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INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E
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MARCELO GONÇALVES DE MARTINO

TEORIA DE CALIBRE EM DIMENSÕES DOIS E QUATRO

DISSERTAÇÃO de MESTRADO APRESENTADA AO INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA DA UNICAMP PARA OBTENÇÃO DO TÍTULO DE MESTRE EM MATEMÁTICA

ORIENTADOR: MARCOS BENEVENUTO JARDIM

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Prof. (a). Dr (a). HENRIQUE NOGUEIRA DE SÁ EARP

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Nota-se, entre os matemáticos, uma imaginação assombrosa...
Repetimos: havia mais imaginação na cabeça de Arquimedes que na de Homero.

Voltaire

Resumo

Este trabalho procurou apresentar os conhecimentos básicos necessários para trabalhar com a teoria de calibre em baixas dimensões e também mostrar algumas aplicações da mesma. Na parte básica da teoria, além de comentar aspectos da teoria de Hodge para variedades compactas, também se discute, com certo nível de detalhes, os conceitos de fibrados vetoriais e conexões, com ênfase dada para os cálculos locais com conexões e curvaturas. Duas aplicações mais concretas da teoria de calibre são apresentadas nesta dissertação. Primeiro, em dimensão quatro, discute-se a equação de Yang-Mills sobre 4-variedades e é apresentada uma solução para a equação anti-auto-dual, solução esta que é conhecida na literatura como *ansatz* de 't Hooft. Por fim, é apresentada a prova, baseado no artigo [DONALDSON, 1983], de um importante teorema devido a M. S. Narasimhan e C. S. Seshadri que relaciona os conceitos de estabilidade com o de existência de conexão unitária satisfazendo certa propriedade, em fibrados vetoriais complexos sobre superfícies de Riemann.

Palavras-chave: Instantons, Fibrados vetoriais, Conexões (Matemática), Campos de calibre (Física), Yang-Mills, Teoria de.

Abstract

In this work it is developed the basic knowledge required to deal with gauge theory in low dimension and it is shown some applications of this theory. Regarding the basic knowledge, apart from discussing some aspects of Hodge theory over compact manifolds, it is also covered, with a certain deal of details, the concepts of vector bundles and connections, paying close attention to the local computations regarding connections and curvature. As for the applications of the theory, we start, in dimension four, by treating the Yang-Mills equation over 4-manifolds and it is showed a solution to the anti-self-dual Yang-Mills equation, solution that is known in the literature as the 't Hooft *ansatz*. At last, it is given a proof, following the paper [DONALDSON, 1983], of an important theorem due to M. S. Narasimhan and C. S. Seshadri that relates the algebro-geometric notion of stability to the differential-geometric notion of existence of unitary connection whose curvature satisfies a certain condition, on vector bundles over Riemann surfaces.

Key words: Instantons, Vector bundles, Connections (Mathematics), Gauge fields (Physics), Yang-Mills theory.

Sumário

Introdução

A geometria diferencial é uma das grandes áreas da Matemática e possui várias aplicações tanto em outras áreas da Matemática como na Física Matemática. A teoria da relatividade geral é o exemplo mais conhecido de aplicação da geometria Riemanniana à Física. Outro exemplo é a teoria de fibrados e classes características, desenvolvida por matemáticos nos anos 50 e 60, que é a linguagem apropriada para lidar com as chamadas teorias de calibre, que foram desenvolvidas por físicos durante a década de 60. Foi aplicando idéias oriundas da Física teórica e técnicas da teoria de fibrados que Atiyah, Bott, Hitchin, Donaldson e outros demonstraram resultados profundos em geometria e topologia diferencial nas décadas de 60, 70 e 80. O objetivo do presente trabalho é descrever alguns aspectos da teoria de calibre em dimensões baixas dando ênfase para os cálculos envolvendo conexões e apresentar algumas aplicações desta teoria.

O corpo desta dissertação está dividido da seguinte maneira:

O primeiro capítulo, contém os aspectos básicos de teoria de calibre. As primeiras duas seções se ocupam das definições e primeiras propriedades de fibrados vetoriais, diferenciáveis e holomorfos, e são apresentadas noções fundamentais ao assunto, tais como mapas entre fibrados, seções locais entre outras. A referência deste início é [GRIFFITH, HARRIS, 1978]. Na terceira seção, encontra-se a noção de conexões em fibrados vetorias, o conceito central da teoria de calibre. Discutem-se como calcular a curvatura de uma conexão e também como construir operadores em formas diferenciáveis, usando a conexão, em todos os fibrados que são construídos a partir de um fibrado dado. Ainda nesta seção, mostram-se como trabalhar no espaço de conexões e como efetuar alguns cálculos interessantes de derivada temporal de uma família a um parâmetro de conexões ou de curvaturas. A seção termina com classes de Chern, em que se discute como calculá-las, da maneira mais sucinta possível, mais no estilo de [NAKAHARA, 2003], focando no cálculo das duas primeiras classes, que são as únicas utilizadas nesta dissertação, visto que este trabalho ateve-se exclusivamente às dimensões dois e quatro. Esta seção foi escrita com o auxílio de diversas fontes, e as principais foram [GRIFFITH, HARRIS, 1978], [NAKAHARA, 2003], [BAEZ, MUNIAN, 1994] e [DONALDSON, KRONHEIMER, 1990]. A última seção do capítulo discute sequências exatas de fibrados e é baseada em [KOBAYASHI, 1987].

O segundo capítulo trata da teoria de Hodge. Aqui é introduzido o conceito do operador de dualidade, ou estrela de Hodge, que é usado extensivamente nos capítulos subsequentes, referentes à teoria de Yang-Mills. Neste capítulo são discutidos aspectos da teoria de Hodge em variedades riemannianas que, em seguida, são estendidos para variedades complexas e, por fim, são especializados para o caso de superfícies de Riemann, que serão essenciais ao quarto capítulo. As referências utilizadas para este capítulo foram [BRYLINSKI, FOTH, 1998] e [NAKAHARA, 2003]. Para as provas dos teoremas principais, no contexto de variedades complexas, recomendam-se [GRIFFITH, HARRIS, 1978] e [WELLS, 1980].

O terceiro capítulo versa sobre teoria de calibre em dimensão quatro. Na primeira seção é feita uma breve recapitulação do eletromagnetismo e é apresentada a equação de Yang-Mills como generalização das equações de Maxwell. Esta parte foi escrita tendo como base o livro [BAEZ, MUNIAN, 1994], que é uma boa ponte entre as linguagens física e matemática da teoria do eletromagnetismo. Ainda nesta seção, é deduzida a equação de Yang-Mills por princípios variacionais, a partir da lagrangiana de Yang-Mills, e mostra-se que as soluções anti-auto-duais (ASD, na sigla em inglês) são os mínimos absolutos do funcional de Yang-Mills. Discutem-se também aspectos da invariância conforme da equação de Yang-Mills. A referência para esta parte é a seção 2.1 do celebrado livro [DONALDSON, KRONHEIMER, 1990]. A segunda seção se ocupa em apresentar soluções explícitas para a equação de Yang-Mills ASD, soluções estas que ficaram conhecidas na literatura como o *ansatz* de ‘t Hooft, físico holandês ganhador do prêmio nobel de 1999 por seus estudos em teoria de calibre, nas interações eletrofracas. Esta seção trata de uma versão mais detalhada do artigo [JACKIW et al., 1977], que descreve bem sucitamente as soluções.

O último capítulo aborda a teoria de calibre em dimensão dois. É estudado um teorema de M. S. Narasimhan e C. S. Seshadri sob a óptica de Simon K. Donaldson. Este capítulo é baseado no artigo [DONALDSON, 1983], em que se estuda uma íntima relação entre estabilidade de fibrados sobre superfícies de Riemann e existência de conexões unitárias cuja curvatura satisfaz uma certa equação. Este teorema mostra como a teoria de calibre permeia diversas áreas da matemática, desde a álgebra linear até a teoria de equações diferenciais parciais. É feita a prova do teorema, basicamente expandindo o que foi feito no artigo [DONALDSON, 1983].

O restante desta dissertação, com exceção da conclusão, está escrito em inglês, pois grande parte deste trabalho foi feito durante o período em que o autor esteve participando do Masterclass sobre Moduli Spaces na universidade de Utrecht, Holanda.

Introduction

Differential geometry is one of the biggest areas of Mathematics, and it has a great deal of applications throughout Mathematics and Mathematical Physics. The theory of general relativity is the best known example of an application of Riemannian geometry to Physics. Another example is the theory of bundles and characteristic classes, developed by mathematicians in the 50's and 60's, that is the proper language to deal with the so-called gauge theories, developed by physicists during the 60's. It was using ideas from theoretical Physics and techniques from bundle theory that Atiyah, Bott, Hitchin and Donaldson, among others, proved deep results in geometry and differential topology during the 60's, 70's and 80's. The purpose of this dissertation is to describe some aspects of gauge theory in low dimensions giving emphasis to the computations involving connections and to show a few applications of the subject.

This dissertation is divided in the following way:

The first chapter contains the basic aspects of gauge theory. The first two sections dwell upon the definitions and immediate properties of smooth and holomorphic vector bundles and fundamental notions as bundle maps, local sections among others are brought forward. The reference for this beginning is [GRIFFITH, HARRIS, 1978]. In the third section, it is found the notion of connections on vector bundles, central to the theory. It is discussed how to calculate the curvature of a connection and also how that the connection induces operators on differential forms in all the vector bundles that can be constructed from a given bundle. Still in this section, it is shown how to work with the space of connections and how to perform some interesting computations regarding derivatives of one-parameter families of connections or curvatures. This section ends with Chern classes, where they are discussed very briefly, following [NAKAHARA, 2003], focussing on the computation of the first two classes, which are the only ones used in this dissertation, given that this work only considered dimensions two and four. This entire section was written using many references, but the main ones were [GRIFFITH, HARRIS, 1978], [DONALDSON, KRONHEIMER, 1990], [BAEZ, MUNIAN, 1994] and [NAKAHARA, 2003]. The last section of the chapter discusses short exact sequences of vector bundles, and it is based on [KOBAYASHI, 1987].

The second chapter treats Hodge theory. Here it is introduced the duality operator, or Hodge star operator, that is extensively used in the subsequent chapters. Still on the second chapter, it is discussed aspects of Hodge theory on Riemannian manifolds that, afterwards, are extended to complex manifold and finally are specialized to Riemann surfaces, that will prove essential to the fourth chapter. The main references to this chapter is [BRYLINSKI, FOTH, 1998] and [NAKAHARA, 2003], and for the proofs of the main theorems in the context of complex geometry, it is recommended [GRIFFITH, HARRIS, 1978] and [WELLS, 1980].

The third chapter goes about gauge theory in dimension four. The first section gives a brief recollection of electromagnetism and the Yang-Mills equation is shown as a generalization of Maxwell's equations. This part was based on [BAEZ, MUNIAN, 1994], which is a book that builds a nice bridge between the mathematical and physical languages. Still on the first section, it is derived the Yang-Mills equation using variational methods starting with the Yang-Mills Lagrangian, and it is shown that the anti-self-dual (ASD) solutions are the absolute minimum of the Yang-Mills functional. It is also discussed some aspects of conformal invariance of the Yang-Mills equation, and a good reference is the section 2.1 of the celebrated book [DONALDSON, KRONHEIMER, 1990]. The second section occupies itself in showing explicit solutions to the Yang-Mills ASD equation, solutions that are known in the literature as the ‘t Hooft *ansatz*, in honor to the dutch physicist Gerard ‘t Hooft, winner of the Nobel prize of 1999 for his studies in gauge theory, specially in the electro-weak interactions. This section is a more detailed version of the article [JACKIW et al., 1977], which describes very briefly the solutions.

The fourth and last chapter discusses gauge theory in dimension two. It is presented a theorem of M. S. Narasimhan and C. S. Seshadri and its proof “a la” Donaldson. This whole chapter is akin to the article [DONALDSON, 1983], that discusses a close relation between stability and existence of special unitary connections on vector bundles over Riemann surfaces. This theorem shows how gauge theory is intertwined with many other areas of mathematics, ranging from linear algebra to partial differential equations.

Capítulo 1

Vector Bundles

The purpose of this chapter is to define vector bundles, the main object that will be dealt with throughout this work. In the beginning, it will be carried out in paralel the definitions for real and complex vector bundles, as they are much similar in nature, until holomorphic vector bundles are defined, a feature which only makes sense in the complex world.

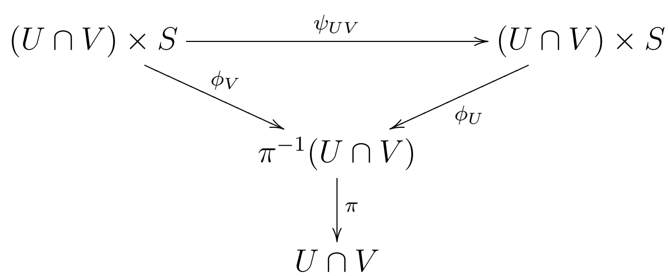
After defining vector bundles, it will be brought forward the definitions and properties of connections on a vector bundle, paying close attention to the computations involved with the theory of connections.

1.1 Vector Bundles

1.1.1 Initial Definitions

1.1 Definitions. Let M be a differentiable manifold. A C^∞ **complex (real) vector bundle** is a family of complex (respectively real) vector spaces, $\{E_p\}$, often called **fibers**, parametrized by the points $p \in M$. Each fiber is isomorphic to a fixed complex (respectively real) vector space S of finite complex (respectively real) dimension, say, k, together with a C^∞ manifold structure on the set $E = \cup_{p\in M} E_p$ that makes the natural projection $\pi : E \to M$, given by $v \in E_p \mapsto p \in M$, a smooth surjection. The dimension of S is called the **rank**, denoted $\mathrm{rk}(E)$. We also ask that the bundle is **locally trivial** in the sense that for all $p \in M$ there exists an open set $U \ni p$, often called **trivializing neighborhood**, such that the bundle E restricted to U, denoted $E|_U := \pi^{-1}(U)$, is trivial, i.e., there exists a diffeomorphism $\phi_U\colon U \times S \to E|_U$ that maps $\{p\} \times S$ as a linear isomorphism into each fiber E_p . Still part of the definition, we ask that, if V is another trivializing neighborhood of $p \in M$ such that $p \in U \cap V$, we have a diffeomorphism $\psi_{UV} := (\phi_U^{-1} \circ \phi_V)|_{U \cap V}$, that when restricted to $\{p\} \times S$ is a linear isomorphism. Hence, it sends $(p, v) \mapsto (p, t_{UV}(p)v)$, where $t_{UV} \colon U \cap V \to GL(S)$, is a smooth map called **transition function**. We often denote the vector bundle E over M by $E \to M$.

1.2 Remark. It is convenient to have the following diagram in mind when dealing with vector bundles:



1.3 Remark. Just for the purposes of the definition we used the vector space S, so we could emphasize the similarities between the real and the complex case. We will often substitute the k-dimensional vector space S by $\mathbb{R}^k$ or $\mathbb{C}^k$ without further mention.

1.4. It is clear from the above definitions that the transition functions satisfy the following properties:

- $t_{UU} \colon U \to GL(S)$, sends every $p \mapsto \mathbb{1}_S$, where $\mathbb{1}_S$ is the identity;
- $t_{UV}(p) = t_{VU}(p)^{-1}, \qquad$ for all $p \in U \cap V$;
- $t_{UW}(p) = t_{UV}(p)t_{VW}(p), \qquad$ for all $p \in U \cap V \cap W$.

Indeed, for the first item, observe that t_{UU} is induced by $\psi_{UU} = \mathbb{1}$, on the second item t_{UV} is induced by ψ_{UV} and t_{VU} is induced by $\psi_{VU} = \psi_{UV}^{-1}$, while for the third, note that

$$\psi_{UW} = \phi_U^{-1} \circ \phi_W = \phi_U^{-1} \circ (\phi_V \circ \phi_V^{-1}) \circ \phi_W = \psi_{UV} \circ \psi_{VW}.$$

These three conditions are often refered as the **cocycle conditions**.

1.5. Reciprocally, assume that we are given an open cover $\{U_\alpha\}$ of M, where for each α we have the trivial bundle $U_\alpha \times S$ and on the intersections $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the cocycle condition mentioned in paragraph 1.4 holds. Then, up to isomorphism, there is a unique vector bundle with all this data. We briefly reconstruct the vector bundle $E \to M$ by setting $\tilde{E} := \sqcup_\alpha (U_\alpha \times S \times \{\alpha\})$ (disjoint union) and $E = \tilde{E}/\sim$, where we identify (p, v, α) with $(p, t_{\alpha\beta}(p)v, \beta)$. Here we used $t_{\alpha\beta}$ instead of $t_{U_\alpha U_\beta}$ for aesthetic purposes. We will do so in what follows without further mention.

1.6 Definition. Let $E \to M$ be a vector bundle, real or complex and $U \subseteq M$ an open subset of M. A **local section** or simply **section**, of the bundle E is a smooth map $s\colon U \to E$ satisfying $\pi \circ s = id$. If $U = M$ we say that s is a **global section** or a **cross-section** of E. The space of sections is denoted $\Gamma(U, E)$ or simply $\Gamma(U)$, whenever the bundle E is clear from the context.

1.7 Remark. Every vector bundle has a natural cross-section, the zero section $0(p) = 0 \in E_p$ for all p.

1.8 Proposition. *Let $E \to M$ be a vector bundle and $U \subseteq M$. The space of local sections $\Gamma(U)$ has the structure of a $C^\infty(U)$-module.*

1.9. Given a finite dimensional real or complex vector space S, we can choose vectors $v_1, \ldots, v_k$ in S that form a basis of the vector space. Since locally a vector bundle is given by $U \times S$, one would expect a similar phenomenon of choice of basis in the vector bundle $E \to M$. We formalize this below.

1.10 Definition. Let $E \to M$ be a vector bundle, real or complex and $U \subseteq M$ an open subset of M. A **local frame** or simply **frame** is a set of smooth sections $e_1, \ldots, e_k$ in $\Gamma(U)$ such that the set $\{e_1(p), \ldots, e_k(p)\}$ is a basis of E_p for each $p \in U$.

1.11. The choice of a local frame on an open subset $U \subseteq M$ is equivalent to a local trivialization of the bundle. Indeed, suppose that we are given a trivialization $\phi_U\colon U \times S \to E|_U$. Pick a basis $\{v_1, \ldots, v_k\}$ of S and set as a local frame, the sections $e_1, \ldots, e_k$ such that $e_i(p) = \phi_U(p, v_i)$, for $p \in U$. Conversely, given a local frame $e_1, \ldots, e_k \in \Gamma(U)$, we obtain a local trivialization by setting $\phi_U(p, v) = \sum_i v^i(p) e_i(p)$, where $v^i(p)$ are the components of the vector $v \in E_p$ with respect to the basis given by the frame in p.

1.12 Example. Let M be a smooth manifold of dimension n with a differentiable structure $\mathscr{F}$. Recall that by a differentiable structure we mean a maximal collection of pairs

$$\mathscr{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\},$$

where $\{U_\alpha\}$ is a covering, A is a set of indices and $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$ is a coordinate chart, meaning that the compositions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth. The maximal condition here says that if (U, φ) is a coordinate chart such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ is C^∞ for all α, then $(U, \varphi) \in \mathscr{F}$. The **tangent bundle** and the **cotangent bundle** of a manifold M are given by

$$TM := \bigcup_{p \in M} T_pM, \qquad T^*M := \bigcup_{p \in M} T_p^*M.$$

The fibers are the tangent and cotangent space on each point that is isomorphic to $\mathbb{R}^n$, thus, their rank is the dimension of the manifold, n. There are natural projections

$$\pi\colon TM \to M, \qquad \pi(v) = x, \text{ if } v \in T_pM,$$

and

$$\pi^\vee\colon T^*M \to M, \qquad \pi^\vee(\omega) = x, \text{ if } \omega \in T_p^*M.$$

We endow these bundles with differentiable structures, such that the natural projections are smooth in the following way. Let $(U, \varphi) \in \mathscr{F}$ with coordinates $\varphi(p) = (x^1(p), \dots, x^n(p))$. Define $\tilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ and $\tilde{\varphi}^\vee\colon (\pi^\vee)^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\begin{aligned}
\tilde{\varphi}(v) &= \big(x^1(\pi(v)), \dots, x^n(\pi(v)), dx^1(v), \dots, dx^n(v)\big) \\
\tilde{\varphi}^\vee(\omega) &= \left(x^1(\pi^\vee(\omega)), \dots, x^n(\pi^\vee(\omega)), \omega\left(\frac{\partial}{\partial x^1}\right), \dots, \omega\left(\frac{\partial}{\partial x^n}\right)\right).
\end{aligned}$$

The differentiable structures are then given by the maximal collections $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}^*$, containing

$$\{(\pi^{-1}(U), \tilde{\varphi}) \mid (U, \varphi) \in \mathscr{F}\}, \qquad \{((\pi^\vee)^{-1}(U), \tilde{\varphi}) \mid (U, \varphi) \in \mathscr{F}\},$$

respectively. We will sketch some proofs that

1. If (U, φ) and (V, ψ) are in $\mathscr{F}$, then $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth.

2. The projection π is smooth.

only for the tangent bundle, as the ones for the cotangent bundle are very similar.

Proofs. We start by saying what will be the topology on TM. For open sets $W \in \mathbb{R}^{2n}$, the collection $\{\tilde{\varphi}^{-1}(W) \mid (U, \varphi) \in \mathscr{F}\}$ forms a basis for a topology that makes TM a Hausdorff, second countable, locally Euclidean space (see

[WARNER, 1983], section of Differentiable manifolds, chapter 1). We will not elaborate on proving this fact, but we claim that $\pi^{-1}(U)$ is open, since

$$\pi^{-1}(U) = \tilde{\varphi}^{-1}(\varphi(U) \times \mathbb{R}^n). \tag{1.1}$$

Indeed, for $v \in \tilde{\varphi}^{-1}(\varphi(U) \times \mathbb{R}^{2n})$, then $\tilde{\varphi}(v) \in \varphi(U) \times \mathbb{R}^n$, which means that $\varphi(\pi(v)) \in \varphi(U)$, i.e., $\pi(v) \in U$, so $v \in \pi^{-1}(U)$ and $\tilde{\varphi}^{-1}(\varphi(U) \times \mathbb{R}^n) \subset \pi^{-1}(U)$. The other inclusion is rather trivial. As a consequence of this, $\tilde{\varphi}$ gives us a homeomorphism between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$. The bijection comes from (1.1), and the continuity comes from the fact that $\tilde{\varphi} = (\varphi, d\varphi)$ is continuous on each coordinate. Now, for (U, φ) and $(V, \psi) \in \mathscr{F}$, then

$$\begin{aligned} \tilde{\psi} \circ \tilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n &\to \psi(U \cap V) \times \mathbb{R}^n \\ (p, u) &\mapsto (\psi \circ \varphi^{-1}(p), d(\psi \circ \varphi^{-1})(u)), \end{aligned}$$

so it is smooth on each coordinate. This settles the first item. Regarding item 2., for each $v \in TM$, take a coordinate chart around v, $(\pi^{-1}(U), \tilde{\varphi})$ and one around $\pi(v)$, (V, ψ), thus the map

$$\psi \circ \pi \circ \tilde{\varphi}^{-1} \colon \varphi(U) \times \mathbb{R}^n \to \psi(V)$$

is clearly smooth, as it is the composition of the projection on the first $\mathbb{R}^n$ with $\psi \circ \varphi^{-1}$, settling the smoothness of π.

Still about the tangent and cotangent bundles, the sections of these bundles are the **vector fields** and 1-**forms** on M. Their sets of global sections are denoted by $\mathscr{X}(M)$ and $\Omega^1(M)$, respectively, and the local ones are analogous changing the M for U. On a chart (U, φ) with coordinates $\varphi(p) = (x^1(p), \ldots, x^n(p))$, the sets

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}, \qquad \{dx^1, \ldots, dx^n\},$$

are local frames for the tangent and cotangent bundle. They are often called **coordinate basis**.

The charts of the base manifold M are also trivializing open sets for TM. Indeed, for a chart (U, φ), we get a diffeomorphism

$$\phi_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^n, \qquad v \mapsto (\varphi^{-1}, id) \circ \tilde{\varphi}(v) = (\pi(v), d\varphi(\pi(v))(v)),$$

so the transition functions are, for charts (U, φ) and (V, ψ)

$$t_{UV} \colon U \cap V \to GL(n, \mathbb{R}), \qquad v \mapsto J(\varphi \circ \psi^{-1})(\pi(v)),$$

where J is the Jacobian matrix of the change of coordinates.

1.13. A **subbundle** $F \subseteq E$ of a rank k complex vector bundle $E \to M$, with M connected, is a collection $\{F_p \subseteq E_p\}_{p \in M}$ of subspaces (of same dimension!) of the fibers E_p such that its union F is a submanifold of E. This is equivalent to saying that for every $p \in M$, there exists a neighborhood $U \ni p$ and a trivialization $\phi_U \colon E|_U \to U \times \mathbb{C}^k$ such that

$$\phi_U|_{F_U} \colon F|_U \to U \times \mathbb{C}^l \subseteq U \times \mathbb{C}^k \quad l \leq k.$$

The matrix representation of a transition function t_{UV} of E is

$$t_{UV}(p) = \begin{pmatrix} s_{UV(p)} & k_{UV(p)} \\ 0 & j_{UV}(p) \end{pmatrix},$$

where s_{UV} will be the transition functions of F and j_{UV} will be the transition functions of the **quotient bundle** E/F, whose fibers are $(E/F)_p = E_p/F_p$.

1.14 Definitions. Let $E \to M$, $F \to M$ be real or complex vector bundles. A **vector bundle morphism** is a smooth map $\Psi \colon E \to F$ that maps fibers to fibers linearly, i.e., if π_E and π_F are the respective projections, we ask $\Psi^p := \Psi|_{E_p} \colon\ E_p \to F_{\Psi(p)}$ is linear and that $\pi_F \circ \Psi = \pi_E$, which is equivalent to say that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & = & M. \end{array}$$

This is easily extended to the case where E and F do not have the same base space, by asking for a similar commuting diagram. If such a Ψ has constant rank, in the sense that each linear map Ψ^p has the same rank, we define the **kernel** of Ψ as the subbundle $\ker \Psi \subseteq E$ whose total space is given by $\{v \in E; \Psi(v) = 0\}$. Similarly, we define the **image** of Ψ as the subbundle $\operatorname{Im} \Psi \subseteq F$ whose total space is given by $\{\Psi(v) \in F; v \in E\}$. We say that two bundles are **isomorphic** if there exists a diffeomorphism $\Psi \colon E \to F$, such that the maps Ψ^p are linear isomorphisms. A bundle is called **trivial** if it is isomorphic to the product $M \times \mathbb{C}^k$

1.15. Short exact sequences of vector bundles become then a natural object in the theory. In particular, consider a short exact sequence with Ψ of constant rank

$$0 \longrightarrow E \xrightarrow{\Psi} F \longrightarrow G \longrightarrow 0.$$

Then the bundle G is the quotient F/E, also called **co-kernel** of the homomorphism Ψ. The total space of G is $F/\sim$, where $v \sim w \iff v - w \in \operatorname{Im}(\Psi)$.

1.1.2 Metrics on Vector Bundles

1.16. The object of study in Riemannian geometry is metrics on the tangent bundle of a manifold M. This consists on defining a smooth symmetric and positive definite $(0, 2)$ tensor field, i.e., assigning to each tangent space a positive definite inner product that varies smoothly with M. We already know that the tangent bundle is a particular vector bundle over M whose fibers are precisely the tangent spaces on a point. For an arbitrary vector bundle E, by a **metric** on E we mean a choice of a positive definite inner product, often denoted as $\langle \cdot, \cdot \rangle$ on each fiber E_p that varies smoothly with $p \in M$, in the sense that the map

$$p \mapsto \langle s(p), t(p) \rangle,$$

is smooth, for any local sections s, t. In what follows, we will be more interested in complex vector bundles.

1.17. Let $E \to M$ be a complex vector bundle. By a **Hermitian metric** on E we mean a choice of a Hermitian inner product denoted $\langle \cdot, \cdot \rangle$ on each fiber E_p that varies smoothly with $p \in M$. Fixed a frame $\{e_1, \dots, e_k\}$, the functions

$$H_{ij}(p) = \langle e_i(p), e_j(p) \rangle$$

are C^∞, and H is matrix of the metric. A frame is called **unitary** if $\langle e_i(p), e_j(p) \rangle = \delta_{ij}$, for every p, that is, the matrix H is unitary. A complex vector bundle equipped with a Hermitian metric is called a **Hermitian vector bundle**.

1.1.3 Gauge Transformations

1.18. Let $E \to M$ be a vector bundle, real or complex. We will be interested in vector bundles with more structure, for instance, when in the presence of a metric, and in particular the so-called $U(k)$-bundles, where $U(k)$ is the Lie group of unitary matrices of rank $k = \mathrm{rk}(E)$. In general, a **G-bundle**, where G is a Lie group, is a vector bundle that is associated to a principal G-bundle, as we can see in appendix B. Or, we can be more practical and construct a G-bundle as a vector bundle over M together with a covering $\{U_\alpha\}$ of M such that E is built by glueing together trivial bundles $U_\alpha \times S$ where S is a vector space in which the group G has a representation ρ, similar to what we saw in 1.5. Assume that $\Psi\colon E \to E$ is a bundle endomorphism. If the induced linear maps $\Psi^p\colon E_p \to E_p$, as we saw in 1.14, live in G for all p, we say that Ψ is a **gauge transformation**, and we denote $\mathcal{G}$ the group of all gauge transformations and call it the **gauge group**, with the operations given by point-wise multiplication, using the group structure of G, i.e., we define

$$\begin{aligned} (gh)(p) &= g(p)h(p) \\ (g^{-1})(p) &= g(p)^{-1}, \end{aligned}$$

where we denoted the elements in $\mathcal{G}$ by small letters instead of capital greek letters, but the elements in $\mathcal{G}$ are still vector bundle endomorphisms, and $p \in M$. The elements in $\mathcal{G}$ acts on sections of the G-bundle by composition.

1.19. Our main interest, as mentioned in the last paragraph, are G-bundles where G is a subgroup of the general linear group. In gauge theories and in many physical applications, one is interested in $U(1)$-bundles, $SU(2)$-bundles or $SU(3)$-bundles.

1.1.4 Vector Bundle Constructions

1.20. Given finite dimensional vector spaces S, T, one can perform several algebraic manipulations such as taking duals, S^*, direct sums, $S \oplus T$, tensor products, $S \otimes T$, exterior powers and so on. It is expected that we can do the same thing with vector bundles over a fixed manifold M. Indeed, if we perform these algebraic manipulations fiberwise, from paragraph 1.5, all we need to do is specify an open cover of M and a recipe on how to glue the fibers on the intersections. We discuss this below.

1.21. Dual Bundle. Let $E \to M$ be a real or complex vector bundle with fibers isomorphic to a vector space S, with a trivializing covering $\{U_\alpha\}$ and a set of transition functions $t_{\alpha\beta}$ on the double intersecions $U_{\alpha\beta}$ satisfying the cocycle conditions as specified in the paragraph 1.4. We construct the **dual bundle** of E, $E^* \to M$, by setting $u_{\alpha\beta} := {}^\tau(t_{\alpha\beta})^{-1}$. We have that the fibers $E_p^* \cong S^*$.

1.22. Direct Sum Bundle. Let $E \to M$, $F \to M$ be real or complex vector bundles with fibers isomorphic to the vector spaces S, T, respectively, with a common trivializing covering $\{U_\alpha\}$ and transition functions given, respectively, by $t_{\alpha\beta}$, $u_{\alpha\beta}$ on the double intersecions $U_{\alpha\beta}$ satisfying the cocycle conditions as specified in the paragraph 1.4. We construct the **direct sum bundle**, $E \oplus F$, by setting $v_{\alpha\beta} := t_{\alpha\beta} \oplus u_{\alpha\beta}$. In matrix representation, we get

$$v_{\alpha\beta} = \begin{pmatrix} t_{\alpha\beta} & 0 \\ 0 & u_{\alpha\beta} \end{pmatrix}.$$

We have that the fibers $(E \oplus F)_p \cong S \oplus T$.

1.23. Tensor product Bundle. Similarly to the previous paragraph, we construct the **tensor product bundle** out of E, F, $E \otimes F$, by setting the transition functions as $v_{\alpha\beta} := t_{\alpha\beta} \otimes u_{\alpha\beta}$. We have that the fibers $(E \oplus F)_p \cong S \otimes T$.

1.24. Hom bundle. Just as in vector spaces where we denote a linear map $L \colon V \to W$ as an element of $W \otimes V^*$, here if E, F are vector bundles, then the **Hom**(E, F) **bundle** is $F \otimes E^*$, and we use the descriptions of 1.21 and 1.23.

1.25. Exterior power bundles. Still in the same way, given a vector bundle E with fiber S and transition functions $t_{\alpha\beta}$, we have that $\wedge^k E$ has transition function $j_{\alpha\beta} = t_{\alpha\beta} \wedge \ldots \wedge t_{\alpha\beta}$ and typical fiber $S \wedge \ldots \wedge S$. In particular, the **determinant bundle** is the exterior power bundle with $k = \operatorname{rk} E$. It is a line bundle with transition function $j_{\alpha\beta} = \det(t_{\alpha\beta})$.

1.2 Holomorphic Vector Bundles

1.26 Definition. Assume that M is a complex manifold (complex manifolds and related topics such as holomorphic maps and so on are treated in the Appendix A). A **holomorphic vector bundle**, $\mathcal{E} \to M$, is a complex vector bundle together with a structure of complex manifold on E such that the local trivializations, ϕ_U, hence the transition functions, t_{UV} are all *holomorphic*.

1.27. We will denote holomorphic vector bundles with calligraphic letters, $\mathcal{E}$, and write the underlying smooth vector bundles with capital letters, E. Everything we have discussed so far for general vector bundles carries out adding the adjective *holomorphic*. To be more precise, following [GRIFFITH, HARRIS, 1978], a holomorphic map of holomorphic vector bundles $\mathcal{E}, \mathcal{F}$ over M is a holomorphic map $\Psi \colon \mathcal{E} \to \mathcal{F}$ with $\Psi^p \colon E_p \to F_p$

linear; a holomorphic subbundle of a holomorphic bundle $\mathcal{E}$ is a subbundle $\mathcal{F} \subset \mathcal{E}$ with $\mathcal{F}$ a complex submanifold of $\mathcal{E}$, and the quotient is again holomorphic. A section s of the holomorphic bundle $\mathcal{E}$ over $U \subset M$ is said to be holomorphic if $s \colon U \to \mathcal{E}$ is a holomorphic map, and a frame $\{e_1, \ldots, e_k\}$ is holomorphic if each e_j is a holohorphic section; in terms of a holomorphic frame, a section $s = \sum s^i e_i$ is holomorphic if, and only if, the functions s^j are.

1.28 Example. A **line bundle** is a vector bundle of rank one. There is a very special holomorphic line bundle over the complex projective space, called the **tautological line bundle** that we describe in what follows. As a set, we get

$$\mathcal{O}(-1) = \{([\ell], z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \; ; \; z \in [\ell]\} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1},$$

where $[\ell] = [u^0 : \ldots : u^n]$. The projection $\pi \colon \mathcal{O}(-1) \to \mathbb{CP}^n$ is given by the projection on the first coordinate. This data defines the tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^n$. Recall the open cover of $\mathbb{CP}^n$, $U_i = [z^i \neq 0]$. This covering $\mathbb{CP}^n = \cup_i U_i$ is a trivializing cover of $\mathcal{O}(-1)$. Indeed, for each i, we define isomorphisms

$$\begin{array}{rcl} \phi_i^{-1} : \pi^{-1}(U_i) & \to & U_i \times \mathbb{C} \\ ([\ell], z) & \mapsto & ([\ell], z^i). \end{array}$$

First, observe that

$$\begin{aligned} \pi^{-1}(U_i) &= \{([\ell], z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \; ; \; [\ell] \in \; U_i \text{ and } z \in [\ell]\} \\ &= \left\{([\ell], z) \; ; \; [\ell] = \left[\frac{u^0}{u^i} : \ldots : \frac{u^{i-1}}{u^i}, 1, \frac{u^{i+1}}{u^i} : \ldots : \frac{u^n}{u^i}\right] \text{ and } z = \lambda \cdot \ell\right\} \\ &= \left\{([\ell], z) \; ; \; z = \left(\frac{z^i}{u^i} \cdot u^0, \ldots, z^i, \ldots \frac{z^i}{u^i} \cdot u^n\right) = \frac{z^i}{u^i} \cdot \ell\right\} \end{aligned} \tag{1.2}$$

so we can define

$$\begin{array}{rcl} \phi_i : U_i \times \mathbb{C} & \to & \pi^{-1}(U_i) \\ ([\ell], \lambda) & \mapsto & \left([\ell], \frac{\lambda}{u^i} \cdot \ell\right). \end{array}$$

This is well defined, since, if $([\ell], \lambda) = ([\ell'], \lambda)$, then we will have $\ell' = \mu \cdot \ell$, that is, if we are consdering $[\ell'] = [u'^0 : \ldots : u'^n]$, we have $u'^k = \mu \cdot u^k$, so

$$\begin{aligned}
\phi_i([\ell'], \lambda) &= \left([\ell'], \frac{\lambda}{u'^i} \cdot \ell'\right) \\
&= \left([\ell'], \left(\frac{\lambda}{u'^i} \cdot u'^0, \ldots, \frac{\lambda}{u'^i} \cdot u'^i, \ldots, \frac{\lambda}{u'^i} \cdot u'^n\right)\right) \\
&= \left([\mu \cdot \ell], \left(\frac{\lambda}{\mu \cdot u^i} \cdot \mu \cdot u^0, \ldots, \lambda, \ldots, \frac{\lambda}{\mu \cdot u^i} \cdot \mu \cdot u^n\right)\right) \\
&= \left([\ell], \left(\frac{\lambda}{u^i} \cdot u^0, \ldots, \lambda, \ldots, \frac{\lambda}{u^i} \cdot u^n\right)\right) = \phi_i([\ell], \lambda)
\end{aligned}$$

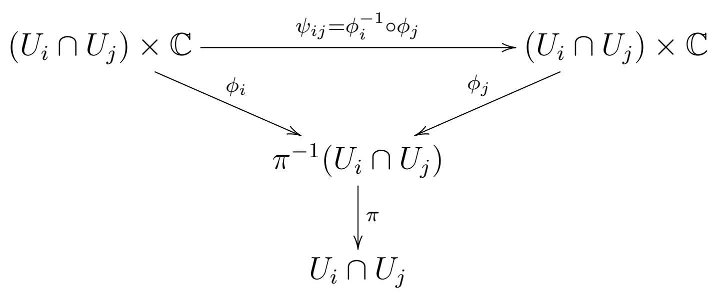
Note that each function is the inverse of the other. Indeed, we have

$$([\ell], z) \mapsto ([\ell], z^i) \mapsto ([\ell], z^i/u^i \cdot \ell) \overset{(1.2)}{=} ([\ell], z),$$

and

$$([\ell], \lambda) \mapsto ([\ell], \lambda/u^i \cdot \ell) \mapsto ([\ell], \lambda).$$

As for the transition functions, we have the following diagram:



From the definition of the local trivializations, the map $\psi_{ij} \colon = \phi_i^{-1} \circ \phi_j$ takes $([\ell], \lambda) \longmapsto ([\ell], \frac{u^i}{u^j} \cdot \lambda)$, thus we define the transition function $g_{ij} \colon U_i \cap U_j \to \mathbb{C}^*$ as

$$g_{ij}([\ell]) = \frac{u^i}{u^j}.$$

All of the maps here defined, ϕ_i, ψ_{ij} and g_{ij} are holomorphic, so this is indeed a holomorphic line bundle.

1.29. Let $E \to M$ be a smooth vector bundle. By an **E-valued p-form** we mean a section of the bundle $\Lambda^p(T^*M) \otimes E$. We denote the set of E-valued p-forms by $\Omega^p(M; E)$. That said, let $\omega \in \Omega^p(M; E)$ and let $\{e_1, \ldots, e_k\}$ be a local frame of E. The bundle-valued form ω is written, with respect to this frame as

$$\omega = \sum \omega^j \otimes e_j,$$

for $\omega^j \in \Omega^p(M)$, of course. From now on, we will make the convention that indices repeated as a superscript and a subscript will be summed over. Now, we could try to define the exterior derivative $d\colon \Omega^p(M;E) \to \Omega^{p+1}(M;E)$ for bundle-valued p-forms by setting

$$d\omega = \sum d\omega^j \otimes e_j. \tag{1.3}$$

Unfortunately, this is not well defined because if we change the local frame to $\{e'_1, \ldots, e'_k\}$ with $e_j = \sum g^i_j e'_i$, we can write ω in two ways:

$$\begin{aligned} \omega &= \sum w^j \otimes e_j \\ \omega &= \sum g^i_j w^j \otimes e'_i. \end{aligned}$$

Hence, the Leibniz rule would make our naive definition of exterior derivative fail, unless $\sum dg^i_j \wedge \omega^j = 0$. On a holomorphic vector bundle, however, the $\bar{\partial}$-operator

$$\bar{\partial}\colon \Omega^{p,q}(M;E) \to \Omega^{p,q+1}(M;E),$$

where $\Omega^{p,q}(M;E)$ are the sections of $\Lambda^{p,q}(T^*M) \otimes E$ (cf. Appendix A, definition A.29), often called **E-valued (p,q)-forms**, is well defined. As is [GRIFFITH, HARRIS, 1978], let $\{e_1, \ldots, e_k\}$ be any holomorphic frame for $\mathcal{E}$ over U and write $\omega \in \Omega^{p,q}(M;E)$ as $\omega = \sum_i \omega^i \otimes e_i$, for $\omega^i \in \Omega^{p,q}(U)$. Define

$$\bar{\partial}\omega := \sum_i \bar{\partial}\omega^i \otimes e_i.$$

So far, everything is similar to the naive definition for exterior derivative we have given in (1.3). The difference is that another holomorphic frame $\{e'_1, \ldots, e'_k\}$ is given by a *holomorphic* change of trivialisation, that is, we have $e_i = \sum_j g^j_i e'_j$, and $\bar{\partial} g^i_j = 0$, for all i and j. Thus, if we write $\omega = \sum g^j_i \omega^i \otimes e'_j$, we will get

$$\bar{\partial}\omega = \sum \bar{\partial}(g^j_i \omega^i) \otimes e'_j = \sum g^j_i \bar{\partial}\omega^i \otimes e'_j = \sum_i \bar{\partial}\omega^i \otimes e_i,$$

which means that $\bar{\partial}$ does not depend on the frame. We say that a section of a holomorphic bundle is **holomorphic** if $\bar{\partial}s = 0$.

1.3 Connections and Curvature

1.3.1 Connections

1.30. Our aim is to work with connections and curvature on vector bundles only. However, sometimes it is also nice to look at these concepts using the geometry of principal bundles. Hence, we intent to give a very brief account on connections and curvature on principal bundles. The next sections are based on the section 2.1 of [DONALDSON, KRONHEIMER, 1990]. We will take as granted some aspects of Lie theory, such as Lie groups, Lie algebras, the exponential map, right translation and others. We have gathered this information together with the notion of principal bundles in an appendix, for convenience. Consider P a principal G-bundle over a manifold M. There are three important ways of seeing a connection, that we describe in what follows.

1.31. (A) First, for an element $p \in P$, by a **vertical subspace**, V_pP, we mean a subspace of the tangent space T_pP that is tangent to the fibers. To construct these vertical spaces, consider an element $Z \in \mathfrak{g}$. From the right action we define a curve γ through p by setting

$$\gamma(t) = p \cdot \exp(tZ).$$

Since $\pi(p) = \pi(p \cdot \exp(tZ))$, this curve lives in the fiber $\pi^{-1}(\pi(p))$. We define a vector $Z^\# \in V_pP$ by

$$Z^\#(p) = \frac{d}{dt}\bigg|_{t=0} (p \cdot \exp(tZ)).$$

Doing this for each $p \in P$ we define a vector field $Z^\#$, called the **fundamental vector field** generated by Z. The map $\#\colon \mathfrak{g} \to V_pP$, that sends Z to $Z^\#$ is actually a vector space isomorphism, and also a **Lie algebra isomorphism**, in the sense that it preserves the brackets

$$[Y^\#, Z^\#] = [Y, Z]^\#.$$

A **horizontal subspace** H_pP is a choice of a complement of V_pP in T_pP. By a **connection** on P, we mean a unique splitting of the tangent space T_pP into vertical subspace V_pP and the horizontal subspace H_pP such that

(a) $T_pP = H_pP \oplus V_pP$;

(b) A smooth vector field v on P is separated as $v = v^H \oplus v^V$;

(c) $H_{p\cdot g}P = d(R_g)(H_pP)$, for arbitrary $p \in P$ and $g \in G$, where R_g denotes the right action, cf. appendix B.

1.32. (B) As an element $A \in \Omega^1(P; \mathfrak{g})$, a 1-form with values in the bundle of Lie algebras $\mathfrak{g}$ of G that is a *projection* in the vertical component $V_pP \cong \mathfrak{g}$, in the sense that

(a) $A(Z^\#) = Z, \qquad Z \in \mathfrak{g}$;

(b) $R_g^* A = \mathrm{Ad}(g^{-1})A$,

that is, for a a vector field v,

$$R_g^* A_{p\cdot g}(v) = A_{p\cdot g}(dR_g\, v) = g^{-1} \cdot A_p(v) \cdot g.$$

We define the horizontal subspace H_pP by the kernel of A, that is

$$H_pP := \{v \in T_pP \mid A(v) = 0\}. \tag{1.4}$$

The following proposition, the proof of which can be found in [NAKAHARA, 2003], gives a relation between the definitions in **(A)** and **(B)**.

1.33 Proposition. *The horizontal spaces defined in (1.4) satisfy*

$$d(R_g)(H_pP) = H_{p\cdot g}P.$$

1.34. (C) Given a representation of G on a vector space S, recall from appendix B that we can construct an associated vector bundle $E = P \times_G S$. Also, using the frame bundle construction, given a vector bundle E, we construct a principal $GL(n)$-bundle, and in this way we pass from principal bundles to vector bundles. Our last notion of connection will be the one we will use the most throughout this work.

1.35 Definitions. A **connection** on a vector bundle $E \to M$ is a map $\nabla\colon \Gamma(E) \to \Omega^1(M; E)$, $\mathbb{R}$-linear, satisfying the *Leibniz rule*

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for sections $s \in \Gamma(E)$ and $f \in C^\infty(E)$. Given a vector field v of M, the smooth section $\nabla s(v) := \nabla_v s$ is often called **covariant derivative** of s along v.

1.36. To pass from **(C)** to **(A)**, observe that ∇ is a local operator, that is, if two sections s_1 and s_2 agree on an open set U, then $\nabla s_1 = \nabla s_2$. This follows by the Leibniz rule and considering ϕs_i, where ϕ is cut-off function, that is, a smooth function that is equal to 1 in a smaller open set inside U and 0 outside U (for the details of the construction of this function, cf. the corollary of the theorem 1.11 in [WARNER, 1983]). Then, we say that a local section σ of the frame bundle is **horizontal** if all of the ∇s_i vanish at p. We then define the horizontal subspace H_pP as the tangent space of the horizontal section, $T_p\sigma(M)$.

1.37. The local description of the connection is rather important. Let $E \to M$ be a vector bundle, set $\sigma = \{e_1, \ldots, e_k\}$ a local frame over an open set U of M, and assume we are given a connection ∇ on E. Then,

$$\nabla e_j = \sum_i (A^\sigma)^i_j \otimes e_i, \tag{1.5}$$

for 1-forms $(A^\sigma)^i_j$ (we will explain shortly in 1.39 the meaning of the superscript σ). The matrix $A^\sigma = ((A^\sigma)^i_j)$ is called **matrix connection**. From now on, we will drop the symbol $\otimes$ in our calculations. Let s be a section of $E|_U$, then we write

$$s = \sum s^j e_j,$$

and

$$\nabla s = \sum \left(ds^j e_j + s^j (A^\sigma)^i_j e_i \right) = \sum \left(ds^i + (A^\sigma)^i_j s^j \right) e_i,$$

or, considering s a column vector, we have, in matrix notation

$$\nabla s = (d + A^\sigma)s. \tag{1.6}$$

Thus, the local frame and the connection matrix on U define the connection locally.

1.38. We ended the last paragraph with the equation $\nabla = d + A^\sigma$, where d was the differential on smooth functions and A^σ was the connection matrix. Let us explore the meaning of the connection matrix A^σ. First, as it is a matrix of 1-forms, locally, on a chart (U, x^μ), with μ ranging from 1 up to the dimension of M, we have

$$A^\sigma = \sum (A^\sigma)_\mu dx^\mu.$$

Denote the coordinate vectors $\frac{\partial}{\partial x^\mu}$ by ∂_μ. If we apply a coordinate-vector to equation

(1.5), we obtain $\nabla e_j(\partial_\mu) = \sum (A^\sigma)^i_j(\partial_\mu) e_i$, or in short, $\nabla_\mu e_j = \sum (A^\sigma)^i_{\mu j} e_i$, a local section of E, but this corresponds, in our matrix notation in (1.6) to $(A^\sigma e_j)(\partial_\mu)$. Therefore, the matrix 1-form A^σ is an **endomorphism-valued** 1-form (which is compatible with the fact of it being a matrix, since we represent a vector space endomorphism by a matrix after we pick basis), i.e., it is a section of the bundle $T^*M \otimes (E \otimes E^*)$, and its local description is

$$A^\sigma = \sum_{\mu,i,j} (A^\sigma)^i_{\mu j} dx^\mu \otimes e_i \otimes e^j.$$

This endomorphism valued 1-form is often called **vector potential**, in physical jargon.

1.39 Remark. Take a local trivialization of E. Then we have a frame $\{e_1, \dots, e_k\}$ on U and thus, we have a local section σ of the principal frame bundle, $\sigma(p) = (p, e_1(p), \dots, e_k(p))$, and this yields a trivialization on P, as we can see in B.36. Then the matrix of the 1-forms A^σ is linked with the connection A on $B(E)$, in the sense of **(B)** as

$$A^\sigma = \sigma^*(A) \in \Omega^1(M; \mathfrak{g}_E),$$

where $\mathfrak{g}_E$ is a subbundle of $\mathrm{End}(E)$, called the **bundle of Lie algebras** of E. Different trivializations yields different matrices, and the choice of σ is called the **choice of a gauge**. We often denote a connection by ∇_A or simply A. From now on, we will drop the reference to a section σ and write also A for the connection matrix. If we have a $U(k)$-bundle, or an $SU(k)$-bundle then the matrix components of its vector potential, A_μ, are $\mathfrak{u}(k)$-valued, or $\mathfrak{su}(k)$-valued. Such connections are called **unitary connections**. In general, for a G-bundle (recall from 1.18), the connection matrices are in $\mathfrak{g}$, and these are called G**-connections**. The set of all G-connections is denoted by $\mathcal{A}$.

1.40. Now, suppose $\sigma' = \{e'_1, \dots, e'_k\}$ is another frame with $e'_i = \sum g^j_i e_j$, or in shorter notation, $e'_i = e_i \cdot g$, where g is a bundle automorphism. Denote by $[\,\cdot\,]$ the matrix representation of a section with respect to the local frame σ and $[\,\cdot\,]'$ with respect to σ'. These matrix representations are related via the automorphism g by the following rule

$$[\,\cdot\,]' = g[\,\cdot\,], \qquad [\,\cdot\,] = g^{-1}[\,\cdot\,]'.$$

Then, on the one hand, we have

$$[\nabla s] = g^{-1}[\nabla s]' = g^{-1}(d + [A]')[s]',$$

while on the other hand,

$$\begin{aligned}
[\nabla s] &= d[s] + [A][s] \\
&= d(g^{-1}[s]') + [A](g^{-1}[s]') \\
&= g^{-1}d[s]' + dg^{-1}[s]' + [A](g^{-1}[s]') \\
&= g^{-1}(d + gdg^{-1} + g[A]g^{-1})[s]',
\end{aligned}$$

from where we conclude that

$$[A]' = gdg^{-1} + g[A]g^{-1}. \tag{1.7}$$

Equation (1.7) shows us how the matrix connection changes with respect to a change of basis.

1.41. Just as we can apply a gauge transformation to a section of a G-bundle, we can do the same to a G-connection. Assume that ∇ is a G-connection and let $g \in \mathcal{G}$ be a gauge transformation. Set $(g \cdot \nabla) := \nabla'$ by $\nabla'(s) = g(\nabla(g^{-1}s))$, where s is a section, or, in short,

$$(g \cdot \nabla) = g \circ \nabla \circ g^{-1}.$$

We have that ∇' is indeed a connection since it satisfies the *Leibniz rule*:

$$\begin{aligned}
\nabla'(fs) &= g(\nabla(g^{-1}fs)) \\
&= g(\nabla(fg^{-1}s)) \\
&= g(df(g^{-1}s) + f\nabla(g^{-1}s)) \\
&= dfs + fg(\nabla(g^{-1}s)) \\
&= dfs + f\nabla'(s),
\end{aligned}$$

and, the vector potential transforms by

$$A \longrightarrow A' = gdg^{-1} + gAg^{-1}.$$

To see this, pick a frame and perform the calculation

$$\begin{aligned}
\nabla' s &= ds + A's \\
&= g(d(g^{-1}s) + A(g^{-1}s)) \\
&= g(g^{-1}ds + dg^{-1}s + A(g^{-1}s)) \\
&= ds + (gdg^{-1} + gAg^{-1})s.
\end{aligned}$$

1.42. No wonder that the vector potential transforms in the same way as in (1.7), since the change of basis is a gauge transformation and, therefore, acts on the connection. It is not hard to prove that if each of the components A_μ of the vector potential A are in $\mathfrak{g}$ and $g \in \mathcal{G}$, then $g\partial_\mu g^{-1}$ and $gA_\mu g^{-1}$ are in $\mathfrak{g}$, and thus, $\nabla' := (g \cdot \nabla)$ will also be a G-connection. This will become clearer below, on proposition 1.67. The connections ∇' and ∇ are called **gauge-equivalent**.

1.43. It follows from the Leibniz rule that a connection is not linear with respect to C^∞ functions. However, the difference of two connections ∇ and ∇' is, as

$$(\nabla - \nabla')(fs) = (dfs + f\nabla s - dfs - f\nabla' s) = f(\nabla - \nabla')s.$$

This means that the difference has the same nature as a vector potential, i.e, it is an element in the vector space $\Gamma(T^*M \otimes E \otimes E^*) = \Omega^1(M; \mathrm{End}(E))$. Actually, if we assume that ∇, ∇' are G-connections, the difference will be on the subbundle $\Omega^1(M; \mathfrak{g}_E)$ of $\Omega^1(M; \mathrm{End}(E))$ that will be precisely $\Omega^1(M; \mathrm{End}(E))$ when the frame bundle has structure group the whole $GL(n)$. Moreover, given any connection ∇_A and an element a in $\Omega^1(M; \mathfrak{g}_E)$, we have that

$$(\nabla_A + a)(fs) = dfs + f(\nabla_A + a)s,$$

where a acts on s via contraction (recall that $\mathrm{End}(E) = E \otimes E^*$), so it satisfies the Leibiniz rule and hence is again a connection, sometimes denoted by $A + a$. Therefore, for every fixed connection $\nabla' \in \mathcal{A}$, there is a bijection between $\Omega^1(M; \mathfrak{g}_E)$ and $\mathcal{A}$ given by

$$a \in \Omega^1(M; \mathfrak{g}_E) \longmapsto (\nabla' + a) \in \mathcal{A},$$

with inverse given by

$$\nabla \in \mathcal{A} \longmapsto \nabla - \nabla' \in \Omega^1(M; \mathfrak{g}_E).$$

Thus, the space of all G-connections of a vector bundle is an affine space modeled on $\Omega^1(X;\mathfrak{g}_E)$, and, in particular, we can endow $\mathcal{A}$ with the topology of $\Omega^1(X;\mathfrak{g}_E)$. We have been denoting a connection by $\nabla = d + A$. The differential d is indeed a connection on $E|_U$, often refered as the **standard flat connection**. It is not canonically defined, since it depends on the choice of a frame. Once we have picked a frame, we have

$$ds = \sum_j ds^j \otimes e_j,$$

and then, in general, we will have $d(fs) = df \otimes s + fds$. This seems to contradict what was written in paragraph 1.29 regarding the definition of an exterior derivative on bundle-valued forms. However, observe that the problem there was when changed the local frame. Here, the connection is defined only after the frame is fixed.

1.44. So far we have derived some local properties of connections and we have seen how they transform with respect to change of basis. In the last paragraph we discussed that locally, the operator d acting on vector valued functions is a connection, and if we sum $d + A$ for any local section $A \in \Omega^1(U;\mathfrak{g}_E)$ we get another connection. Hence, using the fact that our base spaces are paracompact (see [WARNER, 1983], definition 1.7, page 8), we can define connections locally and use a partition of unity argument to extend to a global connection. Hence connections always exist on a vector bundle. For an actual proof in the context of Hermitian vector bundles, see [WELLS, 1980], proposition 1.11.

1.45. On a holomorphic bundle $\mathcal{E}$, using the decomposition of the tangent space of M

$$TM = T^{1,0}M \oplus T^{0,1}M,$$

as it is shown in the appendix A on paragraph A.14, we decompose the connection as $\nabla = \nabla^{1,0} + \nabla^{0,1}$ or $\nabla_A = \partial_A + \bar{\partial}_A$, accordingly. We say that a connection A is **compatible with the complex structure** if $\bar{\partial}_A = \bar{\partial}$ (or $\nabla^{0,1} = \bar{\partial}$), where $\bar{\partial}$ was defined in 1.29. Assuming further that the bundle is Hermitian, the connection is said to be **compatible with the metric** if

$$d\langle s,t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle,$$

for sections s, t.

1.46 Proposition. *In a Hermitian vector bundle $\mathcal{E}$, there is a unique connection ∇_A*

on $\mathcal{E}$ that is compatible with the metric and with the complex structure.

Proof. Cf. [GRIFFITH, HARRIS, 1978], page 73. □

1.47 Definition. This unique connection is called **Chern connection**.

1.48. Chern connections of Hermitian vector bundles are well-behaved with respect to the bundle operations as we saw in subsection 1.1.4.

- On $E_1 \oplus E_2$: $\nabla_1 \oplus \nabla_2$.
- On $E_1 \otimes E_2$: $\nabla_1 \otimes \mathbb{1}_{E_2} + \mathbb{1}_{E_1} \otimes \nabla_2$, where $(\nabla_1 \otimes \mathbb{1}_{E_2})(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2$ and so on. This is the Chern connection with respect to the natural metric on $E_1 \otimes E_2$ given by
$$\langle s_1 \otimes s_2, t_1 \otimes t_2 \rangle = \langle s_1, t_1 \rangle \cdot \langle s_2, t_2 \rangle.$$
- On E^*: The metric on E induces a metric on E^*. For a unitary frame $\sigma = \{e_1, \dots, e_n\}$, let $\sigma^\vee = \{e^1, \dots, e^n\}$ be the dual frame. Declare this frame as unitary, i.e.,
$$\langle e^i, e^j \rangle = \delta^{ij},$$
and the Chern connection $\nabla^\vee$ is defined by the requirement
$$d(t^\vee(s)) = t^\vee(\nabla s) + \nabla^\vee t^\vee(s),$$
for $s \in \Gamma(E|_U)$ and $t^\vee \in \Gamma(E^*|_U)$.

1.49 Definition. A connection ∇ on E is said to **reducible** if there are bundles with connections (E_1, ∇_1) and (E_2, ∇_2) such that
$$(E, \nabla) \cong (E_1 \oplus E_2, \nabla_1 \oplus \nabla_2).$$
The connection is said to be **irreducible**, otherwise.

1.50. Perhaps it is not completely clear from paragraph 1.48 how that connections on Hermitian bundles induce connections on all the bundles that can be constructed from them. In order to clarify some of the aspects, and also to provide a useful result that will be needed later, we shall show how we induce a connection on $\mathrm{End}(E)$, provided that E is a Hermitian bundle with a connection ∇. We start giving a local computation of the connection $\nabla^\vee$ on E^*. Then we construct $\widetilde{\nabla} = \nabla \otimes \mathbb{1} + \mathbb{1} \otimes \nabla^\vee$ as above. Picking a unitary frame $\{e_1, \ldots, e_n\}$, we know from 1.37 that, for $s = \sum s^j e_j$,

$$\nabla s = \sum_i \left(ds^i + \sum_j A^i_j s^j \right) e_i.$$

Let $\{e^1, \ldots, e^n\}$ be the dual frame that we have set as unitary and $t^\vee = \sum t_j e^j$. Then

$$\begin{aligned} \nabla^\vee t^\vee(e_i) &= d(t^\vee(e_i)) - t^\vee(\nabla e_i) \\ &= dt_i - \sum A^j_i t^\vee(e_j) \\ &= dt_i - \sum A^j_i t_j, \end{aligned}$$

or, in matrix notation, we have

$$\nabla^\vee t^\vee = dt^\vee - {}^\tau A t^\vee. \tag{1.8}$$

1.51. Now, consider $E^j_i = e_i \otimes e^j$ a basic section of $\mathrm{End}(E) \cong E \otimes E^*$. We have $E^j_i = \sum (E^j_i)^b_a E^a_b = \sum \delta^j_a \delta^b_i E^a_b$, that is $(E^j_i)^b_a = \delta^j_a \delta^b_i$. Our claim is that

$$\widetilde{\nabla} E^j_i = [A, E^j_i]^b_a E^a_b. \tag{1.9}$$

Indeed,

$$\begin{aligned} \widetilde{\nabla} E^j_i = \widetilde{\nabla} \left(\sum \delta^j_a \delta^b_i \; e_b e^a \right) &= \sum \delta^j_a \delta^b_i A^c_b (e_c e^a) - \delta^j_a \delta^b_i A^a_c (e_b e^c) \\ &= \sum (\delta^j_a \delta^c_i A^b_c - \delta^j_c \delta^b_i A^c_a) \; e_b e^a \\ &= \sum ((A E^j_i)^b_a - (E^j_i A)^b_a) E^a_b \\ &= [A, E^j_i]^b_a E^a_b. \end{aligned}$$

Now observe from the second line of the computation above that

$$[A, E^j_i]^b_a = \sum_c (\delta^j_a \delta^c_i A^b_c - \delta^j_c \delta^b_i A^c_a) = A^b_i \delta^j_a - A^j_a \delta^b_i. \tag{1.10}$$

Hence, for a general section of End(E) given by $g = \sum g^i_j E^j_i$, we have

$$\begin{aligned} \widetilde{\nabla}\left(\sum g^i_j E^j_i\right) &= \sum dg^i_j E^j_i + g^i_j [A, E^j_i]^b_a E^a_b \\ &= \sum (dg^b_a + A^b_i g^i_a - g^b_j A^j_a) E^a_b \\ &= \sum (dg^b_a + [A, g]^b_a) E^a_b, \end{aligned}$$

that is,

$$\widetilde{\nabla} g = dg + [A, g]. \tag{1.11}$$

1.52. As we have seen before, the gauge group acts on G-connections by conjugation (one should compare this situation with the adjoint representation of a Lie group in its Lie algebra), with $g \cdot \nabla = g \circ \nabla \circ g^{-1}$, and we compute the action on a local section s via

$$\begin{aligned} (g \cdot \nabla)(s) &= g \circ \nabla(g^{-1}s) \\ &= g(dg^{-1}s + g^{-1}ds + Ag^{-1}s) \\ &= ds + (gdg^{-1} + gAg^{-1})s. \end{aligned}$$

We claim that we can write the action as

$$g \cdot \nabla = \nabla - \widetilde{\nabla} g g^{-1}. \tag{1.12}$$

To see this, we do the computation locally. From the last paragraph we obtain

$$\begin{aligned} (\nabla - \widetilde{\nabla} g g^{-1})s &= (d + A)s - ((dg + Ag - gA)g^{-1})s \\ &= ds + (-dgg^{-1} + gAg^{-1})s, \end{aligned}$$

which is the same expression as above, if we observe that $dgg^{-1} = -gdg^{-1}$.

1.53. If we are considering unitary connections, we can say even more. But before that, we will make an intermezzo on complexification of Lie groups. Recall that a Lie group G is said to be a **real Lie group** if its correspondent Lie algebra is a real vector space, and **complex** if the corresponding Lie algebra is a complex one. Our favorite Lie group here, the unitary group $U(k)$, is real. To see this, recall that its Lie algebra is

$$\mathfrak{u}(k) = \{A \in \mathfrak{gl}(k, \mathbb{C}) \mid A + A^\dagger = 0\},$$

where $\mathfrak{gl}(k, \mathbb{C})$ is the the Lie algebra of $GL(k, \mathbb{C})$ that is the set of all k by k matrices and $\dagger$ means the transpose conjugate of a matrix. Now observe that for $A, B \in \mathfrak{u}(k)$ and λ a scalar, we have

$$(A + \lambda B)^\dagger = A^\dagger + \overline{\lambda} B^\dagger = -(A + \overline{\lambda} B),$$

therefore $(A + \lambda B)$ is again in $\mathfrak{u}(k)$ only when $\lambda \in \mathbb{R}$. Back to some generality, if G is a real Lie group then the **complexification of** G, $G_{\mathbb{C}}$ is the unique complex Lie group equipped with a map $\varphi : G \to G_{\mathbb{C}}$ such that any map $G \to H$, where H is a complex Lie group, extends to a holomorphic map $G_{\mathbb{C}} \to H$. The respective Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic. Put aside the technicalities, we will show that the complexification of $\mathfrak{u}(k)$ is $\mathfrak{gl}(k, \mathbb{C})$, and, therefore, the complexification of $U(k)$ will be $GL(k, \mathbb{C})$, with the map being the inclusion. To verify our claim, consider the map

$$\begin{array}{ccc} \mathfrak{gl}(k, \mathbb{C}) & \to & \mathfrak{u}(k) \otimes \mathbb{C} := \mathfrak{u}(k) \oplus i \cdot \mathfrak{u}(k) \\ X & \longmapsto & \left(\frac{X - X^\dagger}{2}\right) + i \left(\frac{X + X^\dagger}{2i}\right), \end{array}$$

which is an isomorphism. In our context, the gauge group of a $U(k)$-bundle, $\mathcal{G}$, of unitary automorphisms of the bundle has as its complexification, $\mathcal{G}_{\mathbb{C}}$, the group of all complex linear automorphisms of the bundle.

1.54. Now, considering unitary connections and the action of the gauge group $\mathcal{G}$ as in (1.12), we claim that the action of the complexified gauge group, $\mathcal{G}_{\mathbb{C}}$, which in this case is the group of all complex linear automorphisms is given by

$$g \cdot \nabla = \nabla - (\widetilde{\nabla}^{0,1} g) g^{-1} + ((\widetilde{\nabla}^{0,1} g) g^{-1})^\dagger \tag{1.13}$$

To see this, first some trivial observations:

$$\begin{aligned} [A^{0,1}, g]^\dagger &= [(A^\dagger)^{1,0}, g^\dagger] && (1.14) \\ (\bar{\partial} g)^\dagger &= \partial(g^\dagger) && (1.15) \\ (\widetilde{\nabla}^{1,0} g) g^{-1} &= -g(\widetilde{\nabla}^{1,0} g^{-1}). && (1.16) \end{aligned}$$

Instead of brute forcing and showing equation (1.13), we will simply show that in the case that we consider $g \in \mathcal{G}$, equation (1.13) becomes (1.12). Assuming then that $g \in \mathcal{G}$, i.e., $gg^\dagger = \mathbb{1}$ and recalling that $A + A^\dagger = 0$, we get, applying (1.14) to (1.16) on the last term of (1.13), that

$$\begin{aligned} ((\widetilde{\nabla}^{0,1} g) g^{-1})^\dagger &= ((\bar{\partial} g + [A^{0,1}, g]) g^\dagger)^\dagger \\ &= g(\partial(g^\dagger) - [(A^\dagger)^{1,0}, g^\dagger]) \\ &= g(\partial g^{-1} + [A^{1,0}, g^{-1}]) \\ &= g(\widetilde{\nabla}^{1,0} g^{-1}) \\ &= -(\widetilde{\nabla}^{1,0} g) g^{-1}, \end{aligned}$$

Therefore

$$g \cdot \nabla = \nabla - (\widetilde{\nabla}^{0,1} g) g^{-1} - (\widetilde{\nabla}^{1,0} g) g^{-1} = \nabla - (\widetilde{\nabla} g) g^{-1},$$

as claimed.

1.55. Our last remark is that two unitary connections give isomorphic holomorphic structures on a Hermitian bundle E if they are in the same $\mathcal{G}_{\mathbb{C}}$-orbit. Indeed, assume we are given an element $g \in \mathcal{G}_{\mathbb{C}}$. This is, by definition, an automorphism of the underlying C^∞ vector bundle. Assume further that we are given a unitary connection A on E and let $B = (g \cdot A)$. We claim that the bundle automorphism g is in fact holomorphic, i.e., it preserves the complex structures induced by the connections. Symbolically, this means

$$\bar{\partial}_B(gs) = g(\bar{\partial}_A s), \tag{1.17}$$

for all sections s of $\mathcal{E}_A$. To see this, we construct a connection ∇_{BA} on $\mathcal{E}_B \otimes \mathcal{E}_A^*$, by setting

$$\nabla_{BA} = \nabla_B \otimes \mathbb{1} + \mathbb{1} \otimes \nabla_A^\vee.$$

Recall from the calculation of 1.51 that in matrix notation we have

$$\nabla_{BA} h = dh + Bh - hA.$$

Observe that a map $h : \mathcal{E}_A \to \mathcal{E}_B$ is holomorphic, i. e., satisfies the holomorphic condition (1.17), if, and only if, $\bar{\partial}_{BA}h = 0$. Indeed, for every section s of $\mathcal{E}_A$,

$$\begin{aligned}(\bar{\partial}_{BA}h)(s) &= (\bar{\partial}h + B^{0,1}h - hA^{0,1})(s) \\ &= \bar{\partial}h(s) + (h\bar{\partial}s - h\bar{\partial}s) + B^{0,1}h(s) - hA^{0,1}(s) \\ &= \bar{\partial}_B(hs) - h(\bar{\partial}_A s).\end{aligned}$$

So, in order to prove our claim, all we need is to show that $\bar{\partial}_{BA}g = 0$. But,

$$\begin{aligned}\bar{\partial}_{BA}g &= \bar{\partial}g + B^{0,1}g - gA^{0,1} \\ &= \bar{\partial}g + (-\bar{\partial}g\, g^{-1} + gA^{0,1}g^{-1})g - gA^{0,1} \\ &= 0.\end{aligned}$$

1.3.2 Curvature

1.56. Suppose that we have a vector bundle E over a manifold M and a connection A on E. We start by extending the de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \cdots$$

to bundle-valued forms. To do this, we define exterior derivatives $d_A \colon \Omega^p(M; E) \to \Omega^{p+1}(M; E)$ by requiring that $d_A = \nabla_A$ on $\Omega^0(M; E)$ and the Leibniz rule

$$d_A(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^q \omega \wedge d_A\theta,$$

for $\omega \in \Omega^q(M)$ and $\theta \in \Omega^r(M; E)$. We will, from now on, use either d_A or ∇_A to denote the connection. While $d^2 = 0$ on differential forms, on bundle-valued forms this is not true, in general. Actually, $d_A d_A$ is an algebraic operator on sections, that is, linear with respect to smooth functions, since for f a function and s a section, we have

$$\begin{aligned}d_A d_A(fs) &= d_A(df s + f d_A(s)) \\ &= -df \wedge d_A(s) + df \wedge d_A(s) + f d_A d_A(s) \\ &= f d_A d_A(s).\end{aligned}$$

1.57 Definition. The **curvature** of a connection ∇_A, often denoted F_∇, F_A or $F(A)$, is a section of the bundle $\mathrm{End}(E) \otimes \Lambda^2(T^*M)$ given by

$$F_A(s) = d_A d_A(s).$$

1.58. Let us spell out the curvature in a local trivialisation. Pick a frame $\sigma = \{e_1, \dots, e_n\}$. We have

$$F_A(e_j) = \sum F^i_j e_i,$$

where $F = (F^i_j)$ is the **curvature matrix** with respect to the frame σ. To be more precise, we should write F^σ, where σ, as usual, is the section on the frame bundle given by the frame σ. For another frame $\sigma' = \{e'_1, \dots, e'_n\}$, with $[\,\cdot\,]' = g[\,\cdot\,]$, we have

$$\begin{aligned} [F]'[s]' \; = \; [F_\nabla s]' &= g[F_\nabla s] \\ &= g[F][s] \\ &= (g[F]g^{-1})[s]', \end{aligned}$$

where we see that the matrix of the curvature transforms under change of basis by $[F]' = g[F]g^{-1}$. Now, the action of a gauge transformation g changes the connection ∇ and thus the curvature in the same way as the change of basis, that is, if $\nabla' = (g \cdot \nabla)$

$$F_{\nabla'} = g \circ F_\nabla \circ g^{-1}. \tag{1.18}$$

1.59. We relate the connection matrix and the curvature matrix by the so-called **Cartan structure equation**.

$$F_A = dA + A \wedge A. \tag{1.19}$$

Indeed, observe that

$$\begin{aligned} d_A d_A(e_i) &= d_A\left(\sum_j A_i^j e_j\right) \\ &= \sum_j \left(dA_i^j e_j - A_i^j \wedge d_A(e_j)\right) \\ &= \sum_j dA_i^j e_j - \sum_k A_i^j \wedge A_j^k e_k \\ &= \sum_j \left(dA_i^j + \sum_k A_k^j \wedge A_i^k\right) e_j. \end{aligned}$$

1.60 Remark. One should be careful with the expression in 1.19. The term $A \wedge A$ is interpreted as a matrix multiplication in which the entries, being 1-forms, are multiplied using the wedge product.

1.61. We pause the development of the theory for a word of warning. We have been using the term "local" in distinct ways so far. On the one hand, whenever we pick a local frame on a vector bundle, which is equivalent to picking a local section in the associated principal frame bundle, we mean local trivialization, and we have used this approach every time to do the computations with connections. On the other hand, there is the notion of local coordinates of the base manifold. This is also important, and we have used this, e.g., in paragraph 1.38, where we have written the matrix of 1-forms

$$A^\sigma = \sigma^*(A) = \sum A_\mu dx^\mu.$$

Considering a local coordinate chart U with $\varphi(p) = (x^1(p), \ldots, x^n(p))$, we have the notion of **covariant derivative in the μ direction**, ∇_μ, that is given by

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} + A_\mu,$$

that is, $\nabla = \sum \nabla_\mu dx^\mu$. Using this notion, and writing the curvature of ∇ as $F_\nabla = \frac{1}{2}\sum F_{\mu\nu} dx^\mu dx^\nu$, where the components $F_{\mu\nu}$ are by definition $F(\partial_\mu, \partial_\nu)$, we have

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.20}$$

To establish (1.20), all we need to do is unravel the definitions. Indeed,

$$\begin{aligned} F(\partial_\mu, \partial_\nu) &= (dA + A \wedge A)(\partial_\mu, \partial_\nu) \\ &= \sum \partial_\lambda A_\eta dx^\lambda dx^\eta (\partial_\mu, \partial_\nu) + (A \wedge A)(\partial_\mu, \partial_\nu) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \end{aligned}$$

and also,

$$\begin{aligned} [\partial_\mu + A_\mu, \partial_\nu + A_\nu]s &= (\partial_\mu + A_\mu)(\partial_\nu + A_\nu)s - (\partial_\nu + A_\nu)(\partial_\mu + A_\mu)s \\ &= \partial_\mu \partial_\nu s + \partial_\mu A_\nu s + A_\nu \partial`_\mu s + A_\mu \partial_\nu s + A_\mu A_\nu s - \\ &\qquad - (\partial_\nu \partial_\mu s + \partial_\nu A_\mu s + A_\mu \partial_\nu s + A_\nu \partial_\mu s + A_\nu A_\mu s) \\ &= \big(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\big)s. \end{aligned}$$

1.62 Proposition. *Assuming that we are dealing with a holomorphic Hermitian bundle $\mathcal{E}$ and our connection A is the Chern connection, then the curvature matrix is of type $(1,1)$.*

Proof. First we will find which is the matrix of the Chern connection, as in 1.47. Let us pick a local basis of E consisting of holomorphic sections, $\{e_1, \ldots, e_n\}$. Denote $H = (H_{ij})$, with $H_{ij} = H(e_i, e_j)$, the matrix of smooth functions corresponding to the Hermitian structure. We know that with respect to the local trivialization the connection can be written as

$$\nabla = d + A.$$

As ∇ is compatible with the holomorphic structure, the 1-forms Ae_i are of type $(1,0)$. As ∇ is compatible with the Hermitian structure, we have

$$dH_{ij} = dH(e_i, e_j) = H(Ae_i, e_j) + H(e_i, Ae_j),$$

where $H(Ae_i, e_j)$ is of type $(1,0)$ and $H(e_i, Ae_j)$ is of type $(0,1)$. This implies that

$$\partial H_{ij} = H(Ae_i, e_j) = {}^\tau(Ae_i) H \overline{e_j} = ({}^\tau A H)_{ij}.$$

So we have a matrix equality $\partial H = {}^\tau A H$ or $A = ({}^\tau H^{-1}) \partial ({}^\tau H)$. If we denote $M = {}^\tau H$, the transposed matrix, then $A = M^{-1} \partial M$ and this is the matrix of the metric connection. The curvature of this connection is

$$\begin{aligned} F &= dA + A \wedge A \\ &= -M^{-1}(dM) \wedge M^{-1}\partial M + M^{-1} d\partial M + M^{-1}\partial M \wedge M^{-1}\partial M \\ &= -M^{-1}(\bar\partial M) \wedge M^{-1}(\partial M) + M^{-1}\bar\partial\partial M \\ &= \bar\partial(M^{-1}\partial M), \end{aligned}$$

so we see that F is of type $(1,1)$. □

1.63. The differential forms in $\Omega^p(X; \mathrm{End}(E))$ are locally matrices of p-forms, as we have seen so far. We will use this fact to define a Lie product on the algebra

$$\Omega^*(X; \mathrm{End}(E)) = \oplus_p \Omega^p(X; \mathrm{End}(E)).$$

For $\eta \in \Omega^p(X; \mathrm{End}(E))$ and $\xi \in \Omega^p(X; \mathrm{End}(E))$, set

$$[\eta, \xi] := \eta \wedge \xi - (-1)^{pq} \xi \wedge \eta. \tag{1.21}$$

This graded commutator accounts to the graded commutation law on the ring of differential forms $\eta \wedge \xi = (-1)^{pq} \xi \wedge \eta$. Observe that if we change the frame $e'_i = e_i \cdot g$ and $e'^i = e^i \cdot g$, for $g = \sum g^i_j e_i e^j$ a section of $\Omega^0(X; \mathrm{End}(E))$, the matrix representations changes with conjugation by g^{-1}, as

$$\begin{aligned} \sum \eta^i_j e_i e^j &= \sum \eta^i_j (g^k_i e_k)(g^j_l e^l) \\ &= \sum (g^k_i \eta^i_j g^j_l) e_k e^l \\ &= \sum (g^{-1}\eta g)^k_l e_k e^l, \end{aligned}$$

therefore,

$$[g^{-1}\eta g, g^{-1}\xi g] = g^{-1}\eta g \wedge g^{-1}\xi g - (-1)^{pq} g^{-1}\xi g \wedge g^{-1}\eta g = g^{-1}[\eta, \xi] g,$$

and the expression (1.21) is well defined. With this in mind, observe that the graded commutator of two 1-forms is $[\eta, \xi] = \eta \wedge \xi + \xi \wedge \eta$. Thus, for the vector potential A that is a matrix of 1-forms, we get

$$[A, A] = 2A \wedge A,$$

therefore we can rewrite the Cartan structure equation as

$$F_A = dA + \frac{1}{2}[A, A].$$

With this consideration in mind, if A is a unitary connection, i.e., the the matrix $A \in \mathfrak{u}(n)$, then the curvature 2-form will also take values in $\mathfrak{u}(n)$.

1.64. Given a connection A on a vector bundle E, we know how to induce connections on E^* and in $\mathrm{End}(E)$. Also, we know how to compute the covariant derivative d_A on any bundle valued form. The next propositions are intended to summarize all of our computational knowledge on connections.

1.65 Remark. On paragraphs 1.48 and 1.50 we have distinguished the connections ∇, $\nabla^\vee$, $\widetilde{\widetilde{\nabla}}$ on the bundles constructed from E. However, once we have picked a connection on E, we will denote all the exterior derivative operators by d_A, regardless of if we are considering E-valued, E^*-valued or $\mathrm{End}(E)$-valued forms. The context should be enough to indicate which operator we are dealing with.

1.66 Proposition. *Let A be a connection on a vector bundle E over a manifold M, with extended exterior derivative d_A acting either on E-valued, E^*-valued or* $\mathrm{End}(E)$*-valued forms. Then, locally, the action of the operators d_A and $d_A d_A$ on ξ, a bundle-valued form is given by*

$$\begin{array}{lrcll} (1) & d_A\xi & = & d\xi + A\wedge\xi, & \xi \in \Omega^p(M;E); \\ (2) & d_A d_A\xi & = & F_A \wedge \xi, & \xi \in \Omega^p(M;E); \\ (3) & d_A\xi & = & d\xi - {}^\tau\! A\wedge\xi, & \xi \in \Omega^p(M;E^*); \\ (4) & d_A d_A\xi & = & -{}^\tau\! F_A \wedge \xi, & \xi \in \Omega^p(M;E^*); \\ (5) & d_A\xi & = & d\xi + [A,\xi], & \xi \in \Omega^p(M;\mathrm{End}(E)); \\ (6) & d_A d_A\xi & = & [F_A,\xi], & \xi \in \Omega^p(M;\mathrm{End}(E)). \end{array}$$

Proof. The proof will be given by local computations, so we will assume always that we have pick a gauge σ in our principal frame bundle. For (1), consider

$$d_A\xi = d_A\left(\sum_j \xi^j e_j\right) = \sum_{i,j}(d\xi^j e_j + (-1)^p \xi^j \wedge A^i_j e_i) = \sum_{i,j}(d\xi^i + A^i_j \wedge \xi^j)e_i.$$

For (2), we use the result of (1), and

$$\begin{aligned} d_A(d_A\xi) &= d(d\xi + A\wedge\xi) + A\wedge(d\xi + A\wedge\xi) \\ &= dA\wedge\xi - A\wedge d\xi + A\wedge d\xi + A\wedge A\wedge\xi \;=\; F_A\wedge\xi. \end{aligned}$$

Item (3) is very similar to (1.8).

$$d_A\xi = d_A\left(\sum_i \xi_i e^i\right) = \sum_{i,j}(d\xi_i e^i - (-1)^p \xi_i\wedge A^i_j e^j) = \sum_j \left(d\xi_j - \sum_i ({}^\tau A)^j_i \wedge \xi_i\right) e^j.$$

For item (4), we use the result of (3), and

$$\begin{aligned} d_A(d_A\xi) &= d(d\xi - {}^\tau A\wedge\xi) - {}^\tau A\wedge(d\xi - {}^\tau A\wedge\xi) \\ &= -d\,{}^\tau A\wedge\xi + {}^\tau A\wedge d\xi - {}^\tau A\wedge d\xi + {}^\tau A\wedge{}^\tau A\wedge\xi \\ &= (-d\,{}^\tau A - {}^\tau(A\wedge A))\wedge\xi \\ &= -{}^\tau F_A\wedge\xi. \end{aligned}$$

Item (5) is a little trickier. Recall from (1.10) that $[A, E^j_i]^b_a = A^b_i\delta^j_a - A^j_a\delta^b_i$, hence the sum below, for p-forms ξ^i_j, is

$$\sum_{i,j}\xi^i_j\wedge[A, E^j_i]^b_a = (-1)^p A^b_i\wedge\xi^i_a - \xi^b_j\wedge A^j_a. \tag{1.22}$$

Thus,

$$d_A\xi = d_A\left(\sum_{i,j}\xi^i_j E^j_i\right) = \sum_{i,j,a,b}(d\xi^i_j E^j_i + (-1)^p\xi^i_j\wedge[A, E^j_i]^b_a E^a_b) = \sum_{a,b}(d\xi^b_a + [A,\xi]^b_a)E^a_b,$$

where use have been made of (1.9) and (1.22). Finally, before we prove item (6), observe that

$$\begin{aligned} d[A,\xi] &= d(A\wedge\xi) - (-1)^p d(\xi\wedge A) \\ &= dA\wedge\xi - A\wedge d\xi + (-1)^{p+1}d\xi\wedge A - \xi\wedge dA \\ &= (dA\wedge\xi - \xi\wedge dA) - (A\wedge d\xi - (-1)^{p+1}d\xi\wedge A) \\ &= [dA,\xi] - [A, d\xi]. \end{aligned}\tag{1.23}$$

Also, we have the following

$$\begin{aligned}
[A,[A\wedge\xi]] &= [A, A\wedge\xi-(-1)^p\xi\wedge A]\\
&= A\wedge A\wedge\xi+(-1)^{p+1}A\wedge\xi\wedge A-(-1)^{p+1}(A\wedge\xi\wedge A+\\
&\qquad\qquad +(-1)^{p+1}\xi\wedge A\wedge A)\\
&= A\wedge A\wedge\xi+(-1)^{p+1}A\wedge\xi\wedge A-(-1)^{p+1}A\wedge\xi\wedge A-\xi\wedge A\wedge A)\\
&= [A\wedge A,\xi]. \qquad (1.24)
\end{aligned}$$

Now, using item (5), (1.23) and (1.24), we get

$$d_A(d_A\xi)=d_A(d\xi+[A,\xi])=d[A,\xi]+[A,d\xi]+[A,[A,\xi]]=[dA+A\wedge A,\xi].$$

□

1.67 Proposition. *Let A be a connection on a G-bundle E and g be an element of the gauge group $\mathcal{G}$. Then, the action of g on the covvnt derivative d_A of $\mathfrak{g}_E$-valued forms is given by*

$$d_{g\cdot A}\xi=\mathrm{Ad}(g)d_A(Ad(g^{-1})\xi).$$

Proof. This is just another computation, and we will be a bit terse here. Just recall that $Ad(g)Z=g\cdot Z\cdot g^{-1}$, for g an element on the Lie Group and Z an element in the Lie algebra. Then, on the one hand, using proposition 1.66 and (1.7), we have

$$d_{g\cdot A}\xi=d\xi+[gAg^{-1}-dgg^{-1},\xi],$$

and on the other hand,

$$\begin{aligned}
\mathrm{Ad}(g)d_A(Ad(g^{-1})\xi) &= g(d_A(g^{-1}\xi g))g^{-1}\\
&= g(d(g^{-1}\xi g)+[A,g^{-1}\xi g])g^{-1}\\
&= g(-g^{-1}dgg^{-1}\xi g+g^{-1}d\xi g+(-1)^p g^{-1}\xi dg)g^{-1}+\\
&\qquad\qquad +g[A,g^{-1}\xi g]g^{-1}\\
&= d\xi-[dgg^{-1},\xi]+[gAg^{-1},\xi].
\end{aligned}$$

□

1.68. Accounting for the fact that F_A is an endomorphism-valued 2-form, the induced connection in the End(E)-bundle acts on F_A according to proposition 1.66, and we get

$$\begin{aligned} d_A(F_A) &= dF_A + [A, F_A] \\ &= d(dA + A \wedge A) + [A, (dA + A \wedge A)] \\ &= dA \wedge A - A \wedge dA + [A, dA] + [A, A \wedge A] \\ &= [dA, A] + [A, dA] = 0, \end{aligned}$$

where the commutator in question ais the graded commutator, but since only even forms are involved, what we have here is the usual commutator. This relation is very special and often called the **Bianchi identity**. Some authors refer to the Bianchi identity as

$$dF_A = [F_A, A],$$

which is obviously equivalent to $d_A(F_A) = 0$.

1.3.3 Variations of the Connection

1.69. This brief subsection is entirely based on the subsection 2.1.2 of the acclaimed book [DONALDSON, KRONHEIMER, 1990]. Here, we simply elaborate on how the connection and the curvature vary within a neighborhood of a fixed connection A_0 in the space of connections $\mathcal{A}$. Recall from paragraph 1.43 that the topology on $\mathcal{A}$ is endowed from the topology of $\Omega^1(M; \mathfrak{g}_E)$, as these spaces are in bijection. We start by noting that for $A \in \mathcal{A}$ and $a \in \Omega^1(M; \mathfrak{g}_E)$ then

$$\begin{aligned} F(A + a)s &= (d_A + a)(d_A + a)s \\ &= d_A d_A s + (d_A a)s - a \wedge d_A s + a \wedge d_A s + (a \wedge a)s, \end{aligned}$$

therefore,

$$F(A + a) = F_A + d_A a + a \wedge a. \tag{1.25}$$

1.70. Let A_t be a smooth one-parameter family of connections with $\frac{d}{dt}\big|_{t=0} A_t = a$, or simply a curve in $\mathcal{A}$ passing through A_0 with derivative a. Note that as $\mathcal{A}$ is an affine space modeled on $\Omega^1(M; \mathfrak{g}_E)$, $T_{A_0}\mathcal{A} \cong \Omega^1(M; \mathfrak{g}_E)$, so, indeed, a is a tangent vector to $\mathcal{A}$ at A_0. In order to makes things easier, consider $A_t = A_0 + ta$, for any $a \in \Omega^1(M; \mathfrak{g}_E)$.

From (1.25), we get

$$F_t := F(A_t) = F(A_0) + t d_{A_0} a + t^2 (a \wedge a),$$

thus, $\frac{d}{dt}\big|_{t=0} F_t = d_{A_0} a$. In other words, viewing the curvature as a map $F\colon \mathcal{A} \to \Omega^2(M;\mathfrak{g}_E)$, we have that the derivative of F at A_0 is precisely d_{A_0}, that is,

$$(dF)_{A_0} = d_{A_0} \colon \Omega^1(M;\mathfrak{g}_E) \to \Omega^2(M;\mathfrak{g}_E). \tag{1.26}$$

1.71. Consider now g_t a one parameter family of gauge transformations with $g_0 = id$, the identity of $\mathcal{G}$. The time derivative at 0, $\frac{d}{dt}\big|_{t=0} g_t$ can be viewed as a section ξ of the bundle $\mathfrak{g}_E$. Indeed, observe that for each $p \in M$, $g_t(p) \in G$ and $g_0(p) = e \in G$, therefore $\frac{d}{dt}\big|_{t=0} g_t \in \mathfrak{g}$. For a connection A_0, define

$$A_t = g_t \cdot A_0 = A_0 - (d_{A_0} g_t) g_t^{-1} = g_t A_0 g_t^{-1} - d g_t g_t^{-1},$$

confer (1.7) and (1.12). Thus, denoting the time derivatives with a dot $\cdot$ as is costumary in Newtonian physics, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} A_t &= \left(\dot{g}_t A_0 g_t^{-1} - g_t A_0 (g_t^{-1} \dot{g}_t g_t^{-1}) - d\dot{g}_t g_t^{-1} + d g_t g_t^{-1} \dot{g}_t g_t^{-1}\right)\Big|_{t=0} \\
&= \xi A_0 - A_0 \xi - d\xi + 0 \\
&= -d_{A_0} \xi.
\end{aligned}$$

In other words, the derivative of the map $\alpha_{A_0}\colon \mathcal{G} \to \mathcal{A}$, given by $\alpha_{A_0}(g) = \alpha(g, A_0)$, where α denotes the action $\alpha\colon \mathcal{G} \times \mathcal{A} \to \mathcal{A}$ of the gauge group on the affine space of connections, gives us

$$(d\alpha_{A_0})_{id} = -d_{A_0} \colon \Omega^0(M;\mathfrak{g}_E) \to \Omega^1(M;\mathfrak{g}_E). \tag{1.27}$$

Note again that $T_{A_0}\mathcal{A} \cong \Omega^1(M;\mathfrak{g}_E)$ and $T_{id}\mathcal{G} \cong \Omega^0(M;\mathfrak{g}_E)$.

1.72. Finally, composing (1.26) and (1.27) we have that the curvature of the family $g_t \cdot A_0$ at 0 is given by

$$\frac{d}{dt}\Big|_{t=0} F(g_t \cdot A_0) = -d_{A_0} d_{A_0} \xi = -[F_{A_0}, \xi] = [\xi, F_{A_0}]. \tag{1.28}$$

1.73 Remark. One should compare (1.28) with (1.18), and recall that the infinitesimal version of the adjoint action on a matrix Lie Group is the Lie bracket.

1.3.4 Chern Classes

1.74. Characteristic classes are topological invariants of vector bundles. We will briefly sketch the ideas behind this subject, and more on this subject can be found in [MILNOR, STASHEFF, 1974] and [GRIFFITH, HARRIS, 1978]. Let $E \to M$ be a complex vector bundle, of rank k over an n dimensional manifold M, A a connection on E and $F = F_A$ its curvature. We define the **total Chern class** of E, by

$$c(E) := \det\left(\mathbb{1} + \frac{iF}{2\pi}\right) \in H^{2*}_{dR}(M;\mathbb{R}). \tag{1.29}$$

Since F is a 2-form, $c(E)$ is a direct sum of forms of even degrees, $c(E) = 1 + c_1(E) + c_2(E) + \cdots$, where $c_j(E) \in H^{2j}_{dR}(M;\mathbb{R})$ is called the j**th Chern class**. Note that from the dimension of M, $c_j(E) = 0$ for $2j > n$, and since the determinant is a polynomial of degree k, the rank of the bundle, its series terminates at $c_k(E)$, irrespectively of $\dim M$.

1.75. Some things are to be explained from the definition of Chern classes. First, recall the Taylor expansion of the determinant $\det\colon \mathbb{C}^{k^2} \to \mathbb{C}$, around the identity, given by

$$\det(\mathbb{1} + tB) = 1 + t\det^{(1)}(\mathbb{1})(B) + \frac{t^2}{2!}\det^{(2)}(\mathbb{1})(B,B) + \frac{t^3}{3!}\det^{(3)}(\mathbb{1})(B,B,B) + \cdots .$$

Playing with the partial derivatives of the determinant, we get that $\det^{(j)}(\mathbb{1})(B,\ldots,B)$ is written as a sum of elements of the type

$$\mathrm{Tr}(B^{n_1}) \cdot \mathrm{Tr}(B^{n_2}) \cdots \mathrm{Tr}(B^{n_p}), \tag{1.30}$$

such that $n_1 + n_2 + \cdots + n_p = j$. Now, consider $t = \frac{i}{2\pi}$ and $B = F$, as in (1.29). We saw that the curvature is an element of $\Omega^2(M;\mathrm{End}(E))$, thus, taking n_r wedge products of F we obtain

$$F^{n_r} := F \wedge \ldots \wedge F \in \Omega^{2n_r}(M;\mathrm{End}\,E).$$

Taking the trace of this product yields an element in $\Omega^{2n_r}(M)$. The idea now is to show that the terms $\mathrm{Tr}(F^{n_r})$ are closed so the $c_j(E)$, that are given as sum of elements of the type (1.30), are indeed cohomology classes in $H^{2j}_{dR}(M)$. This follows from the Bianchi identity and the below proposition:

1.76 Proposition. *Let ξ be an* $\mathrm{End}(E)$*-valued p-form and ζ be an* $\mathrm{End}(E)$*-valued q-form. Then,*

$$\mathrm{Tr}(d_A(\xi \wedge \zeta)) = d\,\mathrm{Tr}(\xi \wedge \zeta).$$

Proof. This is just a matter of calculation, where we will strongly use proposition 1.66 and the graded commutator.

$$\begin{aligned}
\mathrm{Tr}(d_A(\xi \wedge \zeta)) &= \mathrm{Tr}(d_A\xi \wedge \zeta + (-1)^p \xi \wedge d_A\zeta) \\
&= \mathrm{Tr}(d\xi \wedge \zeta + (-1)^p \xi \wedge d\zeta) + \mathrm{Tr}([A,\xi] \wedge \zeta + (-1)^p \xi \wedge [A,\zeta]) \\
&= \mathrm{Tr}(d(\xi \wedge \zeta)),
\end{aligned}$$

since

$$\begin{aligned}
\mathrm{Tr}([A,\xi] \wedge \zeta) &= \mathrm{Tr}(A \wedge \xi \wedge \zeta - (-1)^p \xi \wedge A \wedge \zeta) \\
(-1)^p\,\mathrm{Tr}(\xi \wedge [A,\zeta]) &= (-1)^p\,\mathrm{Tr}(\xi \wedge A \wedge \zeta - (-1)^q \xi \wedge \zeta \wedge A),
\end{aligned}$$

therefore,

$$\mathrm{Tr}([A,\xi] \wedge \zeta + (-1)^p \xi \wedge [A,\zeta]) = \mathrm{Tr}(A \wedge \xi \wedge \zeta) - (-1)^{2(p+q)}\,\mathrm{Tr}(A \wedge \xi \wedge \zeta) = 0.$$

□

1.77. Definition in (1.29) apparently depends on the choice a connection A on E, but we claim that this is not the case. Indeed, first we show that the de Rham cohomology class of elements of the type $\mathrm{Tr}(F^{n_r})$ are independent of the connection A. Let A' be other connection on E. Consider $a = A' - A$ and define

$$A_t = A + ta.$$

It is clear that $A_0 = A$ and that $A_1 = A'$. Using (1.25), we get that $F_t = F + t d_A a + t^2 a \wedge a$, therefore

$$\frac{d}{dt}\mathrm{Tr}(F_t^{n_r}) = n_r\,\mathrm{Tr}\left(\frac{d}{dt}F_t \wedge F_t^{n_r-1}\right) = n_r\,\mathrm{Tr}\left((d_A a + 2ta \wedge a) \wedge F_t^{n_r-1}\right)$$

But, as $\mathrm{Tr}\left(a \wedge a \wedge F_t^{n_r-1}\right) = 0$, once a is a 1-form, we get

$$\frac{d}{dt}\mathrm{Tr}(F_t^{n_r}) = n_r\,\mathrm{Tr}\left(d_A a \wedge F_t^{n_r-1}\right) = n_r\,\mathrm{Tr}\left(d_A\left(a \wedge F_t^{n_r-1}\right)\right) = n_r d\,\mathrm{Tr}\left(a \wedge F_t^{n_r-1}\right),$$

where use have been made of the Bianchi identity and the proposition 1.76. Thus, we have

$$\begin{aligned}\operatorname{Tr}(F'^{n_r}) - \operatorname{Tr}(F^{n_r}) &= \operatorname{Tr}(F_1^{n_r}) - \operatorname{Tr}(F_0^{n_r}) \\ &= \int_0^1 \frac{d}{dt}\operatorname{Tr}(F_t^{n_r})dt \\ &= d\left\{\int_0^1 n_r \operatorname{Tr}\left(a \wedge F_t^{n_r-1}\right) dt\right\},\end{aligned}$$

so the difference is exact. Now for the general case of (1.30), we have a similar calculation, and

$$\begin{aligned}\frac{d}{dt}\left(\operatorname{Tr}(F_t^{n_1}) \wedge \ldots \wedge \operatorname{Tr}(F_t^{n_p})\right) &= \sum_{r=1}^{p} \operatorname{Tr}(F_t^{n_1}) \wedge \ldots \wedge \frac{d}{dt}\operatorname{Tr}(F_t^{n_r}) \wedge \ldots \wedge \operatorname{Tr}(F_t^{n_p}) \\ &= \sum_{r=1}^{p} \operatorname{Tr}(F_t^{n_1}) \wedge \ldots \wedge n_r\, d\operatorname{Tr}\left(a \wedge F_t^{n_r-1}\right) \wedge \ldots \wedge \operatorname{Tr}(F_t^{n_p}) \\ &= d\left\{\sum_{r=1}^{p} n_r \operatorname{Tr}(F_t^{n_1}) \wedge \ldots \wedge \operatorname{Tr}\left(a \wedge F_t^{n_r-1}\right) \wedge \ldots \wedge \operatorname{Tr}(F_t^{n_p})\right\},\end{aligned}$$

since in paragraph 1.75, we concluded that $d\operatorname{Tr}(F_t^{n_r}) = 0$.

1.78 Theorem (Properties of Chern Classes)**.** *Let E and E' be smooth complex vector bundles over a manifold M. Then the Chern classes satisfy the following properties:*

(a) (Naturality) *If N is a smooth manifold and $f\colon N \to M$ is a smooth map, then*

$$c(f^*E) = f^*c(E).$$

(b) *$c(E \oplus E') = c(E) \cdot c(E')$, the product in $H^*_{dR}(M;\mathbb{R})$.*

(c) *$c(E)$ depends only on the isomorphism class of the vector bundle E.*

(d) *If E^* is the dual bundle of E, then*

$$c_j(E^*) = (-1)^j c_j(E).$$

1.79. We will deal here only with manifolds of dimension ≤ 4, thus the only relevant Chern classes for us are $c_1(E)$ and $c_2(E)$, and the explicit formulas, calculated using

(1.29) are

$$c_1(E) = \operatorname{Tr}\left(\frac{iF}{2\pi}\right) \tag{1.31}$$

$$c_2(E) = -\frac{1}{8\pi^2}((\operatorname{Tr}(F)^2 - \operatorname{Tr}(F^2)). \tag{1.32}$$

1.4 Exact Sequence of Vector Bundles

1.80. Following [KOBAYASHI, 1987], suppose we have an exact sequence of holomorphic bundles over a base space M,

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Then $\mathcal{S}$ is a subbundle and $\mathcal{Q} = \mathcal{E}/\mathcal{S}$ is the quotient bundle. Assume that $\mathcal{E}$ has a Hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. Restricting the Hermitian structure to $\mathcal{S}$ we endow $\mathcal{S}$ with a Hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{S}}$. Taking the orthogonal complement of $\mathcal{S}$ in $\mathcal{E}$ we obtain another complex subbundle, $S^{\perp}$, which may not be holomorphic, and we have a decomposition $E = S \oplus S^{\perp}$ (we have written E and S without calligraphic letters to recall that we are dealing with the underlying smooth bundles). However, as C^{∞} bundles, we have a natural isomorphism between $S^{\perp}$ and $\mathcal{Q}$, and thus we obtain a Hermitian structure $\langle\cdot,\cdot\rangle_{\mathcal{Q}}$ on $\mathcal{Q}$. Denote by ∇, the metric connection in $\mathcal{E}$. Set

$$\nabla(s) = \nabla_S(s) + \alpha(s), \tag{1.33}$$

where $s \in \Gamma(\mathcal{E})$, $\nabla_S(s) \in \Omega^1(M;\mathcal{S})$ and $\alpha(s) \in \Omega^1(M;S^{\perp})$.

1.81 Proposition. *Under the above considerations, we have:*

1. ∇_S *is the Chern connection of* $(\mathcal{S},\langle\cdot,\cdot\rangle_{\mathcal{S}})$.
2. α *is a* $(1,0)$*-form with values in* $\operatorname{Hom}(\mathcal{S},S^{\perp})$.

Proof. Cf. [KOBAYASHI, 1987], Proposition (6.4), chapter I, section 6. □

1.82. The form $\alpha \in \Omega^{1,0}(M;\operatorname{Hom}(\mathcal{S},S^{\perp}))$ is often called the **second fundamental form** of $\mathcal{S}$ in $\mathcal{E}$. Under the indentification $Q = S^{\perp}$, we consider $\alpha \in \Omega^{1,0}(M;\operatorname{Hom}(\mathcal{S},\mathcal{Q}))$. Similarly to what we saw in 1.80, set

$$\nabla(s) = \beta(s) + \nabla_{S^{\perp}}(s),$$

where $s \in \Gamma(\mathcal{E})$, $\beta(s) \in \Omega^1(M;\mathcal{S})$ and $\nabla_{S^\perp}(s) \in \Omega^1(M;S^\perp)$. From the identification $S^\perp = \mathcal{Q}$, we may consider $\nabla_{S^\perp}$ mapping $\Gamma(\mathcal{Q}) \to \Omega^1(M;\mathcal{Q})$, and we write ∇_Q instead.

1.83 Proposition. *Under the above considerations, we have:*

1. ∇_Q *is the Chern connection of* $(\mathcal{Q}, \langle\cdot,\cdot\rangle_{\mathcal{Q}})$.
2. β *is a* $(0,1)$*-form with values in* $\mathrm{Hom}(\mathcal{Q},\mathcal{S})$.
3. β *is the adjoint of* $-\alpha$, *i.e.,*

$$\langle \alpha s, t\rangle_{\mathcal{E}} + \langle s, \beta t\rangle_{\mathcal{E}} = 0, \qquad s \in \Gamma(\mathcal{S}), \quad t \in \Gamma(\mathcal{Q}).$$

Proof. Cf. [KOBAYASHI, 1987], Proposition (6.6), chapter I, section 6. □

1.84 Proposition. *Write* F, F_S *and* F_Q *for the curvature associated to the metric connection, respectively in* $\mathcal{E}, \mathcal{S}$ *and* $\mathcal{Q}$. *In matrix notations, the curvature* F *is expressed as*

$$F = \begin{pmatrix} F_S - \beta \wedge \beta^\dagger & d\beta \\ -d\beta^\dagger & F_Q - \beta^\dagger \wedge \beta \end{pmatrix}.$$

Proof. Cf. [KOBAYASHI, 1987]. See discussion involving equation (6.12). □

1.85. To finish this section, we will explain the reasons for the terminology second fundamental form used in paragraph 1.82. It is a generalization of the ideas behind the differential geometry of surfaces in $\mathbb{R}^3$. Recall that on a differential surface $S \subset \mathbb{R}^3$ we can define the Gauss map $N\colon S \to S^2$, that attaches to each point $p \in S$ a unit normal vector according to an orientation of S. Identifying the tangent space at p of S, T_pS, with $T_{N(p)}S^2$, we get that the derivative of the Gauss map at p is a self-adjoint linear map (cf. [DO CARMO, 1976] proposition 1 of chapter 3) that gives rise to the quadratic form

$$II_p(v) := -\langle dN_p(v), v\rangle,$$

where $v \in T_pS$ and $\langle\cdot,\cdot\rangle$ is the usual inner product of $\mathbb{R}^3$. This quadratic form is what we call the *second fundamental form* of the surface S at p. There is a generalization of this concept to Riemannian geometry that is as follows. Let M be a submanifold of the Riemannian manifold $(\bar{M}, \langle\cdot,\cdot\rangle)$ under the immersion $\iota\colon M \to \bar{M}$. The metric on $\bar{M}$ induces a metric on M that turns ι an isometric immersion and gives us a decomposition $T_p\bar{M} = T_pM \oplus T_pM^\perp$ for each p that is smooth with respect to p. Hence, any vector field $\bar{v}$ on $\bar{M}$ is decomposed as $\bar{v} = \bar{v}^T + \bar{v}^\perp$. There is a unique connection $\bar{\nabla}$ that is symmetric (cf. [DO CARMO, 2008], definition 3.4 of chapter 2) and compatible with the metric, called the Riemannian or *Levi-Civita* connection (cf. [DO CARMO, 2008], theorem 3.6 of chapter 2). Using the Levi-Civita connection of $\bar{M}$, we endow M with a Riemannian connection by

$$\nabla_v w = (\bar{\nabla}_{\bar{v}} \bar{w})^T,$$

for extensions $\bar{v}$, $\bar{w}$ to $\bar{M}$ of vector fields v, w on M (cf. [DO CARMO, 2008], exercise 3 of chapter 2). With these connections, we define a bilinear and symmetric map $B\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)^\perp$ (cf. [DO CARMO, 2008], proposition 2.1 of chapter 6), by

$$B(v, w) = \bar{\nabla}_{\bar{v}} \bar{w} - \nabla_v w, \tag{1.34}$$

where $\mathscr{X}(M)^\perp$ denotes the vector field in $\bar{M}$ normal to M and $\bar{v}, \bar{w}$ are extension of v, w to $\bar{M}$. Now, for a given point $p \in M$ and a normal vector $\eta \in T_pM^\perp$, we define $S_\eta\colon T_pM \to T_pM$ by asking for

$$\langle S_\eta(v), w\rangle = \langle B(v, w), \eta\rangle.$$

Since B is bilinear and symmetric, S_η is readily seen as a self-adjoint map. We then say that the *second fundamental form* at p with respect to the normal vector η is

$$II_\eta(v) = \langle S_\eta(v), v\rangle = \langle B(v, v), \eta\rangle.$$

In the case when $\bar{M}$ is $\mathbb{R}^{n+1}$ and M is an orientable submanifold of dimension n, also called *hypersurface*, choosing $\eta \in T_pM^\perp$ to be of norm one and to be determined by the orientation of M, we can define a similar Gauss map $N\colon M \to S^n$, where S^n is the sphere of unit vectors of $\mathbb{R}^{n+1}$ (cf. [DO CARMO, 2008] example 2.4 of chapter 6). If $N(p) = \eta$, the derivative of this Gauss map at a point p applied on a vector $v \in T_pM$ is

$$dN_p(v) = -S_\eta(v),$$

cf. [DO CARMO, 2008] example 2.4 of chapter 6, so the second fundamental form is expressed as $II_p(v) = -\langle dN_p(v), v\rangle$, like in the case of surfaces of $\mathbb{R}^3$.

1.86. The link with paragraph 1.82 is the bilinear map $B\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)^\perp$

that we used to construct the map S_η. We can regard it as a section of the bundle $T^*M \otimes \mathrm{Hom}(TM, TM^\perp)$, that is, B is a 1-form with values in the homomorphism of TM to $TM^\perp$, $B \in \Omega^1(\mathrm{Hom}(TM, TM^\perp))$. Now, as $\iota\colon M \to \bar{M}$ is the immersion of M in $\bar{M}$, we have an exact sequence of bundles over M

$$0 \longrightarrow TM \longrightarrow \iota^* T\bar{M} \longrightarrow TM^\perp \longrightarrow 0,$$

that is the situation of the underlying smooth vector bundles treated in paragraph 1.82 in this special case when we consider $\iota^* T\bar{M}$. Then, comparing (1.33) and (1.34), one sees the reasons for the terminology second fundamental form.

Capítulo 2

Harmonic Forms

The aim of this chapter is to introduce the duality operator on forms and state the celebrated Hodge theorems that are important to our theory. We first start on pseudo-Riemannian and Riemannian manifolds, then we pass the ideas to complex manifolds and finish with Riemann Surfaces, since they are vital to chapter 4, where we discuss stability of bundles over Riemann surfaces.

2.1 Hodge Theory on Smooth Manifolds

2.1. Consider M an oriented manifold of dimension n equipped with a semi-Riemannian metric g. In local coordinates we write the metric tensor as

$$g = \sum g_{\mu\nu} dx^\mu \otimes dx^\nu, \qquad g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right).$$

We often abbreviate $\frac{\partial}{\partial x^\mu}$ as ∂_μ, then we have $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$. There is a canonical volume form on M that we construct in this paragraph. First, we cover M with coordinate charts $\{U_\alpha\}$, and on each chart $(U_\alpha, \varphi_\alpha)$, with $\varphi_\alpha(p) = (x^1_\alpha(p), \ldots, x^n_\alpha(p))$, we consider the volume form on U_α

$$\mathrm{vol}_\alpha = \sqrt{|\det(g_\alpha)_{\mu\nu}|}\, dx^1_\alpha \wedge \ldots \wedge dx^n_\alpha, \tag{2.1}$$

where g_α stands for the restriction of the metric g to the submanifold U_α. On the overlaps $U_\alpha \cap U_\beta$, we claim that $\mathrm{vol}_\alpha = \mathrm{vol}_\beta$. In order to see this, recall the change of coordinates

$$dx_\beta^\nu = \sum T_\mu^\nu dx_\alpha^\mu, \qquad T_\mu^\nu = dx_\beta^\nu((\partial_\alpha)_\mu) = \frac{\partial x_\beta^\nu}{\partial x_\alpha^\mu},$$

and

$$(\partial_\beta)_\mu = \sum S_\mu^\nu (\partial_\alpha)_\nu, \qquad S_\mu^\nu = dx_\alpha^\nu((\partial_\beta)_\mu) = \frac{\partial x_\alpha^\nu}{\partial x_\beta^\mu}.$$

Thus, we have that $S = T^{-1}$, and

$$dx_\beta^1 \wedge \ldots \wedge dx_\beta^n = \left(\sum T_\mu^1 dx_\alpha^\mu\right) \wedge \ldots \wedge \left(\sum T_\mu^n dx_\alpha^n\right) = (\det T) dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n,$$

and also

$$(g_\beta)_{\mu\nu} = g\left(\sum S_\mu^\lambda (\partial_\alpha)_\lambda, \sum S_\nu^\eta (\partial_\alpha)_\eta\right) = \sum S_\mu^\lambda S_\nu^\eta (g_\alpha)_{\lambda\eta} = ({}^\tau S (g_\alpha) S)_{\mu\nu},$$

so substituting these last two relations on (2.1) we obtain

$$\mathrm{vol}_\beta = \det(T^{-1}) \sqrt{|\det(g_\alpha)_{\mu\nu}|} (\det T) dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n = \mathrm{vol}_\alpha,$$

as claimed.

2.2. Another interesting use of the metric tensor g is to produce an associated inner product on the differential forms of M. First, we induce inner products on each T_p^*M by considering the inverse of the matrix g, that is, for 1-forms ω, ζ on M , set

$$g(\omega, \zeta) = \sum g^{\mu\nu} \omega_\mu \zeta_\nu,$$

where $g^{\mu\nu}$ stands for the inverse of $g_{\mu\nu}$ in the sense that $\sum g^{\mu\lambda} g_{\lambda\nu} = \delta_\nu^\mu$. Next, we extend this to an inner product on the whole exterior algebra by setting

$$(\omega^1 \wedge \ldots \wedge \omega^p, \zeta^1 \wedge \ldots \wedge \zeta^p) = \det(g(\omega^i, \zeta^j)),$$

where ω^i and ζ^j are 1-forms. This inner product is often considered as the **pointwise inner product** of ω, ζ, where we have written $\omega = \omega^1 \wedge \ldots \wedge \omega^p$ and $\zeta = \zeta^1 \wedge \ldots \wedge \zeta^p$.

We also have an L^2 inner product using the volume form defined as

$$\langle \omega, \xi \rangle = \int_M (\omega, \xi) \, \mathrm{vol},$$

for compact manifolds M, or compactly supported forms.

2.3 Definition. The **Hodge star** or **duality** operator is the unique linear operator on p-forms $*\colon \Omega^p(M) \to \Omega^{n-p}(M)$ that satisfies

$$\omega \wedge *\xi = (\omega, \xi) \, \mathrm{vol}, \qquad \forall \omega, \xi \in \Omega^p(M),$$

where vol is the volume form constructed in paragraph 2.1 and $(\cdot, \cdot)$ is the pointwise inner product defined in 2.2.

2.4. The definition above seems quite mysterious, so it is nice to have a formula to compute the duality operator. Consider $\epsilon_{\mu_1,\ldots\mu_n}$ as a totally anti-symmetric tensor, that takes the value ± 1 according if $(\mu_1 \cdots \mu_n)$ is an even or an odd permutation of $(1 \cdots n)$. If two indices μ_i and μ_j are equal, then $\epsilon_{\mu_1,\ldots\mu_n} = 0$. Using index gymnastics, we raise the indices and write, for instance,

$$\epsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_n} = \sum g^{\mu_1 \nu_1} \cdots g^{\mu_r \nu_r} \epsilon_{\nu_1 \ldots \nu_n}.$$

That said, let (U, φ) be a chart with coordinates $\varphi(p) = (x^1(p), \ldots, x^n(p))$. Then, the action of the duality operator on a basic vector of $\Omega^r(M)$ is

$$*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}) = \frac{\sqrt{|\det g|}}{(n-r)!} \sum \epsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_n} dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_n}.$$

2.5 Remark. Sometimes, it is easier to consider orthonormal non-coordinate basis $\{e^1, \ldots, e^n\}$ of T^*M, or, using vector bundles language, a local orthonormal frame of the cotangent bundle T^*M, or a coframe. We remark, *en passant*, that in general, whenever we have a vector bundle, a local frame of the dual bundle is called a **coframe**. Orthonormal coframes of the cotangent bundle are obtained by using the Gram-Schmidt process on the coordinate basis $\{dx^1, \ldots, dx^n\}$, and we have that

$$e^\nu = \sum E^\nu_\mu dx^\mu,$$

for a matrix of change of basis E. With respect to an orthonormal basis, it is not so hard to prove that the action of the Hodge star operator on r-forms is given by

$$*(e^{\mu_1} \wedge \ldots \wedge e^{\mu_r}) = \pm e^{\mu_{r+1}} \wedge \ldots \wedge e^{\mu_n},$$

where the signal $\pm$ is given by $g^{\mu_1\mu_1} \cdots g^{\mu_r\mu_r} \epsilon_{\mu_1 \ldots \mu_n}$, and $\{\mu_{r+1}, \ldots, \mu_n\}$ are the missing elements of $\{\mu_1, \ldots, \mu_r\}$ in $\{1, \ldots, n\}$ such that $(e^{\mu_1} \wedge \ldots \wedge e^{\mu_r}) \wedge (e^{\mu_{r+1}} \wedge \ldots \wedge e^{\mu_n})$ is a multiple of the volume form $e^1 \wedge \ldots \wedge e^n$.

2.6 Example. Suppose $M = \mathbb{R}^4$, with global coordinates $\{x^0, x^1, x^2, x^3\}$, also denoted as $\{t, x, y, z\}$, equipped with the Minkowski metric

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3.$$

We often denote this $\mathbb{R}^4$ with the Minkowski metric as $\mathbb{R}^{3,1}$, because the metric has signature $(3, 1)$ (it is a convention to write first the number of positive entries on the diagonalized matrix of the metric). Observe that in our case, the global forms $\{dx^0, dx^1, dx^2, dx^3\}$ give us an orthonormal basis for 1-forms, so $|\det g| = 1$. We calculate now the action of the Hodge star on forms of $\mathbb{R}^{3,1}$. Considering 0-forms, that are C^∞ functions, we get

$$*(1) \quad = \quad \frac{1}{4!} \sum \epsilon_{\mu\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

On 1-forms, we have $*dx^\mu = \frac{1}{3!} \sum g^{\mu\alpha} \epsilon_{\alpha\nu\lambda\rho} dx^\nu \wedge dx^\lambda \wedge dx^\rho$, and

$$\begin{aligned} *dx^0 &= -dx^1 \wedge dx^2 \wedge dx^3 \\ *dx^1 &= -dx^0 \wedge dx^2 \wedge dx^3 \\ *dx^2 &= +dx^0 \wedge dx^1 \wedge dx^3 \\ *dx^3 &= -dx^0 \wedge dx^1 \wedge dx^2. \end{aligned}$$

For 2-forms, we have $*(dx^\mu \wedge dx^\nu) = \frac{1}{2}\sum g^{\mu\alpha}g^{\nu\beta}\epsilon_{\alpha\beta\lambda\rho}dx^\lambda \wedge dx^\rho$, so

$$\begin{aligned}
*(dx^0 \wedge dx^1) &= -dx^2 \wedge dx^3 \\
*(dx^0 \wedge dx^2) &= +dx^1 \wedge dx^3 \\
*(dx^0 \wedge dx^3) &= -dx^1 \wedge dx^2 \\
*(dx^1 \wedge dx^2) &= +dx^0 \wedge dx^3 \\
*(dx^1 \wedge dx^3) &= -dx^0 \wedge dx^2 \\
*(dx^2 \wedge dx^3) &= +dx^0 \wedge dx^1.
\end{aligned}$$

Considering 3-forms, we have $*(dx^\mu \wedge dx^\nu \wedge dx^\lambda) = \sum g^{\mu\alpha}g^{\nu\beta}g^{\lambda\gamma}\epsilon_{\alpha\beta\gamma\rho}dx^\rho$, thus

$$\begin{aligned}
*(dx^0 \wedge dx^1 \wedge dx^2) &= -dx^3 \\
*(dx^0 \wedge dx^1 \wedge dx^3) &= +dx^2 \\
*(dx^0 \wedge dx^2 \wedge dx^3) &= -dx^1 \\
*(dx^1 \wedge dx^2 \wedge dx^3) &= -dx^0.
\end{aligned}$$

Finally, on 4-forms we get

$$*(dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) = \sum g^{0\mu}g^{1\nu}g^{2\lambda}g^{3\rho}\epsilon_{\mu\nu\lambda\rho} = -(1).$$

Observe that $*^2\colon \Omega^p(M) \to \Omega^p(M)$ is a linear operator on p-forms. The above results suggest that the star operator squares to the identity up to a sign. This is indeed the case and in general, we have

2.7 Proposition. *Let M be an n-dimensional oriented pseudo-Riemannian manifold with metric g of signature $(s, n-s)$. Then, on p-forms we have*

$$*^2 = (-1)^{p(n-p)+s}.$$

Proof. Consider $\{e^1, \dots e^n\}$ a local orthonormal coframe. Locally, every p-form ω is written as $\sum \omega_{\mu_1 \dots \mu_p} e^{\mu_1} \wedge \dots \wedge e^{\mu_p}$, or in a more condensed way, $\omega = \sum \omega_I e^I$, with $I = \{\mu_1, \dots, \mu_p\}$, and the meanings of ω_I and e^I being the obvious ones. As the Hodge star is linear on p-forms by definition, it suffices only to work on basic forms of type

$$e^{\mu_1} \wedge \dots \wedge e^{\mu_p}.$$

Applying the formula of paragraph 2.4, or of remark 2.5 we obtain

$$\begin{aligned} ** e^{\mu_1} \wedge \ldots \wedge e^{\mu_p} &= *(g^{\mu_1\mu_1} \cdots g^{\mu_p\mu_p} \epsilon_{\mu_1 \ldots \mu_n} e^{\mu_{p+1}} \wedge \ldots \wedge e^{\mu_n}) \\ &= g^{\mu_1\mu_1} \cdots g^{\mu_n\mu_n} \epsilon_{\mu_1 \ldots \mu_n} \epsilon_{\mu_{p+1} \ldots \mu_n \mu_1 \ldots \mu_n} e^{\mu_1} \wedge \ldots \wedge e^{\mu_p} \\ &= (-1)^s (-1)^{p(n-p)} e^{\mu_1} \wedge \ldots \wedge e^{\mu_p}, \end{aligned}$$

where use have been made of the fact that the metric has signature $(n-s, s)$ and that $\epsilon_{\mu_{p+1} \ldots \mu_n \mu_1 \ldots \mu_p} = (-1)^{p(n-p)} \epsilon_{\mu_1 \ldots \mu_n}$. □

2.8 Example. A nice phenomenon occurs in 4-manifolds, namely, the Hodge star operator $*$ is a linear operator on $\Omega^2(M)$. Consider M as the usual $\mathbb{R}^4$, endowed with the Euclidean metric of signature $(4, 0)$, and with global coordinates x^0, x^1, x^2, x^3. From proposition 2.7, the duality squares to the identity, therefore the eigenvalues of $*\colon \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ are $+1$ and -1. Let us compute the eigenspaces $\Omega^2_+(\mathbb{R}^4)$ and $\Omega^2_-(\mathbb{R}^4)$ associated to the respective eigenvalues. Let

$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3\},$$

be an ordered basis of the space $\Omega^2(\mathbb{R}^4)$. The action of $*$ in the basis is

$$*(dx^0 \wedge dx^1) = dx^2 \wedge dx^3, \quad *(dx^0 \wedge dx^2) = -dx^1 \wedge dx^3, \quad *(dx^0 \wedge dx^3) = dx^1 \wedge dx^2,$$

therefore, as $*^2 = \mathbb{1}$,

$$*(dx^2 \wedge dx^3) = dx^0 \wedge dx^1, \quad *(dx^1 \wedge dx^3) = -dx^0 \wedge dx^2, \quad *(dx^1 \wedge dx^2) = dx^0 \wedge dx^3.$$

With respect to this ordered basis we can construct the matrix of the star operator

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using simple tricks of linear algebra, such as writing the elements ω of $\Omega^2(\mathbb{R}^4)$ as column vectors and forcing $*\omega = \omega$, we obtain

$$\Omega^2_+(\mathbb{R}^4) = \text{span}\{(dx^0\wedge dx^1 + dx^2\wedge dx^3), (dx^0\wedge dx^2 - dx^1\wedge dx^3), (dx^0\wedge dx^3 + dx^1\wedge dx^2)\}.$$

Forcing $*\omega = -\omega$, we obtain

$$\Omega^2_-(\mathbb{R}^4) = \text{span}\{(dx^0\wedge dx^1 - dx^2\wedge dx^3), (dx^0\wedge dx^2 + dx^1\wedge dx^3), (dx^0\wedge dx^3 - dx^1\wedge dx^2)\}.$$

From this discussion, we conclude that $\Omega^2(\mathbb{R}^4) = \Omega^2_+(\mathbb{R}^4)\oplus\Omega^2_-(\mathbb{R}^4)$, and any element $\omega\in\Omega^2(M)$ is uniquely written as

$$\omega = \omega_+ + \omega_-, \qquad *\omega_+ = \omega_+, *\omega_- = -\omega_-, \tag{2.2}$$

by setting

$$\omega_+ = \frac{1}{2}(\omega + *\omega), \qquad \omega_- = \frac{1}{2}(\omega - *\omega).$$

Although we have done the above calculations using $\mathbb{R}^4$, we could have used any Riemannian 4-manifold and we would also have the decomposition $\Omega^2(\mathbb{R}^4) = \Omega^2_+(M)\oplus\Omega^2_-(M)$. Forms in $\Omega^2_+(M)$ are called **self-dual** (SD) and those in $\Omega^2_-(M)$ are called **anti-self-dual** (ASD). So, in (2.2), what we have is that any element $\omega\in\Omega^2(M)$ is uniquely written as a sum of its self-dual and anti-self-dual parts. Also, it is clear that if we change the orientation of our manifold, e.g., if in the $\mathbb{R}^4$ case we put $dx^1\mapsto -dx^1$, we send self-dual forms to anti-self-dual forms and vice-versa.

2.9. Using the L^2 inner product we introduced in paragraph 2.2, for closed (i.e., compact and boundaryless) manifolds M, we can define the formal adjoint $d^\dagger\colon \Omega^p(M)\to\Omega^{p-1}(M)$ of the exterior differential d by requiring that

$$\langle d\omega, \eta\rangle = \langle \omega, d^\dagger\eta\rangle,$$

for forms $\omega\in\Omega^{p-1}(M)$ and $\eta\in\Omega^p(M)$. Exploring a bit the equation above, we have,

$$\begin{aligned}\langle d\omega,\eta\rangle &= \int_M d\omega\wedge *\eta \\ &= (-1)^p\int_M \omega\wedge d*\eta \\ &= (-1)^p\int_M \omega\wedge *(*^{-1}d*\eta),\end{aligned}$$

where we used Stokes theorem on the second passage. Comparing the above result with

$$\langle \omega, d^\dagger \eta \rangle = \int_M \omega \wedge * d^\dagger \eta,$$

we obtain that

$$d^\dagger \eta = (-1)^p *^{-1} d * \eta.$$

From the identity $(-1)^{(n-p+1)[n-(n-p+1)]} = (-1)^{np+n+p+1}$ and since $d * \eta \in \Omega^{n-p+1}(M)$, using the result $*^{-1} = (-1)^{p(n-p)+s} *$, from proposition 2.7, we get

$$d^\dagger \eta = (-1)^{(n-p+1)[n-(n-p+1)]+s}(-1)^p d * \eta = (-1)^{np+n+s+1} * d * \eta. \tag{2.3}$$

2.10 Definitions. The **Laplace-Beltrami** operator, or simply **Laplacian** is

$$\Delta := (dd^\dagger + d^\dagger d) \colon \Omega^p(M) \to \Omega^p(M),$$

and elements in the kernel of Δ are called **harmonic forms**. The set of harmonic p-forms on a manifold M is denoted by $\mathscr{H}^p(M)$.

2.11. If M is a compact oriented Riemannian manifold, then the Laplacian is a positive operator, in the sense that

$$\langle \zeta, \Delta \zeta \rangle = \langle \zeta, dd^\dagger \zeta \rangle + \langle \zeta, d^\dagger d \zeta \rangle = \langle d^\dagger \zeta, d^\dagger \zeta \rangle + \langle d\zeta, d\zeta \rangle \geq 0. \tag{2.4}$$

Recall that a p-form ζ is called closed if $d\zeta = 0$, and it is called exact if there exists a $(p-1)$-form ξ such that $d\xi = \zeta$. Very similarly, we say that a p-form ζ is **coclosed** if $d^\dagger \zeta = 0$ and **coexact** if $d^\dagger \xi = \zeta$ for a $(p+1)$-form ξ. It is easy to see that if a form is closed and coclosed it will be harmonic. Conversely, if a form is harmonic, from (2.4) and from the positive definiteness of the L^2 inner product, it will be closed and coclosed, therefore we have proved the following:

2.12 Proposition. *In a compact oriented Riemannian manifold M, a differential form is harmonic if and only if it is closed and coclosed.*

2.13 Theorem (Hodge decomposition theorem)**.** *Consider (M, g) a compact orientable Riemannian manifold. Then $\Omega^p(M)$ is uniquely decomposed as*

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus d^\dagger \Omega^{p+1}(M) \oplus \mathscr{H}^p(M),$$

where the orthogonality is with respect to the L^2 inner product on forms, defined in paragraph 2.2.

Proof. Cf. [NAKAHARA, 2003], Theorem 7.7. □

2.14. From the Hodge decomposition theorem, it is reasonable to define a projection operator on the space of harmonic forms, $P\colon \Omega^p(M) \to \mathscr{H}^p(M)$. Consider a non-zero p-form ζ. As $(\zeta - P\zeta)$ is orthogonal to $\mathscr{H}^p(M)$, it can be proved that there exists another p-form ξ such that $\Delta\xi = (\zeta - P\zeta)$. This is a very technical result and we will not do it here. The form $\xi = \Delta^{-1}(\zeta - P\zeta)$ is granted by existence of the Green function, and more can be found in [GRIFFITH, HARRIS, 1978]. Assuming then that $\Delta\xi = (\zeta - P\zeta)$, for a suitable ξ, the Hodge decomposition of ζ is given by

$$\zeta = d(d^\dagger\xi) + d^\dagger(d\xi) + P\zeta.$$

2.15 Theorem (Hodge theorem)**.** *If (M, g) is a compact orientable Riemannian manifold, then there is an isomorphism between the de Rham cohomology group and the space of harmonic forms. Symbolically,*

$$H^p(M, \mathbb{R}) \cong \mathscr{H}^p(M),$$

and the isomorphism is given by $[\zeta] \in H^p(M, \mathbb{R}) \mapsto P\zeta \in \mathscr{H}^p(M)$.

Proof. Cf. [NAKAHARA, 2003], Theorem 7.8. □

2.16. We conclude this section by discussing the minimizing property of the harmonic representative of a cohomology class. Assume that M is a compact orientable Riemannian manifold and that $\omega \in \Omega^p(M)$ is a closed p-form. Among the forms $\{\omega + d\eta\}$, for $\eta \in \Omega^{p-1}(M)$ that represents the cohomology class $[\omega] \in H^p(M, \mathbb{R})$, we claim that the one which has the smallest norm (the norm here is the one of the L^2 inner product defined in paragraph 2.2) is the harmonic representative. On the one hand, if ω is harmonic, from proposition 2.12, then ω is closed and coclosed, therefore

$$\begin{aligned}
\langle \omega + d\eta, \omega + d\eta \rangle &= \|\omega\|^2 + \|d\eta\|^2 + 2\langle d\eta, \omega \rangle \\
&= \|\omega\|^2 + \|d\eta\|^2 + 2\langle \eta, d^\dagger\omega \rangle \\
&= \|\omega\|^2 + \|d\eta\|^2 \\
&\geq \|\omega\|^2.
\end{aligned}$$

Conversely, if ω has the smallest norm, then for any $\eta \in \Omega^{p-1}(M)$, we have

$$0 = \frac{d}{dt}\bigg|_{t=0} \|\omega + td\eta\|^2 = 2\langle \eta, d^\dagger \omega\rangle,$$

so $d^\dagger\omega = 0$, and ω is closed and coclosed, thus harmonic.

2.2 Hodge Theory on Complex Manifolds

2.17. Let M be a complex manifold endowed with a Hermitian metric h, and fundamental form ω. Considering $g = \Re(h)$, we get a Riemannian metric on its underlying smooth manifold with real coordinates $x^1, y^1, \dots, x^n, y^n$ (cf. A.37 in appendix A), and we can apply the notion of duality operator for real manifolds that we saw in 2.3. We then extend the action of the duality operator to the space of complexified forms as, for instance

$$*dz^\mu = *dx^\mu + i * dy^\mu.$$

Also, the Hermitian extension of the Riemannian metric g to the complexified cotangent space given by

$$g(\xi \otimes \lambda, \zeta \otimes \mu)_{\mathbb{C}} = \lambda\bar{\mu}\; g(\xi, \zeta),$$

when restricted to the holomorphic cotangent bundle $T^{*1,0}M \cong T^*M$ is closely related to the Hermitian inner product on the cotangent space T^*M, see A.41, in appendix A. There is also an extension to the whole exterior algebra of this Hermitian inner product on 1-forms to the **pointwise Hermitian inner product**

$$(\xi \otimes \lambda, \zeta \otimes \mu)_{\mathbb{C}} = \lambda\bar{\mu}(\xi, \lambda).$$

With these considerations, the Hodge star operator satisfies the following equation, similar to the initial definition of the star operator:

$$\zeta \wedge *\bar{\eta} = (\zeta, \eta)_{\mathbb{C}} \operatorname{vol}. \tag{2.5}$$

2.18 Remark. Regarding the pointwise Hermitian inner product, there are two equivalent ways of seeing it. One, is the way we did in the last paragraph: start with a Riemannian metric in T^*M, consider the Hermitian extension to $(T^*M)_{\mathbb{C}}$ and then extend it to the whole exterior algebra $\Lambda\ (T^*M)_{\mathbb{C}}$. But we could also extend the Riemannian metric g first to $\Lambda\, T^*M$, and afterwards, consider the Hermitian extension to $\Lambda\,(T^*M)_{\mathbb{C}} \cong (\Lambda\, T^*M)_{\mathbb{C}}$.

2.19. We claim then that the duality operator sends (p,q)-forms to $(n-q,n-p)$-forms. Indeed, in order to convince yourself of this, all you need is to take a look at equation (2.5). If ζ is a (p,q)-form, then the only way for ζ wedged with anything to be a volume form (an (n,n)-form) is for $*\bar{\eta} = \overline{*\eta}$ be a $(n-p,n-q)$-from, that is $*\eta$ is a $(n-q,n-p)$-form, as we claimed.

2.20. Totally analogous to the pseudo-Riemannian case, we also have a notion of an L^2 Hermitian inner product, when M is compact or for compactly supported forms, given by

$$\langle \zeta, \eta \rangle = \int_M \zeta \wedge *\bar{\eta} \tag{2.6}$$

2.21 Remark. Some authors also denote the star operator $\bar{*}$ defined by $\bar{*}\alpha = *\bar{\alpha} = \overline{*\alpha}$ as the duality operator itself, since, from the above, $*$ sends (p,q)-forms to $(n-q,n-p)$-forms, and $\bar{*}$ sends (p,q)-forms to $(n-p,n-q)$-forms.

2.22. Recall that on a complex manifold M we have that the exterior differential d splits on the sum of the Dolbeault operators as

$$d = \partial + \bar{\partial},$$

where $\partial\colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$, as we can see in the appendix on paragraph A.32. Using the inner product defined in (2.6) we define adjoints $\partial^\dagger\colon \Omega^{p,q}(M) \to \Omega^{p-1,q}(M)$ and $\bar{\partial}^\dagger\colon \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$, by requiring

$$\langle \partial\eta, \zeta \rangle = \langle \eta, \partial^\dagger \eta \rangle \qquad \langle \bar{\partial}\eta, \zeta \rangle = \langle \eta, \bar{\partial}^\dagger \eta \rangle.$$

It follows that $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$, and since a complex manifold is a real even dimensional manifold, from (2.3) we have

$$d^\dagger = - * d *,$$

and thus, also

$$\partial^\dagger = - * \bar{\partial} * \qquad \bar{\partial}^\dagger = - * \partial * .$$

2.23. On a complex manifold, apart from the usual Laplacian, we have operators on

$\Omega^{p,q}(M)$

$$\begin{aligned}\Delta_\partial &:= \partial\partial^\dagger + \partial^\dagger\partial \\ \Delta_{\bar\partial} &:= \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial.\end{aligned}$$

If a (p, q)-form η satisfies $\Delta_{\bar\partial}\eta = 0$, it is called **$\bar\partial$-harmonic**. The set

$$\mathscr{H}^{p,q}_{\bar\partial}(M) = \{\eta \in \Omega^{p,q}(X) \mid \Delta_{\bar\partial}\eta = 0\},$$

denotes the space of all $\bar\partial$-harmonic (p, q)-forms, and we have the following complex analogs of the theorems in Hodge theory:

2.24 Theorem. *On a compact complex manifold M, $\Omega^{p,q}(M)$ is uniquely decomposed as*

$$\Omega^{p,q}(M) = \bar\partial\Omega^{p,q-1}(M) \oplus \bar\partial^\dagger\Omega^{p,q+1}(M) \oplus \mathscr{H}^{p,q}_{\bar\partial}(M).$$

Proof. Cf. [GRIFFITH, HARRIS, 1978], page 84. □

2.25 Theorem. *For a compact complex manifold M, we have an isomorphism*

$$H^{p,q}_{\bar\partial}(M) \cong \mathscr{H}^{p,q}_{\bar\partial}(M).$$

Proof. Cf. [GRIFFITH, HARRIS, 1978], page 100. □

2.26. Similarly to the real case, define the projection $P\colon \Omega^{p,q}(M) \to \mathscr{H}^{p,q}(M)$. Consider a non-zero (p, q)-form ζ. Then, there exists another (p, q)-form ξ such that $\Delta\xi = (\zeta - P\zeta)$. Thus, the Hodge decomposition of ζ is given by

$$\zeta = \bar\partial(\bar\partial^\dagger\xi) + \bar\partial^\dagger(\bar\partial\xi) + P\zeta.$$

2.3 Hodge Theory on Riemann Surfaces

2.27. Throughout this section, we will consider X a compact Riemann surface, and vol the canonical volume form on X that depends on the metric of X, as in 2.1. By a Riemann surface, we mean a complex manifold of complex dimension 1. From A.44 in the appendix, vol $= \omega$, the fundamental form associated to the Hermitian metric. We will briefly specialize Hodge theory to the context of Riemann surfaces, which will be very important in chapter 4.

2.28. With some calculations we can give explicit formulas for the Hodge star in the present case. We have

$$\bar{*}f := \bar{f}\,\mathrm{vol}, \qquad \bar{*}\zeta := -i(\overline{\zeta^{0,1}} - \overline{\zeta^{1,0}}), \qquad \bar{*}\eta := \bar{f},$$

for a function $f\colon X \to \mathbb{C}$, a 1-form $\zeta = \zeta^{1,0} + \zeta^{0,1}$ and a $(1,1)$-form $\eta = f\,\mathrm{vol}$. It is immediate that $\bar{*}^2 = \mathbb{1}$ on $(1,1)$-forms and on functions, while in 1-forms $\bar{*}^2 = -\mathbb{1}$.

2.29. In order to do local computations with the Laplacian, recall that the volume form is given by $\mathrm{vol} = \sqrt{\det g}dx \wedge dy$, and since $dz \wedge d\bar{z} = -2idx \wedge dy$, consider $u = (\sqrt{\det g})^{-1}$, a real valued positive function on the local coordinate z such that

$$u(z)\,\mathrm{vol} = \frac{i}{2} dz \wedge d\bar{z}.$$

For a function f, from the fact that a Riemann surface is automatically Kähler for dimensional reasons, we have

$$\begin{aligned} \Delta f = 2\Delta_\partial f &= 2(- * \bar{\partial} * \partial f) \\ &= -2(*\bar{\partial}(-i)\partial f) \\ &= 2i * (f_{z\bar{z}} d\bar{z} \wedge dz) \\ &= -4u f_{z\bar{z}}. \end{aligned}$$

So, harmonic 0-forms are harmonic functions in the usual sense we are used to. For a $(1,1)$-form $\eta = f\,\mathrm{vol}$, we get

$$\Delta\eta = 2\Delta_\partial(f\,\mathrm{vol}) = 2(-\partial * \bar{\partial} * (f\,\mathrm{vol})) = -2(*^{-1} * \partial * \bar{\partial} f) = *^{-1}(\Delta f) = (\Delta f)\,\mathrm{vol},$$

So a form $\eta = f\,\mathrm{vol}$ is harmonic if and only if the function f is harmonic. Thus the map

$$f \mapsto f\,\mathrm{vol},$$

gives us a $1-1$ correspondence between harmonic $(1,1)$-forms and harmonic functions. We also have the following theorem.

2.30 Theorem. *Global harmonic* $(1,1)$*-forms on a compact Riemann surface* X *are constant multiples of* vol*, and conversely.*

2.31. Consider now $\Omega(X)$ the graded inner product complex vector space of all differential forms

$$\Omega(X) = \Omega^0(X) \oplus \Omega^1(X) \oplus \Omega^{1,1}(X),$$

where the Hermitian inner product is given by

$$\langle f \oplus \alpha \oplus \omega, g \oplus \beta \oplus \theta \rangle := \langle f, g \rangle + \langle \alpha, \beta \rangle + \langle \omega, \theta \rangle,$$

that is, a sum of all Hermitian L^2 inner product as defined in (2.6). Also, let us write

$$\mathscr{H}(X) := \{(f, \alpha, \omega) \ ; \ \Delta f = 0, \ \Delta \alpha = 0, \ \Delta \omega = 0\}.$$

Hodge theory gives us a decomposition

$$\mathscr{H}(X) = \mathscr{H}^0(X) \oplus \mathscr{H}^1(X) \oplus \mathscr{H}^{1,1}(X),$$

and we have the immediate consequence of theorem 2.24:

2.32 Theorem. *There is an orthogonal decomposition that respects the degrees of the forms*

$$\Omega(X) = \Delta(\Omega(X)) \oplus \mathscr{H}(X).$$

2.33 Corollary. *Let $u, f \colon X \to \mathbb{C}$ be smooth functions on a compact Riemann surface. Then the equation $\Delta u = f$ has solution if and only if*

$$\int_X f \operatorname{vol} = \langle f, 1 \rangle = 0.$$

Equivalently, the equation $i\partial\bar{\partial} f = \omega$ has a solution if and only if $\int_X \omega = 0$, for a $(1,1)$-form ω.

Proof. From 2.32, a function f is in the image of the Laplacian operator if, and only if, it is in the orthogonal complement of the harmonic functions. As every harmonic function on X is constant, the condition for f to be in the image of the Laplacian is

$$\int_X f \operatorname{vol} = \langle f, 1 \rangle = 0.$$

Equivalently, a $(1,1)$-form ω is in the image of the Laplace operator if, and only if, it is in the orthogonal complement of the harmonic $(1,1)$-forms. As every global harmonic $(1,1)$-form is a constant multiple of vol, the condition for ω to be in the image of the Laplacian is

$$\int_X \omega = \langle \omega, 1\,\mathrm{vol} \rangle = 0.$$

Observe that $\Delta(f\,\mathrm{vol}) = 2i\partial\bar{\partial} f$. $\square$

Capítulo 3

Instantons

In this chapter we introduce the Yang-Mills equation. We first exhibit it as a generalization to the celebrated Maxwell equations for electromagnetism and afterwards we derive it from variational calculus using the Yang-Mills Lagragian. In the latter, we will make use of many calculations with connections developed in chapter 1. Also, we exhibit explicit solutions to the ASD Yang-Mills equation.

3.1 Yang-Mills Equation

3.1. We start by recalling the Maxwell equations for electromagnetism in Minkowski space-time, written in units for which the speed of light is equal to 1:

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \nabla \cdot \vec{E} &= \rho \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}, \end{aligned}$$

where $\vec{E}\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ is the **electric field**, $\vec{B}\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ is the **magnetic field**, $\vec{j}\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ is the **current density** and $\rho\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is the **electric density**. These equations are written in the classical language where $\nabla\cdot\vec{C}$ and $\nabla\times\vec{C}$ mean the divergent and the curl, respectively, of any time-dependent vector field $\vec{C}(t,\vec{x}) =$

$(C_x(t,\vec{x}), C_y(t,\vec{x}), C_z(t,\vec{x}))$, such as $\vec{E}$ and $\vec{B}$, and are defined by

$$\nabla \cdot \vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} \tag{3.1}$$

$$\nabla \times \vec{C} = \left(\frac{\partial C_y}{\partial z} - \frac{\partial C_z}{\partial y}, \frac{\partial C_z}{\partial x} - \frac{\partial C_x}{\partial z}, \frac{\partial C_x}{\partial y} - \frac{\partial C_y}{\partial x}\right). \tag{3.2}$$

3.2. Using the language of differential geometry, we can give a more elegant description of these equations. Recall that the Minkowski space-time is the Euclidean space of dimension 4 endowed with a metric of signature $(3,1)$. We have global coordinates t, x, y, z, or x^0, x^1, x^2, x^3 (we will interchange between the names of the global coordinates without further mention, since when dealing with sums it is easier to use the latter, and for some definitions it is clearer to use the former), where, obviously, the first coordinate is the time coordinate and the others are the spatial coordinates. Instead of considering the electric and magnetic fields as vector fields, we shall see them as differential forms:

$$\begin{aligned} E &:= E_x dx + E_y dy + E_z dz \\ B &:= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \end{aligned}$$

where $E_\mu(t,\vec{x}), B_\mu(t,\vec{x})$, $\mu = 1,2,3$ are the coordinate functions of the vector fields we had in paragraph 3.1. We unify both fields and define a 2-form called the **electromagnetic field**

$$F := B + E \wedge dt.$$

Taking the exterior derivative, a straightforward computation yields

$$\begin{aligned} dF = \Bigg\{ &\left[\frac{\partial B_x}{\partial t} + \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)\right] dy \wedge dz + \left[\frac{\partial B_y}{\partial t} + \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\right] dz \wedge dx + \\ &+ \left[\frac{\partial B_z}{\partial t} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)\right] dx \wedge dy \Bigg\} \wedge dt + \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx \wedge dy \wedge dz, \end{aligned}$$

Observe that the terms on the right hand side of the equation above are all linearly independent of each other. Therefore, the equation $dF = 0$ is equivalent to saying that each summand vanishes individually, that is,

$$dF = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{cases},$$

thus, we have just rewritten the first pair of equations as $dF = 0$.

3.3. Let us write our 2-form, the electromagnetic field F in coordinates. We have

$$F = \sum \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad F_{\mu\nu} = F\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right).$$

We can represent the coordinates $F_{\mu\nu}$ by the entries of a 4-by-4 anti-symmetric matrix

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Using the Hodge star operator, that was treated in chapter 2, we have that $*F$ is again a 2-form and, using the calculations of example 2.6 we get

$$\begin{aligned} *F &= *B + *(E \wedge dt) \\ &= -(B_x dx + B_y dy + B_z dz) \wedge dt + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy, \end{aligned}$$

or, in coordinates

$$[(*F)_{\mu\nu}] = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

A straightforward computation gives us

$$\begin{aligned} d * F = \Bigg\{ &\left[\frac{\partial E_x}{\partial t} - \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right)\right] dy \wedge dz + \left[\frac{\partial E_y}{\partial t} - \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)\right] dz \wedge dx + \\ &\left[\frac{\partial E_z}{\partial t} - \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)\right] dx \wedge dy \Bigg\} \wedge dt + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx \wedge dy \wedge dz, \end{aligned}$$

Therefore, again using the calculations in example 2.6

$$\begin{aligned} * d * F = &\left\{ \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \frac{\partial E_x}{\partial t} \right\} dx + \left\{ \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - \frac{\partial E_y}{\partial t} \right\} dy + \\ &\left\{ \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - \frac{\partial E_z}{\partial t} \right\} dz - \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dt. \quad (3.3) \end{aligned}$$

Similarly to the unification we have done with the electromagnetic field, we can unify the current and electric density, $\vec{j}$ and ρ in a 1-form called simply **current** defined by

$$J := -\rho dt + j,$$

where $j = \sum j_\mu dx^\mu$, the 1-form whose coordinates are precisely the coordinate functions of the current density $\vec{j} = (j_x, j_y, j_z)$. From (3.3) it is clear that

$$*d*F = J \qquad \Longleftrightarrow \qquad \begin{cases} \nabla \cdot \vec{E} &= \rho \\ \nabla \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} &= \vec{j} \end{cases},$$

thus, we have just rewritten the second pair of equations as $*d*F = J$, so the Maxwell equations are now expressed shortly as

$$\begin{aligned} dF &= 0 \\ *d*F &= J. \end{aligned}$$

3.4. Consider now the Maxwell equations in the vacuum, i.e., $J = 0$, where we have

$$\begin{aligned} dF &= 0 \\ d*F &= 0. \end{aligned} \tag{3.4}$$

They are preserved under $F \mapsto *F$. Recall from example 2.8 that in a Riemannian 4-manifold, $*$ is a linear operator on the vector space $\Omega^2(M)$ that squares to 1 and we have that any element $\omega \in \Omega^2(M)$ is uniquely written as a sum of its self-dual and anti-self-dual parts

$$\omega = \omega_+ + \omega_-, \qquad *\omega_+ = \omega_+, *\omega_- = -\omega_-.$$

A self-dual or anti-self-dual 2-form F satisfying the first pair of the Maxwell equations in the vacuum, $dF = 0$, will automatically satisfy $d*F = 0$, as $d*F = \pm dF = 0$.

3.5. The idea now is to extend this discussion to bundle-valued forms. If we consider the case when the electromagnetic field is the exterior derivative of a 1-form A, often called **vector potential** $F = dA$, the first pair of the Maxwell equations, $dF = 0$, becomes a tautology, so all the physics is concentrated in the second pair (cf.

[BAEZ, MUNIAN, 1994] for more details). Now, fix a vector bundle $E \to M$ and a connection ∇_A. The first pair of equation has a generalization to bundle-valued forms, given by the Bianchi identity

$$d_A F_A = 0.$$

Extending the action of the Hodge star operator to bundle-valued forms as the unique $C^\infty(M)$-linear operator such that

$$*(s \otimes \omega) = s \otimes *\omega, \qquad s \in \Gamma(E), \omega \in \Omega^p(M),$$

where the $*$ on the right is the usual duality operator, then for an $\mathrm{End}(E)$-valued 1-form J on M, called the *current*, the generalization of the second pair of equations is the so-called **Yang-Mills equation**

$$*d_A * F_A = J. \tag{3.5}$$

3.6 Remark. Assuming that we are dealing with a trivial $U(1)$-bundle over a four manifold, e.g., $E = \mathbb{R}^4 \times \mathbb{C}$, given a connection A on E the action of the covariant derivative on $\mathrm{End}(E)$-valued forms is

$$d_A\xi = d\xi + [A, \xi] = d\xi,$$

since we are dealing with matrices of forms of rank 1. Therefore, the Yang-Mills equation reduces to the second pair of the Maxwell equations

$$*d * F = J.$$

3.7. For a connection A, denote $g \cdot A$ as A', and $g \cdot F_A = gF_Ag^{-1}$ as F'. Recalling proposition 1.67, we have that

$$*d_{A'} * F' = *(g(d_A(g^{-1}(*gF_Ag^{-1})g))g^{-1}) = g(*d_A * F_A)g^{-1},$$

thus the Yang-Mills equation is invariant under gauge transformations, since if A satisfies the Yang-Mills equation for a current J, then $g \cdot A$ also satisfies for the transformed current $g \cdot J = gJg^{-1}$

3.8. The case that will be of more interest to us is the generalization of the Maxwell equations in the vacuum. In this case, the Yang-Mills equation takes the form

$$d_A * F_A = 0. \tag{3.6}$$

We now generalize the discussion of example 2.8, to bundle-valued forms. Considering M a Riemannian 4-manifold, the decomposition $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$, that was implicit there extends immediately to bundle valued 2-forms, and in particular, the curvature of a connection A splits as

$$F_A = F_A^+ + F_A^-.$$

We call a connection **anti-self-dual** if $F_A^+ = \frac{1}{2}(F_A + *F_A) = 0$, that is, $*F_A = -F_A$ and **self-dual** if $F_A^- = \frac{1}{2}(F_A - *F_A) = 0$, which implies $*F_A = F_A$. The anti-self-dual, or ASD in short, connections are also called **instantons**.

3.9 Remark. As our theory is over oriented manifolds, otherwise the duality operators would not make sense, reversing the orientation of the base we send self-dual connections to anti-self-dual connections. So it is a mere convention to call the ASD connections as *instantons* and not the SD connections.

3.10. Let $E \to M$ be a Hermitian vector bundle over a compact 4-manifold M with Hermitian metric $\langle \cdot, \cdot \rangle_E$. In this paragraph, we wish to prove that the decomposition $\Omega^2(M; \mathrm{End}(E)) = \Omega^2_+(M; \mathrm{End}(E)) \oplus \Omega^2_-(M; \mathrm{End}(E))$ is actually an orthogonal one, so we need to specify what will be the inner product on $\Omega^2(M; \mathrm{End}(E))$. First, observe that the Hermitian metric on E is extended to the bundle $\mathrm{End}(E)$ by

$$\langle \Psi, \Phi \rangle_{\mathrm{End}(E)} = \mathrm{Tr}(\Psi \Phi^\dagger),$$

where Ψ, Φ are endomorphism of E and $\Phi^\dagger$ denotes the transpose conjugate of Φ. This is a linear algebra fact. If $(V, \langle \cdot, \cdot \rangle)$ is a Hermitian inner product space, with orthonormal basis $\{e_1, \ldots, e_n\}$, then the Hermitian inner product on $\mathrm{End}(V) \cong V \otimes V^*$ is given by

$$\begin{aligned}
\left\langle \sum T^i_j e_i \otimes e^j, \sum S^k_l e_k \otimes e^l \right\rangle &= \sum T^i_j \overline{S^k_l} \langle e_i, e_k \rangle \langle e^j, e^l \rangle \\
&= \sum T^i_j \overline{S^k_l} \delta_{ik} \delta^{jl} \\
&= \sum T^i_j \overline{S^i_j} \\
&= \mathrm{Tr}(T S^\dagger),
\end{aligned}$$

for $T = \sum T^i_j e_i \otimes e^j$ and $S = \sum S^k_l e_k \otimes e^l$ endomorphisms of V. Now, in paragraph 2.2, we defined an L^2 inner product on the space of differential p-forms, so, as a form in $\Omega^2(M; \mathrm{End}(E))$ is a section of $\Lambda^2\, T^*M \otimes \mathrm{End}(E)$, we claim that the L^2 inner product on $\Omega^2(M; \mathrm{End}(E))$ that we are seeking is given by

$$\langle \eta, \zeta \rangle = \int_M \mathrm{Tr}(\eta \wedge *(\zeta^\dagger)). \tag{3.7}$$

Indeed, on a local frame, we write $\eta = \sum \eta^i_j \otimes E^j_i$ and $\zeta = \sum \zeta^k_l \otimes E^l_k$, with $\eta^i_j, \zeta^k_l \in \Omega^2(M)$ and $E^j_i, E^l_k \in \Gamma(\mathrm{End}(E))$ (recall the definition of the E^j_i given in paragraph 1.51), so

$$\begin{aligned}
\langle \eta, \zeta \rangle &= \sum \langle \eta^i_j, \zeta^k_l \rangle \langle E^j_i, E^l_k \rangle \\
&= \sum \langle \eta^i_j, \zeta^k_l \rangle \,\mathrm{Tr}(E^j_i, (E^l_k)^\dagger) \\
&= \sum \langle \eta^i_j, \zeta^k_l \rangle \,\mathrm{Tr}(E^j_i, E^k_l) \\
&= \sum \langle \eta^i_j, \zeta^k_l \rangle \delta^k_i \delta^j_l \\
&= \sum \langle \eta^i_j, \zeta^i_j \rangle \\
&= \sum \int_M \eta^i_j \wedge *\bar{\zeta}^i_j \\
&= \int_M \mathrm{Tr}(\eta \wedge *(\zeta^\dagger)),
\end{aligned}$$

as we claimed. Finally, if $\eta \in \Omega^2_+(M; \mathrm{End}(E))$ and $\zeta \in \Omega^2_-(M; \mathrm{End}(E))$, then

$$\begin{aligned}
\langle \eta, \zeta \rangle &= \int_M \mathrm{Tr}(\eta \wedge *(\zeta^\dagger)) \\
&= -\int_M \mathrm{Tr}(*\eta \wedge (\zeta^\dagger)) \\
&= -\int_M \mathrm{Tr}((\zeta^\dagger) \wedge *\eta) \\
&= -\langle \eta, \zeta \rangle,
\end{aligned}$$

hence $\Omega^2_+(M; \mathrm{End}(E))$ is orthogonal to $\Omega^2_-(M; \mathrm{End}(E))$.

3.11 Example. Consider $M = \mathbb{R}^4$, the Euclidean space, and let us see explicitly what is the ASD equation $F^+_A = 0$ in this case. Recall that locally we write the curvature of a connection A, with connection matrices $A_\mu, \mu = 1, 2, 3, 4$, as $F = \frac{1}{2} \sum F_{\mu\nu} dx^\mu \wedge dx^\nu$, with $F_{\mu\nu} = F(\partial_\mu, \partial_\nu) = [\nabla_\mu, \nabla_\nu]$. In our case there is only one chart so this description is *global*. Even more explicitly, we have (we will omit the wedges here)

$$F = F_{12}dx^1dx^2 + F_{13}dx^1dx^3 + F_{14}dx^1dx^4 + F_{23}dx^2dx^3 + F_{24}dx^2dx^4 + F_{34}dx^3dx^4,$$

thus, applying the $*$, we have

$$*F = F_{34}dx^1dx^2 - F_{24}dx^1dx^3 + F_{23}dx^1dx^4 + F_{14}dx^2dx^3 - F_{13}dx^2dx^4 + F_{12}dx^3dx^4.$$

Hence, for $F_A^+ = 0$, or equivalently $*F = -F$, we obtain a system of non-linear partial diferential equations

$$F_{12} + F_{34} = 0 \tag{3.8}$$

$$F_{13} + F_{42} = 0 \tag{3.9}$$

$$F_{14} + F_{23} = 0. \tag{3.10}$$

3.12. Now recall the expressions for the Chern classes that we have discussed in subsection 1.3.4. If we consider E as an $SU(r)$-bundle, the curvature matrices lie in $\mathfrak{su}(r)$, the traceless and skew-Hermitian matrices. Therefore, the first Chern class of the bundle vanishes, and equation (1.32) boils down to

$$8\pi^2 c_2(E) = \mathrm{Tr}(F_A \wedge F_A) \in H^4_{dR}(M;\mathbb{R}).$$

Then, for compact and oriented base manifolds M, we have

$$c_2(E) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(F_A \wedge F_A) \in \mathbb{Z}.$$

It is a delicate matter that this last integral takes values in the integers, because this means that actually $c_j(E) \in H^{2k}_{dR}(M;\mathbb{Z})$. We refer to [MILNOR, STASHEFF, 1974], chapter 14 and appendix C.

3.13 Lemma. *For a unitary connection A, on a vector bundle E over a Riemannian four-manifold, we have*

$$\mathrm{Tr}(F_A \wedge F_A) = \|F_A^-\|^2 - \|F_A^+\|^2 = (|F_A^-|^2 - |F_A^+|^2)\,\mathrm{vol},$$

where vol *is the invariant volume element and F_A^+, F_A^- denotes the splitting of the 2-forms in the self-dual and anti-self-dual parts.*

Proof. From what was discussed on paragraph 3.10, and since $F_A^\dagger = -F_A$, we have

$$\begin{aligned}
\mathrm{Tr}(F_A \wedge F_A) &= \mathrm{Tr}(F_A \wedge * * F_A) \\
&= -\langle F_A, *F_A \rangle \\
&= -\langle F_A^+ + F_A^-, F_A^+ - F_A^- \rangle \\
&= \|F_A^-\|^2 - \|F_A^+\|^2.
\end{aligned}$$

□

3.14. Now we are in position to see the importance of the anti-self-dual condition in the theory. Still considering $SU(r)$-bundles over compact orientable Riemannian 4-manifolds, we have a linear functional on the space of connections, called **Yang-Mills functional**, given by

$$A \mapsto ||F_A||^2 := \int_M |F_A|^2 \,\mathrm{vol} = -\int_M \mathrm{Tr}(F_A \wedge *F_A). \tag{3.11}$$

Then, we have, from lemma 3.13, that the absolute value of $8\pi^2 c_2(E)$ is a lower bound to the Yang-Mills functional as

$$\begin{aligned}
8\pi^2 c_2(E) &= \|F_A^-\|^2 - \|F_A^+\|^2 && (3.12)\\
\|F_A\|^2 &= \|F_A^-\|^2 + \|F_A^+\|^2, && (3.13)
\end{aligned}$$

so, summing (3.12) and (3.13) we obtain $\|F_A\|^2 = -8\pi^2 c_2(E) + 2\|F_A^-\|^2$, whereas subtracting (3.13) and (3.12) we obtain $\|F_A\|^2 = 8\pi^2 c_2(E) + 2\|F_A^+\|^2$. Whenever $c_2(E)$ is positive, this lower bound is achieved if, and only, if A is ASD. Indeed, $F_A^+ = 0$ implies $|F_A|^2 \,\mathrm{vol} = |F_A^-|^2 \,\mathrm{vol}$, hence $8\pi^2 c_2(E) = \int_M |F_A^-|^2 \,\mathrm{vol} = \int_M |F_A|^2 \,\mathrm{vol} = ||F_A||^2$. On the other hand, if $8\pi^2 c_2(E) = ||F_A||^2$, we have

$$\int_M |F_A^-|^2 \,\mathrm{vol} - \int_M |F_A^+|^2 \,\mathrm{vol} = \int_M |F_A^-|^2 \,\mathrm{vol} + \int_M |F_A^+|^2 \,\mathrm{vol},$$

hence $F_A^+ = 0$. The conclusion here is that the *anti-self-dual connections are the absolute minima to the Yang-Mills functional*, when $c_2(E)$ is positive. The positive value of $c_2(E)$, when finite, is often called **charge** of the instanton A. Similarly, for negative value of $c_2(E)$, this lower bound is achieved if, and only if, A is SD.

3.15. If we recall the way we introduced the Yang-Mills equations as $d * F_A = 0$, these are satisfied by SD or ASD connections. Indeed, if A is SD or ASD, $d{*}F_A = \pm dF_A = 0$, from the Bianchi identity. It is no coincidence that SD or ASD connections satisfy the Yang-Mills equation and are also absolute minima to the Yang-Mills functional. The reason for this, which we will explain in what follows, is that the Yang-Mills equation comprises the Euler-Lagrange equations of the Yang-Mills functional. However, we should observe that not all connections that satisfy the Yang-Mills equation are ASD or SD. There may be other critical points of the Yang-Mills functional that are not absolute minima. For example, in [SADUN, SEGERT, 1991], we can find explicit examples of non-self-dual connections which are solution to the Yang-Mills equation. As is stated in [SADUN, SEGERT, 1991], another interesting point to the ASD and SD connections is that in order to find these, we should solve a first-order system of PDE's $*F_A = \pm F_A$, whereas the Yang-Mills equation $d_A * F_A = 0$ is a system of second-order PDE's.

3.16. We start by recalling the variational principle. Following [BAEZ, MUNIAN, 1994] and consider the celebrated Newton's law of motion $F = ma$, where we want to model the motion of a particle in $\mathbb{R}^3$ of mass m, position $q(t)$, velocity $v(t) = \dot{q}(t)$ and acceleration $a(t) = \ddot{q}(t)$. Assuming that we are given a vector-valued function $F \colon \mathbb{R}^3 \to \mathbb{R}^3$, the path $q(t)$ of a particle subject to the force F will be given by the solution of the differential equation

$$F(q(t)) = m\ddot{q}(t).$$

Another way of describing the motion is with variational methods. Assuming further that $F = -\nabla V$, for a scalar potential $V \colon \mathbb{R}^3 \to \mathbb{R}$ that does not depend on time, the total energy of the system is given by the **Hamiltonian**

$$H = K + V,$$

where $K = \frac{1}{2}m\ddot{q}^2$, called the **kinectic energy**. Observe that the Hamiltonian is independent of time

$$\frac{dH}{dt} = \frac{dK}{dt} + \frac{dV}{dt} = (m\ddot{q}(t) + \nabla V) \cdot \dot{q}(t) = 0.$$

3.17. Now we introduce the **Lagrangian** $L(q, \dot{q}; t) = K - V$. The particle's path now is determined by the **action principle**, that is, the path $q(t)$ is a point that minimizes the **action**, $S(q)$, that is defined, on the space of paths, by

$$S(q) := \int_0^1 L\, dt.$$

We vary q to a nearby path by taking a function $f\colon [0,1] \to \mathbb{R}^3$, with $f(0) = f(1) = 0$, and setting

$$q_s(t) = q(t) + sf(t),$$

for $s \in \mathbb{R}$ small. The function f is the **variation** of q, also written as δq, and it satisfies

$$f(t) = \delta q(t) = \frac{d}{ds}\Big|_{s=0} q_s(t).$$

In general, for any function G on the space of paths, we define its variation by $\delta G = \frac{d}{ds}\big|_{s=0} G(q_s(t))$. In particular, for the action S, that is a function in the space of paths, we get

$$\begin{aligned} \delta S &= \frac{d}{ds}\Big|_{s=0} S(q_s) \\ &= \frac{d}{ds}\Big|_{s=0} \int_0^1 \left(\frac{1}{2}m(\dot{q}_s(t))^2 - V(q_s(t))\right) dt \\ &= \int_0^1 \frac{d}{ds}\Big|_{s=0} \left(\frac{1}{2}m(\dot{q}_s(t))^2 - V(q_s(t))\right) dt \\ &= \int_0^1 \left(m\dot{q}(t)\cdot \dot{f}(t) - \nabla V(q(t))\cdot f(t)\right) dt \\ &= -\int_0^1 \big(m\ddot{q}(t) - F(q(t))\big) f(t)\, dt, \end{aligned} \tag{3.14}$$

where in the last equality, use has been made of integration by parts, and the fact that $f(0) = f(1) = 0$, as

$$\int_0^1 m\dot{q}(t)\dot{f}(t)\, dt = m\dot{q}(t)f(t)\Big|_{t=0}^{t=1} - \int_0^1 m\ddot{q}(t)f(t)\, dt = -\int_0^1 m\ddot{q}(t)f(t)\, dt.$$

To ask that a point minimizes the action, we must have that the point is a critical value, that is, $\delta S = 0$ for all variations $f = \delta q$, and in this case, from equation (3.14) we must have $F = ma$, retrieving Newton's law of motion.

3.18. More generally, the Lagrangian could be any function of the particle's position and velocity, and we obtain

$$\begin{aligned}\delta S &= \delta \int L\, dt = \int \delta L\, dt \\ &= \int \sum_i \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt \\ &= \int \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial q^i} \right) \delta q^i \, dt,\end{aligned}$$

where use has been made of integration by parts, in the last equality. It follows that $\delta S = 0$ for all variations δq vanishing at the endpoints, if and only if the **Euler-Lagrange equations**

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial q^i}, \tag{3.15}$$

holds for all i.

3.19. In our case, we have a slightly different Lagrangian. For us, it will not be a function of the position and the velocity, but a function in the affine space of connections. However, we can do variational calculus using some of the theory we have developed in 1.3.3, of chapter 1. More precisely, the **Yang-Mills Lagrangian** is $L_{YM}(A) = -\operatorname{Tr}(F_A \wedge *F_A)$, and the **Yang-Mills action**, is given by

$$S(A) = \int_M -\operatorname{Tr}(F_A \wedge *F_A). \tag{3.16}$$

To calculate the first-order variation, let $A_s = A + sa$, where $a \in \Omega^1_M(\mathfrak{g}_E)$, thus, from (1.25), $F(A_s) = F_A + s d_A a + s^2 a \wedge a$, hence, $\operatorname{Tr}(F(A_s) \wedge *F(A_s)) = Tr(F_A \wedge *F_A) + s \operatorname{Tr}(d_A a \wedge *F_A + a \wedge d_A * F_A) + o(s^2)$. Therefore,

$$\begin{aligned}\delta S(A) &= \delta \int_M -\operatorname{Tr}(F_A \wedge *F_A) \\ &= -\int_M \delta \operatorname{Tr}(F_A \wedge *F_A) \\ &= -\int_M \operatorname{Tr}(d_A a \wedge *F_A + F_A \wedge *d_A a) \\ &= -2 \int_M \operatorname{Tr}(d_A a \wedge *F_A). \end{aligned} \tag{3.17}$$

Now, observe that for general $\xi \in \Omega^p(M; \mathrm{End}(E))$ and $\zeta \in \Omega^q(M; \mathrm{End}(E))$, we have from proposition 1.76, that $\mathrm{Tr}(d_A(\xi \wedge \zeta)) = d\,\mathrm{Tr}(\xi \wedge \zeta)$. Thus, assuming that M is compact and without boundary we have, by Stokes,

$$0 = \int_M \mathrm{Tr}(d_A(a \wedge *F_A)) = \int_M \mathrm{Tr}(d_A a \wedge *F_A) - \int_M (a \wedge d_A * F_A),$$

thus in (3.17), if we ask that $\delta S(A) = 0$ for all variations a, we must have the Yang-Mills equations

$$d_A * F_A = 0.$$

3.20 Remark. If A is a connection on a unitary bundle $E \to M$ over a compact manifold M, then, we can decompose the curvature orthogonally in a trace-free and central part as

$$F = F^0 + \frac{1}{r}\,\mathrm{Tr}(F) \cdot \mathbb{1},$$

where $F^0 = F - \frac{1}{r}\,\mathrm{Tr}(F) \cdot \mathbb{1}$. It is clear that $\mathrm{Tr}(F^0) = 0$. The fact that this decomposition is orthogonal follows from the inner product given in (3.7)

$$\langle F^0, \mathrm{Tr}(F) \cdot \mathbb{1} \rangle = \int_M \mathrm{Tr}(F^0) \wedge *\overline{\mathrm{Tr}(F)} = 0.$$

Now, if we wish to minimize the norm of F, as the central component represents $-2\pi i c_1(E) \in H^2_{dR}(M; \mathbb{R})$, this part is minimized by the appropriate harmonic-form, as we saw in paragraph 2.16 and we concentrate in minimizing the trace-free part, and that is why we have considered $SU(r)$-bundles.

3.21. So far we have dealt with the theory for bundles over a compact base space M, but we will focus in bundles over the Euclidean space $\mathbb{R}^4$. We will consider connections for which the Yang-Mills action S_{YM} is finite, that is, the integral over $\mathbb{R}^4$ converges. To achieve this, we assume that the field F decays sufficiently fast as we go to infinity, and on a given gauge, this means that the $F_{\mu\nu} \to 0$ sufficiently fast as $x \to \infty$. There are some technical issues here regarding the notion of decay that we will not dwell upon. All we are interested in is that, whenever the action is finite, we compactify $\mathbb{R}^4$ to S^4 and we make the computation of the Lagrangian over S^4, where we can make use of the compactness of S^4, so all of what we have done in paragraph 3.19 holds. But the question that arises is why would this work? To answer this we recall the notion of conformal maps and conformal class of metrics.

3.22 Definition. Let M and N be manifolds of the same dimension, and g_M, g_N be pseudo-Riemannian metrics on M and N, respectively. Recall that a pseudo-Riemannian metric is a smooth and symmetric $(0,2)$-tensor that is non-degenerate everywhere. A smooth map $f\colon M \to N$ is said to be **conformal**, if there exists a positive function $\Omega\colon M \to \mathbb{R}_+$ satisfying

$$f^*(g_N)(x) = \Omega(x) g_M(x), \qquad x \in M.$$

3.23 Proposition. *The stereographic projection $\pi\colon S^n \to \mathbb{R}^n$ is a conformal map.*

3.24. We say that two pseudo-Riemmanian metrics g and $\tilde{g}$ on a manifold M are regarded as conformal if there exists a positive function $\Omega\colon M \to \mathbb{R}_+$ such that $g = \Omega\tilde{g}$. It is not hard to see that the relation "g conformal to $\tilde{g}$" is an equivalence relation on the set of all pseudo-Riemmanian metrics of M. The equivalence class of a metric g is denoted by $[g]$ and is called **conformal class** of g. There are many properties that hold true for the whole conformal class of a metric. We say that this property is **conformally invariant**. For example:

3.25 Proposition. *The Yang-Mills action is conformally invariant in $\mathbb{R}^4$.*

Proof. What we will prove is that for *any* four-manifold, the star operator acting of 2-forms is conformally invariant, therefore, since the only part of the Yang-Mills action over a four-manifold (3.16) that depends on the metric is the star operator, the result will follow. Now, if we scale the metric by a positive factor ρ, the pointwise inner product on 2-forms scales by ρ^{-2}, while the invariant volume form scales by ρ^2. That is, if $\tilde{g} = \rho g$, $\tilde{*}$ and $*$ are the duality operators with respect to $\tilde{g}$ and g, respectively and $\xi = \xi_{\mu\nu} dx^\mu \wedge dx^\nu, \zeta = \zeta_{\mu\nu} dx^\mu \wedge dx^\nu$ are 2-forms, then locally

$$\begin{aligned}
\xi \wedge \tilde{*}\zeta &= (\xi, \zeta)\,\mathrm{vol} \\
&= \left(\sum \det(\tilde{g}(dx^{\mu_i}, dx^{\nu_j}))\xi_{\mu_1\mu_2}\zeta_{\nu_1\nu_2}\right)\sqrt{|\det \tilde{g}|}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
&= \left(\sum \rho^{-2}\det(g(dx^{\mu_i}, dx^{\nu_j}))\xi_{\mu_1\mu_2}\zeta_{\nu_1\nu_2}\right)\sqrt{\rho^4|\det g|}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
&= \left(\sum \det(g(dx^{\mu_i}, dx^{\nu_j}))\xi_{\mu_1\mu_2}\zeta_{\nu_1\nu_2}\right)\sqrt{|\det g|}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
&= \xi \wedge *\zeta.
\end{aligned}$$

□

3.26. The conclusion we draw from propositions 3.23 and 3.25 is that the Yang-Mills action takes the same value whether computed on S^4 or in $\mathbb{R}^4$, thus our trick of using the conformal compactification of $\mathbb{R}^4$ as S^4 is valid, so the theory over $\mathbb{R}^4$ holds.

3.2 't Hooft's Ansatz

3.27. In this section we will be interested in exhibiting a concrete solution to the Yang-Mills equation $d_A * F_A = 0$. Consider an $SU(2)$-bundle of rank 2 over $\mathbb{R}^4$, which necessarily trivializes as $\mathbb{R}^4 \times \mathbb{C}^2$. Our potential A writes as $A = \sum A_\mu dx^\mu$, and

$$A_\mu = \frac{1}{2i} \sum_{b=1}^{3} A_\mu^b \sigma_b,$$

where the σ_b, $b = 1, 2, 3$, are the **Pauli matrices**,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ forms a basis to the real Lie algebra $\mathfrak{su}(2)$ of traceless, skew-Hermitian matrices. These matrices have the following algebraic properties:

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_3\sigma_1 = i\sigma_2, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_1^2 = \sigma_1^2 = \sigma_1^2 = id.$$

Observe that if we define $I = -i\sigma_1$, $J = -i\sigma_2$ and $K = -i\sigma_3$, we obtain precisely the algebraic rules of the imaginary part of the quarternionic numbers. Using the Pauli matrices we define a set of skew-symmetric matrices $\sigma_{\mu\nu}$, for $1 \leq, \mu, \nu \leq 4$ given by

$$\sigma_{jk} := \frac{1}{4i}[\sigma_j, \sigma_k], \qquad \sigma_{j4} := \frac{1}{2}\sigma_j, \qquad j, k = 1, 2, 3. \tag{3.18}$$

3.28 Proposition. *The matrices defined in (3.18) are self-dual, i.e.* $*\sigma_{\mu\nu} = \frac{1}{2}\sum \epsilon_{\mu\nu}{}^{\alpha\beta}\sigma_{\alpha\beta} = \sigma_{\mu\nu}$*, and satisfy:*

$$\sigma_{12} = \tfrac{1}{2}\sigma_3 \qquad \sigma_{34} = \tfrac{1}{2}\sigma_3$$

$$-\sigma_{13} = \tfrac{1}{2}\sigma_2 \qquad \sigma_{24} = \tfrac{1}{2}\sigma_2$$

$$\sigma_{14} = \tfrac{1}{2}\sigma_1 \qquad \sigma_{23} = \tfrac{1}{2}\sigma_1.$$

Also, we have the following table of brackets between them:

$$[\sigma_{12}, \sigma_{13}] = [\sigma_{34}, \sigma_{13}] = [\sigma_{24}, \sigma_{34}] = [\sigma_{24}, \sigma_{12}] = \tfrac{i}{2}\sigma_1$$

$$[\sigma_{34}, \sigma_{14}] = [\sigma_{12}, \sigma_{14}] = [\sigma_{34}, \sigma_{23}] = [\sigma_{12}, \sigma_{23}] = \tfrac{i}{2}\sigma_2$$

$$[\sigma_{13}, \sigma_{14}] = [\sigma_{13}, \sigma_{23}] = [\sigma_{23}, \sigma_{24}] = [\sigma_{14}, \sigma_{24}] = \tfrac{i}{2}\sigma_3$$

Proof. The proof is just straightforward computations. We simply put this information together as a proposition in order to organize the text. □

3.29. The proposed Ansatz for the fields A_μ that satisfies the ASD conditions $*F_{\mu\nu} = -F_{\mu\nu}$ is given by

$$A_\mu = i \sum \sigma_{\mu\lambda} a^\lambda, \tag{3.19}$$

for a field $a = (a_1, a_2, a_3, a_4) \colon \mathbb{R}^4 \to \mathbb{R}^4$, where the components are given by

$$a_\nu = \partial_\nu \ln \rho, \tag{3.20}$$

with respect to a positive (otherwise the logarithm wouldn't be defined) potential $\rho \colon \mathbb{R}^4 \to \mathbb{R}$. The raising of the index in (3.19) is with respect to the Euclidean metric, so $a^\lambda = \sum \delta^{\lambda\nu} a_\nu = a_\lambda$. Let us take a closer look at the field components a_ν as given in (3.20). A small calculation yields $a_\nu = \frac{\partial_\nu \rho}{\rho}$, thus

$$\partial_\mu a_\nu = \frac{\partial_\mu \partial_\nu \rho \cdot \rho - \partial_\nu \rho \cdot \partial_\mu \rho}{\rho^2},$$

so if we define $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$, from the fact that $\partial_\mu \partial_\nu \rho = \partial_\nu \partial_\mu \rho$ and $\partial_\nu \rho \cdot \partial_\mu \rho = \partial_\mu \rho \cdot \partial_\nu \rho$, we have

$$f_{\mu\nu} = 0. \tag{3.21}$$

3.30. Now, let us get our hands dirty and start doing some computations. We start by calculating the fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Using (3.19), these become

$$F_{\mu\nu} = \sum_\alpha i(\sigma_{\nu\alpha}\, \partial_\mu a^\alpha - \sigma_{\mu\alpha}\, \partial_\nu a^\alpha) - \sum_{\alpha,\beta} a^\alpha a^\beta [\sigma_{\mu\alpha}, \sigma_{\nu\beta}].$$

Using the calculations of proposition 3.28, we calculate the fields $F_{\mu\nu}$ by factoring the Pauli matrices σ_b:

$$F_{12} = \frac{i}{2}\big[(-\partial_2 a_4+\partial_1 a_3+a_2a_4-a_1a_3)\cdot\sigma_1+(\partial_1 a_4+\partial_2 a_3-a_1a_4-a_2a_3)\cdot\sigma_2-(\partial_1 a_1+\partial_2 a_2+a_3^2+a_4^2)\cdot\sigma_3\big]$$

$$F_{34} = \frac{i}{2}\big[(\partial_4 a_2-\partial_3 a_1-a_4a_2+a_3a_1)\cdot\sigma_1+(-\partial_4 a_1-\partial_3 a_2+a_4a_1+a_3a_2)\cdot\sigma_2-(\partial_3 a_3+\partial_4 a_4+a_1^2+a_2^2)\cdot\sigma_3\big]$$

$$F_{13} = \frac{i}{2}\big[(-\partial_1 a_2-\partial_3 a_4+a_1a_2+a_3a_4)\cdot\sigma_1+(\partial_1 a_1+\partial_3 a_3+a_2^2+a_4^2)\cdot\sigma_2+(\partial_1 a_4-\partial_3 a_2-a_1a_4+a_3a_2)\cdot\sigma_3\big]$$

$$F_{42} = \frac{i}{2}\big[(\partial_2 a_1+\partial_4 a_3-a_1a_2-a_4a_3)\cdot\sigma_1+(\partial_2 a_2+\partial_4 a_4+a_1^2+a_3^2)\cdot\sigma_2+(-\partial_4 a_1+\partial_2 a_3+a_4a_1-a_2a_3)\cdot\sigma_3\big]$$

$$F_{14} = \frac{i}{2}\big[(-\partial_1 a_1-\partial_4 a_4-a_2^2-a_3^2)\cdot\sigma_1+(\partial_4 a_3-\partial_1 a_2-a_4a_3+a_1a_2)\cdot\sigma_2-(\partial_1 a_3+\partial_4 a_2-a_1a_3+a_4a_2)\cdot\sigma_3\big]$$

$$F_{23} = \frac{i}{2}\big[(-\partial_2 a_2-\partial_3 a_3-a_1^2-a_4^2)\cdot\sigma_1+(\partial_2 a_1-\partial_3 a_4-a_2a_1+a_3a_4)\cdot\sigma_2+(\partial_3 a_1+\partial_2 a_3-a_3a_1+a_2a_4)\cdot\sigma_3\big].$$

3.31. Once we impose the ASD condition, from example 3.11, the equations (3.8), (3.9) and (3.10) become

$$\begin{aligned}
F_{12}+F_{34} &= \frac{i}{2}\left((f_{13}+f_{42})\sigma_1+(f_{14}+f_{23})\sigma_2-\sum_\mu(\partial_\mu a^\mu+(a_\mu)^2)\sigma_3\right)=0\\
F_{13}+F_{42} &= \frac{i}{2}\left((f_{14}+f_{23})\sigma_3-(f_{12}+f_{34})\sigma_1+\sum_\mu(\partial_\mu a^\mu+(a_\mu)^2)\sigma_2\right)=0\\
F_{14}+F_{23} &= -\frac{i}{2}\left((f_{12}+f_{34})\sigma_2+(f_{13}+f_{42})\sigma_3+\sum_\mu(\partial_\mu a^\mu+(a_\mu)^2)\sigma_1\right)=0.
\end{aligned}$$

From (3.21), the ASD condition reduces to an equation for the fields a_μ

$$\sum_\mu(\partial_\mu a^\mu+(a_\mu)^2)=0,$$

and from the definition of the fields a_μ, in (3.20), this reduces even more to an equation for the potential ρ

$$\begin{aligned} 0 &= \sum_\mu \left[\partial_\mu \left(\frac{\partial_\mu \rho}{\rho} \right) + \left(\frac{\partial_\mu \rho}{\rho} \right)^2 \right] \\ &= \sum_\mu \left[\frac{1}{\rho} \partial_\mu^2 \rho - \left(\frac{\partial_\mu \rho}{\rho} \right)^2 + \left(\frac{\partial_\mu \rho}{\rho} \right)^2 \right] \\ &= \frac{1}{\rho} \Delta \rho, \end{aligned}$$

that is, the usual Laplacian on $\mathbb{R}^4$ vanishes, $\Delta\rho = 0$.

3.32. Solutions for $\Delta\rho = 0$ and $\rho > 0$ exist. A simple example is $\rho(x) = 1/||x||$, but the general solution given by ‘t Hooft is

$$\rho(x) = 1 + \sum_{i=1}^{n} \frac{\lambda_i^2}{||x - y_i||^2},$$

that is physically interpreted as a configuration with n instantons, also called *pseudo-particles*, where λ_i are constants that corresponds to the “size” of the instanton, and y_i are the positions of the pseudo-particles.

Capítulo 4

Stable Bundles

This chapter is devoted to the article [DONALDSON, 1983]. We start giving a brief summary of the ideas behind stability of bundles over Riemann surfaces, in the spirit of [ATIYAH, BOTT, 1982] and then we approach the theorem of Narasimhan-Seshadri. Many results from the theory of partial differential equations, in particular some nice inequalities are used, and we will not prove them in this work. We included them in the appendix where we made a brief discussion and gave the references.

4.1 On Moduli of Stable Bundles over Riemann Surfaces

4.1. Following section 7 of [ATIYAH, BOTT, 1982], consider a fixed C^∞ vector bundle E over a Riemann surface X, of rank n, and degree d. It is often denoted by $\mathscr{C}(E)$, or $\mathscr{C}(n,d)$ the space of all holomorphic structures on E. By dimensional reasons, a holomorphic structure is equivalent to the existence of a $\bar\partial$ operator, so that the local holomorphic sections are those such that $\bar\partial\sigma = 0$, cf. [ATIYAH, BOTT, 1982], section 5. In local coordinates, one can write any $\bar\partial$ operator as

$$\bar\partial = \bar\partial_0 + B,$$

where B is an endomorphism valued $(0,1)$ form, $B \in \Omega^{0,1}(X; \mathrm{End}(E))$, and the latter is a vector space, therefore $\mathscr{C}(E)$ is an affine space modeled on $\Omega^{0,1}(X; \mathrm{End}(E))$. Considering the set of automorphism of the bundle E, $\mathrm{Aut}(E)$, and its action on $\mathscr{C}(E)$, the space of orbits is the set of isomorphism classes of holomorphic structures on E, which forms the *moduli space*.

4.2. Just as one finds in other classification problems and moduli spaces, e.g., moduli of stable curves and of stable maps, we consider a restricted set of classes of holomorphic structures in order to have a *good* Moduli Space, which in this case means that the quotient $\mathscr{C}_s(E)/\operatorname{Aut}(E)$ is Hausdorff, where $\mathscr{C}_s(E)$ denotes the space of stable complex structures. Actually, as cited in [ATIYAH, BOTT, 1982], this quotient is a complex manifold and is compact for certain values of n and k. In the next sections, we will only scratch the surface of this subject and present the definition of stability in the case that X is a Riemann surface, and we relate the stability condition with differential geometric aspects as the existence of unitary connections satisfying a certain condition on its curvature. This is a celebrated result due to Narasimhan and Seshadri, and we will follow an alternative proof given in [DONALDSON, 1983].

4.3. It is possible to extend the ideas of stability to higher dimensional complex manifolds, and, in particular, to complex surfaces, i.e., manifolds of complex dimension 2, see for instance [DONALDSON, 1985]. The definition of stability in that case requires the choice of an embedding of the base manifold in projective space and the notion of coherent subsheaves of the underlying sheaf of sections and there is also a nice relation between the algebraic notion of stability of algebraic bundles to the differential geometric notion of irreducible Hermite-Einstein connections. We will not treat this here.

4.2 Narasimham-Seshadri Theorem

4.4. In this section we will consider bundles $E \to X$, with X a compact Riemann surface with a Hermitian metric normalised to unit volume. In what follows we will continue to use the convention introduced in chapter 1 that a smooth vector bundle is written with normal capital letters as in $E \to X$, while a holomorphic bundle will be denoted with caligraphic letter as in $\mathcal{E} \to X$.

4.5 Definition. We define the **degree** of a vector bundle as in 4.4, and denote it $\deg(E)$, by

$$\deg(E) = \int_X c_1(E),$$

where $c_1(E)$ denotes the first Chern class of the bundle.

4.6 Definition. The **normalized degree**, or **slope** $\mu(E)$ of a vector bundle as in 4.4 is given by

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

4.7 Definition. A holomorphic bundle $\mathcal{E} \to X$ is called **indecomposable** if it cannot be writen as a proper direct sum of holomorphic bundles.

4.8 Remark. It is important that we consider the holomorphic structure in the above definition of indecomposability. Recall from paragraph 1.80, that smooth decomposition of Hermitian vector bundles always exists.

4.9 Definition. A holomorphic bundle $\mathcal{E} \to X$ with X as in 4.4 is called **semi-stable** if for all proper subbundles $\mathcal{F} < \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$, and if strict inequality holds, it is called **stable**.

4.10 Remark. A stable bundle is automatically indecomposable. Indeed, assuming that it is decomposable, i.e $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ we would have exact sequences

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}' \oplus \mathcal{E}'' \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E}'' \oplus \mathcal{E}' \longrightarrow \mathcal{E}' \longrightarrow 0.$$

Recall from theorem 1.78 that $c_1(\mathcal{E}' \oplus \mathcal{E}'') = c_1(\mathcal{E}') + c_1(\mathcal{E}'')$, thus $\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'')$, and applying the stability condition we get on the top row $\mu(\mathcal{E}') < \mu(\mathcal{E}'')$ whereas on the bottom row we would have the opposite.

The result we are seeking in this chapter is the following:

4.11 Theorem. *An indecomposable holomorphic bundle $\mathcal{E} \to X$ is stable if and only if there exists a unitary connection on $\mathcal{E}$ having constant curvature*

$$*F = -2\pi i \mu(\mathcal{E})\mathbb{1}, \tag{4.1}$$

where $\mathbb{1}$ is the $\mathrm{rk}(\mathcal{E})$ *identity. Such connection is unique, up to unitary gauge transformation.*

4.12 Proposition. *Any holomorphic line bundle $\mathcal{L} \to X$ over X is stable and admits a unitary connection whose curvature satisfies (4.1).*

Proof. Note that we have no issues with decomposability here and as every line bundle is clearly stable since there is no proper subbundle, what we have to prove is that $\mathcal{L}$ admits a unitary connection with $F = -2\pi i \deg(\mathcal{L})\,\mathrm{vol}$, where vol is the volume form associated with the Hermitian metric on X that is scaled to unit volume. Recall that an effective way of computing the Chern class of the line bundle is picking a connection A and then taking the trace of its curvature, but here, $\mathrm{Tr}(F) = F$, so we have

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F],$$

where $[F]$ denotes the cohomology class of the closed curvature form. Since $\int_X c_1(\mathcal{L}) = \deg(\mathcal{L})$, for every connection A, the difference

$$F(A) - (-2\pi i \deg(\mathcal{L})\,\mathrm{vol}) = d\eta, \tag{4.2}$$

is exact. That said, pick an arbitrary Hermitian metric H on the line bundle $\mathcal{L}$ and a local holomorphic section σ on $\mathcal{L}$. From the proof of 1.62 (set $M = H(\sigma, \sigma)$ and etc.), the vector potential of the Chern connection, ∇_A is locally given by $A = \partial(\log H)$, where $H = H(\sigma, \sigma)$, and associated curvature

$$F = \bar{\partial}\partial(\log H).$$

For a real valued function $\phi\colon X \to \mathbb{R}$, define $H' := e^{\phi}H$ another Hermitian metric. For this new connection, the curvature is given by

$$F' = F + \bar{\partial}\partial\phi,$$

and therefore we have

$$F - F' = -\bar{\partial}\partial\phi.$$

Now, as $\int_X d\eta = 0$, from corollary 2.33 there exists a solution ϕ to the differential equation

$$-\bar{\partial}\partial\phi = d\eta,$$

so, from (4.2), the Chern connection of the Hermitian metric $H' = e^{\phi}H$ for the solution ϕ above meets the requirements. □

4.13 Remark. The proof of theorem 4.11 will be done by induction on the rank of the bundle, thus the above proposition just ensures us that first induction step is valid.

4.14. Recall from chapter 2 that a unitary connection A on a Hermtian vector bundle

$E \to X$ gives an operator $d_A : \Omega^0(X;E) \to \Omega^1(X;E)$ with $(0,1)$-component $\bar{\partial}_A$: $\Omega^0(X;E) \to \Omega^{0,1}(X;E)$ and the latter corresponds to a holomorphic structure $\mathcal{E}_A$ on E. Conversely, if E has holomorphic and Hermitian structures, we also saw that there is a unique connection that is compatible with both structures, called the Chern connection.

4.15. The action of the gauge group $\mathcal{G}$ on the affine space $\mathcal{A}$ of unitary connections can be written, according to 1.52, as

$$u(A) = d_A - d_A u\, u^{-1}, \qquad u \in \mathcal{G},\ A \in \mathcal{A},$$

while the action of the group of general linear automorphism, $\mathcal{G}_{\mathbb{C}}$, is given by

$$g(A) = d_A - (\bar{\partial}_A g)g^{-1} + ((\bar{\partial}_A g)g^{-1})^{\dagger}, \qquad g \in \mathcal{G}_{\mathbb{C}},\ A \in \mathcal{A}, \tag{4.3}$$

where $\dagger$ denotes the adjoint. As we saw in 1.55, connections define isomorphic holomorphic structures precisely when they are on the same $\mathcal{G}_{\mathbb{C}}$-orbit. For a holomorphic bundle $\mathcal{E}$, we will denote by $O(\mathcal{E})$ its orbit of connections on $\mathcal{A}$.

4.16. For the proof of theorem 4.11, we will have to construct a functional J and extract some weakly convergent sequences of connections in $O(\mathcal{E})$. For that, recall from linear algebra, that we have a trace norm on $n \times n$ Hermitian matrices given by $\nu(M) = (\mathrm{Tr}(M^{\dagger}M))^{1/2}$. If we apply ν in each fibre, for any smooth self adjoint-section $s \in \Omega^0(X; E \otimes E^*)$, we get a norm on the space of sections of the bundle of endomorphisms of E given by

$$N(s) = \left(\int_X \nu(s)^2\right)^{1/2}.$$

Below we summarize some properties of the norms ν and N discussed above.

4.17 Proposition. *The trace norm, ν, acting on Hermitian $n \times n$ matrices, satisfies*

$$\nu(M) = \sum |\lambda_j|,$$

where λ_j are the eigenvalues of M, and moreover, if in block representation M is written as $M = \begin{pmatrix} A & B \\ B^{\dagger} & D \end{pmatrix}$, we have

$$\nu(M) \geq |\,\mathrm{Tr}(A)| + |\,\mathrm{Tr}(D)|.$$

Proof. We have a characterization of the trace norm as

$$\nu(M) = \max_{\{e_i\}} \sum_{i=1}^{n} |\langle Me_i, e_i \rangle|, \tag{4.4}$$

where we let $\{e_1, \ldots, e_n\}$ run over all possible orthonormal frames for $\mathbb{C}^n$ (Cf. [ATIYAH, BOTT, 1982], *apud* [DONALDSON, 1983]). Since M is Hermitian, its eigenvalues are real and there is an orthonormal basis that diagonalizes M, thus, by (4.4), $\nu(M) \geq \sum |\lambda_j|$. On the other hand, we have

$$\nu(M)^2 = \sum \lambda_j^2 \leq \left(\sum \lambda_j\right)^2,$$

and therefore, $\nu(M) \leq \sum |\lambda_j|$. For the second statement, again using the fact that M is Hermitian, it follows that the blocks A and D are also Hermitian, so by picking an orthonormal basis of the form $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$, where $\{e_1, \ldots, e_k\}$ diagonalizes A (similar for D), we get the result. □

4.18 Proposition. *The norm N on the space $\Omega^0(X; E \otimes E^*)$ is equivalent to the usual L^2 norm, so it extends to an L^2 section.*

4.19. Now, for any L_1^2 connection A (for information regarding L_1^2 spaces, see appendix C), we define the functional $J\colon \mathcal{A} \to \mathbb{R}$ by

$$J(A) = N\left(\frac{*F(A)}{2\pi i} + \mu \mathbb{1}\right),$$

where $\mathbb{1}$ is the appropriate identity matrix.

4.20 Remark. The functional defined above is well defined. First, $\left(\frac{*F(A)}{2\pi i} + \mu\mathbb{1}\right)$ is self adjoint because the connection A is unitary so the matrix $*F(A) \in \mathfrak{u}(k)$ and also we divide $F(A)$ by the complex scalar i. Now, the norm N extends to an L^2 section and it is not obvious why J would be defined for L_1^2 connections. However, recall that $F(A+a) = F(A) + d_A a + a \wedge a$, so if a is in $L_1^2(X; \mathrm{End}(E))$ then $d_A a$ is in L^2 and also $a \wedge a$, as there is a bounded inclusion $L_1^2 \to L^4$ (cf. appendix C, proposition C.35) and the product of two L^4 sections is in L^2.

4.21 Remark. As in [DONALDSON, 1983], we have $J(A) = 0$ if and only if the connection A is of the required type, i.e., $*F(A) = -2\pi i\mu\mathbb{1}$. Although the functional J is not smooth, it has the property that if a sequence of connections $A_i \to A$ weakly in L_1^2, then $F(A_i) \to F(A)$ weakly in L^2, thus, $J(A) \leq \liminf J(A_i)$, since for all $\epsilon > 0$

we separate $*F(A)/2\pi i$ from the closed convex set $\{\alpha\ ;\ N(\alpha+\mu\mathbb{1})\leq J(A)-\epsilon\}$ by a hyperplane.

4.2.1 Supporting Results

4.22. To arrive at a proof of the theorem 4.11, we will need a series of lemmas and the following proposition due to K. Uhlenbeck that can be found in [UHLENBECK, 1981], theorem 1.5 *apud* [DONALDSON, 1983]. In the next proposition and the subsequent lemma, observe from corollary C.36 on the appendix C, that the L^2_2 gauge transformations are indeed a group and that the action of an L^2_2 gauge transformation on an L^2_1 connection is well defined.

4.23 Proposition. *If $\{A_i\}\subset\mathcal{A}$ is a sequence of L^2_1 connections with $||F(A_i)||_{L^2}$ bounded for each i, then there exists a subsequence $\{A_{i'}\}$ and L^2_2 gauge transformations $u_{i'}$ such that $u_{i'}(A_{i'})\to u(A)$ weakly in L^2_1.*

4.24 Lemma. *Consider a holomorphic vector bundle $\mathcal{E}\to X$. Then, either* $\inf J|_{O(\mathcal{E})}\in O(\mathcal{E})$ *or there exists another holomorphic bundle $\mathcal{F}\ncong\mathcal{E}$, with same rank and degree as $\mathcal{E}$ and* $\mathrm{Hom}(\mathcal{E},\mathcal{F})\neq 0$, *satisfying* $\inf J|_{O(\mathcal{F})}\leq\inf J|_{O(\mathcal{E})}$.

Proof. Let $\{A_i\}$ be a minimizing sequence for $J|_{\mathcal{O}(\mathcal{E})}$. As the norm N is equivalent to the L^2 norm, and $N(*F(A_i)/2\pi i+\mu\mathbb{1})=J(A_i)$ is bounded we have that $N(*F(A_i))$ is bounded and hence $||F(A_i)||_{L^2}$, so we can apply the proposition 4.23 to obtain $A_i\to B$, weakly in L^2_1 and

$$J(B)\leq\liminf J(A_i)=\inf J|_{\mathcal{O}(\mathcal{E})},$$

as we saw in remark 4.21. The connection B is in L^2_1 and is not smooth, however it is shown in [ATIYAH, BOTT, 1982] that every L^2_2-complexified gauge orbit in the space of L^2_1-connections contains a smooth connection. Then, the connection B defines a holomorphic structure on $\mathcal{E}$, via the $\bar\partial_B$ and we claim that $\mathrm{Hom}(\mathcal{E},\mathcal{E}_B)\neq 0$. Indeed, first recall from paragraph 1.50 that for unitary connections A, A', we define a connection $\nabla_{AA'}$ on $\mathcal{E}\otimes\mathcal{E}^*$, by setting

$$\nabla_{AA'}=\nabla_A\otimes\mathbb{1}+\mathbb{1}\otimes\nabla^\vee_{A'},$$

with a corresponding

$$\bar\partial_{AA'}\colon\Omega^0(X;\mathrm{End}\,\mathcal{E})\to\Omega^{0,1}(X;\mathrm{End}\,\mathcal{E}).$$

The solutions to $\bar{\partial}_{AA'}g = 0$ corresponds to elements of $\mathrm{Hom}(\mathcal{E}_{A'}, \mathcal{E}_A)$, as we saw in 1.55. If we had $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) = 0$, $\bar{\partial}_{BA_0}$ (the A_0 corresponds to the first element in the sequence that, by hypothesis is in the orbit of $\mathcal{E}$) would have no kernel and since it is a first order elliptic operator, proposition C.33 in the appendix C, asserts that there exists a constant C such that

$$||\bar{\partial}_{BA_0}\sigma||_{L^2} \geq C||\sigma||_{L^2_1} \qquad \forall \sigma \in \Omega^0(X; \mathrm{End}\,\mathcal{E})). \tag{4.5}$$

Now, we invoke the Sobolev inequality $||\sigma||_{L^2_1} \geq C'||\sigma||_{L^4}$, for a constant C' to obtain a constant C_1 such that in (4.5),

$$||\bar{\partial}_{BA_0}\sigma||_{L^2} \geq C_1||\sigma||_{L^4}. \tag{4.6}$$

On the other hand, as $L^2_1 \hookrightarrow L^4$ is compact (cf. proposition C.35 in appendix C) we get $A_i \to B$ in L^4 (this was proved in a more general setting in C.19). Then, as $\bar{\partial}_{BA_0} - \bar{\partial}_{A_iA_0}$ is the operator that sends a smooth section $\sigma \mapsto (B - A_i)^{0,1}\sigma$, Hölder inequality gives us another constant C_2 such that

$$||(\bar{\partial}_{BA_0} - \bar{\partial}_{A_iA_0})\sigma||_{L^2} \leq C_2||A_i - B||_{L^4}||\sigma||_{L^4}. \tag{4.7}$$

Applying equations (4.6) and (4.7) to the triangle inequality

$$||\bar{\partial}_{BA_0}\sigma||_{L^2} - ||\bar{\partial}_{A_iA_0}\sigma||_{L^2} \leq ||(\bar{\partial}_{BA_0} - \bar{\partial}_{A_iA_0})\sigma||_{L^2},$$

yields

$$||\bar{\partial}_{A_iA_0}\sigma||_{L^2} \geq (C_1 - C_2||A_i - B||_{L^4})||\sigma||_{L^4},$$

and as $A_i \to B$ in the L^4 norm, we have that $\bar{\partial}_{A_iA_0}$ never vanishes for i sufficiently large, which in its turn implies that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_{A_i}) = 0$, contradicting the fact that $\mathcal{E} \cong \mathcal{E}_{A_i}$, as $A_i \in \mathcal{O}(\mathcal{E})$. This finishes the proof of our claim. We then have $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) \neq 0$ and:

$$\begin{cases} \mathcal{E} \cong \mathcal{E}_B \Rightarrow J(B) = \inf J|_{\mathcal{O}(\mathcal{E})}, B \in \mathcal{O}(\mathcal{E}) \\ \mathcal{E} \ncong \mathcal{E}_B \Rightarrow J(B) \leq \inf J|_{\mathcal{O}(\mathcal{E})}, B \notin \mathcal{O}(\mathcal{E}). \end{cases}$$

□

4.25. If we have an exact sequence of holomorphic bundles over a base space X,

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{U} \longrightarrow 0,$$

any unitary connection A on $\mathcal{T}$ has the shape

$$A = \begin{pmatrix} A_S & \beta \\ -\beta^\dagger & A_U \end{pmatrix},$$

where $\beta \in \Omega^{0,1}(X;\mathcal{U}^* \otimes \mathcal{S})$ is the adjoint of minus the second fundamental form, as in 1.83. We can normalize β so that $*\operatorname{Tr}(\beta \wedge \beta^\dagger) = \operatorname{Tr}*(\beta \wedge \beta^\dagger) = 2\pi i|\beta|^2$ and the curvature of A is expressed as

$$F(A) = \begin{pmatrix} F(A_S) - \beta \wedge \beta^\dagger & d\beta \\ -d\beta^\dagger & F(A_U) - \beta^\dagger \wedge \beta \end{pmatrix}.$$

Since we mentioned about exact sequences, we establish the following result that will be used to prove our main theorem.

4.26 Proposition. *In an exact sequence of vector bundles as above,*

$$\deg(\mathcal{T}) = \deg(\mathcal{S}) + \deg(\mathcal{U}).$$

Proof. Recall the definition of the degree of a vector bundle in 4.5. As we saw in (1.31), we have that

$$c_1(\mathcal{T}) = \operatorname{Tr}\left(\frac{iF(A_T)}{2\pi}\right).$$

A quick look in the previous paragraph allows to conclude that

$$\operatorname{Tr}\left(\frac{iF(A_T)}{2\pi}\right) = \operatorname{Tr}\left(\frac{iF(A_S)}{2\pi}\right) + \operatorname{Tr}\left(\frac{iF(A_U)}{2\pi}\right),$$

hence $c_1(\mathcal{T}) = c_1(\mathcal{S}) + c_1(\mathcal{U})$ and from integration, the result follows. □

4.27 Lemma. *Let $\mathcal{F} \to X$ be a holomorphic bundle that sits in an extension*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

If $\mu(\mathcal{F}') \geq \mu(\mathcal{F})$ (so $\mu(\mathcal{F}) \geq \mu(\mathcal{F}'')$), then for all unitary connections A on $\mathcal{F}$ we have

$$J_0 := \operatorname{rk}'(\mu' - \mu) + \operatorname{rk}''(\mu - \mu'') \leq J(A), \tag{4.8}$$

where $\operatorname{rk}' := \operatorname{rk}(\mathcal{F}')$ and so on, and the above equality holds if and only if the extension splits.

Proof. First recall the Hölder inequality, stated in the appendix as equation (C.2), considering $f=\nu$ and $g=1$, $p=q=2$. Then

$$\begin{aligned} J(A) &= \left(\int_X \nu\left(\frac{*F(A)}{2\pi i}+\mu\mathbb{1}\right)^2\right)^{1/2} \\ &\geq \int_X \nu\left(\frac{*F(A)}{2\pi i}+\mu\mathbb{1}\right). \end{aligned}$$

We will omit the metric form vol that is normalised to unit volume to make the calculations. As we saw, e.g., in the proof of 4.12, we have $\operatorname{Tr}(F)=\operatorname{Tr}(*F)\,\mathrm{vol}$ ($\operatorname{Tr}(*F)$ is a function on X). Now, recall from proposition 4.17 that if M has a block composition as $\left(\begin{smallmatrix} A & B \\ B^* & D\end{smallmatrix}\right)$, then $\nu(M)\geq|\operatorname{Tr}(A)|+|\operatorname{Tr}(D)|$. Considering $M=(*F(A)/2\pi i+\mu\mathbb{1})$ and since we normalised X to unit volume,

$$\begin{aligned} J(A) &\geq \int_X\left|\operatorname{Tr}\left(\frac{*F(A')}{2\pi i}+\mu\mathbb{1}'-\frac{*(\beta\wedge\beta^*)}{2\pi i}\right)\right|+\left|\operatorname{Tr}\left(\frac{*F(A'')}{2\pi i}+\mu\mathbb{1}''-\frac{*(\beta^*\wedge\beta)}{2\pi i}\right)\right| \\ &\geq \left|\int_X \operatorname{Tr}\left(\frac{*F(A')}{2\pi i}+\mu\mathbb{1}'\right)-|\beta|^2\right|+\left|\int_X \operatorname{Tr}\left(\frac{*F(A'')}{2\pi i}+\mu\mathbb{1}''\right)+|\beta|^2\right| \\ &= |-\deg(\mathcal{F}')+\mu\operatorname{rk}(\mathcal{F}')-||\beta||^2_{L^2}|+|-\deg(\mathcal{F}'')+\mu\operatorname{rk}(\mathcal{F}'')+||\beta||^2_{L^2}|, \end{aligned}$$

so as $\mu(\mathcal{F}')\geq\mu(\mathcal{F})\geq\mu(\mathcal{F}'')$, we obtain

$$J(A)\geq \operatorname{rk}'(\mu'-\mu)+\operatorname{rk}''(\mu-\mu'')+2||\beta||^2_{L^2}.$$

If the extension splits we get $\beta=0$, hence the equality holds. □

4.28 Lemma. *Suppose that $\mathcal{E}\to X$ is a holomorphic bundle and that theorem 4.11 has been proved for bundles of lower rank. If $\mathcal{E}$ sits in an extension*

$$0\longrightarrow\mathcal{E}'\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}''\longrightarrow 0,$$

then there exists a connection A on $\mathcal{E}$ with

$$J_1:=\operatorname{rk}'(\mu-\mu')+\operatorname{rk}''(\mu''-\mu)>J(A), \tag{4.9}$$

where $\operatorname{rk}':=\operatorname{rk}(\mathcal{E}')$, and so on.

Proof. We start observing that, in general, for a bundle that sits in an exact sequence

$$0\longrightarrow\mathcal{S}\longrightarrow\mathcal{T}\longrightarrow\mathcal{U}\longrightarrow 0,$$

connections in the orbit of $\mathcal{T}$, $\mathcal{O}(\mathcal{T})$, given by triples $(A_S, A_U, t\beta)$ converge in C^∞ when t goes to zero to a connection $A_T \in \mathcal{O}(\mathcal{S}\oplus\mathcal{U})$. Now, it is a fact, due to Harder and Narasimhan (more details can be seen in [ATIYAH, BOTT, 1982], section 7 *apud* [DONALDSON, 1983]) that any holomorphic bundle $\mathcal{P}$ has a *semi-stable filtration*

$$0 = \mathcal{P}_0 < \mathcal{P}_1 < \cdots < \mathcal{P}_k = \mathcal{P},$$

with the quotients $\mathcal{Q}_i := \mathcal{P}_i/\mathcal{P}_{i-1}$ semi-stable and satisfying $\mu(\mathcal{Q}_1) > \mu(\mathcal{Q}_2) > \cdots \mu(\mathcal{Q}_k)$. Each quotient is a holomorphic bundle and in its turn will have a filtration of its own

$$0 = \mathcal{Q}_i^0 < \mathcal{Q}_i^1 < \cdots < \mathcal{Q}_i^{k_i} = \mathcal{Q}_i,$$

with $\mathcal{C}_i^j := \mathcal{Q}_i^j/\mathcal{Q}_i^{j-1}$ semi-stable and satisfying $\mu(\mathcal{C}_i^j) = \mu(\mathcal{Q}_i) < \mu(\mathcal{P}_1) = \mu(\mathcal{Q}_1)$. In our case, we have $\mathcal{E}$ stable, $\mathcal{E}' < \mathcal{E}$ and set $\mathcal{P} = \mathcal{E}'$. From the above, we get, as $\mathcal{P}_1 < \mathcal{E}'$, that

$$\mu(\mathcal{C}_i^j) < \mu(\mathcal{P}_1) < \mu(\mathcal{E}). \tag{4.10}$$

For each i, applying the observation of the first paragraph of this proof to the exact sequence

$$0 \longrightarrow \mathcal{Q}_i^{k_i-1} \longrightarrow \mathcal{Q}_i \longrightarrow \mathcal{C}_i^{k_i} \longrightarrow 0,$$

a 1-parameter family of connections $\{A_{\mathcal{Q}_i,t}\} \in \mathcal{O}(\mathcal{Q}_i)$ converges to a connection in $\mathcal{O}(\mathcal{Q}_i^{k_i-1} \oplus C_i^{k_i})$. Now, we have another exact sequence

$$0 \longrightarrow \mathcal{Q}_i^{k_i-2} \longrightarrow \mathcal{Q}_i^{k_i-1} \longrightarrow \mathcal{C}_i^{k_i-1} \longrightarrow 0,$$

and connections in $\{A_{\mathcal{Q}_i,t}\} \in \mathcal{O}(\mathcal{Q}_i)$ converge to a connection in $\mathcal{O}(\mathcal{Q}_i^{k_i-2}\oplus C_i^{k_i-1})$, therefore, inductively, our 1-parameter family of connections $\{A_{\mathcal{Q}_i,t}\} \in \mathcal{O}(\mathcal{Q}_i)$ converges to a connection in $\mathcal{O}(\oplus_j \mathcal{C}_i^j)$. What we are really interested in is to study 1-parameter familes of connections $\{A'_t\} \in\mathcal{O}\ (\mathcal{E}')$. From the exact sequence

$$0 \longrightarrow \mathcal{P}_{k-1} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{Q}_k \longrightarrow 0,$$

using an analogous argument to the one above we have that $A'_t \to A'_0 \in \mathcal{O}(\mathcal{P}_{k-1}\oplus \mathcal{Q}_k)$, and decomposing further we see that

$$A'_0 \in \mathcal{O}\left(\oplus_{i,j}\mathcal{C}_i^j\right).$$

As $\operatorname{rk}\mathcal{C}_i^j < \operatorname{rk}\mathcal{E}$, we apply the induction hypothesis to obtain

$$*F(A_i^j) = -2\pi i\mu(\mathcal{C}_i^j)\mathbb{1},$$

where $A_i^j := A_{\mathcal{C}_i^j,0}$, then as $A_0' = \oplus_{i,j} A_i^j$, we get

$$*F(A_0') = -2\pi i\Lambda', \tag{4.11}$$

where $\Lambda' = \operatorname{diag}(\mu'_{i,j})$, and, by 4.10 we have $\mu'_{i,j} := \mu(\mathcal{C}_i^j) < \mu(\mathcal{E})$. By a similar procedure for the holomorphic bundle $\mathcal{E}''$, a 1-parameter family of connections in $\mathcal{O}(\mathcal{E}'')$ converges to $A_0'' \in \mathcal{O}(\oplus_{i,j}\mathcal{C}_i^j)$ and

$$*F(A_0'') = -2\pi i\Lambda'', \tag{4.12}$$

where $\Lambda' = \operatorname{diag}(\mu''_{i,j})$. But in this case, we claim that the entries $\mu''_{i,j} := \mu(\mathcal{C})_i^j > \mu(\mathcal{E})$. Indeed,

$$\mu(\mathcal{C}_i^j) = \mu(\mathcal{Q}_i) > \mu(\mathcal{Q}_k) \geq \mu(\mathcal{P}_k) = \mu(\mathcal{E}'') > \mu(\mathcal{E}),$$

where the inequality $\mu(\mathcal{Q}_k) \geq \mu(\mathcal{P}_k)$ comes from the semi-stability of $\mathcal{Q}_k$. Now, for each t, A_t' and A_t'' gives us operators

$$d_t \colon \Omega^*(X) \otimes \Gamma(\operatorname{Hom}(\mathcal{E}'', \mathcal{E}')) \to \Omega^{*+1}(X) \otimes \Gamma(\operatorname{Hom}(\mathcal{E}'', \mathcal{E}')).$$

If we take the $(0,1)$-part of d_t, and call it d_t'', we get $(d_t'')^2 = 0$, as 2-forms on Riemann surfaces are necessarily of type $(1,1)$. Then, it is possible to make a "Dolbeaut cohomology"for this operator and there will be a version of the Hodge theorem is this setting. Thus, For $t \neq 0$, choose the harmonic representative of the extension class of $\mathcal{E}$, β_t, which means that $d_t''\beta_t = (d_t'')^\dagger\beta_t = 0$, but as $(d_t'')^\dagger = d_t'$, we have $d_t\beta_t = 0$. We can assume without loss that β_t is scaled to $||\beta_t||_{L^2} = 1$. As $d_t \to d_0$, and d_t are first order elliptic on $\Omega^{0,1}$, there is a uniform bound

$$\|\beta_t\|_{C^0} < C_t. \tag{4.13}$$

This follows from the inequality on C.33

$$\|\beta_t\|_{L^2_{k+1}} \leq C_k(\|d_t\beta_t\|_{L^2_k} + \|\beta_t\|_{L^2}) = C_t$$

and the Sobolev Embbeding Theorem C.26. Consider now, a 2-parameter family of connections on $A(s,t) \in \mathcal{O}(\mathcal{E})$ given by triples $(A'_t, A''_t, s\beta_t)$ with matrix representation

$$A(s,t) = \begin{pmatrix} A'_t & s\beta_t \\ -s\beta^\dagger & A''_t \end{pmatrix},$$

thus, as $d_t\beta_t = 0$, from 4.25 the curvature is given by

$$F(s,t) = \begin{pmatrix} F(A'_t) - s^2\beta_t \wedge \beta_t^\dagger & 0 \\ 0 & F(A''_t) - s^2\beta_t^\dagger \wedge \beta_t \end{pmatrix}.$$

So, as $s,t \to 0$, we have by (4.11) and (4.12), that

$$J(A(s,t)) \to |\operatorname{Tr}(-\Lambda' + \mu\mathbb{1}')| + |\operatorname{Tr}(-\Lambda'' + \mu\mathbb{1}'')| = J_1.$$

Now we have to check that for a smart choice of s,t we have $J(A(s,t)) < J_1$. As $\Lambda' - \mu\mathbb{1}$ has only negative eigenvalues, the same happens for nearby matrices, M, and for those matrices, we get that $\nu(M) = -\operatorname{Tr}(M)$. Using the uniform bound on β_t of (4.13) as $*F(A'_t) \to -2\pi i\Lambda'$, we have

$$\nu\left(\frac{*F(s,t)}{2\pi i} + \mu\mathbb{1}\right) = J_1 - 2s^2|\beta_t|^2 + \epsilon(t),$$

with $\epsilon(t) \to 0$ with t. So

$$J(A(s,t))^2 = \int_X (J_1 - 2s^2|\beta_t|^2 + \epsilon(t))^2.$$

For small enough s such that $s^4 \int_X |\beta_t|^4$ is much less than $s^2 \int_X |\beta_t|^2 = s^2$, and t also small so that $\epsilon(t)$ is even smaller, we get $J(A(s,t)) < J_1$. □

4.2.2 Proof of Theorem 4.11

4.29. Suppose that an indecomposable holomorphic bundle $\mathcal{E} \to X$ admits a connection A meeting the requirements of the theorem, i.e., $J(A) = 0$, we wish to prove that $\mathcal{E}$ is stable. Indeed, for every subbundle $\mathcal{E}' < \mathcal{E}$, the indecomposability condition tells us that the exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

does not split. Assuming that $\mathcal{E}$ is not stable would imply that for a given subbundle $\mathcal{E}'$ as above we would have $\mu' \geq \mu \geq \mu''$. So now we can apply lemma 4.27 and equation

(4.8) tells us that $J_0 < J(A) = 0$. However, the definition of J_0 and the relations on μ, μ', μ'' imply $J_0 \geq 0$, and we arrive at a contradiction.

4.30. Conversely, suppose that $\mathcal{E}$ is stable and that the theorem has been proved for bundles of lower rank. We claim that $\inf J|_{O(\mathcal{E})} \in O(\mathcal{E})$. Indeed, suppose the contrary. Since $\mathcal{E}$ is holomorphic we can apply lemma 4.24 to construct another bundle $\mathcal{F} \ncong \mathcal{E}$, with same rank and degree as $\mathcal{E}$ with $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$, satisfying

$$\inf J|_{O(\mathcal{F})} \leq \inf J|_{O(\mathcal{E})}. \tag{4.14}$$

In general, for any non-zero holomorphic map of bundles over X, $\alpha \colon \mathcal{E} \to \mathcal{F}$ there are proper extensions and factorizations, as in [NARASIMHAN, SESHADRI, 1965], section 4 *apud* [DONALDSON, 1983]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
 & & & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \\
0 & \longleftarrow & \mathcal{F}'' & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{F}' & \longleftarrow & 0
\end{array}$$

with exact rows and satisfying $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$, $\det \beta \neq 0$ and $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$, thus

$$\mu(\mathcal{F}') \geq \mu(\mathcal{E}'') > \mu(\mathcal{E}) = \mu(\mathcal{F}). \tag{4.15}$$

From (4.15) we can apply lemma 4.27 on the bottom row of the above diagram and get

$$\inf J|_{O(\mathcal{F})} \geq J_0, \tag{4.16}$$

and lemma 4.28 on the upper row

$$\inf J|_{O(\mathcal{E})} < J_1, \tag{4.17}$$

thus combining (4.14), (4.16) and (4.17), we get

$$J_0 < J_1. \tag{4.18}$$

Now, $\operatorname{rk} \mathcal{E} = \operatorname{rk} \mathcal{F}$ and $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$ implies $\operatorname{rk} \mathcal{E}' = \operatorname{rk} \mathcal{F}''$. From the proposition 4.26, $\deg \mathcal{E} = \deg \mathcal{F}$ and $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$ implies $\deg \mathcal{E}' \geq \deg \mathcal{F}''$, thus, from (4.8) and (4.9), we obtain $J_1 \leq J_0$, a contradiction that proves our claim that $\inf J|_{O(\mathcal{E})} \in O(\mathcal{E})$.

4.31. So, $\mathcal{E}$ is stable and $\inf J|_{\mathcal{O}(\mathcal{E})}$ is attained at $A \in \mathcal{O}(\mathcal{E})$. The operator $d_A^\dagger d_A$ (which is similar to the Laplace-Beltrami defined in 2.10, and the dual $d_A^\dagger$ is with respect to the L_2^2 inner product), acting on L_2^2 self-adjoint sections of $\mathrm{End}(E)$, has kernel the constant multiples of the identity, because any other element of the kernel would satisfy

$$0 = (d_A^\dagger d_A u, u) = (d_A u, d_A u),$$

so, in particular, $\bar{\partial}_A u = 0$ and hence the eigenspaces of u would decompose $\mathcal{E}$ holomorphically, which is forbidden by stability. Now, as the projection on the harmonic functions of $\mathrm{Tr}(*F/2\pi i) = *\,\mathrm{Tr}(F/2\pi i)$ is $-\deg(\mathcal{E})$, since $c_1(\mathcal{E}) = \mathrm{Tr}(iF/2\pi)$, we get that the projection of $*F/2\pi i$ on the kernel of $d_A^\dagger d_A$ is $-\mu(\mathcal{E})\mathbb{1}$. Thus, Hodge theory (a slightly different version of 2.32, confer [NARASIMHAN, RAMADAS, 1979] *apud* [DONALDSON, 1983]) implies that there is a self-adjoint section $h \in L_2^2(\Gamma(\mathrm{End}(E))$ such that

$$d_A^\dagger d_A h = 2\pi\mu - i * F(A).$$

For small $t, 1 + th = g_t \in \mathcal{G}_{\mathbb{C}}$. Set $A_t = g_t(A) \in \mathcal{O}(\mathcal{E})$. We compute its curvature using (4.3) and (1.25), and observing that $(\bar{\partial}_A g_t)^\dagger = \partial_A g_t$, we obtain

$$\begin{aligned} F(A_t) &= F(A) - \partial_A((\bar{\partial}_A g_t)g_t^{-1}) + \bar{\partial}_A(g_t^{-1}(\partial_A g_t)) \\ &\quad -\bar{\partial}_A g_t g_t^{-2}\partial_A g_t - g_t^{-1}\partial_A g_t \bar{\partial}_A g_t g_t^{-1} \\ &= F(A) - t(\partial_A\bar{\partial}_A - \bar{\partial}_A\partial_A)h + q(t,h), \end{aligned} \tag{4.19}$$

with $\|q(t,h)\|_{L^2} \leq C(\|h\|_{L_2^2})t^2$, for small t. As $d_A^\dagger d_A = i * (\bar{\partial}_A\partial_A - \partial_A\bar{\partial}_A)$ we get on (4.19)

$$\left(\frac{*F(A_t)}{2\pi i} + \mu\mathbb{1}\right) = \left(\frac{*F(A)}{2\pi i} + \mu\mathbb{1}\right)(1-t) + O(t^2),$$

therefore, taking the norm N on both sides and taking into consideration the fact that $J(A) = \inf J|_{\mathcal{O}(\mathcal{E})}$, we have

$$J(A_t) = J(A) - tJ(A) + O(t^2).$$

Now, if $J(A_t)$ is to be a minimum at $t = 0$, we must have $*F(A)/2\pi i = -\mu\mathbb{1}$, as we wanted. Although A need not be a smooth connection, we can find a unitary gauge transformation $u \in \mathcal{G}$ such that $u(A)$ is actually smooth, as it is shown in [UHLENBECK, 1981] *apud* [DONALDSON, 1983].

4.32. As for the uniqueness, up to unitary gauge transformation, recall that any complex matrix G can be written in the form PU, for P a positive definite Hermitian matrix and U a unitary matrix. Thus $g \in \mathcal{G}_{\mathbb{C}}$ has a factorization as $g = pu$, for p positive Hermitian and u unitary, so, if A, B are distinct solutions, we can put $B = g(A)$, and we can assume $g = g^\dagger$. Since $F(A) = F(B) = -2\pi i \mu \mathbb{1}\,\mathrm{vol}$, where vol is the Hermitian volume form that was normalised to unit volume, we obtain again from (4.3) and (1.25),

$$\partial_A \bar{\partial}_A g^2 = -((\bar{\partial}_A g^2)g^{-1})((\bar{\partial}_A g^2)g^{-1})^\dagger.$$

Taking the trace of the above expression and considering Δ the Laplacian as in 2.10, we obtain $\Delta\,\mathrm{Tr}(g^2) \leq 0$, with equality if, and only if, $\bar{\partial}_A g^2 = 0$. By the maximum principle for subharmonic functions (cf. [EVANS, 1998] section 6.4) the only possibility is $\Delta\,\mathrm{Tr}(g^2) = 0$ everywhere. Since $\mathcal{E}$ is indecomposable, we must have g a constant scalar, thus $A = B$.

Conclusion

To sum up, the notions of connection and curvature on a vector bundle play a central role in gauge theory. That is why close attention was payed to the computations involving these objects throughout this dissertation. Besides the matrix calculations we have just mentioned, it was also treated here how that gauge theory naturally relates with Lie theory, partial differential equations and complex geometry. Nevertheless, what was discussed here was only a small part of what is understood as gauge theory, even in low dimensions. To conclude this work, it will be mentioned some interesting topics that were tangentially related with this work throughout the period in which it was written, but were not included.

Base spaces of dimension four are particularly interesting to Mathematical Physics, in as much as many models for space-time are treated in this dimension. The happy coincidence that the Hodge star operator is an involutive operator ($*^2 = \mathbb{1}$) on the space of 2-forms of a 4-manifold, allows us to produce a wide range of results regarding the topology of 4-manifolds. For instance, an interesting construction in dimension four is the so-called "moduli space of Instantons" associated to a vector bundle. This consists of the space of ASD connections modulo the action of the gauge group. After some hard work using mathematical analysis, it can be shown that this quotient of infinite dimensional spaces turns out to be a finite dimensional manifold, and new topological invariants of the initial base-manifold can be produced using this manifold. More account on this matter can be found at [DONALDSON, KRONHEIMER, 1990].

The close relationship between stability of bundles and existence of special unitary connections on a vector bundle over Riemann surfaces, that was discussed in chapter 4 of the present work, can be extended to base-manifolds of complex dimension higher than 1. As an example, the article [DONALDSON, 1985] treats the case of complex surfaces (complex dimension 2), and in this case, the stability of a bundle is related to connections which the curvature must satisfy the *Hermite-Einstein* condition

$$\Lambda F = \lambda \mathbb{1},$$

where Λ is the dual of the Lefschetz operator (cf. appendix A) and λ is a scalar that depends on the bundle and on the base-manifold. It can be shown that $\Lambda F = *F$ when the base-space is a Riemann surface and the equation used in chapter 4 is nothing else but this *Hermite-Einstein* condition.

Conclusão

Em suma, os conceitos de conexão e curvatura num fibrado vetorial exercem papel central na teoria de calibre. Por isso é que foi dada atenção especial para os cálculos envolvendo conexões no decorrer desta dissertação. Além dos cálculos matriciais mencionados acima, foi visto aqui como a teoria de calibre naturalmente recai sobre a teoria de Lie, a teoria de equações diferenciais parciais e a teoria de variedades complexas. No entanto, o que foi visto neste trabalho reflete apenas uma pequena parte daquilo que compreende a teoria de calibre, mesmo em dimensões baixas. Para finalizar, são mencionados alguns assuntos interessantes que tangenciaram a vida acadêmica do autor durante o período de mestrado, mas que não foram incluídos nesta dissertação.

Espaços-base de dimensão quatro são particularmente interessantes para a Física Matemática, pois muitos modelos de espaço-tempo são feitos nessa dimensão. A feliz coincidência de que a estrela de Hodge é um operador linear involutivo ($*^2 = \mathbb{1}$) no espaço de 2-formas de uma 4-variedade permite produzir uma gama de resultados no estudo da topologia das 4-variedades. Por exemplo, uma interessante construção em dimensão quatro é a dos chamados "moduli Space of Instantons"associados a um fibrado vetorial. Este é o espaço das conexões ASD, módulo a ação do grupo de calibre. Após algumas incursões em análise matemática, pode-se mostrar que este quociente de espaços de dimensão infinita acaba sendo uma variedade diferenciável de dimensão finita, e novos invariantes topológicos da variedade-base inicial podem ser produzidos a partir deste "moduli Space", como consta em [DONALDSON, KRONHEIMER, 1990].

A íntima relação entre estabilidade e existência de conexões unitárias especiais num fibrado vetorial sobre superfícies de Riemann, que foi apresentada no capítulo 4, pode ser estendida para variedades-base de dimensão complexa maior que 1. Por exemplo, o artigo [DONALDSON, 1985] trata do caso de superfícies complexas (dimensão complexa 2), e a estabilidade de um fibrado está relacionada a conexões cuja curvatura deve satisfazer a condição de *Hermite-Einstein*

$$\Lambda F = \lambda \mathbb{1},$$

em que Λ é o dual do operador de Lefschetz (cf. appendix A) e λ é um escalar que depende do fibrado e da variedade-base. Pode-se mostrar que $\Lambda F = *F$ quando o espaço base é uma superfície de Riemann, e a equação usada no capítulo 4 do presente trabalho nada mais é do que esta condição de *Hermite-Einstein*.

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Apêndice A

Complex Manifolds

The theory of complex manifolds is very rich and useful in many areas of Mathematics. In this appendix, we will discuss briefly only the aspects of complex manifolds that are relevant to the present work. The main references for this part are [GRIFFITH, HARRIS, 1978], [NAKAHARA, 2003] and [HUYBRECHTS, 2005].

A.1 Initial Definitions

A.1. Recall the definition of a smooth manifold as a locally Euclidean, second countable topological space M together with a maximal differentiable structure $\mathscr{F}$ (cf. [WARNER, 1983] for a good reference and example 1.12 where we cited the differentiable structure $\mathscr{F}$). The second axiom of countability here is present to ensure that manifolds are paracompact, so they admit partitions of unity. The definition of a complex manifold is very similar to the one of a smooth manifold, basically, the local model for the theory is $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and we change the adjective "smooth" for "holomorphic". As $\mathbb{C} = \mathbb{R}^2$, and being holomorphic implies smoothness, complex manifolds are also smooth manifolds and we shall see here the differences between these objects.

A.2. Consider $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ open, with coordinate functions $z^1, \ldots, z^n \colon U \to \mathbb{C}$. We write $z^\mu = x^\mu + iy^\mu$, $i = \sqrt{-1}$, and $x^\mu, y^\mu \colon U \to \mathbb{R}$, for $\mu = 1, \ldots, n$. For a point $u \in U$, we have the real tangent space

$$T_uU = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n} \right\}.$$

Let $\{dx^1, dy^1, \ldots, dx^n, dy^n\}$ be the dual basis of T_uU, in $T_u^*U = \operatorname{Hom}(T_uU; \mathbb{R})$, also known as the space of 1-forms at $u \in U$, or the cotangent space at u. We now consider the complexified cotangent space

$$\mathrm{Hom}(T_u U; \mathbb{R}) \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(T_u U; \mathbb{C}),$$

and set

$$dz^\mu := dx^\mu + i dy^\mu \qquad d\bar{z}^\mu := dx^\mu - i dy^\mu,$$

for $\mu = 1, \ldots, n$. It is not hard to prove that $\{dz^1, d\bar{z}^1, \ldots, dz^n, d\bar{z}^n\}$ forms a basis to $\mathrm{Hom}_{\mathbb{R}}(T_u U; \mathbb{C})$. We define the symbols $\frac{\partial}{\partial z^\mu}$ and $\frac{\partial}{\partial \bar{z}^\mu}$ by

$$\begin{aligned} \frac{\partial}{\partial z^\mu} &= \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right) \\ \frac{\partial}{\partial \bar{z}^\mu} &= \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right). \end{aligned}$$

It is immediate that

$$dz^\mu\left(\frac{\partial}{\partial z^\mu}\right) = 1, \qquad dz^\mu\left(\frac{\partial}{\partial \bar{z}^\mu}\right) = 0 = dz^\mu\left(\frac{\partial}{\partial z^\nu}\right) = dz^\mu\left(\frac{\partial}{\partial \bar{z}^\nu}\right),$$

and

$$d\bar{z}^\mu\left(\frac{\partial}{\partial \bar{z}^\mu}\right) = 1, \qquad d\bar{z}^\mu\left(\frac{\partial}{\partial z^\mu}\right) = 0 = d\bar{z}^\mu\left(\frac{\partial}{\partial z^\nu}\right) = d\bar{z}^\mu\left(\frac{\partial}{\partial \bar{z}^\nu}\right),$$

for $\nu \neq \mu$. The action of the operators defined above on smooth $\mathbb{C}$-valued functions f is the obvious one.

A.3 Definition. Let f be a smooth $\mathbb{C}$-valued function on U, i.e., $f\colon U \to \mathbb{C}$. We call f **holomorphic** on U if

$$\frac{\partial f}{\partial \bar{z}^\mu} \equiv 0,$$

on U, for every μ.

A.4 Definition. Let $f^1, \ldots, f^n$ be smooth $\mathbb{C}$-valued functions on U, and let $f = (f^1, \ldots, f^n)\colon U \to \mathbb{C}^n$. We say that the map f is **holomorphic** if each f^μ is holomorphic.

A.5 Remark. The definition above is equivalent to the celebrated *Cauchy-Riemann* equations

$$\frac{\partial f}{\partial x^\mu} = -i\frac{\partial f}{\partial y^\mu}.$$

Also, T_uU has a natural structure of a $\mathbb{C}$-vector space, by putting

$$\frac{\partial}{\partial y^\mu} = i\frac{\partial}{\partial x^\mu}, \qquad \forall \mu.$$

A.6. Let f be a smooth $\mathbb{C}$-valued function on $U \subset \mathbb{C}^n$. We define the **total differential** of f as

$$df := \sum_\mu \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial y^\mu} dy^\mu,$$

or in terms of z^μ and $\bar{z}^\mu$, as

$$df := \sum_\mu \frac{\partial f}{\partial z^\mu} dz^\mu + \frac{\partial f}{\partial \bar{z}^\mu} d\bar{z}^\mu.$$

A.7 Proposition. *A smooth $\mathbb{C}$-valued function f on U is holomorphic if, and only if, df is $\mathbb{C}$-linear for every $u \in U$.*

A.8 Corollary. *Let f and g be holomorphic functions on U. Then*

1. *$(f + g)$ and $(f \cdot g)$ are holomorphic;*
2. *if f is nowhere zero, then $1/f$ is holomorphic;*
3. *assume that h is a holomorphic function on an open set of $\mathbb{C}$, and that h is defined in the range of f, then $h \circ f$ is holomorphic.*

A.9 Definitions. A **complex manifold of dimension** n, where the complex dimension is denoted as $\dim_{\mathbb{C}} M = n$, is a Hausdorff, second countable topological space M together with an open cover $\{U_\alpha\}_{\alpha \in I}$ of M and homeomorphisms $\varphi_\alpha \colon U_\alpha \to U \subset \mathbb{C}^n$, for all $\alpha \in I$, such that

$$\varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \tag{A.1}$$

are biholomorphic (holomorphic with holomorphic inverse), for all α, β. The pairs $(U_\alpha, \varphi_\alpha)$ are called **charts** while $\psi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$ are called **transition functions**. In a chart (U, φ), the **coordinate functions on** U are defined by composing $z^\mu \circ \varphi$, where z^μ are the coordinates of $\mathbb{C}^n$. We often abuse the notation and regard z^μ as the coordinates on U and write, for $p \in U$, $\varphi(p) = (z^1(p), \ldots z^n(p))$.

The set of charts $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ is called an **atlas**, and they define a **complex structure** on M. If the union of two atlases $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ and $\{(V_\beta, \psi_\beta) \mid \beta \in J\}$

is again an atlas, in the sense that (A.1) is satisfied, we say that they define the same complex structure.

A.10 Example. The **complex projective space**, $\mathbb{CP}^n$ is perhaps the most important compact complex manifold. We define it as the set of complex lines in $\mathbb{C}^{n+1}$. As a set, we have

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where the $\mathbb{C}^*$ action on $(\mathbb{C}^{n+1} \setminus \{0\})$ is given by

$$\lambda \cdot (z^0, \dots, z^n) = (\lambda z^0, \dots, \lambda z^n) \qquad \lambda \in \mathbb{C}^*, (z^0, \dots, z^n) \in \mathbb{C}^{n+1}.$$

We denote an equivalence class under the action by its **homogeneous coordinates** $[z^0 : \dots : z^n]$, where, we should remember that $[z^0 : \dots : z^n] = [\lambda z^0 : \dots : \lambda z^n]$, for non-zero λ. We will now give an explicit atlas to the projective space. For $\mu = 0, 1, \dots, n$, consider

$$U_\mu := \{[z^0 : \dots : z^n] \mid z^\mu \neq 0\}.$$

If we endow $\mathbb{CP}^n$ with the quotient topology, each U_μ will be an open set, as the inverse image under the quotient map is open in $(\mathbb{C}^{n+1} \setminus \{0\})$. Also, consider the bijective maps

$$\varphi_\mu \colon U_\mu \to \mathbb{C}^n, \quad [z^0 : \dots z^n] \mapsto \left(\frac{z^0}{z^\mu}, \dots \frac{z^{\mu-1}}{z^\mu}, \frac{z^{\mu+1}}{z^\mu}, \dots, \frac{z^n}{z^\mu} \right),$$

with inverse given by inserting "1" in the μ-th coordinate,

$$(w^1, \dots, w^n) \mapsto [w^1 : \dots : w^{\mu-1} : 1 : w^\mu : \dots : w^n].$$

It is not hard to show that these maps are indeed continuous, thus each φ_μ is a homeomorphism. The transition functions (assume without loss of generality $\mu < \nu$) $\psi_{\mu\nu} := \varphi_\mu \circ \varphi_\nu^{-1} \colon \mathbb{C}^n \to \mathbb{C}^n$ are given by

$$\psi_{\mu\nu}(w^1, \dots, w^n) = \left(\frac{w^1}{w^\mu}, \dots, \frac{w^{\mu-1}}{w^\mu}, \frac{w^{\mu+1}}{w^\mu}, \dots, \frac{w^{\nu-1}}{w^\mu}, \frac{1}{w^\mu}, \frac{w^\nu}{w^\mu}, \dots, \frac{w^n}{w^\mu} \right),$$

which are holomorphic in each coordinate, as $w^\mu \neq 0$, since it is in $\varphi_\nu(U_\mu \cap U_\nu)$.

A.11 Remark. It is not obvious from the above example that the complex projective space is a compact space. There is an equivalent way of seeing the projective space as the quotient

$$\mathbb{CP}^n = S^{2n+1}/S^1,$$

where, as usual, S^{2n+1} are the elements of norm 1 in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ with its Euclidean metric, and the action of S^1 in S^{2n+1} is the same as the one in the example above but restricted to $S^1 \subset \mathbb{C}^*$. Namely,

$$\lambda \cdot (z^0, \ldots, z^n) = (\lambda z^0, \ldots, \lambda z^n) \qquad \lambda \in S^1, (z^0, \ldots, z^n) \in S^{2n+1}.$$

Since the sphere S^{2n+1} is compact the projective space will also be compact.

A.12 Definition. Let M be a complex manifold. A map $f\colon M \to \mathbb{C}^m$ is called **holomorphic** if

$$f \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha) \to \mathbb{C}^m$$

is holomorphic for all charts of M.

A.13 Definition. We call a map $f\colon M \to N$ between complex manifolds **holomorphic** if

$$\psi_\beta \circ f\colon M \to \psi_\beta(V_\beta)$$

is holomorphic for all charts (V_β, ψ_β) on N.

A.14. Consider M a complex manifold of dimension $\dim_{\mathbb{C}} M = n$. Following the exposition of [GRIFFITH, HARRIS, 1978], for a point $p \in M$ on a chart U with coordinates $z^\mu(p) = x^\mu(p) + iy^\mu(p)$, $1 \leq \mu \leq n$, we have three notions of tangent space at p:

1. Considering M as smooth manifold of real dimension $2n$, we have the usual **real tangent space** $(T_pM)_{\mathbb{R}}$, or simply T_pM, that is regarded as the space of $\mathbb{R}$-linear derivations on the ring of $\mathbb{R}$-valued smooth functions on U. It is defined similarly to paragraph A.2 as

$$T_pM = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n} \right\}.$$

2. The **complex tangent space** $(T_pM)_{\mathbb{C}} := (T_pM) \otimes \mathbb{C}$ is the complexification of the real tangent space. It is realized as the space of $\mathbb{C}$-linear derivations of the

ring of $\mathbb{C}$-valued smooth functions on U. We write

$$\begin{aligned}(T_pM)_{\mathbb{C}} &= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right\} \\ &= \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^n}\right\}.\end{aligned}$$

3. The **holomorphic tangent space** $(T_pM)^{1,0}$ defined as

$$\operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\right\} \subset (T_pM)_{\mathbb{C}}.$$

It is realized as the set of $\mathbb{C}$-linear derivations that vanish on antiholomorphic functions, i.e., $\mathbb{C}$-valued functions on M such that $\partial \bar{f}/\partial \bar{z}^\mu = 0$, for all μ. Setting $(T_pM)^{0,1} := \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n}\right\}$, we obtain the **antiholomorphic tangent space**, and

$$(T_pM)_{\mathbb{C}} = (T_pM)^{1,0} \oplus (T_pM)^{0,1}.$$

Since $(T_pM)_{\mathbb{C}}$ is given as the complexification of the real tangent space, the operation of conjugation is well-defined, and we have

$$(T_pM)^{0,1} = \overline{(T_pM)^{1,0}}.$$

It follows that the projection

$$T_pM \to (T_pM)_{\mathbb{C}} \to (T_pM)^{1,0} \tag{A.2}$$

is an $\mathbb{R}$-linear isomorphism.

A.15 Proposition. *The separation $(T_pM)_{\mathbb{C}} = (T_pM)^{1,0} \oplus (T_pM)^{0,1}$ is independent of the choice of coordinate chart.*

Proof. See discussion in paragraph A.26. □

A.16. Let M and N be complex manifolds, of complex dimension m and n, respectively, and $f\colon M \to N$ be a smooth map between the underlying smooth manifolds. We have induced $\mathbb{R}$-linear maps between the tangent spaces

$$df_p\colon T_pM \to T_{f(p)}N,$$

and hence, $\mathbb{C}$-linear maps between the complexified tangent spaces

$$df_p\colon (T_pM)_{\mathbb{C}} \to (T_{f(p)}N)_{\mathbb{C}}.$$

A.17 Proposition. *The map $f\colon M \to N$ as in the previous paragraph is holomorphic if and only if $df_p((T_pM)^{1,0}) \subset (T_{f(p)}N)^{1,0}$. In particular, $df_p\colon (T_pM)^{1,0} \to (T_{f(p)}N)^{1,0}$ is $\mathbb{C}$-linear.*

Proof. Consider coordinate charts on M^m and N^n with $\varphi_\alpha(p) = (z^1(p), \ldots, z^m(p))$ and $\psi_\beta(f(p)) = (w^1(f(p)), \ldots, w^n(f(p)))$. Then,

$$df_p\left(\frac{\partial}{\partial z^\mu}\right) = \sum \frac{\partial w^\nu}{\partial z^\mu}\frac{\partial}{\partial w^\nu} + \frac{\partial \bar{w}^\nu}{\partial z^\mu}\frac{\partial}{\partial \bar{w}^\nu},$$

so $df_p((T_pM)^{1,0}) \subset (T_{f(p)}N)^{1,0}$ if and only if $(\partial \bar{w}^\nu/\partial z^\mu) = 0$ for every ν, and this happens if and only if f is holomorphic. □

A.18. Still considering a map between complex manifolds as in A.16, let $\varphi(p) = (z^1(p), \ldots, z^n(p))$ and $\psi(f(p)) = (w^1(f(p)), \ldots, w^m(f(p)))$ be coordinates around p and $f(p)$ on charts of M and N, respectively. Assume that f is now holomorphic. Corresponding to the several notions of tangent spaces seen in A.14, we also have different notions of the **Jacobian** of f.

1. Regarding the the real tangent spaces, write $z^\mu = x^\mu + iy^\mu$ and $w^\nu = u^\nu + iv^\nu$. We have basis for T_pM and $T_{f(p)}N$ given by $\{\partial_{x^\mu}, \partial_{y^\mu}\}$, $1 \le \mu \le m$ and $\{\partial_{u^\nu}, \partial_{v^\nu}\}$, $1 \le \nu \le n$, where we abbreviate $\frac{\partial}{\partial x^\mu}$ by ∂_{x^μ}, and so on. Then, the real Jacobian of f is given by the $2n \times 2m$ matrix

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \dfrac{\partial u^\nu}{\partial x^\mu} & \dfrac{\partial u^\nu}{\partial y^\mu} \\ \dfrac{\partial v^\nu}{\partial x^\mu} & \dfrac{\partial v^\nu}{\partial y^\mu} \end{pmatrix}.$$

2. In terms of the basis $\{\partial_{z^\mu}, \partial_{\bar{z}^\mu}\}$, $1 \le \mu \le \dim_{\mathbb{C}} M$ and $\{\partial_{w^\nu}, \partial_{\bar{w}^\nu}\}$, $1 \le \nu \le \dim_{\mathbb{C}} N$, of $(T_pM)_{\mathbb{C}}$ and $(T_{f(p)}N)_{\mathbb{C}}$, we have the complex Jacobian

$$J_{\mathbb{C}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}, \tag{A.3}$$

where

$$J(f) = \frac{\partial w^\nu}{\partial z^\mu}.$$

A.19 Remark. A complex manifold gets a local canonical orientation using the charts. These local orientations are well-defined since the Jacobian of the transition functions have the shape given by (A.3), and the determinant is given by

$$\det(J_{\mathbb{C}}(\psi_{\alpha\beta})) = \det(J(\psi_{\alpha\beta}))\overline{\det(J(\psi_{\alpha\beta}))} = |\det(J(\psi_{\alpha\beta}))|^2 > 0.$$

Hence, a complex manifold is in particular an oriented manifold.

A.20. Another useful consequence of the Jacobian on holomorphic functions is to derive a formula for the change of coordinates on complex manifolds. Consider overlapping charts (U, φ) and (V, ψ) with coordinates $\varphi(p) = (z^1(p), \ldots, z^n(p))$ and $\psi(p) = (w^1(p), \ldots, w^n(p))$, on a point $p \in U \cap V$ of a complex manifold of complex dimension n. Using (A.3), we have

$$\begin{aligned} \frac{\partial}{\partial z^\mu} &= \sum \frac{\partial w^\nu}{\partial z^\mu}\frac{\partial}{\partial w^\nu} \\ \frac{\partial}{\partial \bar{z}^\mu} &= \sum \frac{\partial \bar{w}^\nu}{\partial \bar{z}^\mu}\frac{\partial}{\partial \bar{w}^\nu} \\ dw^\mu &= \sum \frac{\partial w^\nu}{\partial z^\mu} dz^\nu \\ d\bar{w}^\mu &= \sum \frac{\partial \bar{w}^\nu}{\partial \bar{z}^\mu} d\bar{z}^\nu. \end{aligned} \tag{A.4}$$

A.21 Definition. Let M be a complex manifold. A **complex vector field** is a smooth assignment $p \in M \mapsto v_p \in (T_pM)_{\mathbb{C}}$ of a tangent vector in $(T_pM)_{\mathbb{C}}$ for each p. We denote to the set of all complex vector fields by $\mathscr{X}(M)_{\mathbb{C}}$. Similarly to the decomposition of $(T_p^*M)_{\mathbb{C}}$, we have $\mathscr{X}(M)_{\mathbb{C}} = \mathscr{X}(M)^{1,0} \oplus \mathscr{X}(M)^{0,1}$.

A.2 Almost Complex Structure

A.22. Let M be a complex manifold and define a linear operator on the real tangent space $J_p \colon T_pM \to T_pM$, by

$$J_p\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial}{\partial y^\mu}, \qquad J_p\left(\frac{\partial}{\partial y^\mu}\right) := -\frac{\partial}{\partial x^\mu},$$

for every μ. Note that $J_p^2 = -\mathbb{1}$. Naively, J_p corresponds to the multiplication by i. Although we used coordinates to give an expression to J_p, it does not depend on the chosen coordinates. To see this, consider overlapping charts U, V with coordinates $z^\mu = x^\mu + iy^\mu$ and $w^\mu = u^\mu + iv^\mu$, for $1 \leq \mu \leq \dim_{\mathbb{C}} M$. On $U \cap V$, we have

$$J_p\left(\frac{\partial}{\partial u^\mu}\right) = J_p\left(\sum \frac{\partial x^\nu}{\partial u^\mu}\frac{\partial}{\partial x^\nu} + \frac{\partial y^\nu}{\partial u^\mu}\frac{\partial}{\partial y^\nu}\right) = \sum \frac{\partial y^\nu}{\partial v^\mu}\frac{\partial}{\partial y^\nu} + \frac{\partial x^\nu}{\partial v^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial v^\mu},$$

where use has been made of the Cauchy-Riemann equation $\frac{\partial z^\nu}{\partial u^\mu} = -i\frac{\partial z^\nu}{\partial v^\mu}$. With a similar calculation we obtain that the other expression, $J_p(\partial/\partial v^\mu) = -(\partial/\partial u^\mu)$, is also independent of the chart. The operator J_p has the following matrix representation

$$J_p = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{A.5}$$

Since all the components of J_p are constant at any point, we may define a smooth tensor field J whose components at p are given by (A.5).

A.23 Definition. The tensor field J defined in the previous paragraph is called **almost complex structure** of a complex manifold M.

A.24 Remark. We note that any $2m$-dimensional manifold locally admits a tensor field J which squares to $-\mathbb{1}$. However, J may be patched across charts and defined globally only on a complex manifold. The tensor J completely specifies the complex structure.

A.25. The almost complex structure is extended so that it can be defined on $(T_pM)_{\mathbb{C}}$. Consider a vector $u = v + iw$ in $(T_pM)_{\mathbb{C}}$, with $v, w \in (T_pM)$. Set

$$J_p\left(v + iw\right) = J_p(v) + iJ_p(w),$$

therefore we have $J_p(\partial/\partial z^\mu) = i(\partial/\partial z^\mu)$ and $J_p(\partial/\partial \bar{z}^\mu) = -i(\partial/\partial \bar{z}^\mu)$. With respect to the holomorphic and antiholomorphic basis, the operator J_p has the following matrix representation

$$J_p = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}. \tag{A.6}$$

If a vector in $v \in (T_pM)_{\mathbb{C}}$ is given by $v = \sum v^\mu(\partial/\partial z^\mu)$, then $J_p(v) = iv$. Similarly, if $w = \sum w^\mu(\partial/\partial\bar{z}^\mu)$, then $J_p(v) = -iw$. We then recover the separation $(T_pM)_\mathbb{C} = (T_pM)^{1,0} \oplus (T_pM)^{0,1}$, by identifying $(T_pM)^{1,0}$ with the $(+i)$-eigenspace and $(T_pM)^{0,1}$ with the $(-i)$-eigenspace. We also denote these eigenspaces by $(T_pM)^\pm$, accordingly, and we have $(T_pM)_\mathbb{C} = (T_pM)^+ \oplus (T_pM)^-$. We define projections $\mathcal{P}^\pm : (T_pM)_\mathbb{C} \to (T_pM)^\pm$ by

$$\mathcal{P}^\pm := \frac{1}{2}(\mathbb{1} \mp iJ_p).$$

In fact, observe that $J_p\mathcal{P}^+ = (1/2)(J_p - iJ_p^2) = i(1/2)(\mathbb{1} - iJ_p) = i\mathcal{P}^+$, and similarly $J_p\mathcal{P}^- = -i\mathcal{P}^-$.

A.26. The introduction of the almost complex structure of a complex manifold has some nice features. First, the projection $\mathcal{P}^+$ is precisely the projection that yields the $\mathbb{R}$-isomorphism in (A.2) between the real and holomorphic tangent spaces. Also, we now have an easy way to prove proposition A.15. Since the projections $\mathcal{P}^\pm$ do not depend on the coordinate charts, as $\mathbb{1}$ and J_p do not, dimensional reasons tells us that $(T_pM)_\mathbb{C}$ is generated by $\mathcal{P}^+((T_pM)_\mathbb{C}) + \mathcal{P}^-((T_pM)_\mathbb{C})$, and the vectors lying in the intersection $\mathcal{P}^+((T_pM)_\mathbb{C}) \cap \mathcal{P}^-((T_pM)_\mathbb{C})$ are the ones which are simultaneously in the $(+i)$ and $(-i)$-eigenspace, so it is only the zero vector.

A.3 Complex Differential Forms

A.27. Consider M a smooth manifold of dimension n and recall that the set of r-forms at a point $p \in M$ is given by $\Lambda^r\, T_p^*M$. We define a **complex r-form at** p as an element in the complexification

$$(\Lambda^r\, T_p^*(M))_\mathbb{C} := \Lambda^r\, T_p^*M \otimes \mathbb{C}.$$

More concretely, a complex r-form may be represented as $\xi_p = \omega_p + i\eta_p$, for real p-forms $\omega_p, \eta_p \in \Lambda^r\, T_p^*M$. The conjugation satisfies $\bar{\xi}_p = \omega_p - i\eta_p$. Now, a smooth assignment of a complex r-form at each $p \in M$ is a **complex differential r-form** and the set of complex differential r-forms is denoted by $(\Omega^r(M))_\mathbb{C}$. A complex differential r-form is uniquely decomposed as $\xi = \omega + i\eta$, with $\omega, \eta \in \Omega^r(M)$. Also, the exterior product of two complex differential forms $\xi = \omega + i\eta$ and $\zeta = \varphi + i\psi$ is given by

$$\begin{aligned}\xi \wedge \zeta &= (\omega + i\eta) \wedge (\varphi + i\psi) \\ &= (\omega \wedge \varphi - \eta \wedge \psi) + i(\omega \wedge \psi + \eta \wedge \varphi)\end{aligned}$$

Some properties of real differential forms are valid in complex differential forms. Namely, let $\xi = (\omega + i\eta) \in (\Omega^q(M))_{\mathbb{C}}$, $\zeta = (\varphi + i\psi) \in (\Omega^r(M))_{\mathbb{C}}$ and d be the exterior derivative. Then, we have

$$\begin{aligned} d\xi &= d(\omega + i\eta) = d\omega + id\eta \\ \xi \wedge \zeta &= (-1)^{qr}\zeta \wedge \xi \\ d(\xi \wedge \zeta) &= d\xi \wedge \zeta + (-1)^q \xi \wedge d\zeta. \end{aligned} \tag{A.7}$$

A.28. Now consider M a complex manifold, with complex dimension n. Let $\xi_p \in (\Lambda^m\ T_p^*(M))_{\mathbb{C}}$ and r, q be positive integers such that $m = r + q$. Consider vectors $v_1, \ldots, v_m \in (T_pM)_{\mathbb{C}}$ which are either in $(T_pM)^{1,0}$ or in $(T_pM)^{0,1}$.

A.29 Definition. If $\xi_p(v_1, \ldots, v_m) = 0$ unless r vectors among the $\{v_i\}$ are in $(T_pM)^{1,0}$ and q vectors among $\{v_i\}$ are in $(T_pM)^{0,1}$, we say that ξ_p is an (r,q)**-form**, or a form of **type** (r,q). The set of (r,q)-forms at a point p is denoted by $\Lambda^{r,q}\ T_p^*M$. A smooth assignment of an (r,q)-form on $\Lambda^{r,q}\ T_p^*M$, for each $p \in M$ is called a **differential** (r,q)**-form**. The set of differential (r,q)-forms is denoted by $\Omega^{r,q}(M)$.

A.30. Take a chart (U, φ) with coordinates $\varphi(p) = (z^1(p), \ldots, z^n(p))$, for $p \in U$. In the coordinate basis of $(T_p^*M)_{\mathbb{C}}$, $\{dz^\mu, d\bar{z}^\mu\}$, $1 \leq \mu \leq n$, immediately from the definition we have that dz^μ is of type $(1,0)$ and $d\bar{z}^\mu$ is of type $(0,1)$. We write an (r,q)-form ω locally, on the coordinate basis, as

$$\omega = \sum \omega_{I\bar{J}} dz^I \wedge d\bar{z}^J = \sum \omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_q} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_q}, \tag{A.8}$$

where

$$\omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_q} = \omega \left(\frac{\partial}{\partial z^{\mu_1}}, \ldots, \frac{\partial}{\partial z^{\mu_r}}, \frac{\partial}{\partial \bar{z}^{\nu_1}}, \ldots, \frac{\partial}{\partial \bar{z}^{\nu_q}} \right).$$

We will clearly prefer the notation $\omega_{I\bar{J}} dz^I \wedge d\bar{z}^J$, with $I = \{\mu_1 < \cdots < \mu_r\}$ and $\bar{J} = \{\nu_1 < \cdots < \nu_q\}$ (the "bar" is just to denote the antiholomorphic part). Suppose that (V, ψ) is another chart with coordinates $\psi(p) = (w^1(p), \ldots, w^n(p))$. Using the formula for change of complex variables (A.4), we get

$$\begin{aligned} \omega &= \sum \omega_{I\bar{J}} dz^I \wedge d\bar{z}^J \\ &= \sum \omega_{I\bar{J}} \left(\sum \frac{\partial z^{\mu_1}}{\partial w^\lambda} dw^\lambda \right) \wedge \ldots \wedge \left(\sum \frac{\partial z^{\mu_p}}{\partial w^\lambda} dw^\lambda \right) \wedge \left(\sum \frac{\partial \bar{z}^{\nu_1}}{\partial \bar{w}^\eta} d\bar{w}^\eta \right) \wedge \ldots \wedge \left(\sum \frac{\partial \bar{z}^{\nu_q}}{\partial \bar{w}^\eta} d\bar{w}^\eta \right) \\ &= \sum \omega_{I\bar{J}} \det \left(\frac{\partial z^{\mu_i}}{\partial w^{\lambda_k}} \right)_K \det \left(\frac{\partial z^{\nu_i}}{\partial w^{\eta_l}} \right)_L dw^K \wedge d\bar{w}^L, \end{aligned}$$

where the last sum is over all the multi-indices I, J, K, L, and $K = \{\lambda_1 < \ldots < \lambda_r\}$, $L = \{\eta_1 < \cdots < \eta_q\}$, so the type is independent of the chart chosen. We summarize some properties in the following proposition.

A.31 Proposition. *Consider M a complex manifold of complex dimension n and ω, ξ complex differential forms of M. Then*

1. *If $\omega \in \Omega^{r,q}(M)$, then $\bar{\omega} \in \Omega^{q,r}$;*

2. *If $\omega \in \Omega^{r,q}(M)$ and $\xi \in \Omega^{r',q'}(M)$, then $\omega \wedge \xi \in \Omega^{r+r',q+q'}$;*

3. *A complex differential m-form ω is uniquely written as*

$$\omega = \sum_{r+q=m} \omega^{(r,q)},$$

where $\omega^{(r,q)} \in \Lambda^{r,q}\, T_p^* M$. We thus have a decomposition

$$(\Omega^m(M))_{\mathbb{C}} = \sum_{r+q=m} \Omega^{r,q}(M);$$

4. *For $p \in M$, we have*

$$\dim_{\mathbb{R}} \Lambda^{r,q}\, T_p^* M = \begin{cases} \binom{m}{r}\binom{m}{q}, & \text{if } 0 \le r, q \le m \\ 0, & \text{otherwise,} \end{cases}$$

and $\dim_{\mathbb{R}}(\Lambda^m\, T_p^*(M))_{\mathbb{C}} = \sum_{r+q=m} \dim_{\mathbb{R}} \Lambda^{r,q}\, T_p^* M = \binom{2n}{m}$.

Proof. Cf. [NAKAHARA, 2003], Proposition 8.1. □

A.4 Dolbeault Operators

A.32. Following [NAKAHARA, 2003], consider a differential (p, q)-form ω written in local coordinates as $\omega = \sum \omega_{I\bar{J}} dz^I \wedge d\bar{z}^J$, like in (A.8). The action of the exterior derivative d in ω is

$$d\omega = \sum \left(\frac{\partial}{\partial z^\lambda} \omega_{I\bar{J}} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda} \omega_{I\bar{J}} d\bar{z}^\lambda \right) \wedge dz^I \wedge d\bar{z}^J,$$

a mixture of $(p+1,q)$- and $(p,q+1)$-forms. We separate the action of the exterior differential d according to its destinations,

$$d = \partial + \bar{\partial},$$

with $\partial\colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. These operators are called the **Dolbeault operators**. These differential operators have similar properties as in (A.7).

A.33 Definition. Let M be a complex manifold and $\omega \in \Omega^{p,0}(M)$ a $(p,0)$-form. If $\bar{\partial}\omega = 0$, we say that ω is a **holomorphic p-form**.

A.34. As $0 = d^2 = (\partial+\bar{\partial})(\partial+\bar{\partial}) = \partial^2 + (\partial\bar{\partial}+\bar{\partial}\partial) + \bar{\partial}^2$, we have $\bar{\partial}^2 = 0$, so we can talk about cohomology. A differential (p,q)-form ω is called **$\bar{\partial}$-closed** or a **(p,q)-cocycle** if it satisfies $\bar{\partial}\omega = 0$, and is called **$\bar{\partial}$-exact** or a **(p,q)-coboundary** if there exists a differential $(p,q-1)$-form ξ with $\omega = \bar{\partial}\xi$. As usual, we denote by $Z^{p,q}_{\bar{\partial}}(M)$ to the set of (p,q)-cocycles and $B^{p,q}_{\bar{\partial}}(M)$ to the set of (p,q)-coboundaries, and the complex vector space

$$H^{p,q}_{\bar{\partial}}(M) = \frac{Z^{p,q}_{\bar{\partial}}(M)}{B^{p,q}_{\bar{\partial}}(M)},$$

is called the **(p,q)th $\bar{\partial}$ cohomology group**. The following theorem is an important and rather difficult result on analysis of several complex variables, that can be used on complex manifolds. We will not make much use of it in this work, but it would be a crime if we made no reference on so beautiful a result. A proof can be found in [HUYBRECHTS, 2005], Proposition 1.3.8.

A.35 Theorem ($\bar{\partial}$-Poincaré lemma)**.** *Let U be an open neighborhood of the closure of the polydisc $B_\epsilon \subset \overline{B}_\epsilon \subset U \subset \mathbb{C}^n$. If $\omega \in \Omega^{p,q}(U)$ is $\bar{\partial}$-closed and $q > 0$, then there exists $\xi \in \Omega^{p,q-1}(B_\epsilon)$ such that $\omega = \bar{\partial}\xi$ on B_ϵ.*

A.5 Hermitian and Kähler Manifolds

A.36. When talking about Hermitian metrics on a complex manifold there are three things that are very closely related, and we will discuss in this section. Namely, they are the *Hermitian metric*, the *associated Riemannian metric*, and the *associated* $(1,1)$-*form*. In order to discuss these objects we begin with a bit of linear algebra. Consider

V a complex vector space. Recall that a Hermitian inner product on V is a map $H\colon V \times V \to \mathbb{C}$ that satisfies

1. $H(u+v,w) = H(u,w) + H(v,w)$;
2. $H(\lambda v,w) = \lambda H(v,w)$;
3. $H(v,w) = \overline{H(w,v)}$;
4. $H(v,v) \geq 0$ and is equal to zero iff $v = 0$,

(more details on inner product spaces can be found in [HOFFMAN, KUNZE, 1971]). Given a Hermitian inner product H on V, set $G = \Re(H)$ and $W = \Im(H)$, in other words, we have $H = G + iW$. Then G becomes a real bilinear symmetric form on V and W a real bilinear alternating form on V. Indeed,

$$H(u,v) = G(u,v) + iW(u,v) = G(v,u) - iW(v,u) = \overline{H(v,u)},$$

and clearly $G, W\colon V \times V \to \mathbb{R}$. Now, $H(iu,iv) = H(u,v)$ because of properties 2. and 3. of the definition of Hermitian inner product, hence G and W have the same property. Thus, starting with H, we defined real bilinear forms that remain invariant by the multiplication by i. The not-so-clear feature is that we could start with any of the three, H, G or W and define the other two. Indeed, Given G such that $G(iu,iv) = G(u,v)$, set

$$H_G(u,v) = G(u,v) + iG(u,iv) \quad \text{and} \quad W_G(u,v) = G(u,iv),$$

and given W with the same property, set

$$H_W(u,v) = W(iu,v) + iW(u,v) \quad \text{and} \quad G_W(u,v) = W(iu,v).$$

A.37 Definitions. Let M be a complex manifold. A **Hermitian metric** on M is a smooth choice of a positive definite Hermitian inner product

$$h\colon (T_pM)^{1,0} \times (T_pM)^{1,0} \to \mathbb{C},$$

on each holomorphic tangent space. From the identification of T_pM with $(T_pM)^{1,0}$, the Riemannian metric $g = \Re(h)$ on (T_pM) is called the **associated Riemannian form**, and as $\Im(h)$ is alternating on T_pM, $\omega = -\frac{1}{2}\Im(h)$ is called the **associated $(1,1)$-form** or the **fundamental form**.

A.38 Remark. In the above definition, we used strongly the fact that T_pM and $(T_pM)^{1,0}$ are naturally isomorphic via $\mathcal{P}^+$. Thus, when we say $g = \Re(h)$ is a Riemannian metric on M, we should have written $g(v,w) = \Re\Big(h\big((\mathcal{P}^+)^{-1}v, (\mathcal{P}^+)^{-1}w\big)\Big)$. But we will be sloppy, and just write $g(v,w) = \Re\big(h(v,w)\big)$, and one should understand that the vectors are identified.

A.39 Remark. From (A.6), we get that $Jv = iv$, for every $v \in (T_pM)^{1,0}$, thus the action of the almost complex structure is equivalent to the multiplication by i. Therefore, similarly to what we saw in A.36, we have $h(Jv, Jw) = h(v,w)$, for vector fields v, w on M, and we can also determine g, h, ω, by specifying one of the three. In particular, given $g\colon T_pM \times T_pM \to \mathbb{R}$, with $g(Jv, Jw) = g(v,w)$, set,

$$h(v,w) = g(v,w) + ig(v, Jw). \tag{A.9}$$

A.40. So far, we have a complex manifold M, and for each $p \in M$ we have a vector space T_pM that can be given a complex vector space structure by considering the almost complex structure J. Complexifying, we obtain $(T_pM)_{\mathbb{C}} = T_p^{1,0}M \oplus T_p^{0,1}M$, and we saw that naturally, $T_pM \cong T_p^{1,0}M$. If we have a Riemannian metric g on the underlying real manifold M that is *compatible with the almost complex structure* J, in the sense that $g(Jv, Jw) = g(v,w)$ for any vector fields v, w, we saw that we can define a Hermitian metric h on T_pM and on the holomorphic tangent space $T_p^{1,0}M$ by considering (A.9). Conversely, if we are given a Hermitian metric on M, we find a Riemannian one, by considering $g = \Re(h)$. Now, starting with the Riemannian metric g on T_pM that is compatible with J, we could have defined a Hermitian extension of $g_{\mathbb{C}}$ to the complexified tangent space $(T_pM)_{\mathbb{C}}$ by setting

$$g_{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) = \lambda\bar{\mu} g(v,w), \qquad v, w \in T_pM, \lambda, \mu \in \mathbb{C},$$

and we could ask ourselves if there is any relation between the Hermitian extension of $g_{\mathbb{C}}$ restricted to the holomorphic tangent space and h given by (A.9):

A.41 Proposition. *Let V be a vector space endowed with an inner product $G\colon V \times V \to \mathbb{R}$ and $J\colon V \to V$ be a compatible almost complex structure. Under the isomorphism $V \cong V^{1,0}$,*

$$\frac{1}{2}H = G_{\mathbb{C}}\Big|_{V^{1,0}},$$

where $G_{\mathbb{C}}$ is the Hermitian extension of G to the complexified $V_{\mathbb{C}}$, and H is the Hermi-

tian form associated to G.

Proof. Recall that the isomorphism between V and $V^{1,0}$ is given by $v \mapsto \frac{1}{2}(v-iJv)$. Thus,

$$\begin{aligned} G_{\mathbb{C}}\left(\frac{1}{2}(v-iJv), \frac{1}{2}(w-iJw)\right) &= \frac{1}{4}(G(v,w)+iG(v,Jw)-iG(Jv,w)+G(Jv,Jw)) \\ &= \frac{1}{4}(2G(v,w)+2iG(v,Jw)) \\ &= \frac{1}{2}H(v,w). \end{aligned}$$

□

A.42. Since a complex manifold M is in particular a smooth manifold, standard arguments using partition of unity grants the existence of Riemannian metrics g on M. One could also make use of the Whitney's embedding theorem to embed M in $\mathbb{R}^N$, for large enough N, and then obtain a metric in M by pulling back the Euclidean metric to M. In both ways the existence of Riemannian metrics in complex manifolds is guaranteed. Now, given any Riemannian metric g on M, and $v, w \in T_pM$, define

$$\hat{g}_p(v,w) := \frac{1}{2}\big(g_p(J_pv,w) - g_p(v,J_pw)\big).$$

A small calculation gives us $\hat{g}_p(J_pv, J_pw) = \hat{g}_p(v,w)$, and therefore, every complex manifold M admits a Hermitian metric, from what we saw in (A.9).

A.43. In term of coordinates, let (U,φ) be a chart with $\varphi(p) = (z^1(p), \ldots, z^n(p))$. Let

$$h_{\mu\nu} = h\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right),$$

thus we write h in terms of the basis $\{dz^\mu, d\bar{z}^\nu\}$ of $\left((T_pM)^{1,0}\otimes\overline{(T_pM)^{1,0}}\right)^* = (T_p^*M)^{1,0}\otimes (T_p^*M)^{0,1}$ as

$$h = \sum h_{\mu\nu} dz^\mu \otimes d\bar{z}^\nu.$$

Now, if we consider normal coordinates, i.e., a basis $\{e_1, \ldots, e_n\}$ of $(T_pM)^{1,0}$ and dual basis $\{\theta^1, \ldots, \theta^n\}$ of $(T_p^*M)^{1,0}$ such that $h(e_\mu, e_\nu) = \delta_{\mu\nu}$, and the induced h on the dual also satisfies $h(\theta^\mu, \theta^\nu) = \delta^{\mu\nu}$, then we have

$$h = \sum_\mu \theta^\mu \otimes \bar{\theta}^\mu.$$

Normal coordinates always exists locally, as we can apply Gram-Schmidt to the coordinate basis. Now, writing the elements of the coframe $\{\theta^1, \ldots, \theta^n\}$ as $\theta^\mu = \alpha^\mu + i\beta^\mu$, we obtain

$$h = \sum_\mu \big((\alpha^\mu \otimes \alpha^\mu + \beta^\mu \otimes \beta^\mu) + i(-\alpha^\mu \otimes \beta^\mu + \beta^\mu \otimes \alpha^\mu)\big),$$

therefore,

$$g = \Re(h) = \sum_\mu \alpha^\mu \otimes \alpha^\mu + \beta^\mu \otimes \beta^\mu,$$

and

$$\omega = -\frac{1}{2}\Im(h) = \sum_\mu \alpha^\mu \wedge \beta^\mu = \frac{i}{2}\sum_\mu \theta^\mu \wedge \bar{\theta}^\mu.$$

A.44. For the complex manifold M with Hermitian metric h, associated fundamental form ω, on a local coframe $\{\theta^\mu\}$, $1 \leq \mu \leq \dim_{\mathbb{C}} M$, with $\theta^\mu = \alpha^\mu + i\beta^\mu$, the volume form of the associated Riemannian metric $g = \Re(h)$ is given by

$$\mathrm{vol} = \alpha^1 \wedge \beta^1 \wedge \ldots \wedge \alpha^n \wedge \beta^n.$$

But, we know that $\omega = \sum_\mu \alpha^\mu \wedge \beta^\mu$, thus

$$\omega \wedge \ldots \wedge \omega = \omega^n = n!\,\mathrm{vol},$$

and we find again that a complex manifold is orientable, since ω never vanishes as h is positive-definite.

A.45 Definition. Let M be a Hermitian manifold endowed with a Hermitian metric h whose associated $(1,1)$-form ω satisfies $d\omega = 0$. Such a manifold is called a **Kähler manifold**, and the metric h is also called **Kähler**, as well as the fundamental form ω.

A.46 Example. Let M be a Riemann surface, i.e., a complex manifold of dimension 1 (Riemann surfaces are also called complex curves, because the complex dimension is 1). From paragraph A.42, M can be given a Hermitian metric h, and by dimensional reasons, the associated $(1,1)$-form will be closed, therefore every Riemann surface admits a Kähler structure.

A.47 Example. The complex projective space is also a Kähler manifold. To see this we will construct a global closed $(1,1)$-form, the **Fubini-Study** form, and see that it is a real positive definite form associated to a Hermitian metric, also called the Fubini-Study metric. Recall the definition of $\mathbb{CP}^n$ with the covering $\{U_j\}$, $j = 0, 1, \ldots, n$, given in example A.10 (here we change the subindex from μ to j, since we reserve the greek symbols to spatial indices). For a point $z = [z^0 : \ldots : z^n]$, consider coordinates $\varphi_j(z) = (w^1, \ldots, w^n)$ and $w^\mu = z^\mu / z^j$. We define, for each j, a $(1,1)$-form on U_j given by

$$\omega_j := \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{\lambda=0}^{n}\left|\frac{z^\lambda}{z^j}\right|^2\right).$$

We first observe that with these definitions, if well defined, this form is indeed closed because $d\partial\bar{\partial} = 0$. To see that the partial forms ω_j glues together to form a global form, ω_{FS}, observe that, on $U_j \cap U_k$

$$\log\left(\sum_{\lambda=0}^{n}\left|\frac{z^\lambda}{z^j}\right|^2\right) = \log\left(\left|\frac{z^k}{z^j}\right|^2\sum_{\lambda=0}^{n}\left|\frac{z^\lambda}{z^k}\right|^2\right) = \log\left(\left|\frac{z^k}{z^j}\right|^2\right) + \log\left(\sum_{\lambda=0}^{n}\left|\frac{z^\lambda}{z^k}\right|^2\right).$$

Now, $(z^k/z^j) = w^k$ is the k-th coordinate on the chart U_j, and

$$\partial\bar{\partial}\log(|w^k|^2) = \partial\left(\frac{d\bar{w}^k}{\bar{w}^k}\right) = 0,$$

so $\omega_j = \omega_k$ on the intersections $U_j \cap U_k$. To see that ω is a real form, note that locally

$$\bar{\omega}_j = -\frac{i}{2\pi}\bar{\partial}\partial\log\left(\sum_{\lambda=0}^{n}\left|\frac{z^\lambda}{z^j}\right|^2\right) = \omega_j,$$

as $\bar{\partial}\partial = -\partial\bar{\partial}$. Finally, we must prove that it is positive definite. Working on a chart, observe that

$$\begin{aligned}
\partial\bar{\partial}\log\left(1 + \sum_{\lambda=1}^{n}|w^\lambda|^2\right) &= \partial\left\{\sum_\nu \frac{\partial}{\partial\bar{w}^\nu}\left(\log\left(1 + \sum_\lambda |w^\lambda|^2\right)\right)d\bar{w}^\nu\right\} \\
&= \partial\left\{\sum_\nu \frac{w^\nu d\bar{w}^\nu}{1 + \sum_\lambda |w^\lambda|^2}\right\} \\
&= \sum_{\mu,\nu}\frac{\partial}{\partial w^\mu}\left(\frac{w^\nu}{1 + \sum_\lambda |w^\lambda|^2}\right)dw^\mu \wedge d\bar{w}^\nu \\
&= \sum_{\mu,\nu}\left(\frac{\delta_{\mu\nu}(1 + \sum_\lambda |w^\lambda|^2) - \bar{w}^\mu w^\nu}{(1 + \sum_\lambda |w^\lambda|^2)^2}\right)dw^\mu \wedge d\bar{w}^\nu.
\end{aligned}$$

All we have to do now is to show that the Hermitian matrix $h = (h_{\mu\nu})$ of the coefficients of the form is positive definite. However, from the *Cauchy-Schwarz* inequality on the standard Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$, considering $w = (w^1, \ldots, w^n) \in \mathbb{C}^n$, $u = (u^1, \ldots, u^n) \in \mathbb{C}^n$ a general vector, and $\tilde{h}_{\mu\nu} = \delta_{\mu\nu}(1 + \sum_\lambda |w^\lambda|^2) - \bar{w}^\mu w^\nu$,

$$\begin{aligned} u^t \cdot \tilde{h} \cdot \bar{u} &= \langle u, u \rangle + \langle u, u \rangle \langle w, w \rangle - \langle u, w \rangle \langle w, u \rangle \\ &= \langle u, u \rangle + \langle u, u \rangle \langle w, w \rangle - |\langle u, w \rangle|^2 > 0. \end{aligned}$$

Therefore, $h = (1 + \sum_\lambda |w^\lambda|^2)^{-2} \cdot \tilde{h}$ is positive definite and we are done.

For $n = 1$, the Fubini-Study form has a nice property, namely,

$$\int_{\mathbb{CP}^1} \omega_{FS} = 1.$$

Also, since $S^2 \cong \mathbb{CP}^1$, $H^2(\mathbb{CP}^1, \mathbb{Z}) = H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$, so $[\omega_{FS}] \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is a generator of this cohomology group. We calculate the integral

$$\begin{aligned} \int_{\mathbb{CP}^1} \omega_{FS} &= \int_{\mathbb{C}} \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |z|^2) \\ &= \int_{\mathbb{C}} \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \\ &= \int_{\mathbb{R}^2} \frac{1}{\pi} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} \\ &= \int_0^\infty \frac{2r dr}{(1 + r^2)^2} \\ &= \int_1^\infty \frac{dt}{t^2} = 1. \end{aligned}$$

A.48 Lemma. *Let M be a Kähler manifold with associated form ω and let $\iota \colon N \hookrightarrow M$ be a complex submanifold. Then N is a Kähler manifold with associated form $\iota^*\omega$, the pullback of ω under ι.*

Proof. Consider J as the almost complex structure of the complex manifold M. Recall that the pullback $\iota^* J$ and $\iota^*\omega$ are defined by

$$\iota^* J(v) = j(d\iota\, v), \quad \iota^*\omega(v, w) = \omega(d\iota\, v, d\iota\, w),$$

where v, w are vector fields in N. Then $\iota^* J$ and $\iota^*\omega$ simply are the restrictions of the operator J and the form ω to TN. As the pullback commutes with the exterior differential, $\iota^*\omega$ is closed. From $\omega(v, w) = g(Jv, w)$, it follows that $\iota^*\omega$

is non-degenerate, as g remains non-degenerate when restricted to TN, and also $J(TY) \subset TY$. $\square$

A.49 Corollary. *Every projective manifold, i.e., that can be embedded in* $\mathbb{CP}^N$ *for large enough* N*, is Kähler.*

Proof. Follows from the existence of the Fubini-Study metric on $\mathbb{CP}^N$, as we saw in example A.47 and the lemma above. $\square$

A.50. Let M be a Kähler manifold with Kähler form ω. This form gives rise to an operator, called the **Lefschetz** operator $L\colon \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M)$, given by

$$L\xi = \omega \wedge \xi.$$

Let

$$\Lambda\colon \Omega^{p,q}(M) \to \Omega^{p-1,q-1}(M) \tag{A.10}$$

be the adjoint of the Lefschetz operator with respect to the inner product given in (2.6). We claim that $\Lambda = *^{-1} \circ L \circ *$. Indeed, take two (p,q)-forms η, ζ, and observe that in the one hand

$$\langle \eta, \Lambda\zeta \rangle = \int_M \eta \wedge *\overline{(\Lambda\zeta)},$$

and on the other hand,

$$\langle L\eta, \zeta \rangle = \int_M \omega \wedge \eta \wedge *\bar{\zeta} = \int_M \eta \wedge *(*^{-1}\bar{\omega} \wedge *\bar{\zeta}) = \int_M \eta \wedge *\overline{(*^{-1} \circ L \circ *)\zeta},$$

where use have been made of the commutation $\omega \wedge \eta = \eta \wedge \omega$, since ω has even degree, the fact that $\omega = \bar{\omega}$ and that for any form ξ, $*\bar{\xi} = \overline{*\xi}$. With these operators defined, we are able to discuss the so-called **Kähler identities**, which we sum up in the following proposition:

A.51 Proposition (Kähler identities)**.** *For a Kähler manifold* M*, the following identities hold true:*

1. $[\bar{\partial}, L] = [\partial, L] = 0$ *and* $[\bar{\partial}^\dagger, \Lambda] = [\partial^\dagger, \Lambda] = 0$
2. $[\bar{\partial}^\dagger, L] = i\partial, [\partial^\dagger, L] = -i\bar{\partial}$ *and* $[\bar{\partial}, \Lambda] = i\partial^\dagger, [\partial, \Lambda] = -i\bar{\partial}^\dagger$

3. $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$, *and* Δ *commutes with* $*, \partial, \bar\partial, L, \partial^\dagger, \bar\partial^\dagger, \Lambda$.

Proof. For item 1., take an r-form η. Then

$$\bar\partial L\eta - L\bar\partial\eta = \bar\partial(\omega \wedge \eta) - \omega \wedge \bar\partial\eta = \bar\partial\omega \wedge \eta = 0,$$

since $d\omega = \partial\omega + \bar\partial\omega = 0$. For the second part of item 1., consider

$$\bar\partial^\dagger \Lambda\eta = (- * \partial *)(*^{-1} L *)\eta = -(*\partial L *)\eta,$$

and

$$\Lambda\bar\partial^\dagger\eta = (*^{-1} L *)(- * \partial *)\eta = -(* L \partial *)\eta,$$

since $\partial * \eta$ is a $(2n-r+1)$-form, so $*^2 = (-1)^{r+1}$ and $L\partial * \eta$ is a $(2n-r+3)$-form, thus $*^{-1} = (-1)^{r+1}*$. Hence,

$$[\bar\partial^\dagger, \Lambda]\eta = -(*[\partial, L]*)\eta = 0.$$

Item 2. is rather technical and a bit messy, so we might as well just indicate [GRIFFITH, HARRIS, 1978], page 111, in the section *The Hodge identities and the Hodge Decomposition*. The proof is done by performing the calculation in $\mathbb{C}^n$, and then conclude the result by using the fact that the Kähler metric osculates to order 2 to the Euclidean metric at each $z \in M$. For item 3., first observe that, from item 2., the following holds true,

$$\partial\bar\partial^\dagger + \bar\partial^\dagger\partial = 0 = \bar\partial\partial^\dagger + \partial^\dagger\bar\partial = \overline{\partial\bar\partial^\dagger + \bar\partial^\dagger\partial}, \tag{A.11}$$

as

$$\begin{aligned} \partial\bar\partial^\dagger + \bar\partial^\dagger\partial &= i\big(\partial[\partial, \Lambda] + [\partial, \Lambda]\partial\big) \\ &= i(\partial\Lambda\partial - \partial\Lambda\partial) = 0. \end{aligned}$$

Also, from the anti-commutativity of the wedge product, we have

$$\partial\bar\partial + \bar\partial\partial = 0. \tag{A.12}$$

Now it is just a matter of straightforward computations.

$$\begin{aligned}\Delta_\partial - \Delta_{\bar\partial} &= (\partial\partial^\dagger + \partial^\dagger\partial) - (\bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial)\\ &= i\big(\partial[\Lambda,\bar\partial] + [\Lambda,\bar\partial]\partial - \bar\partial[\partial,\Lambda] - [\partial,\Lambda]\bar\partial\big)\\ &= i\big(\Lambda(\partial\bar\partial + \bar\partial\partial) - (\partial\bar\partial + \bar\partial\partial)\Lambda\big) = 0,\end{aligned}$$

where use have been made, in the last line, of (A.12). Now, from the definition of the Laplacian,

$$\begin{aligned}dd^\dagger + d^\dagger d &= (\partial + \bar\partial)(\partial^\dagger + \bar\partial^\dagger) + (\partial^\dagger + \bar\partial^\dagger)(\partial + \bar\partial)\\ &= \Delta_\partial + \Delta_{\bar\partial} + (\partial\bar\partial^\dagger + \bar\partial^\dagger\partial) + (\bar\partial\partial^\dagger + \partial^\dagger\bar\partial)\\ &= \Delta_\partial + \Delta_{\bar\partial},\end{aligned}$$

where in the last line we have used (A.11). As for the commutations, the ones with $\partial, \bar\partial, \partial^\dagger$ and $\bar\partial^\dagger$ are trivial, and require no more than a line of calculations, since we write $\Delta = 2\Delta_\partial$ or $\Delta = 2\Delta_{\bar\partial}$. The commutations with L and Λ are similar, so we check only for Λ.

$$\begin{aligned}[\Delta,\Lambda] &= 2[\Delta_\partial,\Lambda]\\ &= 2\big([\partial\partial^\dagger,\Lambda] + [\partial^\dagger\partial,\Lambda]\big)\\ &= 2\big(\partial\partial^\dagger\Lambda - \Lambda\partial\partial^\dagger + \partial^\dagger\partial\Lambda - \Lambda\partial^\dagger\partial\big)\\ &= 2\big(\partial[\partial^\dagger,\Lambda] + \partial\Lambda\partial^\dagger - \Lambda\partial\partial^\dagger + \partial^\dagger\partial\Lambda - [\Lambda,\partial^\dagger]\partial - \partial^\dagger\Lambda\partial\big)\\ &= 2\big([\partial,\Lambda]\partial^\dagger + \partial^\dagger[\partial,\Lambda]\big)\\ &= -2i\big(\bar\partial^\dagger\partial^\dagger + \partial^\dagger\bar\partial^\dagger\big) = 0,\end{aligned}$$

where in the last line we have used (A.11). Finally, note that

$$\begin{aligned}[\Delta,*] &= (dd^\dagger + d^\dagger d)* - *(dd^\dagger + d^\dagger d)\\ &= -(d*d* + *d*d)* + *(d*d* + *d*d)\\ &= -d*d*^2 + *^2 d*d = 0,\end{aligned}$$

where in the last line we used the fact that $d*d$ sends an r-form to a $(2n-r)$-form, so the sign of $*^2$ is the same. □

Apêndice B

Lie Groups and Principal Bundles

We give a review in the basic theory about finite dimensional Lie groups and principal bundles that concerns this work. References for this appendix are [WARNER, 1983] and [KOBAYASHI, NOMIZU, 1963].

B.1 Lie Groups and Lie Algebras

B.1 Definitions. A **Lie group** is a smooth manifold G endowed with a group structure such that the map $G \times G \to G$ given by

$$(g,h) \mapsto gh^{-1}$$

is smooth.

B.2 Remark. A more natural definition would be to define a Lie group as a manifold endowed with a group structure such that the group operations are smooth, that is, the product and the inversion are smooth. Actually, the above definition is equivalent. On the one hand, if the product and the inversion is smooth, than the map $(g,h) \mapsto gh^{-1}$ is clearly smooth, and on the other hand, using the map of the definition, one can write the inversion map as a composition of smooth maps

$$g \mapsto (e,g) \mapsto eg^{-1} = g^{-1},$$

and once the inversion is smooth, we also write the product as a composition of smooth maps

$$(g,h) \mapsto (g,h^{-1}) \mapsto g(h^{-1})^{-1} = gh.$$

B.3. Let G be a Lie group and $p\colon G \times G \to G$ be the smooth product. We define smooth maps $L_g, R_g\colon G \to G$ called **left** and **right translation** defined as the partial maps of the product, that is

$$L_g(h) = gh = R_h(g),$$

where gh is the short-hand notation for $p(g,h)$.

B.4 Remark. There is still another way to define a Lie Group. We could only ask for the product $p\colon G \times G \to G$ to be a smooth map, and use the implicit function theorem to conclude that the inverse $i\colon G \to G$ is a smooth map. The sketch of the argument would be as follows. Assume $p(g,h) = e$, and as an implicit function, $h(g) = i(g)$. The partial derivative of the product p with respect to the second G is equal to $(dL_g)_h\colon T_gG \to T_{gh}G$, which is an isomorphism since L_g is a diffeomorphism. Hence, the map $(dp)_{(g,h)}\colon T_{(g,h)}(G \times G) \to T_{gh}G$ is surjective, so we can define

$$(di_g) = -((dL_g)_g)^{-1} \circ (dR_{g^{-1}})_g,$$

and the inverse i is smooth.

B.5 Example. Perhaps the most important example of a Lie Group is the group of invertible matrices $GL(n,\mathbb{R})$. Elements in $GL(n,\mathbb{R})$ are matrices $g = (g^i_j)$, $h = (h^i_j)$, and the product is given by the matrix multiplication

$$(gh)^i_j = \sum g^i_k h^k_j,$$

which is a polynomial, therefore smooth.

B.6 Definition. A **Lie algebra $\mathfrak{g}$** over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ endowed with a bilinear operator called **Lie bracket** or just **bracket**, $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all $X, Y, Z \in \mathfrak{g}$, we have the following properties:

(a) Anti-commutativity: $[X,Y] = -[Y,X]$;
(b) Jacobi identity: $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$.

B.7 Remark. For the purposes of our work, the field $\mathbb{K}$ in the definition above will be either $\mathbb{R}$ or $\mathbb{C}$.

B.8 Example. Consider $\mathfrak{gl}(n,\mathbb{R})$ as the set of all n-by-n matrices and define the bracket between two matrices A, B as the matrix commutator

$$[A,B] = AB - BA.$$

Then $\mathfrak{gl}(n\mathbb{R})$ is a Lie algebra. The only thing one needs to check is the Jacobi identity,

$$\begin{aligned}
[A,[B,C]] &= A(BC-CB)-(BC-CB)A \\
&= A(BC-CB)-(BC-CB)A+BAC-BAC+CAB-CAB \\
&= (AB-BA)C-C(AB-BA)+B(AC-CA)-(AC-CA)B \\
&= [[A,B]C]+[B,[A,C]].
\end{aligned}$$

B.9 Example. On a smooth manifold M, we endow the set $\mathscr{X}(M)$ of smooth vector fields with a Lie algebra structure by defining the Lie bracket between two vector fields v, w as

$$[v,w](f) = v(w(f)) - w(v(f)),$$

for any $f \in C^\infty(M)$. It is quite easy to see that $[v,w] \in \mathscr{X}(M)$. In local coordinates, we have $v = \sum v^\mu \partial_\mu$ and $w = \sum w^\nu \partial_\nu$, for smooth functions v^μ, w^ν and where, as usual, ∂_μ is the shorthand notation for $\frac{\partial}{\partial x^\mu}$. Then, the bracket $[v,w]$ is given locally by

$$\sum \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu},$$

so the bracket of two smooth vector fields is a smooth vector field. The anti-commutativity of this bracket is evident, and all we are left to see is the Jacobi identity, but

$$\begin{aligned}
[u,[v,w]](f) &= u([v,w](f)) - [v,w](u(f)) \\
&= u(v(w(f))) - u(w(v(f))) - v(w(u(f))) + w(v(u(f))) \\
&= u(v(w(f))) - u(w(v(f))) - v(w(u(f))) + w(v(u(f))) + \\
&\qquad + v(u(w(f))) - v(u(w(f))) + w(u(v(f))) - w(u(v(f))) \\
&= v([u,w](f)) - [u,w](v(f)) + [u,v](w(f)) - w([u,v](f)) \\
&= [[u,v],w](f) + [v,[u,w]](f).
\end{aligned}$$

B.10 Definition. A vector field v, not necessarily smooth, a priori, is called **left-invariant** if $(dL_g) \circ v = v \circ L_g$, and **right-invariant** if $(dR_g) \circ v = v \circ R_g$

B.11 Proposition. *Let G be a Lie group and $\mathfrak{g}$ its set of left-invariant vector fields. Then $\mathfrak{g}$ is isomorphic to T_eG as a real vector space and left-invariant vector fields are actually smooth. Moreover, the Lie bracket of two left-invariant vector fields is, itself, a left-invariant vector field, so actually, $\mathfrak{g}$ is a Lie algebra.*

B.12 Definition. We say that the **Lie algebra of a Lie group** G is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G, which, from proposition B.11, is isomorphic to the tangent space of G at the identity.

B.13. Recall that on a manifolds M, N, if we have a vector field $v \in \mathscr{X}(M)$ and diffeomorphism $\varphi\colon M \to N$, we can pushforward the vector field v to a vector field $\varphi_*(v) \in \mathscr{X}(N)$ by setting

$$\varphi_*(v)(x) = d\varphi_{\varphi^{-1}(x)}(v(\varphi^{-1}(x))), \qquad x \in N.$$

That said, observe that we could have defined the Lie algebra of a Lie group G using right-invariant vector fields. Indeed, it is not so hard to prove that if v is a left-invariant vector field on G, then the vector field $i_*(v)$, where i is the inversion $g \mapsto g^{-1}$, is a right-invariant vector field whose value on the identity is $-v(e)$. Therefore, the map $v \to i_* v$ provides us a Lie algebra isomorphism between the Lie algebras of left and right-invariant vector fields.

B.14. Recall also that given a vector field $v \in \mathscr{X}(M)$ on a manifold M, we have the flux of v as $v_t\colon \mathscr{D}_t \to \mathscr{D}_{-t}$ a diffeomorphism that satisfies $v_{t+s} = v_t \circ v_s$, where $\mathscr{D}_t \subset M$ is the domain of v_t defined as the set of points $x \in M$ where the maximal integral curve of the vector field v passing through x exists at the time t. When $\mathscr{D}_t = M$ for every t, we say that the vector field is **complete**. Now, denoting $w = \varphi_*(v)$, we have that the fluxes are related by

$$\varphi \circ v_t = w_t \circ \varphi.$$

Perhaps a diagram is handy in this case

$$\begin{CD} M \supset \mathscr{D}_t @>{\varphi}>> \mathscr{D}_t \subset N \\ @V{v_t}VV @VV{w_t}V \\ M \supset \mathscr{D}_{-t} @>{\varphi}>> \mathscr{D}_{-t}t \subset N. \end{CD}$$

Now, in the case of a Lie Group G, a left-invariant vector field satisfies $(L_g)_* v = v$, or, in terms of the flux

$$L_g \circ v_t = v_t \circ L_g.$$

Elementary results on existence and uniqueness of solutions of ordinary differential equations assert that there exists $\epsilon > 0$ such that the integral curve of v passing through the identity $e \in G$ is defined for $t \in (-\epsilon, \epsilon)$. For any $0 < s < \epsilon$, the left-invariance allows us to define

$$v_s(g) = v_s(L_g(e)) = L_g(v_s(e)),$$

for all $g \in G$, thus, $\mathscr{D}_s = G$, from where we conclude that left-invariant vector fields are complete, since for any t, the map v_t is defined for every $g \in G$, by iterating $v_t = v_{t/k} \circ \ldots \circ v_{t/k}$, for large enough k such that $t/k < \epsilon$.

B.15 Definition. Given $V \in T_eG$ we associate a left-invariant vector $v \in \mathfrak{g}$ using proposition B.11. From paragraph B.14 the flux of v, v_t is complete, so we define the **exponential map** $\exp\colon T_eG \to G$ by

$$\exp(V) := v_1(e).$$

B.16 Proposition. *The exponential map satisfies the following properties:*

1. $\exp(tV) = v_t(e)$, *for* $t \in \mathbb{R}$ *and* $V \in T_eG$*;*
2. $\exp(t+s)V = \exp(tV)\exp(sV)$, *for* $t, s \in \mathbb{R}$ *and* $V \in T_eG$*;*
3. $\exp(-tV) = (\exp(tV))^{-1}$, *for* $t \in \mathbb{R}$ *and* $V \in T_eG$*;*
4. $\exp$ *is smooth and* $d\exp$ *on* 0 *is the identity, so* $\exp$ *gives us a diffeomorphism of a neighborhood of* $0 \in T_eG$ *into a neighborhood of* $e \in G$*.*

B.17. For a Lie group G and its Lie algebra $\mathfrak{g}$, the exponential map provides a powerful link between them, for instance, under reasonable assumptions such as G connected, it is possible to show that every element $g \in G$ is expressed as a finite product of exponentials $g = \exp(X_1)\cdots\exp(X_p)$, where $X_i \in \mathfrak{g}$ Now, given two Lie groups G and H, we define a **Lie group homomorphism** as a map $\varphi\colon G \to H$ that preserves both the differentiable and the group structure of the objects G and H, that is, φ is a smooth group homomorphism. Also, given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a **Lie algebra homomorphism** is a linear map τ from $\mathfrak{g}$ to $\mathfrak{h}$ that preserves the brackets, in the sense that $\tau[X,Y] = [\tau X, \tau Y]$. The next theorems provide us relations between Lie groups and Lie algebras in the presence of a homomorphism. Their proofs can be found in [WARNER, 1983], on chapter 3.

B.18 Theorem. *Let* G*,* H *be Lie groups with respective Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, and* $\varphi\colon G \to H$ *be a Lie group homomorphism. Then,*

$$d\varphi\colon \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

B.19 Theorem. *Let $\varphi\colon G \to H$ be a Lie group homomorphism. Then*

$$\varphi \circ \exp = \exp \circ (d\varphi).$$

B.20 Example. The Lie algebra of $GL(n, \mathbb{C})$ is $\mathfrak{gl}(n, \mathbb{C})$, and the exponential map $\exp\colon \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$ is given by matrix exponentiation, where

$$\exp(tM) = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}.$$

Indeed, observe that the right-hand side of the above expression converges uniformly for M in a bounded region K of $\mathfrak{gl}(n, \mathbb{C})$, for there is $\mu > 0$ such that $|M^i_j| \leq \mu$ for all matrices M, thus, inductively, $|M^i_j|^k \leq n^{(k-1)}\mu^k$, and from the Weierstrass M-test, each of the series

$$\sum_{k=0}^{\infty} \frac{(M^i_k)^k}{k!},$$

converges uniformly for $M \in K$. This settles the smoothness of exp. From the continuity of the product of matrices, it follows that for matrices P, M, we have

$$\begin{aligned} P \exp(M) P^{-1} &= P \left(\lim_{n\to\infty} \sum_{k=1}^{n} \frac{M^k}{k!} \right) P^{-1} \\ &= \lim_{n\to\infty} \left(P \left(\sum_{k=1}^{n} \frac{M^k}{k!} \right) P^{-1} \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{(PMP^{-1})^k}{k!} \right) = \exp(PMP^{-1}) \end{aligned}$$

thus, as for every matrix M we can find P such that PMP^{-1} is upper triangular, using the Jordan form, and denoting by $\lambda_1, \ldots, \lambda_n$ the diagonal entries of PMP^{-1}, it follows that the diagonal entries of $\exp(PMP^{-1})$ are $e^{\lambda_1}, \ldots, e^{\lambda_n}$ (here e^{λ} denotes the usual exponential of complex numbers), so

$$\det(\exp(PMP^{-1})) = \det(P \exp(M) P^{-1}) = \det(\exp(M)) \neq 0,$$

which shows that indeed exp is a map from $\mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$. Using the smoothness of exp, it is easy to see that the Lie algebra of $GL(n, \mathbb{C})$ is $\mathfrak{gl}(n, \mathbb{C})$. We will show that $T_{\mathbb{1}}GL(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C})$, where $\mathbb{1}$ is the identity. One inclusion is obvious from the definition of $\mathfrak{gl}(n,\mathbb{C})$. As for the other, take $M \in \mathfrak{gl}(n,\mathbb{C})$ and observe that $t \mapsto \exp(tM)$ is a curve on $GL(n,\mathbb{C})$ passing through the identity whose tangent vector at $t = 0$ is precisely M.

B.21. Now consider $H \subset G$ an abstract subgroup of a Lie group G, that is, a subset of G that is only a subgroup. Then H is said to be a **Lie subgroup** if H is a submanifold of G such that the restriction of the product $p\colon G \times G \to G$ to H is smooth with respect to the *intrinsic* differentiable structure of H. If H is embedded in G, then the intrinsic and relative topologies agree. In our work, we always consider nice subgroups of the classical matrix groups, that are embedded in $GL(n,\mathbb{C})$, so the relative topology always works for us. A subgroup is called **closed** if H is a closed set in the topology of G. Closed subgroups of a Lie group G are important because they are always Lie subgroups. This is not a trivial result and we cite [WARNER, 1983] for the details.

B.22 Example. Since $SL(n,\mathbb{R})$ is a closed subset given by $\det^{-1}(1)$ in $GL(n,\mathbb{R})$, it follows that it is a Lie group. The condition that defines the orthogonal group $O(n)$ is a closed one, so the set $O(n)$ is a closed subset of $GL(n,\mathbb{R})$, hence a Lie Group. Similarly, $SL(n,\mathbb{C})$, $U(n)$ and $SU(n) = U(n) \cap SL(n,\mathbb{C})$ are Lie groups, and playing with the exponential of matrices, we can find the Lie algebras of these groups.

$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{R}) &= \{M \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr}(M) = 0\} \\
\mathfrak{sl}(n,\mathbb{C}) &= \{M \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr}(M) = 0\} \\
\mathfrak{o}(n,\mathbb{R}) = \mathfrak{so}(n,\mathbb{R}) &= \{M \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr}(M) = 0, M + {}^{\tau}M = 0\} \\
\mathfrak{u}(n,\mathbb{R}) &= \{M \in \mathfrak{gl}(n,\mathbb{C}) \mid M + M^* = 0\} \\
\mathfrak{su}(n,\mathbb{R}) &= \{M \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr}(M) = 0, M + M^* = 0\}
\end{aligned}$$

B.23. Consider G a Lie group and $g \in G$ an arbitrary element. We define the **conjugation** by g as the smooth Lie group diffeomorphism

$$C_g(h) = ghg^{-1} = E_g \circ D_{g^{-1}}.$$

Since $C_g(e) = e$, from theorem B.18, the derivative of C_g at the identity is a Lie algebra homomorphism $(dC_g)\colon T_eG \to T_eG$ and from the identification of T_eG and $\mathfrak{g}$ it is an endomorphism of $\mathfrak{g}$. Since $D_{g^{-1}}$ and E_g are both diffeomorphims, $(dC_g)_e$ is a composition of isomorphisms, thus an isomorphism of $\mathfrak{g}$. As

$$C_{gh}(x) = ghx(gh)^{-1} = g(hxh^{-1})g^{-1} = C_g \circ C_h(x),$$

we have $(dC_{gh})_e = (dC_g)_e (dC_h)_e$.

B.24 Definition. Given a Lie group G with Lie algebra $\mathfrak{g}$, the **adjoint representation** of G in $\mathfrak{g}$, $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$ is given by

$$Ad(g) = (dC_g)_e \colon \mathfrak{g} \to \mathfrak{g}.$$

Here, $\operatorname{Aut}(\mathfrak{g})$ denotes the set of all Lie algebra automorphisms of $\mathfrak{g}$.

B.25. It is not so hard to prove that Ad is actually smooth (cf. [WARNER, 1983], theorem 3.45), so Ad is a Lie group homomorphism. Taking the derivative of Ad at the identity, we have

$$\operatorname{ad} := (d\operatorname{Ad})_e \colon T_e G \to T_{\mathbb{1}} \operatorname{Aut}(\mathfrak{g}),$$

and $T_{\mathbb{1}} \operatorname{Aut}(\mathfrak{g})$ is subset of the set of all linear endomorphisms of $\mathfrak{g}$, called the **derivations of** $\mathfrak{g}$, $\operatorname{Der}(\mathfrak{g}) = \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D[X,Y] = [DX,Y] + [X,DY]\}$. Then, we have the following proposition, the proof of which can be found in [WARNER, 1983]:

B.26 Proposition. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $X, Y \in \mathfrak{g}$. Then*

$$\operatorname{ad}_X(Y) = (d\operatorname{Ad})_e(X)(Y) = [X,Y].$$

B.27. To end this section, assume we have $G = GL(n)$ (it could be $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$). Let $g \in G$ and $M \in \mathfrak{gl}(n)$. Then, to calculate $\operatorname{Ad}(g)(M)$ we ought to find a smooth curve passing through the the identity such that its derivative at time zero is precisely M. This curve could be $t \mapsto \exp(tM)$, so

$$\operatorname{Ad}(g)(M) = \frac{d}{dt}\Big|_{t=0} C_g(\exp(tM)) = \frac{d}{dt}\Big|_{t=0} \exp(g\exp(tM)g^{-1}) = gMg^{-1},$$

where we used the fact argued in example B.20 that $g\exp(M)g^{-1} = \exp(gMg^{-1})$. Sometimes, even when we are not in the group $GL(n)$, we denote the adjoint action as $\operatorname{Ad}(g)(Z) = gZg^{-1}$.

B.2 Principal Bundles

B.28 Definition. Consider M a smooth manifold and G a Lie group. By a **smooth left action** of G on M, we mean a smooth map $\alpha\colon G \times M \to M$ such that the partial application

$$\rho(g) = \alpha(g, \cdot)\colon M \to M$$

is a diffeomorphism for every $g \in G$ and the map $\rho\colon G \to \mathrm{Diff}(M)$ is a smooth group homomorphism, i.e., $\rho(e) = id_M$ and, for $g, h \in G$, and $x \in M$, we have

$$\alpha(g, \alpha(h, x)) = \alpha(gh, x).$$

We often denote the diffeomorphisms $\rho(g)$ only by g, or L_g and specify the action of g on an element $x \in M$ as $g \cdot x$, or $L_g(x)$.

B.29 Remark. A smooth right action is defined analogously as $\alpha(g, \alpha(h, x)) = \alpha(hg, x)$, and we often denote a right action by $x \cdot g$, or $R_g(x)$. Note that this notation is very useful to work with the group operations, since $g\cdot(h\cdot x) = (gh)\cdot x$ and $(x\cdot h)\cdot g = x\cdot(hg)$.

B.30. A bit of terminology comes in handy; the action α is said to be **transitive** if given $x, y \in M$ there exists $g \in G$ with $x = g \cdot y$. The action is said to be **free** if the identity $e \in G$ is the only element with fixed points, i.e., the **stabilizer of** x

$$G_x = \{g \in G \mid g \cdot x = x\},$$

satisfies $G_x = \{e\}$, for all $x \in G$. Finally, the action is said to be **effective** if the group homomorphism $\rho\colon G \to \mathrm{Diff}(M)$ is injective.

B.31 Definition. Consider M a smooth manifold and G a Lie group. A **principal** G**-bundle** is a smooth manifold P on which G acts smoothly, satisfying:

1. G acts freely on the right;

2. $M = P/G$ and the canonical projection on the quotient $\pi\colon P \to M$ is a smooth submersion;

3. P is locally trivial in the sense that for every $x \in M$ there exists an open set of M, $U \ni x$, and an equivariant diffeomorphism, called **trivialization**, $\varphi_U\colon \pi^{-1}(U) \to$

$U \times G$, $\varphi_U(p) = (\pi(p), \phi_U(p))$, $x = \pi(p)$, where equivariance here means that

$$\varphi_U(p \cdot g) = (\pi(p \cdot g), \phi_U(p \cdot g)) = (\pi(p), \phi_U(p) \cdot g) = (\pi(p), \phi_U(p)) \cdot g = \varphi_U(p) \cdot g.$$

B.32 Remark. As $M = P/G$, it follows that $\pi(p) = \pi(p \cdot g)$, so the fiber over a point $x = \pi(p) \in M$ is $\pi^{-1}(x) = \{p \cdot g \mid g \in G, \pi(u) = x\} \cong G$.

B.33 Example. Consider M a smooth n-manifold M. We will define the **frame bundle of** M, the most important principal bundle for the purposes of this work. As a set, we have

$$B(M) = \{p = (x, e_1, \ldots, e_n) \mid x \in M \text{ and } \{e_1, \ldots e_n\} \text{ is a basis of } T_xM\}.$$

We have a natural projection $\pi \colon B(M) \to M$ as the projection on the first factor, and a free right action of $g = (g^i_j) \in GL(n, \mathbb{R})$ on $p \in B(M)$ given by

$$p \cdot g = (x, e_1, \ldots, e_n) \cdot g = (x, e_1 \cdot g, \ldots, e_n \cdot g),$$

where $e_j \cdot g = \sum g^i_j e_i = e'_j$. If $e_j \cdot g = e_j$ for all j, then $g^i_j = \delta^i_j$, so g is the identity. Observe that this is indeed a right action, since

$$(e_j \cdot g) \cdot h = e'_j \cdot h = \sum h^k_j e'_k = \sum h^k_j g^i_k e_i = \sum (gh)^i_j e_i = e_j \cdot (gh).$$

As for the local triviality, consider (U, φ) a chart with $\varphi(x) = (x^1(x), \ldots, x^n(x))$. Then, with respect to the coordinate basis, a frame $\{e_1, \ldots, e_n\}$ on x satisfies

$$e_j = \sum E^i_j \partial_i, \qquad E = (E^i_j) \in GL(n, \mathbb{R}),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, as usual. We define a trivialization

$$\psi_U \colon \pi^{-1}(U) \to U \times GL(n, \mathbb{R}) \cong \varphi(U) \times GL(n, \mathbb{R}),$$

as $\psi_U(x, e_1, \ldots, e_n) = (x, E)$. It is straightforward to see that the inverse of ψ_U is the map $(x, g) \mapsto (x, \partial_1 \cdot g, \ldots, \partial_n \cdot g)$. Note that $E^i_j = dx^i(e_j)$. Now, for overlapping charts (U, φ_U) and (V, φ_V) with coordinates $\varphi_U(x) = (x^1(x), \ldots, x^n(x))$ and $\varphi_V(x) = (y^1(x), \ldots, y^n(x))$, for $x \in U \cap V$, we have that a point $p = (x, e_1, \ldots, e_n) \in \pi^{-1}(U \cap V)$

satisfies

$$e_j = E^i_j \frac{\partial}{\partial x^i} = \tilde{E}^i_j \frac{\partial}{\partial y^i}.$$

Using the Jacobian matrix of change of coordinates, $J = J(x) \equiv J(\varphi_U \circ \varphi_V^{-1})(x)$ and $J = (J^i_j) = \frac{\partial x^i}{\partial y^j}$, we have

$$e_j = \sum E^i_j \frac{\partial}{\partial x^i} = \sum \tilde{E}^k_j \frac{\partial x^i}{\partial y^k}\frac{\partial}{\partial x^i} = \sum (J\tilde{E})^i_j \frac{\partial}{\partial x^i}.$$

Thus, the compositions $\eta_{UV} = \psi_U \circ \psi_V^{-1}$ sends $(x, g) \mapsto (x, J(x)g)$. Using the isomorphisms $U \cap V \times GL(n, \mathbb{R}) \cong \varphi_U(U \cap V) \times GL(n, \mathbb{R}) \cong \varphi_V(U \cap V) \times GL(n, \mathbb{R})$, we define then a differentiable structure on $B(M)$ as the set of

$$(\pi^{-1}(U), (\varphi_U \times id) \circ \psi_U),$$

where (φ_U, U) is a chart of M, and the change of coordinates is smooth as it is given by

$$(u, g) \mapsto (\varphi_U \circ \varphi_V^{-1}(u), Jg),$$

where $u = \varphi_V(x) \in \mathbb{R}^n$. With this differentiable structure, the projection is smooth and the trivializations are diffeomorphisms.

B.34 Remark. The way we have constructed the frame bundle in the previous example used a very specific vector bundle over M, the tangent bundle. We could have done a similar construction with any vector bundle $E \to M$. The frame bundle $B(E)$ will be the set of $(x, e_1(x), \ldots, e_n(x))$ where $\{e_1, \ldots, e_n\}$ is a frame around x.

B.35. Consider now overlapping **trivializing open sets**, that is, open sets U, V such that $\pi^{-1}(U) \cong U \times G$ and $\pi^{-1}(V) \cong V \times G$ via trivializations $\varphi_U = (\pi, \phi_U)$ and $\varphi_V = (\pi, \phi_V)$. For a point $x \in U \cap V$, we have that for every $p \in \pi^{-1}(x)$

$$\phi_U(p) = \phi_U(p)\phi_V^{-1}(p)\phi_V(p),$$

and we define the **transition functions** $g_{UV} \colon U \cap V \to G$, as

$$g_{UV}(x) = \phi_U(p)\phi_V^{-1}(p), \qquad p \in \pi^{-1}(x).$$

Observe that $\phi_U(p \cdot g)\phi_V^{-1}(p \cdot g) = \phi_U(p)gg^{-1}\phi_V^{-1}(p) = \phi_U(p)\phi_V^{-1}(p)$, so the definition above does not depend on the $p \in \pi^{-1}(x)$ chosen. Note that on triple intersections

$U \cap V \cap W$, we have

$$g_{UW} = \phi_U \phi_W^{-1} = \phi_U \phi_V^{-1} \phi_V \phi_W^{-1} = g_{UV} g_{VW},$$

where the product here is given by the product in G, so they satisfy the cocycle conditions, similarly to vector bundles.

B.36. We also have the notion of a **local section** on an open set $U \subset M$ as a smooth map $s \colon U \to P$ such that $\pi \circ s = id_U$. There is a close relation between sections and trivializations. Given $\varphi_U \colon \pi^{-1}(U) \to U \times G$ define $s_U \colon U \to P$ as

$$s_U(x) = \varphi_U^{-1}(x, e).$$

Observe that $\pi s_U(x) = \pi(\varphi_U^{-1}(x, e)) = x$, as $\varphi_U = (\pi, \phi_U)$, and also that s_U is smooth, since it is given by the composition of smooth functions $x \mapsto (x, e) \mapsto \varphi_U^{-1}(x, e)$. Conversely, given a smooth section s_U on U, we define a trivialization $\varphi_U^{-1} \colon U \times G \to \pi^{-1}(U)$ by

$$\varphi_U^{-1}(x, g) = s_U(x) \cdot g. \tag{B.1}$$

To see that this indeed yields a trivialization there are things to be checked. First, the map in (B.1) is smooth since it is the composition of smooth maps $(x, g) \mapsto (s_U(x), g) \mapsto s_U(x) \cdot g$. Then, the transitivity on the fibers and the freeness of the action of G in P ensures that the map is a bijection. Finally, we claim that it is a local diffeomorphism, therefore a diffeomorphism, since it is bijective. Indeed, observe that $\varphi_U^{-1} = \alpha \circ (s_U \times id)$, where α denotes the action. For dimensional reasons, all we need to check is that $(d\varphi^{-1})_{(x,g)} = d(\alpha \circ (s_U \times id))_{(x,g)}$ is injective, and the claim will follow from the inverse function theorem. First, the matrix of the derivative of $(s_U \times id)$ at a point (x, g) has a block matrix representation as

$$d(s_U \times id)_{(x,g)} = \begin{pmatrix} (ds_U)_x & 0 \\ 0 & id \end{pmatrix},$$

and the derivative of the action α at $(s_U(x), g)$ has the shape

$$(d\alpha)_{(s_U(x),g)} = \begin{pmatrix} (\partial_1 \alpha)_{(s_U(x),g)} & (\partial_2 \alpha)_{(s_U(x),g)} \end{pmatrix},$$

where $(\partial_1\alpha)_{(s_U(x),g)}$ denotes the derivative with respect to the coordinates of P and $(\partial_2\alpha)_{(s_U(x),g)}$ with those of G. A little calculation allow us to see that

$$\begin{aligned}(\partial_1\alpha)_{(s_U(x),g)} &= d(\alpha(\cdot,g))_{s_U(x)}\colon T_{s_U(x)}P \to T_{s_U(x)\cdot g}P\\ (\partial_2\alpha)_{(s_U(x),g)} &= d(\alpha(s_U(x),\cdot))_g\colon T_gG \to T_{s_U(x)\cdot g}P,\end{aligned}$$

where $\alpha(\cdot,g) = L_g$ and $\alpha(s_U(x),\cdot)$ are the partial applications of α, hence,

$$d(\alpha\circ(s_U\times id))_{(x,g)} = \begin{pmatrix} (dL_g)_{s_U(x)}(ds_U)_x & d(\alpha(s_U(x),\cdot))_g \end{pmatrix}.$$

As $\pi\circ s_U = id$, it follows that $d\pi\circ ds_U = id$, thus ds_U is injective, and since the partial application L_g is a diffeomorphism, the composition $d(L_g\circ ds_U)_x$ is injective. The map $d(\alpha(s_U(x),\cdot))_g$ is also injective. To see this, take g_t a curve in G with $g_0 = g$ and $\dot g_0 = Z$. If $d(\alpha(s_U(x),\cdot))_g Z = 0$, then,

$$\left.\frac{d}{dt}\right|_{t=0} s_U(x)\cdot g_t = 0,$$

thus, $s_U(x)\cdot g_t$ is constantly equal to $s_U(x)\cdot g$, and $g_t = g$ for all t (the action is free), thence, $Z = \dot g_0 = 0$. Now, pick vectors $v\in T_xM$ and $Z\in T_gG$, and assume that $d(\alpha\circ(s_U\times id))_{(x,g)}(v,Z) = 0$, that is,

$$d(L_g\circ ds_U)_x v + d(\alpha(s_U(x),\cdot))_g Z = 0.$$

Since each map is injective, all we need now is that they are linearly independent, but this is indeed the case, since

$$d\pi_{s_U(x)\cdot g} d(\alpha(s_U(x),\cdot))_g Z = \left.\frac{d}{dt}\right|_{t=0}\pi(s_U(x)\cdot g_t) = \left.\frac{d}{dt}\right|_{t=0} x = 0,$$

and, for a curve γ_t in M such that $\gamma_0 = x$ and $\dot\gamma_0 = v \neq 0$,

$$d\pi_{s_U(x)\cdot g} d(L_g\circ ds_U)_x v = \left.\frac{d}{dt}\right|_{t=0}\pi(s_U(\gamma_t)\cdot g) = \left.\frac{d}{dt}\right|_{t=0}\gamma_t = v.$$

B.37. Two local sections given on overlapping trivializing open sets are related by $s_U(x) = s_V(x)\cdot g$ for $g\in G$. On the one hand, $\varphi_U(s_U(x)) = (x,e)$ and on the other hand, $\varphi_V(s_V(x)\cdot g) = (x,g)$, therefore $\varphi_U\circ\varphi_V^{-1}(x,g) = (x,e)$, so $e = g_{UV}(x)\cdot g$, hence

$g = g_{VU}(x)$, which gives us the relation

$$s_U = s_V \cdot g_{VU}.$$

B.38. Just like in the vector bundle case, we can reconstruct a bundle once we have an open cover of the base manifold and transition functions satisfying the cocycle conditions. Precisely, we have

B.39 Theorem. *Let M be a smooth manifold, $\{U_\alpha\}$ an open covering of M and G a Lie group. Given functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying the cocycle conditions*

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma},$$

on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$, there is a unique, up to isomorphism, G-principal bundle $P \to M$ such that the transition functions are exactly the $g_{\alpha\beta}$.

B.3 Associated Vector Bundles

B.40. Given a principal bundle $P(M, G)$ and a vector space S such that there exists a left action of G by linear automorphism $\rho\colon G \to GL(S)$, we will construct a vector bundle with fiber S. Consider a a right action of G on the product $P \times S$ as

$$(p, v) \cdot g = (p \cdot g, g^{-1} \cdot v),$$

where $g^{-1} \cdot v = \rho(g^{-1})v$, of course. The action is free, since $(p, v) \cdot g = (p, v)$ implies $p \cdot g = p$, and as the action on P is free, $g = e$. Put $E = (P \times S)/G$, also denoted $P \times_G S$ or $P \times_\rho S$, and define a projection $\pi_E([p, v]) = \pi(p)$, where π is the projection of P. This projection is well defined because

$$\pi_E([pg, g^{-1}v]) = \pi(p \cdot g) = \pi(p) = \pi_E([p, v]).$$

B.41. Take $U \ni x$ a trivializing open set for P and s_U a local section with $s_U(x) = \varphi_U^{-1}(x, e)$. Define $\psi_U^{-1}\colon U \times S \to \pi_E^{-1}(U)$ by

$$\psi_U^{-1}(x, v) = [s_U(x), v].$$

This map is bijective. To see this, we will exhibit the inverse $\psi_U\colon \pi_E^{-1}(U) \to U \times S$.

Recall that $[p, v] \in \pi_E^{-1}(U)$ can be written as $[s_U(x) \cdot \phi_U(p), v] = [s_U(x), \phi_U(p) \cdot v]$, so the inverse is given by

$$\psi_U([p, v]) = (x, \phi_U(p) \cdot v), \qquad x = \pi(p).$$

We then define a differentiable structure on E by asking that the $\pi_E^{-1}(U)$ are open submanifolds of E and that the ψ_U are diffeomorphisms. In this way, naturally the projection π_E will be smooth, and also, $\psi_U|_{\pi_E^{-1}(x)}$ induces a linear isomorphism $\pi_E^{-1}(x) \cong S$.

B.42 Definition. The vector bundle $E = P \times_G S$ that we have just constructed is called **associated vector bundle**.

B.43 Example. Given a manifold M, recall the $GL(n, \mathbb{R})$-principal bundle of example B.33, the frame bundle $B(M)$. With the standard action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ given by matrix multiplication

$$\rho(g)v = gv, \qquad g \in GL(n, \mathbb{R}), v \in \mathbb{R}^n,$$

for v seen as a column matrix, we claim that $E = B(M) \times_\rho \mathbb{R}^n = TM$. Indeed, observe that for each $x \in M$, we have $E_x = \pi_E^{-1}(x)$ is T_xM, since there is a bijective linear map $E_x \to T_xM$ given by

$$[p, v] \mapsto \sum v^j e_j,$$

where $\{e_1, \ldots, e_n\}$ is a basis of T_xM, $p = (x, e_1, \ldots, e_n)$ and $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$. Let us see first that this is a well defined map. If $[p, v] = [p', v']$, then $p' = p \cdot g$ and $v' = g^{-1} \cdot v$, thus $e'_j = \sum g^i_j e_i$ and $v'^i = \sum (g^{-1})^i_j v^j$, so

$$[p', v'] \mapsto \sum v'^k e'_k = \sum (g^{-1})^k_j v^j g^i_k e_i = \sum \delta^i_j v^j e_i = \sum v^j e_j = [p, v].$$

The linearity is fairly clear and for a coordinate chart U around x on M, with $\varphi(x) = (x^1(x), \ldots, x^n(x))$, the inverse is given by

$$\sum v^\mu \frac{\partial}{\partial x^\mu} \mapsto \left[\left(x, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right), (v^1, \ldots v^n) \right].$$

B.44 Remark. An element $p \in P_x = \pi^{-1}(x)$, where π is the projection of a principal bundle P can be seen as a smooth map $p\colon S \to \pi_E^{-1}(x)$ with

$$v \mapsto [p, v],$$

and $p \cdot g(v) = [p \cdot g, v] = [p, g \cdot v] = p(g \cdot v)$. The smoothness comes from the composition $v \mapsto (p, v) \mapsto [p, v]$, where the last quotient map is smooth because of the construction of the associated vector bundle. Now, take a section $s_U\colon U \to P$. This map induces a trivialization $\varphi_U^{-1}\colon U \times S \to \pi_E^{-1}(U)$ given by

$$\varphi_U(x, v) = s_U(x)(v) = [s_U(x), v],$$

as we saw in B.41. Hence, local sections on the frame bundle is equivalent to a local frame, i.e., a linearly independent set of sections in the tangent bundle, or in the vector bundle, if we do the construction of the frame bundle regarding an arbitrary vector bundle over M.

Apêndice C

Sobolev Spaces and Elliptic Operators

This brief appendix intends to collect some useful facts from analysis that are used throughout this work, particularly in chapter 4. The references are [WELLS, 1980], [WARNER, 1983], [DONALDSON, KRONHEIMER, 1990] and [EVANS, 1998].

C.1 Linear Functional Analysis

C.1. Consider X a vector space over $\mathbb{C}$ endowed with a norm, that is, a mapping $\|\cdot\| : X \to [0, \infty)$ that satisfies

(a) Triangle inequality: $\|u+v\| \leq \|u\| + \|v\|$, for all $u, v \in X$;
(b) $\|\lambda u\| = |\lambda|\|u\|$, for $\lambda \in \mathbb{C}$ and $u \in X$;
(c) $\|u\| = 0$ if and only if $u = 0$.

Such spaces are called **normed** vector spaces.

C.2 Definitions. We say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ in the normed space X **converges in the norm** or **converges strongly** or simply **converges** to $u \in X$, and write

$$u_n \to u,$$

if $\lim_{n\to\infty} \|u_n - u\| = 0$. A sequence is called a **Cauchy sequence** provided that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\| < \epsilon, \quad \forall n, m \geq N.$$

We then say that a the space X is **complete** if each Cauchy sequence in X converges to an element $u \in X$. Such complete normed vector spaces X are called **Banach spaces**.

C.3. Let X be a normed vector space. Consider the set $\bar{X}$ as the set of equivalence classes of Cauchy sequences under the relation

$$\{u_n\} \sim \{ v_n\} \Longleftrightarrow \lim_{n\to\infty} \|u_n - v_n\| = 0.$$

We endow the set $\bar{X}$ with a normed vector space structure by setting, for equivalence classes $[\{u_n\}]$, $[\{v_n\}]$ in $\bar{X}$ and λ a scalar,

$$\begin{aligned} [\{u_n\}] + [\{v_n\}] &= [\{u_n + v_n\}] \\ \lambda[\{u_n\}] &= [\{\lambda u_n\}] \\ \|[\{u_n\}]\|' &= \lim_{n\to\infty} \|u_n\|. \end{aligned}$$

It is a rather standard fact that the space $(\bar{X}, \| \cdot\| ')$ constructed above is a Banach space that contains a dense subset that is isometric to X.

C.4 Definition. The space $(\bar{X}, \| \cdot \|')$ constructed in paragraph C.3 is called **completion** of X with respect to the norm $\| \cdot\|$.

C.5. Assume now that a vector space X is endowed with an inner product, that is, a mapping $(\cdot,\cdot)\colon X \times X \to \mathbb{C}$ that verifies

(a) $(u, v) = \overline{(v, u)}$, for all $u, v \in X$;
(b) the mapping $u \mapsto (u, v)$ is linear for each $v \in X$;
(c) $(u, u) \geq 0$ for all $u \in X$;
(d) $(u, u) = 0$ if, and only if, $u = 0$

Then, if we denote

$$\|u\| := \sqrt{(u, u)} \quad u \in X,$$

the **Cauchy-Scharz** inequality

$$|(u, v)| \leq \| u\| \|v\| \quad u, v \in X, \tag{C.1}$$

guarantees that $\| \cdot \|$ above is a norm as in paragraph C.1, and we say that the inner product $(\cdot, \cdot)$ **generates** the norm $\| \cdot\|$.

C.6 Definition. A **Hilbert space** X is a Banach space endowed with an inner product that generates the norm.

C.7 Definition. We say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ in the Hilbert space X **converges weakly** to $u \in X$, and write

$$u_n \rightharpoonup u,$$

if $\lim_{n\to\infty}(u_n - u, v) = 0$, or, equivalently, $|(u_n - u, v)| \to 0$, for every $v \in X$.

C.8. Strong convergence implies weak convergence, that is, if $\{u_n\}$ is a sequence in the Hilbert space X and $u_n \to u$, then $u_n \rightharpoonup u$. This is a simple consequence of the Cauchy-Scharz inequality (C.1)

$$|(u_n - u, v)| \leq \| u_n - u\|\|v\| \to 0.$$

Also, if $u_n \rightharpoonup u$, then

$$\|u\| \leq \liminf_{n\to\infty}\|u_n\|.$$

C.9 Definitions. Two elements of a Hilbert space $u, v \in X$ are called **orthogonal** if $(u, v) = 0$. A countable basis $\{v_k\}_{k=1}^{\infty} \subset X$ is called an **orthonormal basis** if v_j, v_k are orthogonal for every $j \neq k$, and $\|v_k\| = 1$ for all k. If S is a subset of X, then

$$S^{\perp} = \{u \in X \mid (u, v) = 0, \text{ for all } v \in S\},$$

is the **subspace orthogonal to** S.

C.10 Definitions. Let X and Y be Banach spaces. A map $A\colon X \to Y$ is a **linear operator** provided that

$$A(\lambda u + v) = \lambda A(u) + A(v),$$

for λ a scalar and $u, v \in X$. The **range** of A is the set $R(A) := \{Au \in Y \mid u \in X\}$, and the **null space** or **kernel** of A is $\ker(A) := \{u \in X \mid Au = 0\}$. A linear operator $A\colon X \to Y$ is called **bounded** if

$$\|A\| := \sup\{\|Au\| \mid \| u\| \leq 1\} < \infty.$$

C.11 Proposition. *Every bounded linear operator $A\colon X \to Y$ is continuous.*

Proof. Take $\{u_n\}$ a sequence that converges, $u_n \to u \in X$. Then,

$$\|Au_n - A_u\| = \|A(u_n - u)\| \leq \|A\|\|u_n - u\| \to 0.$$

□

C.12 Theorem (Uniform Boundedness Principle)**.** *Let X be a Banach space and Y be any normed space. Suppose that $\mathcal{C}$ is a collection of linear operators from X to Y. If for every $u \in X$ we have*

$$\sup_{A\in\mathcal{C}} \|Au\| < \infty,$$

then

$$\sup_{A\in\mathcal{C}} \|A\| < \infty.$$

C.13. If we consider X a Hilbert space and $Y = \mathbb{C}$ in the previous theorem, then it follows that every weakly convergent sequence in X is bounded. Indeed, let $\{u_n\}$ be a weakly convergent sequence converging to u, and set $U_n \colon X \to \mathbb{C}$ by $U_n(v) = (v, u_n)$. Then, $\sup |U_n(v)| < \infty$, for all v, hence the Uniform Boundedness Principle says that $\|u_n\| \leq \sup \|U_n\| < \infty$, so u_n is bounded.

C.14 Definition. A linear operator $A \colon X \to Y$ is called **closed** if for every sequence $\{u_n\}$ that converges to $u \in X$ and also satisfies $Au_n \to v \in Y$, we have

$$Au = v.$$

C.15 Theorem (Closed Graph Theorem)**.** *Let $A \colon X \to Y$ be a closed linear operator. Then A is bounded.*

C.16 Definition. If a linear operator $A \colon X \to X$ on a Hilbert space X is bounded, its **adjoint** $A^\dagger \colon X \to X$ satisfies

$$(Au, v) = (u, A^\dagger v), \quad u, v \in X,$$

and A is called **self adjoint** if $A^\dagger = A$.

C.17 Proposition. *Let X be a Hilbert space and $A \colon X \to X$ a bounded operator. Then, A is* **weakly continuous***, in the sense that for each weakly convergent sequence $u_n \rightharpoonup u$, we have $Au_n \rightharpoonup Au$.*

Proof.

$$(Au_n, v) = (u_n, A^\dagger v) \to (u, A^\dagger v) = (Au, v).$$

□

C.18 Definition. A bounded linear operator $K\colon X \to Y$ between two Banach spaces is called **compact** if for each bounded sequence $\{u_n\}$ in X the sequence $\{Ku_n\}$ is **precompact** in Y, in the sense that there exists a subsequence $\{u_{n_k}\}$ such that $\{Ku_{n_k}\}$ converges.

C.19 Proposition. *Let X, Y be Hilbert spaces and $K\colon X \to Y$ be a compact operator. If $\{u_n\}$ is a weakly convergent sequence converging to u, then $Ku_n \to Ku$.*

Proof. If $u_n \rightharpoonup u$, then from the Uniform boundedness Principle we get that $\{u_n\}$ is bounded (see paragraph C.13), thus, as K is compact, there is a subsequence that converges in norm $Ku_{n_k} \to u'$, for $u' \in Y$. Now, in paragraph C.8, we saw that strong convergence implied weak convergence, therefore, $\{Ku_{n_k}\}$ converges weakly to u'. But, from proposition C.17, bounded operators are weakly continuous, hence $\{Ku_n\}$, and thus, $\{Ku_{n_k}\}$ converges weakly to Ku, so $\{Ku_{n_k}\}$ converges weakly both to u' and Ku. We claim that $u' = Ku$. In fact, for any $v \in Y$

$$0 = [(Ku_{n_k}, v) - (Ku_{n_k}, v)] \to [(Ku, v) - (u', v)],$$

thence $(Ku - u', v) = 0$ for all v, so the claim and the proposition follows. $\square$

C.20 Theorem (Compactness of adjoints)**.** *If $K\colon X \to X$ is compact operator on a Hilbert space, so is the adjoint $K^\dagger\colon X \to X$.*

C.21 Theorem (Fredholm alternative)**.** *Let $K\colon X \to X$ be a compact operator on a Hilbert space X. Then*

1. $\ker(Id - K)$ *is finite dimensional;*
2. $R(Id - K)$ *is closed;*
3. $R(Id - K) = \ker(Id - K^\dagger)^\perp$*;*
4. $\ker(Id - K) = \{0\}$ *if and only if* $R(Id - K) = H$*;*
5. $\dim\ker(Id - K) = \dim\ker(I - K^\dagger)$.

C.2 Sobolev Spaces and Elliptic operators

C.22. Recall that on $U \subset \mathbb{R}^n$, we regard the space $L^2(U)$ as the completion with respect to the norm

$$\|f\|_{L^2} = \left(\int_U |f|^2\right)^{1/2},$$

of the space of all smooth compactly-supported functions on U. For general p we have that the L^p spaces are defined similarly but with the slightly different norm

$$\|f\|_{L^p} = \left(\int_U |f|^p \right)^{1/p}.$$

Some famous inequalities appear in the theory of L^p spaces such as the Hölder inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}, \qquad \text{for } 1 < p, q \leq \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \tag{C.2}$$

$f \in L^p$, $g \in L^q$, and the Minkovski inequality

$$\|fg\|_{L^p} \leq \|f\|_{L^p} + \|q\|_{L^p}, \qquad 1 \leq p < \infty, ; \text{ and } f, g \in L^p.$$

The latter ensures us that the triangle inequality holds for the various norms $\| \cdot \|_{L^p}$, and in its proof it is used inequality (C.2).

C.23 Definitions. For $k \geq 0$ the space $L^2_k(\mathbb{R}^n)$ is defined as the completion of the space of smooth compactly-supported functions on $\mathbb{R}^n$ under the norm

$$\|f\|^2_{L^2_k} = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha f|^2 = \sum_{|\alpha| \leq k} \|D^\alpha f\|^2_{L^2_k},$$

where $D^\alpha f$ means the muliti-index notation for partial derivatives of order $|\alpha|$, that is, $\alpha = (\alpha_1, \ldots, \alpha_n)$, every $\alpha_i \in \mathbb{N}$, and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}},$$

for $|\alpha| = \alpha_1 + \ldots + \alpha_m$ the order of the derivative. If M is a compact manifold and E is a vector bundle over M, we define the spaces $L^2_k(M; E)$ of L^2_k **sections of** E by taking a completion of the space of smooth section $\Gamma(M; E)$ with respect to the norm

$$\|s\|^2_{L^2_k} = \sum_{i=0}^{k} \int_M |\nabla^i_A s|^2 \,\mathrm{vol},$$

for ∇_A a connection compatible with a fiber metric on E and vol a volume form with respect to a metric on the base manifold M.

C.24 Remark. The more general L^p_k spaces arise if we consider the L^p_k norm given by

$$\|f\|^p_{L^p_k} = \sum_{|\alpha|\le k} \int |D^\alpha f|^p = \sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p_k}.$$

C.25 Theorem (Rellich Lemma)**.** *For bundles over a compact base space M, the natural inclusion $L^2_{k+1}(M;E) \hookrightarrow L^2_k(M;E)$ is compact, for every k.*

C.26 Theorem (Sobolev Embedding Theorem)**.** *Let $n = \dim_{\mathbb{R}} M$ and assume $k > r + n/2$. Then there is a bounded inclusion map $L^2_k(M;E) \hookrightarrow C^r(M;E)$. Hence, a function that lies on $L^2_k(M;E)$ for all k is smooth.*

C.27 Definitions. On $\mathbb{C}^n$, a **differential operator** is a complex linear operator $L\colon C^\infty(\mathbb{C}^n) \to C^\infty(\mathbb{C}^n)$ given as

$$Lf = \sum a_\alpha D^\alpha f,$$

where $a_\alpha \in C^\infty(\mathbb{C}^n)$ are smooth complex-valued functions. If E and F are smooth vector bundles over M of rank r and q, respectively, we say that a linear map $L\colon \Gamma(M;E) \to \Gamma(M;F)$ is a **differentiable operator** if, for any choice of local coordinates and local trivializations, writing a section of E as $s = (s^1, \ldots, s^r)$, there is a differential operator $\tilde{L}$ on $\mathbb{C}^n$ acting on s as

$$(\tilde{L}s)^i = \sum (a_\alpha)^i_j D^\alpha s^j$$

such that the below diagram commutes.

$$\begin{array}{ccc} \Gamma(U; U \times \mathbb{R}^r) & \xrightarrow{\tilde{L}} & \Gamma(U; U \times \mathbb{R}^q) \\ \uparrow & & \uparrow \\ \Gamma(U; E|_U) & \xrightarrow{L} & \Gamma(U; F|_U) \end{array}$$

The operator is said to have **order** l if there are no derivatives of higher orders. We denote as $\mathrm{Diff}_l(E,F)$ the vector space of all differential operators of order l from sections of E to sections of F. If we are dealing with compact base space M, we define $\mathrm{Op}_l(E,F)$ as the vector space of all linear mappings $A\colon \Gamma(M;E) \to \Gamma(M;F)$ that has a continuous extension $\tilde{A}\colon L^2_k \to L^2_{k-l}$ which are called **operators of order** l.

C.28 Proposition. $\mathrm{Diff}_l(E,F) \subseteq \mathrm{Op}_l(E,F)$.

C.29. Let M be a smooth manifold. Let $T'(M) = T^*(M) \setminus \{\bar{0}(M)\}$ be the bundle of nonzero 1-forms ($\bar{0}$ denotes the zero section), with projection π. We will denote elements of $T'(M)$ as ω or ω_x, if we want to emphasize the base point for which $\omega_x \in E_x^* \setminus \{0\}$. Let E, F be $\mathbb{C}$-vector bundles over M and let π^*E, π^*F denote the pullbacks of E and F to $T'(M)$. Recall that, as a set,

$$\pi^*E = \{(\omega, e) \in T'(M) \times E \mid \pi(\omega) = \pi_E(e)\} \subseteq T'(M) \times E.$$

Then, set

$$\mathrm{Smbl}_l(E, F) := \{\sigma \in \mathrm{Hom}(\pi^*E, \pi^*F) \mid \sigma(\rho\omega, \cdot) = \rho^l \sigma(\omega, \cdot),\ \rho > 0,\ \omega \in T'(M)\}.$$

Now, define a linear map

$$\sigma_l \colon \mathrm{Diff}_l(E, F) \to \mathrm{Smbl}_l(E, F),$$

where $\sigma_l(L)$ is called the **l-symbol** of L. Since $\sigma_l(L)$ is in $\mathrm{Smbl}_l(E, F)$, $\sigma_l(L)(\omega)$ must be a linear map from $E_x \to F_x$ for every $x \in M$. For $(\omega) \in T'(M)$ and $e \in E_x$, find a function $g \in C^\infty(M)$ and a section $s \in \Gamma(M; E)$ such that $dg_x = \omega$ and $s(x) = e$. Then, define

$$\sigma(L)(\omega)e = L\left(\frac{i^l}{l!}(g - g(x))^l s\right)(x) \in F_x.$$

This construction defines a linear mapping $\sigma_l(L)(\omega_x) \colon E_x \to F_x$, which then defines an element in $\mathrm{Smbl}_l(E, F)$. It can be shown that $\sigma_l(L)$ is independent of the choices made.

C.30 Proposition. *The symbol map σ_l gives rise to an exact sequence*

$$0 \longrightarrow \mathrm{Diff}_{l-1}(E, F) \longrightarrow \mathrm{Diff}_l(E, F) \longrightarrow \mathrm{Smbl}_l(E, F) \longrightarrow 0.$$

C.31 Definition. Let E, F be vector bundles over a compact base manifold M and $L \colon \Gamma(E) \to \Gamma(F)$ be a differential operator of order l. We say that the operator is **elliptic**, if, for every non-zero cotangent vector ξ at $x \in M$ the linear map $\sigma_l(L)(\xi)$ constructed in C.29 is invertible.

C.32 Example. Consider the second order differential operator $\Delta = \sum_j \partial_j^2$, where ∂_j is the short-hand notation for $\frac{\partial}{\partial x^j}$, acting on differential forms of $\mathbb{R}^n$. Let us compute the symbol of Δ for an arbitrary non-zero cotangent vector ω. From the definition of the symbol, take a function g such that $dg_x = \omega$. Thus, for any differential form ξ, we have

$$\begin{aligned}
\sigma(\Delta)(\omega)(\xi) &= \Delta\left(\frac{i^2}{2!}(g-g(x))^2\xi\right)(x) \\
&= -\frac{1}{2}\left\{\left(\sum_j 2(g-g(x))\partial_j^2 g + 2(\partial_j g)^2\right)\xi + (g-g(x))^2\Delta\xi\right\}(x) \\
&= -\left(\sum_j (\partial_j g)^2\right)\xi \\
&= -|\omega|^2\xi.
\end{aligned}$$

Thus, the Laplacian is elliptic as $\omega \neq 0$. If we consider a smooth manifold instead of $\mathbb{R}^n$, the computation is similar because we use local coordinates.

C.33 Proposition. *If $D \in \mathrm{Diff}_l(E,F)$ is an elliptic operator, then for each $k \geq 0$, there is a constant C_k so that for all sections s of E we have*

$$\|s\|_{L^2_{k+l}} \leq C_k(\|Ds\|_{L^2_k} + \|s\|_{L^2}).$$

If s is L^2 orthogonal to the kernel of D, then we can omit the term $\|s\|_{L^2}$ in the right.

C.34. So far we have been treating the spaces L^2_k, but we could also consider the L^p_k spaces by replacing the L^2 norm for the L^p norm. If n is the real dimension of the base manifold we define a **scaling weight**

$$w(k,p) = k - \frac{n}{p}.$$

We have

C.35 Proposition. *Let $E \to M$ be a vector bundle over a compact base space M. If $k > l$ and $w(k,p) \geq w(l,q)$, there is a bounded inclusion map*

$$L^p_k(M;E) \to L^q_l(M;E).$$

If strict inequality holds, that is, $w(k,p) > w(l,q)$, then this embedding is compact.

C.36 Corollary. *In dimension $n = 2$, the multiplication gives us a bounded bilinear map*

$$L^2_2 \times L^2_2 \to L^2_2.$$

In particular, the space $L^2_2(\mathrm{End}(E))$ of L^2_2 sections of gauge transformations of a vector bundle E over a compact M is a group. Furthermore, $L^2_2(\mathrm{End}(E))$ acts on $L^2_1(\mathrm{End}(E))$, as there exists a bounded bilinear map

$$L^2_2 \times L^2_1 \to L^2_1.$$

Proof. From the Sobolev Embedding Theorem in C.26, we have a bounded inclusion $L^2_k \to C^0$ for $k \geq 2$. Expanding by the Leibnitz rule gives us that for $k \geq l$ and $k \geq 2$ the multiplication

$$L^2_k \times L^2_l \to L^2_l$$

is a bounded bilinear map. Therefore, the result follows if we use $k = 2$ and $l = 1$ or $l = 2$. □

C.37. We finish by generalizing the the Sobolev Embedding Theorem C.26 for arbitrary L^p spaces. The basic result, in dimension n is:

C.38 Theorem. *If M is compact of dimension n, then there is a bounded inclusion map $L^p_1 \hookrightarrow C^0$, for $p > n$, i.e., $w(1,p) > 0$.*

C.39 Proposition. *Let D be an elliptic first order operator acting on sections of a bundle E over a compact space M. Let p, q be related as $q = np/(n-p)$. Then, there is a constant C such that*

$$\|f\|_{L^q} \leq C(\|Ds\|_{L^p} + \|s\|_{L^p}),$$

and if s is L^p-orthogonal to the kernel of D we can omit the term $\|s\|_{L^p}$.