Universidade Estadual de Campinas
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David Quispe Aruquipa

An analogue model of gravity based on a radial fluid flow: the case of AdS and its deformations

Um modelo análogo à gravitação baseado em um fluxo radial: o caso do espaço-tempo AdS e suas deformações

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Para mi madre, Catalina.

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"Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world."

A. Einstein

Abstract

Analogue models are a useful tool when one wants to understand or probe phenomena in one physical system in terms of concepts from another, which may be more familiar or more easily accessed by experiments. This work explores this framework in the context of analogue models of gravity based on fluid dynamics. Particularly, we are interested in providing an analogue model for a radial fluid flow with a point source/sink at the origin.

We start by considering the case where the (radial) flow velocity is constant. The resulting analogue model is then the Anti-de Sitter spacetime (AdS) which is known to be non-globally hyperbolic. As a result, the dynamics of fields in this background is not well defined until extra boundary conditions at its spatial boundary are prescribed. The fluid dynamics counterpart of these extra boundary conditions provide an effective description of the point source/sink at the origin.

After that, we consider regularizations of this model near the source/sink at the origin. We then impose conditions on them in order that the dynamics is well defined so that no extra boundary conditions are required. We calculate how physical quantities, like the phase difference between ingoing and outgoing scattered waves are affected by the regularizations. These results are then compared with the AdS case to understand the main implications of the regularization, which has the effect of deforming the AdS space near its spatial infinity. We also show that, under certain conditions, the phase difference obtained for these deformed AdS spaces agrees with that obtained in the AdS case.

Key-words: Analogue models. General Relativity. Fluid dynamics.

Resumo

Os modelos análogos são uma ferramenta muito útil quando se quer entender ou testar fenômenos de um sistema físico em termos e conceitos de outro, que podem ser mais familiares ou mais facilmente reproduzíveis por experimentos. Este trabalho explora esta questão no contexto de modelos análogos à gravitação baseados na mecânica dos fluidos. Particularmente, estamos interessados em fornecer um modelo análogo para um fluxo radial com uma fonte/sorvedouro na origem.

Começamos por considerar o caso em que a velocidade do fluxo (radial) é constante. O modelo análogo resultante é então o espaço-tempo Anti-de Sitter (ou AdS) que é conhecido por ser não-globalmente hiperbólico. Como resultado, a dinâmica dos campos neste contexto não está bem definida até que sejam estabelecidas condições adicionais na fronteira no infinito espacial do espaço–tempo AdS. A contrapartida destas condições de fronteira extras na mecânica dos fluidos proporciona uma descrição efetiva da fonte/sorvedouro que está na origem.

Depois disso, nós consideramos regularizações para o modelo análogo perto da fonte/sorvedouro na origem. Logo, impomos condições sobre eles, a fim de que a dinâmica esteja bem definida de modo que não sejam mais necessárias as condições na fronteira. Calculamos como as quantidades físicas, como a diferença de fase entre as ondas que entram e saem, são afetadas pelas regularizações. Estes resultados são então comparados com o caso AdS para compreender as principais implicações do processo de regularização, que tem o efeito de deformar a região perto do infinito espacial do AdS. Mostramos também que, sob certas condições, a diferença de fase obtida para esses espaços deformados do AdS coincide com a obtida no caso do espaço–tempo AdS.

Palavras-chaves: Modelos análogos. Relatividade Geral. Mecânica dos fluidos.

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1 Fluid dynamics

1.1 Introduction

In this chapter we present the basic aspects and equations which govern all phenomena of fluids within the classical limit. This review contains some considerations which will be necessary in order to derive hydrodynamic analogue models of gravity. We begin our description of fluids by reviewing the physical ideas that are needed to understand the general behavior of fluids. First, we can represent a fluid as a set of “fluid particles” [1]. These are to be considered as a mathematical idealization of an entity with a sufficiently large amount of fluid particles so that a classical (i.e, non-quantum) description of this object is valid. The atoms inside a fluid particle do not have to be same as time evolves, but the exchange of atoms between fluid particles should not drastically change the number of atoms inside them. The last requirement we make is that the size of these fluid particles is very small compared with the length scale of the system (the fluid flow), so that they can be assigned particle-like characteristics.

Mathematically, a fluid can be characterized by three main quantities: its density ρ which tells how the fluid mass is distributed over the fluid, the velocity $\vec{v}$ of fluid particles, and the pressure p. All these quantities are defined at position $\vec{r}$ and at time t. When they are known or given, the state of the fluid is entirely determined.

In order to find ρ, $\vec{v}$ and p, we need five (scalar) equations to relate them. As we mentioned before, we restrict ourselves to classical physics so that those equations can be derived from Newton laws (or more generally by using the principle of least action). After that, such equations are identified as the continuity equation, the Navier–Stokes equations and an additional equation which comes from thermodynamics (a constitutive relation).

With respect to the energy of the fluid, in fluid dynamics we can also have conservative and non-conservative systems. The case of non-conservative fluids are well described in terms of another quantity, the viscosity. We are interested here in fluids with zero viscosity and this kind of fluids are said to be ideal. We describe this case in more detail in what follows.

1.2 Mathematical aspects of ideal fluids

Although there are no ideal fluids in nature, within certain regimes and conditions a real fluid can be considered ideal and this is often useful for simplifying its description or approximating its behavior.

To proceed with the derivation of the fundamental equations for ideal fluids, we first need to consider some conservation principles. To do that, consider some volume V within the fluid. Given the mass density ρ, the total mass in this volume is

$$\int_V \rho \mathrm{d}V.$$

The mass of fluid which passes through an element $\mathrm{d}S$ of surface ∂V is $\rho\vec{v} \cdot \mathrm{d}\vec{S}$, where the module of $\mathrm{d}\vec{S}$ is $\mathrm{d}S$ and its direction is pointing along the normal direction of ∂V. Consequently, the total mass per unit time is

$$\oint_{\partial V} \rho\vec{v} \cdot \mathrm{d}\vec{S}.$$

This mass flow changes the total mass in V at the rate

$$-\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V.$$

Equating the last two expressions and using the divergence theorem, we get the continuity equation [1]

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = 0. \tag{1.1}$$

The quantity $\rho\vec{v}$ is called the *mass flux density* and it is denoted by $\vec{j}$. In this way, equation (1.1) has the same form as the conservation of charge, just as expected.

The continuity equation, as we said before, is one of the five equations needed to describe a fluid. The next three equations come from considering the fluid particle as a point particle and applying the Newton second law. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \mathrm{d}V\vec{v}) = -\vec{\nabla}p\mathrm{d}V + \vec{f}\mathrm{d}V, \tag{1.2}$$

where p is the pressure and $\vec{f}$ is the external force density applied to the infinitesimal fluid element. Considering $\rho = \rho(\vec{r}(t), t)$, and applying the chain rule on the left hand side of equation (1.2) *Euler equations* arise

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \frac{1}{\rho}\vec{f}. \tag{1.3}$$

This equation was first obtained by Euler in 1755 [2].

To explore further the meaning of the above equation, let us recall the fact that there is no heat transfer in the ideal fluid case. Because of that, the motion of the fluid is adiabatic. From thermodynamics, this fact implies that the entropy must remain constant. Denoting by s the entropy per unit mass, we get

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0. \tag{1.4}$$

When $s =$ constant, the motion is said to be *isentropic* [1]. Once more, from thermodynamics, the enthalpy H in terms of entropy and pressure is given by

$$\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}p,$$

where p is the pressure and T the temperature of the system. Dividing this relation by the mass contained in V, the above expression can be written in terms of s, ρ, and p as follows:

$$\mathrm{d}h = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p, \tag{1.5}$$

where h is the specific enthalpy (enthalpy per unit mass). Relation (1.4) implies

$$\vec{\nabla}h = \frac{1}{\rho}\vec{\nabla}p. \tag{1.6}$$

This can be used to rewrite the Euler equations as

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}h + \frac{1}{\rho}\vec{f}. \tag{1.7}$$

To simplify this expression, we can use the following relation which comes from vector analysis:

$$\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right) = \left(\vec{a}\cdot\vec{\nabla}\right)\vec{b} + \left(\vec{b}\cdot\vec{\nabla}\right)\vec{a} + \vec{a}\times\left(\vec{\nabla}\times\vec{b}\right) + \vec{b}\left(\vec{\nabla}\times\vec{a}\right).$$

Making $\vec{a} = \vec{b} = \vec{v}$, substituting into equation (1.7) and taking the curl then leads to

$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{v} = \vec{\nabla}\times\left(\vec{v}\times\vec{\nabla}\times\vec{v}\right) + \vec{\nabla}\times\left(\frac{1}{\rho}\vec{f}\right), \tag{1.8}$$

which are the Euler equations expressed only in terms of $\vec{v}$ and the external force $\vec{f}$.

Another interesting result arises when $\vec{f}/\rho$ comes from a potential. Then, we can write $\vec{f} = -\rho\vec{\nabla}\varphi$, with φ a scalar field. Then, the above relation is expressed solely in terms of the fluid flow velocity. Using the previous vector relation for $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$ in equation (1.7) then leads to

$$\frac{\partial}{\partial t}\vec{v} + \left(\vec{\nabla}\times\vec{v}\right)\times\vec{v} = -\vec{\nabla}\left(\frac{1}{2}v^2 + \varphi + h\right). \tag{1.9}$$

We identify the expression $\frac{1}{2}v^2 + \varphi + h$ as the *Bernoulli function* and denote it by $\mathcal{H}$. This function plays and important role in the study of energy of fluid flows.

The continuity equation along with the Euler equations give us four of the five needed equations to describe the fluid. The fifth equation is a constitutive relation which generally comes from thermodynamics.

1.3 Energy and momentum flux

To find the energy flux of a fluid, we consider once again a volume V in the fluid with no external forces acting upon it. The energy density is then given by

$$\frac{1}{2}\rho v^2 + u,$$

where the first term represents the fluid kinetic energy and the second term its internal energy. This internal energy can also be written in terms of the energy density per unit mass ε, i.e. $u = \rho\varepsilon$. In this way, we can write energy flux as follows

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) &= \frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + \varepsilon\right)\right] \\ &= \frac{\partial \rho}{\partial t}\left(\frac{1}{2}v^2 + \varepsilon\right) + \rho\vec{v}\cdot\frac{\partial \vec{v}}{\partial t} + \rho\frac{\partial \varepsilon}{\partial t}.\end{aligned} \tag{1.10}$$

To simplify this expression, we appeal to the continuity equation and the equations of motion to obtain the derivatives $\frac{\partial \rho}{\partial t}$ and $\frac{\partial \vec{v}}{\partial t}$. To calculate $\frac{\partial \varepsilon}{\partial t}$, we use the first law of thermodynamics, which in terms of specific quantities becomes

$$\mathrm{d}\varepsilon = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho = \mathrm{d}h - \frac{1}{\rho}\mathrm{d}p + \frac{p}{\rho^2}\mathrm{d}\rho = \mathrm{d}h - \mathrm{d}\left(\frac{p}{\rho}\right).$$

As a result, we get $\rho\varepsilon = \rho h - p$ and, therefore, $\rho\frac{\partial \varepsilon}{\partial t}$ is given by

$$\rho\frac{\partial \varepsilon}{\partial t} = \frac{\partial \rho}{\partial t}(h - \varepsilon) - \rho\vec{v}\cdot\vec{\nabla}h + \vec{v}\cdot\vec{\nabla}p = \frac{\partial \rho}{\partial t}(h - \varepsilon) - \rho\vec{v}\cdot\left(\vec{\nabla}h - \frac{1}{\rho}\vec{\nabla}p\right),$$

where we used the equation (1.5). We realize that the second term will vanish for ideal fluids leaving

$$\rho\frac{\partial \varepsilon}{\partial t} = \frac{\partial \rho}{\partial t}(h - \varepsilon).$$

Putting all together, the energy flux then becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) = \frac{\partial \rho}{\partial t}\left(\frac{1}{2}v^2 + h\right) + \rho\vec{v}\cdot\frac{\partial \vec{v}}{\partial t}.$$

We need one last step to give the final expression for the energy flux. Obtaining the time derivatives of the density and the velocity flow from the continuity equation and Euler equations, we write the energy flux as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + h\right)\right]. \tag{1.11}$$

To understand the physical meaning of this equation, we integrate it over the volume V, i.e.,

$$\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right)\mathrm{d}V = -\int_V\vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + h\right)\right]\mathrm{d}V = -\oint_{\partial V}\rho\vec{v}\left(\frac{1}{2}v^2 + h\right)\cdot\mathrm{d}\vec{S},$$

where the divergence theorem was used. The right hand side of the last expression can be interpreted as the amount of energy flowing throughout ∂V. On the other hand, the left hand side, as mentioned before, is the energy rate that passes throughout V per unit of time. The quantity $\rho\vec{v}\left(\frac{1}{2}\rho v^2 + h\right)$ is called the *energy flux density* vector.

We can also identify a momentum flux (similar to the energy flux) in the fluid. To derive its mathematical expression, we first write the Euler equations in a simpler way using index notation. We denote a vector $\vec{A}$ as A_i, where i represents the ith component of $\vec{A}$. Equations (1.3) then take the form

$$\partial_t v_i + (v_k \partial_k) v_i = -\frac{1}{\rho}\partial_i p,$$

where, for simplicity, the derivatives were written as $\frac{\partial}{\partial x_i} \to \partial_i$ and $\frac{\partial}{\partial t} \to \partial_t$, and the convention of summation over repeated indices was considered. After this, Euler equations can be written as

$$\partial_t(\rho v_i) = -\partial_k \Pi_{ik}, \tag{1.12}$$

where Π_{ij} is called the *momentum flux density tensor*, and is given by

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j, \tag{1.13}$$

where δ_{ij} is the Kronecker delta.

Similarly to the energy flux, we can obtain the physical meaning of Π_{ij} by integrating the above expression over a volume V, i.e.,

$$\partial_t \int_V \rho v_i \mathrm{d}V = -\int_V \partial_j \Pi_{ij} \mathrm{d}V = -\oint_{\partial V} \Pi_{ij} \mathrm{d}S_j.$$

In contrast to the energy flux, we realize that there is a second rank tensor related to this flux. Physically, Π_{ij} represents the momentum flux through a perpendicular surface to the x_j-axis per unit time. To sum up, we realize that the momentum flux is represented by a second rank tensor and the energy flux by a vector (or first rank tensor). This notation and expressions are useful when one needs to generalize the basic ideas we developed about fluids. Moreover, these two quantities play an important role in the covariant formulation of fluid dynamics.

1.4 Velocity gradient

Since fluids do not have a definite shape, concepts like deformation or rotation have interesting implications. Consider once again a fluid flow with velocity v_i and some point X_i in the fluid. The way how v_i changes near this point gives rise to the concept of velocity gradient. Consider a series expansion near X_i of v_i:

$$v_i(x_i) = v_i(X_i) + \partial_j v_i \delta x_j + \mathcal{O}(\delta x_i^2),$$

where $\delta x_i = x_i - X_i$. The constant term $v_i(X_i)$ could be identified with the velocity of a rigid translation of a fluid particle. The velocity gradient is the quantity defining the second term, $\partial_j v_i$. To obtain a physical interpretation for it, the velocity gradient can be split into symmetric and antisymmetric part, i.e.,

$$\partial_i v_j = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) + \frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right).$$

Let

$$D_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right)$$

and

$$\Omega_{ij} = \frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right)$$

be the symmetric and antisymmetric part of the velocity gradient respectively. Consider the distance s between X_i and x_i,

$$s^2 = \delta x_i \delta x_i.$$

The rate of change of this distance with time is

$$\frac{\mathrm{d}s^2}{\mathrm{d}t} = 2\frac{\mathrm{d}\delta x_i}{\mathrm{d}t}\delta x_i = 2\delta x_i\left[v_i(x_i) - v_i(X_i)\right] = 2\left(D_{ij} + \Omega_{ij}\right)\delta x_i \delta x_j = 2D_{ij}\delta x_i \delta x_j,$$

where in the last equality we used the fact that the contraction between symmetric $(\delta x_i \delta x_j)$ and antisymmetric (Ω_{ij}) tensors vanish.

As δx_i is arbitrary, D_{ij} then encodes the variation in the distance between fluid particles. Physically it is interpreted as a deformation on the fluid. The symmetric part of velocity gradient (D_{ij}) is usually called *deformation tensor*.

For the antisymmetric part, we note that for any cyclic indexes i, j, k the quantity Ω_{ij} is the kth component of a vector $\omega_k = \epsilon_{ijk}\left(\partial_i v_j - \partial_j v_i\right)$, and that $\partial_i v_j - \partial_j v_i$ in terms of ω_k is $\frac{1}{2}\epsilon_{ijk}\omega_k$. Moreover, the contraction of ω_{ij} with δx_i is

$$\Omega_{ij}\delta x_i = \frac{1}{2}\epsilon_{ijk}\omega_k \delta x_i.$$

Now this term, in vector representation, reads

$$\Omega \cdot \delta\vec{x} = \frac{1}{2}\vec{\omega} \times \delta\vec{x}. \tag{1.14}$$

Note that in the left hand side of the above equation the dot does not mean scalar product, since Ω is not a vector (it is a second rank tensor); this dot represents a contraction and as result we get a cross product. We are all familiar with this expression in the context of classical mechanics as it represents a rotation with angular velocity equals to $\frac{1}{2}\vec{\omega}$.

The vector $\vec{\omega}$ is called the *vorticity* and, physically, it quantifies the local spinning motion of the fluid at some point. Furthermore, by using equation (1.8), we easily find the evolution equation for the vorticity. By means of a simple substitution, we get

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{\omega}\right) + \vec{\nabla} \times \left(\frac{\vec{f}}{\rho}\right)$$

or equivalently

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{v} \times \left(\vec{\nabla} \times \vec{\omega}\right) + \vec{\nabla} \times \left(\frac{\vec{f}}{\rho}\right). \tag{1.15}$$

A fluid with zero vorticity ($\vec{\omega} = 0$) is called an irrotational flow or potential flow, because zero vorticity implies $\vec{\nabla} \times \vec{v} = 0$ and therefore $\vec{v}$ can be obtained from a scalar potential field Ψ, i.e.,

$$\vec{v} = \vec{\nabla}\Psi. \tag{1.16}$$

Substituting both Ψ and $\vec{\nabla} \times \vec{v} = 0$ in equation (1.9), we obtain

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}v^2 + h = g(t), \tag{1.17}$$

where $g(t)$ is some function of t.In the case of a steady flow, Ψ is independent of t. This reads $\frac{1}{2}v^2 + h = \textit{constant}$, which physically means that this quantity will be constant along a streamline.

When the density remains constant along the fluid and throughout its motion, the fluid is said to be incompressible. In this case the continuity equation becomes $\vec{\nabla} \cdot \vec{v} = 0$ and the equation of motion can be reduced to one single scalar equation, i.e.,

$$\nabla^2\Psi = 0. \tag{1.18}$$

1.5 Lagrangian formulation for ideal fluids

We now consider a Lagrangian formalism for fluid dynamics. Sometimes this perspective allows us to simplify the process of solving the equation of motion for fluids.

1.5.1 Incompressible fluid case

We restrict ourselves to isentropic fluids, which are characterized by the relation $\vec{\nabla}p = \rho\vec{\nabla}h$. According to the variational principle, the equations of motion for incompressible fluids can be derived from an action [3]

$$S = \int L \mathrm{d}t^1 ... \mathrm{d}t^N,$$

where L is the Lagrangian and t^i are parameters that describe the evolution of the system. In the presence of constrains, the Lagrangian must be modified accordingly. The complete Lagrangian can then be written as $L + \sum_n \lambda_n F_n$, where F_n is a function that represents the constraints and λ_n are the Lagrange multipliers.

To construct the Lagrangian for fluids, let $x_i = x_i(t, \alpha_1, \alpha_2, \alpha_3)$ be the coordinates of a fluid particle. The equation of motion for a given fluid particle can be found by fixing the values of α_1, α_2, α_3. Then, the kinetic energy per unit mass is

$$T = \frac{1}{2}\frac{\partial x_i}{\partial t}\frac{\partial x_i}{\partial t}, \tag{1.19}$$

where, once again we use the summation convention. The constraint imposed on the fluid, according to [3], has to be

$$\mathrm{d}m = \rho \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3. \tag{1.20}$$

This can be interpreted as the relation between point masses and fluid particles developed in previous sections. For convenience, the constraint equation can also be written as

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} = \frac{1}{\rho}.$$

The action is then given by

$$\int \left[T + p \left(\frac{\partial(x_1, y_2, z_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} - \frac{1}{\rho} \right) \right] \mathrm{d}t \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3,$$

where p is the Lagrange multiplier of (1.20), which will become the pressure. Consequently, the Euler-Lagrange equations become

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}\alpha_j} \left[p \frac{\partial(x_1, x_2, x_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right] \frac{\mathrm{d}\alpha_j}{\mathrm{d}x_i} = 0,$$

which can be reduced to

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} + \frac{\partial(p, x_j, x_k)}{\partial(\alpha_1, \alpha_2, \alpha_3)} = 0, \tag{1.21}$$

with i, j, k cyclic.

To identify the Euler equations from this expression, it is necessary to rewrite it as follows

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} + \frac{\partial(p, x_j, x_k)}{\partial(\alpha_1, \alpha_2, \alpha_3)} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial p}{\partial \alpha_1} \frac{\partial \left(x_j, x_k\right)}{\partial \left(\alpha_2, \alpha_3\right)} + \frac{\partial p}{\partial \alpha_2} \frac{\partial \left(x_j, x_k\right)}{\partial \left(\alpha_3, \alpha_1\right)} + \frac{\partial p}{\partial \alpha_3} \frac{\partial \left(x_j, x_k\right)}{\partial \left(\alpha_1, \alpha_2\right)} = 0.$$

Next, we replace $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ by v_i and $\frac{\partial p}{\partial \alpha_l}$ by $\frac{\partial p}{\partial x_i} \frac{\partial x_i}{\partial \alpha_l}$ to obtain

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = 0, \tag{1.22}$$

which are the Euler equations since $v_i = v_i(t, x_j(t))$.

Having a Lagrangian for the ideal fluid, it is straightforward to write down the energy-momentum tensor and then explore symmetries to obtain conservation laws. This will be developed in the following chapters.

2 Analogue models of gravity

2.1 Introduction

The aim of this chapter is to review analogue models based on the theory of General Relativity (GR) and to discuss the main concepts and uses of this topic. There are many instances wherein analogue models of gravity arise as an effective method for describing physical phenomena. For instance, analogue models play an important role in fluid dynamics, Bose-Einstein condensates and quantum field theory in curved space-times. Here we focus our attention on those models that can be at least partially reproduced by fluid flows.

Given that mathematics is the main language used for describing nature, it is not to be unexpected that phenomena in different areas of physics have a similar (or equivalent) theoretical description. Mathematically speaking, an analogue model can be thought of as a framework wherein a set of tools developed in a given context is used to solve problems in another background. This gives problems a new perspective once they are solved. Analogue models are also useful when they are used to probe phenomena that cannot be directly accessed by means of experiments. Perhaps the most popular example is the case of phenomena associated with black holes [4] (for instance, the Hawking radiation) for which analogue models based on classical fluids [5, 6, 7, 8] and condensates [9, 10, 11, 12] are employed.

2.2 Analogue models in physics

The first analogue model in general relativity was developed by Gordon in 1923 [13] who presented an analogue space-time based on optics. People rapidly lost interest in it and it took many years for it to be revisited. This happened with Unruh in 1981 with the article [5], where an analogue black hole model was developed. The model was based on fluid dynamics and its aim was to emulate aspects of the physics of black holes. In a supersonic fluid, something similar to an event horizon takes place, wherein acoustic excitations are not able to escape a horizon. The associated phenomenon was therefore referred to as a “dumb hole”. It was not until one decade later that such models became dominant and that interesting topics about dumb holes started to be studied.

The main idea here is to model a physical system as a background which is represented by an effective metric. Then, excitations can be studied perturbatively as they propagate on this background. These excitations can be broadly classified into *rays* and *waves*. Ray-like phenomena have as excitations particles/quasi-particles which propagate along geodesics of the effective metric. On the other hand, to describe wave excitations we need to derive a field equation which is compatible with that background. As we will show later, in many cases this equation can be reduced to a massless Kein-Gordon equation,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right)=0,$$

where $g = \det g_{\mu\nu}$ is the determinant of the effective metric. The scalar field ψ represents the excitation (which could be sound waves or quantum fields, for example).

Since particles have less structure than waves, analogue models within the ray-limit are simpler. We therefore consider them first in the following sections, and we reserve the discussion about the wave-limit to the next chapter.

2.3 Analogue models in optics

We start by defining the notation that we are going to use in what follows. The Minkowski metric is represented by

$$\eta_{\mu\nu}=\left(\begin{array}{c|c}-1 & 0\\ \hline 0 & \delta_{ij}\end{array}\right).$$

Greek indices run from 0 to 3 while latin from 1 to 3. The 4-velocity, denoted by u^μ, can be written as

$$u^\mu=\gamma\left(c_{\text{light}},\vec{v}\right),$$

where c_{light} is the speed of light (that will be set to 1 for simplicity), $\vec{v}$ is the velocity of the medium, and $\gamma=\left(1-v^2/c_{\text{light}}^2\right)^{-1/2}$.

Let n be the refractive index of the medium. The Gordon metric is then defined as

$$g_{\mu\nu}=\eta_{\mu\nu}+\left(1-\frac{1}{n^2}\right)u_\mu u_\nu. \tag{2.1}$$

When the medium is at rest, this metric reduces to

$$g_{\mu\nu}=\left(\begin{array}{c|c}-\frac{1}{n^2} & 0\\ \hline 0 & \delta_{ij}\end{array}\right), \tag{2.2}$$

which yields

$$\mathrm{d}s^2=-\frac{1}{n^2}\mathrm{d}t^2+\mathrm{d}\vec{x}^2. \tag{2.3}$$

In the case of geometric optics the (light) rays move at light speed and, as such, they satisfy $\mathrm{d}s^2=0$. Therefore

$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right|=\frac{1}{n}, \tag{2.4}$$

which is a well-known relation from optics.

The effective metric (2.2) describes light-rays in a medium with refractive index n. It is also possible to substitute $\eta_{\mu\nu}$ by an arbitrary non-flat spacetime with metric $h_{\mu\nu}$. From the point of view of geometrical optics, we could then write the Gordon metric as follows:

$$g_{\mu\nu} = \Omega^2 \left[h_{\mu\nu} + \left(1 - \frac{1}{n^2} \right) u_\mu u_\nu \right], \tag{2.5}$$

where Ω is a conformal factor. This expression, with all parameters depending on space and time is the more general form of the Gordon metric.

Interesting applications of (2.5) and other effective metrics appear in metamaterials (materials obtained artificially in a laboratory with properties not usually found in Nature). These materials can be synthetically produced in a way that their corresponding effective metric emulates interesting examples of "physical spacetimes".

As an application of the metric (2.5), let us set the conformal factor Ω as the refractive index of the medium $\Omega = n(\vec{x})$. When the physical metric is the Minkowski spacetime and the medium is at rest ($u_\mu = (1, \vec{0}) = \delta^0_\mu$), the Gordon metric then becomes

$$g_{\mu\nu} = n^2 \left(\eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu \right) - \delta^0_\mu \delta^0_\nu, \tag{2.6}$$

which leads to the following line element:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + n^2 \mathrm{d}x_i \mathrm{d}x^i. \tag{2.7}$$

From here, we can emulate different metrics by manipulating the refractive index n. For instance, if the refractive index takes the form

$$n(r) = \frac{n_0}{1 + r^2/a^2},$$

then the line element (in spherical coordinates) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{n_0^2}{1 + r^2/a^2} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \right). \tag{2.8}$$

This is the metric of the Einstein static universe [14]. This opens the possibility of studying several interesting examples of media with varying refractive index in terms of effective metrics (see, for instance, [15]).

2.4 Analogue models in acoustics

Similarly to optics, acoustics can be described from two perspectives: geometrical acoustics and physical acoustics. Here we focus on the first case (geometrical acoustics); the second case will be discussed in detail in the following chapters.

Within the realm of geometrical acoustics, we can consider two cases: the classical and the relativistic regimes.

2.4.1 Classical regime

Most of the phenomena observed in the laboratory rarely reach relativistic velocities. In fact, relativistic acoustics are only manifested in astrophysical and cosmological situations. In this way, an analogue model for acoustics within the classical limit is enough for many experiments in laboratory. Here we make two considerations. First, let c be the speed of sound measured by an observer moving with the fluid, and $\vec{v}$ the fluid velocity measured in the laboratory. Using the Galilean velocity addition law, the sound velocity in the laboratory will be

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = c\hat{n} + \vec{v}, \tag{2.9}$$

where $\hat{n}$ defines the direction of propagation of sound.

In the ray-limit, condition $\hat{n}^2 = 1$ allows one to write equation (2.9) as

$$(\mathrm{d}\vec{x} - \vec{v}\mathrm{d}t)^2 = c^2\mathrm{d}t^2,$$

which can be rearranged to

$$-\left(c^2 - \vec{v}^2\right)\mathrm{d}t^2 - 2\vec{v}\mathrm{d}t\cdot\mathrm{d}\vec{x} + \mathrm{d}\vec{x}^2 = 0. \tag{2.10}$$

This expression shows that sound rays will travel along null geodesics of the effective metric

$$g_{\mu\nu} = \Omega^2 \left(\begin{array}{c|c} -\left(c^2 - v^2\right) & -v^j \\ \hline -v^i & \delta_{ij} \end{array}\right), \qquad \mathrm{d}x^\mu = (\mathrm{d}t, \mathrm{d}\vec{x}). \tag{2.11}$$

Then, as expected

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = 0.$$

We note that, by inverting $[g_{\mu\nu}]$, we obtain the following expression for $g^{\mu\nu}$

$$g^{\mu\nu} = \frac{1}{c^2\Omega^2}\left(\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c^2\delta^{ij} - v^iv^j \end{array}\right), \tag{2.12}$$

so that $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$.

The metric defined above is called the *acoustic metric* or *Unruh's metric* [16]. We note that the conformal factor Ω present in the acoustic metric cannot be determined in the geometrical acoustics limit. As we will see later, this is no longer the case when we consider the physical acoustics limit.

2.4.2 Relativistic regime

The simplest way to obtain the general form for the acoustic metric is by noting that it can also be written in a similar way to the Gordon metric. In order to do that we must make some sense of the“refractive index” in acoustics, by relating n and c in a kind of “Gordon metric” for fluid mechanics. After tedious algebra considering the relativistic equations of motion for fluids, one can show that that n should be substituted by c_{light}/c. In this way, the Gordon metric for geometrical acoustics has the following expression [16]

$$g_{\mu\nu} = \Omega^2 \left[h_{\mu\nu} + \left(1 - \frac{c^2}{c_{\text{light}}^2} \right) u_\mu u_\nu \right]. \tag{2.13}$$

Once again, the physical metric $h_{\mu\nu}$ is arbitrary but when $h_{\mu\nu} \to \eta_{\mu\nu}$ and $c \ll c_{\text{light}}$, one can show that the above expression reduces to (2.11).

The acoustic metrics (2.13) and (2.11), with Ω undetermined, is essentially all that can be done within the geometrical acoustics limit. As mentioned before, in order to determine Ω we must enter the physical acoustics realm. We will show later that the conformal factor is important for the study of sound waves in the medium.

2.5 Bose-Einstein condensates in analogue models

Leaving classical systems aside, Bose-Einstein condensates (BECs) become relevant when we want to illustrate some effective metrics in laboratory. BECs are relatively accessible when it comes to their generation and manipulation. An interesting characteristic about sound waves in BECs is that they have a very small speed compared with the speed found in classical fluids. This fact facilitates, in principle, the production of horizons and ergo–regions. The high degree of quantum coherence and low temperatures of BECs show that they are also attractive to mimic aspects of semiclassical gravity such as Hawking radiation and particle production (more details about this topic can be found in [17, 18, 19]).

The derivation of an analogue model based on BECs starts by splitting the wave function (of BEC) Ψ into a background part plus a fluctuation, i.e., $\Psi = \langle \Psi \rangle + \psi$. However, ψ by itself, will not behave like an excitation over a background. Since

$$\psi = e^{-i\theta/\hbar} \left(\frac{1}{2\,|\Psi|} \eta - i \frac{|\Psi|}{\hbar} \xi \right), \tag{2.14}$$

(where η and ξ are quantum fields) one can prove that ξ satisfy the massless Klein Gordon equation, i.e.,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \xi \right) = 0.$$

The effective metric $g_{\mu\nu}$ here is similar to (2.11), with the difference that Ω, $\vec{v}$ and c are now given by

$$\Omega^2 = \frac{\rho}{mc},$$
$$\vec{v} = \vec{\nabla}\theta/m,$$
$$c^2 = \frac{\kappa(a)\rho}{m}.$$

where $\rho = |\Psi|$, m is the mass and $\kappa(a)$ is a characteristic parameter of BECs.

We refer the reader to [20] for additional information and references on analogue models based on BECs.

2.6 Surface waves

Surface waves on interfaces have a complex theory. Their physical properties in general have a strong dependence on the wave frequency. Some aspects of the theory may be simplified by using analogue models wherein the effective metric has the frequency as a parameter. The first works on this topic are [8, 21]. It is also possible to obtain a description of surface waves within the geometrical limit, but some assumptions must be made to that end. Once again, if the wavelength and period of the surface wave are small compared to the system scale, we can successfully adopt the metric

$$\mathrm{d}s^2 = \Omega^2 \left[-c_{\mathrm{sw}}^2 \mathrm{d}t^2 + (\mathrm{d}\vec{x} - \vec{v}\mathrm{d}t)^2 \right],$$

where c_{sw} is the speed of surface waves and $\vec{v}$ is the horizontal velocity of the surface. We see that this metric is defined in a (2+1)-spacetime and that surface waves also travel along null geodesics of that spacetime. Although the above expression for the effective metric seems to be simple it might get increasingly complicated depending on the speed of surface waves c_{sw}. In the physical limit case (i.e., when one considers wave properties of surface waves), the effective metric also contributes to wave properties through the conformal factor Ω. With those considerations and after algebraic procedures, the expressions for the effective metric for surface waves are similar in form to those obtained for the acoustics case. This implies that they will have many features in common such as horizons and ergo–regions. Some experimental results about this topic can be found in [22, 23].

2.7 Slow light in fluids

The electromagnetically induced transparency (EIT) is a technique which allows one to manipulate the refractive index of a medium to drastically reduce the speed of light. With this, event horizons and ergo–regions (for the slowed light) are easier to reproduce. Since the EIT technique is complicated, the theory to derive an analogue model for those cases is quite involved. However, in the end, the theory can be shown to yield the following effective metric

$$g^{\mu\nu} = \left(\begin{array}{c|c} -(1+\alpha v^2/c^2_{\text{light}}) & -\alpha v_j/c^2_{\text{light}} \\ \hline -\alpha v_i/c^2_{\text{light}} & \delta_{ij} - 4\alpha v_i v_j/c^2_{\text{light}} \end{array} \right), \tag{2.15}$$

where α is a characteristic parameter of the experiment. Some exciting results based on this field are found in [24, 25, 26].

The cases presented in this chapter are, in our view, the most relevant/interesting examples that can be developed by means of analogue models of gravity. The remaining models can be derived (or extended) from these fundamental examples. To obtain additional information about other analogue models we refer to the reader to [20].

As a last remark, we note that among the effective metrics we mentioned, the acoustic metric is more popular than others because of its simplicity and applicability to a wide variety of experiments in the laboratory. Since our focus are analogue models based on fluids, we provide a more detailed characterization of the metric (2.11) in the next chapter.

3 Analogue models in fluid dynamics

3.1 Introduction

As discussed in the previous chapters, analogue models are useful in many areas of physics. They give a simplified description of phenomena and also illustrate aspects of general relativity in the laboratory. Since Unruh's first publication on the subject, many authors studied different kinds of analog models in different contexts.

The analogue models considered before follow a certain general pattern. This can be summarized in a two-step procedure: (1) the equations which characterize the system's background are obtained and (2) small perturbations on the main quantities involved are considered (depending on the physical system, sometimes additional conditions are needed in order to obtain the effective metric). For instance, in the vicinity of a black hole, the accretion disc is described using relativistic hydrodynamics and the analogue description of this system is still made by means of an effective metric. In this case, this effective metric incorporates the physical metric due to the black hole and an acoustic metric similar to (2.13). A detailed study of this subject can be found in [27]. In the same way, one could construct analogue models that describe a wide variety of systems of almost any area of physics.

As mentioned before, to derive an analogue model in fluid dynamics, whether relativistic or not, there are different perspectives that could be adopted. Depending on how one would like to study a physical system, a given perspective will be more convenient than another. The simple perspective given by the case of the geometrical acoustic limit was already considered in the previous chapter. Now we consider the case of physical acoustics, which incorporates the wave properties of the perturbations in the analogue model. For convenience, we restrict ourselves to non-relativistic velocities for the fluid flow. Different properties of the acoustic metric will be discussed below, such as horizons, ergo-regions and singularities. We also clarify an ambiguous definition of energy that appears in this context and that will be important for the next chapter.

3.2 Effective field theory of ideal fluid

In order to construct an analogue model for fluids, we will first obtain a description of fluid dynamics from the effective field theory perspective. Consider an ideal isentropic fluid. We once again denote the density by ρ and the velocity of the flow by $\vec{v}$. According to Eckart's variational principle [28], the Lagrangian density of a fluid flow with these characteristics may be written as

$$\mathcal{L} = \frac{1}{2}\rho\vec{v}^2 - \rho\varepsilon + \Psi\left[\partial_0\rho + \vec{\nabla}\cdot(\rho\vec{v})\right], \tag{3.1}$$

where $\varepsilon = \varepsilon(\rho)$ is the internal energy per unit mass, Ψ is the Lagrange multiplier associated with the mass conservation constraint, and $\frac{\partial}{\partial t} \to \partial_0$. Variation of $\mathcal{L}$ with respect to $\vec{v}$ yields

$$\vec{v} = \vec{\nabla}\Psi, \tag{3.2}$$

which implies that $\vec{v}$ is in fact irrotational. The Lagrange multiplier Ψ turns out to be the scalar potential associated with the velocity of the fluid.

In this way, the Lagrangian density in terms of Ψ is given by

$$\mathcal{L} = -\rho\left[\partial_0\Psi + \frac{1}{2}\left(\vec{\nabla}\Psi\right)^2 + \varepsilon\right]. \tag{3.3}$$

Variation of $\mathcal{L}$ with respect to ρ then yields

$$\partial_0\Psi + \frac{1}{2}\left(\vec{\nabla}\Psi\right)^2 + \frac{\partial\rho\varepsilon}{\partial\rho} = 0, \tag{3.4}$$

where the third term on the left hand side is identified with the specific enthalpy h, i.e.,

$$h = \frac{\partial\rho\varepsilon}{\partial\rho}. \tag{3.5}$$

Taking the gradient of the expression (3.4) and using standard thermodynamics to relate the specific enthalpy to pressure, one arrives to the Euler equations

$$\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla}p,$$

as expected. With the Lagrangian density already defined, the energy-momentum tensor and the energy of the fluid may be obtained in a straightforward way, by usual field theoretical procedures.

3.3 Perturbations of the fluid flow and the Unruh's metric

The second step to derive an analogue model is to split the system in question into two sectors: a background part and a perturbative part. Let us write

$$\begin{cases} \Psi = \Psi_0 + \psi, \\ \rho = \rho_0 + \varrho, \end{cases} \tag{3.6}$$

where Ψ_0 and ρ_0 represent the background and ψ and ϱ are the perturbations. Before substituting these quantities in equation (3.3), we need to expand the internal energy $\rho\varepsilon(\rho)$ around ρ_0. A Taylor series expansion applied $\rho\varepsilon(\rho)$ leads to

$$\rho\varepsilon = \rho_0\varepsilon(\rho_0) + h(\rho_0)\varrho + \frac{1}{2}\left.\frac{\partial h}{\partial\rho}\right|_{\rho_0}\varrho^2 + \mathcal{O}\left(\varrho^3\right).$$

To find $h(\rho_0)$ we use equations (3.2) and (3.4). The derivative of h is obtained through the relation $\mathrm{d}p = \rho \mathrm{d}h$. Writing

$$\frac{\partial h}{\partial \rho} = \frac{1}{\rho}\frac{\partial p}{\partial \rho},$$

and evaluating at $\rho = \rho_0$ gives

$$h'(\rho_0) = \frac{c_0^2}{\rho_0}, \tag{3.7}$$

where, for convenience, we define c_0 as

$$c_0^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}. \tag{3.8}$$

Substituting all these expressions (including the perturbed quantities) back into the Lagrangian density, and retaining terms up to second order, we get

$$\mathcal{L} \approx \mathcal{L}_0 - \varrho \partial_0 \psi - \frac{1}{2}\frac{c_0^2}{\rho_0}\varrho^2 - \frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 - \varrho \vec{\nabla}\Psi_0 \cdot \vec{\nabla}\psi, \tag{3.9}$$

where the term $\mathcal{L}_0$ represents the Lagrangian of the background. The remaining terms can be identified with the Lagrangian for the perturbative terms, i.e.,

$$\delta\mathcal{L} \equiv -\varrho \partial_0 \psi - \frac{1}{2}\frac{c_0^2}{\rho_0}\varrho^2 - \frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 - \varrho \vec{v} \cdot \vec{\nabla}\psi, \tag{3.10}$$

where $\vec{v} = \vec{v}_0 = \vec{\nabla}\Psi_0$. Consequently, the equation of motion for ϱ is

$$\varrho = -\frac{\rho_0}{c_0^2}\left(\partial_0 \psi + \vec{v} \cdot \vec{\nabla}\psi\right). \tag{3.11}$$

Substituting this back into the Lagrangian density yields

$$\delta\mathcal{L} = -\frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 + \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\partial_0 \psi + \vec{v} \cdot \vec{\nabla}\psi\right)^2. \tag{3.12}$$

In this form, the Lagrangian has only ψ as a parameter and its equation of motion becomes

$$\left(\partial_0 + \vec{\nabla} \cdot \vec{v}\right)\frac{\rho_0}{c_0^2}\left(\partial_0 + \vec{v} \cdot \vec{\nabla}\right)\psi - \vec{\nabla}\left(\rho_0 \vec{\nabla}\psi\right) = 0. \tag{3.13}$$

After tedious algebra this equation can be written in matrix form,

$$\begin{pmatrix} \partial_0 & \partial_j \end{pmatrix} \begin{pmatrix} -\rho_0/c_0^2 & -\rho v_j/c_0^2 \\ -\rho v_i/c_0^2 & \rho_0 \delta_{ij} - \rho_0 v_i v_j/c_0^2 \end{pmatrix} \begin{pmatrix} \partial_0 \psi \\ \partial_i \psi \end{pmatrix} = 0. \tag{3.14}$$

This matrix representation suggests that we introduce the so-called metric density

$$\mathcal{F}^{\mu\nu} \equiv \frac{\rho_0}{c_0^2}\left(\begin{array}{c|c} -1 & -v_j \\ \hline -v_i & c_0^2 \delta_{ij} - v_i v_j \end{array}\right). \tag{3.15}$$

It then follows from equation (3.13) that

$$\partial_\mu \left(\mathcal{F}^{\mu\nu}\partial_\nu \psi\right) = 0.$$

Now we note that, if we make the identification

$$\mathcal{F}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}, \tag{3.16}$$

then the equation of motion for ψ becomes the wave equation in a curved space-time with metric $g_{\mu\nu}$, i.e.,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0. \tag{3.17}$$

To obtain the explicit form of $g_{\mu\nu}$, we first need to compute $g^{\mu\nu}$. Taking the determinant of the metric density

$$|\mathcal{F}^{\mu\nu}| = -\frac{\rho_0^4}{c_0^2} = (-g)^{(1+N)/2}\,|g^{\mu\nu}| = -(-g)^{(N-1)/2},$$

where N represents the number of spatial dimensions, the acoustic metric $g_{\mu\nu}$ reduces to

$$g^{\mu\nu} = \frac{1}{\sqrt{-g}}\mathcal{F}^{\mu\nu} = \left(\frac{\rho_0}{c_0}\right)^{-\frac{2}{N-1}}\left(\begin{array}{c|c} -1/c_0^2 & -v_j/c_0^2 \\ \hline -v_i/c_0^2 & \delta_{ij} - v_iv_j/c_0^2 \end{array}\right). \tag{3.18}$$

Therefore, the acoustic metric takes the form

$$g_{\mu\nu} = \left(\frac{\rho_0}{c_0}\right)^{\frac{2}{N-1}}\left(\begin{array}{c|c} -(c_0^2 - v^2) & -v_j \\ \hline -v_i & \delta_{ij} \end{array}\right), \tag{3.19}$$

where, as usual, we took $g_{\mu\nu} = [g^{\mu\nu}]^{-1}$ so that $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$. As a result, the line element is given by

$$\mathrm{d}s^2 = \left(\frac{\rho_0}{c_0}\right)^{\frac{2}{N-1}}\left[-\left(c_0^2 - v^2\right)\mathrm{d}t^2 - 2v_i\mathrm{d}t\mathrm{d}x^i + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]. \tag{3.20}$$

As expected, this last result agrees with the one found in the geometrical acoustics limit. Moreover, the conformal factor can now be unequivocally determined, ans is given by

$$\Omega = \left(\frac{\rho_0}{c_0}\right)^{\frac{1}{N-1}}. \tag{3.21}$$

We note in passing that, until now, we only considered non-relativistic fluids. To obtain a general form of the acoustic metric including the relativistic case, one could start from the relativistic energy-momentum tensor and follow the above steps. As a result, one would find that the acoustic metric is changed only by its conformal factor, which becomes

$$\Omega = \left[\frac{n^2}{c_0(\rho_0 + p_0)}\right]^{\frac{1}{N-1}},$$

where n is the particle density and ρ_0 now represents the energy density. In addition, the new acoustic metric will reduce to (3.19) when $p_0 \ll \rho_0$ and $\rho_0 \approx \bar{m}n$, where $\bar{m}$ is the average mass. For the present work, we are only interested in non-relativistic fluids, so the expression (3.21) for Ω will be enough.

The exciting part of constructing an analogue model appears when the effective metric (the acoustic metric in this case) is viewed as a "physical metric" of some spacetime model along with its symmetries, possible ergo-regions, etc. Here we briefly consider the meaning and implications of these concepts. Some of those topics will be further developed in the following sections.

3.4 Some properties of the acoustic metric

Let us point out some features of the acoustic metric discussed above:

- It is Lorentzian, i.e., it has signature $(-, +, +, +)$.
- In general relativity, a $(3+1)$ Lorentzian metric has 6 degrees of freedom but the acoustic metric is only defined by three scalar quantities, ρ_0, Ψ_0 and c_0. This implies that it is not possible to reproduce all aspects of a generic physical metric obtained from Einstein's field equations by means of analogue models.
- There is a correspondence between "steady flow" and "static metric" in general relativity.
- Similarly to general relativity, it is possible to define the 4-velocity and proper time as
$$u^\mu = \left(\frac{c_0}{\rho_0}\right)^{\frac{1}{N-1}} (1/c_0, \vec{v}/c_0),\ \tau = \int \frac{1}{c_0}\left(\frac{c_0}{\rho_0}\right)^{\frac{1}{N-1}} \mathrm{d}t,$$
respectively. However, these are not as relevant concepts as in the theory of relativity. This is because the relevant physical quantities here are the perturbations of the system and these are represented by the quantities ψ and ϱ.
- Notice that the "real physical metric of the system" is the Euclidean (or Minkowski) metric. However, everything happens as if the perturbations propagate as waves in an effective metric described by $g_{\mu\nu}$.
- It is important to emphasize that a physical metric obtained by solving the Einstein's field equations relates the mass distribution to the spacetime curvature. However, the acoustic metric does not relate any physical quantity with the space geometry; it appears here only as a mathematical tool.

3.5 Killing vectors and the acoustic metric

Other interesting concepts related to Lorentzian spacetimes are horizons and ergosurfaces. In the case of a physical system such as a spinning black hole, these regions play an important role because they change the spacetime structure in a radical way. In this way, it is to be expected that in the case of the acoustic metric these regions should play an important role.

We start by classifying a metric according to how it depends on the time coordinate. A metric is said to be **stationary**, if there is a reference frame where the metric is time independent, i.e.,

$$\partial_0 g_{\mu\nu} = 0. \tag{3.22}$$

Otherwise it is said to be **non-stationary**.

We know that invariance of a physical quantity under some coordinate change often gives rise to a conservation law. Consider an infinitesimal coordinate transformation $x^{\mu'} = x^\mu + \xi^\mu$. Under this coordinate change the metric transforms to

$$g^{\mu'\nu'}(x^{\sigma'}) = \frac{\partial x^{\mu'}}{\partial x^\alpha}\frac{\partial x^{\nu'}}{\partial x^\beta} g^{\alpha\beta}(x^\sigma) \approx g^{\mu\nu}(x^\sigma) + g^{\mu\alpha}\partial_\alpha \xi^\nu + g^{\nu\alpha}\partial_\alpha \xi^\mu. \tag{3.23}$$

To evaluate the change in the metric we compute the numerical value of both $g^{\mu'\nu'}(x^\sigma)$ and $g^{\mu\nu}(x^\sigma)$ in the coordinates x^σ, i.e.,

$$g^{\mu'\nu'}(x^{\sigma'}) = g^{\mu'\nu'}(x^\sigma + \xi^\sigma) \approx g^{\mu\nu}(x^\mu) + g^{\mu\alpha}\partial_\alpha \xi^\nu + g^{\nu\alpha}\partial_\alpha \xi^\mu. \tag{3.24}$$

Taylor expanding $g^{\mu'\nu'}$ in terms of x^σ then gives

$$\begin{aligned} g^{\mu'\nu'}(x^\sigma) &\approx g^{\mu\nu}(x^\sigma) + g^{\mu\alpha}\partial_\alpha \xi^\nu + g^{\nu\alpha}\partial_\alpha \xi^\mu - \xi^\sigma \partial_\sigma g^{\mu\nu} \\ &= g^{\mu\nu} + \delta g^{\mu\nu}, \end{aligned} \tag{3.25}$$

where $\delta g^{\mu\nu} \equiv g^{\mu\alpha}\partial_\alpha \xi^\nu + g^{\nu\alpha}\partial_\alpha \xi^\mu - \xi^\sigma \partial_\sigma g^{\mu\nu}$. This can also be expressed in a covariant form (using $\nabla^\mu \xi^\nu = g^{\mu\sigma}\partial_\sigma \xi^\nu + g^{\mu\sigma}\Gamma^\nu_{\sigma\alpha}\xi^\alpha$) as

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu. \tag{3.26}$$

Or, equivalently,

$$\delta g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \tag{3.27}$$

We note that the form of the metric will not change under infinitesimal transformations if $\delta g^{\mu\nu} = 0$. If $g_{\mu\nu}$ does not depend on time, it does not change under the transformation

$$\begin{cases} t \to t + \epsilon, \\ \vec{x} \to \vec{x}, \end{cases} \tag{3.28}$$

i.e., $x^\mu \to x^\mu + \epsilon\delta^\mu_0$. Taking

$$\xi^\mu = \delta^\mu_0, \tag{3.29}$$

it is straightforward to check that this vector in any coordinate then satisfies

$$\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = 0. \tag{3.30}$$

In this way, we get $\delta g^{\mu\nu} = 0$ as we expected.

Any vector ξ^μ satisfying (3.30) is then called a *Killing vector field*. We see that the symmetries of space are manifested mathematically by means of Killing vector fields. Hence, the Killing vector shown in (3.29) encodes the time translation symmetry of the acoustic metric.

Returning to stationary metrics, there is an additional classification to consider. The metric can be "static" or "non-static". Mathematically speaking, if there is a coordinate transformation for which the metric assumes the form

$$\mathrm{d}s^2 = (\xi_\mu \xi^\mu)^{-1} \mathrm{d}\tau^2 + g_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{3.31}$$

where ξ^μ is a timelike Killing vector, then the metric is said to be static.

For the acoustic effective metric, we see that if it is possible to find a coordinate transformation

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\vec{v} \cdot \mathrm{d}\vec{x}}{c_0^2 - v^2}, \tag{3.32}$$

then the acoustic metric is static.

Looking closely at equation (3.32), we note that the new time coordinate τ mixes t and $\vec{x}$. This leads to an important implication when we talk about conservation of energy in this context. In fact, since energy is not a (Lorentz) scalar, conservation of energy in one frame does not imply that it is conserved in all frames. We note, however, that in the acoustic context not all frames of reference are (physically) equivalent: unlike in general relativity, the laboratory reference is a preferred frame in this case.

The acoustic metric also allows us to define, just as in general relativity, the concepts of ergo–regions and horizons. These are defined as follows:

- An ergo-region is formed when the timelike Killing vector ξ^μ becomes spacelike, and this occurs when $|\vec{v}| \geq c_0$.
- Horizons are surfaces from where null geodesics cannot escape, i.e., for which $|\vec{v}_\perp| = c_0$.

3.6 Conservation laws in acoustics

Noether's Theorem relates conservation laws to symmetries of the action. Here we use this to derive conservation laws for sound waves.

Let us rewrite the Lagrangian density for sound waves as

$$\mathcal{L} = -\frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 + \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\partial_0\psi + \vec{v}\cdot\vec{\nabla}\psi\right)^2,$$

where, for the sake of simplicity, the term δ of the left hand side of (3.12) was dropped. Rearranging this Lagrangian, we get

$$\mathcal{L} = \sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi, \tag{3.33}$$

where $g^{\mu\nu}$ is the contravariant acoustic metric and $g = \det g_{\mu\nu}$. As a result, the action associated to this Lagrangian density is given by

$$S = \int \frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi\sqrt{-g}\mathrm{d}^4x. \tag{3.34}$$

This shows that the term

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$$

is indeed the Lagrangian of a scalar field in curved space-time with zero potential energy.

A general expression for the action will depend on $g_{\mu\nu}$, ψ, and $\partial_\mu\psi$, i.e., $S = S\left[g_{\mu\nu}, \psi, \partial_\mu\psi\right]$. The principle of least action ($\delta S = 0$) then leads to

$$\delta S = \int \mathrm{d}^4x \left(\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S}{\delta\psi}\delta\psi + \frac{\delta S}{\delta\partial_\mu\psi}\partial_\mu\delta\psi\right) = 0. \tag{3.35}$$

For convenience, we split the terms of the above equation as follows:

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right) + \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\psi}\delta\psi\right) + \int \mathrm{d}^4x\frac{\delta S}{\delta\partial_\mu\psi}\partial_\mu\delta\psi.$$

Integrating by parts, using the divergence theorem, and imposing that $\delta\psi$ vanishes on the boundary, the third term will vanish. The second term vanishes by virtue of the equation of motion for ψ, i.e., $\frac{\delta S}{\delta\psi} = 0$. Therefore, the variation of the action reduces to

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right). \tag{3.36}$$

Now we explore the symmetries of the action. Consider an infinitesimal translation expressed as a coordinate transformation $x^{\mu'} = x^\mu - \epsilon^\mu$. From the previous discussion, the metric $g_{\mu'\nu'}$ changes from $g_{\mu\nu}$ to $g_{\mu\nu} + \delta g_{\mu\nu}$, where

$$\delta g_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu. \tag{3.37}$$

Substituting this back into equation (3.36) yields

$$\int \mathrm{d}^4x\sqrt{-g}\left(\nabla_\mu\epsilon_\nu\right)\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \int \mathrm{d}^4x\sqrt{-g}\epsilon_\nu\nabla_\mu\left(\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right) = 0, \tag{3.38}$$

where we used the fact that

$$\nabla_\mu\left(\epsilon_\nu\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right) = \left(\nabla_\mu\epsilon_\nu\right)\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} + \epsilon_\nu\nabla_\mu\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right).$$

The second part of equation (3.38) leads to a first conservation laws. Defining

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}, \tag{3.39}$$

which is called the pseudo energy-momentum tensor, it follows from (3.38) that

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.40}$$

This expression encodes local conservation of energy-momentum. Such statement is not enough to obtain a global conservation law unless the spacetime has a Killing vector ξ^μ. In that case it follows at once that if $T^{\mu\nu}$ is conserved, then the quantity $T^{\mu\nu}\xi_\nu$ is also conserved, i.e.,

$$\nabla_\mu \left(T^{\mu\nu}\xi_\nu\right) = 0. \tag{3.41}$$

Moreover,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^{\mu\nu}\xi_\nu\right) = 0. \tag{3.42}$$

If we denote the 4-vector density $\sqrt{-g}T^{\mu\nu}\xi_\nu$ by $\mathcal{Q}^\mu$, the above expression reduces into

$$\partial_\mu \mathcal{Q}^\mu = 0, \tag{3.43}$$

And the following quantity is then conserved in time

$$\int_S \mathcal{Q}^\mu n_\mu \mathrm{d}^3x, \tag{3.44}$$

where n_μ denotes the normal to S (which is a spacelike surface).

We now proceed to compute the conservation laws for sound waves. Equation (3.34) represents the action of a scalar field for which the Lagrangian is given by

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi.$$

Its corresponding canonical momentum is

$$\pi^\mu = \frac{\partial L}{\partial(\partial_\mu \psi)} = \partial^\mu \psi. \tag{3.45}$$

The energy–momentum tensor for this Lagrangian can then be written as follows [29]

$$T^{\mu\nu} = \pi^\mu \partial^\nu \psi - g^{\mu\nu} L, \tag{3.46}$$

where $g^{\mu\nu}$ is the acoustic metric. Substituting π^μ and L into this energy–momentum tensor we get

$$T^{\mu\nu} = \partial^\mu \psi \partial^\nu \psi - g^{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi\right). \tag{3.47}$$

In the case of the acoustic metric, the Killing vector associated to the conservation of energy was (3.29). Thus

$$\mathcal{Q}^\mu = \sqrt{-g} T^{\mu\alpha} g_{\alpha\beta} \delta_0^\beta = \frac{\rho_0^2}{c_0} T^{\mu\alpha} g_{\alpha 0}. \tag{3.48}$$

Substituting the expression for $T^{\mu\nu}$ we get

$$\mathcal{Q}^0 = \frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 + \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\partial_0\psi\right)^2 - \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\vec{v}\cdot\vec{\nabla}\psi\right)^2, \tag{3.49}$$

and we identify this as the energy density of sound waves. Thus

$$E = \int \mathcal{Q}^0 \mathrm{d}^3x \tag{3.50}$$

is conserved and

$$E = \int \left[\frac{1}{2}\rho_0 \left(\vec{\nabla}\psi\right)^2 + \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\partial_0\psi\right)^2 - \frac{1}{2}\frac{\rho_0}{c_0^2}\left(\vec{v}\cdot\vec{\nabla}\psi\right)^2\right]\mathrm{d}^3x \tag{3.51}$$

will be the total energy.

In this last expression, we note that there are additional second order terms contributing to the total energy such as $\partial_0\psi$. It could be difficult to figure out that $\partial_0\psi$ or $\vec{v}\cdot\vec{\nabla}\psi$ would contribute to the total energy had we not followed the action approach considered above.

The energy is a crucial quantity which identifies in many cases whether solutions of specific systems make sense physically. These criteria will be useful ahead to impose conditions on solutions we find for the physical systems considered here.

4 Analogue Model for Radial Flow

4.1 Introduction

We now use the tools developed in the previous chapters to study the case of an ideal fluid flow with cylindrical symmetry, with velocity field given by $\vec{v} = v(r)\hat{e}_r$. We will limit our analysis to the subsonic case, i.e., to $v(r) < c$, where c is the speed of sound when the fluid is at rest.

The case of a constant radial velocity was studied in [30]. This is the case where $\vec{v} = \alpha c\hat{e}_r$, with constant $|\alpha| < 1$. The resulting effective metric then turns out to be that of $AdS_2 \times S^1$, where AdS_2 is the two-dimensional Anti-de Sitter (AdS) space. As well known, the AdS space is nonglobally hyperbolic [4]. On nonglobally hyperbolic spacetimes, the evolution of the wave equation is not completely determined by initial conditions (since, by definition, the space does not admit a Cauchy surface in this case). Some kind of boundary conditions is then needed in order to one obtain a deterministic solution to the wave equation. In the case of AdS, extra boundary conditions should be specified at its spatial infinity. As we shall see, this spatial infinity is mapped to the origin of the space in the fluid dynamics analogy. In this way, the lack of global hyperbolicity of AdS can be interpreted as the need to provide a description of how sound waves interact with the point source/sink of the fluid. Such description is made by means of a phase difference between ingoing and outgoing scattered waves.

Here we propose an alternative solution to this issue of extra boundary conditions. This is done by regularizing the velocity profile of the fluid near the origin. After this regularization, the resulting effective metric can be interpreted as that of a deformed AdS spacetime.

4.2 Wave equation for sound waves

We start by studying sound waves in two dimensional ideal fluid flows. As discussed above, we are interested in the case where the velocity can be written as

$$\vec{v} = v(r)\hat{e}_r. \tag{4.1}$$

The acoustic metric for the fluid flow then takes the form

$$g_{\mu\nu} = \frac{\sigma^2}{c^2}\begin{pmatrix} -(c^2 - v(r)^2) & -v(r) & 0 \\ -v(r) & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \tag{4.2}$$

where $\sigma = \sigma(r)$ is the mass density and c the speed of sound.

The continuity equation provides and additional relation between the mass density and the velocity of the fluid:

$$\frac{\partial}{\partial r}\left[r\sigma(r)v(r)\right] = 0, \tag{4.3}$$

$$\sigma(r) = \frac{A}{rv(r)}, \tag{4.4}$$

where A is an integration constant.

As noted before, the continuity equation requires the mass density to have the form $\sigma(\rho) \propto \frac{1}{\rho v(\rho)}$. Although this flow is somehow exotic, it will perfectly work for us as a toy model in our context [30]. The line element is then given by

$$\mathrm{d}s^2 = \frac{\sigma^2}{c^2}\left[-\left(c^2 - v^2\right)\mathrm{d}t^2 - 2v\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right]. \tag{4.5}$$

As shown in previous chapters, when the fluid is irrotational, we can obtain the fluid velocity from the gradient of a scalar field. We observe that the fluid velocity (4.1) is indeed irrotational, so that we can represent it in terms of a scalar field Ψ.

The next step is to consider a perturbation on this scalar field, i.e.,

$$\Psi(t, r, \theta) = \Psi_0(r) + \psi(t, r, \theta), \tag{4.6}$$

where $\psi(t, r, \theta)$ is the scalar field associated with the perturbation velocity.

According to (3.17), it is straightforward to write down the wave equation for $\psi(t, r, \theta)$ with $g_{\mu\nu}$ defined by equation (4.2). Before doing that, we take advantage of the fact that the acoustic metric is static and use equation (3.32) to construct the transformation

$$\begin{cases} \mathrm{d}\tau = \mathrm{d}t + \frac{v}{c^2 - v^2}\mathrm{d}r, \\ \mathrm{d}\rho = \mathrm{d}r, \\ \mathrm{d}\phi = \mathrm{d}\theta, \end{cases} \tag{4.7}$$

In this way, we can express the acoustic metric in static coordinates as

$$g_{\mu'\nu'} = \frac{\sigma(\rho)^2}{c^2}\begin{pmatrix} -\left(c^2 - v(\rho)^2\right) & 0 & 0 \\ 0 & \frac{c^2}{c^2 - v(\rho)^2} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}. \tag{4.8}$$

Using relations (3.17) and (4.4) we see that the wave equation in the new coordinates is then given by

$$\frac{\partial^2\psi}{\partial\tau^2} = c^2\left(1 - v^2/c^2\right)^2\frac{\partial^2\psi}{\partial\rho^2} - c^2\frac{1 - v^4/c^4}{v}\frac{\mathrm{d}v}{\mathrm{d}\rho}\frac{\partial\psi}{\partial\rho} - \frac{c^2 - v^2}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}, \tag{4.9}$$

where $v = v(\rho)$ and $\psi = \psi(\tau, \rho, \phi)$.

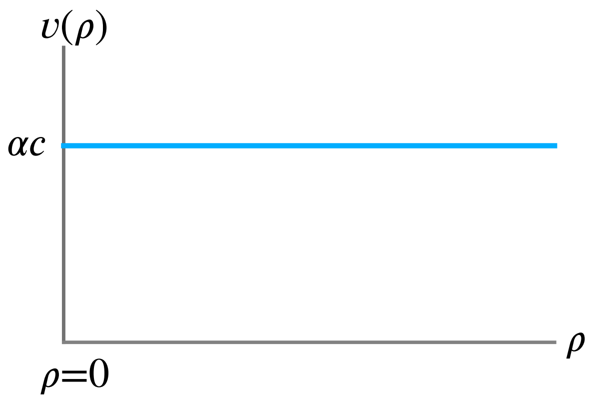

Figure 1 – Fluid flow with constant subsonic velocity, $|\alpha| < 1$. c is the speed of sound.

In order to solve this equation we use separation of variables and consider $\psi(\tau, \rho, \phi) = T(\tau)R(\rho)P(\phi)$. In this way, (4.9) is split into three equations:

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}T(\tau) + \omega^2 T(\tau) = 0, \\ \left(1 - v^2/c^2\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\rho^2}R(\rho) - \frac{1-v^4/c^4}{v}\frac{\mathrm{d}v}{\mathrm{d}\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}R(\rho) + \left(\frac{c^2-v^2}{\rho^2}\frac{\kappa^2}{c^2} + \frac{\omega^2}{c^2}\right)R(\rho) = 0, \\ \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}P(\phi) + \kappa^2 P(\phi) = 0, \end{cases} \tag{4.10}$$

where ω and κ are the separation constants.

It immediately follows that the solutions for $T(\tau)$ and $P(\phi)$ are $e^{\pm i\omega\tau}$ and $e^{\pm i\kappa\phi}$ respectively. To proceed with the calculation we must specify the form of the background velocity $v(r)$ in the differential equation for $R(\rho)$.

The simplest case corresponds to a constant velocity profile, i.e., $v(\rho) = \alpha c$, with $|\alpha| < 1$ (see figure (1)). In this case, the effective metric reduces to

$$\mathrm{d}s^2 = \frac{A^2}{\alpha^2 c^4 \rho^2}\left[-c^2\left(1-\alpha^2\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1-\alpha^2} + \rho^2\mathrm{d}\phi^2\right]. \tag{4.11}$$

Before continuing this analysis we pause to discuss the geometry of the Anti-de Sitter space, for reasons that will soon become obvious.

4.3 The Anti-de Sitter spacetime

We start by considering the n-dimensional Anti-de Sitter spacetime AdS$_n$. This can be defined as a hyperboloid of radius R [31]

$$-\left(X^0\right)^2 + \sum_{k=1}^{n-1}\left(X^k\right)^2 - \left(X^n\right)^2 = -R^2, \tag{4.12}$$

embedded in $\mathbb{R}^{n+1}$ endowed with the (flat) metric,

$$\mathrm{d}s^2 = -\left(\mathrm{d}X^0\right)^2 + \sum_{k=1}^{n-1}\left(\mathrm{d}X^k\right)^2 - \left(\mathrm{d}X^n\right)^2. \tag{4.13}$$

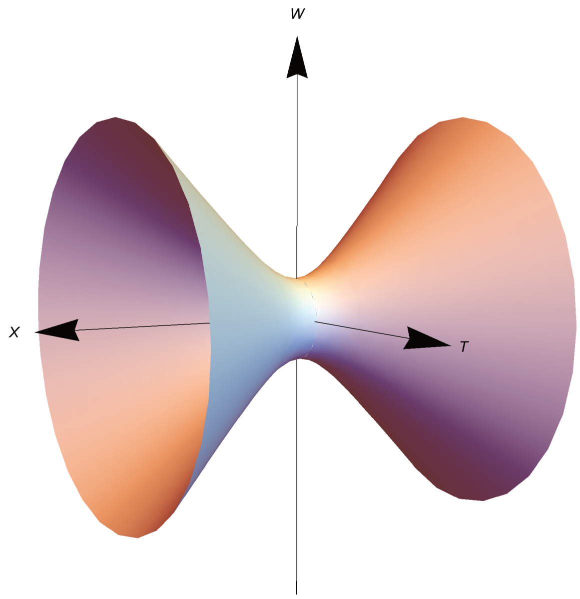

Figure 2 – The AdS_2 space embedded in a Minkowski-like spacetime $M^{1,2}$.

It is important to note that the AdS spacetime still have Lorentzian signature, despite the appearance of two minus signs in the expression above. To clarify this fact, we focus on the AdS_2 space with unit radius ($R = 1$) (see figure (2)). The line element then becomes

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X^2 - \mathrm{d}W^2.$$

If we use the so-called Poincaré coordinates,

$$\begin{aligned} u &= T - X, \\ v &= T + X, \end{aligned} \tag{4.14}$$

we can define the time coordinate in terms of u and W as

$$t = \frac{W}{u}. \tag{4.15}$$

Taking into account (4.12), we see that the coordinates (T, X, W) can be expressed solely in terms of t and u. Therefore, we get

$$\begin{aligned} T &= \frac{1}{2u}\left[1 + u^2\left(1 - t^2\right)\right], \\ X &= \frac{1}{2u}\left[1 - u^2\left(1 + t^2\right)\right], \\ W &= ut. \end{aligned}$$

Since $u = \frac{1}{r}$, the line element finally becomes

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(-\mathrm{d}t^2 + \mathrm{d}r^2\right). \tag{4.16}$$

We see from the above expression that we have a single time, t, in AdS, as expected. We note that $r = 0$ is a singular point in these coordinates.

We see that the spatial boundary of the AdS spacetime is mapped to $r = 0$ in the above coordinates. It is a well-known fact that the AdS spacetime in nonglobally hyperbolic, so that extra boundary conditions must be specified at its spatial boundary in order for one to solve the wave equation [32]. Therefore, we already expect some kind of problem in solving the wave equation near $r = 0$ in the coordinates above. We will arrive at this same conclusion, by an independent analysis, in the sections that follow.

Returning to the analogue model of the previous sections and comparing the metrics (4.11) and (4.16), we can identify that the underlying spacetime in this case is given by $AdS_2 \times S^1$. Therefore, the acoustic metric which represents the radial fluid flow with constant radial velocity has the above mentioned properties of AdS. In particular, sound waves will now need the above mentioned extra boundary condition at the origin ($\rho = 0$). Physically speaking, the specification of an extra condition at the origin can be interpreted as the need to provide a description of how sound waves interact with the point source/sink.

4.4 The extra condition problem in the analogue model

We now return to the radial equation (4.10) which, for the constant radial velocity profile $v(\rho) = \alpha c$), reduces to

$$\left(1-\alpha^2\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} R(\rho) + \left(\frac{1-\alpha^2}{\rho^2}\kappa^2 + \frac{\omega^2}{c^2}\right) R(\rho) = 0. \tag{4.17}$$

Let us consider its axisymmetric solutions, which correspond to the case when κ is equal to zero. The differential equation (4.17) then has as solutions $e^{\pm i \frac{\omega\rho}{c(1-\alpha^2)}}$. The complete solution for $\psi(\tau, \rho)$ is thus

$$\psi(\tau, \rho) = \left(\eta e^{-i\frac{\omega\rho}{c(1-\alpha^2)}} + \xi e^{i\frac{\omega\rho}{c(1-\alpha^2)}}\right) e^{-i\omega\tau}, \tag{4.18}$$

where η and ξ are constants. We note that this solution is composed by ingoing and outgoing circular waves.

This solution in coordinates (t, r, θ) can be written as $\psi = \eta\psi_- + \xi\psi_+$, where

$$\psi_\pm = e^{\pm i \frac{\omega r}{c(1\pm\alpha)}} e^{-\omega t}.$$

Computing the energy for each solution (using the relation (3.51)), we note that both ψ_+ and ψ_- have finite energy. In this way there is no canonical choice of a unique linear combination of ψ_+ and ψ_- that determines a solution to this problem. In order to specify such a distinguished linear combination of $\psi_\pm$ we need to provide a relation between η and

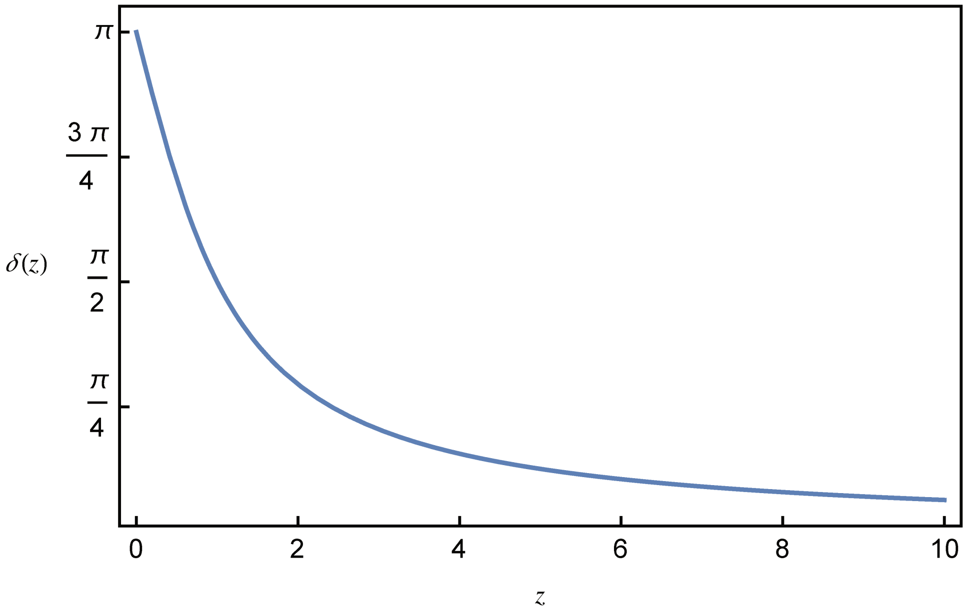

Figure 3 – Phase difference between ingoing and outgoing waves according to (4.21) for the case $v(\rho) = \alpha c$.

ξ. This is intimately related to the fact that the spatial boundary of AdS needs an extra boundary condition for the wave equation since this space is non-globally hyperbolic.

We would like to relate this extra boundary condition for ψ at the origin with a physical property of the wave. In order to do that we consider the phase difference $\delta(\omega)$ between ingoing and outgoing waves. From (4.18) we get

$$\delta(\omega) = \arg\left(\frac{\eta}{\xi}\right). \tag{4.19}$$

We see that, in order to determine $\delta(\omega)$, it is necessary to use an extra condition to relate ξ and η. This can be done by defining a parameter β as follows [33]

$$\psi(0) + \beta\psi'(0) = 0. \tag{4.20}$$

Substituting $\psi(\tau, \rho)$ in the above condition, we obtain

$$\xi\left[\frac{i\omega\beta}{c\,(1-\alpha^2)} + 1\right] - \eta\left[\frac{i\omega\beta}{c\,(1-\alpha^2)} - 1\right] = 0.$$

Therefore

$$\delta(z) = \arg\left(\frac{iz-1}{iz+1}\right), \tag{4.21}$$

where $z = \frac{\beta\omega}{c(1-\alpha^2)}$. Figure (3) shows $\delta(z)$ as a function of z. We emphasize that the parameter β is still arbitrary, so that the dynamical evolution of sound waves is not uniquely determined until we specify β.

Physically speaking, an ambiguous dynamical evolution does not make sense and that is why one needs to impose an extra condition on the system so that a unique,

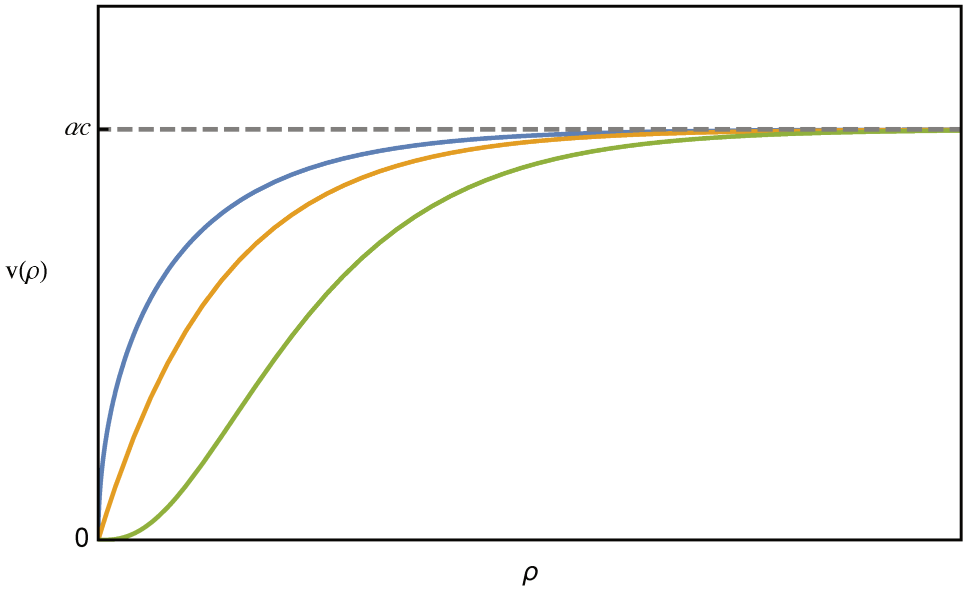

Figure 4 – Arbitrary smooth transitions near the origin for the fluid flow to regularize $v(\rho)$.

deterministic, evolution for fields (ψ in our case) is obtained. As we mentioned above, the AdS spacetime in coordinates (τ, ρ, ϕ) has its boundary at the origin and, therefore, this extra condition should be related to the behavior of the fluid flow near $\rho = 0$.

4.5 Regularization of $v(\rho)$

We now consider regularizations of this model near the source/sink at the origin and impose conditions on them in order that no extra boundary conditions are required at the origin.

As we shall see, there are many options that can be chosen to regularize $v(\rho)$. We investigate, in what follows, if removing the singularity of $v(r)$ at the origin is enough to obtain a unique dynamical evolution for sound waves. Note that $v(r)$ is not well defined at the origin in the model above and that a radial velocity profile can only be continuous at $r = 0$ if $v(0) = 0$.

Therefore, we consider regularizations of $v(\rho)$ by setting $v(\rho)$ to zero at the origin. For $\rho > 0$, the velocity should then increase until it becomes constant. For the moment, the consider the transition of $v(\rho)$ from $v(0) = 0$ to $v(\rho) = \alpha c$ as being arbitrary. Figure (4) shows different possible choices of velocity profiles of this kind.

Let us then consider a velocity profile which can be Taylor expanded at the origin as

$$v(\rho) = \alpha c \left(\frac{\rho}{\rho_0}\right)^n ; \; n > 0, \; \rho \ll \rho_0, \tag{4.22}$$

where ρ_0 and n are (still undetermined) parameters. Notice that this profile must still be matched to another expression, valid for instance for $\rho > \rho_0$, which approaches the constant value αc for v for large ρ. We see that ρ_0 is related with the width of the region wherein $v(\rho)$ is not constant, and that n determines how fast $v(\rho)$ grows near the origin.

4.5.1 Solutions to wave equation near the origin

To simplify future expressions, let us introduce a dimensionless radial coordinate $x = \rho/\rho_0$ and a dimensionless frequency $m = \omega\rho_0/c$. With $v(\rho)$ given by (4.22), the radial part of the wave equation (4.10) in terms of x and m is then given by

$$\left(1 - \alpha^2 x^{2n}\right)^2 R''(x) - \frac{n}{x}\left(1 - \alpha^4 x^{4n}\right) R'(x) + m^2 R(x) = 0. \tag{4.23}$$

Notice that the above equation is meaningful only for $x < 1$. We also note that $x = 0$ is a singular regular point of this ordinary differential equation. We can then use the Frobenius method to write a solution of (4.23) as follows

$$R(x) = x^s \sum_{k=0}^{\infty} a_k x^k. \tag{4.24}$$

Substituting this into equation (4.23) we find that $s = n + 1$ or $s = 0$. For $s = n + 1$ we obtain a first solution:

$$R_1(x) = x^{n+1} \sum_{k=0}^{\infty} a_k x^k. \tag{4.25}$$

The case when $s = 0$ does not correspond to a solution in the form (4.24). The reason is that this is the tricky case of the Frobenius method for which the difference between the two roots of the indicial polynomial is an integer. However, in this case one can construct a second solution from $R_1(x)$ as follows [34]

$$: R_2(x) = pR_1(x) \ln x + \sum_{k=0}^{\infty} b_k x^k, \tag{4.26}$$

where p is a constant.

Recall that n is fixed but arbitrary. For each value of n the coefficients of the above series expansions may be then computed in recursive form. However, there is no closed form for them in general.

As an example, for $n = 1$ the solutions can be shown to take the form:

$$R_1(x) = x^2 + \left(\tfrac{\alpha^2}{2} - \tfrac{m^2}{8}\right) x^4 + \tfrac{1}{164}\left(m^4 - 28m^2\alpha^2 + 64\alpha^4\right) x^6 + \mathcal{O}\left(x^7\right),$$

$$R_2(x) = 1 - \tfrac{1}{2} m^2 \ln(x) x^2 - \tfrac{m^2}{64}\left(3m^2 + 12\alpha^2 - m^2 \ln x + 16\alpha^2 \ln x\right) x^4 + \mathcal{O}\left(x^5\right),$$

and their corresponding plots are shown in figure (5). We chose the parameters in these solutions so that the values of $R_1(0)$ and $R_2(0)$ are zero and one respectively.

Having solved the above equation near the origin we can write the complete solution of ψ as a function of τ and $x = \rho/\rho_0$ as

$$\psi(\tau, x) = A\psi_1(\tau, x) + B\psi_2(\tau, x), \tag{4.27}$$

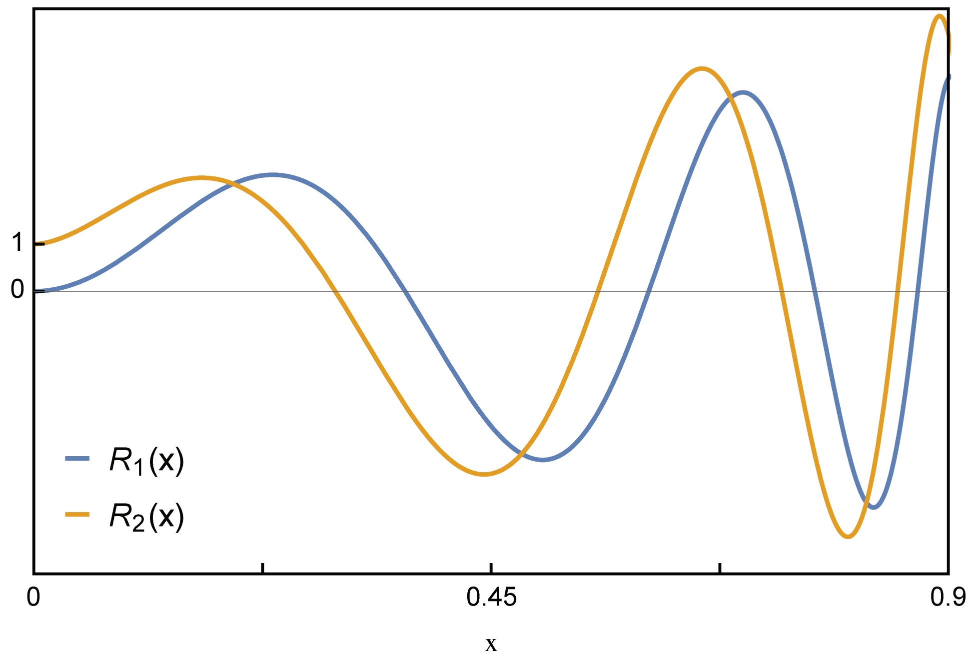

Figure 5 – Solutions of spatial part of the wave equation (4.23). The plot corresponds to the case $n = 1$, but its behavior at $x = 0$ is similar for any other value of n.

where $\psi_k(\tau, x) = R_k(x)e^{-i\omega\tau}$, $k = 1, 2$.

We see that $R_1(x)$ and $R_2(x)$ are fairly well-behaved functions near the origin. However, since we are dealing with a physical system, another consideration to be kept in mind is that sound waves must carry a finite energy.

4.5.2 Finite energy condition

We start by considering the acoustic energy derived in the previous chapter. According to the transformation laws for a tensor density, the expression for $\mathcal{Q}^\mu$ in the coordinates $x^{\mu'} = (\tau, \rho, \phi)$ is

$$\mathcal{Q}^{\mu'} = \left|\frac{\partial x^\alpha}{\partial x^{\beta'}}\right| \frac{\partial x^{\mu'}}{\partial x^\nu} \mathcal{Q}^\nu, \tag{4.28}$$

where $x^\mu = (t, r, \theta)$. Once again, x^μ and $x^{\mu'}$ are related through the transformation (4.7).

The quantity $\mathcal{Q}^0$ represents the acoustic energy density.Using the above transformation law, we get the corresponding quantity in the $x^{\mu'}$ coordinates:

$$\begin{aligned} \mathcal{Q}^{0'} &= \frac{\partial x^{0'}}{\partial x^0}\mathcal{Q}^0 + \frac{\partial x^{0'}}{\partial x^1}\mathcal{Q}^1 + \frac{\partial x^{0'}}{\partial x^2}\mathcal{Q}^2 = \frac{\partial \tau}{\partial t}\mathcal{Q}^0 + \frac{\partial \tau}{\partial r}\mathcal{Q}^1 + \frac{\partial \tau}{\partial \theta}\mathcal{Q}^2 \\ &= \mathcal{Q}^0 + \frac{v(\rho)}{c^2 - v(\rho)^2}\mathcal{Q}^1. \end{aligned} \tag{4.29}$$

This shows, once more, that the acoustic energy density is not the same in coordinates (τ, ρ, ϕ). For this reason, the energy may or may not be conserved in a generic reference frame; this will depend on the coordinates in which it is measured. The "physical frame" in which we should measure the energy, in our case, is the laboratory frame.

Another consequence of having a non-homogeneous transformation for the time coordinate is that τ does not represent the real (laboratory) time anymore. Quantities in this coordinate system do not represent a direct physical meaning. For instance, $\mathcal{Q}^0$ represents the acoustic energy density, but $\mathcal{Q}^{0'}$ does not because, in these coordinates, $\vec{\nabla}'\psi$ is not the sound wave velocity. This will be important when we examine the energy of the sound waves.

From the results of the previous section, we can now write ψ_k back in the coordinates (t, r, θ) by using (4.7). Next we compute the total energy for each solution ($\psi_1(t, r)$ and $\psi_2(t, r)$) using equation (3.51)[1]. By doing this, we arrive at the conclusion that for $n < 1$, both solutions $\psi_1(t, r)$ and $\psi_2(t, r)$ carry finite energy. On the other hand, only $\psi_1(t, r)$ carries a finite energy for $n \geq 1$. We refer the reader to appendix B for approximated expressions of this energy.

In this way, for $n \geq 1$ the second solution of the wave equation does not fulfill the finite energy condition and should be disregarded. This eliminates the ambiguity related to the need of an extra boundary condition at the origin. We are therefore compelled to work with regularizations for which n lies in the interval $[1, \infty)$. It is interesting to note, in passing, that this is precisely the condition that makes the velocity profile not only continuous but also differentiable at the origin.

To sum up, a regularized $v(\rho)$ provides the fluid flow with an adequate behavior at $\rho = 0$, but this is not sufficient to provide a unique dynamical evolution for sound waves because of the arbitrariness of the regularization. The finite energy condition plays an important role to limit that arbitrariness. Particularly, only the regularizations which behave as (4.22) near the origin and with $n \geq 1$ are capable to determine uniquely the dynamical evolution of sound waves.

4.6 Global solution to the wave equation

With the results found above, we are ready to solve the wave equation in the whole space. We already noticed that, when $n \geq 1$, $R_1(x)$ (or equivalently $\psi_1(\tau, x)$) is the only physically allowed solution. For instance, if $n = 1$, then $v(x) \approx \alpha c x$ for $x \ll 1$. We note that n could also take non-integer values such as $\frac{3}{2}$ or $\frac{5}{2}$.

There are, of course, many transitions that satisfy a fluid flow behavior in such a way that $v(x) = \alpha c x^n$ $(n \geq 1)$ near the origin and $v(x) = \alpha c$ far from the origin. Some of those could lead to an analitic solutions of the wave equation and some could not. The

[1] The energy density was integrated over a disc of radius r_0 in such a way $r_0 \approx 0$ just as ρ_0.

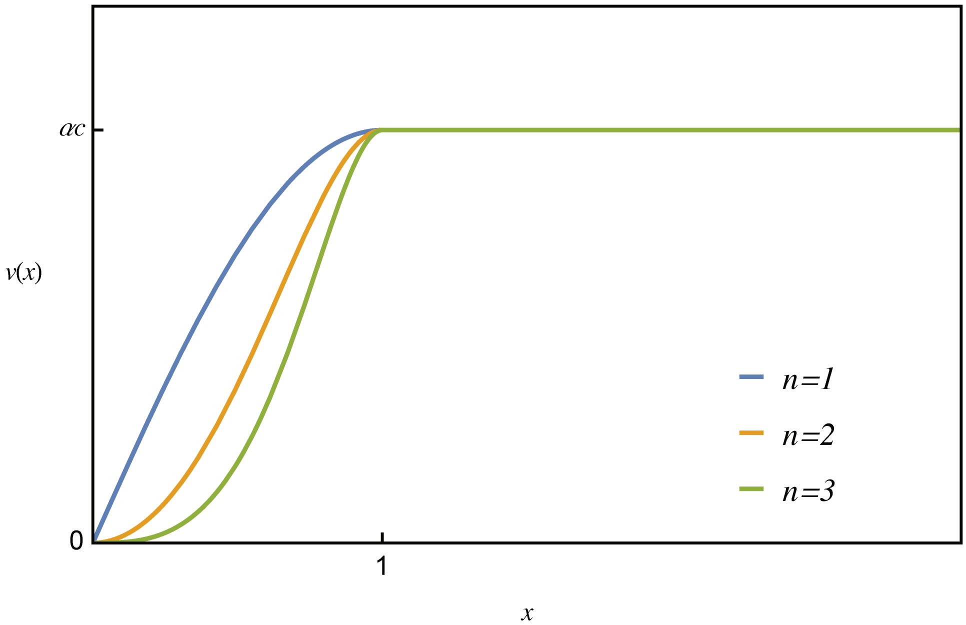

Figure 6 – Fluid flow according to equation (4.30) for different values of n.

easiest way to find one of those family of transitions is to define $v(x)$ in a piecewise way. For instance, the following fluid flow is a good candidate to be considered

$$v(x) = \begin{cases} \alpha c \sin\left(\frac{\pi}{2}x^n\right) & x < 1, \\ \alpha c & x \geq 1. \end{cases} \tag{4.30}$$

As another example, one could replace the sine term with $v(x) = \alpha c\,(1 - e^{-x})^n$. The advantage of defining $v(x)$ in a piecewise way is that for $x > 1$ the solutions are already known.

4.6.1 Numerical solution

The profiles considered above do not admit analytic, closed form, solutions. So we turn to a numeral analysis of the solutions for the wave equation in this context. In order to do that, we consider the velocity profile to be given by (4.30). This is shown in figure (6). Since the solutions in the $x > 1$ region are already given by ingoing and outgoing waves as found in section 4.4, we now focus on solving the wave equation in the $x < 1$ region.

There is a complication associated with the numerical solution here. The point $x = 0$ cannot be directly used to provide an initial condition to the differential equation for the differential equation is singular there. To circumvent this difficulty we use the series solution developed above to obtain initial conditions at $x = \epsilon$ with ϵ is a very small quantity. The differential equation can then be solved numerically with initial conditions at this point. We therefore split the domain into the following three regions:

Region	$v(x)$	**Solution for** $R(x)$
$0 \leq x < \epsilon$	$\frac{\pi}{2}\alpha c x^n$	$R_1(x)$
$\epsilon \leq x < 1$	$\alpha c \sin\left(\frac{\pi}{2}x^n\right)$	$R_{\mathrm{N}}(x)$, to be solved numerically
$1 \leq x < \infty$	αc	$R(x) = \eta e^{-i\frac{mx}{1-\alpha^2}} + \xi e^{i\frac{mx}{1-\alpha^2}}$, $m = \omega\rho_0/c$

In this way, for the second region ($\epsilon \leq x < 1$), we should impose the following initial conditions

$$\begin{cases} R_{\mathbf{N}}(\epsilon) = R_1(\epsilon), \\ R'_{\mathbf{N}}(\epsilon) = R'_1(\epsilon). \end{cases} \tag{4.31}$$

Likewise, the solution for the region $1 \leq x < \infty$ should satisfy

$$\begin{cases} R_{\mathbf{N}}(1) = \eta e^{-i\frac{m}{1-\alpha^2}} + \xi e^{i\frac{m}{1-\alpha^2}}, \\ R'_{\mathbf{N}}(1) = -i\frac{m}{1-\alpha^2}\eta e^{-i\frac{m}{1-\alpha^2}} + i\frac{m}{1-\alpha^2}\xi e^{i\frac{m}{1-\alpha^2}}. \end{cases} \tag{4.32}$$

Once again, we can use the phase difference between ingoing and outgoing waves to analyse the physical process that occurs near the origin. Since this requires one to know the solution for the $x > 1$ region, the phase difference must be computed in that region. The phase difference as a function of the dimensionless frequency $m = \omega\rho_0/c$ is $\delta(m) = \arg(\xi/\eta)$. To find the coefficients η and ξ, we consider the relations (4.32). The phase difference then becomes

$$\delta(m) = \arg\left[e^{-i\frac{2m}{1-\alpha^2}}\left(\frac{\frac{imB}{1-\alpha^2} - 1}{\frac{imB}{1-\alpha^2} + 1}\right)\right], \tag{4.33}$$

where

$$B \equiv -\frac{R_{\mathrm{N}}(1)}{R'_{\mathrm{N}}(1)}. \tag{4.34}$$

Using (4.31) we can then compute (numerically) the solution $R_{\mathbf{N}}(x)$ and then B. Figure (7) shows a particular solution to the wave equation (for $m = 10$ and $\alpha = 1/2$). We note $R_{\mathrm{N}}(x)$ fits nicely to the other regions and that the amplitude and frequency of sound waves are constant outside the transition region of $v(x)$.

If we look at the definition of B, we note it depends (implicitly) on the frequency m. For this reason, the phase difference defined in (4.33) cannot be plotted directly. In this way, we must construct B by computing $R_{\mathrm{N}}(1)$ and $R'_{\mathrm{N}}(1)$ for different values of m and only then we can obtain an interpolated function for $B(m)$. Once calculated $B = B(m)$ numerically, we can then plot the phase difference $\delta(m)$, The result is shown in figure (8).

To summarize, the regularized $v(r)$ we considered led to a certain behavior for the solution near the origin. This yielded a unique dynamical evolution for the sound waves. In figure (8) we observe that this leads to a phase difference (between ingoing and outgoing waves) with periodic behavior.

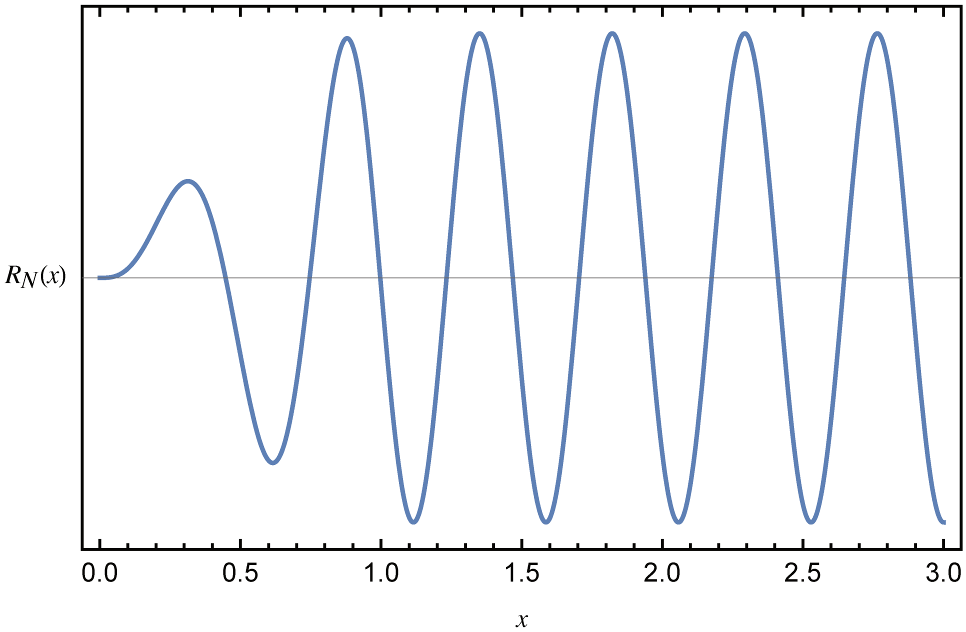

Figure 7 – Numerical solution of sound waves moving along the fluid flow shown in (4.30), here $m = 10$ and $\alpha = 1/2$.

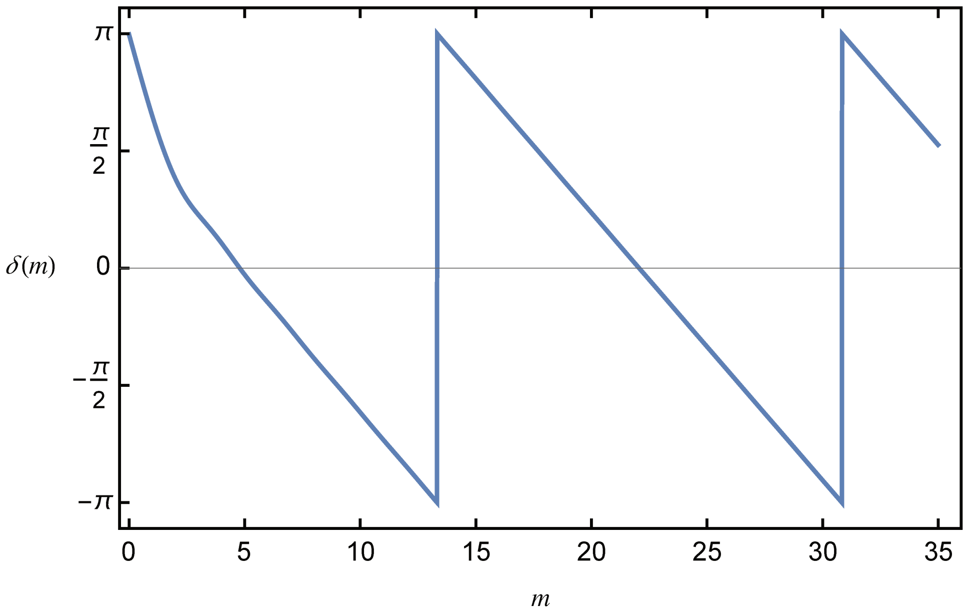

Figure 8 – Phase difference between ingoing and outgoing scattered waves as a function of m. As can be seen, there is a periodic behavior which not appears for the case $v(\rho) = \alpha c$.

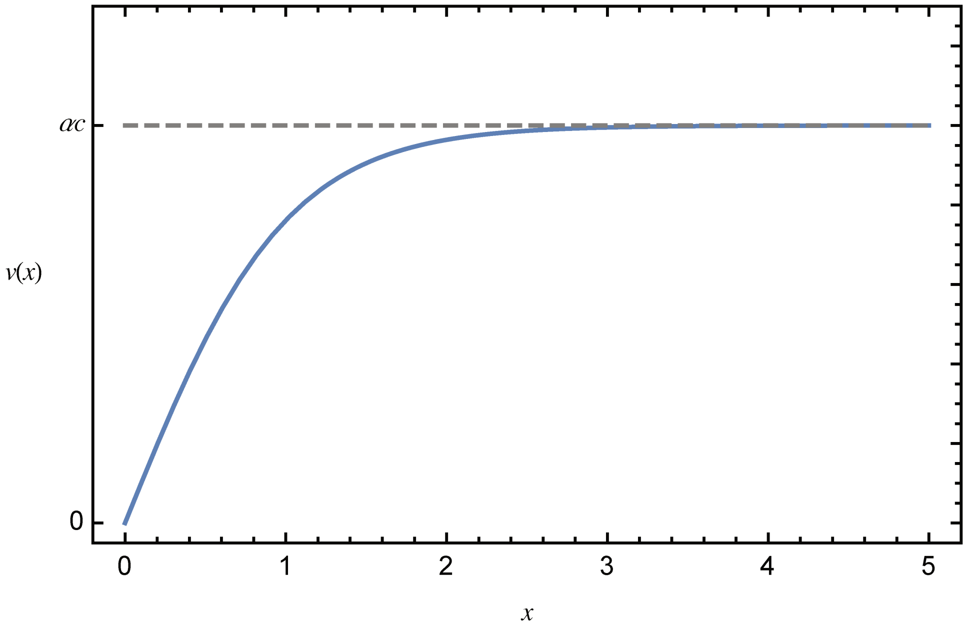

Figure 9 – Plot of $v(x) = \alpha c \tanh x$, this case solves the wave equation analytically. Near the origin, this can also be approximated by $\alpha c x$ which corresponds to the case $n = 1$ in the Frobenius solution ($x \ll 1$).

The problem with numerical procedures is that, besides the fact that their solutions are hard to manipulate, from them it is not easy to see how the solutions change under a variation of the parameters. On the other hand, the only way to obtain an analytic description for the problem is by finding a fluid flow which:

- Leads to a wave equation with analytic solutions.
- Has the required behavior (near the origin) that leads to sound waves with finite energy.

4.6.2 An analytic solution

The advantages of analytical solutions over their numerical counterparts are numerous. Besides allowing one to easily probe for interesting physical limits, they often provide a qualitative understanding on how each parameter affects the physics of the system.

Finding a fluid flow that leads to an analytical solution for the wave equation with the characteristics described above is not an easy task. After an extensive investigation, we could find one that fits perfectly our needs. This is given by

$$v(x) = \alpha c \tanh x \tag{4.35}$$

(see figure (9).

According to results from the previous sections, the fluid flow (4.35) belongs to the regularization class where $n = 1$, so that it fulfills the necessary conditions near the origin.

Going back to the analogue model, the velocity profile given by (4.35) leads to the following radial equation (4.10):

$$\left(1-\alpha^2\tanh^2 x\right)^2\frac{d^2R}{dx^2}-\frac{1-\alpha^4\tanh^4 x}{\sinh x\cosh x}\frac{dR}{dx}+m^2R=0. \tag{4.36}$$

The solutions to this equation can be analytically obtained and are given by

$$R_\pm=\exp\left[-\frac{im\alpha\kappa(x)}{2(1-\alpha^2)}\right]\exp\left[\pm i\frac{m\chi(x)}{1-\alpha^2}\right]{}_2F_1\left(a_\mp+\frac{1}{2},a_\mp-\frac{1}{2},c_\mp,-\frac{\operatorname{csch}^2 x}{1-\alpha^2}\right). \tag{4.37}$$

We see that its first term is composed of two exponential functions while its second term is given by a hypergeometric function [35]. The functions $\kappa(x)$, $\chi(x)$ and the parameters $a_\pm$, $c_\pm$ are given by

$$\begin{aligned}\kappa(x)&=\ln\left(1-\alpha^2+\operatorname{csch}^2 x\right),\\ \chi(x)&=\ln\sinh x,\\ a_\pm&=\frac{1}{2}\pm\frac{im/2}{1\pm\alpha},\\ c_\pm&=1\pm\frac{im}{1-\alpha^2},\end{aligned} \tag{4.38}$$

where, as before, $x=\rho/\rho_0$ and $m=\omega\rho_0/c$.

As already discussed, these solutions must reduce to ingoing and outgoing waves when $x\to\infty$. It is straightforward to show that, indeed, $R_\pm \propto \exp\left(\pm i\frac{mx}{1-\alpha^2}\right)$ for x large.

Let us write the complete solution to the wave equation as a linear combination of R_+ and R_-, i.e.,

$$R(x)=AR_+(x)+DR_-(x), \tag{4.39}$$

where A and D are constants to be determined. Here we appeal to the Frobenius solutions found in previous sections to select the physically accepted solution. From the analysis of section 4.5, we see that

$$R(0)=AR_-(0)+DR_+(0)=0 \tag{4.40}$$

is the condition that must be used to obtain the desired solution. Thus, without considering global constant phases, A and D can then be written as follows

$$\begin{aligned}A=+R_+(0)&=\frac{\exp\left[-\frac{im\ln(1-\alpha^2)}{2(1-\alpha)}\right]\Gamma\left(1-\frac{im}{1-\alpha^2}\right)}{\Gamma\left(1-\frac{im/2}{1-\alpha}\right)\Gamma\left(1-\frac{im/2}{1+\alpha}\right)},\\ D=-R_-(0)&=-\frac{\exp\left[\frac{im\ln(1-\alpha^2)}{2(1+\alpha)}\right]\Gamma\left(1+\frac{im}{1-\alpha^2}\right)}{\Gamma\left(1+\frac{im/2}{1-\alpha}\right)\Gamma\left(1+\frac{im/2}{1+\alpha}\right)}.\end{aligned} \tag{4.41}$$

These constants encode the information about the regularization made at the origin.

Having obtained these expressions for A and D, the solution $R(x)$ now provide a complete description for circular waves moving on this fluid. In particular, the way by which the waves interact with the source/sink located at $x = 0$ (or equivalently $r = 0$) is also encoded in $R(x)$.

Our next step is to obtain the phase difference $\delta(m)$ for this solution. We must, therefore, obtain an asymptotic expression for $R(x)$ for large x. In order to do that, we define

$$\epsilon \equiv \operatorname{csch} x, \tag{4.42}$$

which goes to zero when $x \to \infty$, and take the asymptotic limit for the hypergeometric function as $\epsilon \to 0$. Expanding the hypergeometric function in terms of this quantity we get

$${}_2F_1\left(a_\mp + \frac{1}{2}, a_\mp - \frac{1}{2}, c_\mp, -\frac{\epsilon^2}{1-\alpha^2}\right) = 1 + \frac{1-4a_\mp^2}{4\left(1-\alpha^2\right)c_\mp}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right). \tag{4.43}$$

Next, we compute the asymptotic limit for $\kappa(x)$ and $\chi(x)$. Using again the parameter ϵ, we obtain

$$\begin{aligned} \exp\left[-i\frac{m\kappa(x)}{2(1-\alpha)}\right] &= \left(1-\alpha^2+\epsilon^2\right)^{-i\frac{m}{2(1-\alpha)}} \\ &\approx \exp\left[-i\frac{m\ln\left(1-\alpha^2\right)}{2(1-\alpha)}\right]\left(1 - \frac{im\epsilon^2}{2(1-\alpha)(1-\alpha^2)} + \mathcal{O}\left(\epsilon^4\right)\right) \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \exp\left[\pm i\frac{m\chi(x)}{1-\alpha^2}\right] &\approx \exp\left[\pm i\frac{m(x-\ln 2)}{1-\alpha^2}\right] \\ &= \exp\left[\mp i\frac{m\ln 2}{1-\alpha^2}\right]\exp\left[\pm i\frac{mx}{1-\alpha^2}\right]. \end{aligned} \tag{4.45}$$

As a result, when $x \to \infty$ we can write the asymptotic solution as

$$\psi(\tau, x) \to \left[B_+(\epsilon)A\exp\left(\frac{-imx}{1-\alpha^2}\right) + B_-(\epsilon)D\exp\left(\frac{imx}{1-\alpha^2}\right)\right]e^{-i\omega\tau}, \tag{4.46}$$

where global phases were dropped and the parameters $B_\pm$ are

$$B_\pm(\epsilon) = \exp\left[\pm i\frac{m\ln 2}{1-\alpha^2}\right]\left(1 + \frac{\left(1+4a_\mp^2\right)\epsilon^2}{4\left(1-\alpha^2\right)c_\mp} - \frac{im\epsilon^2}{(1-\alpha)\left(1-\alpha^2\right)} + \mathcal{O}\left(\epsilon^4\right)\right).$$

These parameters encode all the asymptotic limit results. From (4.46) we are able to identify the coefficients of ingoing and outgoing scattered waves and write the phase difference. This yields

$$\begin{aligned} \eta &\equiv B_+(\epsilon)A, \\ \xi &\equiv B_-(\epsilon)D. \end{aligned} \tag{4.47}$$

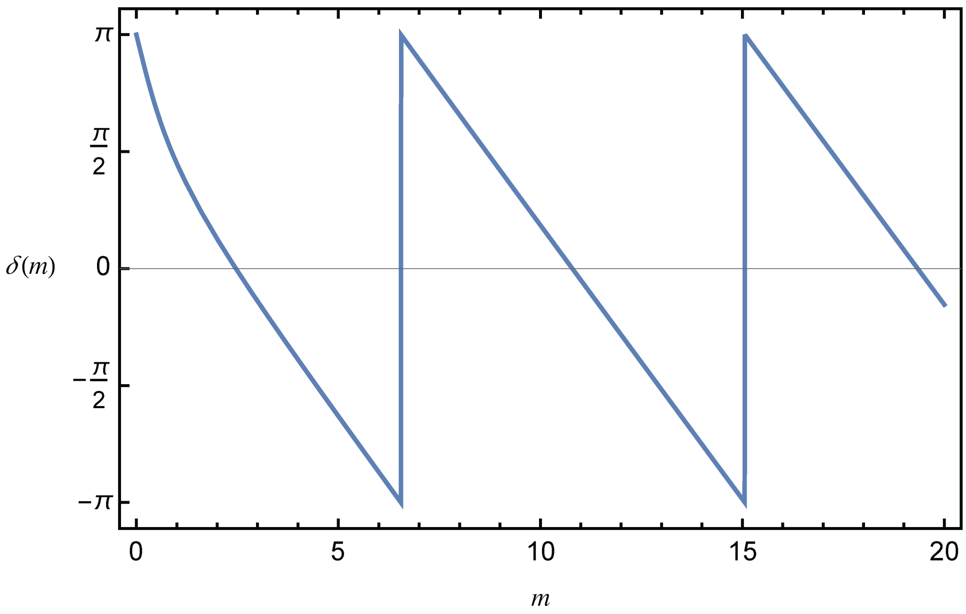

Figure 10 – Phase difference $\delta = \arg(\xi/\eta)$between ingoing and outgoing waves, here ξ and η are given by (4.47).

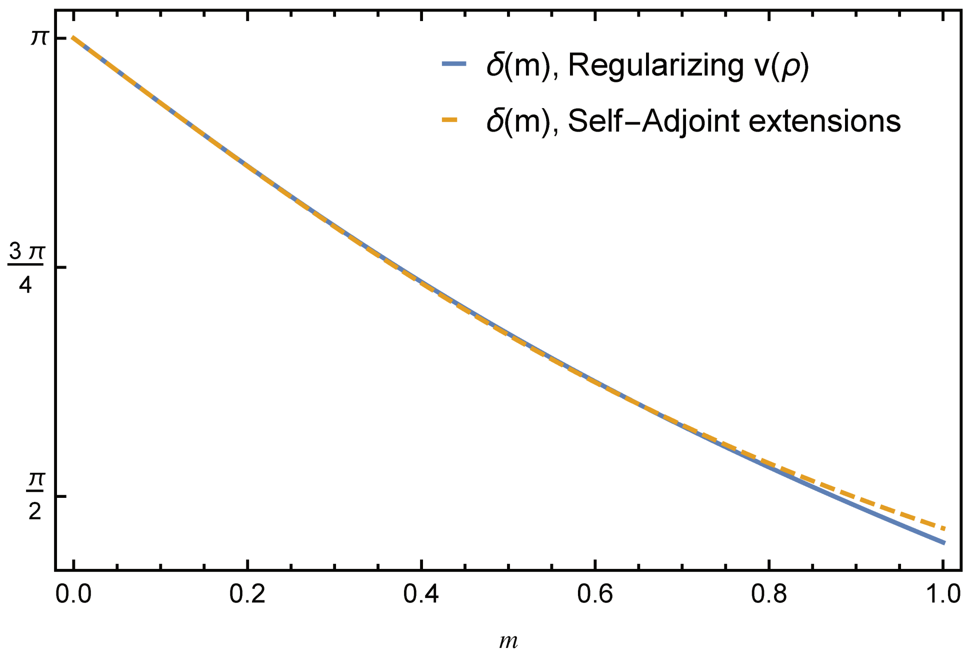

Figure 11 – $\delta(m)$ obtained by two different methods, $m \ll 1$.

Therefore, in the limit of $x \to \infty$ and using the coefficients given in (4.47) the phase difference becomes

$$\delta(m) = \arg\left\{-\exp\left[i\frac{m\ln\left(\frac{1-\alpha^2}{4}\right)}{1-\alpha^2}\right]\frac{\Gamma\left(1+\frac{im}{1-\alpha^2}\right)\Gamma\left(1-\frac{im/2}{1-\alpha}\right)\Gamma\left(1-\frac{im/2}{1+\alpha}\right)}{\Gamma\left(1-\frac{im}{1-\alpha^2}\right)\Gamma\left(1+\frac{im/2}{1-\alpha}\right)\Gamma\left(1+\frac{im/2}{1+\alpha}\right)}\right\}. \tag{4.48}$$

Figure (10) shows the phase difference given by the above relation. We observe that its behavior is similar to the one found numerically. We can now go a step further and find the limit of small m that should be compared to the AdS case. We note that the behavior of $\delta(m)$ agrees with the result found in [30]. If we compare both results up to first order in m, we may then conclude that this regularization attributes a certain definite value to β (defined in (4.20)), which is given by

$$\beta = \ln\left(\frac{2}{\sqrt{1-\alpha^2}}\right). \tag{4.49}$$

Figure (11) shows an agreement between the phase difference obtained from regularizing $v(x)$ and that obtained from using the extra boundary condition (4.21). In order for us to understand physically the implications of such agreement, let us go back to dimensionful quantities. The dimensionless frequency m is then written as

$$m = \omega\rho_0/c = 2\pi\rho_0/\lambda,$$

where we used the fact that $\lambda c = 2\pi\omega$, λ being the wavelength of the sound wave. Recall that the parameter ρ_0 represents the region where $v(\rho)$ raises from zero to its constant value. We see that, for small m, the sound wavelength is much larger than ρ_0 (the radius which is related with the region where $v(\rho)$ is not constant). Physically speaking, within this limit, the sound waves cannot properly probe the region for which $v(\rho)$ is not constant and, as a result, the results found in [30] are approximatelly valid.

5 Conclusions

Analogue models of gravity have become an important tool for exploring different aspects of general relativity in terms of concepts associated with other physical systems. Here we studied an analogue model based on classical fluid dynamics. We were specially interested in the case of a constant radial fluid flow with a point source/sink at the origin. We showed that the resulting effective metric corresponds in this case to an $AdS_2 \times S^1$ spacetime. It is well known that the AdS spacetime is nonglobally hyperbolic. This implies that the dynamics of fields in this background is not well defined unless extra boundary conditions are prescribed (in this case at the spatial boundary of AdS). On the analogue model end this implies that one needs to specify extra boundary conditions at the origin. This corresponds to an effective description of how the field interacts with the point source/sink of the flow.

We also considered regularizations of the fluid velocity near the source/sink at the origin. By imposing physical conditions on the system (a finite energy condition for sound waves), we found that a certain class of regularizations leads to a well defined dynamics for sound waves without the need of extra boundary conditions. On the effective spacetime end this corresponds to the introduction of a deformation of AdS near its spatial infinity.

When compared to the results found in the AdS case, we found that the regularization attributes a periodic behavior to the phase difference between ingoing and outgoing waves. We also showed that, when the wavelength of the sound is much larger than the effective radius set by the regularization, the effects of the latter are negligible, as expected.

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APPENDIX A – Viscosity in fluids

The ideal fluid model is the simplest description one could adopt for studying fluid dynamics. Sometimes this is good enough, sometimes we need more. The next simplest model arises when we take into account another property of fluids, which is their viscosity.

Consider the volume V shown in figure (12). In the case of an ideal fluid, the force density $\vec{\tau}$ is always normal to the surface ∂V. In the case of real fluids, one also has a tangential component, which gives rise to a momentum transfer between two adjacent surfaces similar to ∂V. Mathematically, this momentum transfer is described by means of a tensor τ_{ij} (the viscosity stress tensor) which is added to the stress tensor of the ideal fluid:

$$\sigma_{ij} \equiv -p\delta_{ij} + \tau_{ij}. \tag{A.1}$$

The momentum flux density tensor then becomes

$$\Pi_{ij} = -\sigma_{ij} + \rho v_i v_j. \tag{A.2}$$

As the viscosity is related to the momentum transfer, it should be expressed in terms of the velocity gradient $\partial_i v_j$. It can be shown that, under appropriate physical assumptions [1],

$$\tau_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \xi\delta_{ij}\partial_k v_k, \tag{A.3}$$

where η and ξ (independent of v_i) are called *coefficients of viscosity* and they are non-negative.

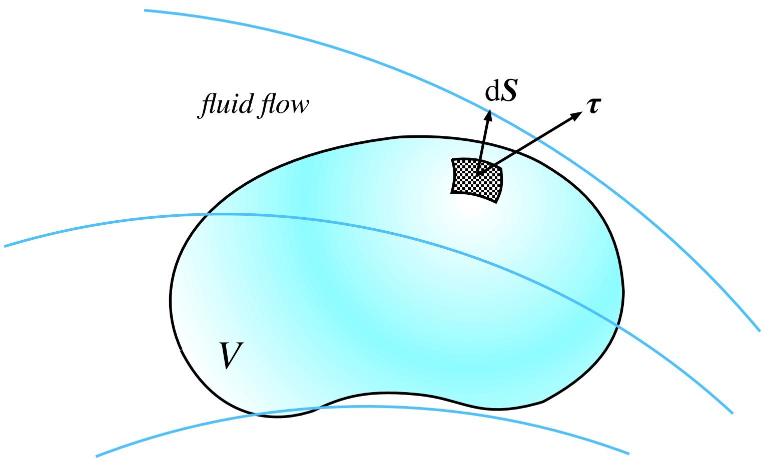

Figure 12 – Some region of volume V inside a fluid flow. There is a force per unit area $\vec{\tau}$ (called *shear stress*) acting on a surface element dS of ∂V.

It is straightforward to find the equations of motion once we have the expression for Π_{ij}. It follows from (1.12) that

$$\partial_t v_i + (v_k \partial_k)\, v_i = -\frac{1}{\rho}\partial_i p + \frac{\eta}{\rho}\partial_l \partial_l v_i + \frac{1}{\rho}\left(\eta + \frac{1}{3}\xi\right)\partial_i \partial_k v_k. \tag{A.4}$$

If the fluid is incompressible, the Navier–Stokes equations arises:

$$\partial_t v_i + (v_k \partial_k)\, v_i = -\frac{1}{\rho}\partial_i p + \frac{\eta}{\rho}\partial_l \partial_l v_i, \tag{A.5}$$

i.e.,

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla} p + \frac{\eta}{\rho}\nabla^2 \vec{v}. \tag{A.6}$$

We note that when the fluid has zero viscosity, the above equations reduce to the Euler equations (1.3).

APPENDIX B – An approximation for computing the energy for sound waves near the origin

Before computing the energy for sound waves, we need to derive expressions for $\psi_k(t,x)$ near the origin. To do that, we first solved the radial part of the wave equation shown in (4.23). The solutions can be expressed in a simple way if we consider only the first two terms in powers of x:

$$\begin{aligned} R_1(\tau,x) &= x^{n+1} + a_1 x^{n+2} + \mathcal{O}\left(x^{n+3}\right), \\ R_2(\tau,x) &= 1 + (b_1 + c_1 \ln x)\, x^2 + \mathcal{O}\left(x^3\right). \end{aligned} \tag{B.1}$$

As long as x is small, this approximation is valid.

We note that, no matter what is the value of n, at $x=0$ we get

$$\begin{aligned} R_1(0) &= 0, \\ R_2(0) &= 1, \end{aligned} \tag{B.2}$$

which is also an important result for one to identify the solutions only by checking its value at origin.

The solutions to the wave equation near the origin are then written as $\psi(\tau,x) \approx A\psi_1(\tau,x) + D\psi_2(\tau,x)$, where

$$\begin{aligned} \psi_1(\tau,x) &= \left(x^{n+1} + a_1 x^{n+2}\right) e^{-i\omega\tau}, \\ \psi_2(\tau,x) &= \left[1 + (b_1 + c_1 \ln x)\, x^2\right] e^{-i\omega\tau}. \end{aligned} \tag{B.3}$$

As we mentioned before, the energy is supposed to be measured in the laboratory frame, i.e., in coordinates (t,r,θ). Thus, we should express the solutions of sound waves back into coordinates (t,r,θ). Since the relation between x and r is linear, for simplicity, we could maintain x as the radial coordinate (where $x = \rho/\rho_0 = r/r_0$). For the time coordinate, however, we need to transform it through the relations (4.7). For the fluid flow shown in (4.22), this transformation for τ yields

$$\tau = t + \int \frac{\alpha c x^n}{c^2 - \alpha^2 c^2 x^{2n}} r_0 \mathrm{d}x. \tag{B.4}$$

Therefore

$$\tau = t + \frac{\alpha r_0 x^{n+1}}{c(n+1)}\, {}_2F_1\left(1, \frac{n+1}{2n}, \frac{3n+1}{2n}, \alpha^2 x^{2n}\right). \tag{B.5}$$

Using the same approximation as that used for $R_k(x)$, the above relation can be written as

$$\tau \approx t + \frac{\alpha r_0 x^{n+1}}{c(n+1)}.$$

We emphasize that this approximations are used only to illustrate how the results behave near the origin. The solutions (B.3) in coordinates (t, x, θ) then reduce to

$$\begin{aligned} \psi_1(t,x) &= \left(x^{n+1} + a_1 x^{n+2}\right) \exp\left[-i\frac{\omega\alpha r_0 x^{n+1}}{c(n+1)}\right] e^{-i\omega t}, \\ \psi_2(t,x) &= \left[1 + (b_1 + c_1 \ln x)\, x^2\right] \exp\left[-i\frac{\omega\alpha r_0 x^{n+1}}{c(n+1)}\right] e^{-i\omega t}. \end{aligned} \tag{B.6}$$

We can also obtain, in a similar way, formal results for the total energy for different values of n:

- For $n = 1/2$

$$\begin{aligned} E_1 &\approx \frac{3A\cos^2(\omega t)}{20\alpha c r_0}\, x^{3/2}\left(3\alpha^2 x + 5\right)\Big|_0^1, \\ E_2 &\approx \frac{A r_0 \omega^2 (1 - \cos(2\omega t))}{6\alpha c^3}\, \sqrt{x}\left(\alpha^2 x + 3\right)\Big|_0^1. \end{aligned}$$

- For $n = 1$

$$\begin{aligned} E_1 &\approx \frac{A(1 + \cos(2\omega t))}{2\alpha c r_0}\, x^2\Big|_0^1, \\ E_2 &\approx \frac{A r_0 \omega^2 \sin^2(\omega t)}{2\alpha c^3}\, \ln x\Big|_0^1. \end{aligned}$$

- For $n = 3/2$

$$\begin{aligned} E_1 &\approx \frac{5A(1 + \cos(2\omega t))}{8\alpha c r_0}\, x^{5/2}\Big|_0^1, \\ E_2 &\approx \frac{A r_0 \omega^2 (\cos(2\omega t) - 1)}{2\alpha c^3 \sqrt{x}}\Bigg|_0^1. \end{aligned}$$

- For $n = 2$

$$\begin{aligned} E_1 &\approx \frac{3A(1 + \cos(2\omega t))}{4\alpha c r_0} x^3\Bigg|_0^1, \\ E_2 &\approx \frac{A r_0 \omega^2 (\cos(2\omega t) - 1)}{4\alpha c^3 x}\Bigg|_0^1. \end{aligned}$$

- For $n = 3$

$$\begin{aligned} E_1 &\approx \frac{A\left[1 + \cos\left(2\omega t + \frac{\pi\omega r_0}{3\sqrt{3}\,\sqrt[3]{\alpha c}}\right)\right]}{\alpha c r_0}\, x^4\Big|_0^1, \\ E_2 &\approx -\frac{A r_0 \omega^2 \sin^2\left(\omega t + \frac{\pi\omega r_0}{6\sqrt{3}\,\sqrt[3]{\alpha c}}\right)}{4\alpha c^3 x^2}\Bigg|_0^1. \end{aligned}$$

Looking at these results for the energy, we observe that when $n \geq 1$ the energy for the second solution ($\psi_2(t, x)$) is not finite near the origin, so that $\psi_2(t, x)$ does not fulfill the finite energy condition. In this way, the second solution should be discarded and, as a result, no extra boundary conditions are required.