to the kernel of $p_* : TB \longrightarrow TM$, then the restriction $p : L \longrightarrow M$ is a local diffeomorphism. Now, we affirm that the map p is proper. Let K be a compact subset of M. Since K is compact and $\mathcal{B}$ is a fiber bundle, we can take a cover of K by compact sets $K_1, \ldots, K_l$ such that $p^{-1}(K_i)$ is diffeomorphic to $K_i \times F$. Then, $\bigcup p^{-1}(K_i)$ is compact set containing $p^{-1}(K)$. It follows that $p^{-1}(K)$ is compact, hence p is a proper map. Now, any proper local homeomorphism is a covering map ([35], lemma 3). Therefore $p : L \longrightarrow M$ is a convering map. Now, let $x \in M$ and a U be an open contractible neighborhood of x in M. Define

$$\phi : U \times p^{-1}(x) \longrightarrow p^{-1}(U)$$

as follows. Given $(z, y) \in U \times p^{-1}(U)$, let $\alpha : [0,1] \longrightarrow U$ path conecting x to z and $\tilde{\alpha}$ unique path covering α concerning to the covering map $p : L_y \longrightarrow M$. Define $\psi(z, y) = \tilde{\alpha}(1)$. Since U is contractible, this function is well-defined. Now, fixed $y \in p^{-1}(x)$, the function $\psi^y : U \longrightarrow L_y$ given by $\psi^y(z) = \psi(y, z)$ has inverse given by the restriction of p to $p^{-1}(U) \cap \mathcal{L}_y$, hence a local diffeomorphism. For fixed $z \in U$, we can show that $\psi_z : p^{-1}(x) \longrightarrow p^{-1}(z)$

[11] For example, every metric space (X, d) satisfies this property. Indeed, if $x, y \in X$, $x \neq y$, then the continuous functions $f : X \longrightarrow \mathbb{R}$ given by $f(z) = \frac{d(z,x)}{d(z,x)+d(z,y)}$ separates x from y.

given by $\psi_z(y) = \psi(z, y)$ is a diffeomorphism. This is made proceeding as in the definition of germinal holonomy (see [9, 8]). Thus, ψ is a diffeomorphism. To finish, note that $\mathfrak{F}|_{p^{-1}}(U)$ is the trivial foliation $\{U \times \{y\}_{y\in p^{-1}(x)}\}$. $\square$

Theorem 1.4.7. Let G be a Lie group and M and F be connected manifolds. Given a homomorphism $\varphi : \pi_1(M) \longrightarrow G$, the bundle $\mathcal{B}_\varphi$ is a differentiable principal fiber bundle and the foliation $\mathfrak{F}(\varphi)$ determines a (flat) connection for $\mathcal{B}_\varphi$[12].

Proof's skecth. A local trivialization of $\mathcal{B}_\varphi$ is given as follows. Let $\rho : \tilde{M} \longrightarrow M$ the universal covering of M. Let U be an open set of M evenly covered by ρ. For each path connected component V of $\rho^{-1}(U)$ we have that

$$\phi_V : U \times G \longrightarrow p^{-1}(U)$$

given by $\phi_V(x, g) = [(\rho|_V^{-1}(x), g)]$ is a local trivialization of $\mathcal{B}_\varphi$ (see [8] chapter 5). Let U' be another open set of M evenly covered by ρ and W be a path connected component of $\rho^{-1}(U')$. If $U \cap U' \neq \varnothing$, we can take $\alpha \in \pi_1(M)$ such that $\alpha\rho|_V^{-1}(U \cap U') = \rho|_W^{-1}(U \cap U')$. Thus

$$\phi_V(x, g) = [(\rho|_V^{-1}(x), g)] = [(\alpha\rho|_V^{-1}(x), \varphi(\alpha)^{-1}g)] = [(\rho|_W^{-1}(x), \varphi(\alpha)^{-1}g)] = \phi_W(x, \varphi(\alpha)^{-1}g),$$

concluding that $\phi_{W,x}^{-1}\phi_{V,x}(g) = \varphi(\alpha)^{-1}g$. From this and Theorem 1.2.29 we conclude that $\mathcal{B}_\varphi$ is a differentiable principal fiber bundle. The right action of G on B_φ given in Theorem 1.2.29 has the form

$$[(\tilde{x}, g)]h = [(\tilde{x}, h^{-1}g)].$$

Thus, since the leaves of $\mathfrak{F}(\varphi)$ are given by $\mathcal{L}_g = [\tilde{M} \times \{g\}] = \{[(\tilde{x}, g)]/\tilde{x} \in \tilde{M}\}$, we have $R_h(\mathcal{L}_g) = \mathcal{L}_{L_{h^{-1}(g)}}$, concluding that R_h is an automorphism of $\mathfrak{F}$ for each $h \in G$ (that is, take leaf on leaf). Since $\mathfrak{F}$ is transversal to the fibers, $T\mathcal{F}$ satisfies $T_xP = T_x\mathcal{F}(\varphi) \oplus G_x$. Therefore $T\mathcal{F}$ is a (flat) connection in $\mathcal{B}_\varphi$. $\square$

1.5 Orientability

In this section, we address some notions of orientability.

Definition 1.5.1. A smooth manifold M is said to be orientable provided there exists a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ for M satisfying $\det J(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0$ for all $\alpha, \beta \in \Lambda$.

Remark 1.5.2. Let M be a smooth manifold, and Ω be a top degree differential form defined on M. If Ω does not vanish anywhere in M (that is, $\Omega(x) \neq 0$ for each $x \in M$) then we call Ω a volume form of M and we say that $\Omega > 0$. It is easy to prove that M is orientable if, and only if, M admits a volume form.

[12] Conversely, every flat bundle has the form $\mathcal{B}_\varphi$; see Lemma 1 in [50].

Definition 1.5.3. Let $\mathcal{B} = \{B, p, M, F, G\}$ be a smooth fiber bundle with F be an orientable manifold. The fiber bundle $\mathcal{B}$ is said to be orientable provided there exist local trivializations $\{(\psi_\alpha, U_\alpha)\}$ to $\mathcal{B}$ such that $B = \bigcup \pi^{-1}(U_\alpha)$ and the transitions functions $g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} : U_\alpha \cap U_\beta \longrightarrow G$ are always orientation-preserving.

Remark 1.5.4. Using Definition 1.5.3 we may introduce the idea of positive (correct) and negative orientation on the fibers. Let Ω_F be a volume form on F that orientate F positively. Let $\{(\psi_\alpha, U_\alpha)\}$ as in Definition 1.5.3. For each index α, the differential form $\omega_\alpha = \psi_\alpha^* \Omega_F$ induces an orientation on the fiber that is called positive. Indeed, given $x \in M$ and indexes α, β, we have that $\omega_\beta = g_{\alpha\beta}^* \omega_\alpha$. Since the transition functions $g_{\alpha\beta}$ are orientation-preserving, it follows that ω_α and ω_β must determine the same orientation in the fibers. The definition of negative orientation on the fibers is analogous.

Remark 1.5.5. The usual notion of orientability of a vector bundle coincides with Definition 1.5.3 because the transition functions in vector bundles behave like linear maps on the fibers.

Definition 1.5.6. Let M be a smooth manifold with P a continuous k-plane field on it. P is said to be orientable provided for each $x \in M$ there exists an orientation $\mathcal{O}(x)$ of $P(x)$ that is continuous in the following sense. It is possible obtain an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M such that the restriction $P \mid_{U_\alpha}$ is defined by k-continuous vector fields $X_1^\alpha, \dots, X_k^\alpha$, $\mathcal{O}(x)$ is equal the orientation determined by $\{X_1^\alpha(x), \dots, X_k^\alpha(x)\}$ and for all $\alpha, \beta \in \Lambda$ the orientation determined by $\{X_1^\alpha, \dots, X_k^\alpha\}$ and the one determined by $\{X_1^\beta, \dots, X_k^\beta\}$ coincides in $U_\alpha \cap U_\beta$. A continuous k-plane field is said to be *tranversely orientable* provided there exists a field complementary to P that is continuous and orientable. A C^r $(r \geqslant 1)$ foliation $\mathfrak{F}$ is said to be orientable (transversely orientable) provided $T\mathcal{F}$ is orientable (transversely orientable).

Remark 1.5.7. A C^r k-plane field on an orientable manifold M is orientable if and only if it is transversely orientable. A $C^r(r \geqslant 1)$ foliation is transversely orientable if and only if there exists a foliated atlas $\mathcal{U} = \{U_\alpha, x_\alpha, y_\alpha\}$ for M such that $\det(Jg_{\alpha\beta}) > 0$ for all $g_{\alpha\beta} \in \Gamma_{\mathcal{U}}$ (see [8], page 38).

Theorem 1.5.8. Let P be a C^k $(k \geqslant 1)$ p-plane field defined on a smooth manifold M. Then

(1) P is orientable if and only if there exists a p-form ω defined on M satisfying the following condition. For any $x \in M$ and basis $\{X_1, \dots, X_p\}$ of $P(x)$ we have $\omega(X_1, \dots, X_{n-p}) \neq 0$ (in this case, we say that ω is positive on P);

(2) P is transversely orientable if and only it is given by the kernel of a p-form of class C^{k-1}.

Proof. Suppose that there exists a p-form ω such that for all $x \in M$ and any basis $\{X_1, \ldots, X_p\}$ of $P(x)$ we have $\omega(X_1, \ldots, X_p) \neq 0$. Define an orientation of P as follows. A basis $\{v_1, \ldots, v_p\}$ of $P(x)$ gives a positive orientation of P when $\omega_x(v_1, \ldots, v_p) > 0$. Let $\{X_1, \ldots, X_p\}$ defined on U and $\{Y_1, \cdots, Y_p\}$ defined on V be C^k set of vector fields that determine P locally and are positively oriented. If $Y_i = \sum a_{ij} X_j$ then

$$\omega(Y_1, \cdots, Y_p) = \det(a_{ij}) \omega(X_1, \cdots, X_p).$$

Since both $\omega(Y_1, \cdots, Y_p)$ and $\omega(X_1, \cdots, X_p)$ are positive, then $\det(a_{ij})$ is positive. It follows that $\{Y_i\}$ determines the same orientation of $\{X_i\}$, concluding that P is orientable.

Suppose now that P is orientable. Let Q be a complementary $(n-p)$-plane field to P. Let $\{X_1^\alpha, \ldots, X_p^\alpha\}$ and $\{Y_1^\alpha, \ldots, Y_q^\alpha\}$ sets of C^k vector fields that determine locally P and Q, respectively, in a domain common U_α such that $\{X_1^\alpha, \ldots, X_p^\alpha\}$ is positively oriented. Define on U_α the p-form

$$\omega_\alpha(X_1^\alpha, \ldots, X_p^\alpha) = 1 \text{ and } i_{Y_j^\alpha} \omega_\alpha = 0.$$

Cover M by open sets U_α and take a partition of unity $\{\lambda_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Define $\omega = \sum \lambda_\alpha \omega_\alpha$. It is easy to conclude that $i_X \omega = 0$ for all vector field X tangent to Q (that is, $Q \subset \ker \omega$). We will now proof that ω is positive in any positively oriented basis of P. If $U_\alpha \cap U_\beta \neq \varnothing$, then $(X_1^\beta, \ldots, X_p^\beta) = A_{\alpha\beta}(X_1^\alpha, \ldots, X_p^\alpha)$ for some matriz $A_{\alpha\beta}$ with positive determinant. It follows that

$$\omega_\alpha(X_1^\beta, \ldots, X_p^\beta) = \det(A_{\alpha\beta})$$

and $\omega(X_1^\beta, \ldots, X_p^\beta) = \sum \lambda_\alpha \det A_{\alpha\beta} > 0$. In particular, since $Q \subset \ker \omega$ and ω is nowhere-vanishing form on M, we conclude by dimension argument that $Q = \ker \omega$. Thus, if P is orientable, there exists a p-form of rank p such that ω is positive in P; if Q is transversely orientable, there exists a p-form ω of rank p such that $Q = \ker \omega$.

To finish, let ω be a p-form of rank p and $P(x) = \ker \omega_x$. Let g be a Riemannian metric on M. The q-plane field

$$Q(x) = \{v \in T_x M / g(u, v) = 0 \text{ for all } u \text{ tangent to } P\}$$

is C^k and complementay to P. Given $x \in M$ and basis $\{X_1, \ldots, X_q\}$ of $Q(x)$, we have that $\omega(X_1, \ldots, X_q) \neq 0$, since $\omega_x \neq 0$ for all $x \in M$. It follows that Q is orientable and P transversely orientable. □

Remark 1.5.9. By this theorem, a manifold M admitting a transversely orientable and orientable p-plane field is orientable. Indeed, in this case, given ω a positive form on P and η a form with $P = \ker \eta$, the form $\omega \wedge \eta$ is a nowhere-vanishing top degree form defined on M, hence M is orientable.

Theorem 1.5.10. Let $\mathcal{B} = \{B, p, M, F, G\}$ be a smooth fiber bundle with fiber be an orientable manifold. Consider the following statements:

(1) M is an orientable manifold;

(2) E is an orientable manifold;

(3) $\mathcal{B}$ is orientable.

Then the validity of any two of these statements implies the same for the other.

Proof. First of all, we will obtain an equivalent condition to the orientability of a fiber bundle. Suppose that $\mathcal{B}$ is orientable. Let Ω be an orientation of F and $\{(\psi_\alpha, U_\alpha)\}$ be local trivializations of $\mathcal{B}$ such that $B = \bigcup \pi^{-1}(U_\alpha)$ and the transitions functions

$$g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Diff}(F)$$

are always orientation-preserving. Let $\{\lambda_\alpha\}$ be a partition of the unity subordinate to the cover $\{\pi^{-1}(U_\alpha)\}$. The differential form $\omega = \sum \lambda_\alpha \psi_\alpha^* \Omega$ restricts to a volume form in each fiber. Conversely, suppose that there is a $(\dim F)$-differential form ω restricting to a volume form in each fiber. We will demonstrate that $\mathcal{B}$ is orientable. Let (ψ, U) be local trivialization of $\mathcal{B}$. Since ω is volume in each fiber, there exists a smooth function $f : \pi^{-1}(U) \longrightarrow \mathbb{R}$ such that $\psi^*\Omega_F = f\omega$. Necessarly we have that f is always positive or always negative. Consider $\{(\psi_\alpha, U_\alpha)\}$ the collection of all local trivializations such that the respective function f_α are positive. We have that

$$g_{\alpha\beta}^*\Omega = (\psi_\alpha \circ \psi_\beta^{-1})^*\Omega = (\psi_\beta^{-1})^* f_\alpha \omega = f_\alpha((\psi_\beta^{-1})^*\omega) = (f_\alpha/f_\beta)\Omega.$$

Since $(f_\alpha/f_\beta) > 0$, we conclude that $g_{\alpha\beta}$ is orientation-preserving. We will now prove the theorem.

(1) Suppose that $\mathcal{B}$ and M are orientable. Let Ω_M be an orientation of M and ω be a $(\dim F)$-differential form that restricts to each fiber is a volume form. It is easy to see that the form $\eta = \omega \wedge \pi^*\Omega_M$ define a nowhere-vanishing top degree form on B. It follows that B is an orientable manifold.

(2) Suppose that B and M are orientable manifolds. Let Ω_M and Ω_B orientations for M and B respectively. If there exists a differential form ω such that $\omega \wedge \pi^*\Omega_M = \Omega_B$, the form ω restricts to a volume form in the fibers. Hence $\mathcal{B}$ is orientable. We can obtain a solution for this equation showing the existence of local solutions. Indeed, suppose that ω_α are local solutions defined on U_α with $B = \bigcup U_\alpha$. Let $\{\lambda_\alpha\}$ be a partition of the unity subordinate to the cover $\{U_\alpha\}$ and $\omega = \sum \lambda_\alpha \omega_\alpha$. We have that

$$\omega \wedge \pi^*\Omega_M = (\sum \lambda_\alpha \omega_\alpha) \wedge \Omega_M = \sum \lambda_\alpha(\omega_\alpha \wedge \Omega_M) = \sum \lambda_\alpha \Omega_B = \Omega_B$$

is a global solution. To obtain local solutions, let $x = (x^1, \ldots, x^n)$ be a local coordinate system to M. Complete this set to obtain $(x, y) = (x_1 \circ \pi, \ldots, x_n \circ \pi, y_1, \ldots, y_m)$ a local coordinate system to B. We have that, locally, $\pi^* \Omega_M = f \pi^* dx$ and $\Omega_B = g(\pi^* dx \wedge dy)$ for some never null functions f and g. A local solution is given by $\omega = \pm (g/f) dy$.

(3) Suppose that B and $\mathcal{B}$ are orientable. Let ω be an orientation of $\mathcal{B}$ and Ω_B be an orientation of B. Given local trivialization $\psi : \pi^{-1}(U) \longrightarrow U \times F$ of $\mathcal{B}$ and orientation Ω_U of U, the form $\omega \wedge \psi^* \Omega_U$ is a nowhere-vanishing top degree form on $\pi^{-1}(U)$. It follows that $\omega \wedge \psi^* \Omega_U = f \Omega_B$ for some never null function $f : \pi^{-1}(U) \longrightarrow \mathbb{R}$. Hence the form $\eta_U = (1/f) \Omega_U$ satisfies $\omega \wedge \psi^* \eta_U = \Omega_B$. Noting that $\psi^* \eta_U = \pi^* \eta_U$, we can use a partition of unity to obtain a form η defined on M satisfying $\omega \wedge \pi^* \eta = \Omega_B$. The differential form η give us an orientation of M.

□

Example 1.5.11. Let $M \subset \mathbb{R}^3$ be the usual Möbius strip. We claim that the normal bundle $\mathcal{B}$ of M is non-orientable. Indeed, the total space of this bundle can be viewed as an open subset of $\mathbb{R}^3$. Thus, B is an orientable manifold. Hence, if we assume that $\mathcal{B}$ is orientable, it follows that so is M. But it is well-known that the Möbius strip is non-orientable. This proves the claim.

Theorem 1.5.12. Let Σ be a codimension one embedding orientable submanifold of an orientable manifold M. The normal bundle of Σ is trivial.

Proof. Let g be a Riemannian metric on M. To prove the theorem, it enough to show the existence of a smooth normal vector field $N : \Sigma \longrightarrow TM$. Given $x \in S$, let $\{v_1, \ldots, v_{n-1}\}$ any ortonormal set of vectors tangent to Σ at x and positively oriented. Let $N \in T_x M$ such that $\{N, v_1, \ldots, v_{n-1}\}$ is an ortonormal set and positively oriented concerning to the oriention of M. The association

$$x \in \Sigma \longrightarrow N(x) \in T_x M$$

give us the required smooth nonvanishing normal vector field. Indeed, let $\{w_1, \ldots, w_{n-1}\}$ another ortonormal set tangent to Σ at x and positively oriented. Let $N' \in T_x M$ such that $\{N', w_1, \ldots, w_{n-1}\}$ is an ortonormal set and positively oriented concerning to the oriention of M. We have

$$w_1 \wedge \ldots \wedge w_{n-1} = v_1 \wedge \ldots \wedge v_{n-1}$$

and

$$N' \wedge w_1 \wedge \ldots \wedge w_{n-1} = N \wedge v_1 \wedge \ldots \wedge v_{n-1},$$

concluding that $N = N'$. It follows that the assosiation $x \longrightarrow N(x)$ is well-defined. It is easy to see that N is smooth by using smooth local orthonormal frames. □

Remark 1.5.13. Let $M = \mathbb{R}^n$ and Σ a codimension one embedding submanifold without boundary of M. We have that Σ must be orientable (see [71]). When $M \neq \mathbb{R}^n$, this result is not valid in general. For example, the two-dimensional real projective space $\mathbb{RP}^2$ is non-orientable, but it is a closed embedded submanifold of the three-dimensional real projective space $\mathbb{RP}^3$ that is orientable.

Theorem 1.5.14. Let M be an orientable manifold and $f : \mathbb{S}^1 \longrightarrow M$ be an embedding. The normal bundle of $f(\mathbb{S}^1)$ in M is trivial.

Proof. Since $\pi_1(G_n) = \mathbb{Z}_2$ for $n \geqslant 2$, there exists precisely two n-plane bundle over $\mathbb{S}^1$ (see [51] page 70). On the other hand, it is always possible to construct a non-orientable vector bundle over $\mathbb{S}^1$ (typically a Möbius stripe). Thus, the two possible vector bundles over $\mathbb{S}^1$ are the trivial vector bundle and the non-orientable vector bundle (since equivalent bundles are both orientable or both non-orientable).

Since M is orientable, the total space of the normal bundle of $f(\mathbb{S}^1)$ in M is orientable. Hence, since $\mathbb{S}^1$ is orientable, Theorem 1.5.10 ensures that the normal bundle of $\mathbb{S}^1$ in M is orientable, therefore trivial by the preceding paragraph. $\square$

1.6 De Rham theory of currents

In this section, we present some tools needed to prove some theorems important used in this work. Namely, Sullivan's criterion to a foliation admits a transversal closed form, Schwartzamnn's one for the existence of a global cross-section for flows, and Tischler's argument by an approximation on the current space. The most significant references for this section are [13, 67, 68, 73].

1.6.1 The space of smooth differential forms

Let M be a compact manifold (with boundary or not). Denote by $\mathcal{D}_p^k$ the space of all C^k p-forms defined on M. For $k = \infty$, we symple write $\mathcal{D}_p$[13]. We now define a norm in the space $\mathcal{D}_p^k$. For $J = (i_1, \ldots, i_l)$ $0 \leqslant l \leqslant k$, or $J = \varnothing$, denote ∂_J the operator $\partial_J = \frac{\partial}{\partial x_{i_1} \ldots \partial x_{i_l}}$, where for $J = \varnothing$ we set $\partial_J f = f$. The set of all possible such J's with $0 \leqslant l \leqslant k$ have the description $\{J/\#J \leqslant k\}$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a finite coordinate atlas for M. Let $\omega \in \mathcal{D}_p^k$. For each $\alpha \in I$ we know that ω has unique local representation

$$\omega|_{U_\alpha} = \sum_J a_J^\alpha dx_J^\alpha,$$

[13] This notation for arbitrary manifold was introduced by De Rham to denote the space of all compactly supported C^∞-differential forms defined on a manifold M.

where $a_J^\alpha : U_\alpha \longrightarrow \mathbb{R}$ are C^k-functions. The function $\rho_k : \mathcal{D}_p^k \longrightarrow \mathbb{R}$ defined by

$$\rho_k(\omega) = \sum_{\alpha \in I} \sup_{x \in U_\alpha} \sum_{\#J \leqslant k} |\partial_J a_J^\alpha(x)|$$

is a norm in $\mathcal{D}_p^k$. Denote by τ_k the topology on $\mathcal{D}_p^k$ genereted by this norm. Clearly $\tau_k \subset \tau_{k+1}$ for each $k \in \mathbb{N}$. We may equip $\mathcal{D}_p$ with the topology τ genereted by $\bigcup_{k \geqslant 0} \tau_k$. In what follows, we will highlight some properties of such spaces.

Theorem 1.6.1. The space $\mathcal{D}_p$ with the topology τ become a convex and Hausdorff topological vector space.

This theorem is a particular case of a more basic general result that states the topology of a convex space[14] is described by a collection of seminorms (see [68], theorem 1.37).

Theorem 1.6.2. The space $\mathcal{D}_p$ is a Fréchet space[15].

Proof's sketch. The topology of $\mathcal{D}_p$ has a countable basis since it is generated by an enumerable collection of norms ([68], theorem 1.37). Any topological vector space with an enumerable basis of open sets admits a translation-invariant metric generating its topology ([68], theorem 1.24). Hence $\mathcal{D}_p$ is metrizable. The completeness follows by a well-known theorem concerning the term-by-term differentiation of a sequence of functions. □

Corollary 1.6.3. The differential operator $d : \mathcal{D}_p \longrightarrow \mathcal{D}_{p+1}$ is continuous.

Proof. Since $\mathcal{D}_p$ is a metrizable topological vector space, it is enough to show that if $\omega_m \longrightarrow 0$ in $\mathcal{D}_p$ then $d\omega_m \longrightarrow 0$ in $\mathcal{D}_{p+1}$. In general, if $\omega \in \mathcal{D}_p$ have a local representation $\omega_U = \sum_J a_J dx_J$ concerning to a coordinate neighborhood (U, φ), $\varphi = (x_1, \ldots, x_n)$, then the local representation for $d\omega$ in U is

$$d\omega|_U = \sum_{i=1}^{n} \frac{\partial a_J}{\partial x_i} dx_i \wedge dx_J.$$

From this, it is clear that $\omega_m \longrightarrow 0$ implies $d\omega_m \longrightarrow 0$. □

Definition 1.6.4. Let E be a topological vector space with a topology τ. A subset B of E is said to be τ-*bounded* provided given a τ-neighborhood U of the origin, there exists $t > 0$ such that $B \subset tU$ [16].

[14] A topological vector space E is said to be a *convex space* if there exists a basis of convex neighborhoods of the origin. Some authors use the terminology *locally convex space* instead of convex space.

[15] By a *Fréchet space* we mean a convex, complete, and metrizable topological vector space.

[16] This notion is equivalent to the following condition: a subset B of a topological vector space E is bounded provided given any sequence (x_n) in E and any sequence of real numbes (λ_n) converging to zero, the sequence $(\lambda_n x_n)$ converges to zero [38].

Remark 1.6.5. Let E be a convex Hausdorff topological vector space with a topology given by a family of seminorms $\mathcal{P}$. A subset B of E is bounded if and only if $p(B)$ is bounded for each $p \in \mathcal{P}$ (see [68], theorem 1.37). It follows that a subset B of $\mathcal{D}_p$ is bounded if and only if $p_k(B)$ is bounded for all k.

Theorem 1.6.6. The spaces $\mathcal{D}_p$ satisfy the Heine-Borel property[17].

Proof. Let B be a bounded closed subset of $\mathcal{D}_p$. Since $\mathcal{D}_p$ is metrizable, to show that B is compact, it is enough to show that every sequence (ω_n) contained in B has a convergent subsequence. Let (ω_n) be a sequence of elements in B. Since B is bounded, there exists a sequence of positive real numbers M_k such that $p_k(\omega) \leqslant M_k$ for all $\omega \in B$. Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^{l}$ be a finite atlas for M and $\{\lambda_\alpha\}$ be a partition of the unity subordinate to the cover $\mathcal{U}$. Suppose we have shown that given α and sequence (ω_n) contained in B, then there exists a subsequence (ω_{n_k}) such that the restriction $\lambda_\alpha\omega_{n_k}$ converges to some $\omega_\alpha \in \mathcal{D}_p$. From this, we can obtain a common subsequence (ω_{n_k}) such that $(\lambda_\alpha\omega_{n_k})$ converges to ω_α for all $\alpha = 1, \ldots, l$. It follows that

$$\omega_{n_k} = \sum \lambda_\alpha\omega_{n_k} \longrightarrow \sum \omega_\alpha,$$

concluding that B is compact. We will now show that given a sequence (ω_n) containing in B, there exists a subsequence of (ω_{n_k}) and $\omega \in \mathcal{D}_p$ with $\lambda_\alpha\omega_{n_k} \longrightarrow \omega$. Since $\omega_n = \sum \lambda_\alpha\omega_n$, we have $p_k(\lambda_\alpha\omega_n) \leqslant M_k$ for all n and α. Writing $\lambda_\alpha\omega_n = \sum a^\alpha_{J,n} dx^\alpha_J$, then $\mid \partial_J a^\alpha_{n,J} \mid \leqslant M_k$ when $\#J \leqslant k$. This implies that the collection of functions $\partial_J a^\alpha_{n,J}$ is equicontinuous in the compact set $\text{supp}\lambda_\alpha$. It now follows by the Ascoli's theorem and Cantor diagonal process that there exists a subsequence $(\lambda_\alpha\omega_{n_k})$ such that $\partial_J a^\alpha_{n_k,J}$ converge uniformly in $\text{supp}\lambda_\alpha$. It follows that $(\lambda_\alpha\omega_{n_k})$ converge in $\mathcal{D}_p$. □

Remark 1.6.7. It is easily seen a normed vector space possesses a bounded neighborhood of origin. The converse is also true: a topological vector space with a bounded neighborhood of the origin is a normed space (see [38]). We also know that a normed space admits a compact neighborhood of the origin, if and only if it is finite-dimensional. It then follows from Theorem 1.6.6 that $\mathcal{D}_p$ does not admit a norm generating its topology.

Definition 1.6.8. Let E be a topological vector space and A and B subsets of E. A is said *absorb* B, if there exists $t > 0$ such that $B \subset tA$. The set A is said to be *balanced* if $\alpha C \subset C$ for all $\mid \alpha \mid \leqslant 1$. The set A is said to be *absolutely convex* if it is convex and balanced. The topological space E is said to be *bornological space* provided any absolutely convex set that absorbs any bounded set is a neighborhood of the origin[18].

[17] A topological vector space E is said to satisfy the *Heine-Borel property* provided every closed bounded subset of E is compact.

[18] Bornological spaces were studied first by G.W. Mackey. The name *bornological* coined by Bourbaki is a reference to the French word *borné* for bounded.

Remark 1.6.9. The spaces $\mathcal{D}_p$ are bornological since they are Fréchet. In general, a convex Hausdorff topological space E is a bornological space if, and only if satisfies the following condition. Every linear transformation T from E into a convex topological space that is bounded is also continuous. Some authors use this equivalent condition to define bornological spaces (for example, [5] section 4 and [67] page 81).

1.6.2 The space of currents

In [13], G. De Rham defined the notion of *current*[19] as a linear functional T in the vector space $\mathcal{E}_p$ of all C^∞-forms with compact support that are continuous in the following sense: if (ω_h) is a sequence of differential forms with support contained in a single local coordinate system such that each derivative of the coefficients of ω_h in this local representation tends uniformly to zero, then $T(\omega_h) \longrightarrow 0$. This notion of convergence is one we call *weak convergence* on the topological dual of E (the topology on E given implicitly by the notion of convergence in it). The notion of continuity of currents presented by G. De Rham follows as in L. Schwartz [73], chapter III.

The topology on $\mathcal{D}_p$ did is not given explicitly in De Rham's book. However, G. De Rham makes the following observation: let E be a vector space where the notion of a *bounded* set is defined. Then, one can equip E with a topology τ that makes E a topological vector space. This topology is defined as one with a basis of the neighborhood of the origin given by a collection $\mathcal{C}$, generated by all sets that *absorbs* any bounded set. On the other hand, L. Schwartz defined explicitly the topology on $\mathcal{E}' = \mathcal{E}_0'$ in terms of the bounded subsets of $\mathcal{E}$ and showed the reflexivity of such spaces. In what follows, we describe the topology for such spaces explicitly and show Schwartz's reflexivity theorem. For this purpose, we present some results in topological vector spaces' theory. For simplicity, we restrict attention to compact manifolds, where $\mathcal{E}_p$ and $\mathcal{D}_p$ coincides.

Definition 1.6.10. Let E and F be vector spaces. We call (E, F) a *dual pair* if there exists a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : E \times F \longrightarrow \mathbb{R}$.

The interesting case is when F is the set of all continuous linear functionals defined on E'. The fact of (E, E') be a dual pair follows by Hahn-Banach's theorem for convex spaces (see [68]) chapter 3). In particular, (E, E^*) is a dual pair, where E^* is the algebraic dual of E.

Let (E, F) be a dual pair. Each $y \in F$ can be viewed as a function $y : E \longrightarrow \mathbb{R}$ given by $y(x) = \langle x, y \rangle$. The smallest (coarsest) topology in E so that every function y, $y \in F$, is continuous is called *weakly topology* and denoted by $\sigma(E, F)$. Analogously one can define the weakly topology $\sigma(F, E)$ on F.

[19] The choice of the term "current" is motivated by the fact that in ordinary 3-space, "1- dimensional currents" can be interpreted as electrical currents (De Rham).

Definition 1.6.11. Let (E, F) be a dual pair. The *polar* of a subset A of E is the set

$$A^0 = \{y \in F / \mid \langle x, y\rangle \mid \leqslant 1\}.$$

The polar of a subset B of F is defined analagously.

Next, we will highlight some properties of taking dual of a subset. One can find proof of these facts in many references, for example [5], [43] and [67].

Theorem 1.6.12. Let (E, F) be a dual pair. Then

(1) $A \subset B$ implies $B^0 \subset A^0$;

(2) $(\bigcup A_\alpha)^0 = \bigcap A_\alpha^0$;

(3) $A \subset A^{00}$;

(4) $(\lambda A)^0 = \frac{1}{\lambda} A^0$;

(5) A^0 is absolutely convex and $\sigma(F, E)$-closed;

(6) The polar of a subset A absorbs every one-point set if and only if A is $\sigma(E, F)$-bounded;

(7) [Bipolar theorem] If A is an absolutly convex subset of E, then $A^{00} = \overline{A}^{\sigma(E,F)}$[20].

Let (E, F) be a dual pair and A a $\sigma(E, F)$-bounded subset of E. Since each $y \in F$ gives a continuous linear functional on E, we have that $\langle A, y\rangle$ is a bounded subset of $\mathbb{R}$. It follows that the rule

$$p_A(y) = \sup_{x \in A}\{\mid \langle x, y\rangle \mid\}$$

define a seminorm on F (see 1.6.4 and 1.6.5).

Definition 1.6.13. Let (E, F) be a dual pair and $\mathcal{A}$ a collection of $\sigma(E, F)$-bounded subsets of E. The collection $\mathcal{A}$ is said to be *saturated* provided

(1) Given $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ with $A \cup B \subset C$;

(2) $\lambda A \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $\lambda > 0$;

(3) $\bigcup \mathcal{A} = E$.

[20] Given X topological space and $Y \subset X$, the symbol $\overline{Y}$, as usual, denotes the closure of Y in X.

Let $\mathcal{A}$ be a saturated collection of $\sigma(E,F)$-bounded subsets of E. The family of seminorms $\{p_A/A \in \mathcal{A}\}$ defines a convex topology on F called *topology of uniform convergence on the sets of* $\mathcal{A}$ or *topoloy of* $\mathcal{A}$*-convergence*. Such topology has as a basis of the neighborhoods of the origin the collection $\mathcal{B}_{\mathcal{A}} = \{A^0/A \in \mathcal{A}\}$. By considering different saturated collections $\mathcal{A}$ we obtain different topologies on F . The smallest (coarsest) one is the *weak topology* $\sigma(E,F)$ and it is precisely the topology of uniform convergence on all finite subsets of E. The largest (finest) one is obtained by considering the collection of all $\sigma(E,F)$-bounded subsets of E. This topology is denoted by $\beta(E.F)$ and is called *strong topology*.

Let (E,F) be a dual pair. Then, the topological dual of E with the weak topology if equal to F , that is, $(E,\sigma(E,F))' = F$. A fundamental problem in the theory of convex topological vector spaces is to characterize the topologies τ in E for which $(E,\tau)' = F$. If τ is a such topology, we say that τ is a *topology for the dual pair* (E,F). The Mackey-Arens Theorem characterizes all topologies of a dual pair (E,F).

Theorem 1.6.14 (Mackey-Arens)**.** Let (E,F) be a dual pair with E convex and Hausdorff. Then the topological dual of E is F if and only if the topology on E is the topology of uniform convergence on a collection of absolutely convex $\sigma(F,E)$-compact subsets of F.

By this theorem, there exists the largest topology of a dual pair (E,F). It is sometimes called *Mackey topology* and denoted by $\tau(E,F)$. A space with the Mackey topology is called *Mackey space*. To show this theorem, we need to state some results.

Theorem 1.6.15 (Banach-Alaoglu-Bourbaki[21])**.** Let E be a convex Hausdorff topological vector space. Then $U^0 \subset E'$ is $\sigma(E',E)$-compact for each neighborhood U of the origin in E.

Lemma 1.6.16. Let (E,F) be a dual pair. Let A a convex subset of E. The closure of A is the same for every topology of the dual pair (E,F).

Proof. Let τ a topology of the dual pair (E,F). Since $\sigma(E,F) \subset \tau$, we have the inclusion $\overline{A}^{\tau} \subset \overline{A}^{\sigma(E,F)}$. Now, let $x \in E - \overline{A}^{\sigma(E,F)}$. By Hahn-Banach theorem, there exists a continuous linear functional f defined on E such that $f(x) \notin \overline{f(A)}$. Since $(E,\tau)' = F$, we have that $f = \langle \cdot, y\rangle$ for some $y \in Y$. It follows that there exists $\epsilon > 0$ such that $|\langle x-a, y\rangle| \geqslant \epsilon$ for all $a \in A$. Let $U = \{z \in E/ \mid \langle z, y\rangle \mid < \epsilon\}$. Then, $x + U$ is a $\sigma(E,F)$-neighborhood of the x not meeting A. Thus, $x \in E - \overline{A}^{\sigma(E,F)}$ concluding that $E - \overline{A}^{\tau} \subset E - \overline{A}^{\sigma(E,F)}$. Hence $\overline{A}^{\sigma(E,F)} \subset \overline{A}^{\tau}$, concluding the proof. □

[21] A proof of this theorem for separable normed vector spaces was published in 1932 by Stefan Banach. The first proof for the general case was published in 1940 by the mathematician Leonidas Alaoglu. Some authors attribute the generalization to convex space to the mathematician's group N. Bourbaki; for example, [43] and [49].

Definition 1.6.17. Let E and F topological vector spaces. A collection $\mathcal{L}$ of linear maps of E to F is said to be *equicontinuous* provided given any neighborhood V of the origin in F, there exists a neighborhood U of the origin in E, depending on V, such that $T(U) \subset V$ for all $T \in \mathcal{L}$.

Lemma 1.6.18. Let E be a convex Hausdorff topological vector space with topological dual $E^{'}$. Let $\mathcal{A}^{'}$ the collection of all equicontinuous subsets of $E^{'}$. The topology of E is the topology of $\mathcal{A}^{'}$-convergence.

Proof. Let $\Phi \in \mathcal{A}^{'}$ and U be a neighborhood of the origin in E. Since Φ is an equicontinuous set of linear functionals on E, there exists $t > 0$ such that $\Phi(U) \subset (-t, t)$. Thus, $\Phi \subset (\frac{1}{t}U)^0$. It follows that $\frac{1}{t}U \subset (\frac{1}{t}U)^{00} \subset \Phi^0$ concluding that Φ^0 is a neighborhood of the origin in E.

Now, let V be an absolutely convex and closed neighborhood of the origin in E. By Lemma 1.6.16, V is weakly closed. By Bipolar Theorem 1.6.12, $V^{00} = V \subset U$. Since $V^0 \in \mathcal{A}^{'}$ (it follows because $V^0(V) \subset [-1, 1]$), we have that U is a neighborhood of the origin concerning the topology of uniform convergence on the sets of $\mathcal{A}^{'}$. □

Lemma 1.6.19. Let (E, F) be a dual pair and $\mathcal{A}$ a collection of saturated $\sigma(E, F)$-bounded subsets of E. Then the topological dual of F under the topology of $\mathcal{A}$-convergence is $\bigcup_{A \in \mathcal{A}} A^{00}$, the bipolars being taken in F^*.

Proof. Let E be a convex Hausdorff topological vector space. Firstly, we will show that the topological dual of E is $\bigcup_{U \in \mathcal{B}} U^0$ (the polar being taken in E^*), where $\mathcal{B}$ is any basis for the topology on E. Let $x^{'} \in E^{'}$. By the continuity of $x^{'}$, there exists $U \in \mathcal{B}$ such that $x^{'}(U) \subset [-1, 1]$. It follows that $x^{'} \in U^0$. Conversely, let $U \in \mathcal{B}$ and $x^* \in U^0$. We will show that x^* is continuous. Let V be a symmetric neighborhood of the origin such that $V + y \subset U$ for some $y \in U$. Given $x \in V$, since $-x \in V$ and $\sup_{x \in U} \mid x^*(x) \mid \leqslant 1$, we have

$$\mid x^*(x) \mid = \mid x^*(y) - x^*(y - x) \mid \leqslant \mid x^*(y) \mid + \mid x^*(y - x) \mid = 2.$$

Thus, given $\epsilon > 0$ then $x^*(\epsilon/2V) \subset (-\epsilon, \epsilon)$ concluding that $x^* \in E^{'}$.

Now, a basis for the topology of $\mathcal{A}$-convergence on F is given by $\{A^0 / A \in \mathcal{A}\}$. Thus, the dual of F for the topology of $\mathcal{A}$-convergence is given by $\bigcup_{A \in \mathcal{A}} A^{00}$, the bipolars being taken in F^*. □

Proof of Mackey-Arens Theorem. Suppose that E has F as topological dual. Then, by 1.6.18 the topology of E is the topology of uniform convergence concerning the collection of all equicontinuous subsets of F. In particular, given a neighborhood U of the origin in E, since the set U^0 is equicontinuous, then $U^{00} \subset E$ is a neighborhood of the origin. On the other hand, Banach-Alaoglu-Boubaki's theorem ensures that U^0 is $\sigma(F, E)$-compact.

It follows that the topology of E is a topology of uniform convergence on a collection of absolutely convex $\sigma(F, E)$-compact subsets of F.

Conversely, suppose the topology of E is a topology of uniform convergence on a collection $\mathcal{A}$ of absolutely convex $\sigma(F, E)$-compact subsets of F. Since (F, E) is a dual pair, Lemma 1.6.19 ensures that the dual of E is $\bigcup_{A\in\mathcal{A}} A^{00}$, the bipolars being taken in E^*. Now, compact sets are closed in a Hausdorff space. Hence any $A \in \mathcal{A}$ is an absolutely convex and closed set. We can apply Bipolar Theorem 1.6.12 to conclude that the $A^{00} = A$. It follows that the dual of E is F. □

Definition 1.6.20. Let E be a convex Hausdorff topological vector space. The topological *bidual* of E is the topological dual of $E^{'}$ concerning the strong topology. It is denoted by $E^{''}$. In symbols, $E^{''} = (E^{'}, \beta(E^{'}, E))^{'}$.

For each $x \in E$ we have a well-defined element $x \in E^{''}$ given by $x(x^{'}) = x^{'}(x)$. Indeed, note that

$$x^{-1}(-\epsilon, \epsilon) = \{x^{'} \in E^{'} / \mid x^{'}(x) \mid < \epsilon\}$$

is a neighborhood of the origin concerning the $\sigma(E^{'}, E)$-topology. In particular, a neighborhood of the origin concerning the strong topology $\beta(E^{'}, E)$. Therefore $x : (E^{'}, \beta(E^{'}, E)) \longrightarrow \mathbb{R}$ is continuous linear functional. Thus, we can identify E with a vectorial subspace of $E^{''}$.

Definition 1.6.21. Let E be a convex Hausdorff topological vector space.

(1) E is said to be *semi-reflexive* provided the inclusion $E \hookrightarrow E^{''}$ is surjective; that is, if every continuous linear functional on $E^{'}$ has the form J_x for some $x \in E$.

(2) E is said to be *reflexive* provided the inclusion $E \hookrightarrow E^{''}$ is a topological isomorphism.

Consider the dual pair $(E^{'}, E)$. Note that the following inclusions are always true

$$\sigma(E^{'}, E) \subset \xi \subset \tau(E^{'}, E) \subset \beta(E^{'}, E).$$

Suppose that E is semi-reflexive. Then we can identifie E with $E^{''}$ (identification as set). Since the topology already defined on $E^{'}$ is the stronger one and, on the other hand, $(E^{'})^{'} = E$, then the topology of $E^{'}$ is contained in the Mackey topology. It follows that $\tau(E^{'}, E) = \beta(E^{'}, E)$. Conversely, if $\beta(E^{'}, E) = \tau(E^{'}, E)$, then $(E^{'})^{'} = E$ by Mackey-Arens' theorem. In short, we have:

Theorem 1.6.22. Let E be a convex Hausdorff topological vector space. Then E is semi-reflexive if and only if the Mackey topology on $E^{'}$ coincides with the stronger topology.

Below is given a more workable characterization of semi-reflexive spaces.

Theorem 1.6.23. A convex Hausdorff topological vector space E is semi-reflexive if and only if the weakly closure of any bounded subset of E is weakly compact.

Proof. Suppose that E is semi-reflexive. Let A be a bounded subset of E. Then, A^0 is a neighborhood of the origin for $\beta(E', E)$. By 1.6.22, $\beta(E', E) = \tau(E', E)$. Since $\tau(E', E)$ is the topology of uniform convergence on all absolutely convex $\sigma(E, E')$-compact subsets of E, there exists an absolutely convex $\sigma(E', E)$-compact subset B of E with $B^0 \subset A^0$. It follows that $A \subset A^{00} \subset B^{00} = B$ concluding that the closure of A is $\sigma(E, E')$-compact.

Conversely, suppose that the closure of all bounded subset of E is weakly compact. Let A a bounded subset of E. Since A is bounded, the smalest absolutely convex set containing A is bounded. Did is not difficult to see that this set is characterized by

$$\Gamma(A) = \{x \in E / x = \sum a_i x_i, a_i \in A, \sum \mid a_i \mid \leqslant 1\}.$$

Since the weakly closure $\overline{\Gamma(A)}$ of $\Gamma(A)$ is weakly compact, we have that $(\overline{\Gamma(A)})^0 \in \tau(E', E)$. On the other hand, $A^0 \in \beta(E', E)$ and $(\overline{\Gamma(A)})^0 \subset A^0$. It follows that $\beta(E', E) \subset \tau(E', E)$ and, consequently, $\beta(E', E) = \tau(E', E)$, concluding that E is semi-reflexive. □

Definition 1.6.24. A *barrel (tonneau)* in a topological vector E space is absolutely convex closed subset of E that absorb every one-point set. A convex topological vector space is said to be *barelled (tonnelé)* provided every barrel is a neighborhood of the origin.

Banach-Steinhaus's theorem is one of the most powerful tools in functional analysis. A version of this theorem holds in barreled spaces, and it is deeply associated with the notion of reflexivity, as we will see now.

Theorem 1.6.25. Let E be a convex Hausdorff topological vector space with topological dual E'. Then E is barreled if and only if every $\sigma(E', E)$-bounded subset of E' is equicontinuous.

Proof. Firstly, given Φ be a subset of E', if the polar Φ^0 in E is a neighborhood, then Φ is equicontinuous. Of course, in this case, we have $\Phi(\Phi^0) \subset [-1, 1]$.

Suppose now that E is barreled. Let Φ be a $\sigma(E', E)$-bounded subset of E'. Then, Φ^0 absorb any one-point set (1.6.12 iten 6). It follows that Φ^0 is a barrel in E. Since E is barreled, then Φ^0 is a neighborhood of the origin in E. Hence Φ is equicontinuous.

Conversely, suppose that every $\sigma(E', E)$-bounded subset of E' is equicontinuous. Let B a barrel in E. Since B is (absolutely) convex and closed, it is $\sigma(E, E')$-closed by 1.6.16. It follows from Bipolar Theorem 1.6.12 that that $B^{00} = B$. Again by 1.6.12, since $B^{00} = B$ absorbs every one-point set, then B^0 is $\sigma(E', E)$-bounded, hence equicontinuous by hypothesis. Thus, there exists a neighborhood U of the origin in E with $B^0(U) \subset [-1, 1]$. It follows that $U \subset U^{00} \subset B^{00} = B$, concluding that B is a neighborhood of the origin. □

Lemma 1.6.26. In a convex space E, a barrel absorbs every convex compact set.

Proof. Let B be a barrel in E and K a compact convex subset of E. We will show the existence of a positive integer n, an open set U, and a point $x \in K$ with

$$K \cap (x + U) \subset nB.$$

Firstly, note that compact sets are bounded (it is easy to show) and that any convex set A containing the origin satisfies $A \subset \lambda A$ for all $\geqslant 1$. Thus, there exists $\lambda \geqslant 1$ such that $K - x \subset \lambda U$. Hence

$$K - x \subset \lambda(K - x) \cap \lambda U \subset \lambda(nB - x).$$

Since B is absorbing, there exists $\mu \geqslant 1$ such that $\lambda nB \subset \mu B$. It follows that

$$K \subset \lambda nB - (\lambda - 1)x \subset \mu B.$$

Suppose then that no n, U, x can be found to satisty the condition $K \cap (x + U) \subset nB$. Then, taking $n = 1$, any $x_0 \in K$ and any open set U_0, there is some

$$x_1 \in K \cap (x_0 + U_0) \cap (E - B).$$

Since $(x_0 + U_0) \cap (E - B)$ is an open set, there exists some open set U_1 with $x_1 + \overline{U_1} \subset (x_0 + U_0) \cap (E - B)$. Take $n = 2, x = x_1, U = U_1$ and repeat the process. So, we obtain a decreasing sequence $(K \cap (x_n + \overline{U_n}))$ of closed non-empty sets contained in K. Since K is compact, they have a common point $y \in K$. For each n, $y \notin nB$, and so B is not absorbent, wich is a contradiction. □

Theorem 1.6.27 (Mackey [46], theorem 7)**.** The same sets are bounded in every topology of a dual pair.

Proof. Let ξ be any topology of the dual pair (E, E'). Then the ξ-bounded sets are certainly $\sigma(E, E')$-bounded. Conversely, let A be a $\sigma(E, E')$-bounded set and U be a closed absolutely convex ξ-neighborhood. By 1.6.12 item 6, A^0 absorbs any one-point set. Since A is absolutely convex and closed, it follows that A^0 is a barrel in E' (under $\sigma(E, E')$). Since U^0 is convex and $\sigma(E', E)$-compact by 1.6.15, then A^0 absorbs U^0 by 1.6.26. Therefore U^{00} absorbs A^{00}. But $U^{00} = U$ by the bipolar theorem. It follows that $A \subset A^{00}$, hence U absorbs A. Thus A is ξ-bounded. □

Theorem 1.6.28. Let E be a convex Hausdorff topological vector space. Then E is reflexive if and only if is semi-reflexive and barreled.

Proof. Each equicontinuous subset of E' is bounded in E'. On the other hand, the collection of all equicontinuous subsets of E' is saturated. Hence we can consider on E'' the topology of uniform convergence on all equicontinuous subsets of E'. Denote this topology by $\epsilon(E'', E')$.

Suppose that E is barreled. Let Φ be a bounded subset of E. Then, Φ is $\sigma(E', E)$-bounded, since the same sets are bounded in every topology of a dual pair 1.6.27. Since E is barreled, we have that Φ is equicontinuous subset of E'. It follows that a bounded subset of a barreled space is equicontinuous. Thus, $\beta(E'', E') = \epsilon(E'', E')$.

As observed in 1.6.18, E has the topology of uniform convergence on the collection of all equicontinuous subsets of E'. Thus, given an absolutely convex closed neighborhood U of the origin in E we have that U^{00}, the polar taken in E'', is a neighborhood of the origin to $\epsilon(E'', E') = \beta(E'', E')$. Then, if E is barreled and semi-reflexive, we have that $U = U^{00}$ is a neighborhood of the origin for the topology of E and for the topology of E''. It follows that a barreled and semi-reflexive space is reflexive.

Conversely, suppose that E is reflexive. Then E is trivially semi-reflexive. So remains to show that E is barreled. Let B be a barrel in E. Since B absorbs every one-point set and $B^{00} = B$, we have by 1.6.12 that B^0 is $\sigma(E', E)$-bounded. Since E is reflexive, the topology of E is the topology of uniform convergence on the collection of all $\sigma(E', E)$-bounded subsets of E'. Consequently, $B^{00} = B$ is a neighborhood of the origin in E, concluding that E is barreled.

□

Theorem 1.6.29. Every convex topological vector space E, which is a Baire space[22], is barreled.

Proof's sketch. Let B be a barrel in E. Given $x \in E$, there is $t > 0$ such that $x \in tB$. Since B is totally convex given $s > t$ we have that $x \in sB$. In particular, $x \in nB$ for some integer $n \geqslant t$. Thus $E = \bigcup_{n \geqslant 1} nB$. Since E is a Baire space, there exists $n \in \mathbb{N}$ such that $\mathrm{int}(nB) \neq \varnothing$. It follows that $\mathrm{int}(B) \neq \varnothing$. Let $y \in \mathrm{int}(B)$. A general fact about convex sets in a topological vector space is that it contains the interior of lines; that is, if B is convex and $x \in \mathrm{int}(B)$, $y \in \overline{B}$, then the open line segment joining x to y is interior to B. From this, we have

$$0 = \frac{1}{2}y - \frac{1}{2}y \in \mathrm{int}(B).$$

Then every barrel is a neighborhood of the origin concluding that E is barreled. □

Definition 1.6.30 (Currents)**.** A continuous linear functional $\varphi : \mathcal{D}_p \longrightarrow \mathbb{R}$ will be called a *p-currents* on M. The space $\mathcal{D}'_p$ with the strong topology $\beta(E', E)$ will be called *space of p-currents* on M.

The strength of the current theory resides in the following beautiful theorem, due to L. Schwartz [23].

[22] A topological space X is said to be Baire if the intersection of any countable collection of dense open subsets of X is dense in X. See [68] chapter 2.

[23] Schwartz established this property for $p = 0$ and $M = \mathbb{R}^n$. See also [13], page 89.

Theorem 1.6.31 (Schwartz, [73])**.** The space $\mathcal{D}_p$ is reflexive.

Proof. By 1.6.6 the space $\mathcal{D}_p$ satisfies the Heine-Borel property. Therefore the closure of each bounded subset of $\mathcal{D}_p$ is compact. Since compact subsets are certainly weakly compact and the same sets are closed in any topology of a dual pair, we conclude from 1.6.23 that $\mathcal{D}_p$ is semi-reflexive. On the other hand, $\mathcal{D}_p$ being metrizable is a Baire Space by the Baire Theorem. Therefore, by 1.6.29, $\mathcal{D}_p$ is barrelled. So, since $\mathcal{D}_p$ is semi-reflexive and barreled, it is reflexive by 1.6.28. □

Remark 1.6.32. A barreled Hausdorff topological vector space E satisfying the Heine-Borel property is called a Montel space. Historically, this nomenclature honors the French mathematician Paul Antoine Aristide Montel (29 April 1876 – 22 January 1975), who showed that the space of holomorphic functions has this property. It follows from Theorem 1.6.31 that any Montel space is reflexive. In the following, we will highlight a property important in these spaces.

Theorem 1.6.33. The dual of a Montel space is Montel.

Proof. Let E be a Montel space. So, just like in Theorem 1.6.31, one can show that E is reflexive. It follows that $E^{'}$ is reflexive, hence barreled by 1.6.28.

We will now demonstrate that $E^{'}$ satisfies the Heine-Borel property. First of all, since $E^{'}$ is barreled, it follows from 1.6.25 and 1.6.27 that every bounded subset of $E^{'}$ is equicontinuous. On the other hand, since the bounded subsets of E are relatively compact and the topology on $E^{'}$ is one of uniform convergence on the collection of all bounded subsets of E, we conclude that the topology on $E^{'}$ is one of compact convergence[24].

Let X be an equicontinuous subset of $E^{'}$. We claim that the topology of compact convergence and the weakly topology coincides on X. We have already said that the topology of the $\mathcal{A}$-convergence is the same as the generated by the family of seminorms $\{p_A/A \in \mathcal{A}\}$. This topology as a subbasis for it the collection of all sets

$$V(p_A, x, n) = \{y/p_A(x-y) < 1/n\}, A \in \mathcal{A}, x \in E, n \in \mathbb{N}.$$

Let then C be a compact subset of E, $f_0 \in X$ and $n_0 \in \mathbb{N}$. The set $V(p_C, f_0, 1/n_0) \cap X$ is a subbasis open set for the topology of compact convergence on X. Since X is equicontinuous, there is a neighborhood U of origin with $\sup_{f \in X, x \in U} |(f_0 - f)x| < 1/n_0$. Since C is compact, there are finitely many $x_i \in C$ with $C \subset \bigcup_{i=1}^{k}(x_i + U)$. Each $y \in C$ therefore takes the form $y = x_i + x$, $x \in U$. Then, given $f \in X$ with $f \in \bigcap_{i=1}^{k} V(p_{x_i}, f_0, 1/2n_0)$ (a weakly neighborhood of f_0), we have

$$p_C(f - f_0) = \sup_{y \in C} |(f - f_0)y| \leqslant \sup_{i=1,\dots k} |(f - f_0)x_i| + \sup_{x \in U} |(f - f_0)x| < 1/n_0.$$

[24] This topology is one of uniform convergence on the collection of all compact subsets of E.

It follows that a subbasis open set for the topology of compact convergence on X is an open set for the weakly topology on X. Since the weakly topology is contained in the topology of compact convergence, this proves the claim.

Now, ler X be a closed bounded subset of $E^{'}$ (hence equicontinuous). Let U be a neighborhood of the origin in E such that $X \subset U^0$ (it follows from equicontinuity). By Banach-Alaoglu-Bourbaki theorem, U^0 is weakly compact. Since X is closed, it follows by 1.6.16 that X is also weakly closed, hence weakly compact (because then X is a weakly closed subset of the weakly compact set U^0). Now, on the one hand, the topology of compact convergence and the weakly topology coincides on X, on the other the topology on $E^{'}$ is the topology of compact convergence. Therefore, X is compact. □

Remark 1.6.34. Let $\omega \in \mathcal{D}_p$. We can see ω as current $\omega : \mathcal{D}_q \longrightarrow \mathbb{R}$ by the rule $\omega(\eta) = \int_M \omega \wedge \eta$. Such currents are called *diffuse currents*. We claim that the inclusion $\iota : \mathcal{D}_p \hookrightarrow \mathcal{D}_q^{'}$ is injective and continuous. Indeed, let B be a bounded subset of $\mathcal{D}_p$. Firstly, we will show that $\iota(B)$ is weakly bounded (hence bounded by 1.6.27). Let $\varphi : \mathcal{D}_q^{'} \longrightarrow \mathbb{R}$ be an arbitrary continuous linear functional. By 1.6.31, $\varphi = \eta$ for some $\eta \in \mathcal{D}_q$. Thus,

$$\varphi(\iota(B)) = \eta(\iota(B)) = \iota(B)(\eta) = \int_M B \wedge \eta = \pm \int_M \eta \wedge B = \pm\eta(B)$$

is bounded, since $\eta : \mathcal{D}_q \longrightarrow \mathbb{R}$ is a continuous linear functional. It follows by 1.6.5 that $\iota(B)$ is weakly bounded. Since $\mathcal{D}_p$ is a bornological space and the inclusion $\iota : \mathcal{D}_p \longrightarrow \mathcal{D}_q^{'}$ takes bounded subsets of $\mathcal{D}_p$ onto bounded subsets of $\mathcal{D}_q^{'}$, it follows by 1.6.9 that ι is continuous. Suppose now that ω is a nonzero p-form. Let $x \in M$ with $\omega_x \neq 0$ and (U, ϕ) be a local coordinate neighborhood around $x \in M$ such that $\omega \mid_U \neq 0$. Let $\omega = \sum_{\#J=p} a_J dx^J$ the local representation of ω in this local coordinate neightborhood and J_0 such that $a_{J_0} \neq 0$. Let $V \subset U$ be a neighborhood of the origin such that a_{J_0} not change the signal on V. Let $f : M \longrightarrow \mathbb{R}$ be a nonnegative smooth function with support contained in V satisfying $f(x) = 1$. If $\eta = a_{J_0} dx^{J_0}$, then

$$\omega(\eta) = \int_M \omega \wedge \eta = \pm \int_V f a_{J_0} dx^1 \wedge \ldots \wedge dx^n \neq 0.$$

Therefore, if $\omega \neq 0$ the inclusion of ω in $\mathcal{D}_q^{'}$ gives a nonzero element. It follows that the inclusion $\mathcal{D}_p \hookrightarrow \mathcal{D}_q^{'}$ is injective. Another significant fact established by L. Schwartz is the density of $\mathcal{D}_p$ in $\mathcal{D}_q^{'}$ ([73] page 75).

Remark 1.6.35. Let $L : E \times F \longrightarrow Z$ be a bilinear function. L is said to be *separately continuous* provided it is continuous in each factor. Let E be a barreled topological vector space and F and Z be any topological vector spaces. Then a separately continuous bilinear function $L : E \times F \longrightarrow Z$ is sequentially continuous. In particular, if E and F are metrizable, then L is continuous. Indeed, let (x_n, y_n) be a sequence converging to (x_0, y_0). We claim that $L(x_n, y_n)$ converges to $L(x_0, y_0)$. In fact, let V, W be neighborhoods of the origin in Z with $V + V \subset W$. Define $f_n(x) = L(x, y_n)$. So, $f_n \in E'$. Hence, for each $x \in E$, $\sup_n\{f_n(x)\}$ is bounded. Denote $\Phi = \{f_n, n \in \mathbb{N}\}$. Since the seminorms $p_x(f) = |f(x)|$, $x \in E$, generate the topology $\sigma(E', E)$ and $p_x(\Phi)$ is bounded for each $x \in E$, it follows that Φ is $\sigma(E', E)$-bounded (see 1.6.5). On the other hand, since E is barreled 1.6.28, the set Φ is equicontinuous by 1.6.25. Thus, there exists a neighborhood U of the origin in E with $\Phi(U) \subset V$. So, for sufficiently large n we have that $x_n \subset x_0 + U$. Hence $f_n(x_n - x_0) \in V$, and $L(x_0, y_n - y_0) \in V$ by the continuity of L at y. It follows that

$$L(x_n, y_n) - L(x_0, y_0) = f_n(x_n - x_0) + L(x_0, y_n - y_0) \in V + V \subset W$$

for large n. Thus $L(x_n, y_n) \longrightarrow B(x_0, y_0)$ when $(x_n, y_n) \longrightarrow (x_0, y_0)$, concluding that L is sequentially continuous.

Remark 1.6.36. Let E be a convex Hausdorff topological vector space. Consider the following separately continuous bilinear map $e : E' \times E \longrightarrow \mathbb{R}$ defined by $e(x', x) = x'(x)$. We claim that this bilinear map is continuous if and only if E is normable. Indeed, suppose that there is a neighborhood U of origin in E and a neighborhood V of the origin in E' satisfying $V(U) \subset [-1, 1]$. It follows that $V \subset U^0$, therefore, U^0 absorbs every one-point set (because every neighborhood of the origin absorbs them). Hence, by 1.6.12, U is $\sigma(E, E')$-bounded. It follows by 1.6.27 that U is bounded. But by Remark 1.6.7, normed topological vector spaces are the only ones that admit a bounded neighborhood of the origin. Hence, E must be normed.

1.6.3 The De Rham Theorem

In his doctoral thesis, G. De Rham showed that for each $p \geqslant 0$, the singular homology group $H_p(M; \mathbb{R})$ and the differential cohomology group $H^p_{DR}(M)$ are dual real vector spaces via the operation $(\omega, c) \longrightarrow \int_c \omega$. In a posterior work, he demonstrated that chains and differential forms are two aspects of currents. He introduced the notion of the *De Rham homology* and identified it with the singular homology. Today it is well-known that any homology (cohomology) theory with the same value in an one-point set is equivalent in the category of finite CW-complexes. It follows that the singular cohomology with coefficients in $\mathbb{R}$ of a manifold is equivalent to De Rham's cohomology[25]. This axiomatization was made by S. Eilenberg and N. Steenrod after the work of G. De Rham (see [84] chapter 3).

Theorem 1.6.37 (De Rham)**.** The bilinear map

$$k : H^p_{DR}(M) \longrightarrow H^p(M; \mathbb{R})$$

[25] Since one can triangulate any smooth manifold, each of them is equivalent to a finite CW-complex. For example, see [52] theorem 3.5 and corollary 6.7.

given by $k([\omega])([\sigma]) = \int_\sigma \omega$ is an isomorphism between the De Rham cohomology and the singular chomology[26].

Definition 1.6.38. Let E and F be vector spaces and $T : E \longrightarrow F$ be a linear map. The *transpose* of T, denoted by $T^{'}$, is the linear map $T^{'} : F^{'} \longrightarrow E^{'}$ defined by

$$T(y^{'})(x) = y^{'}(T(x)).$$

Lemma 1.6.39. Let E and F reflexive Hausdorff topological vector spaces. Then a linear map $T : E \longrightarrow F$ is continuous if and only if $T^{'} : F^{'} \longrightarrow E^{'}$ is continuous.

Proof. We claim that $(T(A))^0 = T^{'^{-1}}(A^0)$ for any vector spaces E, F and linear function $T : E \longrightarrow F$. Indeed:

$$\varphi \in T^{'^{-1}}(A^0) \Longleftrightarrow T^{'}(\varphi) \in A^0 \Longleftrightarrow \varphi(T(A)) \subset [-1, 1] \Longleftrightarrow \varphi \in (T(A))^0.$$

Suppose now that E and F are Hausdorff reflexive topological vector spaces. Then $E^{'}$ has the topology of uniform convergence on the collection of all bounded subsets of E and $F^{'}$ has one of uniform convergence on the collection of all bounded subsets of F. Suppose that T is continuous linear function from E to F. Let V be a neihgborhood of the origin in $E^{'}$. There exists a bounded subset A of E with $A^0 \subset V$. Since T is continuous and A bounded, then $T(A)$ is bounded. It follows that $U = T^{'^{-1}}(A^0) = (T(A))^0$ is a neighborhood of the origin in $F^{'}$. Hence, $T^{'}(U) = A^0 \subset V$, concluding that $T^{'}$ is continuous. Suppose now that $T^{'}$ is continuous. So, since $E^{'}$ and $F^{'}$ are reflexives Hausdorff spaces, from what we already showed follows that the function $T^{''} : E^{''} \longrightarrow F^{''}$ is continuous. That is, $T^{''}$ is a continuous linear function from E to F. Since $\varphi(T^{''}(x)) = \varphi(T(x))$ for all $\varphi \in F^{'}$, it follows from Hahn-Banach Theorem that $T = T^{''}$ and, therefore, T is continuous. $\square$

Definition 1.6.40. let M be a manifold. The *boundary operator*

$$\partial : \mathcal{D}^{'}_p \longrightarrow \mathcal{D}^{'}_{p-1}$$

is defined as the transpose of the differential operator, that is, $\partial = d^{'}$. Since the differential operator is continuous and $\mathcal{D}_p$ reflexive for all p, then the boundary operator is continuous. The set $\mathcal{Z}_p = \ker \partial$ is called *space of p-cycles* of M and the space $\mathcal{B}_p = \mathrm{im}\partial$ is called *space of p-boundaries* of M. The ph-*De Rham homology* of M is defined by $H^{DR}_p(M) = \mathcal{Z}_p/\mathcal{B}_p$.

For each smooth p-chain c, one can associate a p-current by the rule $c(\omega) = \int_c \omega$. Since any continuous map between smooth manifolds is homotopic to a smooth map, one can define the homology of M by using smooth singular chains. So, we can work in the smooth category. Denote by ∂_s the usual boundary operator defined on the complex of

[26] For a modern proof see [87].

smooth singular p-chains. The combinatorial version of the Stokes' theorem states that for a singular p-chain c and a smooth $(p-1)$-form ω,

$$\int_{\partial_s c} \omega = \int_c d\omega.$$

It follows that the singular boundary operator ∂_s and the boundary operator $\partial = d'$ coincides in the set $C_p(M;\mathbb{R})$ of all smooth singular p-chains. Thus, the inclusion

$$i : C_p(M;\mathbb{R}) \longrightarrow \mathcal{D}_p'$$

induces a homomorphism

$$i_* : H_p(M;\mathbb{R}) \longrightarrow H_p^{DR}(M).$$

Now, the evaluation map $e : \mathcal{D}_p' \times \mathcal{D}_p \longrightarrow \mathbb{R}$ induces a well-defined bilinear map

$$\begin{aligned} &H_p^{DR}(M) \times H_{DR}^p(M) \longrightarrow \mathbb{R} \\ &([\omega],[\varphi]) \longrightarrow \varphi(\omega). \end{aligned}$$

This bilinear map induces an homomorphism

$$j_* : H_p^{DR}(M) \longrightarrow (H_{DR}^p(M))'.$$

By the De Rham Theorem, $(H_{DR}^p(M))'$ is canonically identified with $H_p(M;\mathbb{R})$ and the homomorphism j_* can be viewed as a homomorphism of $H_p^{DR}(M)$ into $H_p(M;\mathbb{R})$.

Theorem 1.6.41 (De Rham [13], *Chapiter* IV)**.** The homomorphisms i_* and j_* are mutually inverse.

Theorem 1.6.42. The subspaces $\mathcal{B}_p$ and $\mathcal{Z}_p$ are closed.

Proof. Since $\mathcal{D}_p'$ is a Hausdorff space, any one-point set in $\mathcal{D}_p'$ is closed. Hence, since the boundary operator is continuous, it follows that $\mathcal{Z}_p = \partial^{-1}(0)$ is closed.

By Theorem 1.6.41, a cycle $\varphi \in \mathcal{D}_p'$ is a boundary if and only if take zero value in every closed p-form $\omega \in \mathcal{D}_p$. Thus

$$\mathcal{B}_p = \bigcap_{\omega \in \mathcal{D}_p,\ d\omega = 0} \omega^{-1}(0),$$

where $\omega : \mathcal{D}_p' \longrightarrow \mathbb{R}$ is the continuous linear functional $\omega(\varphi) = \varphi(\omega)$. It follows that $\mathcal{B}_p$ is a closed subset in $\mathcal{D}_p'$ since it is an intersection of closed subsets in $\mathcal{D}_p'$. $\square$

Remark 1.6.43. By De Rham Theorem, given any set $c_1, \ldots, c_k$ of p-cycles in a manifold M where among which there is no homology (that is, such that $\{[c_1], \ldots, [c_k]\}$ is a linearly independent subset of $H_1(M;\mathbb{R})$) and real number $r_1, \ldots, r_k$, there exists a closed p-form ω such that $\int_{c_i} \omega = r_i$. This was stated by H. Poincaré in his famous *Analysis Situs* (1895) without proof (see [40] for a historical account).

1.7 The Hodge Theorem

The Laplace operator on the Euclidian space has a generalization for Riemannian manifolds observed for the first time by E. Beltrami and defined for differential forms by W.V.D. Hodge. This operator now is called *Laplace-Beltrami operator*. A differential form living in the kernel of the Laplace-Beltrami operator is called *harmonic*. In his book [36], W.V.D. Hodge shows that the space of forms has an orthogonal decomposition, where a factor is the space of harmonic forms, which he showed to be a finite-dimensional vector space. Hodge's demonstration was not entirely correct, and the correction appears in Weyel paper [88]. A detailed historical account and proof of Hodge's decomposition theorem appears in [13].

Suppose now that M is an oriented Riemannian manifold. Here, to avoid confusion with the preceding section, we denote the space of smooth p-forms on M by $\Omega^p(M)$ instead of $\mathcal{D}_p$. We can induce a scalar product in $\Lambda^p T_x^* M$ as follows. Let $\{v_1, \ldots, v_n\}$ be a positive ortonormal basis for $T_x M$ and $\{\omega_1, \ldots, \omega_n\}$ be the dual basis for $T_x^* M$, that is, $\omega_i(v_j) = \delta_{ij}$. Set $\langle \omega_i, \omega_j \rangle = \delta_{ij}$ and

$$\langle \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \omega_{j_1} \wedge \cdots \wedge \omega_{j_p} \rangle = \det(\langle \omega_{i_k}, \omega_{i_s} \rangle) = \operatorname{sgn}(i_1, ..., i_p, j_1, ..., j_p).$$

Let $\{\omega_1, \ldots, \omega_n\}$ be a positive orthonormal basis for $T_x^* M$. The *Hodge star operator*, $* : \Lambda_x^p \longrightarrow \Lambda_x^q$, is defined as the linear extension of

$$*(\omega_{i_1}, \wedge \cdots \wedge \omega_{i_p}) = \omega_{j_1}, \wedge \cdots \wedge \omega_{j_q},$$

where $\{i_1, \ldots, i_p, j_1, \ldots j_q\}$ is an even permutation of $\{1, \ldots, n\}$. Clearly there is an induced map, still denoted by the same symbol,

$$* : \Omega^p(M) \longrightarrow \Omega^q(M).$$

Since

$$\langle \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \omega_{j_1} \wedge \cdots \wedge \omega_{j_q} \rangle = \operatorname{sgn}(i_1, ..., i_p, j_1, ..., j_q),$$

we can rewrite the scalar product in $\Lambda^p T_x^* M$ as

$$\langle \omega, \eta \rangle = *(\omega \wedge *\eta).$$

From these pointwise scalar products we obtain a global scalar product in $\Omega^p(M)$ by integration:

$$(\omega, \eta) = \int_M \omega \wedge *_g \eta.$$

Note that this scalar product induces a L^2-norm in $\Omega^p(M)$, as usual, putting $\| \omega \| = (\omega, \omega)^{1/2}$. We will now obtain the adjoint of the differential operator concerning the scalar product $(\cdot, \cdot)$ (for a closed manifold M). Since $*^2 = (-1)^{pq}$, we have

$$\begin{aligned} &d(\omega \wedge *\eta) = d\omega \wedge *\eta + (-1)^p \omega \wedge d\eta = \\ &d\omega \wedge *\eta - \omega \wedge \pm * * d * \eta \end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_M d(\omega \wedge *\eta) = \int_M d\omega \wedge *\eta + (-1)^p \omega \wedge d\eta = \\ &\textstyle\int_M d\omega \wedge *\eta - \int_M \omega \wedge \pm * * d * \eta = \\ &(d\omega, \eta) - (\omega, \pm * d * \eta).\end{aligned}$$

It follows by Stokes' theorem that

$$(d\omega, \eta) = (\omega, \pm * d * \eta),$$

where the signal is $(-1)^{n(p+1)+1}$. Thus, $\delta = (-1)^{n(p+1)+1} * d*$ is the adjoint of d. In this situation we define:

(1) The *codifferential*

$$\delta : \Omega^p(M) \longrightarrow \Omega^{p-1}(M), \quad \delta = (-1)^{n(p+1)+1} * d* \ := -\mathrm{div}$$

where div is the *divergence operator* acting on p-forms.

(2) The (geometers) *Laplacian* or *Laplace-Beltrami operator*

$$\Delta : \Omega^p(M) \longrightarrow \Omega^p(M), \quad \Delta = d\delta + \delta d.$$

Remark 1.7.1. Let $M = \mathbb{R}^n$ with the flat metric and $f : M \longrightarrow \mathbb{R}$ be a smooth function (0-form). Then

$$\Delta f = -\mathrm{div}\ df = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

is the opposite of the usual Laplacian.

Definition 1.7.2. A differential form ω defined on a Riemannian manifold M is said to be *co-closed* if $\delta\omega = 0$. The form ω is said to be *harmonic* if it is a zero of the Laplace-Beltrami operator. We will denote the space of all harmonic p-forms of a Riemannian manifold (M, g) by $\mathbb{H}^p(M)$.

Instead of using the above definition of harmonic forms, we will use one due to W.V.D. Hodge. These coincide on compact Riemannian manifolds.

Definition 1.7.3. Let ω be a differential form defined on a Riemannian manifold (M, g). The form ω is said to be harmonic if both ω and $*_g\omega$ are closed forms.

Remark 1.7.4. Let M be a closed Riemannian manifold. The following equalities hold:

$$(\triangle\omega, \omega) = (d\delta\omega + \delta d\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega) = \| \delta\omega \|^2 + \| d\omega \|^2 .$$

Hence, a differential form on a closed manifold is harmonic if and only if it is closed and co-closed. It follows that the two definitions of harmonic form coincide on compact Riemannian manifolds. However, if M is a noncompact manifold, there are harmonic forms that are not closed or co-closed. For example, $x \in \Omega^0(\mathbb{R})$ is harmonic but not closed.

Theorem 1.7.5 (Hodge's decomposition theorem[27]). For each integer p we have that $\mathbb{H}^p(M)$ is finite dimsensional vector space and $\Omega^p(M)$ admit the following ortogonal decomposition

$$\Omega^p(M) = \triangle(\Omega^p(M)) \oplus \mathbb{H} = d(\Omega^{p-1}(M)) \oplus \delta(\Omega^{p+1}(M)) \oplus \mathbb{H}^p.$$

Let us mention three significant consequences of Hodge's decomposition theorem.

Corollary 1.7.6 (Hodge's theorem). Let M be a closed Riemannian manifold. In each cohomological class $[\omega] \in H^p_{DR}(M)$ there exist a unique harmonic representative.

Proof. Let ω be a closed p-form. By Theorem 1.7.5 ω have a unique representation $\omega = d\eta + \delta\psi + \tau$, where τ is harmonic. Applying the differential operator in both side we obtain $d\delta\psi = 0$. It follows that $\| \delta\psi \|^2 = (\delta\psi, \delta\psi) = (d\delta\psi, \psi) = 0$, hence $\delta\psi = 0$. Let $\tau^{'}$ be a harmonic form in the cohomological class of ω. Then $\tau - \tau^{'} = d\xi$ for some ξ and

$$\| \tau - \tau^{'} \| = (d\xi, \tau - \tau^{'}) = (\varphi, \delta\tau - \delta\tau^{'}) = 0.$$

Therefore τ is unique. □

Corollary 1.7.7 (Poincaré duality). Let M be a closed Riemannian manifold. Then the map

$$h : H^p_{DR}(M) \longrightarrow H^q_{DR}(M)$$
$$[\omega] \in H^p_{DR}(M) \longrightarrow [*\omega_h] \in H^q_{DR}(M),$$

where ω_h denotes the unique harmonic representative in the cohomological class determined by ω, is an isomorphism.

Proof. Suppose that $h[\omega] = 0$. Then $*\omega_h$ is harmonic and represents a null cohomological class. Since the zero-form is a harmonic form representing the zero cohomological class, then $*_h\omega = 0$. It follows that $\omega_h = 0$ and $[\omega] = 0$, hence h is injective. This map is surjective since given a class $[\eta] \in H^q_{DR}(M)$ with harmonic representative η_h, then $\eta_h = *(*(-1)^{pq}\eta_h)$. □

Corollary 1.7.8 (De Rham). The bilinear map

$$H^p_{DR}(M) \times H^q_{DR}(M) \longrightarrow \mathbb{R}$$
$$([\omega], [\eta]) \longrightarrow \int_M \omega \wedge \eta$$

is nondegenerated.

[27] For a proof see [87].

Proof. Suppose $[\omega] \neq 0$. Let τ the harmonic representative in the cohomological class determined by ω. Then, by 1.7.6 τ and $*\tau$ are nonzero. Since τ is harmonic, then τ and $*\tau$ are closed forms (here M is a closed manifold). It follows that

$$\int_M \omega \wedge *\tau = \int_M \tau \wedge *\tau = \| \tau \| \neq 0.$$

□

Remark 1.7.9. A form ω is exact if, and only if, as an element of $\mathcal{D}_q'$ it is a boundary. Indeed, let $\omega = d\xi$. By Stokes' theorem

$$\partial\omega(\eta) = \omega(d\eta) = \int_M d\xi \wedge d\eta = \int_M d(\xi \wedge d\eta) = 0.$$

Hence $\omega \in \mathcal{Z}_q$. On the other hand, given a closed form $[\eta] \in H^p_{DR}(M)$ we have

$$[\omega]([\eta]) = \int_M \omega \wedge \eta = \int_M d\xi \wedge \eta = \int d(\xi \wedge \eta) = 0,$$

again by Stokes' theorem. It follows that $[\omega] = 0 \in H_q^{DR}(M)$, that is, $\omega \in \mathcal{B}_q$. Conversely, suppose $\omega = \partial\psi$. Then $d\omega(\eta) = \partial\omega(\eta) = 0$, with imples $d\omega = 0$ in $\mathcal{D}_{q-1}'$. Since the inclusion $\mathcal{D}_{p+1} \longrightarrow \mathcal{D}_{q-1}'$ is injective by 1.6.34, it follows that $d\omega = 0$ in $\mathcal{D}_{p+1}$. This proves the claim. Let (ω_n) be a convergente sequence of exact forms and $\psi_n \in \mathcal{D}'_{q+1}$ with $\omega_n = \partial\psi_n$, $n \in \mathbb{N}$. We claim that $\omega = \lim \omega_n$ is an exact form. Let $[\eta] \in H^p_{DR}(M)$. Then, looking ω as a diffuse current (see 1.6.34), we have

$$\int_M \omega \wedge \eta = \omega(\eta) = \lim \omega_n(\eta) = \lim \partial\psi_n(\eta) = \lim \psi_n(d\eta) = 0.$$

Since the form η was arbitrarily chosen and the bilinear map $([\omega], [\eta]) \longrightarrow \int_M \omega \wedge \eta$ is nondegenerated by 1.7.8, we concludes that ω is exact.

1.8 The Tischler argument

In the beautiful paper [82], D. Tischler showed that any closed n-dimensional manifold supporting a set of m everwhere linearly independent closed 1-forms must be a fiber bundle over the m-dimensional torus. The principal idea is to approximate each of such forms by forms with integral periods, thus obtaining submersions $M \longrightarrow \mathbb{S}^1$. By Eheresmann's lemma 1.2.3, such submersion gives a fiber bundle. The approximation used by D. Tischler uses a metric in the space of 1-forms (that is, a L^2-approximation). For a later application, we demonstrate that it is possible to obtain the same result by an approximation concerning the topology of $\mathcal{D}_p$.

Lemma 1.8.1. Let $[\omega] \in H^1_{DR}(M;\mathbb{Z})$[28]. There exists a smooth function $f : M \longrightarrow \mathbb{S}^1$ such that $\omega = f^*d\theta$. Conversely, given $f : M \longrightarrow \mathbb{S}^1$ be a smooth function, we have that $f^*d\theta$ represents a class in $H^1_{DR}(M;\mathbb{Z})$.

Proof. Define $f : M \longrightarrow \mathbb{S}^1$ by

$$f(x) = \exp\left(2\pi i \int_{x_0}^{x} \omega\right),$$

where $x_0 \in M$ is fixed and the integral is calculated in some smooth path connecting x_0 to x. This function is well-defined since $\int_K \omega \in \mathbb{Z}$ when K is a closed path. Let K any smooth path. Then

$$\begin{aligned} f^*d\theta(\partial_t|_0 K(t)) &= d(\theta \circ f)(\partial_t|_0 K(t)) \\ &= \partial_t|_0 (\theta \circ f)(K(t)) \\ &= \partial_t|_0 \theta\left(\exp\left(\left(\int_{x_0}^{K(t)} \omega\right)\right)\right) \\ &\partial_t|_0 \int_{x_0}^{K(t)} \omega \\ &= \omega(\partial_t|_0 K(t)), \end{aligned}$$

concluding that $[\omega] = [f^*d\theta]$.

Conversely, given $f : M \longrightarrow \mathbb{S}^1$ be a smooth function and a closed curve $K : \mathbb{S}^1 \longrightarrow M$, we have

$$\int_{K(\mathbb{S}^1)} f^*d\theta = \int_{\mathbb{S}^1} (f \circ K)^* d\theta.$$

Since $\int_{\mathbb{S}^1} d\theta = 1$, then

$$\int_{K(\mathbb{S}^1)} f^*d\theta = \operatorname{degree}(f \circ K) \in \mathbb{Z}.$$

Therefore, $f^*d\theta$ represents a class in $H^1_{DR}(M;\mathbb{Z})$[29]. $\square$

Lemma 1.8.2. The subset of all nowhere-vanishing forms in $\mathcal{D}_p$ is an open set.

Proof. Let ω be a nowhere-vanishing p-form. We want to show that there exists a neighborhood U of ω in $\mathcal{D}_p$ such that $\eta \in U$ implies that η is a nowhere-vanishing form. Suppose

[28] The set $H^1_{DR}(M;\mathbb{Z})$ is defined as being the subset of $H^1_{DR}(M)$ of all cohomological class with integral period. That is, $\omega \in H^1_{DR}(M;\mathbb{Z})$ if and only if $\omega(K) \in \mathbb{Z}$ for any closed smooth curve $K : \mathbb{S}^1 \longrightarrow M$. By the Universal Coefficient Theorem, $H^1_{DR}(M) = H^1(M,\mathbb{Z}) \otimes \mathbb{R}$. From this, it is easily seen that $H^1_{DR}(M,\mathbb{R}) = H^1_{DR}(M,\mathbb{Z}) \otimes \mathbb{R}$. It follows that $H^1_{DR}(M;\mathbb{Z})$ has a natural structure of a free $\mathbb{Z}$-module with dimension equal the *Betti number* of M.

[29] For every smooth map $f : X \longrightarrow Y$ between oriented closed manifolds having the same dimension, if $([\omega],[Y]) = 1$, then $\int_X f^*\omega$ is the number of points, counted with multiplicity ± 1, in the inverse image of any regular point in N (see [4]).

that it is impossible to obtain such a neighborhood. Then we can contruct a sequence ω_n in $\mathcal{D}_p$ converging to ω such that each ω_n is zero at some $x_n \in M$. Since M is compact, taking a subsequence if necessary, we can suppose that $x_n \longrightarrow x$. Let $v_x \in \Lambda^p(TM)$ be any p-vector in M. Let $v_{x_n} \in \Lambda^p(T_{x_n}M)$ be a sequence of p-vectors converging to v_x (we can obtain such a sequence working with a local trivialization to $\Lambda^p(TM)$ at x). Since each p-vector v_x define an element in $\mathcal{D}_p'$ by rule $v_x(\omega) = \omega(v_x)$, and the bilinear evaluation form $(\varphi, \omega) \in \mathcal{D}_p' \times \mathcal{D}_p \longrightarrow \varphi(\omega) \in \mathbb{R}$ is sequentially continuous by 1.6.35, then $\omega(v_x) = \lim v_{x_n}(\omega) = 0$. Now, v_x was arbitrarily chosen. Hence ω is zero at x, a contradiction. The lemma is proven. □

Theorem 1.8.3 (D. Tischler, [82])**.** Let M be a closed manifold. Suppose that there is on M a set $\omega_1, \ldots, \omega_m$ of m everwhere linearly independent closed 1-forms. Then M is the total space of a fiber bundle over the m-dimensional torus.

Proof. Let $\eta_1, \ldots, \eta_k$ be a basis of the free part in $H^1(M;\mathbb{Z})$. By 1.8.1, we can suppose that $\eta_i = f_i^*(d\theta)$, for certain smooth functions $f_i : M \longrightarrow \mathbb{S}^1$, $i = 1, \ldots, k$. Given an 1-form ω in M, we can write

$$[\omega] = \sum r_j [f_j^*(d\theta)]$$

for certain constants $\alpha_j \in \mathbb{R}$. Let $g_{ij} : M \longrightarrow \mathbb{R}$ be smooth funtions such that

$$\omega = \sum \alpha_j f_j^*(d\theta) + \sum dg_j.$$

Consider the identification $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with natural projection $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Let $dt = \pi^*(d\theta)$. Given $f : M \longrightarrow \mathbb{S}^1$ and $h : M \longrightarrow \mathbb{R}$, we can rewrite a form $\eta = f^*(d\theta) + dg$ as

$$\eta = f^*(d\theta) + dg \circ dt = f^*(d\theta) + dg \circ \pi^*(d\theta) = (f + \pi \circ dg)^*(d\theta),$$

where the last sum is from the group structure in $\mathbb{S}^1$. By this observation, we can suppose that

$$\omega = \sum \alpha_j f_j^*(d\theta).$$

Fixed $\eta_1, \ldots, \eta_m \in \mathcal{D}_1$, the association

$$(r_1, \ldots, r_m) \in \mathbb{R}^m \longrightarrow \sum r_i \eta_i \in \mathcal{D}_1$$

is continuous, since $\mathcal{D}_1$ is a topological vector space. Thus, there exist rational numbers r_{in} such that $\omega = \lim_n \sum r_{jn} f_j^*(d\theta)$. Since the expressions $\sum n_j f_j$, with $f_j : M \longrightarrow \mathbb{S}^1$ and $n_j \in \mathbb{Z}$, give us functions $h = \sum n_j f_j : M \longrightarrow \mathbb{R}$, there are integers d_n such that

$$\omega = \lim_n \frac{1}{d_n} h_n^*(d\theta),$$

where $h_n = \sum r_{jn} f_j$ and the convergence is in $\mathcal{D}_1$. Now, the bilinear mapping $(\omega, \eta) \in \mathcal{D}_p \times \mathcal{D}_q \longrightarrow \omega \wedge \eta \in \mathcal{D}_{p+q}$ being separately continuous must be continuous by Remark 1.6.35. It follows that there exist integers d_n and functions $h_{in} : M \longrightarrow \mathbb{S}^1$ such that

$$\omega_1 \wedge \ldots \wedge \omega_m = \lim_n \frac{1}{d_n} h_{1n}^*(d\theta) \wedge \ldots \wedge h_{mn}^*(d\theta)$$

in $\mathcal{D}_m$. Since $\omega_1 \wedge \ldots \wedge \omega_m$ is nowhere-vanishing, it follows by 1.8.2

$$h_{1n}^*(d\theta) \wedge \ldots \wedge h_{mn}^*(d\theta)$$

is nowhere-vanishing for sufficiently large n. Equivalenty, the functions

$$h_n = h_{1n} \times \ldots \times h_{1m} : M \longrightarrow \mathbb{T}^m$$

are submersions for sufficiently large n . By Ehresmann's lemma 1.2.3, a submersion defined on a closed manifold gives a fiber bundle. $\square$

Remark 1.8.4. A continuous p-form is defined as being a continuous section $\omega : M \longrightarrow \Lambda^p(T^*M)$. A continuous p-form is said to be closed provided $\int_c \omega = 0$ for all p-cycle $[c] \in H_p(M;\mathbb{R})$. J.F. Plante [65] showed that any smooth manifold admitting a continuous nowhere-vanishing closed 1-form fiber over $\mathbb{S}^1$, generalizing Tischler's result.

2 The problem of to obtain an intrinsic characterization of harmonic forms

The problem to obtain an intrinsic characterization of harmonic forms was first placed by E. Calabi [7]. More precisely, we can present the question in the following way. Let C be a set formed by closed forms defined on M. What are the topological (global) and local conditions on C such that each member of C is harmonic for a suitable Riemannian metric on M? For the set formed by closed 1-forms possessing only nondegenerate zeros[1], he obtained the following theorem.

Theorem 2.0.1 (E. Calabi, [7])**.** Let M be a closed smooth manifold and ω be a closed 1-form on M having only nondegenerate zeros of finite order different from 0 or n. Then, a necessary and sufficient condition so that M admits a Riemannian metric g such that ω become harmonic is ω be transitive.

Under the above assumptions, E. Calabi showed the existence of a Riemannian metric g_1 on a neighborhood U of the zero-set S of ω such that ω_U is g_1-harmonic. Furthermore, one can take U in such a way the form $*_{g_1}\omega|_U$ is an exact form, an important step in Calabi's proof. The topological (global) condition in Calabi's theorem was called *transitivity*. This hypothesis enables one to show the following assertion. Given a neighborhood U of the zero-set S of ω, there is a closed $(n-1)$-form η defined on M satisfying: $\omega \wedge \eta > 0$ in $M - U$ and $\eta = 0$ outside a neighborhood of S. Hence, for a suitable small neighborhood U of S, by using a standard gluing argument, E. Calabi constructs a Riemannian metric g such that ω becomes g-harmonic. A sharpened version of this theorem was proved by E. Volkov in paper [85]. To formulate it, we make use of the following definition.

Definition 2.0.2. A closed form ω is said to be *locally intrinsically harmonic* provided there exists a neighborhood U of the zero-set of ω and a Riemannian metric g defined on U such that $\omega|_U$ is g-harmonic.

Theorem 2.0.3 (E. Calabi-E. Volkov)**.** Let M be a closed smooth manifold and ω be a closed 1-form on M. A necessary and sufficient condition so that M admits a Riemannian metric g such that ω become g-harmonic is it be locally intrinsically harmonic and transitive.

An essential fact to demonstrate this result appears in Bär's paper [3]: the zero-set of solutions of a *semilinear elliptic system of first-order* is contained in a countable

[1] A zero x of a closed 1-form ω is said to be nondegenerate provided the following holds. Let U be a neighborhood of x such that $\omega = df$ for some smooth function $f : U \longrightarrow \mathbb{R}$. Then the Hessian of f at x is nonsingular.

union of smooth $(n-2)$-dimensional submanifolds. By using this, Volkov shows that $*_g\omega$ is an exact form in a suitable neighborhood of its zero-set and follows the argument in the same way as in Calabi's paper. The transitivity condition is obtained in a different way, by using The Poincaré Recurrence Theorem, a classical result from dynamical systems.

To date, Ko Honda, in your doctoral thesis [37], made the only advance in the problem of characterizing intrinsically harmonic forms of degree $(n-1)$. He got an analogous theorem as that Calabi.

Theorem 2.0.4 (Ko Honda, [37]). Let M be a closed smooth manifold and ω be a closed $(n-1)$-form on M having only nondegenerate zeros. Then, a necessary and sufficient condition so that M admits a Riemannian metric g such that ω become g-harmonic is it be transitive and locally intrinsically harmonic[2].

As we will see, forms of degree strictly between 1 and $(n-1)$ seem to present considerable additional difficulties. Next, we extend a definition due to E. Calabi that we will use throughout this thesis.

Definition 2.0.5. Let ω be a closed form defined on a manifold M. We call ω *intrinsically harmonic* if there exists a Riemannian metric on M such that $*_g\omega$ is a closed form.

2.1 Tautologically characterization of intrinsically harmonic p-forms of rank p

We are interested in finding conditions to forms of degree strictly between 1 and n be intrinsically harmonic. We consider firstly closed p-forms of rank p since, for these, there is a well-defined foliation induced by the integrable distribution $\ker \omega$ 1.3.4. So, maybe some aspect of foliation theory developed so far can be useful. For such a class of forms, we provide a straightforward characterization of when they are intrinsically harmonic.

Lemma 2.1.1. Let E be an oriented real vector space with a scalar product and $\omega \in \Lambda^p(E)$ be an exterior form of rank p. Then $\ker \omega \oplus \ker *\omega$ provides an orthogonal decomposition of E.

Proof. Let Ω the volume form induce by the scalar product on E. Since $\omega \wedge *\omega = \| \omega \| \Omega$, given $v \in \ker \omega \cap \ker *\omega$ we have $i_v\Omega = 0$. It follows that $v = 0$.

[2] In the formulation of this theorem, we have omitted some additional conditions in the zero-set of ω. The notion *having only nondegenerated zeros* means that ω as a section of the bundle $\Lambda^{n-1}(T^*M) \longrightarrow M$ is transverse to the zero-section.

Now, let $\{v_1, \cdots, v_{n-p}\}$ be an ortonormal basis for $\ker\omega$ and $\{v_1, \cdots, v_{n-p}, \cdots, v_n\}$ an ortonormal basis for E. For any integer $k > 0$ we have

$$i_{u_{n-p+k}} * \omega(v_{i_1}, \cdots, v_{i_{n-p-1}}) = *\omega(u_{n-p+k}, v_{i_1}, \cdots, v_{i_{n-p-1}}) = \pm\omega(v_{j_1}, \cdots, v_{j_p}),$$

where $\{n-p+k, i_1, \cdots, i_{n-p-1}, j_1, \cdots, j_p\} = \{1, \cdots, n\}$. If some v_i with $i \leqslant n-p$ not appear in the set $\{v_{i_1}, \cdots, v_{i_{n-p-1}}\}$, the latter expression vanishes. Indeed, in this case, v_i will appear in $\{v_{j_1}, \cdots, v_{j_p}\}$ and $\omega(v_{j_1}, \cdots, v_{j_p}) = 0$ is zero if j_k is less or equal than $n-p$. Now, in the choice of $(n-p-1)$ elements among $(n-p)$ possible, there will be at least one left by the *pigeonhole principle*. Therefore

$$i_{u_{n-p+k}}(*\omega) = 0.$$

The lemma follows now by a dimension argument. □

Theorem 2.1.2. Let M be a manifold with a closed p-form ω of rank p. Then ω is intrinsically harmonic if and only if there exists a closed form η of rank q such that $\ker\omega \cap \ker\eta = \{0\}$.

Proof. Let ω be a closed p-form of rank p. Suppose that there is on M a Riemannian metric g such that ω is g-harmonic. Then, by Lemma 2.1.1 the form $\eta := *\omega$ is a closed q-form with the desired property. Conversely, let η be a q-form with the property above. Choose Riemannian metrics $g_1,\ g_2$ on the bundles $\ker\omega$ and $\ker\eta$ and let g be the Riemannian metric on M which is the orthogonal sum of g_1 and g_2. The forms $*\omega$ and η are volume forms in $\ker\eta^{\perp}$ and have the same nullity space. Hence

$$*\omega = s\eta$$

where $s : M \longrightarrow \mathbb{R}$ is a smooth function. We have

$$\omega \wedge *\omega = \|\omega\|^2\Omega, \qquad \omega \wedge \eta = k\Omega,$$

where Ω is the volume form of g and $k : M \longrightarrow \mathbb{R}$ is a smooth function. It follows that

$$\|\omega\|^2\Omega = \omega \wedge *\omega = \omega \wedge s\eta = sk\Omega.$$

Therefore $sk = \|\omega\|^2$. Let $\{E_i\}$ and $\{F_j\}, i = 1, \ldots q,\ j = 1, \ldots, p$ be orthonormal bases for $\ker\omega,\ \ker\eta$, respectively, and let $f : M \longrightarrow \mathbb{R}$ be a positive function. Setting $E_i' = f^{\frac{-1}{2}}E_i$, then $\{E_i', F_j\}$ is an orthonormal basis concerning

$$g_f = fg_i \oplus g_2.$$

We look for a function f which is a solution of

$$*_{g_f}\omega = \eta. \tag{2.1}$$

If f is such a solution, ω is harmonic with respect to g_f, since η is closed and we have:

$$\omega(F_1, \dots, F_p) = *_{g_f}\omega(E'_1, \dots, E'_q) = \eta(E'_1, \dots, E'_p) = f^{\frac{p-n}{2}}\eta(E_1, \dots, E_p).$$

Hence

$$f = (\omega(F_, \dots, F_p)/\eta(E_1, \dots, E_p))^{\frac{2}{p-n}} = s^{\frac{2}{p-n}}$$

is a solution of (2.1). □

Remark 2.1.3. Note that any closed p-form ω of rank p is locally intrinsically harmonic, in sense that given $x \in M$, there is an open set U containing x such that $\omega|_U$ is harmonic. Indeed, let $x \in M$ and $\varphi = (u, v)$ be a foliated chart around x concerning the foliation $\mathfrak{F}_\omega$ (the foliation induced by ω), where the leaves of it are given by $v =$ constant. Then, $\eta = dv$ is a closed form of rank q satisfying $\omega \wedge \eta > 0$. It follows by 2.1.2 that $\omega|_U$ is harmonic for a suitable Riemannian metric on U.

2.2 The Calabi's transitivity condition

We already said that E. Calabi introduced a global restriction to closed 1-form ω be intrinsically harmonic. Where this condition holds, it is possible construct a closed differential $(n-1)$-form η satisfying $\omega \wedge \eta > 0$. Generalizing literally the transitivity condition introduced by E. Calabi for forms of arbitrary degree, the latter conclusion is still true for certain kind of differential forms (Lemma 2.2.2).

Definition 2.2.1. Let ω be a differential p-form defined on M. The form ω is said to be *transitive* at $x \in M$ provided there exists an embedding closed smooth submanifold Σ of M containing x such that $\omega|_\Sigma > 0$. The form ω is said to be transitive if it is transitive at every $x \in M$ where it does not null.

Lemma 2.2.2. Let M be a closed manifold and ω a p-form defined on M. Suppose that for every $x \in M$ with $\omega_x \neq 0$, there is a closed p-dimensional submanifold Σ with normal trivial bundle containing x and such that $\omega|_\Sigma > 0$. Let U be a neighborhood of S. Then, there exists a closed form ψ defined on M satisfying $\omega \wedge \psi > 0$ where $\psi \neq 0$. Moreover, we can take $\psi = 0$ in some closed neighborhood W of S contained in U. If, in addition, $S = \varnothing$, we can take ψ with $\omega \wedge \psi > 0$ everywhere, and the compactness of M can be dropped.

Proof. Let $x \in M$ and Σ be a closed submanifold containing x with normal trivial bundle. Let

$$\phi : \Sigma \times \mathbb{D}^{n-p} \longrightarrow V$$

diffeomorphism satisfying $\phi(x, 0) = x$ for $x \in \Sigma$ (given by trivializing the normal bundle of Σ). Denoting $\phi = (x, y)$, we have $\omega_x(\partial x_1, \cdots, \partial x_p) > 0$ for $x \in \Sigma$. By compactness, $\omega \wedge dy > 0$ in an open set V_0 with $\Sigma \subset V_0 \subset V$. Let $\epsilon > 0$ such that $V_{2\epsilon} \subset V_0$ and $h : [0, 2\epsilon] \longrightarrow \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} h(t) = 1 \text{ if } t < \epsilon \\ h(t) = 0 \text{ if } t \geqslant 2\epsilon \\ h \geqslant 0 \text{ everwhere.} \end{cases}$$

The differential form

$$\eta = h(\|y\|^2)dy$$

is closed on M and satisfies: $\omega \wedge \eta \geqslant 0$, $\omega \wedge \eta > 0$ where $\eta \neq 0$ and $\omega \wedge \eta = \omega \wedge dy > 0$ in V_ϵ. Since $S \cap \overline{V_{2\epsilon}} = \varnothing$, the vanishing of η near of the boundary of $V_{2\epsilon}$ implies that η vanishes in some neighborhood of S. The set $M - U$ is a closed subset of the compact M, hence compact. By using the preceding argument we can obtain a finite cover $V_1, \ldots, V_m$ of $M - U$ and respective closed forms $\eta_1, \ldots, \eta_m$ such that:

(1) $\omega \wedge \eta_i \geqslant 0$ everwhere;

(2) $\omega \wedge \eta_i > 0$ in V_i;

(3) $\eta_i = 0$ in some a closed neighborhood U_i of S.

To finish the first part of the lemma, take $\psi = \sum \eta_i$ and $W = \cap U_i$. Note that if $\psi_x = 0$, then

$$0 = \omega_x \wedge \psi_x = \sum \omega_x \wedge (\eta_i)_x$$

with $\omega_x \wedge (\eta_i)_x > 0$ when $(\eta_i)_x \neq 0$. It follows that $x \in \cap U_i$ hence $\ker \psi = W$. Suppose now that $S = \varnothing$ with M be merely a boundaryless manifold. Consider an enumerable locally finite cover of M by opens sets V_i with respective closed forms η_i satisfying the three conditions above. The expression

$$\psi = \sum_k \eta_k$$

gives a well-defined closed differential form satisfying $\omega \wedge \eta > 0$ in M. □

2.3 Outline of demonstration of Calabi-Volkov-Honda's theorems

The assumption about the zero-set made in 2.0.1 ensures the existence of a neighborhood U of it such that $H^{n-1}_{DR}(U) = 0$. Since those zeros are finite number one because they are isolated and M is compact, this fact follows trivially by the well-known Poincaré's lemma. More effort is needed to deal with the general case, as shown in the following theorem that needs a nontrivial result from Bär's paper [3].

Theorem 2.3.1 (E. Volkov, [85])**.** Let ω be a locally intrinsically harmonic closed 1-form defined on a closed manifold M and S the zero-set of it. Then, there exists a neighborhood U of S such that $H^{n-1}_{DR}(U) = 0$.

The latter theorem is the main contribution of Volkov to the problem of characterizing intrinsically harmonic 1-forms. We will now give the outline of the demonstration of Calabi-Volkov-Honda's theorems.

Lemma 2.3.2. Let ω be a transitive and locally intrinsically harmonic closed 1-form defined on a closed orientable manifold M. Let S the zero-set of ω. Then there exists a closed $(n-1)$-form η on M satisfying the following conditions:

(1) there is a neighborhood U of S and a Riemannian metric g_U on U such that $\eta|_U = *_{g_U}\omega$;

(2) $\omega \wedge \eta > 0$ everwhere on $M - S$.

Proof. Since ω is locally intrinsically harmonic, Theorem 2.3.1 ensure the existence of a neighborhood U_0 of the zero-set of ω and a Riemannian metric g defined on U_0 such that $*_{g_{U_0}}\omega$ is an exact form, say $*_{g_{U_0}}\omega = d\eta$. Let V be a neighborhood of S with $\overline{V} \subset U_0$. By the transitivity of ω, given x outside S there exists an embedding circle containing x where ω restricting to a volume form. Since M is orientable, any embedding circle has a trivial normal bundle in M by 1.5.14. By this, concerning V, we can take ψ and W as in Lemma 2.2.2. Remember the following properties of ψ and W:

(1) $\omega \wedge \psi \geqslant 0$ everwhere;

(2) $\omega \wedge \psi > 0$ in $M - V$;

(3) W is a neighborhood of S and $\psi|_W = 0$.

Since W and $M - U_0$ are closed disjoint sets, there exists a smooth bump function $\alpha : M \longrightarrow \mathbb{R}$ satisfying $\alpha|_{\overline{U}} = 1$ and $\alpha|_{M-U_0} = 0$. For each $K > 0$, set

$$\psi_K = K\psi + d(\alpha\eta).$$

Setting $U = \text{int}(W)$, given any $K > 0$, we have

$$(\psi_K)|_U = d\eta = *g_U\omega.$$

It follows that the first statement in the lemma is satisfied for any K. We claim that the second statement in the lemma follows by choosing K sufficiently large. Indeed, in $U - S$ we have that $\omega \wedge \psi_K = \omega \wedge d\eta = \omega \wedge *_{g_{U_0}}\omega > 0$. Outside of U_0 we have $\omega \wedge \psi_K = \omega \wedge K\psi > 0$.

In the region where $0 \leqslant \alpha \leqslant 1$, we have that $\omega \wedge \psi > 0$ and $\omega \wedge d(\alpha\eta)$ is bounded since $\{0 \leqslant \alpha \leqslant 1\}$ is a compact set. So, for sufficiently large K we have

$$\omega \wedge \psi_K|_{M-S} > 0.$$

The proof is completed by taking $\eta = \psi_K$ for a sufficiently large positive constant K. □

Proof's sketch of 2.0.3. Let η, U, g_U and g_V as in Lemma 2.3.2. Let $\{\phi_U, \phi_V\}$ be a partition of the unity subordinate to the cover $\{U, V\}$ of M satisfying $\phi_U = 1$ in a neighborhood U' of S. Then $h = \phi_U g_U + \phi_V g_V$ define a Riemannian metric on M such that $h|_V = h|_{\ker\omega} \oplus h|_{\ker\eta}$. As in the proof of 2.1.2, there exists a function $f : M - S \longrightarrow \mathbb{R}$ such that $g = fh$ define a Riemannian metric on $M - S$ satisfying $*_g\omega = \eta$. Note that $h = g_U$ on $U' - S$. On the other hand, $\eta = *_{g_U}\omega = *_h\omega$ on $U' - S \subset U$. It follows that $f = 1$ on $U' - S \subset U$. Thus, we can extend the metric g on M putting $g = g_U$ on S. Therefore $*_g\omega = \eta$, concluding that ω is g-harmonic.

The last argument is due to Calabi. The following transitivity condition obtained by Volkov differs from one presented by E. Calabi (Calabi's necessity proof does not apply to the general case). To prove the necessity, note that there exists on $M - S$ a nowhere-vanishing flow transversal to ω and preserving a volume of M. Such a flow can be obtained in the equation $i_X\Omega = *_g\omega$, where g is a Riemannian metric that makes ω harmonic and Ω is any volume form on M. By Poincaré Recurrence Theorem concerning the natural measure determined by Ω, given any open set $V \subset M - S$, one can obtain an arc of the flow generated by X going from V and returning to V after a finite time. By an usual concatenation argument, we obtain an embedding circle such that ω restrict to a volume form. □

Proof's sketch of 2.0.4. The zeros of ω are isolated since are nondegenerate. Because M is compact, it follows that the zero-set of ω is a finite set. Thus, we can obtain a neighborhood U of the zero-set of ω satisfying $H^{n-1}_{DR}(U) = 0$. Each embedded submanifold $\Sigma \subset M$ satisfying $\omega|_\Sigma > 0$ is orientable, since in such case ω is a top degree form without zeros defined on Σ. Since M is orientable, any orientable embedding $(n-1)$-closed submanifold admit trivial normal bundle in M by 1.5.12. By the same proof as in Lemma 2.3.2 we can obtain a closed 1-form η satisfying the conditions stated in the lemma. It follows that the demonstration of 2.0.3 applies literally for this case. To the transitivity of harmonic $(n-1)$-form having only nondegenerate zeros, see Honda's thesis. □

Although the Lemma 2.2.2 is holds for forms of degree p in general only with the more weak hypothesis that each Σ have normal neighborhood diffeomorphic to a product by a function ϕ satisfying $\phi(x, 0) = x$ for any $x \in \Sigma$, Calabi's argument does not generalize for intermediate degrees. As we will see, any nowhere-vanishing transitive closed differential p-form ω of rank p satisfies $\omega \wedge \eta > 0$ for some closed differential q-form η. However, by Lemma 2.1.2, ω is intrinsically harmonic if, and only if, it is possible to obtain η with rank $(n-p)$. Concrete examples of this phenomenon are presented in the next chapter. Let us mention to other works about the problem of characterizing intrinsically harmonic forms, the works [1, 47, 44].

2.4 An intrinsic characterization of harmonic nowhere-vanishing $(n-1)$-forms

This section addresses the problem of characterizing when a nowhere-vanishing closed $(n-1)$-form is intrinsically harmonic. This problem was motivated by an observation made by Calabi. He argues that any closed nowhere-vanishing 1-form on a closed manifold is intrinsically harmonic (see [7], §5). We begin by proving this geometrically by showing firstly that such forms are transitive.

Lemma 2.4.1. Let M be a closed manifold and ω be a nowhere-vanishing closed 1-form on M. Then ω is transitive.

Proof. Let X be the smooth vector field defined by the equation $\omega(X) = 1$. Let $x \in M$ and $\phi_x = (x_1, \cdots, x_n)$ be a chart defined on a neighborhood U_x of x such that the leaves of $\mathfrak{F}_\omega$ are given by $x_n =$ constant. Since the flow $\Phi_t(x)$ induced by X is complete and transversal to $\mathfrak{F}_\omega$, the curve $t \longrightarrow \Phi_t(x)$ stays in U_x only by finite time. If the image of this curve is compact, then it and $\mathbb{S}^1$ are diffeomorphic (see [54]). Otherwise, if the curve $t \longrightarrow \Phi_t(x)$ is noncompact, since M can be covered by finite collection of open sets U_x, this curve returns to some open set of this cover, lets say, returns to (ϕ, U). We now can construct an embedding $J : [0, 1] \longrightarrow M$ transversal to $\mathfrak{F}_\omega$ such that $\partial J = \{x_0, x_1\}$ is contained in the plaque $\{x_n = 0\}$. Since $\mathfrak{F}_\omega$ is transversely orientable by 1.5.8 and the form ω is positive in the image of J, we can construct a closed transversal to $\mathfrak{F}_\omega$ as in the lemma 3.3.7 of [9] (see figure below). The proof is completed.

Theorem 2.4.2. Let M be an orientable manifold. Suppose that ω is a nowhere-vanishing transitive closed 1-form (or $(n-1)$-form) on M. Then ω is intrinsically harmonic.

Proof. Since M is an orientable manifold, any closed 1-dimensional or any closed orientable $(n-1)$-dimensional submanifold of M have a normal trivial bundle in M (see 1.5.12 and 1.5.14). Thus, in the case that ω is a transitive 1-form or a transitive $(n-1)$-form, we can apply Lemma 2.2.2 to obtain a closed form ψ defined on M such that $\omega \wedge \psi > 0$. The proof is completed by applying Theorem 2.1.2 observing that in both cases ω and ψ have the correct rank. □

Let M be an orientable manifold (with boundary or not). Let ω be a $(n-1)$-form on M and Ω a volume form on M. There exists a unique vector field X such that $\omega = \iota_X \Omega$. To prove this, let $\{U_i\}$ be a cover of M and $\{\lambda_i\}$ be a partition of the unity subordinate to it. For each i we can suppose that there exist linearly independent vector fields $\{X_1^i, X_2^i, \cdots, X_n^i\}$ defined on U_i such that X_1 is tangent to $\ker \omega$ and

$$\Omega(X_1^i, X_2^i, \cdots, X_n^i) = 1.$$

In U_i then we have

$$\omega = \omega(X_2^i, \cdots, X_n^i)\iota_{X_1^i}\Omega$$

and, therefore,

$$\omega = \sum \lambda_i \omega = \sum \lambda_i \omega(X_2^i, \cdots, X_n^i)\iota_{X_1^i}\Omega = \iota_{\left[\sum \lambda_i \omega(X_2^i, \cdots, X_n^i) X_1^i\right]}\omega = \iota_X \Omega.$$

The unicity follows since Ω is a volum form on M.

Suppose now that ω is a closed nowhere-vanishing $(n-1)$-form defined on an orientable manifold M. Let Ω be a volume form on M and X the unique vector field given in the equation $\omega = i_X \Omega$. The flow generated by X is volume-preserving. That is, $\mathcal{L}_X \Omega = 0$, where $\mathcal{L}_X$ denotes the Lie's derivative in the X direction. Conversely, if a flow generated by a vector field X preserves a volume form Ω on M, then $d(\iota_X \Omega) = 0$, by Magic Cartan Formula. Thus, we have a 1-1 correspondence between closed nowhere-vanishing $(n-1)$-forms and nowhere-vanishing volume-preserving flows. We will now prove the following theorems.

Theorem 2.4.3. Let M be a closed orientable manifold. A nowhere-vanishing volume-preserving flow defined on M admits global cross-section if and only if the induce closed nowhere-vanishing $(n-1)$-form is intrinsically harmonic.

By a *global cross section* we mean a closed $(n-1)$-dimensional submanifold everywhere transverse to the flow and cutting every orbit. Necessary and sufficient conditions for the existence of a global cross-section for a flow given in [24] and [74], for example. We will discuss the latter in the context of smooth manifolds in Chapter 4. The next theorem enables us to approach the problem of obtaining an intrinsic characterization of nowhere-vanishing harmonic $(n-1)$-forms by another direction.

Theorem 2.4.4. Let M be a closed orientable manifold. A nowhere-vanishing C^r-volume-preserving flow on M admits a C^r-global cross-section if and only if it admits a C^r-transversal foliation ($r \geqslant 2$).

Remark 2.4.5. The geodesic flow of a Riemannian manifold with negative sectional curvature gives us an example of a nowhere-vanishing volume-preserving flow admitting complementary C^1 foliation (is an Anosov flow). However, the inducing closed nowhere-vanishing $(n-1)$-form cannot be intrinsically harmonic. In this case, we cannot obtain a transversal foliation induced by the kernel of a closed 1-form. We will clarify this remark in Chapter 5.

Remark 2.4.6. From a result of Palais ([65], corollary 2.11), the latter theorem follows for C^1 flows admitting complementary foliation given by the kernel of a nowhere-vanishing continuous closed 1-form. This result is the main ingredient in the preprint [76], which states the same one as here. It is worth pointing out that the characterization of intrinsically harmonic nowhere-vanishing (n-1)-forms is implicit in Honda's thesis. Our proof here follows the spirit of Calabi's work and does not use Plante's result above.

proof of 2.4.3. Let ω be a closed nowhere-vanishing harmonic $(n-1)$-form defined on an orientable Riemanian manifold (M, g) and Φ be a volume-preserving flow induced by ω. Then $\omega \wedge *_g\omega$ is a volume form on M, with $*_g\omega$ being a closed nowhere-vanishing 1-form. By Tischler's argument 1.8.3, there is a 1-form η with integral periods such that $\omega \wedge \eta$ is a volume form on M. Each of such forms have the form $f^*d\theta$ for some smooth function $f : M \longrightarrow \mathbb{S}^1$, where $d\theta$ is the obvious volume form on $\mathbb{S}^1$. Since $\eta = f^*d\theta$ is nowhere-vanishing, the function f is a submersion. By Eheresmann's lemma 1.2.3, $\mathcal{B} = \{M, f, \mathbb{S}^1, F\}$ defines a fiber bundle with fiber F diffeomorphic to $f^{-1}(\theta)$, $\theta \in \mathbb{S}^1$. Since Φ induces a 1-dimensional foliation of M transversal to the fibers of this fiber bundle, we have by 1.4.6 that $\mathcal{B} = \{M, f, \mathbb{S}^1\}$ is a foliated bundle, the foliation given by the orbits of Φ, of course. In particular, every fiber of this bundle intercept every orbit of Φ, concluding that $f^{-1}(\theta)$ is a global cross-section to Φ for all $\theta \in \mathbb{S}^1$.

Conversely, suppose that the flow induced by ω admits a global cross-section Σ. We can assume $\omega|_\Sigma > 0$. By a standard argument from foliation theory (sliding Σ transversally in the flow direction), it follows that the flow has a global cross-section at every point $x \in M$. Thus ω is transitive. From Theorem 2.4.2, we conclude that ω is intrinsically harmonic. □

Lemma 2.4.7. Let X be a nowhere-vanishing C^r-vector field admitting complementar C^r-foliation $\mathfrak{F}$, $r \geqslant 2$. Then, $\mathfrak{F} = \mathfrak{F}_\eta$ for some closed 1-form η of class C^{r-1}.

Proof. The foliation $\mathfrak{F}$ is transversely orientable since it is transversal to the nowhere-vanishing vector field X. Thus, by 1.5.8 there is a differential C^{r-1}-form ω such that $T\mathfrak{F} = \ker\omega$. We claim that $\eta = \frac{1}{\omega(X)}\omega$ is closed. Consider (x, y) and (u, z) be local coordinates with common domain such that $X = \partial_x$ and $\mathfrak{F}$ is given locally by z =constant. By transversality, (x, u) defines a coordinate neighbohoord. In this coordinate, since ∂_{u_i} is tangent to $\mathfrak{F}$, $\eta(X) = 1$ and $[\partial_x, \partial_{u_i}] = 0$, then

$$d\eta(\partial_x, \partial_{u_i}) = \partial_x(\eta(\partial_{u_i})) - \partial_{u_i}(\eta(\partial_x)) + \omega[\partial_x, \partial_{z_i}] = 0.$$

It follows that η is closed. Now, note that $\ker\eta = \ker\omega$, hence η is a closed form and $\mathfrak{F} = \mathfrak{F}_\eta$.

□

proof of 2.4.4. Let Ω be a volume form on M and Φ a flow preserving it. Let X be a vector field generating Φ. Set $\omega = \iota_X\Omega$. If X admits a complementary foliation, by 2.4.7 there exists a closed 1-form η such that $\eta(X) = 1$. Thus, $\omega \wedge \eta > 0$ with $d\omega = 0$ and $d\eta = 0$. Hence, Theorem 2.1.2 applies, concluding that ω is intrinsically harmonic and, therefore, by 2.4.3 Φ admits a global cross-section. Conversely, if Φ admits a global cross-section, then again by 2.4.3 there is a Riemannian metric g on M such that $*_g\omega$ is a closed 1-form. The foliation generated by the distribution $\ker *_g\omega$ is transversal to Φ. The proof is completed. □

3 Forms of intermediate degrees

The only affirmatives answers in the problem of obtaining an intrinsic characterization of harmonic forms occur in degree 1 and $(n-1)$, synthesized in Calabi-Volkov-Honda's theorems, and the class of symplectic forms (that appear in Honda's thesis [37]). We have already noted that even in the case o closed p-forms of rank p the problem is equivalent to one very hard in foliation theory (see 2.1.2). As shown in an example due to J. Latshev, generalized here, even we add the transitivity hypothesis, it has no affirmative answer.

In this chapter, we will be concerned with the study closed p-forms of rank p. We have clarified some aspects of obtaining an intrinsic characterization of harmonic forms in this class, although we do not provide any significant results. We organized it as follows. Section 3.1 provides a detailed generalization of the mentioned Latshev's example. In Section 3.2, we give a brief exposition of foliation currents' theory to establish Sullivan's characterization of foliations with closed transversal forms. From this, we showed that any transitive closed p-form of rank p has many closed transversal forms. However, this not implies that these are intrinsically harmonic by concrete examples. In Section 3.3, we show that a closed manifold with a harmonic p-form of rank p admits two complementary foliations, where the leaves of both are minimal surfaces for a suitable metric. Also, are given another characterization of intrinsically harmonic p-forms of rank p and examples of these. In Section 3.4, we deal with the study of transitivity condition.

3.1 Non intrinsically harmonic forms examples

Theorem 3.1.1. Let $\mathcal{B} = \{B, p, M, F\}$ be a smooth fiber bundle with compact total space and simply connected base. Let Ω be a volume form on M. Then $\mathcal{B}$ is trivializable if and only if $p^*\Omega$ is intrinsically harmonic.

Proof. Set $\alpha = p^*\Omega$. Suppose that α is a g-harmonic form. It is easily seen that

$$\ker \alpha_y \oplus \ker(*\alpha)_y = T_y B.$$

Therefore, $*\alpha$ has a constant rank. Since $*\alpha$ is closed, the regular distribution $\ker(*\alpha)$ induces a foliation in B transversal to the fibers (see 1.3.4). The compactness of the fibers and Theorem 1.4.6 implies that we have a foliated bundle. By 1.4.3 this kind of bundle is differentiable equivalent to

$$\mathcal{B}_\varphi = (\tilde{M} \times F)/\pi_1(M),$$

where $\varphi : \pi_1(M) \longrightarrow \mathrm{Diff}(F)$ is the holonomy homomorphism characterizing it. Since $\pi_1(M) = 0$, it follows that $\mathcal{B}$ is trivializable.

Conversely, suppose that $\mathcal{B}$ is a trivializable (now, M and F can be any boundaryless oriented manifolds). There exists a diffeomorphism $f : B \longrightarrow M \times F$ such that $p_2(f(x)) = p(x)$, where $p_2 : M \times F \longrightarrow F$ is the natural projection onto the second factor. Let Ω_F be a volume form on F. It is easy to see that the closed form $(f \circ p_2)^*(\Omega_F)$ has degree and rank equal to $(\dim F)$ and, moreover, it is complementary to $p^*(\Omega_M)$. It follows by 2.1.2 that $p^*(\Omega_M)$ is intrinsically harmonic. □

The hypothesis $\pi_1(M) = 0$ cannot be dropped. Otherwise, since any closed 1-form without singularities on closed manifolds is intrinsically harmonic (see 2.4.1 and 2.4.2), any orientable bundle over a circle would be trivial (this happens only for vector bundles). We can construct concrete examples of no trivializable circle bundles as follows. Let Σ_g be an orientable surface and $\phi \in \mathrm{Diff}^+(\Sigma_g)$. Then, $S_\phi = (\Sigma_g \times [0,1])/\{(x,0) \sim (x,1)\}$ fibers over $\mathbb{S}^1$ with fiber Σ_g and projection $\pi_g(x,t) = e^{2\pi i t}$.

In the argument to prove Theorem 3.1.1, the compactness hypothesis of the fiber cannot be dropped. Generally, a submersion $\pi : E \longrightarrow M$ with compact fibers and a transversal foliation give us a foliated bundle (see 1.2.3 and 1.4.6). However, without the compactness of the fibers, we cannot ensure that a transversal foliation gives us a compatible one. With a transversal foliation, each restriction $\pi : L \longrightarrow M$, L leaf, is a surjective local diffeomorphism. Despite that, the injectivity follows in general only when those functions are proper maps (see [35]). In what follows, we provide examples of bundles with transversal foliations that do not admit any compatible foliation.

Theorem 3.1.2. There exists transversal foliations to the vector bundle $\tau_{\mathbb{S}^n}$ for all $n \geqslant 1$. Every such foliations are incompatible for $n \neq 1, 3, 7$.

Proof. J.W. Milnor showed in [53] that every tangent bundle τ_M of a smooth manifold M has a codimension $n = \dim M$ transversal foliation. He obtained a *microfoliation* in the τ_M from the pullback of the trivial foliation $M = \bigcup_{x \in M}\{x\}$ by using the exponential map and extend it to the whole tangent space (see the demonstration of Proposition 6.1 in referred Milnor's paper). We conclude from a particular case of Theorem 1 in [14] that if $T\mathbb{S}^n$ and $\mathbb{S}^n \times \mathbb{R}^n$ are diffeomorphic, then $T\mathbb{S}^n$ is trivializable. Thus, if some foliation guaranteed to exist by Milnor's paper were compatible with $\tau_{\mathbb{S}^n}$, then $T\mathbb{S}^n$ and $\mathbb{S}^n \times \mathbb{R}^n$ would be diffeomorphic, hence $\tau_{\mathbb{S}^n}$ would be trivializable. However, by a well-known result of J.F. Adams, the tangent bundle of $\mathbb{S}^n$ is trivializable if, and only if, $n \in \{0, 1, 3, 7\}$.

□

This theorem is similar to “using a bazooka to kill a fly”. The mentioned Milnor's result is elementary, but the others not. By using the following observation, we provide elementary examples as ones in Theorem 3.1.2.

Theorem 3.1.3. Let M be a compact manifold. Suppose that there exists a homeomorphism $f : TM \longrightarrow M \times \mathbb{R}^n$ satisfying $p_1(f(v)) = p(v)$, where p_1 is the natural projection onto the first factor and p is the projection of the fiber bundle τ_M. Then the Euler characteristic $\chi(M)$ of M is zero.

Proof. Given $y \in \mathbb{R}^n$, let $s_y : M \longrightarrow TM$ the global cross-section defined by $s_y(x) = f^{-1}(x, y)$ (it is a cross-section since $p = p_1 \circ f$). Let (y_i) be a sequence in $\mathbb{R}^n$ with $\| y_i \| \longrightarrow \infty$. Suppose that for each i there exists $x_i \in M$ with $s_{y_i}(x_i) = 0_{x_i}$, that is, $s_{y_i}(x_i)$ is the origin of $T_{x_i}M$. Since the image of the zero cross-section in TM is compact, taking a subsequence if necessarily, we can suppose that 0_{x_i} converge to 0_x for some $x \in M$. Thus, letting $i \longrightarrow \infty$ we obtain

$$0_{x_i} = f^{-1}(x_i, y_i) \longrightarrow 0_x \text{ and } (x_i, y_i) = f(0_{x_i}) \longrightarrow f(0_x).$$

Since (x_i, y_i) is a no convergent sequence, this leads a contradiction. It follows that there exists some $y \in \mathbb{R}^n$ such that the cross-section s_y no intercept the zero cross-section. By Poicaré-Hopf Theorem (see [54], page 35) we concludes that $\chi(M) = 0$. □

Corollary 3.1.4. Let M be a compact manifold. There exist transversal foliations to τ_M. If $\chi(M) \neq 0$ any such a foliation is incompatible with τ_M.

Proof. The existence of such foliations is given in Milnor's paper [53]. If some such a foliation is compatible with τ_M, then there will exist a diffeomorphism $f : TM \longrightarrow M \times \mathbb{R}^n$ satisfying $p_1(f(v)) = p(v)$ (see 1.4.3). The latter theorem and the hypothesis $\chi(M) \neq 0$ leads to a contradiction. This proves the corollary. □

3.1.1 Concrete transitive non intrinsically harmonic forms examples

Let $\mathbb{S}^{n-1}$ be a great $(n-1)$-sphere of $\mathbb{S}^n$, E_1, E_2 be the closed hemisphere determined by $\mathbb{S}^{n-1}$ and V_1, V_2 be contractible open disjoint neighbohood containing E_1 and E_2, respectively. Then, V_1, V_2 cover $\mathbb{S}^n$ and $V_1 \cap V_2$ is an equatorial band containning $\mathbb{S}^{n-1}$. Since that V_1, V_2 are contractible, there exist bundle maps $\phi_1' : \mathcal{B}_{V_i} \longrightarrow V_i \times G$. Fixed $x_0 \in \mathbb{S}^{n-1}$ we can obtain a bundle strictely equivalent to $\mathcal{B}$ putting $g_{12}(x) = g_{12}'(x_0)^{-1} g_{12}'(x)$. The map

$$T = g_{12}|_{\mathbb{S}^{n-1}} : (\mathbb{S}^{n-1}, x_0) \longrightarrow (G, e)$$

is called *characteristic map of* $\mathcal{B}$. A bundle over $\mathbb{S}^n$ is said to be in *normal form* if it coordinate neighborhoods are V_1, V_2 and $T(x_0) = e$.

Theorem 3.1.5 ([78], theorem 18.3)**.** Let $\mathcal{B}, \mathcal{B}'$ be bundles over $\mathbb{S}^n$ in normal form and having the same fiber and group. Let T, T' be their characteristic maps. Then $\mathcal{B}, \mathcal{B}'$ are equivalent if and only if there exists an element $a \in G$ and a homotopy between T' and

aTa^{-1}. If G is arcwiseconnected, then $\mathcal{B}, \mathcal{B}'$ are equivalent if and only if there there exists a homotopy between T' and T.

Corollary 3.1.6. Let F be an arcwise connected smooth manifold such that $\pi_i(F) = 0$ for all $i = 1, \ldots n-1$. Suppose that $\mathrm{Diff}^+(F)$ is arcwise connected and $\pi_{n-1}(\mathrm{Diff}^+(F)) \neq 0$. Let $\mathcal{B} = \{B, p, \mathbb{S}^n, F, \mathrm{Diff}^+(F)\}$ be a nontrivial fiber bundle. Then $p^*(\Omega_{\mathbb{S}^n})$ is transitive but not an intrinsically harmonic form.

Proof. Suppose $p^*(\Omega_{\mathbb{S}^n})$ intrinsically harmonic. Thus, by 3.1.1, the fiber bundle $\mathcal{B} = \{B, p, \mathbb{S}^n, F, \mathrm{Diff}^+(F)\}$ is $\mathrm{Diff}(\mathbb{S}^n)$-trivializable, that is, there exists a diffeomorphism $\varphi : B \longrightarrow \mathbb{S}^n \times F$ satisfying $\varphi(x) = (p(x), \phi(x))$. Changing orientation if necessary, we can suppose that ϕ is always orientation preserving, since $x \longrightarrow \det(\phi_*)_x$ is never zero. It follows that $\mathcal{B}$ is trivial, a contradiction. Thus by 1.2.36 is a transitive no intrinsically harmonic form. $\square$

The hypothesis in latter theorem is no vacuous. In fact, S. Smale [77] showed that the inclusion $SO(3) \hookrightarrow \mathrm{Diff}^+(\mathbb{S}^2)$ is a homotopy equivalence. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, it follows from 3.1.5 that there exists a nontrivial bundle over $\mathbb{S}^2$ with group $\mathrm{Diff}^+(\mathbb{S}^2)$ and fiber $\mathbb{S}^2$. According to E. Volkov, this example is due to J. Latschev.

3.2 The Sullivan Theorem

Theorem 2.1.2 gives us a sufficient condition to a closed p-form ω of rank p to be intrinsically harmonic. Naturally, two questions come up:

(1) If the foliation $\mathfrak{F}_\omega$ admit merely a complementary foliation, is ω an intrinsically harmonic form?

(2) If there exists a closed form η with $\omega \wedge \eta > 0$, is ω an intrinsically harmonic?

The category of the foliated bundle is the more simple one where we can discuss the first question. The latter has a negative answer by using the results of Sullivan's paper [80]. D. Sullivan and D. Ruelle introduced in [69] the notion of *foliation currents*. These are real homological classes associated with a partial or total orientable foliation $\mathfrak{F}$. The relative position between the set of all foliations currents and the spaces of cycles and boundaries determine whether $\mathfrak{F}$ admits closed forms or exact forms transversal, respectively. D. Sullivan also characterized foliations cycles, currents that are also cycles, in terms of holonomy invariant measures. In this section, we present shortly the basic definitions and the results stated in Sullivan's paper. From this, we demonstrate any transitive closed p form of rank p admits a closed transversal form.

Let $\mathfrak{F}$ be a smooth orientable foliation defined on a closed manifold M. Fix a regular foliated atlas $\mathcal{U} = \{(U_\alpha, x_\alpha, y_\alpha)\}$ for M such that for each $x \in M$, the p-tuple $(x_1, \cdots, x_p)$ determines the correct orientation for $\mathfrak{F}|U_\alpha$. Give a positive basis $(v_{1x}, \cdots, v_{px})$ of the $T_x L_x$, where L_x is the leaf containing x, the p-vector

$$v_x = v_{1x} \wedge \cdots \wedge v_{px}$$

determines a p-current called *Dirac foliation current for* $\mathfrak{F}$.

Definition 3.2.1. Let $\mathfrak{F}$ be a smooth orientable foliation defined on a closed manifold M. Define $C_{\mathfrak{F}}$ the set of all finite linear combinations with positive coefficients of the Dirac foliation currents for $\mathfrak{F}$. We define:

(1) the set $\mathcal{C}_{\mathfrak{F}}$, as being the topological closure of $C_{\mathfrak{F}}$ in $\mathcal{D}'_p$. The elements of it are called *foliation currents* (for $\mathfrak{F}$);

(2) the set $\mathcal{Z}_{\mathfrak{F}} = \mathcal{Z}_p \cap \mathcal{C}_{\mathfrak{F}}$, whose elements are called *foliation cycles* (for $\mathfrak{F}$);

(3) and the set $\mathcal{B}_{\mathfrak{F}} = \mathcal{B}_p \cap \mathcal{C}_{\mathfrak{F}}$, whose elements are called *foliation boundaries* (for $\mathfrak{F}$).

Definition 3.2.2. Let $\mathfrak{F}$ be a foliation of M. A p-form ω on M is said to be *transversal* to $\mathfrak{F}$ provided $v_x(\omega) > 0$ for every Dirac foliation current for $\mathfrak{F}$. In particular, a transversal form to $\mathfrak{F}$ restricts to a volume form on each leaf of it.

Any orientable foliation admits many transversal differential forms. It is easy to construct such differential forms locally by using a foliated chart. Hence, by the usual gluing method from a partition of unity subordinate to a foliated atlas, one can construct examples of transversal forms (see 1.5.8). When a foliation admits, a transversal closed or an exact form is a more engaging question and was answered by D. Sullivan as follows.

Lemma 3.2.3 (Sullivan, [80])**.** Let $\mathfrak{F}$ be an orientable foliation. There exists an exact form transversal to $\mathfrak{F}$ if, and only if, 0 is the unique foliation cycle for $\mathfrak{F}$. There exists a closed form transversal to $\mathfrak{F}$ if, and only if, 0 is the unique foliation boundary for $\mathfrak{F}$.

Proof. Let ω be a differential form transversal to $\mathfrak{F}$. First of all, we will show the set $\widehat{\mathcal{C}_{\mathfrak{F}}} = \mathcal{C}_{\mathfrak{F}} \cap \omega^{-1}(1)$ is compact in $\mathcal{D}'_1$ (where here $\omega : \mathcal{D}'_1 \longrightarrow \mathbb{R}$ is given by $\omega(\varphi) = \varphi(\omega)$). Since $\mathcal{D}'_1$ satisfies the Heine-Borel propertie 1.6.33, it is enough to show that $\widehat{\mathcal{C}_{\mathfrak{F}}}$ is bounded. The proof of the boundedness of $\widehat{\mathcal{C}_{\mathfrak{F}}}$ will be divided into two parts.

(1) Let E be a topological vector space and $\omega : E \longrightarrow \mathbb{R}$ be a continuous linear functional. For any cone $C \subset E$ we have the equality

$$\overline{C} \cap \omega^{-1}(1) = \overline{C \cap \omega^{-1}(1)}.$$

Indeed, let $x \in \overline{C} \cap \omega^{-1}(1)$ and U be a neighborhood of x. There exists $\epsilon > 0$ and neighborhood V of x with $(1-\epsilon, 1+\epsilon)V \subset U$ (because the scalar multiplication of E is continuous). Now, since ω is continuous, $\omega(x) = 1$ and $x \in \overline{C}$, for each $n \in \mathbb{N}$ there exists $a_n \in V \cap \omega^{-1}(1 - 1/n, 1 + 1/n)$. If follows that there exists a sequence $(a_n) \subset C \cap V$ such that $\omega(a_n) \longrightarrow 1$. Therefore $\frac{1}{\omega(a_n)} a_n \in U$ for large n. Since C is a cone, $\frac{1}{\omega(a_n)} a_n \in C$. Thus, for an arbitrary neighborhood U of x we showed that there exists $a \in C \cap U$ with $\omega(a) = (1)$, with implies $x \in \overline{C \cap \omega^{-1}(1)}$. Since is always true that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, this proves the claim.

(2) We will now prove that $\widehat{\mathcal{C}_{\mathfrak{F}}}$ is bounded. Put on M a Riemannian metric. It induces a metric in the space $\Lambda^p(TM)$ of all p-vectors tangent to M. Since $\omega : \Lambda^p(TM) \longrightarrow \mathbb{R}$ is continuous and the set of all unit p-vectors tangents to M is compact, there exists $\lambda > 0$ such that $0 < \omega(v_x) \leqslant \lambda$ for all unit p-vector. By Theorem 1.6.27, it is enough to show that $C_{\mathfrak{F}} \cap \omega^{-1}(1)$ is $\sigma(\mathcal{D}_p^{'}, \mathcal{D}_p)$-bounded. Equivalentely, it is sufficent to show that $\eta(C_{\mathfrak{F}} \cap \omega^{-1}(1))$ is bounded for all differential form $\eta \in \mathcal{D}_p$. Let $\eta \in \mathcal{D}_p$ and α the maximum value of η on any unit p-vector. Let $\varphi \in C \cap \omega^{-1}(1)$. We can write $\varphi = \sum a_i v_{x_i}$, with $a_i \geqslant 0$ for all i and each v_{x_i} being a Dirac foliation current with norm 1. Then,

$$1 = \omega(\varphi) \leqslant \lambda \sum a_i$$

and $\sum a_i \leqslant 2/\lambda$. Thus,

$$|\eta(\varphi)| = |\sum a_i \eta(v_{x_i})| \leqslant 2\alpha/\lambda.$$

It follows that $C \cap \omega^{-1}(1)$ is bounded. This proves the claim.

Since a subset A of a topological vector space E is bounded if, and only if, $\overline{A}$ is bounded, we conclude from the above item that

$$\widehat{\mathcal{C}_{\mathfrak{F}}} = \overline{C_{\mathfrak{F}}} \cap \omega^{-1}(1) = \overline{C_{\mathfrak{F}} \cap \omega^{-1}(1)}$$

is bounded.

We will now prove the first statement of the lemma. by hypothesis we have that $\mathcal{C}_{\mathfrak{F}} \cap \mathcal{Z}_p = \{0\}$. Since the vectorial subspace $\mathcal{Z}_p$ and the compact convex set $\widehat{\mathcal{C}_{\mathfrak{F}}}$ are disjoints, by Hahn-Banach Theorem there exists a continuous linear functional $\varphi : \mathcal{D}_p^{'} \longrightarrow \mathbb{R}$ such that

$$\varphi(\mathcal{Z}_p) = 0 \text{ and } \varphi(\widehat{\mathcal{C}_{\mathfrak{F}}}) > 0.$$

By Schwartz's theorem 1.6.31, there exists a differential form ω such that $\varphi = \omega$. Since $\omega(\mathcal{Z}_p) = \varphi(\mathcal{Z}_p) = 0$, given any current $c \in \mathcal{D}'_{p+1}$ we have

$$d\omega(c) = \omega(\partial c) = 0.$$

It follows that ω is closed. In fact, ω is exact, since as a linear functional $\omega : H^p_{DR}(M) \longrightarrow \mathbb{R}$ is identically zero. Furthemore,

$$\omega(\widehat{\mathcal{C}}_{\mathfrak{F}}) = \theta(\widehat{\mathcal{C}}_{\mathfrak{F}}) > 0,$$

concluding that ω is an exact form transversal to $\mathfrak{F}$. Conversely, if $c \in \mathcal{C}_{\mathfrak{F}} \cap \mathcal{Z}_p$ and $\omega = d\eta$ is transversal to $\mathfrak{F}$, then

$$\omega(c) = d\eta(c) = \eta(\partial c) = 0.$$

Since $c \in \mathcal{C}_{\mathfrak{F}}$ and ω is strictly positive on $\widehat{\mathcal{C}}_{\mathfrak{F}}$, it follows that that $c = 0$.

To proof the second statement of the lemma, suppose $\mathcal{C}_{\mathfrak{F}} \cap \mathcal{B}_p = \{0\}$. Analougously as in the proof of the first part of the lemma, there exists one form ω such that

$$\omega(\mathcal{B}_p) = 0 \text{ and } \omega(\widehat{\mathcal{C}}_{\mathfrak{F}}) > 0.$$

Thus, given any current $c \in \mathcal{D}'_{p+1}$ we have $d\omega(c) = \omega(\partial c) = 0$. It follows that ω is a closed form transversal to $\mathfrak{F}$. Conversely, if ω is a closed form transversal to $\mathfrak{F}$, given $c \in \mathcal{C}_{\mathfrak{F}} \cap \mathcal{B}_p$, say $c = \partial z$, then $\omega(c) = \omega(\partial z) = d\omega(z) = 0$. It follows that $c = 0$, since ω is transversal to $\mathfrak{F}$. The proof is completed.

□

Remark 3.2.4. By a theorem of J. Dieudonné, every Fréchet Montel topological vector space E is separable (see [43], page 370). Let E be a separable topological vector space. Then, a convex $\sigma(E', E)$-compact subset K of E' is metrizable with the $\sigma(E', E)$-topology (see [68], page 70). Thus, with the notation of the latter theorem, since $\mathcal{D}_p$ is Fréchet Montel space and $C \cap \omega^{-1}(1)$ is dense in the compact convex set $\widehat{C}_{\mathfrak{F}}$, we have that every element of $\widehat{C}_{\mathfrak{F}}$ is a weakly limit of members of $C \cap \omega^{-1}(1)$. On the other hand, by the demonstration of 1.6.33, a sequence in $\mathcal{D}'_p$ is strongly convergent if, and only if, is weakly convergent. It follows that the elements of $\widehat{C}_{\mathfrak{F}}$ are limits of finite linear combinations with positive coefficients of Dirac foliations currents.

We will now describe Sullivan's characterization of foliations cycles in terms of holonomy invariant measures.

Definition 3.2.5. Let Γ be a pseudogroup of homeomorphisms of a space X and μ be a nonnegative, finite, σ-additive measure, defined on the ring of the subsets of X generated by compact sets. We will say that μ is Γ-*invariant* provided for every $\gamma \in \Gamma$ and every mensurable set A in the domain of γ we have $\mu(\gamma(A)) = \mu(A)$.

Definition 3.2.6. A foliation $\mathfrak{F}$ on a manifold M is said to have a holonomy invariant measure (or transversal measure) if their holonomy pseudogroup admits an invariant measure.

Let $\mathfrak{F}$ be an orientable foliation of a closed manifold M. Let $\mathcal{U}$ be a regular foliated atlas for M and λ_α be a partition of the unity subordinate to it. Given $\omega \in \mathcal{D}_p$, for each plaque $P_y \in S_\alpha$ we can integrate $\lambda_\alpha\omega$ in P_y getting a continuous real function on S_α. Integrating on S_α concerning to a holonomy invariant measure μ and adding, we obtain

$$\phi_\mu(\omega) = \sum_\alpha \int_{S_\alpha} \left(\int_{P_y} \lambda_\alpha \omega \right) d\mu(y).$$

By Stokes' theorem,

$$\partial\phi_\mu(\omega) = \phi_\mu(d\omega) = \sum_\alpha \int_{S_\alpha} \left(\int_{P_y} d(\lambda_\alpha \omega_\alpha) \right) d\mu(y) = 0.$$

Therefore, ϕ_μ is a cycle. Indeed, it is a foliation current and independent of the partition of unity chosen, see [69] for details.

Theorem 3.2.7 (D. Sullivan, [80])**.** Every foliation cycle has the form ϕ_μ for some holonomy invariant measure μ.

3.2.1 Forms admitting complementary forms

We will now show that any transitive p-form of rank p admits complementary closed forms.

Definition 3.2.8 (Plante, [66])**.** Let μ be a $\mathfrak{F}$-invariant measure. Its support (denoted suppμ) is the set of points $x \in M$ with the following property: if Σ is a submanifold transverse to $\mathfrak{F}$ with dimension equal to the codimension of $\mathfrak{F}$ and that contains x as an interior point, then $\mu(\Sigma) > 0$. The support of an $\mathfrak{F}$-invariant measure is closed and $\mathfrak{F}$-saturated, that is, it is an union of leaves of $\mathfrak{F}$.

Theorem 3.2.9 (Plante [66])**.** Let $\mathfrak{F}$ be an orientable codimension k foliation. If a closed submanifold Σ transverse to $\mathfrak{F}$ intersepts the support of a $\mathfrak{F}$-invariant measure μ, then Σ represents a nonzero element in $H_k(M;\mathbb{R})$ and $[\phi_\mu] \neq 0$ in $H^{n-p}(M;\mathbb{R})$.

This proposition is proved building a closed form η, Poincaré dual to Σ, such that $\phi_\mu(\eta) = \mu(\Sigma) > 0$.

Theorem 3.2.10. Let ω be a transitive closed p-form of rank p on a closed manifold M. Then ω admits a closed transversal form.

Proof. Since ω is transitive, for each point $x \in M$ there exists a closed submanifold Σ containing x such that $\omega|_\Sigma > 0$. It follows that for each holonomy $\mathfrak{F}_\omega$-invariant measure μ, there exists a submanifold Σ transverse to $\mathfrak{F}_\omega$ and contained in support of μ. The Lemma 3.2.3 and theorems 3.2.7 and 3.2.9 together ensure the existence of a closed form η transversal to $\mathfrak{F}_\omega$. That is, satisfying $\omega \wedge \eta > 0$. □

The latter theorem and the existence of concrete examples of transitive non intrinsically harmonic p-forms of rank p show that the condition of admitting a complementary form, evidently a necessary condition for a form to be intrinsically harmonic, is not sufficient. In the intermediate degrees, considering a p-form of rank p, we cannot ensure that $\omega \wedge \eta > 0$ implies that η has rank equal to $(n - p)$.

Corollary 3.2.11. Let M be a closed 4-dimensional manifold with a transitive 2-form ω of rank 2. Then M admits a structure of the symplectic manifold.

Proof. By 3.2.10, there exist a closed 2-form η such that $\omega \wedge \eta > 0$. Since M is compact, the differential forms $\omega_k = \omega + k\eta$ have null kernel for k sufficiently large. □

3.3 Foliations induced by harmonic p-forms of rank p

Denote by $T\mathfrak{F}$ the subbundle of the TM formed of all vectors tangent to $\mathfrak{F}$.

Lemma 3.3.1. Let ω be a closed p-form of rank p on an orientable manifold M. Then $\mathfrak{F}_\omega$ is orientable.

Proof. It follows by 1.5.8. □

Let (M, g) be a Riemannian manifold and $\mathfrak{F}$ be an orientable codimension p foliation of M. A differential form on M is said to be a *characteristic form* for $\mathfrak{F}$ provided it induces the Riemannian volume on the leaves of $\mathfrak{F}$. The characteristic form for a foliation $\mathfrak{F}$ is denoted by $\chi_{\mathfrak{F}}$. Another description for this form is given as follows. Fixe for each point $x \in M$ a positive orthonormal frame $\{\xi_1, \dots, \xi_p\}$ for $T_x\mathfrak{F}$. Then,

$$\chi_{\mathfrak{F}}(\eta_1, \dots, \eta_p) = \det(g(\xi_i, \eta_j)),$$

$\eta_1 \dots, \eta_p \in TM$.

Lemma 3.3.2. Let ω be a harmonic p-form of the rank p on a Riemannian manifold (M, g). Let $g = g_1 \oplus g_2$ the decomposition of g in the ortogonal sum $TM = \ker \omega \oplus \ker *_g\omega$. Then there exist strictly positive smooth functions $f, k : M \longrightarrow \mathbb{R}$ such that ω is characteristic form for $\mathfrak{F}_{*_g\omega}$ and $*_g\omega$ is characteristic form for $\mathfrak{F}_\omega$ concerning to the Riemannian metric $h = f^2 g_1 \oplus k^2 g_2$.

Proof. Let θ be a smooth section of $\Lambda^p(T^*\mathfrak{F}_\omega)$ such that restrict to each leaf of $\mathfrak{F}_\omega$ it give the volume induced by g. Since $*\omega$ is positive in the leaves of $\mathfrak{F}_{*\omega}$, there exist strictly postive function $k : M \longrightarrow \mathbb{R}$ such that $k^p\theta = *\omega|_{\Lambda^p(T(\mathfrak{F}))}$. Analogously, let $f : M \longrightarrow \mathbb{R}$

be a smooth such that $f^{n-p}\kappa = *\omega|_{\Lambda^p(T(\mathfrak{F}))}$, where κ is the volume form induced by g in $\Lambda^p(T^*\mathfrak{F}_{*\omega})$. The required metric is given by

$$h = f^2 g_1 \oplus k^2 g_2.$$

$\square$

Definition 3.3.3. A form η is said to be *relatively closed* concerning to a foliation $\mathfrak{F}$ provided

$$d\eta(X_1, \ldots, X_p, Y) = 0$$

whenever $X_1, \ldots, X_p$ are tangent to $\mathfrak{F}$.

Theorem 3.3.4 (H. Rummler, [70])**.** Let $\mathfrak{F}$ be a foliation of a manifold M. The characteristic form $\chi_{\mathfrak{F}}$ of $\mathfrak{F}$ is relatively closed (concerning to $\mathfrak{F}$) if, and only if, each leaf of $\mathfrak{F}$ is a minimal submanifold[1].

We then have the following structure on a manifold supporting a harmonic p-form of rank p.

Theorem 3.3.5. Let (M, g) be a Riemannian manifold. Suppose that ω is a g-harmonic p-form of rank p on M. Then there exist a Riemannian metric h on M such that the tangent space of M split orthogonally as $TM = T\mathfrak{F}_\omega \oplus T\mathfrak{F}_{*\omega}$. Furthemore, $\chi_{\mathfrak{F}} = *_g\omega$, $\chi_{\mathfrak{F}_{*_g\omega}} = \omega$ and all leaves of the foliations $\mathfrak{F}_\omega$ and $\mathfrak{F}_{*\omega}$ are minimal submanifolds (concerning to h).

Proof. It follows by 3.3.2 and 3.3.4. $\square$

Definition 3.3.6 ([83], page 49)**.** Let G be a Lie group. A foliation $\mathfrak{F}$ is said to be a *$G-$foliation* if

$$(\gamma_{\alpha\beta})_* : U_\alpha \cap U_\beta \longrightarrow G \subset GL(q, \mathbb{R})$$

for every generator of the $\Gamma_{\mathcal{U}}$, the holonomy pseudgroup of $\mathfrak{F}$ concerning to the cover $\mathcal{U}$.

Theorem 3.3.7 ([83], theorem 6.32)**.** Let $\mathfrak{F}$ be a transversally orientable foliation defined on a Riemannian manifold (M, g). Then $\mathfrak{F}$ is $SL(q)$-foliation if and only if $d\chi_{\mathfrak{F}} = 0$.

Therefore, if ω is harmonic $p-$form of rank p, then $\mathfrak{F}_\omega$ is a $SL(q)$-foliation and $\mathfrak{F}_{*\omega}$ is a $SL(p)$-foliation.

Corollary 3.3.8 (Local structure of harmonic p-form of rank p)**.** Let (M, g) be a Riemanninan manifold and ω be a harmonic p-form of rank p on M. Then there exists an atlas $\mathcal{U} = \{U_\alpha, x_\alpha, y_\alpha\}$ for M such that the equations $x =$ *constant* and $y =$ *constant* determine locally $\mathfrak{F}_{*\omega}$ and $\mathfrak{F}_\omega$, respectively. Furthemore, $(\gamma_{\alpha\beta})_* \in SL(q)$ and $(\theta_{\alpha\beta})_* \in SL(p)$. Besides this, the rules $\eta = dy^1_\alpha \wedge \ldots \wedge dy^p_\alpha$ and $\xi = dx^1_\alpha \wedge \ldots \wedge dx^q_\alpha$ fit together in a well-defined closed form which determines globally $\mathfrak{F}_\omega$ and $\mathfrak{F}_{*\omega}$, respectively.

[1] See [31] for a short proof of this theorem.

Proof. If $\mathcal{V} = \{U_\alpha, x_\alpha, u_\alpha\}$ and $\mathcal{W} = \{U_\alpha, y_\alpha, v_\alpha\}$ are foliated atlas for $\mathfrak{F}_\omega$ and $\mathfrak{F}_{*\omega}$, respectively, then $\mathcal{U}_\alpha = \{U_\alpha, x_\alpha, y_\alpha\}$ is the required atlas. By the previous discussion, we have $(\gamma_{\alpha\beta})_* \in SL(q)$ and $(\theta_{\alpha\beta})_* \in SL(p)$. Therefore

$$dy_\alpha^1 \wedge \ldots \wedge dy_\alpha^p = \det(\gamma_{\beta\alpha})_* dy_\beta^1 \wedge \ldots \wedge dy_\beta^p = dy_\beta^1 \wedge \ldots \wedge dy_\beta^p.$$

Analogously,

$$dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^p = \det(\theta_{\beta\alpha})_* dx_\beta^1 \wedge \ldots \wedge dx_\beta^p = dx_\beta^1 \wedge \ldots \wedge dx_\beta^p,$$

concluding the corollary. □

The next theorem follows from the sequence given in [83] to prove Theorem 3.3.7.

Theorem 3.3.9. Let ω be a closed p-form of rank p on a manifold M. Suppose there exists a Riemannian metric g on M such that every leaf of the foliation induced by ω is a minimal submanifold. If $(\ker \omega)^\perp$ is an integrable distribution, then ω is intrinsically harmonic.

Proof. Throughout the proof, $\mathfrak{F}$ denotes the foliation induced by ω, η denotes $\chi_{\mathfrak{F}}$ and $\mathfrak{F}^\perp$ denotes $(\ker \omega)^\perp$. We claim that η is a closed form. The proof of this will be divided into four steps.

(1) Let $Y \in \mathfrak{F}^\perp$. We have $i_Y\eta = 0$. By Theorem 3.3.4, $(i_Y d\eta)|_{\mathfrak{F}} = 0$. Then,

$$(\mathcal{L}_Y\eta)|_{\mathfrak{F}} = (di_Y\eta + i_Y d\eta)|_{\mathfrak{F}} = 0.$$

(2) Let $Z \in \mathfrak{F}^\perp$ and $X, Y \in TM$. Using the rule $[\mathcal{L}_X, i_Y]\alpha = i_{[X,Y]}\alpha$, we have

$$i_Z(\mathcal{L}_Y\eta) = \mathcal{L}_Y i_Z\eta - i_{[Y,Z]}\eta = -i_{[Y,Z]}\eta.$$

By the involutivity of $\mathfrak{F}^\perp$ then $[Y, Z] \in \mathfrak{F}^\perp$. It follows that $i_Z(\mathcal{L}_Y\eta) = 0$. Since Z was arbitrarily chosen, then $(\mathcal{L}_Y\eta)|_{\mathfrak{F}^\perp} = 0$.

(3) By 1 and 2, given any $Y \in \mathfrak{F}^\perp$, we have $(\mathcal{L}_Y\eta)|_{\mathfrak{F}} = 0$ and $(\mathcal{L}_Y\eta)|_{\mathfrak{F}^\perp} = 0$. Therefore

$$i_Y d\eta = \mathcal{L}_Y\eta - di_Y\eta = 0$$

for all $Y \in \mathfrak{F}^\perp$.

(4) We will now prove that $i_X d\eta = 0$ for all X tangent to $\mathfrak{F}$. By item 3, $i_Y d\eta = 0$ for all Y tangent to $\mathfrak{F}^\perp$. It follows that $i_X i_Y d\eta = -i_Y i_X d\eta = 0$. Hence $i_X d\eta$ is determined completely by its values in the tangent vectors to $\mathfrak{F}$. But, since η is $\mathfrak{F}$-closed, then $i_X d\eta|_{\mathfrak{F}} = 0$. Thereofre $i_X d\eta = 0$ for all X tangent to $\mathfrak{F}$. Since $TM = T\mathfrak{F} \oplus T\mathfrak{F}^\perp$, $i_X d\eta = 0$ for all X tangent to $\mathfrak{F}$ and $i_Y d\eta = 0$ for all Y tangent to $\mathfrak{F}^\perp$, we conclude that η is closed.

To finish the proof of the theorem, note that η have rank q (since it is the characteristic form of $\mathfrak{F}$), it is complementary to ω and closed as have shown above. It follows by 2.1.2 that ω is intrinsically harmonic. $\square$

We show that in any smooth fiber bundle $\mathcal{B} = \{B, p, M, F\}$ with $\chi(F) \neq 0$, the form $\pi^*(\Omega_M)$ determines a nontrivial cohomological class (see Lemma 5.2.7). Thus, it is suspicious that we can weaken theorem 2.1.2, assuming that $\mathfrak{F}_\omega$ admits only a complementary foliation. If this theorem holds merely with this hypothesis, then any foliated bundle with $[\pi^*(\Omega_M)] \neq 0$ admits a $SL(q)$-transversal foliation by the results presented in this section so far. We end this section by providing a class of examples of transitive intrinsically harmonic forms.

Theorem 3.3.10. Let $\mathcal{B} = \{B, p, M, F, \mathfrak{F}\}$ be a foliated bundle with total space and base base being oriented manifolds. If the fundamental group of M is finite, then $\pi^*\Omega$ is a transitive intrinsically harmonic form for all volume form Ω on M.

Proof. First fo all, we observe the following general fact. Let G be a finite subgroup of $\mathrm{Diff}(F)$. Since for any element $g \in G$ we have $g^{|G|} = 1$, then $\det(g_*) = \pm 1$.

As in the proof of Theorem 1.5.10, we can obtain a differential form that restricts to each fiber give us a volume form. It follows by 1.5.8 that the foliation given by the fibers of $\mathcal{B}$ is orientable. Since $\mathfrak{F}$ is transversal to the fibers, then $\mathfrak{F}$ is transversely orientable. Hence, we can take the holonomy homomorphism

$$\varphi : \pi_1(M) \to \mathrm{Diff}(F)$$

with image in $\mathrm{Diff}^+(F)$. By the preceding paragraph, since $\pi_1(M)$ is finite, we have

$$\det(\varphi(\alpha))_*) = 1$$

for all $\alpha \in \pi_1(M)$. Thus, the foliation $\mathfrak{F}$ is a $SL(q)$-foliation (the demonstration of the first assertion in the theorem could stop here). Let $\mathcal{U} = \{U_\alpha, x_\alpha, u_\alpha\}$ be a foliated atlas for B concerning to $\mathfrak{F}$ such that $\det \gamma = 1$ for all $\gamma \in \Gamma_{\mathcal{U}}$. Since $du_\alpha = \det(\gamma_{\beta\alpha})_* du_\beta$ and $\mathfrak{F}$ is a $SL(q)$-foliation, the rule $\eta_\alpha = du_\alpha$ fit together in a well-defined closed form η which determines the foliation $\mathfrak{F}$ globally . It follows by 2.1.2 that $\pi^*\Omega_M$ is intrinsically harmonic. Now, each leaf of $\mathfrak{F}$ is compact, since that cover the compact manifold M with finite sheets. Hence, each leaf of $\mathfrak{F}$ is a global cross-section for the fibers. From this, we conclude that $\pi^*(\Omega_M)$ is transitive. $\square$

3.4 The transitivity condition

Let M be a manifold and $x \in M$. Denote by $\mathcal{T}_p^x$ the subset of $\mathcal{D}_p$ consinting of all p-forms transitive at x. Let $\omega \in \mathcal{T}_p^x$ and (ω_n) be a sequence converging to ω. Let Σ be a closed embendding submanifold containing x such that $\omega|_\Sigma$ is a volume form. Fixed a Riemannian metric on M, let

$$S = \{v \in \Lambda^p(T\Sigma); \| v \| = 1\}.$$

Suppose for each $n \in \mathbb{N}$ there exists $v_{x_n} \in S$ such that $\omega_n(v_{x_n}) = 0$. Since Σ is compact, we can suppose that $x_n \longrightarrow y \in \Sigma$ and $v_{x_n} \longrightarrow v_y$, for some $v_y \in S$. Thus, for each p-form $\eta \in \mathcal{D}_p$, $\eta(v_{x_n}) \longrightarrow \eta(v_y)$. It follows that $v_{x_n} \longrightarrow v_y$ in $\mathcal{D}_p'$. Since the evaluation map $e : \mathcal{D}_p \times \mathcal{D}_p' \longrightarrow \mathbb{R}$ is sequentially continuous by 1.6.35, then $\omega(v_x) = \lim \omega_n(v_{x_n}) = \lim(e(\omega_n, v_{x_n})) = 0$, a contradiction. Thus, for sufficiently large n, we have $\omega_n \in \mathcal{T}_x^p$. From the fact of $\mathcal{D}_p$ is a metrizable space, we conclude that $\mathcal{T}_x^p$ is an open set[2]. Note we have proved the subset of all differential forms that are volume in Σ is an open set.

Theorem 3.4.1. Let $\mathfrak{F}$ be a smooth foliation of a manifold M. The set T_Σ of all points in M contained in a closed submanifold transverse to $\mathfrak{F}$ and diffeomorphic to Σ is $\mathfrak{F}$-saturated and open.

Proof's sketch. Suppose there exists a submanifold containing x, diffeomorphic to Σ, and transverse to $\mathfrak{F}$. By deformation in the leaves direction, we conclude that the leaf containing x is contained in T_Σ (the proof of this is analogous to the given in J.W. Milnor [54] to show the Homogeneity Lemma, page 22). Working in a foliated chart around x, it is easily seen that T_Σ contains a neighborhood of x. Since x was arbitrarily chosen, the proof is completed. □

Corollary 3.4.2. Let ω be a closed p-form of rank p on a manifold M. Suppose that there exists a closed p-submanifold of M transversal to every leaf of $\mathfrak{F}_\omega$. Then ω is transitive.

In some situations, we can prove that the transitivity condition is *homogeneous*, in the sense that transitivity at one point implies transitivity at any point. We will highlight some examples.

Theorem 3.4.3. Let M be a closed manifold and $\mathfrak{F}$ be a foliation of M such that $M/\mathfrak{F}$ satisfy the Hausdorff property. If Σ is a closed submanifold intercepting $\mathfrak{F}$ transversally, then Σ intercept every leaf of $\mathfrak{F}$.

Proof. Denote by $\pi : M \longrightarrow M/\mathfrak{F}$ the natural projection. It is easily seen that $\pi(\Sigma)$ is an open set since Σ is an embedding submanifold without boundary. Since $\pi(\Sigma)$ is compact and $M/\mathfrak{F}$ Hausdorff, then it is closed. Thus, $\pi(\Sigma)$ is both open and closed in the connected space $M/\mathfrak{F}$. Therefore, $M/\mathfrak{F} = \pi(\Sigma)$, concluding the theorem. □

[2] Let X be a metric space. Let U be a subset of X satisfying the following condition: given $x \in U$ and a sequence $x_n \longrightarrow x$, then there exists $n \in \mathbb{N}$ such that $x_k \in U$ for all $k \geqslant n$. Under this condition, if (x_n) is a sequence in $X - U$ converging to x, then $x \notin U$. Thus, $X - U$ is a sequentially closed subset of X, hence closed since X is metric.

Definition 3.4.4 (J.F. Plante, [64])**.** A *polisection* of a fiber bundle $\mathcal{B} = \{B, p, X, Y, G\}$ is a covering map $\pi : \tilde{X} \longrightarrow X$ with a continuous function $f : \tilde{M} \longrightarrow B$ such that $p \circ f = \pi$.

Theorem 3.4.5. Let $\mathcal{B} = \{B, p, M, Y\}$ be a smooth fiber bundle with closed total space and orientable base space. Let Ω be a volume form on M. If $p^*\Omega$ is transitive at one point, then B admits a compact polisection. In particular, $p^*\Omega$ is transitive.

Proof. Let $\mathfrak{F}$ the foliation given by the fibers of $\mathcal{B}$. By hypothesis, there exists a closed submanifold Σ of B intercepting $\mathfrak{F}$ transversally. Since $E/\mathfrak{F} = M$, then Σ intercept every fiber of $\mathcal{B}$ (every leaf of $\mathfrak{F}$). By transversality, it is easy to check that $p|\Sigma$ is a local diffeomorphism. Since the fibers of $\mathcal{B}$ are compact, then $p|_\Sigma$ is a proper map, hence a covering projection (see [35]). $\square$

Definition 3.4.6. An automorphism of a foliated manifold $(M, \mathfrak{F})$ is a diffeomorphism $\phi : M \longrightarrow M$ preserving the foliation. That is, the image of any leaf by ϕ is a leaf of $\mathfrak{F}$ too. A foliation is said to be *homogeneous* provided the group of automorphisms of $(M, \mathfrak{F})$ acts transitively on M.

If ω is a closed p-form of rank p and $\mathfrak{F}_\omega$ is homogeneous, then the transitivity condition for ω at one point implies the transitivity of ω. For example, if ω is a *closed-decomposable* p-form, that is, a form described globally by

$$\omega_1 \wedge \ldots \wedge \omega_p,$$

where each ω_i is closed nowhere-vanishing 1-form, then $\mathfrak{F}_\omega$ is homogenous (see [57], chapter 4). We do not know if a harmonic form is transitive in general. G. Katz proposed this problem in [39].

4 Asymptotic cycles on smooth manifolds

In Chapter 2, we proved a characterization of nowhere-vanishing intrinsically harmonic $(n-1)$-forms on closed manifolds. We have shown this in terms of the existence of a global cross-section for a canonical associated volume-preserving flow, Theorem 2.4.3. The first criterion in the literature for a flow admits a global cross-section is due to S. Schwartzmann [74]. He introduced the notion of the asymptotic cycle that is a 1-dimensional real homological class (in the sense of De Rham for the smooth context) associated with an invariant measure. Global cross-sections are determined by an integral one-dimensional cohomology class that is positive on all these asymptotic cycles. Since Schwartzmann's results are for flows defined on compact metric spaces, the notion of a global cross-section is slightly different, and a translation for the differentiable context was necessary. In this chapter, we present the theory developed by S. Schwartzmann. Are discussed two criteria for a flow admits a global cross-section. One of them appears rigorously proved in [74]. For the other case, we provide a detailed demonstration.

4.1 Asymptotic cycles

Let M be a compact manifold and $C = C^{\infty}(M, \mathbb{S}^1)$ the set of all smooth functions $f : M \longrightarrow \mathbb{S}^1$ with group structure given by $(f \cdot g)(x) = f(x) \cdot g(x)$. Set R the subgroup of C

$$\{f \in C; f(x) = \exp(2\pi i H(x)), H : M \longrightarrow \mathbb{R} \text{ smooth}\}.$$

Every element $f \in C$ is given locally by $f(x) = \exp 2\pi i H(x)$, where $H(x)$ is some smooth real function determined up to an additive constant. If H_1 and H_2 are determined as above in two different coordinate systems, dH_1 and dH_2 agree in the region of overlapping. Then for each $f \in C$ there is a closed 1-form ω_f given locally by dH. Consider the local parametrization of the circle given by $\theta(\exp(2\pi i t)) = t$ and the usual volume form on $\mathbb{S}^1$ described locally by $d\theta$. Let $\alpha : I \longrightarrow M$ be a smooth curve with $X(x) = \partial_t|_0 \alpha(t)$. If $f = \exp 2\pi i H(x)$ in a neighborhood of $\alpha(0)$, then

$$\begin{aligned} f^*d\theta(X) &= d\theta(f_*X) \\ &= d\theta(\partial_t|_0 f(\alpha(t))) \\ &= \partial_t|_0 \theta(f(\alpha(t))) \\ &= \partial_t|_0 \theta(\exp(2\pi i H(\alpha(t)))) \\ &= \partial_t|_0 H(\alpha(t)) \\ &= dH(X) \\ &= \omega_f(X) \end{aligned}$$

concluding that $\omega_f = f^*d\theta$. The association $f \longrightarrow \omega_f$ give us a natural homomorphism

$$[f] \in C/R \longrightarrow [\omega_f] \in H^1_{DR}(M;\mathbb{Z}).$$

By Lemma 1.8.1, this homomorphism is surjective. Furthemore, if $fg^{-1}(x) = \exp(2\pi i H(x))$ everywhere, then it is easily seen that $\omega_f - \omega_g = dH$. Therefore, the forms ω_f and ω_g determine the same cohomological class. We synthesize these facts in the next theorem.

Theorem 4.1.1. The rule that associates each $[f] \in C/R$ to the closed 1-form $[\omega_f] = [f^*d\theta] \in H^1_{DR}(M;\mathbb{Z})$ is an isomorphism.

Proof. It remains only to show that the association $[f] \longrightarrow [\omega_f]$ is a homomorphism. Let $f, g \in C$. Let U be an open subset of M where f and g have local representaion $f = \exp(2\pi i H)$ and $g = \exp(2\pi i L)$. In the open set U we have

$$\omega_{f\cdot g} = d(H + L) = \omega_f + \omega_g.$$

This concludes the theorem. $\square$

Notation 4.1.2. Let K be a parametrized curve in M and $f \in C$. We will use the symbol $\Delta_K \arg f(x)$ to denote the change in the angular variable of f along K.

Remark 4.1.3 (A geometric interpretation of C/R)**.** Let K be a smooth closed curve in M. The number $\frac{1}{2\pi}\Delta_K \arg f(x)$ is an integer depending only on the element of the first homology group corresponding to K and the coset determined by f in C/R. Furthemore, setting $\delta_f([K]) = \frac{1}{2\pi}\Delta_K \arg f(x)$, the function

$$\delta : C/R \longrightarrow H^1(M;\mathbb{Z})$$

defined by $\delta([f]) = \delta_f$ is an isomorphism. The proof of this claim will be divided into four steps.

(1) Fixe $[f] \in C/R$. We claim that $\delta_f : H^1(M;\mathbb{Z}) \longrightarrow \mathbb{R}$ is a $\mathbb{Z}$-linear function. Firstly, given $K, L : [0,1] \longrightarrow M$ be smooth closed curves with the same base point, then

$$[K] + [L] = [K \cdot L] \text{ in } H_1(M, \mathbb{Z}),$$

where

$$K \cdot L(t) = \begin{cases} K(2t), & \text{if } 0 \leqslant t \leqslant 1/2 \\ L(2t-1), & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

(see [32], page 166). From this, it is easy to check that

$$\delta_f([K] + [L]) = \delta_f([K]) + \delta_f([L]),$$

concluding the claim.

(2) Let $f \in C$ and $K : [0,T] \longrightarrow M$ be a smooth path. If $f(x) = \exp 2\pi i H(x)$ for some smooth function $H : M \longrightarrow \mathbb{R}$, then

$$\frac{1}{2\pi}\Delta_K \arg f(x) = H(x_2) - H(x_1),$$

where p_1 and p_2 are the initial and terminal point of K. Thus, using a suitable partition of $[0,T]$, it follows that

$$\Delta_K \arg(f \cdot g)(x) = \Delta_K \arg f(x) + \Delta_K \arg g(x).$$

Hence $\delta(f \cdot g) = \delta_f + \delta g$, concluding that δ is a homomorphism.

(3) We will now show that $\delta_f([K]) \in \mathbb{Z}$. Let $K : [0,T] \longrightarrow M$ be a smooth closed curve and $0 = t_0 < \ldots < t_m = T$ be a partition of $[0,T]$ such that $f(K(t)) = \exp(2\pi i H_i(K(t)))$ when $t_i \leqslant t < t_{i+1}$. Then,

$$\begin{aligned} \frac{1}{2\pi}\Delta_K \arg f &= \sum (H_i(K(t_{i+1})) - H_i(K(t_i))) \\ &= \sum \int_{t_i}^{t_{i+1}} dH_i(\partial_t(K(t))) \\ &= \sum \int_{t_i}^{t_{i+1}} \omega_f(\partial_t(K(t))) \\ &= \int_{[0,T]} K^*\omega_f \\ &= \int_{[0,T]} K^* f^* d\theta. \end{aligned}$$

Consider the natural projection $\pi : [0,1] \longrightarrow \mathbb{S}^1$ where 0 is identified with 1. Since $K(0) = K(T)$, this curve induces a smooth function $\tilde{K} : \mathbb{S}^1 \longrightarrow M$ such that

$\tilde{K} \circ \pi = K$. It follows that

$$\begin{aligned}
\frac{1}{2\pi}\Delta_K \arg f &= \int_{[0,T]} \pi^* \tilde{K}^* f^* d\theta \\
&= \int_{[0,T]} \pi^*((f \circ \tilde{K})^* d\theta) \\
&= \int_{\pi[0,T]} (f \circ \tilde{K})^* d\theta \\
&= \int_{\mathbb{S}^1} (f \circ \tilde{K})^* d\theta \\
&= \text{degree}(f \circ \tilde{K}) \in \mathbb{Z}
\end{aligned}$$

(for the last equality see 1.8.1).

(4) From the previous items it follows that $\delta : C/R \longrightarrow H^1(M;\mathbb{Z})$ is a homomorphism. It remains only to show that δ is a bijective function. It is sufficient to show that δ injective, since C/R and $H^1(M;\mathbb{Z})$ are free modules with the same dimension equal the Betti number of M (see the footnote in Lemma 1.8.1). Suppose that $\delta_f = 0$. With the same notation as in item (2), it follows that degree$(f \circ \tilde{K}) = 0$ for all closed curve K. Since

$$\text{degree}(f \circ \tilde{K}) = \int_{\tilde{K}(\mathbb{S}^1)} f^* d\theta,$$

then $f^* d\theta$ represent the zero class. From 4.1.1 we conclude that $[f] = 0$.

Notation 4.1.4. Let X be a vector field on M. We will denote by Φ_X (or Φ for short) the flow generated by X and $(x,t) \longrightarrow xt$ the $\mathbb{R}$-action induced by it (this flow is complete since M is compact).

Definition 4.1.5. A measure μ defined in the σ-algebra generated by all Borel subsets of M is called *invariant* provided that is normalized ($\mu(M) = 1$) and for every mensurable set S and every real number t we have $\mu(St) = \mu(S)$.

Definition 4.1.6. A point $x \in M$ is called *quasi-regular* provided that

$$\lim_{T \longrightarrow \infty} 1/T \int_0^T f(xt)dt$$

exists for every real-valued continuous function defined on M.

Theorem 4.1.7 (Oxtoby, [62])**.** For every quasi-regular point x, there exist a unique normalized invariant positive Borel measure μ_x such that

$$\lim_{T \longrightarrow \infty} 1/T \int_0^T f(xt)dt = \int_M f(p)d\mu_x(p).$$

Proof. Let $C^0(M)$ the topological vector space of all continuous real valued functions defined on M with usual norm $\| f \| = \sum_{x \in M} |f(x)|$. For each $n \in \mathbb{N}$, set I_n the linear functional on $C^0(M)$ by the rule

$$I_n(f) = 1/n \int_0^n f(xt)dt.$$

We have

$$\| I_n \| = \sup_{\|f\|=1} |I_n(f)| = 1.$$

Thus, by Banach-Alaoglu-Bourbaki Theorem, the sequence (I_n) admits a weakly convergent subsequence. Suppose that $I_{n_k} \longrightarrow I$ weakly. Since the linear functional I is nonnegative, the Riesz Representation Theorem ensures that there exists a unique nonnegative Borel measure μ_x which represent I in sense that $I(f) = \int_M f(p)d\mu_x(p)$ for every $f \in C^0(M)$. Then

$$\begin{aligned} \lim_{T \longrightarrow \infty} 1/T \int_0^T f(xt)dt &= \\ \lim_{n_k \longrightarrow \infty} 1/n_k \int_0^{n_k} f(xt)dt &= \\ \int_M f(p)d\mu_x(p). \end{aligned}$$

It remains only to show that μ_x is normalized and invariant. It is normalized:

$$\mu_x(M) = \int_M d\mu_x(p) = \lim_{T \longrightarrow \infty} 1/T \int_0^T dt = 1.$$

To conclude that μ_x is invariant, it is enough to show that $\int_M (f(pt) - f(p))d\mu_x(p) = 0$ for every $f \in C^0(M)$. Denote by R_t the function $p \longrightarrow pt$. We have

$$\begin{aligned} \int_M f(pt)d\mu_x(p) = \int_M fR_t(p)d\mu_x(p) &= \\ \lim_{T \longrightarrow \infty} 1/T \int_0^T fR_t(xs)ds &= \\ = \lim_{T \longrightarrow \infty} 1/T \int_0^T f(x(t+s))ds. \end{aligned}$$

Using the change $u = t + s$,

$$\begin{aligned} \lim_{T \longrightarrow \infty} 1/T \int_t^{T+t} f(xu)du &= \\ = \lim_{T \longrightarrow \infty} 1/T \left(\int_0^T f(xu)du - \int_0^t f(xu)du + \int_T^{T+t} f(xu)du \right) &= \\ = \int_M f(p)d\mu_x(p) + \lim_{T \longrightarrow \infty} 1/T \int_T^{T+t} f(xu)du. \end{aligned}$$

Thus,

$$\int_M (f(pt) - f(p))d\mu_x(p) = \lim_{T \longrightarrow \infty} 1/T \int_T^{T+t} f(xu)du.$$

But

$$\lim_{T \longrightarrow \infty} 1/T \int_T^{T+t} f(xu)du \leqslant \lim_{T \longrightarrow \infty} 1/T \int_T^{T+t} \sup_{p \in M} |f(p)|du = \lim_{T \longrightarrow \infty} 1/T \sup_{p \in M} |f(p)|t = 0.$$

The proof is completed. □

Theorem 4.1.8 (Oxtoby, [62])**.** The set of all not quasi-regular points has zero measure concerning any invariant measure.

Notation 4.1.9. The change in the angular variable of f along the orbit going from x to xt will be denoted by $\Delta_{(x,xt)}\arg f$.

Let $f : M \longrightarrow \mathbb{S}^1$ be a smooth function and α be a smooth curve on M with $\alpha(0) = x$. We have that

$$f_*(\partial_t|_0 \alpha(t)) = \partial_t|_0 f(\alpha(t))$$

can be interpreted as a complex number.

Notation 4.1.10. The symbol $f'(x)$ will denote the derivative of f in the $X(x)$-direction, that is,

$$f'(x) = \partial_t|_0 f(xt) = f_*(X(x)). \tag{4.1}$$

Note that if $f(x) = \exp 2\pi i H(x)$, then $f'(x) = 2\pi i H_*(\partial_t|_0(xt))f(x)$ hence $f'(x)/(2\pi i f(x)) = \partial_t|_0 H(xt) = \omega_f(X(x))$.

Theorem 4.1.11. Let x be a quasi-regular point. For any $f \in C$ the limit

$$\lim_{T \longrightarrow \infty} \Delta_{(x,xT)}\arg f$$

exists and depends only on the class determined by f. The induced mapping of C/R into the real line is a group homomorphism.

Proof. It follows analogously as in 4.1.3. □

Thus, we have associated each quasi-regular point x a homomorphism from C/R to $\mathbb{R}$. By the Universal Coefficient Theorem, any member of the real cohomology is expressible as a finite linear combination of members in the integral cohomology. Since C/R may be identified with the elements of the first cohomology group with integral periods, then we can extend this homomorphism uniquely to a linear functional A_x on the first cohomology group with real coefficients. To make it clear, we will write the definition of A_x.

Definition 4.1.12. Let x be a quasi-regular point. The *asymptotic cycle* associated with x is defined as being the linear extension of $A_x : H^1(M;\mathbb{Z}) \longrightarrow \mathbb{R}$ to $H^1(M;\mathbb{R})$, where A_x is given by

$$A_x[f] = \lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\int_0^T f^{'}(xt)/f(xt).$$

Theorem 4.1.13. Let μ be an invariant measure for a flow Φ. For each $f \in C$

$$A_\mu([f]) = \int_M A_x[f]d\mu(x) = \frac{1}{2\pi i}\int_M f^{'}(x)/f(x)dt.$$

Proof. We have

$$\int_M A_x[f]d\mu(x) = \int_M \left(\lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\int_0^T f^{'}(xt)/f(xt)dt\right) d\mu(x).$$

Since the quantity inside the parenthesis is uniformly bounded and μ is finite measure, this equals

$$\lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\int_M \left(\int_0^T f^{'}(xt)/f(xt)dt\right) d\mu(x)$$

which by the Fubini's theorem equals

$$\lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\int_0^T \left(\int_M f^{'}(xt)/f(xt)d\mu(x)\right) dt$$

wich by the invariance of μ equals

$$\begin{aligned}
\lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\int_0^T \left(\int_M f^{'}(x)/f(x)d\mu(x)\right) dt \\
= \lim_{T\longrightarrow\infty} \frac{1}{2\pi i T}\left(\int_M f^{'}(x)/f(x)d\mu(x)\right) T \\
= \frac{1}{2\pi i}\int_M f^{'}(x)/f(x)d\mu(x).
\end{aligned}$$

□

Thus, to each invariant measure μ we have associated (by linear extension) a linear functional $A_\mu : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$, hence A_μ determine an element in $H_1(M;\mathbb{R})$.

Definition 4.1.14. Given an invariant measure μ, the μ-asymptotic cycle associated with this measure is defined by

$$A_\mu[f] = \frac{1}{2\pi i}\int_M (f^{'}(x)/f(x))d\mu(x).$$

For certain kinds of flows, the μ-asymptotic cycle is independent of μ. S. Schwartzman called those flows *spectrally determinate.*

Definition 4.1.15. Let (X, d) be a metric space. Let $\Phi : X \times \mathbb{R} \longrightarrow X$ be a flow on X (that is, a $\mathbb{R}$-action on X). It is said to be *recurrent* provided there exists a sequence $t_k \longrightarrow \infty$ such that

$$\lim_k \sup_{x \in X} d(x, xt_k) \longrightarrow 0.$$

Let us mention a theorem fundamental for us, which will be used in the next chapters. For detailed proof of it, we refer the reader to Schwartzman's paper.

Theorem 4.1.16 (S. Schwartzmann, [74])**.** A recurrent flow on a compact metric space is spectrally determinate.

Remark 4.1.17. Implicitly we have the following definition of μ-asymptotic cycles. Let M be a closed manifold and X be a smooth vector field on M. The asymptotic cycle associated with an invariant measure μ is the homological class determined by the continuous linear functional $A'_\mu : \mathcal{D}'_1 \longrightarrow \mathbb{R}$ given by

$$A'_\mu(\omega) = \int_M \omega(X) d\mu.$$

Indeed, let $[f] \in C/R$. Since $f'(x)/f(x) = \omega_f(X)$, then

$$A_\mu([f]) = \int_M (f'(x)/f(x)) d\mu = \int_M \omega_f(X) = A'_\mu(\omega_f).$$

On the other hand, a linear functional is determined completely by its action on a basis, and $H^1(M; \mathbb{Z})$ generates $H^1(M; \mathbb{R})$. It follows that $A_\mu = [A'_\mu]$.

4.2 Global cross-section to flows

The main result in Schwartzman's paper is a criterion to a flow admits a global cross-section. In what follows, we discuss two forms of it for the smooth context.

Definition 4.2.1. Let Φ be a smooth flow on a closed manifold M. A closed codimension one submanifold Σ of M is said to be a global cross-section for Φ provided intercept transversally every orbit under Φ.

Let Φ be a smooth flow on a closed manifold M. A regular value θ (or t) of a smooth function $f : M \longrightarrow \mathbb{S}^1$ (or $f : M \longrightarrow \mathbb{R}$) give us a closed codimension one submanifold $f^{-1}(\theta)$ (or $f^{-1}(t)$). When $\ker f_*$ is transversal to Φ and intercepts every orbit under this flow, then the preimage of any regular value of f is a global cross-section for it. In virtue of the theory developed in the previous section, it is natural to study functions f in C. We will start by answering the following question: Let $[f] \in C/R$. What are the conditions to exist a global cross-section of the form $g^{-1}(\theta)$ for some $g \in [f]$?

Let $g \in C$ and $g = \exp(2\pi i H(x))$ be a local representation of it. Let $\alpha : I \longrightarrow M$ be a smooth curve with $\alpha(0) = x$. Then

$$
\begin{aligned}
g_*(\alpha^{'}(0)) &= \partial_t|_0 \exp(2\pi i H(\alpha(t))) \\
&= 2\pi i \partial_t|_0 H(\alpha(t)) g(\alpha(0)) \\
&= 2\pi i g(x) H_*(\alpha^{'}(0)).
\end{aligned}
$$

It follows that g is submersion if and only if for each $x \in M$ there exists a curve starting in x with

$$g_*(\alpha^{'}(0))/(2\pi i g(x)) = H_*(\alpha^{'}(x)) \neq 0.$$

By transversality, this condition is satisfied when

$$g_*(\partial_t|_0(xt)) \neq 0.$$

These observations lead to the next lemma.

Lemma 4.2.2. Let Φ be a smooth flow on a closed manifold M. Let $g \in C$ and suppose that $g^{'}(x)/(2\pi i g(x)) > 0$ for all $x \in M$. Then g is submersion and $g^{-1}(\theta)$ is a global cross-section for Φ for every $\theta \in \mathbb{S}^1$.

Proof. Let $\theta \in \mathbb{S}^1$. By the above discussion, showing that $g^{-1}(\theta)$ intercepts every orbit under the flow, we will have completed the proof (we have proved it in Section 2.4, but we will give straightforward proof for this particular case). Since M is compact, there exists $\epsilon > 0$ such that

$$g^{'}(x)/(2\pi i g(x)) > \epsilon > 0 \text{ for every } x \in M.$$

The expression $g^{'}(x)/(2\pi i f(x))$ is the variation of the argument of g in the orbit of x. Since this variation is a positive number greater than ϵ, and the orbit of x is a convex set, there exist $t \in \mathbb{R}$ such that $f(xt) = \theta$. From this, we conclude that $g^{-1}(\theta)$ is a global cross-section to Φ. □

Let Φ be a smooth flow on M and X the vector field generating Φ. Given $g \in [f]$, there exists a smooth function $H : M \longrightarrow \mathbb{R}$ such that $g = f(x)\exp(2\pi i H(x))$. Calculating the derivative of g in the flow direction, we obtain

$$
\begin{aligned}
g^{'}(x) &= f^{'}(x)\exp(2\pi i H(x)) + 2\pi i dH(X(x))g(x) \\
&= \exp(2\pi i H(x))(f^{'}(x) + 2\pi i dH(X(x)))
\end{aligned}
$$

hence

$$g^{'}(x)/(2\pi i g(x)) = f^{'}(x)/(2\pi i f(x)) + dH(X). \tag{4.2}$$

Denote by D the set of all functions $K : M \longrightarrow \mathbb{R}$ such that $K(x) = dH(X)$ for some smooth function $K : M \longrightarrow \mathbb{R}$. The Equation 4.2 gives us a condition for the existence of a function g in the class of f satisfying the positivity condition of Lemma 4.2.2. Indeed, if the subspace $E = \langle f'(x)/(2\pi i f(x))\rangle \oplus D$ of $C^0(M)$ have nontrivial intersection with the cone of positive functions, namely, if for some H, the expression

$$f'(x)/(2\pi i f(x) + dH(X)$$

represents a positive function, then $g(x) = f(x)\exp(2\pi i H(x))$ satisfies the positive condition of Lemma 4.2.2 and, therefore, the flow has a global cross-section. On the other hand, if it no intercept the cone of positive functions, we can use functional analysis tools, as Hahn-Banach Theorem and Riesz Representation Theorem, to obtain a Borel measure μ satisfying

$$\int_M \Big(f'(x)/(2\pi i f(x)\Big)\, d\mu(x) = 0.$$

As we will see, this measure is invariant and, consequently, gives us an asymptotic cycle. Observe when the positivity condition of Lemma 4.2.2 is satisfied, we have trivially

$$\int_M \Big(f'(x)/(2\pi i f(x)\Big)\, d\mu(x) > 0$$

for every nonnegative measure μ. When the measure is invariant, we can rewrite the last equation as

$$A_\mu([f]) > 0.$$

These observations lead to the following criterion for the existence of a global cross-section to a flow.

Theorem 4.2.3. Let M be a closed manifold and Φ be a flow on M. Let $f \in C$. There exists a global cross-section to Φ in the class of $f \in C$ if and only if $A_\mu([f]) > 0$ for every invariant measure μ.

To show this theorem, we need a characterization of invariant measure. It is easily seen a measure μ is invariant if and only if

$$\int_M (H(xt) - H(x))d\mu(x) = 0$$

for every real-valued continuous function H on M and every real number t. Note that it is sufficient to require that the above equality hold for merely smooth functions since any continuous function can be approximated uniformly by such ones. Define D the set of all functions $K : M \longrightarrow \mathbb{R}$ satisfying $K(x) = dH(X)$ for some smooth function H and E the set of all functions which have the form

$$H(xt) - H(x),$$

where $H(x)$ ranges in the space of smooth functions on M and t ranges over all possilble real numbers.

Lemma 4.2.4. The closure of E and the closure of D are identical.

Proof. For each $t \in \mathbb{R}$ and $H : M \longrightarrow \mathbb{R}$, set $K_t(x) = H(xt)$. Note that $K_t(xt) - K_t(x) \in E$ for each $t \in \mathbb{R}$ and $H \in E$. Given $K \in D$, we have

$$K(x) = \partial_t|_{t=0} H(xt)$$
$$= \lim_{t \longrightarrow 0} \left(\frac{1}{t} H(x) - \frac{1}{t} H(x) \right)$$
$$= \lim_{t \longrightarrow 0} (K_t(xt) - K_t(x)).$$

Hence $K \in \overline{E}$, concluding that $\overline{D} \subset \overline{E}$. Given now $(H(xt) - H(x)) \in E$, we have

$$H(xt) - H(x) = \int_0^t H^{'}(xs)ds.$$

By use of Riemannian's sums to approximate the integral, it is easily seen that we have a uniform approximation on M. Now, observe that each term in a Riemannian's sum referent to one partition $0 = t_0 < \ldots < t_l = T$ is given by

$$(t_{i+1} - t_i)\partial_t|_{t=t_0} H(xt) = \partial_t|_{t=0}(t_{i+1} - t_i)H(x(t + t_0)).$$

Since $\partial_t|_{t=0}(t_{i+1} - t_i)H(x(t + t_0))$ belongs to E for each $i = 1, \ldots, l$, it follows that $\overline{E} \subset \overline{D}$, concluding the lemma. □

Lemma 4.2.5. A measure μ is invariante if and only if $\int_M H(x)d\mu(x) = 0$ for every $H \in \overline{D}$.

Proof. By Lemma 4.2.4, the closure of D and E are identical. Now, the set E is one of all function which have the form $H(xt) - H(x)$ where $H(x)$ ranges in the space of smooth real functions on M and t ranges over all possible real numbers. On the other hand, a measure μ is invariante if and only if

$$\int_M (H(xt) - H(x))d\mu(x) = 0$$

for every smooth real function H on M and every real number t. Thus, a measure μ is invariant if and only if $\int_M K(x)d\mu(x) = 0$ for all $K \in E$. Since the topology on $C^0(M)$ is one of uniform convergence, it follows that any element in $\overline{E}$ can be uniformly approximated by elements in E. Thus, $\int_M K(x)d\mu(x) = 0$ for all $K \in E$ if and only if $\int_M K(x)d\mu(x) = 0$ for all $K \in \overline{E} = \overline{D}$. The proof is completed. □

Proof of 4.2.3. Set $F = D \oplus \langle f^{'}(x)/(2\pi i f(x))\rangle$. We claim that $\overline{F}$ interspet the cone of positive functions. Otherwise, by Hahn-Banach Theorem, there exists a continuous linear functional $L : C^0(M) \longrightarrow \mathbb{R}$ such that $L \neq 0$ and $L|_F = 0$. By the Riesz Representation Theorem, this functional is represented by a finite positive Borel measure μ on M satisfying

$$L(g) = \int_M g(x) d\mu(x)$$

for every $g \in C^0(M)$. The condition $L|_{\overline{D}} = 0$ implies that μ is invariant (see Lemma 4.2.5). Since $L(f'(x)/(2\pi i f(x))) = 0$, this leads to the following contradiction

$$0 < A_\mu([f]) = \int_M f'(x)/(2\pi i f(x)) d\mu(x) = 0.$$

It follows that $\overline{F}$ intersept the cone of positive functions in $C^0(M)$, concluding the theorem. ☐

4.3 A condition ensuring the existence of a global cross-section

Given a finite Borel measure μ and a continuous vector field X, the rule

$$\varphi_{\mu,X}(\omega) = \int_M \omega(X) d\mu$$

define an element in $\mathcal{D}_1'$. Now, let Φ be a flow on a closed manifold M and X be a vector generating it. If μ is an invariant measure, then $A_\mu = [\varphi_{\mu,X}]$. Throughout this section the symbol A_μ represent the linear functional $A_\mu(\omega) = \int_M \omega(X) d\mu$ (see 4.1.17).

Lemma 4.3.1. Let Φ be a smooth flow on a closed manifold M. The set

$$\mathcal{C} = \{A_\mu, \mu \text{ is an invariant measure}\}$$

is convex and $\sigma(\mathcal{D}_1', \mathcal{D}_1)-$compact.

Proof. It is easy to check that $\mathcal{C}$ is convex. The proof of the second statement of the lemma will be divided into three steps.

(1) Let $L : C^0(M) \longrightarrow \mathbb{R}$ be a continuous linear functional. By Riesz Representation Theorem, there exists a unique finite measure μ defined on the σ-algebra of the Borel subsets of X such that

$$L(f) = \int_M f d\mu$$

for all $f \in C^0(M)$. It follows that we can identifie the topological dual to $C^0(M)$ with the set of all finite measure μ defined on the σ-algebra of Borel subsets of M. Via this identification, the set of all probability measures on M is characterized as $P = \{\mu / \parallel \mu \parallel = 1\}$, where the norm considered is the usual norm in the topological dual of the Banach space $C^0(M)$. By Banach-Alaoglu-Bourbaki Theorem, we have that P is weakly-compact.

(2) We will now show that the set of all Φ-invariant measures is a closed subset of P. Let $\mu_n \longrightarrow \mu$ be a sequence with μ_n invariant for all $n \in \mathbb{N}$. Given $f \in C^0(M)$ we have

$$\int_M (f(xt) - f(x))d\mu = \mu(f \circ t - f) = \lim \mu_n(f \circ t - f) = 0,$$

concluding that μ is invariant. It follows that the set of all invariant measures is closed (since $(C^0(M))'$ is metric, a subset of $(C^0(M))'$ is closed if and only if it is sequentially closed).

(3) Let X be a vector field generating Φ. Let

$$\iota : (C^0(M))' \longrightarrow \mathcal{D}_1'$$

given by $\iota(\mu)(\omega) = \int_M \omega(X)d\mu$. We will give straightforward proof that ι is weakly continuous. Remember that for a dual pair (E, F), the weak topology on F has as subbasis of neighborhoods of the zero the sets $U_{x,\epsilon} = \{y \in F/|\langle x, y\rangle| < \epsilon\}$, where x range over E and ϵ range over all positive real numbers. Let $\omega \in \mathcal{D}_1$ and $\epsilon > 0$. Concerning the dual pair $(\mathcal{D}_1, \mathcal{D'}_1)$ we have $\mu \in \iota^{-1}(U_{\omega,\epsilon})$ if and only if $|\int_M \omega(X)d\mu| < \epsilon$. Since $\omega(X) \in C^0(M)$, we have that $\iota^{-1}(U_{\omega,\epsilon}) = U_{\omega(X),\epsilon}$ is weakly open (of course, concerning the dual pair $(C^0(M), (C^0(M))')$). It follows that ι is weakly continuous.

To finish, note that for an invariant measure μ we have

$$\iota(\mu)(\omega) = A_\mu(\omega).$$

It follows from itens (1), (2) and (3) that $\mathcal{C}$ is the image of a weakly compact set by a weakly continuous function. Therefore, $\mathcal{C}$ is weakly compact. □

Theorem 4.3.2. Let Φ be a smooth flow on a closed manifold M. There exist a global cross-section for Φ if and only if for every invariant measure μ the homological class determined by A_μ is nonzero.

Proof. By Lemma 4.3.1, the set $\mathcal{C} = \{A_\mu, \mu \text{ invariant measure}\}$ is convex and weakly compact. Since $\mathcal{B}_1$ is $\sigma(\mathcal{D}_1', \mathcal{D}_1)$-closed and $\mathcal{C} \cap \mathcal{B}_1 = \emptyset$, by Hahn-Banach Theorem there exists a $\sigma(\mathcal{D}_1', \mathcal{D}_1)$-continuous linear functional $\varphi : \mathcal{D}_1' \longrightarrow \mathbb{R}$ such that

$$\varphi|_{\mathcal{C}} > 0 \text{ and } \phi|_{\mathcal{B}_1} = 0.$$

This linear functional is strongly continuous (because $(\mathcal{D}_1, \mathcal{D}_1')$ is a dual pair by Schwartz's theorem 1.6.31 and in any dual pair (E, F) a weak continuous linear functional $f : E \longrightarrow \mathbb{R}$ is strongly continuous). Again by 1.6.31, there exist a 1-form η such that $\varphi = \eta$. Hence

$$d\eta(\phi) = \eta(\partial\phi) = \varphi(\partial\phi) = 0,$$

concluding that η is closed. The positivity condition of φ on $\mathcal{C}$ implies that

$$A_\mu(\eta) = \eta(A_\mu) = \varphi(A_\mu) > 0$$

for every invariant measure μ. Now, the set of all invariant measures is weakly compact, and by 1.8.3 η can be approximated in $\mathcal{D}_1$ by forms with integral periods. It follows that there exists a form ω with integral periods such that $\omega(A_\mu) > 0$ for all invariant measure μ. Since $\omega = f^* d\theta$ for some smooth function $f : M \longrightarrow \mathbb{S}$, it follows by Scwartzmann's Theorem 4.2.3 that Φ admit a global cross-section. □

Corollary 4.3.3. Let Φ be a smooth flow on a closed manifold M. Suppose that the homological class determined by A_μ is nonzero for any invariant measure μ. Then there exists a submersion $f : M \longrightarrow \mathbb{S}^1$ with all fibers transverse to Φ.

Proof. The hypothesis and Theorem 4.3.2 implies the existence of a global cross-section to Φ. Hence, Theorem 4.2.3 implies the existence of a smooth function $f : M \longrightarrow \mathbb{S}^1$ satisfying the positivity condition in the Lemma 4.2.2. The proof is completed. □

Example 4.3.4 (Hamiltonian vector fields)**.** Let (M^{2n}, ω) be a symplectic closed manifold. Given any 1-form α, there exists a vector field X_α such that $\alpha = \iota_{X_\alpha}\omega$, since ω is nonsingular two-form. When α is closed, then the $(n-1)$-form $\iota_{X_\alpha}\omega^n$ is closed and the vector field X_α preserves the volume form ω^n. Those vector fields are called *Hamiltonian vector fields*. Suppose μ is an invariant measure for the flow generated by X_α. The μ-asymptotic cycle associated with μ is given by

$$A_\mu(\beta) = \int_M \beta(X) d\mu.$$

Denote now μ the measure determined by the volum form ω^n. We have

$$A_\mu(\beta) = \int_M \beta(X_\alpha)\omega^n = \int_M \beta \wedge \iota_{X_\alpha}\omega^n.$$

In general, $\iota_X\omega^k = k\iota_X\omega \wedge \iota_X\omega^{k-1}$. If X_β is such that $\beta = \iota_{X_\beta}\omega$, then

$$\begin{aligned} \beta \wedge \iota_{X_\alpha}\omega^n = n\iota_{X_\beta}\omega \wedge \iota_{X_\alpha}\omega \wedge \omega^{n-1} = \\ \iota_{X_\alpha}\omega \wedge n(\iota_{X_\beta}\omega) \wedge \omega^{n-1} = \\ \alpha \wedge \iota_{X_\beta}\omega^n. \end{aligned}$$

It follows that

$$A_\mu(\beta) = \int_M \alpha \wedge \iota_{X_\beta}\omega^n.$$

On the other hand, there is an isomorphism $H^1_{DR}(M;\mathbb{R}) \longrightarrow H^{n-1}_{DR}(M;\mathbb{R})$ given by $[\beta] \longrightarrow [\iota_{X_\beta}\omega^n]$. Thus, the μ-asymptotic cycle associated with ω^n concerning X_α is homologous to zero if and only if the class determined by α cohomologous to zero. It follows by 4.1.16 and 4.3.2 that a recurrent Hamiltonian vector field X_α admits a global cross-section if and only if $[\alpha] \neq 0$ in $H^1_{DR}(M)$.

4.4 Correspondence between asymptotic cycles and foliation cycles

Many references in the literature indicate that the foliation currents theory is a generalization of the asymptotic cycle theory. For example, Schwartzamnn itself in [75] says: "using the notion of a transversal invariant measure as defined in [80], a generalization of the μ-asymptotic cycle was given that applies to arbitrary smooth oriented foliations of a compact manifold".

This section deal with the "equivalence" between asymptotic cycles and foliation cycles for the case of 1-dimensional foliations. Namely, let Φ be a smooth nonsingular flow on a closed manifod M (a 1-dimensional foliation). Let c be a 1-dimensional current. Then $c(\omega) = \int_M \omega(X) d\nu$ for some measure ν and, furthemore, c is a cycle if, and only if, ν is an Φ-invariant measure.

Theorem 4.4.1 (Sullivan, [80])**.** Let $\mathfrak{F}$ be a 1-dimensinal foliation on a closed manifold M. Let X be a nowhere-vanishing vector field tangent to $\mathfrak{F}$. Given a foliation current c (for $\mathfrak{F}$) there exists one, and only one, positive, finite, Borel measure ν_c in M such that

$$c(\eta) = \int_M \eta(X) d\nu_c$$

for all $\eta \in \mathcal{D}_1$.

Proof's sketch. Let ω be a 1-form satisfying $\omega(X) = 1$. The idea is to use Riesz Representation Theorem to obtain a measure νc such that the linear functional

$$\begin{aligned} J_c : C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\to c(f\omega) \end{aligned}$$

be given by $J_c(f) = \int_M f d\nu_c$. Suppose $c = aX_x$, a Dirac foliation current in the X_x direction. Then, denoting ν_c the atomic measure concentred in $\{x\}$, we have

$$c(f\omega) = (aX_x)(f\omega) = af(x)\omega(X_x) = af(x) = \int_M f d\nu_c.$$

It follows that if c is a finite sum with positive coefficients of Dirac foliation currents, then the unique measure which satisfies the condition in the theorem is given by the sum of atomic measures concentred in each x_i.

Now, let c any foliation current. Then $c = \lim c_n$, where each c_n is a finite linear combination with positive coeficients of Dirac foliation currents (convergence in the $\sigma(\mathcal{D}'_1, \mathcal{D}_1)$-topology, see 3.2.4). For each $n \in \mathbb{N}$, let $\nu_{_n}$ as in the preceding paragraph. The bounded sequence $\{\nu_{c_n}\}$ has a subsequence converging for a measure ν with the same proprieties. By Riesz Representation Theorem, the measure ν is uniquely determined. Given a 1-form η, writing

$$\eta = \eta_0 + \eta(X)\omega$$

with $\eta_0(X) = 0$, we have

$$c_n(\eta) = c_n(\eta_0 + \eta(X)\omega) = c_n(\eta(X)\omega) = \int_M \eta(X) d\nu_{c_n}.$$

Therefore,

$$c(\eta) = \lim c_n(\eta) = \lim \int_M \eta(X) d\nu_{c_n} = \int_M \eta(X) d\nu_c.$$

□

Theorem 4.4.2. With the notations of the previous theorem, a foliation current c is a cycle if and only if is Φ-invariant. Furthemore, every invariant measure ν determines a foliation cycle by the rule $\eta \longrightarrow A_\nu(\eta)$. In particular, there is an one-to-one correspondence between foliations cycles and μ-asymptotic cycles.

Proof's skecth. Suppose that c is a cycle. Then, for every smooth function $f : M \to \mathbb{R}$ we have

$$0 = \partial\phi_\mu(f) = \phi_\mu(df) = \int_M df(X) d\nu.$$

Given $t \in \mathbb{R}$, $f \in C^\infty(M)$, using Fubbini Theorem, we have

$$\int_M (f(xt) - f(x)) d\nu(x) = \int_M \left(\int_0^t df(X(xs)) ds \right) d\nu(x) = \\ \int_0^t \left(\int_M df(X(xs)) d\nu(x) \right) ds = 0.$$

Since $\int_M (f(xt) - f(x)) d\nu(x) =$ for every $t \in \mathbb{R}$ and $f \in C^\infty(M)$, we conclude that ν is invariant. Conversely, if ν is invariant, A_ν have sense and is a cycle. Hence $c = A_\nu$ is a cycle.

To prove that invariant measures determine foliation cycles requires a little more effort. This is clear in the proof of Sullivan's theorem. An invariant measure ν can be "disintegrated" in each foliated chart obtaining two measures, a measure ν_p in each plaque (q.t.p.) and a transversal measure that is holonomy invariant. Thus, for a foliated atlas $\mathcal{U}$, and a transversal $S_\mathcal{U}$ associated, we can obtain a holonomy invariant measure μ such that $\phi_\mu = A_\nu$. It follows that invariant measures determine foliations cycles.

□

5 Characterization of flat circle bundles

E. Calabi showed that any nowhere-vanishing closed 1-form is intrinsically harmonic (see section 2.4). Thus, it is natural to raise the question about the dual case. Is a closed nowhere-vanishing $(n-1)$-form intrinsically harmonic? We must exclude exact forms since by 1.7.6 they are harmonic only if they are identically zero. In contrast with the case of nowhere-vanishing closed 1-forms that determines a nonzero cohomological class, there exists exact nowhere-vanishing closed $(n-1)$-forms, as the natural pullback form given in the Hopf's fibration $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$. In Chapter 2, we provide a condition to a closed nowhere-vanishing $(n-1)$-form on a closed manifold to be intrinsically harmonic, namely, when an associated canonical flow admits a global cross-section. This and the next chapter, where is given an improvement of this characterization, originated in an attempt to provide an example of a nowhere-vanishing closed $(n-1)$-form, non-exact and non intrinsically harmonic. Unfortunately, we did not get such an example nor manage to show that obtaining it is impossible.

We started this search in the class of forms given in circle bundles $\mathcal{B} = \{B, p, \mathbb{T}^{n-1}, \mathbb{S}^1\}$. By Thishler's argument and characterization of the covering of the torus, if the form $p^*\Omega_{\mathbb{T}^{n-1}}$ is intrisically harmonic, then B and $\mathbb{T}^n$ are diffeomorphic. Hence, a possible counterexample would arise from a circle bundle $p : B \longrightarrow \mathbb{T}^{n-1}$ with total space not diffeomorphic to the n-torus and $p^*_{\Omega_{\mathbb{T}^{n-1}}}$ a non-exact form. The fact that this attempt fails more generally, in any circle bundle with closed total space, is the main result of this chapter. We provide the following results.

Theorem 5.0.1. Let M be a closed smooth manifold with a closed $(n-1)$-form ω that induces a compact foliation[1]. If each orbit of the flow induced by ω is homologous to each other and $[\omega] \neq 0$, then ω is intrinsically harmonic. If ω is intrinsically harmonic, then there exists a smooth $\mathbb{S}^1$-action on M with the same orbits as the one from the flow induced by ω.

Theorem 5.0.2. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a differentiable principal bundle over a closed orientable manifold M. Then $\mathcal{B}$ admit a flat connection if and only if $p^*(\Omega_M)$ is intrinsically harmonic.

By another direction, in Subsection 5.2.1 we provide the following characterization of the foliated circle bundle:

Theorem 5.0.3. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a differentiable circle bundle over a closed manifold M. Then $\mathcal{B}$ is smooth foliated if and only if the fiber determines a nontrivial

[1] A foliation $\mathfrak{F}$ is said to be compact provided every leaf of $\mathfrak{F}$ is compact.

homological class. When M is orientable, this happens if and only if $p^*\Omega_M$ is nonzero in cohomology.

We finish this chapter by providing the following characterization of the torus.

Theorem 5.0.4. The n-torus is the unique closed n-dimensional manifold admitting a set of $(n-1)$ everywhere linearly independent nowhere-vanishing closed 1-forms $\{\omega_i\}$, such that the product $\omega = \omega_1 \wedge \cdots \wedge \omega_{n-1}$ determines a nonzero cohomological class.

5.1 Examples of intrinsically harmonic $(n-1)$-forms

Lemma 5.1.1. Let M be a closed orientable manifold. Let Ω be a volume form on M and X be a smooth vector field on M preserving Ω. Set $\omega = i_X\Omega$ and μ the measure determined by Ω. Then μ is invariant by the flow determined by X and $A_\mu = [\omega]$, ω viewed as a diffuse current. In particular, if ω determines a nonzero cohomological class, then $A_\mu \neq 0$ in $H_1^{DR}(M)$.

Proof. Let η be a 1-form on M. Using the identity $i_X(\eta \wedge \Omega) = i_X\eta \wedge \Omega + \eta \wedge i_X\Omega$ and the definition of μ, then

$$\begin{aligned}\int_M \eta(X)d\mu = \int_M \eta(X)\Omega =\\ \int_M i_X\eta \wedge \Omega =\\ \int_M (i_X(\eta \wedge \Omega) + \eta \wedge i_X\Omega) =\\ \int_M \eta \wedge \omega.\end{aligned}$$

Thus, since $d(f\omega) = df \wedge \omega - fd\omega$ and $d\omega = 0$, by Stokes's theorem

$$\int_M df(X)d\mu = \int_M df \wedge \omega = \int_M d(f\omega) = 0$$

for all $f \in C^\infty(M)$. Using this, we will prove that μ is invariant. Let $t \in \mathbb{R}$ and $f \in C^\infty(M)$. By Fubini's theorem

$$\begin{aligned}\int_M (f(xt) - f(x))d\mu(x) = \int_M \left(\int_0^t df(X(xs))ds\right) d\mu(x) =\\ \int_0^t \left(\int_M df(X(xs))d\mu(x)\right) ds = 0.\end{aligned}$$

Hence, since $\int_M (f(xt) - f(x))d\mu(x) = 0$ for every $t \in \mathbb{R}$ and $f \in C^\infty(M)$, it follows that μ is invariant. Therefore, have sense A_μ and

$$A_\mu(\eta) = \int_\eta (X)d\mu = \int_M \eta \wedge \omega = \omega(\eta),$$

where ω is seen as a diffuse current. Now, by 1.7.9, the form ω determine a nonzero cohomological class if and only if, as a diffuse current, determine a nonzero homological class. The proof is completed. □

Lemma 5.1.2. Let M be a manifold and Φ be a continuous flow on M with a periodic point x under Φ. Then x is quasi-regular and $\lambda(x)A_x = [C_x]$, where $\lambda(x)$[2] is the period of x and $[C_x]$ represents the integral homological class determined by the orbit of x under Φ.

Proof. If x is a fixed point, the assertion follows. Suppose x is not a fixed point. For each smooth function $f : M \longrightarrow \mathbb{R}$, it is easily seen that

$$\lim_{t \longrightarrow \infty} 1/T \int_0^T f(xt)dt = 1/\lambda(x) \int_0^{\lambda(x)} f(xt)dt.$$

Therefore every periodic point $x \in M$ is quasi-regular. Set $\alpha : [0, T] \longrightarrow M$ given by $\alpha(t) = xt$. Let $f \in C$. Then

$$\begin{aligned} A_x[f] &= \lim_{T \longrightarrow \infty} \frac{1}{2\pi i T} \int_0^T f^{'}(xt)/f(xt) \\ &= \lim_{T \longrightarrow \infty} \frac{1}{2\pi i T} \int_0^T 2\pi i \left(\omega_f(\alpha^{'}(t))dt\right) \\ &= \lim_{T \longrightarrow \infty} \frac{1}{T} \int_{\alpha([0,T])} \omega_f \\ &= 1/\lambda(x) \int_{\alpha[0,\lambda(x)]} \omega_f. \end{aligned}$$

Denoting by $[C_x]$ the integral homological class determined by the orbit of x parametrized by α, then

$$[C_x]([f]) = \int_{\alpha[0,\lambda(x)]} \omega_f = \lambda(x)A_x[f].$$

Therefore,

$$\lambda(x)A_x = [C_x].$$

□

To prove the next theorem, let us mention a result of D. Montgomery and, for later use, a mild generalization of it mentioned by D.B. Epstein in [21] (Theorem 7.3).

[2] The period function is defined by $\lambda(x) = \inf_{t \geqslant 0}\{xt = x\}$.

Theorem 5.1.3 (D. Montgomery, [58])**.** Let X be a connected metric space locally homeomorphic to $\mathbb{R}^n$. If T is a pointwise periodic homeomorphism of X into itself then T is periodic.

Theorem 5.1.4 (K.C. Millett)**.** Let M be a connected manifold and let G be a group of homeomorphisms of M, such that the orbit of any point of P is finite. Then G is finite.

Proof of Theorem 5.0.1: Let Ω be a volume form on M and X be the vector field given in the equation $\omega = i_X\Omega$. Denote by Φ the flow on M induced by X. Suppose every orbit under Φ determines the same element in homology. Let μ the measure induced by Ω. By Lemma 5.1.1, $A_\mu = [\omega]$. By 1.7.9 the homological class of ω as a diffuse current is zero if and only if the cohomological class of ω is zero. It follows that $A_\mu \neq 0$. Now, given $f \in C/R$ and ν be a Φ-invariant measure, it follows by Lemma 5.1.2 that

$$\begin{aligned} A_\nu[f] &= \int_M A_x[f]d\nu(x) \\ &= \int_M \frac{1}{\tau_x}([C_x],[f])d\nu(x). \end{aligned}$$

Therefore, $[C_x] \neq 0$ for all $x \in M$, since $[C_x]$ independ on x by hypothesis and $A_\mu \neq 0$. Let $x_0 \in M$ and $f \in C/R$ such that $([C_{x_0}],[f]) \neq 0$. Then

$$A_\nu[f] = ([C_{x_0}],[f])\int_M \frac{1}{\lambda(x)}d\nu(x)$$

for all Φ-invariant measure ν. Now, since $x \longrightarrow \lambda(x)$ is a positive function[3], we conclude that $A_\nu \neq 0$ for all Φ-invariant measure ν. The theorems 2.4.3 and 4.3.2 together implies that ω is intrinsically harmonic.

Conversely, suppose that ω is intrinsically harmonic. Then there exists a closed 1-form η such that $\eta(X) > 0$. By 1.8.3, we can take $\eta = f^*d\theta$ with $f : M \longrightarrow \mathbb{S}^1$ be a submersion. By Ehresmann's lemma 1.2.3 and Theorem 1.4.6, we have that $f : M \longrightarrow \mathbb{S}^1$ determines a foliated bundle $\mathcal{B}$, with the foliation given by the orbits under Φ. Such a bundle is characterized by the holonomy homomorphism

$$\varphi : \pi_1(\mathbb{S}^1) \longrightarrow \mathrm{Diff}(F).$$

Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, the homomorphism φ is determined completely by the diffeomorphism $g = \varphi(1)$. On the other hand, since each orbit under Φ is compact, the diffeomorphism g is pointwise periodic. By Theorem 5.1.3, there exists $k \in \mathbb{Z}$ such that $\varphi^k(1) = 1$, concluding that Φ is periodic. It follows easily that there is an smooth action $\Psi : M \times \mathbb{S}^1 \longrightarrow M$ with the same orbits as Φ.

□

[3] The function λ is lower-semicontinuous (see 6.5.3), hence Lebesgue measurable. It follows that $\int_M f d\nu \neq 0$ for any measure ν defined on the σ-algebra generated by all Borel subsets of M.

5.2 Flat circle bundles

Every orientable smooth circle bundle $\mathcal{B}$ admits principal bundle structure. It is a well-known fact that a principal bundle is trivial if, and only if, it admits a global cross-section. In the case of circles bundles, there is a unique obstruction to the existence of a global cross-section, an element

$$\chi(\mathcal{B}) \in H^2(M;\mathbb{Z})$$

called *Euler characteristic* of the bundle $\mathcal{B}$ ([59], page 346). Hence, an orientable circle bundle $\mathcal{B}$ has a global cross-section if and only if $\chi(\mathcal{B}) = 0$. This cohomological class carries not only information about the triviality of the circle bundle. As we will see, it is possible to characterize when a circle bundle is smooth foliated in terms of this class. As already said (see Section 1.4), these are classified by their holonomy homomorphisms. Namely, there is a natural bijection between the set of all smooth foliated orientable circle bundles over M modulo leaf preserving bundle isomorphism and the set of all homomorphisms $\varphi : \pi_1(M) \longrightarrow \mathrm{Diff}^+(\mathbb{S}^1)$ modulo conjugacy. Many papers were concerned about the problem of the existence of a codimension-one foliation transverse to the fibers of a given circle bundle, that is, the question of when is foliated a circle bundle. For circle bundles with the base space being a surface, a necessary and sufficient condition for the existence of a transverse foliation was obtained by J. Milnor [50] and J. W. Wood [90]. Circle bundles with base space being a 3-manifold were studied by S. Miyoshi [55]. An answer to when is smooth foliated a circle bundle was given in [56, 61].

In this section, the main result is to characterizing a flat circle bundle over an orientable base. Also are given results on non-orientable circle bundles and circle bundles with a non-orientable base similar to the previous one. As a consequence, but by a different approach, we obtain the same results as in [56, 61].

Theorem 5.2.1. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a differentiable orientable circle bundle over a closed orientable manifold M. Let Ω be a volume form on M. Then $\mathcal{B}$ is smooth foliated if and only if $p^*(\Omega)$ is intrinsically harmonic.

Proof. Suppose that $p^*(\Omega)$ determines a nontrivial cohomological class. Since the orbits of the flow induced by $p^*(\Omega_M)$ are homologous to each other, Theorem 5.0.1 ensures that $p^*(\Omega)$ is an intrinsically harmonic form.

Conversely, suppose $p^*(\Omega_M)$ is intrinsically harmonic. Then there exists a closed 1-form η transversal to the flow generated by $p^*(\Omega_M)$. This form induces a foliation transversal to the fibers of $\mathcal{B}$. Since $\mathcal{B}$ has compact fibers, it follows by 1.4.6 that this foliation is compatible and $\mathcal{B}$ becomes a smooth foliated bundle. The proof is completed. □

Lemma 5.2.2. The isometry group of the sphere with the canonical metric is the orthogonal group.

Proof. It is well-known that an isometry of a connected Riemannian manifold is characterized by its value at a point x and the evaluation of its differential at x. That is, if f, g are isometries of a connected Riemannian manifold M, $f(x) = g(x)$, and $df_x = df_x$ then $f = g$ (for example, see [63] page 143). Observe now that any element in $O(n+1)$ is an isometry of $\mathbb{S}^n$. On the other hand, using linear algebra we can show that given $x, y \in \mathbb{S}^n$ and α, β ortonormal basis for $T_x\mathbb{S}^n$ and $T_y\mathbb{S}^n$, respectively, there exists $g \in O(n+1)$ such that $g(x) = y$ and df_x take the basis α in the basis β. Thus, $O(n+1)$ contains all possibles isometries of $\mathbb{S}^n$. $\square$

Lemma 5.2.3. Let g and h be Riemannian metrics on $\mathbb{S}^1$. Then there exists an isometry $f : (\mathbb{S}^1, g) \longrightarrow (\mathbb{S}^1, h)$.

Proof. Consider the usual covering map $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. The Riemannian metrics π^*g π^*h are related by a positive function, that is, there exists a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\pi^*g = e^f\pi^*h$. The function f is necessarily invariant by translation of elements in $\mathbb{Z}$. It follows that the isometry

$$\phi(t) = \int_0^x e^{\frac{1}{2}f(t)}dt$$

between $(\mathbb{R}, \pi^*g)$ and $(\mathbb{R}, \pi^*h)$ induces an isometry between $(\mathbb{S}^1, g)$ and $(\mathbb{S}^1, h)$. $\square$

Remark 5.2.4. If two Riemannian manifolds (M, g) and (N, h) are isometric by a function $f : M \longrightarrow N$, then the groups $\mathrm{Iso}_g(M)$ and $\mathrm{Iso}_h(N)$ are related by the isomorphism $\Phi_f : \mathrm{Iso}_g(M) \longrightarrow \mathrm{Iso}_h(N)$ given by $\Phi_f(\theta) = f \circ \theta \circ f^{-1}$. In particular, if $f : (\mathbb{S}^1, g) \longrightarrow (\mathbb{S}^1, \mathrm{can})$ is an isometry, each element of $\mathrm{Iso}_g(\mathbb{S}^1)$ have the form $f \circ L_\theta \circ f^{-1}$, where L_θ denotes the left translation by $\theta \in \mathbb{S}^1$ (compare with [25], proposition 4.1).

Theorem 5.2.5. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a smooth orientable circle bundle over a closed orientable manifold. If $p^*(\Omega_M)$ determines a nontrivial cohomological class, then $\mathcal{B}$ is $\mathrm{Diff}^+(\mathbb{S}^1)$-equivalent to a flat circle bundle.

Proof. In the proof of Theorem 5.2.1 we can take η with integral periods by Tischler's argument 1.8.3. Then $\eta = f^*d\theta$, for some smooth function $f : B \longrightarrow \mathbb{S}^1$. It follows that $\mathcal{B}$ is foliated by a compact foliation. Let

$$\varphi : \pi_1(M) \longrightarrow \mathrm{Diff}^+(\mathbb{S}^1)$$

be the holonomy homomorphism which characterizes this foliated bundle. Since each leaf of $\mathfrak{F}_\eta$ is compact and transversal to the fibers of $\mathcal{B}$, then the orbit of any point in the fiber B_x over x under the group $\varphi(\pi_1(M))$ is finite (indeed, it is equal to $\#(f^{-1}(f(x)) \cap B_x)$). This and Theorem 5.1.4, implies that $\varphi(\pi_1(M))$ is a finite group. Now, let g be a Riemannian metric on $\mathbb{S}^1$. Set

$$h(X, Y) = \frac{1}{|\varphi(\pi_1(M))|} \sum_{k \in \varphi(\pi_1(M))} g(d\varphi(k)(X), d\varphi(k)(Y)).$$

Then, h is a Riemannian metric on $\mathbb{S}^1$ such that $\varphi(k)$ is an isometry for all $k \in \mathrm{Im}(\varphi)$. It follows by 5.2.3 that there exists a diffeomorphism $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $f^{-1} \circ \varphi(\alpha) \circ f \in \mathbb{S}^1$ for all $\alpha \in \pi_1(M)$. The fiber bundle obtained considering as holonomy homomorphism the map

$$\varphi^{'} : \pi_1(M) \longrightarrow \mathbb{S}^1$$

given by $\varphi^{'}(\alpha) = f^{-1}\varphi(\alpha)f$ is a differentiable principal flat bundle by 1.4.7. Furthemore, this bundle is $\mathrm{Diff}^+(\mathbb{S}^1)$-equivalent to $\mathcal{B}$ by 1.4.2. The proof is completed. $\square$

Theorem 5.2.6. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a differentiable principal circle bundle with M closed and orientable. Then $\mathcal{B}$ admits a flat connection if and only if the form $p^*(\Omega_M)$ is intrinsically harmonic.

Proof. Suppose that $\pi^*\Omega_M$ is intrinsically harmonic. Then, by Theorem 5.2.5 there exists a differentiable principal flat circle bundle which is $\mathrm{Diff}^+(\mathbb{S}^1)$-equivalent to $\mathcal{B}$. It follows by 1.2.39 that $\mathcal{B}$ is differentiable isomorphic to $\mathcal{B}^{'}$ as $\mathbb{S}^1$-bundles. To finish, a flat connection in $\mathcal{B}^{'}$ can be transferred to a flat connection for $\mathcal{B}$ by 1.2.31. The converse follows by 5.2.1. The proof is completed. $\square$

Another characterization of flat circle bundles in terms of the real homological class represented by the fiber is possible and depends on the next lemma.

Lemma 5.2.7. Let $\mathcal{B} = \{B, p, M, F\}$ be a differentiable fiber bundle with compact total space and base be an orientable manifold. Let Ω be a volume form on M with $\int_M \Omega = 1$. Then, the Poincaré dual to the fiber of $\mathcal{B}$ can be represented by $p^*\Omega$. In particular, $[F] = 0$ in $H_{\dim F}(B; \mathbb{R})$ if and only if $p^*\Omega$ is an exact form.

Proof. Let $x \in M$. Denote by F the fiber over x. Let U be a sufficiently small open set diffeomorphic to $\mathbb{R}^n$ such that there exists a trivialization

$$\phi : p^{-1}(U) \longrightarrow U \times F,$$

for which the normal bundle of F is equivalent to the trivial vector bundle $\tau : F \times U \longrightarrow F$. Denote by

$$\pi_1 : U \times F \longrightarrow U$$

the projection on the first factor. Given a n-form ω in U generating $H^n_c(U) = \mathbb{R}$ (the compactly supported cohomology), the closed form

$$\eta = \pi_1^*\omega$$

represents the *Thom class* of the bundle τ, because generates $H^n_c(\{y\} \times U)$ for each $y \in F$ (proposition 6.18 of [4]). On the other hand, the Thom class of τ can be represented by the Poincaré dual of the null section, which in this case is F (proposition 6.24 of the [4]). Then, η represents the Poincaré dual of F in $U \times F$ and, therefore,

$$\phi^*\eta = \phi^*(\pi_1^*\omega) = (\pi_1 \circ \phi)^*\omega = p^*\omega$$

represents the Poincaré dual of F in $p^{-1}(U)$. The trivial extension of $p^*\omega$ to B gives the Poincaré dual of F in B. Clearly ω defines a form in M, by trivial extension, and

$$([\omega], [M]) = \int_M \omega = \int_U \omega = 1 = ([\Omega_M], [M]),$$

concluding that Ω_M and ω are cohomologous. To finish, $p^*\Omega_M$ and $p^*\omega$ are cohomologous with $\pi^*\omega$ representing the Poincaré dual of F in B. Therefore, we have $[F] = 0$ in $H_{\dim F}(B; \mathbb{R})$ if, and only if, $p^*\Omega_M$ is an exact form. □

Example 5.2.8. Let $\mathcal{B} = \{B, p, M, F\}$ be a differentiable fiber bundle with compact total space and base be an orientable manifold. Suppose that $\chi(F) \neq 0$ (the Euler characteristic of F is nonzero). Let Ω be a volume form on M. Then $\pi^*\Omega$ determines a nonzero cohomological class by 5.2.7. Indeed, since $\chi(F) \neq 0$ it is not possible for F to be a boundary and, therefore, we have $[F] \neq 0$ in $H_{\dim F}(B; \mathbb{R})$.

Theorem 5.2.9. A differentiable principal circle bundle $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ over a closed orientable manifold admits a flat connection if, and only if, $[\mathbb{S}^1] \neq 0$ in $H_1(B, \mathbb{R})$.

Proof. It follows by 5.2.1 and 5.2.7. □

Example 5.2.10 (Plante, [64])**.** Given a smooth orientable sphere bundle $\mathcal{B} = \{B, p, M, \mathbb{S}^k\}$ we have from the Gysin cohomology sequence (real coefficients)

$$\ldots \longrightarrow H^k(M) \xrightarrow{p^*} H^k(B) \longrightarrow H^0(M) \xrightarrow{\Psi} H^{k+1}(M) \longrightarrow \ldots$$

where $\Psi(1) \in H^{k+1}(M)$ is just the Euler class of the bundle. Hence, the fiber is null-homologous in $H_k(E)$ if and only if $p*$ is surjective which is true if and only if the Euler class is nonzero.

Theorem 5.2.11. A differentiable principal circle bundle $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ with closed orientable base space admits a flat connection if and only if one (and consequently all) of the four conditions below holds:

(1) $p^*(\Omega_M)$ is intrinsically harmonic;

(2) $p^*(\Omega_M)$ determines a nonzero cohomological class;

(3) the fiber represents a nonzero class in $H_1(B; \mathbb{R})$;

(4) $\chi(\mathcal{B})$ is a torsion element.

Proof. It follows by 5.2.6, 5.2.7 and 5.2.10 and from the fact that the real Euler class $\chi_{\mathbb{R}}(\mathcal{B})$ of $\mathcal{B}$ is equal to $\chi(\mathcal{B}) \otimes \mathbb{R}$. □

5.2.1 Smooth foliated circle bundle

Theorem 5.2.12. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a circle bundle over a non-orientable closed manifold. The fiber determines a nonzero real homological class if and only if $\mathcal{B}$ is smooth foliated. In the case of $\mathcal{B}$ be smooth foliated, then B is a non-orientable manifold and $\mathcal{B}$ is an orientable fiber bundle.

Proof. Since the real homological class determined by the fiber is nonzero, the flow Φ induced by the it admits a global cross-section by 4.3.2 (this flow has only an assymptotic circle given by the class of the fiber which is nonzero). By Corollary 4.3.3, there exists a closed differential form ω transverse to the orbits under Φ. Thus, $\mathcal{B}$ is smooth foliated. Conversely, suppose that $\mathcal{B}$ is smooth foliated. A compatible foliation to $\mathcal{B}$ being transversal to the nowhere-vanishing vector field X given by the fibers of $\mathcal{B}$ must be induced from a closed differential 1-form η by 2.4.7. Since the form η restricts to a nowhere-vanishing form in the fibers, then each of these represents a nonzero homology class. Furthemore, $\mathcal{B}$ is orientable since ω restricted to each fiber is a volume form. It follows now by 1.5.10 that B is non-orientable, since M is non-orientable and $\mathcal{B}$ is orientable. □

Remark 5.2.13. By the latter theorem, a non-orientable circle bundle over a non-orientable closed manifold cannot be smooth foliated.

Theorem 5.2.14. Let $\mathcal{B} = \{B, p, M, \mathbb{S}^1\}$ be a non-orientable circle bundle over an orientable closed manifold. Then $\mathcal{B}$ is smooth foliated if and only if the fiber represents a nonzero real homological class (or the pullback of the volume form is intrinsically harmonic).

Proof. The proof is similar to the one given in Theorem 5.2.12. In this case, in contrast to the previous one, the fibers are the integral submanifolds of the distribution $\ker p^*\Omega$, where Ω is any volume form on M. If this form is intrinsically harmonic, then $\mathcal{B}$ is smooth foliated. Conversely, if $\mathcal{B}$ is smooth foliated, this foliation is given by the kernel of a closed 1-form by 2.4.7, necessarily transversal to $p^*\Omega$. Hence $p^*\Omega$ is an intrinsically harmonic form. □

5.3 Characterization of the torus

By Tischler's argument and the characterization of the covering spaces over the n-torus, ti is the unique closed manifold admitting a set consisting of n closed forms of degree one everywhere linearly independent. Moreover, if in a closed n-dimensional manifold M there exists a set consisting of $(n-1)$ closed forms of degree one everywhere linearly independent, say $\omega_1, \ldots, \omega_{n-1}$, such that the product $\omega_1 \wedge \ldots \wedge \omega_{n-1}$ is a nonzero cohomological class, then M and $\mathbb{T}^n$ are diffeomorphic.

Proof of Theorem 5.0.4. By the Tischler's argument 1.8.3, there exist integers d_l and a collection of smooth fiber bundles

$$\mathcal{B}_l = \{M, p_l, \mathbb{T}^{n-1}\}$$

such that

$$\eta_l = \frac{1}{d_l} p_l^*(\Omega_{\mathbb{T}^{n-1}}) \longrightarrow \omega$$

in $\mathcal{D}_{n-1}$. Since the limit of exact forms is an exact form by 1.7.9, at most a finite number of η_l can be cohomologous to zero. If in $\mathcal{B}_k$ we have η_k not cohomologous to zero, then η_k is intrinsically harmonic by 5.2.1. Thus, there exist a closed 1-form η such that

$$\eta_l \wedge \eta > 0.$$

From the last equality, we can obtain a set with n linearly independet closed 1-forms defined on M. Again by Tischler's argument, there exists a surjective submersion $g : M \longrightarrow \mathbb{T}^n$. The map g is proper, because M is compact. By Lemma 3 in [35], we conclude that g is a covering map. A covering of $\mathbb{T}^n$ is diffeomorphic to $\mathbb{R}^n/H$, where H is subgroup of $\pi_1(T^n) = \mathbb{Z}^n$. They are compact only when H have the form $m_1\mathbb{Z} \times \cdots \times m_n\mathbb{Z}$, $m_i \in \mathbb{Z}$. Therefore, M and $\mathbb{T}^n$ are diffeomorphic. $\square$

Remark 5.3.1. Let M be a manifold with a closed-decomposable form $\omega = \omega_1 \wedge \cdots \wedge \omega_p$ defined on M. By Tischler's argument, there exist a sequence of integers (d_i) and a collection of smooth fiber bundles

$$\mathcal{B}_l = \{M, p_l, \mathbb{T}^p, F_l\}$$

such that $\omega_l = \frac{1}{d_l} p_l^* \Omega_{\mathbb{T}^p} \longrightarrow \omega$ in $\mathcal{D}_p$. In particular, if ω determines a nonzero cohomological class, then the forms $\pi_l^* \Omega_{\mathbb{T}^n}$ are eventually not cohomologous to zero by 1.7.9. Suppose ω is harmonic concerning to a Rimannian metric g on M. Then $\eta = *_g\omega$ is a closed form with rank $(n-p)$. Since $\omega_l \longrightarrow \omega$ in $\mathcal{D}_p$ and $\omega \wedge \eta > 0$, eventually we have $\omega_l \wedge \eta > 0$. Theorem 2.1.2 ensures that ω_l is intrinsically harmonic. From these observations, the study of intrinsical harmonicity of closed-decomposable p-forms is equivalent to the study about closed-decomposable p-forms $\pi^*\Omega_{\mathbb{T}^p}$ given in fiber bundles $\mathcal{B} = \{M, p, \mathbb{T}^p, F\}$ over the p-torus. We can start weakening the hypothesis in Theorem 2.1.2, considering only foliated bundles. By a result of J.F. Plante in [64], the fiber of a foliated bundle having a torus as base detemines a nonzero cohomological class. Indeed, in this case the group $\pi_1(\mathbb{T}^p)$ is abelian group, hence it has polinomial growth. On the other hand, Plante showed that in a foliated bundle where the fiber determines a zero homological class, the fundamental group of the base has polinomial growth. Consequently, in a foliated bundle having a torus as base, we have $[\pi^*(\Omega_{\mathbb{T}^p})] \neq 0$ by 5.2.7. More generally, using 3.3.8, we can obtain the following equivalence: given a foliated bundle $\mathcal{B} = \{M, p, \mathbb{T}^p, F\}$, the form $\pi^*(\mathbb{T}^p)$ is intrinsically harmonic if, and only if, $\mathcal{B}$ admits a $SL(p)$-transversal foliation.

6 Pointwise periodic flows on locally compact metric spaces

In the preceding chapter, we gave a condition to a pointwise periodic volume-preserving flow admits a global cross-section, Theorem 5.0.1. It was sufficiently strong to show that every asymptotic cycle for the flow is a nontrivial homological class if the induced $(n-1)$-form is not cohomologous to zero. We can extend this theorem for recurrent volume-preserving flows since, for these, the existence of a global cross-section will depend only on the class of the induced closed $(n-1)$-form.

It is well-known from dynamical systems' theory that a volume-preserving flow is pointwise recurrent. Hence it is natural to ask if we can obtain a uniform recurrence for them. We begin by analyzing what happens with a pointwise periodic flow (without singularity). Is such kind of flow recurrent? Since we want to obtain a global cross-section to flows and this condition is topological, then, to our purpose, we can ask if the orbits under this flow are the orbits under a (continuous) recurrent flow. Given a pointwise periodic flow Φ with period function λ, since the flow $\Psi(x,t) = \Phi(x, \lambda(x)t)$ (not necessarily continuous) has same orbits as the flow Φ and also is periodic (hence recurrent), this lead us to study the set of continuity of λ. Since D.B. Epstein [20] it is well-known that this set is invariant, open, and dense. On the other hand, if a flow is recurrent on a dense invariant subset, it is easily seen that it is recurrent (on full space). This fact leads to the study of pointwise periodic flows with continuous period function (in this case, flows on noncompact manifolds).

We noted that recurrence is equivalent to existing a suitable function sequence converging uniformly to the identity. In general, a function sequence on a compact metric space itself that converges uniformly determines an equicontinuous set. Thus, it is natural to ask when, given a flow Φ, the collection $\{\Phi_t; t \in \mathbb{R}\}$ is an equicontinuous set of functions. Such flows are called *equicontinuous*. For example, flows induced by Killing vector fields on Riemannian manifolds are equicontinuous. If a flow is equicontinuous, we can show the recurrence property for it. Thus, using Carrière's characterization of a geodesible flow, the above observations enable us to provide one of nowhere-vanishing intrinsically harmonic $(n-1)$-forms, improving the one given in Chapter 2 (see Theorem 6.3.8).

Let X be a locally compact metric space and Φ be a continuous pointwise periodic flow on X without singularity. We showed that the set of continuity of the period function is open and dense as follows. We showed the existence of a local cross-section through any point of X. Using these, we built Poincaré's first return map. From then on, the proof is identical as in Epstein's paper [20]. For some kinds of flows, namely, *locally* *weakly almost periodic flows*, we showed that Poincaré's first return is a homeomorphism for a suitable local cross-section. We established this result using the characterization of a locally weakly almost periodic flow given by W.H. Gottschalk [29], namely, flows whose decomposition of the phase space by the closure-orbits under the flow is a *closed decomposition* (equivalently, for compact metric spaces, the *orbit space* satisfies the Hausdorff property). This fact is fundamental for the main results stated here, namely, Theorem 6.8.1.

This chapter is organized as follows. In Section 6.1, we specialize the recursive properties of general transformation groups on arbitrary uniform spaces for flows on metric spaces. In Section 6.2, we use the Arzelá-Ascoli Theorem to prove that an equicontinuous flow on a compact metric space is recurrent. In general, any equicontinuous transformation group on a compact metric space is *almost periodic*, which implies recurrence [26]. Also it is presented a *inheritance theorem* for flows on compact metric spaces. In Section 6.3, we show that a volume-preserving flow on a closed manifold has a global cross-section if, and only if it is geodesible and the induced closed $(n-1)$-form determines a nontrivial cohomological class. In Section 6.4, we will establish the existence of a local cross-section to continuous flows around periodic points not fixed when the metric space is locally compact. From this, we built Poincaré's first return map. This construction is used in Section 6.5 to show the openness of the period function for pointwise periodic flows without singularity. In Section 6.6, we present some conditions for equicontinuity and the relation with the decomposition of the phase space by the orbit-closure under the transformation group. In sections 6.7 and 6.8, we will study pointwise periodic flows and show that a C^1-pointwise periodic and equicontinuous flow on a compact manifold is periodic, generalizing a theorem of A.W. Wasdley when the manifold is compact. In Section 6.9, we use the theory of asymptotic cycle for compact metric spaces and the results of the preceding sections to show that recurrent pointwise periodic C^1-flow with bounded period function is periodic. We finished this chapter with some problems for further study.

6.1 Recursive properties to flows

A *flow* on a topological space X is a continuous function $\Phi : X \times \mathbb{R} \longrightarrow X$ such that the folowing axioms are satisfied:

1 $\Phi(x, 0) = x$ for all $x \in X$.

2 $\Phi(xt, s) = \Phi(x, (t + s))$ for all $x \in X$ and $s, t \in \mathbb{R}$.

The second axiom can be written more concisely by $(xt)s = x(t + s)$. We will denote by Φ_t the function $x \in X \longrightarrow \Phi(x, t) \in X$. Since $\Phi_t \circ \Phi_{-t} = \Phi_0 = 1$, given any

$t \in \mathbb{R}$, the function Φ_t is a homeomorphism of X into itself with inverse given by Φ_{-t}. Let $A, B \subset \mathbb{R}$ and $X \subset M$. The set AB and XA are defined by $AB = \{a + b; a \in A, b \in B\}$ and $XA = \{xa; x \in X, a \in a\}$.

The *orbit* of a point $x \in X$ under Φ is defined by $x\mathbb{R} = \{x\}\mathbb{R}$. The *orbit space* X/Φ is defined as the image of the natural function $\pi : X \longrightarrow X/\Phi$, where $\pi(x) = \pi(y)$ if and only if $x = yt$ for some $t \in \mathbb{R}$. The orbit space has the smallest topology such that the function π is continuous. W.H. Gottschalk introduced in [27] many recursivity notions for topological transformations group on uniform spaces. The definition below is a specialization of some of these for metric spaces.

Definition 6.1.1. Let (X, d) be a metric space.

1) [Periodicity] A flow on X is said to be *periodic* provided there exists a nonzero real number t such that $xt = x$ for all $x \in X$.

2) [Almost periodicity] A flow on X is said to be *almost periodic* provided given $\epsilon > 0$ there exists a *relatively compact* subset A of $\mathbb{R}$ such that $d(x, xt) < \epsilon$ for all $t \in A$ (a subset A of $\mathbb{R}$ is said to be relatively compact if there exists a compact subset K of $\mathbb{R}$ such that $\mathbb{R} = KA$).

3) [Locally almost periodicity] A flow on X is said to be *locally almost periodic* at $x \in X$ provided given a neighborhood U of x there exists a neighborhood V of x and a relatively compact subset A of $\mathbb{R}$ such that $VA \subset U$. A flow is said to be locally almost periodic if it is locally almost periodic at each point of X.

4) [Weakly almost periodicity] A flow on X is said to be *weakly almost periodic* provided given $\epsilon > 0$ there exists a compact subset K of $\mathbb{R}$ such that for each $x \in X$ and $t \in \mathbb{R}$, there exists $s \in K$ with $d(xt, xs) < \epsilon$.

5) [Locally weakly almost periodicity] A flow on X is said to be *locally weakly almost periodic* at $x \in M$ provided given a neighborhood U of x there exists a neighborhood V of x and a compact subset K of $\mathbb{R}$ such that $V\mathbb{R} \subset UK$. A flow is said to be locally weakly almost periodic if it is locally weakly almost periodic at each point of X.

6) [Equicontinuity] A flow on X is said to be *equicontinuous* at $x \in M$ provided given $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \epsilon$. The flow is said to be *uniformly equicotinuous* provided given $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \epsilon$ for all $t \in \mathbb{R}$.

7) [Recurrence] A flow on X is said to be *recurrent* provided there exists a sequence $t_n \longrightarrow \infty$ such that $\lim_{n \longrightarrow \infty} \sup_{x \in X} d(x, xt_n) = 0$. Equivalently, a flow is recurrent if given $s, \epsilon > 0$, there exists $t > s$ such that $d(x, xt) < \epsilon$ for all $x \in X$.

8) [Distality] A flow on X is said to be *distal* provided given $x \neq y$ there exists $\epsilon > 0$ such that $d(xt, yt) \geqslant \epsilon$ for all $t \in \mathbb{R}$.

Remark 6.1.2. Let X be a compact metric space and Φ be a continuous flow on X. It is relatively easy to check that:

(1) if Φ is equicontinuous, then Φ is uniformly equicontinuous;

(2) if Φ is locally weakly almost periodic, then Φ is weakly almost periodic.

Remark 6.1.3 (Transformation group in general)**.** A *transformation group* is a triple (X, T, π) consisting of a nonvacuous topological space X, a topological group T and a continuous map $\pi : X \times T \longrightarrow X$ satisfying the same conditions of a flow, namely, $xe = x$ and $(xt)s = x(ts)$ for all $x \in X$, $s, t \in T$, where e denotes the identity element of T. We will now see as we can extend the recursive properties given in 6.1.1 for transformation groups in general. For this aim, we need to define a class of subsets of T. A subset A of T is said to be $\{$*left*$\}$ $\{$*right*$\}$ *syndetic* in G provided that $\{T = AK\}\{T = KA\}$ for some compact subset K of T. If A is an invariant subgroup, that is, $gAg^{-1} \subset A$ for all $g \in T$, then A is left syndetic if, and only if, is right syndetic, hence for invariant subgroups for short we say only syndetic. If a flow Φ is periodic with period t, then $xt = x$ for all $x \in X$. Hence, $H = \{mt \in \mathbb{Z}\}$ is a syndetic subset of $\mathbb{R}$ and $xs = x$ for all $x \in X$ and $s \in H$. Thus, we have a reasonable definition of periodic transformation group: a transformation group is said to be periodic if there exists a syndetic subset P of T such that $xt = x$ for all $t \in P$. A transformation group is said to be *almost periodic* if given $\epsilon > 0$, there exist a syndetic subset A of T such that $d(x, xt) < \epsilon$ for all $t \in A$. The notions of weakly almost periodicity, locally almost periodicity, locally weakly almost periodicity, distality, and equicontinuity for transformation groups follow analogously. The notion of recurrence for a transformation group requires more definitions. A subset A of T is said to be *replet* provided S contains some bilateral translate of each compact subset of T (for instance, the sets $A = [t, \infty)$ are replete in $\mathbb{R}$). A subset of A of T is said to be *extensive* provided that intersept every replet *semigroup* in T (a semigroup of T is a subset S satisfying $SS \subset S$). For instance, if $T = \mathbb{R}$, then a subset A of T is extensive if, and only if, contains a sequence marching to $+\infty$ and a sequence marching to $-\infty$. A transformation group T is said to be *recurrent* if given $\epsilon > 0$, there exists an extensive subset A of T such that $d(x, xt) < \epsilon$ for all $t \in A$.

6.2 Recurrence from equicontinuity

Let X be a topological space and a (Y, d) be a metric space. The space $C(X, Y)$ of all continuous functions of X to Y has a natural topology induced by the metric d. It is called *the topology of uniform convergence* and has as subbasis for open subsets the sets

$$B_\epsilon(f) = \{g \in C(X, Y) / \sup_{x \in X} d(f(x), g(x)) < \epsilon\}$$

where f range over $C(X,Y)$ and ϵ range all the real numbers. It and the co.o.-topology coincides when X is compact [1]. When X is noncompact, the latter is coarsest than the former.

Theorem 6.2.1 (S. Myers, [60])**.** Let C be the family of all continuous functions from a regular, locally compact, and Hausdorff topological space into a metric space and C has the co.o.-topology. Then a subfamily T of C is compact if and only if

(1) T is closed in C,

(2) xT has compact closure for each $x \in X$, and

(3) the family T is equicontinuous[2].

.

Comparing Theorem 6.2.1 with the original in Mayers' paper, the hypothesis that $\overline{xT}$ is compact for all $x \in X$ replaces the connectedness of X and completeness of Y; see also [41] pages 233-234. S. Myers also concluded the following. Let G be an equicontinuous group of homeomorphisms on metric space into itself. If G has the co.o.-topology, then it is a topological group. It follows that $\overline{G}$ equipped with the co.o.-topology is also a (compact) group of transformations on X.

Lemma 6.2.2. Let G be a compact topological group. Given $g \in G$, there exists a sequence of integers $n_k \longrightarrow \infty$ such that $g^{n_k} \longrightarrow e$.

Proof. Since G is compact, given $g \in G$, the sequence $(g^n)_{n\in\mathbb{N}}$ has a convergente subsequence (g^{m_s}). Given any integer k, let $m_r \in \{m_s\}$ such that $n_k := m_r - m_k > k$, $m_k \in \{m_s\}$. Using the continuity of the product and inversion operations of G and letting $k \longrightarrow \infty$, then

$$g^{n_k} = g^{m_r} g^{-m_k} \longrightarrow e.$$

The proof is completed. □

Theorem 6.2.3. An equicontinuous flow on compact metric space is recurrent.

Proof. Let Φ be an equicontinuous flow on a compact metric space X. If Φ is periodic, there is nothing to be done. Suppose Φ not periodic. Then $G = \{\Phi_t\}$ is an equicontinuous group of homeomorphisms of X. It follows by 6.2.1 and 1.2.19 that $\overline{G}$ with the co.o.-topology is a compact topological group. Hence, from 6.2.2, given $t \in \mathbb{R}$, there exists a sequence

[1] More generally, when Y is a uniform space, one can define this notion analogously. See [41] page 230 for the equivalence between the topology of uniform convergence and the co.o.-topology for compact uniform spaces.

[2] A set T of continuous functions from a topological space X into a metric space (Y,d) is called *equicontinuous* provided corresponding to each $x_0 \in X$ and each $\epsilon > 0$, there is a neighborhood U of x_0 in X such that $d(tx_0, tx) < \epsilon$ for all $t \in T$ and all $x \in U$.

$n_k \longrightarrow \infty$, $n_k \in \mathbb{Z}$, such that $\Phi_{tn_k} \longrightarrow 1$ in the co.o.-topology. Since X is compact, the co.o.-topology coincides with the uniform topology. It follows that $\Phi_{tn_k} \longrightarrow 1$ uniformly, hence Φ is recurrent. □

The *inheritance theorem* consists of proving the following. If the flow admits a recursive property, then some category of subgroups admits this property. Such a kind of theorem is treated extensively in the book Topological Dynamics [29]. R. A. Christiansen provided in [12] an inheritance theorem for recurrent transformation group (X, T, π) where X is a compact uniformizable space. Next, we establish an inheritance theorem for flows.

Theorem 6.2.4. Let Φ be a continuous flow on a compact metric space (X, d). Let H be a nontrivial subgroup of $\mathbb{R}$. Then Φ is recurrent if and only if the restriction $\Phi : X \times H \longrightarrow X$ is a recurrent flow.

Proof. Suppose there exists a sequence $t_n \longrightarrow \infty$ such that $\Phi_{t_n} \longrightarrow 1_X$ uniformly. Let $h \in H$ be a positive number. For each $n \in \mathbb{N}$, write $t_n = k_n h - \alpha_n$, with $k_n \in \mathbb{Z}$ and $\alpha_n \in [0, h]$. Set $F = \{\Phi_t / t \in [0, h]\}$ with the topology of $[0, h]$. This topology on F is compact and admissible. Hence, by 1.2.22, it coincides with the co.o.-topology. Taking a subsequence if necessary, we have that $\Phi_{\alpha_n} \longrightarrow \Phi_\alpha$ in the co.o.-topology for some $\alpha \in [0, h]$. Since the group of homeomorphisms of X with the co.o.-topology is a topological group by 1.2.19, then $\Phi_{k_n h} \longrightarrow \Phi_\alpha$ in this topology. For each $n \in \mathbb{N}$, let k_m such that $k_m - k_n > n$. Then, $(k_m - k_n)h \in H$ and letting $n \longrightarrow \infty$ we have

$$(k_m - k_n)h \longrightarrow \infty$$

and

$$\Phi_{(k_m - k_n)h} = \Phi_{k_m h}\Phi_{-k_n h} \longrightarrow 1.$$

Since M is compact, the co.o.-topology and the uniform topology coincides. It follows that the restriction $\Phi : X \times H \longrightarrow X$ is a (possibly discrete) recurrent flow. □

6.3 A characterization of volume-preserving flows with a global cross-section

If a closed manifold supports a smooth flow with a global cross-section, then there exists a Riemannian metric so that each orbit under the flow is a geodesic (see 6.3.7). In this section, we provide the converse for volume-preserving flows. We have shown that it depends only on the cohomological class determined by the induced form. From this, by proving Theorem 6.3.8, we improve the characterization of nowhere-vanishing intrinsically harmonic $(n-1)$-forms presented in Chapter 2. Throughout this section, the flows and vector fields considered are smooth.

Definition 6.3.1. Let M be a manifold. A flow on M is said to be *geodesible* if there exists a Riemannian metric g on M such that every orbit under the flow is a geodesic.

Theorem 6.3.2. Let M be a manifold and X be a nowhere-vanishing vector field on M. Are equivalents:

(1) there exists a 1-form ω with $\omega(X) = 1$ and $\mathcal{L}_X\omega = 0$;

(2) there exists a 1-form ω with $\omega(X) = 1$ and $i_X d\omega = 0$;

(3) the flow generated by X is geodesible by a Riemannian metric g satisfying $g(X, X) = 1$;

(4) there exists a $(n-1)$-plane bundle $P \subset TM$, complementary to X, such that $[X, Y] \in P$ for all $Y \in P$.

Proof. First of all, let g be a Riemanninan metric on M satisfying $g(X, X) = 1$. Let ω be the 1-form given by $\omega(Y) = g(X, Y)$. We claim that

$$\mathcal{L}_X\omega(Y) = g(\nabla_X X, Y),$$

where ∇ is the Levi-Civita connection associated to g. Let $(x_1, \ldots, x_n)$ be a local coordinate system with $\partial_{x_1} = X$. Denoting $Y = \partial_{x_i}$, we have

$$\mathcal{L}_X\omega(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X(Y)).$$

Since $\mathcal{L}_X(Y) = [\partial_{x_1}, \partial_{x_i}] = 0$, then

$$\mathcal{L}_X\omega(Y) = \mathcal{L}_X(\omega(Y)) = \mathcal{L}_X(g(X, Y)) = g(\nabla_X X, Y) + g(X, \nabla_X Y).$$

Using the equations $\nabla_X Y = \nabla_Y X + \mathcal{L}_X Y$ and $\mathcal{L}_X Y = 0$, we have

$$\mathcal{L}_X\omega(Y) = g(\nabla_X X, Y) + g(X, \nabla_Y X).$$

By the equality $Yg(X, X) = 2g(X, \nabla_Y X)$ then

$$\mathcal{L}_X\omega(Y) = g(\nabla_X X, Y) + \frac{1}{2}Yg(X, X).$$

Hence, since $g(X, X) = 1$, it follows that

$$\mathcal{L}_X\omega(Y) = g(\nabla_X X, Y),$$

concluding the claim.

(1)$\Rightarrow$(2) If $\omega(X) = 1$ and $\mathcal{L}_X\omega = 0$, then

$$0 = \mathcal{L}_X\omega = di_X\omega + i_X d\omega.$$

Since $d(\omega(X)) = 0$, it follows that $i_X d\omega = 0$. Thus, (1) implies (2).

(2)⇔(3) Suppose now that ω is a 1-form satisfying $\omega(X) = 1$ and $\mathcal{L}_X\omega = 0$. Let h be a Riemannian metric on M such that X and $\ker\omega$ are orthogonal concerning h. Setting $g = \frac{1}{h(X,X)}h$, we have $g(X,X) = 1$ and $\omega(Y) = g(X,Y)$ (since any Y is expressible as $\omega(Y)X + Z$, with $Z \in \ker\omega$). Now, since $\mathcal{L}\omega(Y) = g(\nabla_X X, Y)$ for all Y, and $\mathcal{L}_X = 0$, it follows that $\nabla_X X = 0$, hence X is geodesible. Conversely, if all orbits under the flow generated by X are geodesic concerning to a Riemannian metric g on M, then the form $\omega(Y) = g(X,Y)$ satisfies $\mathcal{L}_X\omega = 0$. Thus (2) and (3) are equivalents.

(2)⇒(4) Suppose there exists a 1-form ω with $\omega(X) = 1$ and $i_X d\omega = 0$. Set $P = \ker\omega$. Given $Y \in P$ we have

$$\omega([X,Y]) = -d\omega(X,Y) + X\omega(Y) - Y\omega(X) = 0.$$

Hence $[X,Y] \in P$ whenever $Y \in P$. Thus, (2) implies (4).

(4)⇒(1) Suppose (4) holds. Let g be a Riemannian metric on M such that X and P are orthogonal and $g(X,X) = 1$. Set $\omega(Y) = g(X,Y)$. Note that $P = \ker\omega$ and $\omega(X) = 1$. Given $Y \in P$, we have

$$i_X d\omega(Y) = X(\omega(Y)) - Y(\omega(X)) + \omega([X,Y]).$$

Since $[X,Y] \in P$ by hypothesis, it follows that $i_X d\omega = 0$. Hence

$$\mathcal{L}_X\omega = di_X\omega + i_X d\omega = 0.$$

□

Remark 6.3.3. The Theorem 6.3.2 follows from Rummler's criterion 3.3.4 since a 1-dimensional submanifold is minimal if, and only if, it is a geodesic.

Definition 6.3.4. A flow $\Phi : (-\epsilon,\epsilon) \times M \longrightarrow M$ on a Riemannian manifold (M,g) is said to be *isometric* provided Φ_t is an isometry for all $t \in (-\epsilon,\epsilon)$. A vector field X on (M,g) is said to be *Killing* if the flow generated by X is isometric. A flow Φ on M is said to be *isometrisable*, if there exists a Riemannian metric g on M such that Φ is given by a nowhere-vanishing Killing vector field.

Theorem 6.3.5 (Y. Carrière, [11])**.** A nowhere-vanishing vector is geodesible if and only if it is isometrisable.

Proof. Suppose that X is a nowhere-vanishing Killing vector on a Riemannian manifold (M,g). Denote by Φ the flow generated by X. Given $x \in M$, we have

$$[X,Y](x) = \mathcal{L}_X Y(x) = \lim_{t\longrightarrow 0} 1/t((\Phi_{-t})_* Y_{\Phi_t(x)} - Y_x).$$

Hence $\mathcal{L}_X Y(x)$ is orthogonal to $X(x)$ since $Y(x)$ and $X(x)$ are orthogonal and Φ_t is isometry for all t where Φ_t have sense. Thus, the $(n-1)$-plane bundle set as the complement orthogonal to X satisfies the condition (4) of Theorem 6.3.2. Therefore, X is geodesible.

Conversely, suppose there exists a Riemannian metric g on M such that $\nabla_X X = 0$. Since $g(X,X)$ is constant, we can suppose $g(X,X) = 1$. We will use the well-known characterization: a vector field X is Killing if and only if $\mathcal{L}_X g = 0$. Since

$$\mathcal{L}_X g(Y,Z) = 1/2\{\mathcal{L}_X g(Y+Z, Y+Z) - \mathcal{L}_X g(Y,Y) - \mathcal{L}_X g(Z,Z)\},$$

it is enough to show that $\mathcal{L}_X g(Y,Y) = 0$ for all $Y \in TM$. Now, $\mathcal{L}_X g$ is a tensor. Hence it is determined only by its value at X and $X^{\perp}$. Since

$$\mathcal{L}_X g(Y,Z) = Xg(Y,Z) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z),$$

we have

$$\mathcal{L}_X g(Y,Y) = Xg(Y,Y) - 2g(\mathcal{L}_X Y, Y).$$

Using the equality $\mathcal{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X$, then

$$\begin{aligned} \mathcal{L}_X g(Y,Y) - Xg(Y,Y) = -2g(\mathcal{L}_X Y, Y) = \\ -2g(\nabla_X Y - \nabla_Y X, Y) = \\ -2\{1/2 Xg(Y,Y) - g(\nabla_Y X, Y)\} = \\ 2g(\nabla_Y X, Y) = \\ 2\{Yg(X,Y) - g(X, \nabla_Y Y)\} = \\ -2g(X, \nabla_Y Y). \end{aligned}$$

It follows from these calculations that

$$\mathcal{L}_X(Y,Y) = Xg(Y,Y) - 2g(X, \nabla_Y Y) \text{ for all } Y \in TM. \tag{6.1}$$

Let $x \in X$. Given $v_i \in T_x M$ such that $\{X(x), v_2, \ldots, v_n\}$ is an ortonormal basis for $T_x M$, and open set $U \subset T_p M$ such that $\exp_x : U \longrightarrow \exp(U)$ is a diffeomorphism, the function

$$\psi(x_1, x_2, \ldots, x_n) = \exp_x(x_1 X(x) + x_2 v_2 + \ldots + x_n v_n)$$

is a local coordinate neighborhood safistying $\nabla_{\partial_{x_i}} \partial_{x_i}(x) = 0$ for all $i = 1, \ldots, n$. Then, using the equation 6.1 we have

$$\mathcal{L}_X(\partial_{x_i}, \partial_{x_i})(x) = Xg(\partial_{x_i}, \partial_{x_i})(x) - 2g(X(x), \nabla_{\partial_{x_i}} \partial_{x_i}(x)) = Xg(\partial_{x_i}, \partial_{x_i})(x).$$

Since the curve $\alpha(t) = \exp_x(tX(x))$ is a geodesic with $\alpha'(0) = X(x)$, and by hypothesis $\nabla_X X = 0$, it follows by unicity of geodesics that $\alpha'(t) = X(\alpha(t))$ for all t. Thus,

$$Xg(\partial_{x_i}, \partial_{x_i})(x) =$$

$$\alpha^{'}(t)g(\partial_{x_i}, \partial_{x_i})|_{t=0} =$$
$$2g(\nabla_{\partial_{x_1}}\partial_{x_i}(x), \partial_{x_i}(x)) = 0.$$

Therefore X is a Killing vector field. □

Theorem 6.3.6. Let M be an orientable closed manifold. A nowhere-vanishing volume-preserving flow on M has a global cross-section if it is geodesible and the induced closed $(n-1)$-form determines a nontrivial cohomological class.

Proof. Let X be a geodesible vector field. Let Φ the flow generated by X. By Carrière's theorem 6.3.5, there exists a Riemannian metric g on M such that Φ_t is isometry for all $t \in \mathbb{R}$. The Riemannian metric g induces a metric d on M generating the topology of M. Since M is compact, given $x, y \in M$, there exists a (minimizant) geodesic γ on M with $d(x, y) = \text{lenght}(\gamma)$ (see [10], chapter 7). Now, an isometry $f : M \longrightarrow M$ carries geodesic in geodesic and preserves the length of curves. Hence, if $d(x, y) = l(\gamma)$, then

$$d(fx, fy) \leqslant l(f\gamma) = l(\gamma).$$

Given now $\gamma^{'}$ be a minimizant geodesic from fx to fy, then

$$d(x, y) \leqslant l(f^{-1}\gamma^{'}) = l(\gamma^{'}).$$

Therefore $d(fx, fy) = d(x, y)$ for all $x, y \in M$. It follows that a Riemannian isometry preserves the induced metric d. From this we conclude that Φ is a (uniformly) equicontinuous flow on (M, d). The Theorem 6.2.3 then ensures that Φ is a recurrent flow. We know by Theorem 4.1.16 a recurrent flow has only one asymptotic cycle. Since the class of the closed $(n-1)$-form induced by X is an asymptotic cycle by 5.1.1 and also determines a nontrivial cohomological class by hypothesis, it follows by Theorem 4.3.2 that there exists a global cross-section to Φ. The proof is completed. □

Theorem 6.3.7. A flow Φ on a closed manifold M has a global cross-section if, and only if, it is geodesible and admits a nontrivial asymptotic cycle.

Proof. It remains only to show that flows with global cross-section are geodesible (since geodesible flows have the same orbits as a recurrent one). Let Φ be a flow on M with a global cross-section. Then every asymptotic cycle of Φ is nontrivial. Hence by Corollary 4.3.3, there exists a closed 1-form ω transversal to Φ. Let X be the vector field generating Φ. Then $\omega(X) > 0$ and, trivially, $i_X d\omega = 0$. Thus, the vector field $Y = \frac{1}{\omega(X)}X$ generates a flow with the same orbits under Φ and satisfies $\omega(Y) = 1$ and $i_Y d\omega = 0$. It follows by 6.3.2 that Φ is geodesible. □

We finish this section with a characterization of nowhere-vanishing intrinsically harmonic $(n-1)$-forms.

Theorem 6.3.8. Let ω be a nowhere-vanishing closed $(n-1)$-form on a closed manifold M and X be a vector field generating the foliation induced by ω. Suppose $[\omega] \neq 0$ in $H^{n-1}_{DR}(M)$. Are equivalentes:

(1) there exists a 1-form η satisfying $\eta \wedge \omega > 0$ and $i_X d\eta = 0$;

(2) X is geodesible;

(3) X is Killing for some Riemannian metric on M;

(4) ω is intrinsically harmonic.

Proof. It remains only show that the condition given in (1) implies (2). In this case, we have that $\eta(X) > 0$. Thus, $Y = \frac{1}{\eta(X)}X$ induce a flow with the same orbits as the flow generated by X. Furthermore, it satisfies

$$\eta(Y) = 1 \text{ and } i_Y d\eta = 0.$$

Hence Y is geodesible by 6.3.2. It follows that X is geodesible. The proof is completed. □

It is reasonable the conjecture that a closed manifold is the total space of a fiber bundle over the circle if, and only if, it admits a nowhere-vanishing closed $(n-1)$-form determining a nonzero cohomological class. This assertion is the "dual" of Tischler's theorem, which asserts a closed manifold is the total space of a fiber bundle over $\mathbb{S}^1$ if and only if admits a nowhere-vanishing closed 1-form. The necessity condition follows since if $p : M \longrightarrow \mathbb{S}^1$ is a submersion, M closed, then the closed-form $p^*d\theta$ is intrinsically harmonic, hence there exists a closed $(n-1)$-form ω satisfying $\omega \wedge p^*d\theta > 0$. Furthemore, a volume-preserving flow is pointwise recurrent, then, maybe, it is recurrent in sense of definition 6.1.1.

6.4 Local cross-section to flows

In this section, we establish the existence of local cross-section to flows on locally compact metric spaces around periodic points (such constructions are possible for recurrent points). From this, we built Poincaré's first return map and the map of the first return.

Definition 6.4.1 (Local cross-section)**.** Let X be a topological space and $\Phi : X \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous flow on X. A subset K of X is said to be a *local cross-section* to Φ through $x \in X$ provided there exist $\epsilon > 0$ such that the natural map $\Phi : K \times [-\epsilon, \epsilon] \longrightarrow S[-\epsilon, \epsilon]$ is bijective and x is an interior point of $K[-\epsilon, \epsilon]$. We will denote this configuration by (K, ϵ).

Remark 6.4.2. Let X be a Hausdorff topological space and Φ be a continuous flow on X. Suppose that there exists a local cross-section (K,ϵ) through $x \in X$, with K compact. Then the natural map $K \times [-\epsilon,\epsilon] \longrightarrow K$ being a continuous bijective map between a compact space and a Hausdorff space become a homeomorphism. From this, it is easily seen that for every t, with $|t| < \epsilon$, the point xt is an interior point of $K[-\epsilon,\epsilon]$.

Remark 6.4.3 (The local cross-section for built the Poincaré's first return map)**.** Let X be a metric space and Φ be a continuous flow on X periodic at $x \in X$. Suppose there exists a local cross-section (K,ϵ) through x, with K compact. Then, given a positive number ϵ with $\epsilon < \lambda(x)/2$, there exists a neighborhood K' of x in K such that $(K', \lambda(x)/2 - \epsilon)$ is a local cross-section through x. By Remark 6.4.2, it is suffice to show that for some neighborhood U of x in K, the natural map $U \times [-\lambda(x)/2 + \epsilon, \lambda(x)/2 - \epsilon] \longrightarrow X$ is injective. Otherwise, there exist sequences $y_n, z_n \longrightarrow x$ with $y_n \neq z_n$ and t_n, s_n with $|t_n|, |s_n| \leqslant \lambda(x)/2 - \epsilon$ and $y_n t_n = z_n s_n$. Taking a subsequence if necessary, we can suppose that $t_n \longrightarrow t$ and $s_n \longrightarrow s$. Hence, $xt = xs$ and, therefore, $s - t = m\lambda(x)$ for some $m \in \mathbb{Z}$. Since $|t_n - s_n| \leqslant \lambda(x) - 2\epsilon$ for all n, it follows that $|t - s| \leqslant \lambda(x) - \epsilon$, hence $m = 0$. Thus, given n with $|t_n - s_n| < \epsilon$ we have $y_n(t_n - s_n) = z_n$ with $|t_n - s_n| < \epsilon$, $y_n, z_n \in K$. Since (K,ϵ) is a local cross-section, then $t_n - s_n = 0$ and $y_n = z_n$, a contradiction.

6.4.1 Poincaré's first return map

Let X be a metric space and $\Phi : X \times \mathbb{R} \longrightarrow X$ be a continuous flow periodic at $x \in X$. Suppose there exists a local cross-section (K,ϵ) through x with K compact (necessarily we have $\lambda(x) > 0$). We can take ϵ with $4\epsilon < \lambda(x)$. By 6.4.3 there exists K such that $(K, \lambda(x)/2 - \epsilon)$ and $(K, 2\epsilon)$ are also local cross-section through x. The function $g : K[-\epsilon,\epsilon] \longrightarrow M$ given by $g(y) = y\lambda(x)$ is continuous. Since $g(x) = x$ and x is an interior point of $K[-\epsilon,\epsilon]$, there exists a neighborhood U of x in M such that $g(U) \subset K[-\epsilon,\epsilon]$. Then $U\lambda(x) \subset K[-\epsilon,\epsilon]$. We will make use of the following constructions and facts.

(1) (Poincaré's first return map) Let $y \in U$. Since $g(y) \in K[-\epsilon,\epsilon]$ and (K,ϵ) is a local cross-section, there exist $s \in [-\epsilon,\epsilon]$ and $k \in K$ with $y\lambda(x) = ks$. Furthemore, k and s are uniquely determined. Set

$$f : U \longrightarrow \mathbb{R} \text{ and } T : U \longrightarrow K$$

by $f(y) = \lambda(x) - s$ and $T(y) = k = yf(y)$. The function f is called *the time for the first return*. Note that T is injective, since if $T(y) = T(z) = k$, then $y\lambda(x) = ks = kl = z\lambda(x)$, with $k \in K$ and $s, l \in [-\epsilon,\epsilon]$. Since $(K, 2\epsilon)$ is also a local cross-section, $y, z \in K$, $|l - s| \leqslant 2\epsilon$ and $y = z(l - s)$, it follows that $s = l$ and $y = z$.

(2) T and f are continuous. Denote by p_1 and p_2 the natural projections $K \times \mathbb{R} \longrightarrow K$ and $K \times \mathbb{R} \longrightarrow \mathbb{R}$, respectively. Since $y\lambda(x) = \Phi|_{K\times[-\epsilon,\epsilon]}(k,s)$, and $\Phi : K \times [-\epsilon,\epsilon] \longrightarrow$

$K[-\epsilon, \epsilon]$ is a homeomorphism (see 6.4.2), then

$$T(y) = p_1 \left(\Phi|_{K\times[-\epsilon,\epsilon]}\right)^{-1} (y\lambda(x)) \text{ and } f(y) = p_2 \left(\Phi|_{K\times[-\epsilon,\epsilon]}\right)^{-1} (y\lambda(x))$$

are continuous. If X is a smooth manifold, those maps have the class of differentiability of Φ.

(3) Suppose now that Φ is pointwise periodic. Denote by λ the period function of Φ. We will show that $f \leqslant \lambda$ for small U. Since λ is lower semincontinuous (see Lemma 6.4.5 below), there exists a neighborhood W of x in M such that $y \in W$ implies $\lambda(y) > \lambda(x) - \epsilon$. Taking U smaller if necessary, we can suppose that $U \subset W$. Suppose that $f(y) = \lambda(y) + r$, $r \geqslant 0$. Then $f(y) = \lambda(x) - s = \lambda(y) + r$. This implies that $0 \leqslant r = \lambda(x) - \lambda(y) - s < 2\epsilon$, since $\lambda(y) > \lambda(x) - \epsilon$ and $-\epsilon \leqslant s \leqslant \epsilon$. Then $k = yf(y) = y(\lambda(y) + r) = yr$. But the function $\Phi : K \times [-2\epsilon, 2\epsilon] \longrightarrow K[-2\epsilon, 2\epsilon]$ is bijective and $k = \Phi(k, 0) = \Phi(y, r)$. This implies that $k = y$ and $r = 0$, concluding that $f \leqslant \lambda$.

(4) Suppose now that λ is continuous at $x \in X$. Then $f = \lambda$ for a small U. Therefore, the Poincaré first return map is the identity for a small U and λ is continuous in U. In the construction, take W a neighborhood of x such that $y \in W$ implies $-\epsilon < \lambda(y) - \lambda(x) < \epsilon$. If $f(y) = \lambda(y) - r$, with $r \geqslant 0$, then $r = \lambda(y) - \lambda(x) + s < 2\epsilon$. As in the item (3), we conclude that $r = 0$. It follows that $T(y) = yf(y) = y\lambda(y) = y$.

Remark 6.4.4. In the contruction of the Poincaré's first return map, the map of first return f has this name due to the following characterization:

$$f(y) = \inf_{t>0}\{yt \in K\}.$$

Hence why the name "Poincaré's first return map". Indeed, suppose that given $y \in K$, there exists $0 < t \leqslant f(y)$ with $yt \in K$. Since $(K, \lambda(x)/2 - \epsilon)$ is local cross-section, then $t > \lambda(x)/2 - \epsilon$. Now, $f(y) = \lambda(x) - s$, $|s| \leqslant \epsilon$, and $yf(y) = k$, $yt = k'$, for some $k, k' \in K$. It follows that $k' = k(t + s - \lambda(x))$. We have

$$\begin{aligned} \lambda(x)/2 - \epsilon < t \leqslant \lambda(x) - s \Longrightarrow \\ -\lambda(x)/2 - \epsilon + s < t + s - \lambda(x) \leqslant 0 \Longrightarrow \\ -\lambda(x)/2 + \epsilon \leqslant -\lambda(x)/2 - \epsilon + s + 3\epsilon \leqslant t + s - \lambda(x) + 3\epsilon \leqslant 3\epsilon < \lambda(x)/2 - \epsilon. \end{aligned}$$

Thus, $k'(3\epsilon) = k(t + s - \lambda(x) + 3\epsilon)$ with $3\epsilon, |t + s - \lambda(x) + 3\epsilon| \leqslant \lambda(x)/2 - \epsilon$ and $k, k' \in K$. Since $(K, \lambda(x)/2 - \epsilon)$ is a local cross section, then $k = k'$ and $t + s - \lambda(x) = 0$. This proves the claim.

Lemma 6.4.5. Let Φ be a continuous pointwise periodic flow without singularity[3] on a metric space X. Then the period function λ is lower semicontinuous.

[3] By a flow Φ on topological space X *without singularity* we mean one *without fixed point*, that is, if $xt = x$ for some $x \in X$ and $t \in \mathbb{R}$ then $t = 0$.

Proof. Let $x \in X$. It is enough to show that if $\lambda(x) > t$, $t \in \mathbb{R}$, then there exists a neighborhood U of x such that $\lambda(y) > t$ for all $y \in U$. Suppose $\lambda(x) = 0$ and $\lambda(x) > t$. Then t is a negative number. It follows that $\lambda(y) > t$ for all $y \in X$, since λ is a nonnegative function. Hence, there is no loss of generality in assuming $t > 0$ and $\lambda(x) > 0$. Set $f : X \times [-t, t] \longrightarrow \mathbb{R}$ the function given by $f(y, s) = d(y, ys)$. Since $\lambda(x) > t$ and $\lambda(x) \neq 0$, we have that $f(x, \delta) > 0$ for all $\delta \in [-t, t]$. By the continuity of f, there exist neighborhoods U_δ of x and I_δ of δ such that f is positive on $U_\delta \times I_\delta$. Let $\delta_1, \cdots, \delta_k$ such that $[-t, t] = \bigcup I_{\delta_i}$ and set $U = \bigcap U_{\delta_i}$. It follows that U is neighborhood of x and f is positive on $U \times [-t, t]$. In particular, given $y \in U$ we have that $\lambda(y) > t$, concluding that λ is lower semicontinuous. $\square$

6.4.2 Existence of local cross-section to continuous flows on locally compact metric spaces

Let X be a metric space and Φ be a continuous flow on X. Let $f : X \longrightarrow \mathbb{R}$ be a continuous function. We call f be diferentiable with respect to the flow provided that the limit $\lim_{t \longrightarrow 0} f(xt)$ exists for all $x \in X$. This limite will be denoted by $f'(x)$.

Theorem 6.4.6. Let X be a locally compact metric space and Φ be a continuous flow on X. Let $f : X \longrightarrow \mathbb{R}$ be a continuous function differentiable with respect to the flow. Suppose that $f'(x) > 0$ and the function f' is continuous at $x \in X$. Then there exists a compact local cross-section through x.

Proof. We claim that there exists $\epsilon > 0$ and a compact neighborhood U of x in X such that the function $t \in [-2\epsilon, 2\epsilon] \longrightarrow f(yt)$ is increasing for each $y \in U$. Indeed, let V be a neighborhood of x in X such that $f'(y) > 0$ for each $y \in V$. By continuity of Φ and the local compactness of X, there exists a compact neighborhood U of x in X and $\delta > 0$ such that $\Phi(U, [-2\epsilon, 2\epsilon]) \subset V$. If $y \in U$, then $f'(yt) > 0$ for $|t| < 2\epsilon$. Thus, the function $t \in [-2\epsilon, 2\epsilon] \longrightarrow f(yt)$ is an increasing function as claimed.

We will now show that the compact set $K = U \cap f^{-1}(x)$ is a local cross-section through x. The proof will be divided into two steps.

(1) The natural function $\Phi : K \times [-\epsilon, \epsilon] \longrightarrow K[-\epsilon, \epsilon]$ is a continuous bijection. Indeed. This function is continuous since it is the restriction of such one. Now, if $y_1 t_1 = y_2 t_2$, then $y_1 = y_2(t_2 - t_1)$ and $f(y_1) = f(y_2(t_2 - t_1))$ with $f(y_1) = f(y_2) = f(x)$. Since the function $t \in [-\epsilon, \epsilon] \longrightarrow f(y_2 t)$ is one-to-one for $|t| < 2\epsilon$ and $f(y_2) = f(y_1) = f(y_2(t_2 - t_1))$ with $|t_2 - t_1| < 2\epsilon$, then $t_2 - t_1 = 0$. Therefore $y_1 = y_2$.

(2) The point x is an interior of $K[-\epsilon, \epsilon]$. Indeed, since X is a locally compact metric space, f is continuous at x, and the flow is continuous, for each $n \in \mathbb{N}$, we can obtain a compact neighborhood U_n of x such that $y \in U_n$ implies $-\frac{1}{n} < f(x) - f(y) < \frac{1}{n}$.

Moreover, we can take such a neighborhoods satisfying $U_{n+1} \subset U_n$ and $U_n[-\epsilon, \epsilon] \subset U$ for all $n \in \mathbb{N}$. Set

$$s_n = \sup\{f(y(-\epsilon)); y \in U_n\} \text{ and } t_n = \inf\{f(y\epsilon); y \in U_n\}.$$

Then $\{s_n\}$ is a monotone limited non-increasing sequence and $\{t_n\}$ is a monotone limited nondecreasing sequence. For each $n \in \mathbb{N}$, let $y_n \in U_n$ such that $f(y_n(-\epsilon)) = s_n$. By continuity of f at x and the continuity of Φ we have

$$\lim s_n = f(\lim y_n(-\epsilon)) = f(x(-\epsilon))$$

and, analogously, $\lim t_n = f(x\epsilon)$. Thus, since $f(x(-\epsilon)) < f(x) < f(x\epsilon)$, there exists m such that $s_m < f(x) < t_m$. Taking $m > 1/\epsilon$, If $y \in U_m$, we have that

$$f(y(-\epsilon)) \leqslant s_m < f(x) < t_m \leqslant f(y\epsilon).$$

Using the continuity of $t \in [-\epsilon, \epsilon] \longrightarrow f(ty)$ and the connectedness of $[-\epsilon, \epsilon]$, we can ensure the existence of $|t| < \epsilon$ such that $f(yt) = f(x)$. By construction of the sequence U_n, $yt \in U$. Therefore, $yt \in U \cap f^{-1}(x) = K$. It follows that $y \in K[-\epsilon, \epsilon]$. Hence $U_m \subset K[-\epsilon, \epsilon]$, concluding that x is an interior point of $K[-\epsilon/2, \epsilon/2]$.

□

Lemma 6.4.7. Let Φ be a continuous flow without singularity on a locally compact metric space X. Then there exists a compact local cross-section through each point $x \in X$[4] .

Proof. Let $x \in X$. If x is a periodic point under Φ, let t_0 any real number with $0 < t_0 \leqslant \lambda(x)$. If x is nonperiodic, let t_0 any real positive number. The function

$$\theta(y) = \int_0^{t_0} d(ys, x)ds$$

is continuous and satisfy

$$\theta(yt) - \theta(y) = \int_0^{t_0} d(y(t+s), x)ds - \int_0^{t_0} d(ys, x)ds = \int_t^{t+t_0} d(ys, x)ds + \int_0^{t_0} d(ys, x)ds.$$

It follows that θ is diferentiable with respect to the flow. The derivative of f in the flow direction is given by

$$\theta'(y) = d(\Phi(y, t_0), x) - d(y, x).$$

It follows that θ' is continuous and positive near x, since $\theta'(x) = d(\Phi(x, t_0), x) > 0$. The theorem follows by 6.4.6. □

[4] The idea for proof of this lemma is due to Whitney [89].

6.5 The continuity set of λ

D.B. Epstein [20], showed that the continuity set of the period function of a pointwise periodic flow on a manifold M is an open and dense subset of M. We will extend this result for such flows on locally compact metric spaces. The idea is the same as in Epstein's proof, based on the existence of local cross-sections.

Theorem 6.5.1. Let Φ be a continuous pointwise periodic flow on a metric space X. Suppose that Φ admits a local cross-section through $x \in X$. If λ is continuous at x, then λ is continuous in a neighborhood of x. In particular, the continuity set of the period function is an open set.

Proof. The period function is invariant by the flow. Hence, the continuity and discontinuity sets of λ decompose X in two disjoint sets. Indeed:

(1) the function λ is invariant: given $x \in X$, and $t \in \mathbb{R}$, then

$$\lambda(xt) = \inf\{s > 0; (xt)s = xt\} = \inf\{s > 0; xs = x\} = \lambda(x).$$

It follows that λ is constant in the orbit of x;

(2) the continuity set of λ is saturated (therefore the same holds for the discontinuity set of λ). Suppose that λ is continuous at $x \in X$. Let $t \in \mathbb{R}$ and (y_n) be a sequence converging to xt. Then

$$\lim \lambda(y_n) = \lim \lambda(y_n t) = \lambda(x) = \lambda(xt).$$

It follows that λ is continuous at xt for all $t \in \mathbb{R}$.

Now, by construction given in 6.4.1, we can obtain a local cross-section (U, ϵ) through x such that $\lambda|_U = f$ is continuous, where f is the time for the first return. Hence, since λ is continuous on U and also invariant, then λ is continuous on $\Phi(U, \mathbb{R})$. But x is an interior point of $\Phi(U, \mathbb{R})$, since that (U, ϵ) is local cross-section through x. The proof is completed. □

Lemma 6.5.2. Let X be a metric space X and $f : X \longrightarrow \mathbb{R}$ be a lower semicontinuous function. Then the continuity set of f is a dense subset of X.

Proof. A function is continuous at x if and only if it is both upper and lower semicontinuous at x. Since f is lower semicontinuous, the discontinuity set of f is characterized as one where f is not lower semicontinuous. Hence, f is discontinuous at $x \in X$ if and only if there exists $\epsilon > 0$ such that for each neighborhood U of x, there exists $y \in U$ with $f(y) \geqslant f(x) + \epsilon$. Given then $x \in X$ a discontinuity point for f, we can construct a sequence $x_n \longrightarrow x$ with $f(x_n) \geqslant f(x) + \epsilon$. For each rational number r, set $B_r = \{y \in X / f(y) \leqslant r\}$.

Let q a rational number with $f(x) < q < f(x) + \epsilon$. We have that $x \in B_q - \text{int}(B_q)$ (indeed, given an open subset U of B_q, if $x \in U$, then for large n we have $x_n \in U$, since x_n converge to x and $f(x_n) \leqslant q$. But, by construction $f(x_n) \geqslant f(x) + \epsilon > q$, a contradiction). It follows that $x \in B_q - \text{int}(B_q)$. Since f is lower semicontinuous, $F_q = B_q - \text{int}(B_q)$ is a closed subset of X. Therefore the discontinuity set of f is contained in $\bigcup_{q \in \mathbb{Q}} F_q$, hence the continuity set of f contains the intersection $\bigcap (X - F_q)$. To finish, note that $X - F_q$ is dense in X. Since X is a Baire space, an enumerable intersection of open subsets dense in X is dense in X. It follows that the set of continuity of f is dense in X, since contains a dense subset. □

Theorem 6.5.3. Let X be a locally compact metric space and Φ be a pointwise periodic flow on X without singularity. The continuity set of λ is an invariant, open, and dense subset of X.

Proof. It follows by 6.4.5, 6.5.1, and 6.5.2 togheter. □

6.6 Conditions ensuring equicontinuity

In this section, we will see some conditions for a flow to be equicontinuous. The starting point was Gottschalk's one for equicontinuity given in [28]. Namely, a distal transformation group on a compact, uniform space X that induces a locally weakly almost periodic flow in the diagonal is equicontinuous (we provide a similar result for locally compact metric spaces, Corollary 6.6.2). W.H. Gottschalk concludes that every distal and locally almost periodic transformation group on a compact and uniform space is equicontinuous. In what follows, we restrict our attention only to metric spaces.

It is easily seen that a locally almost periodic transformation group induces a transformation group that is locally weakly almost periodic in the diagonal. Thus, by the Gottschalk's result, a transformation group T on a compact metric space that is locally almost periodic and distal is also equicontinuous. Since a pointwise periodic flow is distal (Lemma 6.7.2 below), we try to show that these flows are locally almost periodic. As we will see in the next sections, this condition is very restrictive.

Lemma 6.6.1 ([28], Lemma 1)**.** Let X be compact metric space and T a transformation group on X. Suppose there exists $x \in X$ such that T is locally weakly almost periodic at (x, x) but T is not equicontinuous at x. Then T is not distal on X.

Corollary 6.6.2. Let X be locally compact metric space and T a transformation group on X. Suppose there exists $x \in X$ such that T is locally weakly almost periodic at x and (x, x) but T is not equicontinuous at x. Then T is not distal on X.

Proof. Let $x \in X$ and U be a compact neighborhood of x. Since T is locally weakly almost periodic at x, there exists a compact subset K of T and a neighborhood V of x with

$VT \subset UK$. Now, UK is compact, since U and K are compact and T is a transformation group ($UK = \pi(U \times K)$ and $\pi : X \times T \longrightarrow X$ is continuous). Let $Y = \overline{VT}$. Then,

(1) Y is compact and we have a well-defined transformation group $Y \times T \longrightarrow Y$, since Y is invariant (if A is any invariant set, given $t \in T$, since $t : X \longrightarrow X$ is a homeomorphism, then $\overline{A}t = \overline{At} = \overline{A}$).

(2) T is locally weakly almost periodic at (x, x). Indeed, let U' be a neighborhood of (x, x) in Y. Since $(x, x) \in \operatorname{int}(Y \times Y)$ (interior in $X \times X$, of course), U' is a neighborhood of (x, x) in $X \times X$. Then, there exists a neighborhood V' of x in $X \times X$ and a compact subset K' of T such that $V'\mathbb{R} \subset U'K$. Since $U'K \subset Y \times Y$, it follows that the action of T in $Y \times Y$ is locally weaklly almost periodic at (x, x).

(3) Since Y is an invariant neighborhood of x, then T is equicontinuous at $x \in X$ if, and only if, the restriction of T to Y is equicontinuous at $x \in Y$.

By the above itens and Lemma 6.6.1, T is not distal on Y. Since T distal on X implies T distal on any invariant subset of X, it follows that T is not distal on X. □

Lemma 6.6.3 ([29], Lemma 5.16)**.** Let X be a compact metric space and T be a continuous transformation group on X. Suppose that if $x, y \in X$ with $x \neq y$, then there exist neighborhoods U and V of x and y, respectively, and $\epsilon > 0$ such that $t \in T$ implies $d(zt, wt) \geqslant \epsilon$. Then T is uniformly equicontinuous.

Proof. Given $\epsilon > 0$ we want to show that there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \epsilon$ for all $t \in T$. Setting $\alpha_\delta = \{(x, y) \in X \times X; d(x, y) < \delta\}$ and for $A \subset X \times X$, $AT = \{(xt, yt); (x, y) \in A\}$, this condition is equivalent to the existence of $\delta > 0$ such that $\alpha_\delta T \subset \alpha_\epsilon$. For each $z = (x, y) \in X \times X - \alpha_\epsilon$ there exists $\epsilon_z > 0$, a neighborhood U_z of x, and V_z of y, such that $(U_z \times V_z)T \cap \alpha_{\epsilon_z} = \varnothing$. Hence

$$\alpha_{z_\epsilon} T \subset X \times X - (U_z \times U_z)$$

(in general, $AT \cap B = \varnothing$ if, and only if, $BT \subset X - A$). By the compactness of $X \times X - \alpha_\epsilon$, we can obtain a finite subset E of $X \times X$ such that $X \times X - \alpha_\epsilon \subset \bigcup_{z \in E} U_z \times V_z$. Hence,

$$\bigcap_{z \in E} (X \times X - U_z \times V_z) \subset \alpha_\epsilon.$$

Let $\delta > 0$ such that $\alpha_\delta \subset \bigcap_{z \in E} \alpha_z$. Since

$$\alpha_\delta T \subset \bigcap_{z \in E} \alpha_{\epsilon_z} T \subset \bigcap_{z \in E} (X \times X - U_z \times V_z) \subset \alpha_\epsilon,$$

it follows that $\alpha_\delta T \subset \alpha_\epsilon$, concluding that T is uniformly equicontinuous. □

We will now describe the characterization of locally weakly almost periodicity given by W.H. Gottschalk.

Definition 6.6.4. Let X be a set. A *partition* of X is defined as a disjoint class $\mathscr{A}$ of nonvacuous subsets of X such that $X = \bigcup \mathscr{A}$. A partition of a topological sapace X is said to be {open}{closed} provided the natural projection $p : X \longrightarrow \mathscr{A}$ is {open}{closed}. A *decomposition* of a topological space X is a partition $\mathscr{A}$ of X such that every member of $\mathscr{A}$ is compact. A partition $\mathscr{A}$ of X is said to be *upper semi-continuous* if, and only if, for each $A \in \mathscr{A}$ and for each open neighborhood U of A there exists a saturated open neighborhood V of A contained in U.

Theorem 6.6.5. Let X be a topological space and $\mathscr{A}$ be a partition of X. Then

(1) the natural map $\pi : X \longrightarrow \mathscr{A}$ is closed if, and only if, for any closed subset F of X, the saturation of F is closed;

(2) $\mathscr{A}$ is upper semi-continuous if and only if it is closed;

(3) If $\mathscr{A}$ is upper semi-continuous decompositiion of X, then $\mathscr{A}$ is, respectively, Hausdorff, regular, locally compact or satisfies the second axiom of countability, provided X has the corresponding property;

(4) If X is compact and $\mathscr{A}$ is a Hausdorff partition of X, then π is a closed map.

Proof's skecth. (1) Suppose π is a closed map. Let F be a closed subset of X. Since the saturation of F is equal to $\pi^{-1}(\pi(F))$, then the saturation of F is closed. Conversely, if $\pi(F)$ is closed whenever F is a closed subset of X, then $\pi^{-1}(\mathscr{A} - \pi(F)) = X - \pi^{-1}(\pi(F))$ is an open set. Hence the saturation $\pi^{-1}(\pi(F))$ of F is a closed set.

(2) [41], Theorem 12, page 99.

(3) [41], Theorem 20, page 148.

(4) Suppose X is compact and $\mathscr{A}$ is a Hausdorff partition of X. Let F be a closed subset of X. The subspace F is compact since it is a closed subspace of such one. It follows that $\pi(F)$ is a compact set in a Hausdorff space, hence closed. Therefore π is a closed map.

□

Remark 6.6.6. In Kelley's book [41], the words *partition* and *decomposition* are synonymous.

Definition 6.6.7. Let (X, π, T) be a transformation group. The orbit of x under T or the T-orbit of x is defined to be the subset xT of X. The orbit-closure of x under T or the T-orbit-closure of x is defined to be the subset $\overline{xT}$ of X.

Theorem 6.6.8 ([29], Theorem 4.17)**.** Let X be a locally compact Hausdorff space and T be a transformation group on X. Let $\mathscr{A}$ the partition of X given by the all orbit-closure under T. Then T is locally weakly almost periodic if and only if $\mathscr{A}$ is a closed decomposition of X.

Proof. The proof will be divided into four steps.

(1) Suppose X be a compact Hausdorff space and T be a locally weakly almost periodic transformation group on X. We claim that $\mathscr{A}$ is a closed partition of X. Let $A \in \mathscr{A}$ and U be an open neighborhood of A in X. By Theorem 6.6.5 item 2, we need only to show the existence of a saturated neighborhood W of A contained in U. We will now show this. Since X is also a regular space (it is a Hausdorff compact space; see [41], page 146) and A is a compact subset of X, there exists a closed neighborhood W of A with $W \subset U$. For each $x \in X - U$, there exists a neighborhood N_x of x and a compact subset K_x of T such that

$$N_xT \subset (X - W)K.$$

Since $X - U$ is a closed subset of the compact space X, then we can select a finite subset E of $X - U$ so that

$$X - U \subset \bigcup_{x\in E} N_x.$$

Set $K = \bigcup_{x\in E} K_x$. It follows that K is a compact subset of T and

$$(X - U)T \subset (\bigcup_{x\in E} N_x)T \subset \bigcup_{x\in E}(X - W)K_x = (X - W)K.$$

Let V be a neighborhood of A such that $V \subset WK$. Then

$$VK^{-1} \cap (X - W) = \varnothing \Longrightarrow V \cap (X - W)K = \varnothing \Longrightarrow$$

$$\Longrightarrow V \cap (X - U)T = \varnothing \Longrightarrow VT \cap (X - U) = \varnothing.$$

Hence $VT \subset U$, concluding the claim.

(2) Suppose X be a locally compact Hausdorff space and T be a locally weakly almost periodic transformation group on X. Let F be a compact subset X. We claim that $\overline{FT}$ is compact. Indeed, given $x \in F$ and U_x be a compact neighborhood of x, there exists a neighborhood V_x of x and a compact subset K_x of T such that $V_xT \subset U_xK_x$. By compacteness of F there exists a finite subset E of F such that $F \subset \bigcup_{x\in E} V_x$. Then

$$FT \subset \bigcup_{x\in E} V_xT \subset \bigcup_{x\in E} U_xK_x,$$

hence $\overline{FT}$ is compact as required.

(3) Suppose X be a locally compact Hausdorff space and T be a locally weakly almost periodic transformation group on X. Let $A \in \mathscr{A}$. By item (2) above, there exists a compact neighborhood V of A in X. Hence, given U be an open neighborhood of A in X, since the restriction of T to V is still locally weakly almost periodic and V is compact, by item (1) above there exists a saturated neighborhood W of A in V such that $W \subset (U \cap V)$. Necessarily W is a saturated neighborhood of A in X. It follows by Theorem 6.6.5 item 2 that $\mathscr{A}$ is a closed partition of X. The fact that this partition is actually a decomposition follows from item (2) above.

(4) Suppose now that the class $\mathscr{A}$ of all orbit-closures under T is a closed decomposition of X. Let $x \in X$ and U be an open neighborhood of x. Since $\overline{xT} \subset UT$, and $\overline{xT}$ is compact, there exist a finite subset K of T with $\overline{xT} \subset UK$. Since $\mathscr{A}$ is closed, then it is upper semi-continuous by 6.6.5. Hence, there exists a saturated open neighborhood V of $\overline{xT}$ with $V \subset UK$ (then $VT = V \subset UK$). It follows that T is locally weaklly almost periodic.

□

Remark 6.6.9. Let (X, π, T) be a transformation group. The decomposition of X by the orbits under T is open. Indeed, given U an open subset of X, then $UT = \bigcup_{t \in T} Ut$ is an open subset of X. It follows from $\pi^{-1}(\pi(U)) = UT$ that $\pi(U)$ is an open set.

We finish this section with another characterization of uniformly equicontinuous transformation group.

Theorem 6.6.10 (W.H. Gottschalk, [26])**.** Let X be a totally bounded metric space and T a transformation group on X such that $t : X \longrightarrow X$ is uniformly continuous for all $t \in T$. Then T is uniformly equicontinuous if and only if T is almost periodic.

6.7 Pointwise periodic flows

Let X be a locally compact metric space. Let Φ be a pointwise periodic and locally weakly almost periodic flow Φ on X. One can prove that the condition given in Lemma 6.6.3 holds for $x, y \in X$ whenever $x\mathbb{R} \cap y\mathbb{R} = \varnothing$. Nevertheless, it can fail for x and y in the same orbit under Φ. Analyzing such failure, we obtain Lemma 6.7.1 below, which gives a very restrictive condition to a pointwise periodic flow to be equicontinuous. Essentially this lemma says that an equicontinuous pointwise periodic flow is locally constant where the period function is continuous.

Lemma 6.7.1. Let Φ be a continuous pointwise periodic flow on a locally compact metric space X. Suppose for some $x \in X$ with $\lambda(x) > 0$, there exists a sequence $y_n \longrightarrow x$ with $\lambda(x) \neq \lambda(y_n)$ for all n and $|\lambda(x) - \lambda(y_n)| \longrightarrow 0$. Then T is not equicontinuous at x.

Proof. Let s with $0 < s < \lambda(x)$ and U, V be compact disjoint neighborhoods of x and xs, respectively. Set $\epsilon = d(U, V)$. Note that $\epsilon > 0$. Given $\delta > 0$ we will show that there exists $y \in X$ and $t \in \mathbb{R}$ with $d(x, y) < \delta$ but $d(xt, yt) \geqslant \epsilon$. Let $f : X \times \mathbb{R} \longrightarrow X$ given by $f(y, t) = y(s + t)$. Since f is continuous and $f(x, 0) = xs \in V$, there exists $\gamma > 0$ and $W \subset U$ a neighborhood of x such that if $y \in W$ and $|t| < \gamma$ then $f(y, t) = y(s + t) \in V$. For each $y \in X$, write $\lambda(x) = \lambda(y) + \gamma_y$. We want to obtain y and m such that $y(m\lambda(x))$ is close to ys. Since

$$y(m\lambda(x)) = y(m(\lambda(y) + \gamma_y)) = y(m\gamma_y), \tag{6.2}$$

then, with this aim, it is enough to obtain $y \in W$ and $m \in \mathbb{Z}$ such that $|m\gamma_y - s| < \gamma$. By hypothesis, there exists a sequence $y_n \longrightarrow x$ with $0 < |\lambda(y_n) - \lambda(x)| \longrightarrow 0$. So there exists $y \in W$ with $d(x, y) < \delta$ and $0 < |\gamma_y| \leqslant \gamma$. We can suppose that $\gamma_y < s$. Hence, there exists $m \in \mathbb{Z}$ with $m\gamma_y \leqslant s < (m+1)\gamma_y$ ($m = \max_{k \in \mathbb{Z}}\{m\gamma_y \leqslant s\}$). It follows that

$$-\gamma < 0 \leqslant s - m\gamma_y < \gamma_y \leqslant \gamma. \tag{6.3}$$

Set $t = m\lambda(x)$. By equations 6.2 and 6.3, since $y \in W$ and $z(s + t) \in V$ whenever $z \in W$ and $|t| < \gamma$, we have

$$y(m\lambda(x)) = y(m\gamma_y) = f(y, m\gamma_y - s) \in V.$$

It follows that $y \in U$, $yt \in V$, $d(x, y) < \delta$ but

$$d(xt, yt) = d(x, yt) \geqslant \epsilon,$$

concluding that Φ is not equicontinuous at x. $\square$

Let us now provide a consequence of this lemma. For this, we need to show that pointwise periodic flows are distal.

Lemma 6.7.2. Let Φ be a pointwise periodic flow on a metric space X. Then Φ is distal.

Proof. Let $x \neq y$ and $s \in \mathbb{R}$. If x and y live in the same orbit under Φ, then it is easily seen that

$$d(xs, ys) \geqslant \inf_{t \in [0, \lambda(x)]} d(xt, yt) > 0.$$

Suppose now that $x\mathbb{R} \cap y\mathbb{R} = \varnothing$. Since $z\mathbb{R} = \Phi(z, [0, \lambda(z)])$ is a compact set for all $z \in X$, then $d(x\mathbb{R}, y\mathbb{R}) > 0$. Therefore, in both cases, there exists $\epsilon > 0$ such that $d(xs, ys) \geqslant \epsilon$ for all $s \in \mathbb{R}$. The proof is completed. $\square$

Theorem 6.7.3. Let Φ be a pointwise periodic flow without fixed points on a connected, locally compact, and locally connected metric space X. Suppose Φ weakly almost periodic in the diagonal of $X \times X$. If the period function λ is continuous, then Φ is periodic.

Proof. It follows from lemmas 6.7.2 and 6.6.2 that Φ is equicontinuous on X. Let $x \in X$ and U be a connected neighborhood of x in X. Since λ is continuous, $\lambda(U)$ is a connected subset of $\mathbb{R}$, hence contains a set of the form $[\lambda(x)-\epsilon, \lambda(x)+\epsilon]$, $\epsilon \geqslant 0$. By the equicontinuity of Φ, the Lemma 6.7.1 ensures that $\epsilon = 0$. Therefore, λ is a locally constant and continuous function on a connected space X. Hence is constant, concluding that Φ is periodic. □

Example 6.7.4. Let $X = [1,2] \times \mathbb{S}^1$. Set $\Phi(s, e^{2\pi i\theta}) = (s, e^{2\pi i(\theta + t/s)})$. Then Φ is a continuous pointwise periodic flow with continuous period function given by $\lambda(s, e^{2\pi i\theta}) = s$. Since locally almost periodicity implies locally weakly almost periodicity in the diagonal (see proof of 6.8.4), it follows by 6.7.3 that Φ cannot be locally almost periodic.

We finish this section with two sufficient conditions to a flow Φ on a locally compact metric space X to be locally weakly almost periodic. Unfortunately, it is not clear whether any of these conditions provide some relevant information about the behavior of flow in the diagonal of $X \times X$.

Lemma 6.7.5. Let Φ be a pointwise periodic flow on a locally compact metric space X. If the period function λ is continuous or bounded, then the flow Φ is locally weakly almost periodic.

Proof. In general, given an Φ-invariant function $f : \mathbb{X} \longrightarrow \mathbb{R}$, we can define a flow Ψ from f and Φ as follows. Set $\Psi(x,t) = \Phi(x, f(x)t)$. Writting $x * t = \Psi(x,t)$, we have

$$x * 0 = \Phi(x, 0) = x$$

and

$$\begin{aligned}(x * t) * s = (x * t)f(x * t)s &= \\ (xf(x)t)f(xf(x)t)s &= \\ x(f(x)t + f(x)s) &= \\ x(f(x)(t+s)) &= \\ x * (t+s).\end{aligned}$$

Hence Ψ is a flow. We will now prove that Φ is locally weakly almost periodic. Suppose λ continuous. Set $\Psi : X \times \mathbb{R} \longrightarrow X$ $\Psi(x,t) = \Phi(x, \lambda(x)t)$. As above, Ψ is a flow. Since λ and Φ are continuous functions, Ψ is a continuous flow on X. Note that Ψ has period 1, hence is, in particular, locally weakly almost periodic. By Theorem 6.6.8, the decomposition of X by the orbits under Ψ is closed (note each orbit under Φ is closed in X). Since this decomposition is the same as the decomposition of X by the orbits under Φ, we conclude again by 6.6.8 that Φ is locally weakly almost periodic.

Suppose now that λ is a bounded function on X. Then there exists a compact subset K of $\mathbb{R}$ such that $\lambda(x) \in K$ for all $x \in X$. It follows that $Y\mathbb{R} = YK$ for all $Y \subset X$ and, therefore, Φ is locally weakly almost periodic.

□

6.8 Periodicity of equicontinuous pointwise periodic C^1-flows on closed manifolds

Smooth pointwise periodic flows on compact three-manifolds have been completely analysed by D. B. Epstein in the paper [20]. Essentially, he shows that all flows arise as a decomposition of the manifold by the orbits of a smooth circle action. The same result is still true for an arbitary smooth manifold, compact or not, if the flow is geodesible (see [86]). Another fundamental question in the theory of pointwise periodic flow is to obtain conditions so that the period function is locally bounded. The *Periodic Orbit Conjecture* was one that every pointwise periodic flow on compact manifolds has the period function bounded. This conjecture is known to be false ([79, 81, 22]). In particular, not all pointwise periodic flows arise as the orbits of some circle-action.

In this section, we characterize when a pointwise periodic flow of class C^1 on a closed manifold has the period function bounded. Furthemore, we provide a class of flows arising as the orbits of a smooth circle action. The main result is:

Theorem 6.8.1. Let M be a closed manifold. Let Φ be a pointwise periodic flow without singularity of class C^1 on M. Let N the continuity set of the period function λ. Suppose that Φ is locally weakly almost periodic. If Φ is equicontinuous in N for some metric inducing its topology, then Φ is periodic.

This theorem does not ensure that that λ is constant on M. For example, the natural flow in the Klein bottle K^2 induced by the flow generated by $X = \partial_x$ on $[-1,1] \times [-1,1]$, has period function $\lambda(x) = 1$, if $x = [(t,0)]$ and $\lambda(x) = 2$ if $x \neq [(t,0)]$, where $K^2 = ([-1,1] \times [-1,1])/\{(-1,t) \sim (1,-t); (t,-1) \sim (t,1)\}$. This flow has global period 2. Before starting the proof of the theorem, we highlight some consequences.

Corollary 6.8.2. Let M be a closed manifold. Let Φ be a pointwise periodic flow without singularity of class C^1 on M. Suppose that Φ preserves some metric generating the topology of M. Then Φ is periodic.

Proof. Let d be a metric generating the topology of M and preserved by Φ. Then Φ is uniformly equicontinuous concerning the metric d. It follows by Theorem 6.6.10 that Φ is almost periodic. Hence, in particular, Φ is locally weakly almost periodic. We conclude from Theorem 6.8.1 that Φ is periodic. □

Corollary 6.8.3. Let M be a closed manifold. Let Φ be a pointwise periodic flow without singularity of class C^1 on M. If Φ is almost periodic then it is periodic.

Proof. Since Φ is almost periodic it is, in particular, locally weakly almost periodic. By 6.6.10 Φ is uniformly equicontinuous. It follows by 6.8.1 that it is periodic. □

Corollary 6.8.4. Let M be closed manifold. Let Φ be a pointwise periodic flow without singularity of class C^1 on M. If Φ is locally almost periodic, then it is periodic.

Proof. The proof will be divided in two steps:

(1) Let T be a locally almost periodic transformation group on a metric space X. We claim that it induces a locally weakly almost periodic transformation group on the diagonal of $X \times X$. Let $x \in X$ and U be a neighborhood of x. There exists a neighborhood V of x and a syndetic subset A of T such that $VA \subset U$. Let K compact subset of T with $T = AK$. Then

$$(V \times V)T \subset (V \times V)AK \subset (VA \times VA)K \subset (U \times U)K,$$

concluding the claim.

(2) Now, let Φ be a flow as in the corollary. By the above paragraph, Φ is locally weakly almost periodic in the diagonal of $M \times M$. Since a pointwise periodic flow is distal by 6.7.2, it follows by 6.6.1 that Φ is equicontinuous on M. Therefore, by Theorem 6.8.1, that flow is periodic.

□

Corollary 6.8.5 (A.W. Wadsley, [86])**.** Let Φ be a pointwise periodic C^3-flow on a compact manifold M. Suppose that there exists a Riemannian metric g on M such that every orbit under Φ is a geodesic. Then Φ is periodic.

Proof. Let g be a Riemannian metric on M making all orbits under Φ geodesics. Concerning the metric on M induced by g, Φ is uniformly equicontinuous (see 6.3.6). It follows by 6.8.1 that Φ is periodic. □

The original Wadsley's theorem asserts a pointwise periodic geodesible C^3-flow on a manifold M, compact or not, must be periodic. The latter corollary holds for C^1-flows, but our argument, as stated, holds only for compact manifolds. We now deal with the proof of Theorem 6.8.1. To this aim, we need to prove some preliminary results.

Lemma 6.8.6. Let X be a locally connected topological space and $\mathscr{A}$ be an open partition of X by connected sets. Given an open connected subset W of $\mathscr{A}$ then $\pi^{-1}(W)$ is connected.

Proof. Let W be an open connected subset of $\mathscr{A}$. Write $\pi^{-1}(W) = \bigcup_{\alpha \in J} U_\alpha$, where each U_α is a connected component of $\pi^{-1}(W)$. We have that each U_α is open in $\pi^{-1}(W)$, hence open in X. Let $A \in \mathscr{A}$. Suppose that $A \cap U_\alpha \neq \varnothing$. Then $A \subset \pi^{-1}(W)$, since $\pi^{-1}(W)$ is invariant. On the other hand, since A is a connected subset of $\pi^{-1}(W)$ intersepting the component U_α then $A \subset U_\alpha$. Therefore $\pi^{-1}(W)$ is a disjoint union of open invariant subsets of X. From this and the hypothesis that π is an open map, we conclude that W is a disjoint union of open subsets. It is possible only if J has only one element. This proves the lemma. □

Lemma 6.8.7. Let X be a locally compact metric space. Let Φ be a continuous pointwise periodic and locally weakly almost periodic flow on X. Given $x \in X$, there exists a local cross-section K through x such that the Poincaré's first return map is a homeomorphism of K. In addition, we have

(1) K can be obtained as an open subset of a local cross-section given previously;

(2) if X is a locally connected space, we can obtain K being a connected space.

Proof. Let $x \in X$ and (K_0, ϵ_0) be a local cross-section through x. Let $(K_1, 2\epsilon)$ be a local cross-section through x with K_1 an open subset of K_0. Let $U \subset K_1$ be a neighborhood of x in K_1 such that Poincaré's first return map $T : U \longrightarrow K_1$ is defined (see 6.4.1; here we can begin supposing K_0 compact). By 6.6.8, the partition $\mathscr{A}$ of X by the orbit-closures under Φ, which in this case is the same as the partition of X by the orbits under Φ, is a Hausdorff space, hence upper semi-continuous. Therefore, by item 3 in Theorem 6.6.5, $\mathscr{A}$ is regular. Since (U, ϵ) is a local cross-section and π is an open map, the point $[x] \in \mathscr{A}$ is an interior point in $\pi(U)$. From the fact that $\pi(K_1 - U)$ is closed, $[x] \notin \pi(K_1 - U)$ (because $T(x) = x$), and $\mathscr{A}$ is a regular space, there exists disjoint neighborhood W and W' in $\mathscr{A}$ of $[x]$ and $\pi(K - U)$, respectively[5]. Note that given $y \in K_1 \cap \pi^{-1}(W)$, then $y \in U$. Indeed, otherwise $\pi(y) \in \pi(K_1 - U) \cap W = \varnothing$. It follows that for $K = K_1 \cap \pi^{-1}(W)$ then the Poincaré's first return map $T|_K \longrightarrow K_1$ is defined (as the restriction of $T : U \longrightarrow K_1$ to K) and, moreover, $T(K) \subset K$. Indeed, given $y \in K$, and t with $yt \in K_1$, then $yt \in \pi^{-1}(W) \cap K_1 = K$. Thus, we can consider Poincaré's first return map as a map of K into itself,

$$T|_K : K \longrightarrow K.$$

[5] An equivalent condition to regularity is that given closed set A and $x \notin A$, we can separate x and A by disjoint neighborhoods.

In the construction of Poincaré's first return map 6.4.1, we consider the function $g(y) = y\lambda(x)$, and an open set U containing x with $g(U) \subset K[-\epsilon, \epsilon]$. We can restrict U to obtain the "inverse" of T as follows. Let $h(y) = y(-\lambda(x))$ and take U with $g(U), h(U) \subset K[-\epsilon, \epsilon]$. In this case, given $k \in U$, there exists uniquely determined $y \in K$ and $s \in [-\epsilon, \epsilon]$ with $k(-\lambda(x)) = ys$. Hence, the map $S : U \longrightarrow K$ given by $S(k) = y$ is continuous. This map is the first return in the "negative direction". With this construction and the choice of K, we have that $S|_K = (T|_K)^{-1}$. Indeed, let $y \in K$. Then, $T(y) = k \in K$ where $y\lambda(x) = ks$ and $S(k) = z \in K$ where $k(-\lambda(x)) = zt$, $s, t \in [-\epsilon, \epsilon]$. Hence

$$y(\lambda(x) - s) = k = z(t + \lambda(x)) \Rightarrow y = z(s + t).$$

Since $(K_1, 2\epsilon)$ is local cross section through x, the points y and z are in K_1, and $s + t \in [-2\epsilon, 2\epsilon]$, thus $y = z$ and $s = t$, concluding that $S|_K(T|_K(y)) = y$ for all $y \in K$. Analogously, we can show that $T|_K(S|_K(y)) = y$ for all $y \in K$. We conclude that $T|_K$ is a homeomorphism.

Suppose now that X is a locally connected space. With the notation of the previous paragraph, we can assume without loss of generality that K_0 is a connected space. Therefore, we can take K_1 as a connected subspace of K_0. Indeed, let V be a connected neighborhood of x contained in $K_0[-\epsilon_0, \epsilon_0]$. Let U be an open subset of K and $\delta > 0$ such that $U[-\delta, \delta] \subset V$. Then (U, δ) is a local cross-section through x contained in K_0 with U being a connected space, concluding the claim. Now, since $\mathscr{A}$ is also locally connected, we take W in $\mathscr{A}$ being a connected neighborhood of $[x]$. By Lemma 6.8.6, it follows that $\pi^{-1}(W)$ is connected. Thus, $K = \pi^{-1}(W) \cap K_1$ is a connected space. The proof is completed. □

Lemma 6.8.8. Let X be a metric space and Φ be a continuous pointwise periodic flow on X. Suppose that for each $x \in X$, there exists a local cross-section (K, ϵ) through x, with K being compact, such that Poincaré's first return map is a well-defined map $T : K \longrightarrow K$ and also a periodic homeomorphism. Then the period function λ is locally bounded.

Proof. Let $k \in \mathbb{Z}$ with $T^k = 1$. Since $T(y) = yf(y)$, where $f : K \longrightarrow \mathbb{R}$ is the time for the first return, then

$$\begin{aligned} Ty &= y(fy) \\ T^2y &= Ty(fTy) = y(fy + fTy) \\ T^3y &= T^2y(fT^2y) = y(fy + fTy + fT^2y) \\ &\dots \\ T^ky &= y\left(\sum_{i=1}^{k} fT^{i-1}y\right). \end{aligned}$$

It follows that $\sum_{i=1}^{k} fT^{i-1}y = m_y\lambda(y)$ for some $m_y \in \mathbb{Z}$, $m_y \geqslant 1$. In particular,

$$\lambda(y) \leqslant \sum_{i=1}^{k} fT^{i-1}y$$

for all $y \in K$. Hence, for all $y \in K\mathbb{R}$. Since K is compact and $y \longrightarrow \sum_{i=1}^{k} fT^{i-1}y$ is continuous, it follows that λ is bounded in $K\mathbb{R}$, hence locally bounded on X. □

Lemma 6.8.9. Let M be a smooth manifold and Φ be a pointwise periodic flow on M of class C^1 . Suppose that for each $x \in M$, there exists a local cross-section (K, ϵ) through x, with K being a manifold, $K(-\epsilon, \epsilon)$ being an open subset of X, and such that Poincaré's first return map is a well-defined map $T : K \longrightarrow K$ and also a homeomorphim. Suppose that λ is locally constant where it is continuous. Then Φ is locally periodic. If, in addition, M is compact, then Φ is periodic.

Proof. Let $x \in X$ and (K, ϵ) be a local cross-section through x as stated in the lemma. Since Φ is pointwise periodic, then $T : K \longrightarrow K$ is a pointwise periodic homeomorphism. On the other hand, K being a manifold (connected, by our convetion of manifold), Montgomery's theorem 5.1.3 ensures that T is periodic. Thus, by Lemma 6.8.6, for each $y \in K$, there exists an integer m_y such that

$$m_y\lambda(y) = \sum_{i=1}^{k} fT^{i-1}y := g(y),$$

$g : K \longrightarrow \mathbb{R}$. We claim that the function g is constant. Indeed, let N the continuity set of the period function λ. By 6.5.3, N is an invariant, open, and dense subset of M. Hence $N \cap K$ is open dense subset of K (indeed, given an open subset V of K, then $V\mathbb{R}$ has nonempty interior. Therefore, it contains some point of N, say $yt \in N$. Since N is invariant, then $y \in N$, concluding that $N \cap V \neq \varnothing$).

Let V be a connected component of $K \cap N$. Note that V is open in K (because N is open in X). By hypothesis, λ is locally constant where it is continuous. Thus, $\lambda(y) = c$ for all $y \in V$ and $g(y) = m_y c$ for all $y \in V$. Since the function g is continuous, V is connected and $g(V) \subset \mathbb{Z}c$, then g is constant on V. Hence, g is locally constant on a dense open subset of K. Now we have known that Poincaré's first return map and the time for the first return map are C^1-functions when induced by a C^1-flow. It follows that g is a differentiable function with differential dg also a continuous one. Since g is locally constant on a dense open subset of K, then $dg = 0$ in an open dense subset of K, hence $dg = 0$ on K. Therefore g is constant in K as claimed, say $g(y) = \alpha$ for $y \in K$.

By the previous paragraph, $g(y) = \alpha$ for all $y \in K$. Then

$$\Phi(y, \alpha) = \Phi(y, m_y\lambda(y)) = y$$

for all $y \in K$. Hence Φ is periodic on $K\mathbb{R}$. Now let U and V open connected subsets of M with $U \cap V \neq \varnothing$ and $s, t > 0$ such that $ys = y$ for all $y \in U$ and $zt = y$ for all $y \in V$. We have that $s = m_y \lambda(y)$ for each $y \in U$ and $t = n_y \lambda(y)$ for each $y \in V$, with $m_y, n_y \in \mathbb{Z}$. Thus, for $z \in U \cap V$ we have

$$s = m_z \lambda(z) \text{ and } t = n_z \lambda(z)$$

concluding that $s/t \in \mathbb{Q}$. Write $s/t = m/n$, $m, n \in \mathbb{Z}$. Then $y(mt) = y$ for $y \in V$ and $y(mt) = y(ns) = y$ for $y \in U$. It follows that Φ is periodic on $U \cup V$. Suppose now that M is compact. We can cover M by finite open sets $\{U_i; i = 1, \ldots, l\}$ such that Φ is periodic in each U_i and $U_i \cap U_{i+1} \neq \varnothing$. By the the preceeding argument, there exists a global period for Φ. The proof is completed. □

Proof of Theorem 6.8.1. Since M is a smooth manifold and Φ is flow on M of class C^1, given $x \in M$ there exists a local cross-section K_0 homeomorphic to $\mathbb{R}^{n-1}$ through x. By Lemma 6.8.6 there exists a local cross-section K such that the Poincaré's first return map $T : K \longrightarrow K$ is a homomorphism. In addition, we can take K being an open connected subset of K_0. It follows that K is a connected manifold. By hypothesis, there exists a metric inducing the topology of the continuity set N of the period function λ for which Φ is equicontinuous. By Theorem 6.7.3 λ is constant in each connected component of N. It follows that λ is locally constant where it is continuous. Since M is compact, the Lemma 6.8.9 ensures that Φ is periodic.

□

Note in the proof of Theorem 7 we cannot ensure, at first, that Φ is periodic without the compactness hypothesis. However, weakening only this, the flow Φ should be at least locally periodic. We will end this section by highlighting this and characterization of pointwise periodic C^1-flows with locally bounded period function. This characterization is similar to results already known, since a flow on a closed manfiold is locally weakly almost periodic if, and only if, the orbit space is Hausdorff (see [21, 16]).

Theorem 6.8.10. Let Φ be a pointwise periodic C^1-flow on a manifold M, compact or not. Then the period function λ is locally bounded if, and only if, Φ is locally weakly almost periodic.

Proof. Suppose that Φ is locally weakly almost periodic flow of class C^1. Given $x \in X$, there exists a local cross-section K_0 through x homeomorphic to $\mathbb{R}^{n-1}$. By Lemma 6.8.7 there exists a connected local cross-section K through x being an open subset of K_0 such that Poincaré's first return map $T : K \longrightarrow K$ is a homeomorphism. Since K is, therefore, a connected manifold, it follows by Lemma 6.8.8 that λ is bounded in a neighborhood of x. Since x was arbitrarily chosen, we conclude that λ is locally bounded.

Conversely, suppose that λ is locally bounded (here, M can be any metric space). Let $x \in X$ and U be a neighborhood of x. Let V be a neighborhood of x contained in U such that $\sup_{x\in V} \lambda(x) < \infty$. Say, $\lambda(x) < t_0$ for all $x \in V$. Set $C = [0, t_0]$. Then

$$V\mathbb{R} = VC \subset UC,$$

concluding that Φ is locally weakly almost periodic at x. Since x was arbitrarily chosen, the proof is completed. □

6.9 Pointwise periodic recurrent flows

We already showed that a pointwise periodic, equicontinuous flow on a closed manifold has a global period. The equicontinuity property implies the boundedness of the period function and recurrence of the flow. Therefore it is natural to weak the initial hypothesis, supposing the flow to be merely recurrent with bounded period function. In this section, we prove that the same result still holds with these weaker hypotheses.

Theorem 6.9.1. Let X be a locally connected and compact metric space. Let Φ be a pointwise periodic recurrent flow without singularity on X. Suppose that there exists a nontrivial asymptotic cycle for Φ. Then the period function λ is locally constant where it is continuous.

Proof. Let $x \in X$. Then x is a quasi-regular point for Φ and $\lambda(x)A_x = [C_x]$, where A_x is the asymptotic cycle associated with x and $[C_x]$ is the element of the first Betti group determined by the orbit of x (the proof for metric space is similiar to one given in Lemma 5.1.2). Since Φ is recurrent, A_x is independent of x by Theorem 4.1.16. So, $A_x \neq 0$ for all x. It follows that $1/\lambda(x)[C_x] = 1/\lambda(y)[C_y]$ for all $x, y \in M$. Since for all $x \in X$, $[C_x]$ represents an element in $H_1(M;\mathbb{Z})$, it follows that $\lambda(x)$ and $\lambda(y)$ are rationally dependet for each $x, y \in X$. Denote by Y the continuity set of λ. Then Y is open by Theorem 6.5.3 and therefore, locally connected space since X is one. Now, the function $(x, y) \in Y \times Y \longrightarrow \lambda(x)/\lambda(y) \in \mathbb{R}$ is continuous with image contained in $\mathbb{Q}$. Thus $\lambda(x)/\lambda(y)$ is locally constant in each connected component of $Y \times Y$. Let U be a connected component of Y. Then $U \times U$ is a connected component of $Y \times Y$. Since $\lambda(x)/\lambda(y)$ is constant for each $x, y \in U$, making $x = y$ we concludes that $\lambda(x) = \lambda(y)$ for all $x, y \in U$. It follows that λ is locally constant where is continuous. □

Theorem 6.9.2. Let Φ be a pointwise periodic recurrent C^∞-flow without singularity on a closed manifold M. If there exists a nontrivial asymptotic cycle, then Φ is periodic.

Proof. In manifolds, any recurrent C^∞-flow with a nontrivial asymptotic cycle has a global cross-section K being a smooth manifold (theorems 4.1.16 and 4.3.2). In this case, it is well-known from dynamical systems' theory that the Poincaré's first return map is well-defined smooth diffeomorphisms $T : K \longrightarrow K$. Hence, if Φ is pointwise periodic then T is pointwise periodic diffeomorphism, hence a periodic hemeomorphism by Theorem 5.1.3. Since the flow is recurrent by hypothsis, it follows from 6.9.1 that the period function is locally constant where it is continuous. The Lemma 6.8.9 ensures that Φ is periodic. $\square$

We can improve the latter theorem using the concept of a global cross-section for flow on compact metric spaces as given in Schwartzamnn's paper [74]. He showed that every asymptotic cycle of a flow on a compact metric space admitting a global cross-section determines a nontrivial homological class. The orbit of any continuous point for the period function admits something similar to a normal bundle that determines an invariant compact "torus". Thus, we can ensure from what was demonstrated by S. Schwartzmann and Theorem 6.9.1 that the period function is locally constant where it is continuous provided the flow is recurrent. We need to clarify what means a global cross-section in this more general context. A global cross-section for a flow on a topological space X mean a closed subset K of X such that the natural map $K \times \mathbb{R} \longrightarrow X$ is a local homeomorphism. This notion is the same as what was called a *surface of section* by G.D. Birkoff in his book *Dynamical Systems* and is the notion of a global cross-section in Schwartzamnn's paper [74].

Lemma 6.9.3. Let X be a metric space. Let Φ be a continuous pointwise periodic flow on X and $x \in X$. Then there exists a compact saturated neighborhood U of x such that the restriction $\Phi : U \times \mathbb{R} \longrightarrow U$ has a global cross-section when

(1) X is locally compact and Φ is locally weakly almost periodic on X or

(2) X is compact and the period function λ is continuous at x.

Proof. Suppose that X is locally compact and Φ locally weaklly almost periodic. By Lemma 6.8.7, there exists a local cross section K through x such that Poincaré's first return map is a homeomorphism $T : K \longrightarrow K$. Set $U = K\mathbb{R}$. Then, since K is compact and Φ is locally weaklly almost periodic, we concludes from the proof of ite 2 in Theorem 6.6.8 that U is compact. Since the natural map $\Phi : K \times [-\epsilon, \epsilon] \longrightarrow K[-\epsilon, \epsilon]$ is a homeomorphism for some $\epsilon > 0$ and Φ_t is a homeomorphism for each $t \in \mathbb{R}$ it is easy to check that $\Phi : K \times \mathbb{R} \longrightarrow U$ is a local homeomorphism. Therefore K is a global cross-section for the flow $\Phi : U \times \mathbb{R} \longrightarrow U$.

Suppose now that X is compact and λ continuous at x. Let K be a compact local cross-section through x such that the Poincaré's first return map is the identity and λ is continuous in K (it is possible since λ is continuous at x; see item 4 in Section 6.4.1). We claim that $K\mathbb{R}$ is compact. Let $x_n t_n$ be a sequence in $K\mathbb{R}$, $x \in K$, $t_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Since X and K are compact, there is no loss of generality in supposing that $x_n t_n$ converge to some $y \in X$ and x_n converge to some $z \in K$. Write

$$t_n = m_n \lambda(x_n) + r_n, \text{ with } 0 \leqslant r_n < \lambda(x_n).$$

Since the Φ-invariant function λ is continuous on the compact set K, then it is bounded in K, hence, it is bounded in $K\mathbb{R}$. Thus, we can suppose that r_n converge to some $r \in \mathbb{R}$. Letting $n \longrightarrow \infty$ we have

$$x_n t_n = x_n(m_n\lambda(x_n) + r_n) = x_n r_n \longrightarrow zr = y$$

concluding that $y \in K\mathbb{R}$. It follows that $K\mathbb{R}$ is compact (since X is a metric space, a subspace of it is compact if, and only if, it is sequentially compact). As in the above paragraph we conclude that K is a global cross-section for the flow $\Phi : K\mathbb{R} \times \mathbb{R} \longrightarrow K\mathbb{R}$. The proof is completed. □

Theorem 6.9.4. Let X be a locally connected metric space. Let Φ be a continuous pointwise periodic recurrent flow on X. Then

(1) If X is locally compact and Φ is locally weakly almost periodic on X then λ is locally constant where it is continuous;

(2) if X is compact then λ is locally constant where it is continuous.

Proof. By Lemma 6.9.3, in both cases, given $x \in X$ be a continuous point for λ, there exists a saturated compact neigborhood U of x such that the restriction $\Psi = \Phi|_{U\times\mathbb{R}}$: $U \times \mathbb{R} \longrightarrow U$ has a global cross-section. Hence, by the main Schwartzamnn's theorem [74], every asymptotic cycle under Ψ determines a nontrivial homological class. Since Φ is recurrent, then Ψ is recurrent. It follows by 6.9.1 that the period function λ' concerning to Ψ is locally constant where is continuous. But the $\lambda|_U = \lambda'$. Since U is a neighborhood of x in X, it follows that λ is locally constant where it is continuous. □

Theorem 6.9.5. Let Φ be a pointwise periodic C^1-flow without singularity with bounded period function on a manifold M. Then Φ is recurrent if and only if Φ it is locally periodic. If, also, M is compact, then Φ is recurrent if, and only if, it is periodic.

Proof. By Theorem 6.8.10, Φ is locally weakly almost periodic. Since by Theorem 6.9.4 λ is locally constant where it is continuous, the theorem follows by Lemma 6.8.9. □

Since pointwise periodic flows are distal, from the results of this section, we obtain a class of distal not recurrent flows: the ones where the period function is not locally constant where it is continuous (for instance, see 6.7.4). These are more simple examples than the ones presented in [12].

6.10 Problems

In this section we will list some problems that can lead to future work.

- Is the transitivity condition homogeneous?
- Are harmonic forms transitive?
- Is it suffice a complementary foliation in Theorem 2.1.2?
- Are closed-decomposable forms intrinsically harmonic?
- Are volume-preserving flows recorrent?
- Does a flow that admits a closed submanifold transversal to any orbit has a global cross-section (for example, this always holds if the orbit space satisfies the Hausdorff property)?
- Do equicontinuous C^1-flows on noncompact manifolds has a global period?
- Do pointwise periodic volume-preserving flows have bounded period function?
- Does a recurrent pointwise periodic flow has the period function bounded?

We concluded this work with the certainty that obtain a general intrinsic characterization of harmonic forms is a great challenge.

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