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Quantum fluctuations in non-globally hyperbolic spacetimes

Flutuações quânticas em espaços-tempos não-globalmente hiperbólicos

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Aos meus pais, Solange e Daniel.

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We can forgive a man for making a useful thing as long as he does not admire it.
The only excuse for making a useless thing is that one admires it intensely.
All art is quite useless.
- Oscar Wilde

Resumo

A quantização de campos em backgrounds providos pela Relatividade Geral é a pedra angular da Teoria Quântica de Campos em espaços curvos. Usualmente, tais espaços-tempos são globalmente hiperbólicos, i.e., espaços-tempos em que o problema de Cauchy é bem posto. Por outro lado, em espaços-tempos não-globalmente hiperbólicos, a perda de previsibilidade compromete o procedimento de quantização. Entretanto, Wald argumenta que uma dinâmica evolutiva razoável por ser recuperada encontrando as extensões auto-adjuntas e positivas da componente espacial do operador de onda diferencial. E ainda, Ishibashi e Wald mostram que essa prescrição para a dinâmica é a única cujos desfechos são fisicamente coerentes. De acordo com a prescrição deles, em espaços-tempos tais como o Monopolo Global e anti-de Sitter - onde a hiperbolicidade global está ausente -, uma dinâmica razoável é obtida através da imposição de um conjunto de condições de contorno na singularidade nua ou na fronteira conforme, respectivamente. Em nosso trabalho, examinamos os efeitos dessas condições de contorno que não são de Dirichlet sobre quantidades físicas relevantes, e.g., o valor esperado do tensor de energia-momento. Nossos dois primeiros toy models consistem em examinar o espalhamento de campos escalares no espaço-tempo tridimensional do cone e no Monopolo Global. Obtemos contribuições para a amplitude de espalhamento dependentes exclusivamente do parâmetro da condição de contorno - e independente das propriedades topológicas das respectivas variedades. Ainda no Monopolo Global, encontramos contribuições analíticas para as flutuações do tensor de energia-momento e o campo ao quadrado. Novamente, nossas contribuições dependem somente do parâmetro da condição de contorno. Além disso, nossos resultados assemelham-se àqueles de campos escalares em Minkowski com um ponto removido, o que confirma que quaisquer contribuições provenientes das condições de contorno não tem relação com a topologia do espaço-tempo. Por fim, consideramos a propagação de campos escalares no espaço-tempo de anti-de Sitter. A imposição de condições de contorno de Robin no infinito para somente um dos modos da equação de onda resulta em valores esperados que não respeitam as simetrias do adS. Um preço é pago ao se adotar a prescrição para uma dinâmica razoável em adS, a saber, a violação de condições de energia e a quebra da invariância do espaço-tempo. Basicamente, nesta dissertação, exploramos o desenvolvimento de uma Teoria Quântica de Campos fisicamente consistente em espaços-tempos não-globalmente hiperbólicos. No entanto, tal prescrição dá origem a resultados não triviais. Dentre as infinitas dinâmicas evolutivas possíveis que os campos podem admitir, não temos presente nenhuma restrição da natureza que nos force a escolher uma específica. Em todo caso, nossos resultados revelam que, se de alguma forma a natureza determinar uma condição de contorno em particular, tal escolha deve influenciar quantidades físicas relevantes.

Palavras-chave: Teoria Quântica de campos em espaços curvos. Gravitação Semi-clássica. Flutuações do vácuo. Defeitos topológicos. Anti-de Sitter.

Abstract

Quantization of fields on backgrounds provided by General Relativity is the primary cornerstone of Quantum field theory in curved spaces. Such spacetimes are usually globally hyperbolic, i.e., spacetimes where the Cauchy problem associated with hyperbolic equations - in particular, the Klein-Gordon equation - is well-posed. Conversely, in non-globally hyperbolic spacetimes, lost of predictability jeopardizes the quantization procedure. Although, Wald argues that one recovers a sensible dynamical evolution of fields by finding the positive self-adjoint extensions of the spatial component of the differential wave operator. Moreover, Ishibashi and Wald show that this prescription for dynamics is the only whose outcomes are physically consonant. According to their prescription, in spacetimes such as the Global Monopole or anti-de Sitter - where global hyperbolicity is absent -, a sensible dynamics is obtained through the imposition of a set of boundary conditions at the naked singularity or the conformal boundary, respectively. In our work, we examine the effects that these non-Dirichlet boundary conditions have on physically relevant quantities, e.g., the expectation value of the stress-energy tensor. Our first two toy models consist of examining the scattering of scalar fields on the three-dimensional conical spacetime and the Global Monopole. We obtain contributions to the scattering amplitude depending on the boundary condition parameter exclusively - and independent of topological features of the respective manifolds. Additionally, in the Global Monopole, we find analytical contributions to the fluctuations of the stress tensor and the field squared. Once again, our contributions depend solely on the boundary condition parameter. Besides, our results resemble those of scalar fields in Minkowski spacetime with a point removed, which confirm that any contributions from boundary conditions are unrelated to the topology of the spacetime. Finally, we consider the propagation of scalar fields on the anti-de Sitter spacetime. The imposition of Robin boundary conditions at infinity for one of the modes of the wave equation results in expectation values that do not respect all symmetries of adS. We pay the price for adopting a prescription for sensible dynamics in adS, namely the violation of energy conditions and the breakdown of spacetime invariance. Ultimately, in this dissertation, we exploit the development of a physically consistent Quantum Field Theory on non-globally hyperbolic spacetimes. Nevertheless, such prescription gives rise to non-trivial outcomes. Amongst the infinitely many dynamical evolutions that fields may admit, we are not aware of any constraints in nature that force us to choose a specific one. In any case, our results reveal that, if somehow nature determines a particular boundary condition, such a choice shall influence physically relevant quantities.

Keywords: Quantum field theory in curved spacetimes. Semiclassical Gravity. Vacuum fluctuations. Topological defects. Anti-de Sitter.

List of Figures

List of abbreviations and acronyms

QM	Quantum Mechanics
QFT	Quantum Field Theory
GR	General Relativity
QFTCS	Quantum Field Theory in Curved Spaces
PDE	Partial Differential Equation
v.e.v.	Vacuum expectation value
w.r.t.	with respect to
b.c.	boundary condition

List of symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers, i.e., $\{x \in \mathbb{R} \vert x \geqslant 0\}$.
$\mathbb{R}_{++}$	Set of positive real numbers, i.e., $\{x \in \mathbb{R} \vert x > 0\}$.
$\mathbb{N}_0$	Set of natural numbers and zero.
$\mathbb{C}$	Set of Complex numbers
$C^k(\Omega)$	Set of functions with k continuous derivatives on a set Ω. $C^\infty(\Omega)$ denotes the set of smooth functions on Ω.
$C_0^\infty(\Omega)$	Set of smooth functions that vanish at infinity of Ω.
$AC(\mathrm{I})$	Set of absolutely continuous functions in an interval I of $\mathbb{R}$.
$\mathbb{I}_n$	Identity matrix of order $n \times n$.
$\varnothing$	Empty set.
$\mathrm{diag}(\cdots)$	Denotes a diagonal matrix with components specified inside the parentheses.
$\eta_{\mu\nu}$	Metric tensor of the n-dimensional Minkowski spacetime given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$, unless otherwise stated.
$g_{\mu\nu}$	Metric tensor of a general spacetime.
g	Denotes the determinant of the covariant metric tensor $g_{\mu\nu}$, i.e., $g := \det(g_{\mu\nu})$.
$f'(x)$	Derivative of the function f with respect to x, also denoted by $\dfrac{df}{dx}$.
$\partial_\mu f$ or $f_{,\mu}$	Partial derivative of f w.r.t. x^μ.
$\nabla_\mu f$ or $f_{;\mu}$	Covariant derivative of f w.r.t. x^μ.
$\Gamma^\alpha_{\mu\nu}$	Connection coefficients (or Christoffel symbols of the second kind).
$\vec{v}$	Vector with spatial components only.

Contents

Introduction

Nowadays, General Relativity and Quantum Mechanics may be considered the two pillars of modern Physics. Both theories marked the 19th century with surprising, almost whimsical descriptions of the most fundamental properties of Nature, employing intricate mathematical apparatuses to do so. Now, state-of-art experiments confirm their predictions and level both theories up to other remarkably successful ones in Physics.

With the advent of Quantum Mechanics in the early decades of the 19th century, a significant change occurred on the general comprehension of phenomena taking place at scales as small as the size of an atomic nucleus. The proposition of the wave-particle duality compelled us to reimagine our very own understanding of the Nature of light. Since the Einstein-Planck relations for the new-born concept of the photon, Quantum Theory went through a series of extraordinary discoveries and enhancements that have been withstanding for over a century.

Parallel to the developments of Quantum Theory, in the early 1900s, Einstein published his two postulates of Special Relativity. Among other breakthroughs, this new theory led to a reformulation of Classical Mechanics, to the correspondence between matter and energy, and settled inconsistencies on Maxwell's theory of electromagnetism. Later on, through his Equivalence Principle, Einstein showed Special Relativity would be a restricted version of a broader theory, known as General Relativity. John A. Wheeler summarizes the content of it in his famous quote:

> Matter tells spacetime how to curve; spacetime tells matter how to move.
> -Wheeler, John A., in *Geons, Black Holes, and Quantum Foam: A Life in Physics*[3]

His statement refers to the way that Einstein's field equations intrinsically relate the geometric description of the Universe and the energy-matter content in it.

Before the full establishment of General Relativity, a natural question arose amongst physicists: is there a way to confront and reconcile two of the most successful theories of the 19th century: Quantum Mechanics and Special Relativity? As A. Zee argues in his book, the motion of a rocket ship flying close to the speed of light constitute a problem of relativistic dynamics, not of quantum mechanics. Conversely, a 'slow' moving electron orbiting a proton does not require any special relativity considerations; indeed, Schrödinger's equation would do the job just fine. Could such - apparently - distant theories converge to a description of a system requiring both of them? Well, empirical results suggested so. Non-relativistic quantum mechanics failed in describing particular, however relevant physical phenomena.

A simple process such as an electron scattering off a proton reveals the inconsistencies between quantum and relativistic descriptions. From a non-relativistic quantum mechanical point of view, the electron must remain one electron only throughout the entire process. However, from a relativistic point of view, its energy is free to vary; thus, a sufficiently energetic scattered electron could have its energy converted to matter - e.g., through the appearance of an electron-positron pair. Indeed, the states described by Schrödinger's equation have a fixed number of particles, hence shall not allow for the production-annihilation of particles.

Furthermore, as one takes electromagnetism into account, the primary subjects of the theory are classical fields. In the early attempts of describing the interactions between light (electromagnetic field) and the electron (quantum particle), the former would be regarded as a classical background field interacting with the latter. Even in this approach, photons could be destroyed and created; while, electrons could not. This situation was in direct conflict with experimental data showing the existence of electron-positron pairs. One would expect then a procedure that extends the quantization procedure to matter fields.

This extended quantization procedure appeared in the context of many-body systems and became known as second (or canonical) quantization. It was then adjusted to relativistic frameworks and consisted of promoting a classical matter field to a quantum field operator, written in terms of creation/annihilation operators inspired by the quantum mechanical harmonic oscillator. The coefficients of this decomposition are a complete set of eigenfunctions of a wave-like equation, known as the Klein-Gordon (KG) equation. In comparison to Schrödinger's equation, the KG equation preserves the mass-energy-momentum constraints of relativistic mechanics.

Past the mid-1900s, several researchers had perfected this novel theory that became known as Quantum Field Theory. Its accuracy in describing High-Energy systems is remarkable, and its primary role in the development of the Standard Model of fundamental particle interactions is outstanding. It is safe to say that QFT reached the podium of successful theories in Physics in terms of its theoretical predictions and empirical confirmations.

The thriving of QFT naturally led physicists and mathematicians to go further and investigate the feasibility of a quantum portrayal of gravity. Ultimately, they were looking for a unifying theory of GR and QM. This Quantum Gravity had numerous developments and approaches during the last decades. Through the efforts of several researchers, String Theory gained notability as a strong candidate for a quantum field description of the fundamental forces of Nature.

Notoriously, Juan Maldacena proposed in 1997 the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, also known as the holographic principle. His work brought us the concept of this holographic reality, in which a CFT developed on the boundary of AdS is in equivalence with the string theory taking place in the bulk of AdS. As yet, String Theory has still a long way to come in terms of experimental confirmation of its theoretical predictions.

Back in the 1950s, QFT in curved spacetimes (QFTCS) arose with the promise of circumventing the issues involved in the development of a theory of Quantum Gravity. In this semiclassical approach, the spacetime - and its gravitational structure described by GR - is treated as a classical background over which quantum fields propagate under the rules of QFT. Our work relies solidly on such a theoretical framework and its previous improvements.

In their reference book on QFTCS, Birrell and Davies recognize Stephen Hawking's results on quantum black holes and their thermal emission as a "cornerstone of the theory" of quantum fields in curved spacetimes. Although the feasibility of experimental probes of QFTCS is dubious, the authors point out that Hawking's results revealed a path towards the possibility of observational outcomes emerging from purely theoretical predictions. Such circumstances enclose our work, as we are looking for the effects that subtle theoretical considerations bring on physically relevant quantities.

Here, we are interested in a particular category of spacetimes whose causal structure is pathological for QFT, namely non-globally hyperbolic spacetimes. For instance, this class includes all solutions of Einstein's equations featuring naked singularities, e.g., negative mass Schwarzchild and extremal Reissner-Nordstrom solutions, and the Global Monopole. Another significant component of such a category is the anti-de Sitter solution. In all before-mentioned spacetimes, the equations of motion dictating the dynamics of quantum fields do not have a unique evolution for given initial data.

The uncertain history of a quantum field poses a critical issue on its quantization. We can understand this problem by considering that information could arbitrarily flow in or out of the spacetime through a region or point in space, such as a naked singularity or a spatial boundary. Hence, it would be impracticable to describe the entire history of a quantum field everywhere in spacetime. Nevertheless, Robert Wald proposed a prescription for dynamics that guarantees a unique evolution and, thus, allows for the study of non-globally hyperbolic spacetimes in the context of QFTCS. Later, together with Akihiro Ishibashi, they showed that Wald's prescription was the only one satisfying some pre-established conditions of physical consistency.

In static spacetimes, one can readily verify that the wave equation (Klein-Gordon equation) separates into temporal and spatial components. Wald examined the latter in the context of Functional Analysis and obtained his prescription for sensible dynamics by requiring the self-adjointness of the spatial differential operator. This procedure corresponds to finding boundary conditions at the singularities or spatial boundaries that renders the operator self-adjoint.

Our work resides on finding the effects that Wald's sensible dynamical prescription may cause to physically relevant quantities. In other words, we aim to find the presence of those boundary conditions in mathematical quantities computed from the fields and, thus, carrying physical content. In the examples treated in this dissertation, other than analyzing the self-adjointness of diverse differential operators for scalar fields, we obtain the scattering cross-section, the quadratic expectation values, and the v.e.v. of the stress-energy tensor of scalar fields propagating in non-globally hyperbolic spacetimes.

We organized this dissertation as follows. The first two Chapters serve as an overview of the theoretical and mathematical frameworks over which our work is established. By use of the leading textbooks on the topic, we review fundamental properties of QFTCS in Chapter 1. We provide a brief overview of the quantization procedure of free scalar fields in (globally hyperbolic) spacetimes of GR, and the computation of their expectation values, such as Green's Functions. Then, we recall the renormalization schemes of QFTCS, which are necessary to obtain finite quantities, as they will be carrying all physical content of the fields following the dynamical prescription by Wald and Ishibashi.

Elliptic differential operators play a central role in Wald's procedure for sensible dynamics of scalar fields in non-globally hyperbolic spacetimes. Chapter 2 summarizes the mathematical foundation required for the study of these operators and their analytical properties. We provide the reader with a series of useful, however simplified, definitions, and refer to more complete and formal textbooks in Functional Analysis for further reading. In Section 2.3, some illustrative examples are shown in order to draw the usefulness of considering the self-adjointness of differential operators.

In the following Chapters, 3 and 4, we abandon the global hyperbolicity of the spacetime and modify the theoretical basis presented in Chapters 1 and 2 accordingly. First, Section 3.1 familiarizes the reader with the concept of (non-)global hyperbolicity. Then, in Section 3.2, we briefly introduce the prescription for sensible dynamics of scalar fields in non-globally hyperbolic spacetimes, as proposed by Wald and Ishibashi. Finally, in Chapter 4, we present a systematic procedure to apply all previous discussions in three spacetimes, namely: the three-dimensional conical spacetime, the Global Monopole, and the Anti-de Sitter solution. Chapter 5 is reserved for our final remarks.

Conventions

On what follows, we adopt 'natural' units, in which the speed of light c, the reduced Planck constant $\hbar$ and the Boltzmann constant k_B are all set to one, i.e., $c = \hbar = k_B = 1$. Greek letter indices $(\alpha, \beta, \gamma, \delta, \cdots)$ label quantities with temporal and spatial components and range from 0 to the number of spatial dimensions. While Roman letters $(j, k, l, m, n, \cdots)$ indicate the spatial components and range from 1 to the number of spatial dimensions. Covariant quantities are identified with lower indices (e.g., $A_\mu, T_{\mu\nu}$), while contravariants, with upper indices (e.g., $A^\mu, T^{\mu\nu}$). All n-vectors v^μ are standardly defined as contravariant vectors, i.e., $v^\mu = (v^0, \vec{v})$. Einstein's summation convention over repeated indices is adopted for all types of indices, i.e.,

$$\sum_\mu x^\mu x_\mu \equiv x^\mu x_\mu, \qquad \sum_k x^k x_k \equiv x^k x_k.$$

We follow Misner, Thorne and Wheeler in [4] for all geometrical conventions. The metric signature is always $(-, +, +, \cdots, +)$, unless otherwise stated. For instance, the n-dimensional Minkowski metric tensor is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, \mathbb{I}_{n-1}).$$

The Riemann curvature tensor is defined as

$$\mathrm{R}^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha}.$$

The Ricci tensor is defined as

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^\alpha_{\mu\alpha\nu}.$$

The scalar curvature is given by

$$\mathrm{R} \equiv \mathrm{R}^\mu{}_\mu.$$

All authors agree that positive energy density requires

$$T_{00} > 0.$$

In this convention, Einstein field equations are

$$G_{\mu\nu} = \mathrm{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathrm{R} = \kappa T_{\mu\nu},$$

where $\kappa \equiv 8\pi G$.

1 Quantum Field Theory in curved spacetimes

Quantum Field Theory in curved spaces is a well-established description of the propagation of quantum particles on classical backgrounds of GR. An extensive literature (see Refs. [5, 6, 7, 8]) provides a complete guide on this semiclassical approach, and conduct us through a generalized quantization procedure based on that of QFT in Minkowski spacetime. As we will see in this chapter, quantizing a field in a curved spacetime of GR is a procedure filled with subtleties and incertitudes. We will show that issues appear right at the beginning when dealing with the ambiguity on defining a vacuum state and its associated normal modes expansion of the field operator. Even though we may borrow some regularization techniques from the usual QFT, renormalization procedures will change considerably. In our text, we restrict the discussion to free scalar fields coupled to the geometric structure of the spacetime (see Refs. [5] and [9] for a complete discussion on non-zero spin fields). In this context, the physically relevant quantities will be the expectation values of quadratic combinations of the field operator, instead of scattering amplitudes, as it would be in standard QFT. Perhaps the central quantity to be evaluated in QFTCS is the expectation value of the energy-momentum tensor, which appears in the semiclassical Einstein equations. Naturally, we shall present a systematic way of obtaining a finite, well-behaved form for that quantity.

1.1 Scalar fields in spacetimes of GR

Consider a general spacetime defined as an n-dimensional smooth, globally hyperbolic[1], pseudo-Riemannian manifold $\mathcal{M}$ provided with metric tensor $g_{\mu\nu}$ with signature $(-,+,+,\cdots,+)$. Let us now take a a massive scalar field $\phi(x)$ free of interactions, except for the non-minimal coupling with the metric tensor. The functional of action of this system is given by[2]

$$S[\phi, g_{\mu\nu}] = -\frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\phi\nabla\nu\phi + m^2\phi^2 + \xi\mathrm{R}(x)\phi^2\right], \tag{1.1}$$

in which m is the mass of the field, R is Ricci scalar curvature, and ξ is a coupling parameter. The Euler-Lagrange equation associated with the Lagrangian density in Eq. (1.1) with

[1] We shall address this property in detail later in Chap. 3 and show that it may be dropped under some specific conditions.

[2] Like in standard QFT, the Lagrangian density in Eq. (1.1) is not unique in that it may not be the only one that recovers Klein-Gordon equation. For an argument on the choice of the term $\xi\mathrm{R}\phi^2$, we refer to Chap. 6, p.117 in Ref. [7].

respect to ϕ yields the Klein-Gordon equation in a general spacetime

$$\Box\phi - m^2\phi - \xi \mathrm{R}\phi = 0, \tag{1.2}$$

where $\Box$ denotes the covariant d'Alembertian operator, i.e.,

$$\Box\phi = \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\partial_\nu \phi\right). \tag{1.3}$$

There are two values of ξ of particular relevance, namely $\xi = 0$ and $\xi = \frac{n-2}{4(n-1)} \equiv \xi_c$. The former indicates *minimally coupled* scalar fields, meanwhile the latter indicates *conformally coupled* ones. Minimal coupling occurs when the field does not couples with the curvature of the spacetime. In the case of $\xi = \xi_c$, the equations of motions exhibit conformal invariace, that is why it is regarded as conformal coupling of the field.

Since S is a functional of both, the field and the metric, i.e. $S \equiv S[\phi, g_{\mu\nu}]$, we can also compute the variation of the action with respect to the latter, yielding a tensorial quantity, namely the Energy-Momentum Tensor (EMT)[3]. Its tensorial components include physically relevant quantities, as the energy density and the momentum flux of the field, and its functional form is

$$\begin{aligned} T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} &= (1-2\xi)\phi_{;\mu}\phi_{;\nu} + (2\xi - 1/2) g_{\mu\nu} g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - 2\xi\phi_{;\mu\nu}\phi \\ &+ \frac{2}{n}\xi g_{\mu\nu}\phi \Box \phi - \xi\left[\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R} g_{\mu\nu} + \frac{2(n-1)}{n}\xi \mathrm{R} g_{\mu\nu}\right]\phi^2 \\ &+ 2\left[\frac{1}{4} - \frac{n-1}{n}\xi\right] m^2 g_{\mu\nu}\phi^2. \end{aligned} \tag{1.4}$$

Similarly, in General Relativity, we may obtain Einstein's field equations

$$G_{\mu\nu}(x) \equiv \mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R} g_{\mu\nu} = \kappa T_{\mu\nu}(x), \tag{1.5}$$

as the Euler-Lagrange equations associated with variations with respect to $g_{\mu\nu}$ of a more general action S_{GR} given by

$$S_{\mathrm{GR}} = S_{\mathrm{EH}} + S_{\mathrm{matter}}, \tag{1.6}$$

where S_{matter} refers to the action bearing all matter content, except the gravitational field, and has the form of Eq. (1.1), for scalar fields. On the other hand, the so-called Einstein-Hilbert action, S_{EH}, carries all gravitional content, and it is given by

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int d^n x \sqrt{-g}\, \mathrm{R}(x). \tag{1.7}$$

Meanwhile Eq. 1.5 classically encodes all dynamical properties of the gravitational field, Eq. (1.2) determines the dynamics of scalar fields propagating on the classical background of GR. The eigenfunctions of Eq. (1.2), denoted $u_k(x)$ (where k represents all labels

[3] Some authors employ the term Stress-Energy tensor or simply Stress tensor interchangeably with EMT, we shall do the same here.

necessary to identify the modes), compose a complete set of eigenvectors orthonormalized, i.e.,

$$(u_k, u_{k'}) = \delta_{k,k'}, \tag{1.8}$$
$$(u_k, u^*_{k'}) = 0, \tag{1.9}$$

by the scalar product defined as follows[4]

$$(\phi_1, \phi_2) = i\int_\Sigma \Big[\phi_1^*(\partial_\mu\phi_2) - (\partial_\mu\phi_1^*)\phi_2\Big]\sqrt{-h}\; n^\mu d\Sigma \equiv i\int_\Sigma \phi_1^* \overleftrightarrow{\partial}_\mu \phi_2 \sqrt{-h}\; n^\mu d\Sigma, \tag{1.10}$$

where Σ is a spacelike hypersurface of $\mathcal{M}$ with volume element $d\Sigma$ and metric tensor h_{ij}, n^μ is a future-directed null vector orthogonal to Σ, and $h \equiv \det(h_{ij})$. The canonical quantization follows by promoting $\phi(x)$ to a field operator defined on a Fock space. Such an operator admits an expansion in terms of the modes $u_k(x)$ given by

$$\phi(x) = \sum_k [a_k u_k(x) + a_k^\dagger u_k^*(x)], \tag{1.11}$$

in which the set of annihilation a_k and creation $a_k^\dagger$ operators respect canonical commutation relations, as follows

$$[a_k, a_{k'}] = 0, \tag{1.12}$$
$$[a_k^\dagger, a_{k'}^\dagger] = 0, \tag{1.13}$$
$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}; \tag{1.14}$$

and possess a vacuum eigenstate, denoted $|0\rangle$, satisfying

$$a_k |0\rangle = 0. \tag{1.15}$$

The expansion (1.11) is useful in a setting where the modes are constructed in a 'box' following a quantized fashion for the eigenvalues of Eq. (1.2) (for instance, this is the case of AdS spacetime, as we will see in Sec. 4.3). However, those wave packets in Eq. (1.11) may be replaced by orthonormal modes $u_{\vec{k}}(x)$ with continuum spectrum, which is the usual procedure in QFT in Minkowski (e.g., see Secs. 2.3 and 2.4 of Ref. [10]). In this last situation, the field is expanded as follows

$$\phi(x) = \int d^{n-1}k\big[a(\vec{k})u_{\vec{k}}(x) + a^\dagger(\vec{k})u^*_{\vec{k}}(x)\big], \tag{1.16}$$

and the commutation relations are replaced by

$$[a(\vec{k}), a(\vec{k}')] = 0, \tag{1.17}$$
$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0, \tag{1.18}$$
$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}'). \tag{1.19}$$

[4] For a derivation of this inner product, see Chap. 3 in Ref. [7].

One might transit between these two descriptions by proceeding with the quantization in a box of volume L^{n-1} with periodic boundary conditions at the walls. In the end, it suffices to take $L \to \infty$ to recover the appropriate results (see Refs. [5] and [9]).

In opposition to Minkowski, the quantization described above is inherently ambiguous in generic spacetimes, since there is no 'natural' choice of solutions of the wave equation[5]. Indeed, Minkowski spacetime has an usual coordinate system - namely, Euclidean rectangular coordinates - with which we can solve Eq. (1.2), hence providing a natural choice for the set of modes $u_j(x)$. Such modes are also eigenvectors of the temporal Killing field ∂_t, which allows us to expand the field operator in terms of positive energy solutions, which are eigenfunctions of the energy operator $i\partial_t$ with positive eigenvalues, $\omega > 0$. The prescription that we have presented above is the usual adopted in QFT textbooks, e.g., Refs. [11, 10].

Let us address the before-mentioned ambiguity in the quantization by considering another complete set of orthonormal solutions $\bar{u}_j(x)$ to Eq. (1.2), and following the same decomposition of the field operator

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)], \tag{1.20}$$

with a new set of annihilation operators $\bar{a}_j$ possessing a new vacuum state $|\bar{0}\rangle$, such that

$$\bar{a}_j |\bar{0}\rangle = 0. \tag{1.21}$$

By a Bogoliubov transformation, we can establish a relation between both Fock spaces, yielding

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*), \tag{1.22}$$

with α_{ij} and β_{ij} being Bogoliubov coefficients to be determined in each case. Thus, the operator become

$$\bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger). \tag{1.23}$$

From which we notice that, for $\beta_{ij} \neq 0$, the vacuum states $|0\rangle$ and $|\bar{0}\rangle$ do not coincide, as $\bar{a}_j$ does annihilate $|0\rangle$. Furthermore, if we consider the number operator $N_j = a_j^\dagger a_j$ and compute its vacuum expectation value in the corresponding Fock space, we get to

$$\langle \bar{0}| N_j |\bar{0}\rangle = \sum_i |\beta_{ij}|^2 \neq \langle 0| N_j |0\rangle = 0. \tag{1.24}$$

The equation above shows that we detect $\sum_i |\beta_{ij}|^2$ particles of the mode u_i in the vacuum state of $\bar{u}_i$.

Accordingly, non-correspondent Fock spaces provoke the appearance of non-trivial physical outcomes, such as the Unruh effect or the production of particles in a cosmological model of an expanding universe[5]. As a result, observers in different reference frames may disagree on the existence of particles. This surprising effect indicates the lack of a clearly defined particle concept in general spacetimes. In such conditions, based on the grounds of physical consistency, we accept that a given quantum state $|\Psi\rangle$ is emptied of particle content. Instead, we infer all physical properties from expectation values taken with respect to $|\Psi\rangle$, leaving the blurry particle concept behind[12]. In the next sections, we will discuss the general approach to obtain these expectation values and will establish the main distinctions between QFT in Minkowski and general spacetimes.

1.2 Green's functions and the Stress Tensor

In standard QFT, Green's functions play a fundamental role when computing scattering amplitudes and obtaining corrections to physical quantities[10]. Such objects are expectation values of combinations of field operators and provide correlation functions between different positions in spacetime. From a mathematical point of view, their construction as products of operators designates them as operator-valued distributions. Thus, one would anticipate their expectation values to be formally infinite. Indeed, when handling with Green's functions, several issues regarding divergences appear.

One may consistently deal with those issues through renormalization procedures, which results in finite correlation functions. By way of a process called regularization, divergences can be isolated and subtracted off. However, one may pay the price when following this scheme: some quantities of the theory, such as the electronic mass or the normalization factor of the fields, will receive perturbative corrections, according to the diagram level that was chosen to describe the interactions.

In the context of semiclassical gravity, we face infinities arising from a different problem. Because our approach is based on geometrical objects - all of them locally defined throughout the spacetime - we need to relate them to local quantities from QFT, such as the expectation values of the field squared or of the EMT. We will show that the process of obtaining such quantities involves taking a coincidence limit in the end, i.e., $x' \to x$. However, Green's functions are transitional amplitudes hence non-local objects by construction. Additionally, its distributional nature gives rise to the appearance of infinities.

A familiar example of this situation is the Dirac delta-function, $\delta(x-x')$, whose value is formally infinite as one takes $x' \to x$ since it is mathematically defined as a distribution. Nevertheless, we usually make sense of $\delta(x-x')$ by applying it to a test function with compact support on a given domain. Despite being aware of such technicalities, we do not follow a formal distributional treatment but refer to the discussion in Chap. 4 of Ref [7] by Fulling for guidance. Later in this chapter, we discuss the standard procedure adopted in QFTCS to address the divergent behavior of Green' functions.

For a given normalized quantum state $|\Psi\rangle$ of the field ϕ, the Feynman propagator is defined by the time ordering of the fields, i.e.,[5]

$$G_F(x,x') := -i\langle\Psi|\, T\{\phi(x)\phi(x')\}\, |\Psi\rangle\,, \tag{1.25}$$

and must be a solution of

$$\left[-\Box + m^2 + \xi R(x)\right] G_F(x,x') = -\frac{1}{\sqrt{-g}}\delta^n(x,x'). \tag{1.26}$$

By considering a normalized vacuum state, we can rewrite Eq. 1.25 in terms of a set of eigenfunctions of the wave equation, as follows

$$G_F(x,x') = -i\sum_j \left(\Theta(t-t')u_j(x)u_j^*(x') + \Theta(t'-t)u_j(x')u_j^*(x)\right), \tag{1.27}$$

or

$$G_F(x,x') = -i\int d^{n-1}k\,\left(\Theta(t-t')u_{\vec{k}}(x)u_{\vec{k}}^*(x') + \Theta(t'-t)u_{\vec{k}}(x')u_{\vec{k}}^*(x)\right). \tag{1.28}$$

In some cases, the summation of modes in Eq. (1.27) and the integral of modes in Eq. (1.28) provide us with an explicit form of the Green's function. Nevertheless, in several cases, that may not be the case. In such conditions, let us recall that Green's functions may be regarded as the integral kernels of the inverse Klein-Gordon operator, i.e., if

$$f(x) \equiv \int d^n x' G(x,x')g(x'), \tag{1.29}$$

then

$$\left[-\Box + m^2 + \xi R(x)\right] f(x) = g(x). \tag{1.30}$$

In a distributional sense, we may express the Green's function as

$$G_F(x,x') = -\left[-\Box + m^2 + \xi R(x)\right]^{-1}\frac{1}{\sqrt{-g}}\delta^n(x,x'), \tag{1.31}$$

and use the Schwinger representation

$$G_F(x,x') = -\int_0^\infty ds\; e^{-s(\Box - m^2 - \xi R(x))}\frac{\delta^n(x,x')}{\sqrt{-g}}. \tag{1.32}$$

In Eq. (1.32), we evoke the completeness relation of the Delta function,

$$\frac{\delta^n(x,x')}{\sqrt{-g}} = \int_\lambda \sum_n u_{\lambda,n}(x)u_{\lambda,n}^*(x'), \tag{1.33}$$

for a complete set of eigenfunctions $u_{\lambda,n}$ of Klein-Gordon equation, identified by the appropriate labels λ and n. In Sections 4.1 and 4.2, we will display examples in which Eq. (1.32) is employed in the computation of G_F.

[5] This is the definition adopted by Birrell & Davies in Ref. [5].

Another possible set of Green's functions is given by the commutator and anticommutator of the field, as follows

$$iG(x, x') = \langle \Psi | \left[\phi(x), \phi(x') \right] | \Psi \rangle , \tag{1.34}$$

$$G_{(1)}(x, x') = \langle \Psi | \left\{ \phi(x), \phi(x') \right\} | \Psi \rangle , \tag{1.35}$$

respectively. They satisfy the homogeneous differential equation

$$\left[-\Box + m^2 + \xi R(x) \right] G(x, x') = 0, \tag{1.36}$$

and allow us to decompose Feynman's function in terms of them,

$$G_F(x, x') = \frac{1}{2} \left[\Theta(t - t') - \Theta(t' - t) \right] G(x, x') - \frac{i}{2} G_{(1)}(x, x'). \tag{1.37}$$

From which we can see that the first term vanishes when approaching the coincidence limit, i.e.,

$$G_F(x, x') \sim -\frac{i}{2} G_{(1)}(x, x'), \quad x' \to x. \tag{1.38}$$

This equation shows that the behavior of G_F in the coincidence limit is dictated entirely by $G_{(1)}$, known as the Hadamard function. Thus, if any divergences were to appear in G_F, they should appear in $G_{(1)}$ as well. We will see in the next section that this is precisely the case here.

Let us now turn our attention back to the Stress Tensor, which, after quantizing the field in Eq. (1.4) becomes an operator. Its expectation value, $\langle T_{\mu\nu} \rangle$, plays a fundamental role in semiclassical gravity. Since all matter content is now encoded in expectation values of fields, it is rather natural to propose a description of the coupling between the quantum fields and the underlying structure of spacetime, given by the gravitational field $g_{\mu\nu}$. Such conjecture translates into the semiclassical version of Einstein equations, given by

$$G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle . \tag{1.39}$$

Here, the expectation value $\langle T_{\mu\nu} \rangle$ acts as the source of matter. The procedure of finding new solutions to Eq. (1.39) is regarded as the *back-reaction* of matter fields on the spacetime structure.

It may be convenient to write the expectation value $\langle T_{\mu\nu} \rangle$ in terms of Green's functions, since we may have calculated them previously. We can do so as follows

$$\langle T_{\mu\nu} \rangle (x) \equiv \langle \Psi | T_{\mu\nu} | \Psi \rangle (x) = \lim_{x' \to x} \mathcal{T}_{\mu\nu}(x, x') G_{(1)}(x, x'), \tag{1.40}$$

in which $\mathcal{T}_{\mu\nu}(x, x')$ is a non-local differential operator with tensorial structure. Like Green's functions, each component of $T_{\mu\nu}$ is an operator-valued distribution. Also, its expectation value $\langle T_{\mu\nu} \rangle (x)$ is a local quantity obtained from a non-local one, namely $G_{(1)}(x, x')$. Hence, $\langle T_{\mu\nu} \rangle$ is formally infinite and can have no place in Eq. (1.39). To make sense out of this quantity and employ it in back-reaction, we must follow some kind of renormalization procedure that yields a finite $\langle T_{\mu\nu} \rangle$.

1.3 Renormalization in QFTCS

One of the issues of the canonical quantization commonly addressed by QFT in Minkowski textbooks is the divergence of the vacuum energy (see [10, 13]). In that situation, one may write down the Hamiltonian operator of the fields and calculate its expectation value w.r.t. the usual Minkowski vacuum, $|0_{\mathrm{M}}\rangle$, in order to find its energy. All terms depending on the annihilation/creation operators vanish, and the remaining term is an integral of the zero-point energy of the field over the entire momentum space. Such integral diverges, indicating that the energy of the Minkowski vacuum is infinite. In non-gravitational physics, we can shamelessly discard this infinite vacuum energy by taking the *normal ordering*. In this way, we renormalize the vacuum energy since only energy differences are measurable. Conversely, when gravity comes to play, energy is itself a source in Einstein equations; hence it intrinsically influences the spacetime gravitational structure. We are not allowed to rescale the vacuum energy freely.

The most commonly used renormalization scheme in QFTCS is called *point-splitting* and we refer to Ref. [5] as a complete guide on that. However, on what follows, we adopt an extension of the point-splitting method, usually regarded as *Hadamard renormalization*(see Wald[14] and Fulling[7]). In such procedure, it is assumed that our quantum field ϕ has normalized states $|\Psi\rangle$ - including a vacuum state $|0\rangle$ - of the Hadamard type. Accordingly, expectation values taken with respect to $|\Psi\rangle$ will reproduce the Hadamard form on short-distance limits. Hence, one could eliminate the divergences appearing on the Green's functions when taking the coincidence limit by subtracting off their corresponding Hadamard representation.

As most developments regarding the Hadamard renormalization scheme were done in four spacetime dimensions, Décanini and Folacci present in [12] a systematic way to extend it to arbitrary dimensions. Given Feynman Green's function (1.25) defined with respect to a normalized Hadamard state of ϕ, the authors propose a decomposition of G_F as follows

$$G_F(x,x') = G_F^{\mathbf{div}}(x,x') + G_F^{\mathbf{reg}}(x,x'), \tag{1.41}$$

where $G_F^{\mathbf{div}}$ and $G_F^{\mathbf{reg}}$ are biscalars containing the divergent and regular components of G_F, respectively. On the other hand, the Hadamard form of G_F dictates the singular behavior by $G_H^{\mathbf{div}}$. The renormalized Green's function is then defined as

$$[G_F]_{\mathrm{ren}}(x,x') = G_F(x,x') - G_H^{\mathbf{div}}(x,x'). \tag{1.42}$$

In these conditions, Décanini and Folacci identify the expectation value of the quadratic field fluctuations by

$$\langle \phi^2 \rangle (x) = \lim_{x' \to x} i \, [G_F]_{\mathrm{ren}} (x,x'). \tag{1.43}$$

Through some manipulations, they show the renormalized expectation value of the energy momentum tensor (1.40) can be written in an arbitrary spacetime dimension as

$$\langle T_{\mu\nu}\rangle_{\rm ren} = -[G]_{\mu\nu} + \frac{1}{2}(1-2\xi)[G]_{;\mu\nu} + \frac{1}{2}\left(2\xi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\nabla^\sigma[G] + \xi R_{\mu\nu}[G]\right) + \Theta_{\mu\nu}, \quad (1.44)$$

where

$$[G](x) := \lim_{x'\to x} i\,[G_F]_{\rm ren}\,(x,x'), \quad (1.45)$$

$$[G]_{\mu\nu}(x) := \lim_{x'\to x} i\,[G_F]_{\rm ren}\,(x,x')_{;\mu\nu}, \quad (1.46)$$

and $\Theta_{\mu\nu}$ is a purely geometric tensor constructed to be conserved. Also, the renormalized energy-momentum tensor will be conserved, as expected,

$$\nabla_\nu \langle T_\mu{}^\nu\rangle_{\rm ren} = 0. \quad (1.47)$$

2 Elliptic differential operators and self-adjointness

From the classical equations of motion in Newtonian Mechanics to geodesic equations in General Relativity. From Maxwell's electromagnetic field equations to Schrödinger equation in Quantum Mechanics. Partial differential equations appear throughout the description of the most various physical setups and are of high relevance in Physics. In the particular case of time-independent linear PDE theory, we shall be able to perform the usual separation of variables and develop an expansion in eigenfunctions of elliptic partial differential operators. Perhaps the most familiar of such operators is the Laplace operator, which, for instance, appears in the wave and heat equations, and Schrödinger equation. Indeed, the Fourier transform and the Fourier series - both commonly employed when solving a wide range of linear PDEs - rely on the decomposition of arbitrary functions in terms of those sets of eigenfunctions of the Laplace operator.

Another quite recurrent concept in Physics is self-adjointness. Indeed, a standard procedure in Quantum Mechanics is to impose that an operator is Hermitian (or self-adjoint[1]). Such property guarantees physical interpretations of the expectation values and the spectrum of those 'observable' operators. However, the mathematical subtleties are not always addressed accordingly in most textbooks, which ends up leaving an open door for eventual contradictions and inconsistencies (see Ref. [17]). Despite overlooking such mathematical issues might not always cause significant difficulties in Quantum Mechanics, we will discuss further in the text that rigorous considerations on the self-adjointness of operators play a primary role in our work. In this chapter, we aim to address some properties of self-adjoint, elliptic differential operators briefly. We will provide a few definitions and some useful theorems, but we shall not prove them.

2.1 Second-order, linear elliptic differential operators

Let us denote the partial derivative operator as follows:

$$\mathrm{D}_j \equiv \partial_j = \frac{\partial}{\partial x^j}, \tag{2.1}$$

[1] In Quantum Mechanics textbooks, the term *self-adjoint* is usually employed interchangeably with *Hermitian*. However, Kreyszig[15] reserves the term Hermitian to *bounded*, self-adjoint operators. Meanwhile, the standard reference in *Functional Analysis*, Reed & Simon[16], regards *unbounded*, *symmetric* operators as Hermitian. In order to avoid any ambiguities, we do not use the term Hermitian anywhere in our work and adopt the mathematical terminology.

and consider a linear PDE of the form

$$\mathrm{L}[\phi] = w(x), \tag{2.2}$$

where L is a linear differential operator of the form

$$\mathrm{L} = a^{jk}(x)\mathrm{D}_j\mathrm{D}_k + b^j(x)\mathrm{D}_j + c(x) \tag{2.3}$$

acting on functions $\phi : \Omega \to \mathbb{C}$, where Ω is an open subset of $\mathbb{R}^n$. We assume $a^{jk}, b^j, c : \Omega \to \mathbb{C}$, $a^{jk}, b^j, c \in C^\infty(\Omega)$ and $a^{jk} = a^{kj}$. The operator L is *elliptic* if the matrix-valued function (a^{jk}) is positive-definite (or negative-definite)[18, 7]. This definition may be extended to higher-order differential operators (see Ref. [19]), but we shall not present it as it will not be useful for us in this dissertation.

The simplest of the second-order, elliptic differential operators is the Laplace operator, ∇^2, familiar to several problems in Physics. Its form in Euclidean coordinates is

$$\nabla^2 w = \frac{\partial^2 w}{\partial(x^1)^2} + \cdots + \frac{\partial^2 w}{\partial(x^n)^2} \equiv (\mathbb{I}_n)^{jk}\mathrm{D}_j\mathrm{D}_k w. \tag{2.4}$$

Upon comparison with Eq. (2.3), we see that the Laplacian is indeed elliptic, since $(a^{jk}) = (\mathbb{I}_n)^{jk}$, which is clearly positive-definite.

Another useful way of writing the operator L is in its *divergence form*[18], whose simplest case is

$$\mathrm{L}[u] = \mathrm{D}_j(a^{jk}\mathrm{D}_k u). \tag{2.5}$$

Again, the operator L is *elliptic* if the matrix (a^{jk}) is positive-definite (or negative-definite) in Ω. The subset Ω might be whether a coordinate chart for a particular manifold or the manifold itself. We may identify the matrix (a^{jk}) as the metric tensor on the manifold. Indeed, a relevant example of the form (2.5) is the Laplace-Beltrami operator on a (pseudo-)Riemannian manifold Σ provided with covariant metric tensor h_{ij}, given by

$$\Box_\Sigma w = \frac{1}{\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j w), \tag{2.6}$$

where $h := \det(h_{ij})$. It is clear that the matrix (a^{jk}) is proportional to the contravariant metric tensor h^{jk}. This remark shall play a significant role in the context of semiclassical gravity since the spacetime manifold will act as a background for the propagation of fields and its metric tensor will appear in the equations of motion of them. We will return to this point in the next chapter to relate it to our work. In the upcoming section we will provide an overview on the self-adjointness of linear operators.

2.2 Self-adjoint operators in Hilbert Spaces

On what follows, we refer to the texts of Kreyszig in [15] and Reed & Simon in [16] when presenting some of the definitions and theorems - which we shall not prove - that are necessary to proceed with our discussion.

Let us consider a complex Hilbert space $\mathcal{H}$ provided with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$,and a linear operator $A:\mathcal{D}(A)\subseteq\mathcal{H}\to\mathcal{H}$.

Definition 1. The operator A is *densely defined* in $\mathcal{H}$ if $\mathcal{D}(A)$ is dense in $\mathcal{H}$, i.e., $\overline{\mathcal{D}(A)}=\mathcal{H}$.

Theorem 1. (Closed operator) A is *closed* if and only if

$$\{\phi_n\}\to\phi,\ \text{for}\ \{\phi_n\}\in\mathcal{D}(A),\ \text{and}\ A\phi_n\to\psi$$

together imply that $\phi\in\mathcal{D}(A)$ and $A\phi=\psi$. (Cf. 10.3-2(a) of Ref. [15])

Definition 2. (Closable operator) If A has an extension which is a closed linear operator, then A is said to be *closable*. If there exists a minimal closed extension of A, denoted $\bar{A}$, it is called *closure* of A.(Cf. 10.3-4 of Ref. [15])

Definition 3. (Adjoint operator) Let $A:\mathcal{D}(A)\to\mathcal{H}$ be a linear operator densely defined in $\mathcal{H}$. The subset $\mathcal{D}(A^{\dagger})$ consists of all $\psi\in\mathcal{H}$ such that there is a $\psi^{\dagger}\in\mathcal{H}$ satisfying $\langle A\phi,\psi\rangle=\langle\phi,\psi^{\dagger}\rangle$. We define the *adjoint* $A^{\dagger}$ of A for all $\psi\in\mathcal{D}(A^{\dagger})$ by

$$\psi^{\dagger}:=A^{\dagger}\psi.$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, the formula above uniquely determines $\psi^{\dagger}$. (Cf. 10.1-2 of Ref. [15])

Definition 4. (Symmetric operator) Let $A:\mathcal{D}(A)\to\mathcal{H}$ be a linear operator densely defined in $\mathcal{H}$. A is said to be *symmetric* if, for all $\phi,\psi\in\mathcal{D}(A)$, the following relation is satisfied

$$\langle A\phi,\psi\rangle=\langle\phi,A\psi\rangle.$$

A Lemma of this definition is that A is symmetric if and only if $\mathcal{D}(A)\subset\mathcal{D}(A^{\dagger})$ and $A=A^{\dagger}|_{\mathcal{D}(A)}$ $(A\phi=A^{\dagger}\phi$, for $\phi\in\mathcal{D}(A))$.

Definition 5. (Self-adjoint operator) A linear operator A is said to be *self-adjoint* if and only if it is symmetric and $\mathcal{D}(A^{\dagger})=\mathcal{D}(A)$, that is, if $A^{\dagger}=A$. Accordingly, self-adjoint operators are symmetric, but the converse may not be true. (Cf. p.255 of Ref. [16])

Definition 6. (Essentially self-adjoint operator) A symmetric operator A is said to be *essentially self-adjoint* if its closure $\bar{A}$ is self-adjoint. (Cf. p.256 of Ref. [16])

At this point, one may ask: given a symmetric operator, is it possible to make it self-adjoint? The answer is: sometimes, yes. Indeed, in some cases, it is possible to extend the domain of a symmetric operator to obtain its so-called self-adjoint extensions, as we show on what follows. Additionally, it is guaranteed that if A is symmetric and positive[2], then there exists at least one positive self-adjoint extension, namely the Friedrichs extension(Cf. X.3 in Ref. [20]).

Let us begin considering a self-adjoint operator A. Consider a function $\phi \in \mathcal{D}(A^\dagger) = \mathcal{D}(A)$ such that $A^\dagger\phi = \pm i\phi$. Self-adjointness of A guarantees that $A\phi = \pm i\phi$ and

$$\mp i \langle \phi, \phi \rangle = \langle \pm i\phi, \phi \rangle = \langle A\phi, \phi \rangle = \langle \phi, A^\dagger \phi \rangle = \langle \phi, A\phi \rangle = \pm i \langle \phi, \phi \rangle, \qquad (2.7)$$

which leads to $\langle \phi, \phi \rangle = 0$ (Cf. p.256 of Ref. [16]). This brief deduction shows that $A^\dagger\phi = \pm i\phi$ can have no solutions if A is self-adjoint. We may formalize this requirement by defining the so-called *deficiency subspaces* as follows:

Definition 7. (Deficiency Subspace) We define the deficiency subspaces $\mathcal{N}_\pm$ and their deficiency indices (n_+, n_-) by

$$\mathcal{N}_\pm = \{\psi \in \mathcal{D}(A^\dagger),\ A^\dagger\psi = \pm i\lambda\psi,\ \lambda \in \mathbb{R}_{++}\} \quad \text{and} \quad n_\pm = \dim(\mathcal{N}_\pm). \qquad (2.8)$$

The indices $n_\pm$ are independent of the choice of λ.

Weyl[21] and von Neumann[22] established a theorem showing whether a symmetric linear operator is self-adjoint or not, and if it admits any self-adjoint extensions. Their theorem states the following:

Theorem 2. Let A be a symmetric operator with deficiency indices (n_+, n_-), then there are three possible cases:

1. If $n_+ = n_- = 0$, then A is *essentially self-adjoint* (indeed, this condition is necessary and sufficient).

2. If $n_+ = n_- = n \geqslant 1$, then A has infinitely many self-adjoint extensions. They are in one-to-one correspondence to the isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$ parametrized by an $n \times n$ unitary matrix, U.

3. If $n_+ \neq n_-$, then A has no self-adjoint extensions.

Clearly, the second case is more complicated than the others, and we must follow a systematic procedure for obtaining the self-adjoint extensions. For that, we employ the following theorem.

Theorem 3. Let A be a closable symmetric operator with closure $\bar{A}$. Consider an isometry U between the deficiency subspaces $\mathcal{N}_+$ and $\mathcal{N}_-$. We define the closed self-adjoint extensions A_E of A by

$$\mathcal{D}(A_E) = \{\Phi_0 + \Phi_+ + U\Phi_+ |\ \Phi_0 \in \mathcal{D}(\bar{A}), \Phi_+ \in \mathcal{N}_+\}, \qquad (2.9)$$

[2] An operator A defined on a Hilbert space $\mathcal{H}$ is positive if the quadratic form $\psi \mapsto \langle \psi, A\psi \rangle_{\mathcal{H}}$ is non-negative, for all $\psi \in \mathcal{H}$.

and

$$A_E\Phi = \bar{A}\Phi_0 + i\Phi_+ - iU\Phi_+, \tag{2.10}$$

for all $\Phi \in \mathcal{D}(A_E)$. (Cf. Theorem X.2 of Ref. [20])

Now, A_E is self-adjoint, since $A_E = A_E^\dagger$ and $\mathcal{D}(A_E) = \mathcal{D}(A_E^\dagger)$. This procedure can always be followed to find whether an operator has self-adjoint extensions and what they are when they exist. Another question may arise naturally at this point: what is the meaning of these self-adjoint extensions? The short, not-so-obvious answer is that to each extension (or extended domain), there is a correspondent boundary condition at determined regions of space. However, this statement makes much more sense after looking at practical examples. In the next section, we present two cases that illustrate applications of the definitions and theorems that we have presented so far, and clear out the relation between self-adjoint extensions and boundary conditions.

2.3 Illustrative examples

In this section, we restrict our discussion to a particular Hilbert space, namely $\mathcal{H} = \mathcal{L}^2(\Omega; \rho d^d x)$, which is the set of all square-integrable functions on Ω with measure $\rho(x)d^d x$. The space $\mathcal{L}^2(\Omega; \rho d^d x)$ is defined by the inner product

$$\langle \phi_1, \phi_2 \rangle \equiv \int_\Omega \phi_1^*(x)\phi_2(x)\rho(x)d^d x. \tag{2.11}$$

This is the usual Hilbert space of QM and appears in QFT and QFTCS when defining the Fock space, i.e., the abstract space of a multiple-particles system. We shall present an example of an operator in QM that is not self-adjoint: the momentum operator for a quantum particle in a box. Our second example includes a quite useful analysis of the three-dimensional Laplace operator in spherical coordinates, which appears in problems of QM and QFT.

2.3.1 Momentum operator on a finite interval (particle in a box)

Let us consider the one-dimensional momentum operator

$$p = -i\tfrac{d}{dx} : \mathcal{D}(p) \to \mathcal{L}^2(0, L),$$

for

$$\mathcal{D}(p) = \{\psi \in AC(0, L) \subset \mathcal{L}^2(0, L) | \psi(0) = \psi(L) = 0\}.$$

This operator is densely defined in $\mathcal{L}^2(0, L)$, closed and unbounded (see Ref. [15] for proofs). Also, we can verify that it is symmetric; for that, consider the functions $\phi_1, \phi_2 \in \mathcal{D}(p)$, then using integration by parts it follows that

$$\langle \phi_1, p\phi_2 \rangle = \int_0^L \phi_1^* \left(-i\frac{d\phi_2}{dx}\right) dx = -i\cancelto{0}{[\phi_1\phi_2]_0^L} + \int_0^L \left(-i\frac{d\phi_1}{dx}\right)^* \phi_2 \, dx = \langle p\phi_1, \phi_2 \rangle, \tag{2.12}$$

where the boundary terms vanished because ϕ_1 and ϕ_2 both satisfy $\phi(0) = \phi(L) = 0$. Hence p is symmetric by definition 4. Its adjoint is given by $p^\dagger = -i\frac{d}{dx} : \mathcal{D}(p) \to \mathcal{L}^2(0, L)$ defined on $\mathcal{D}(p^\dagger) = \mathcal{L}^2(0, L)$. It is clear that the domain of the operator is smaller than that of its adjoint, i.e., $\mathcal{D}(p^\dagger) \supset \mathcal{D}(p)$, thus, p is not self-adjoint. Indeed, for instance, take $\Psi(x) = \exp(\pm ikx) \in \mathcal{D}(p^\dagger)$; it follows that $\Psi^\dagger = p^\dagger\Psi = \pm k \exp(ikx)$, so $\Psi^\dagger \in \mathcal{L}^2(0, L)$ but $\Psi \notin \mathcal{D}(p)$, which shows that $\mathcal{D}(p) \subset \mathcal{D}(p^\dagger)$.

From theorem 2, we may determine whether p has self-adjoint extensions or not. First, we must find the deficiency indices as in definition 7, i.e. by solving

$$p^\dagger\phi_\pm = -i\frac{d}{dx}\phi_\pm = \pm i\lambda\phi_\pm, \tag{2.13}$$

whose solutions are

$$\phi_\pm(x) = \frac{\sqrt{2\lambda}e^{\mp\lambda x}}{\sqrt{\pm(1 - e^{\mp 2\lambda L})}}. \tag{2.14}$$

Both $\phi_\pm$ are in $\mathcal{L}^2(0, L)$ and the deficiency indices are $(n_+, n_-) = (1, 1)$. Theorem 2 tells that the operator p has infinitely many self-adjoint extensions parametrized by a set of $U(1)$ isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$, i.e., $U_\beta\phi_+ = e^{i\beta}\phi_-$. Now, the extensions $p_\beta : \mathcal{D}(p_\beta) \to \mathcal{L}^2(0, L)$ can be determined by Eq. (2.9) and $\mathcal{D}(p_\beta)$ will be the set of all functions $\phi_\beta \in \mathcal{L}^2(0, L)$ of the form

$$\phi_\beta(x) \equiv \phi_0(x) + \phi_+(x) + U_\beta\phi_+(x) = \phi_0(x) + \phi_+(x) + e^{i\beta}\phi_-(x), \quad \phi_0 \in \mathcal{D}(p). \tag{2.15}$$

At the boundaries 0 and L of the domain, the functions ϕ_β behave as follows

$$\phi_\beta(0) = \cancelto{0}{\phi_0(0)} + \frac{\sqrt{2\lambda}}{\sqrt{(1 - e^{-2\lambda L})}} + e^{i\beta}\frac{\sqrt{2\lambda}}{\sqrt{(e^{2\lambda L} - 1)}} = \frac{\sqrt{2\lambda}e^{-\lambda L}(e^{\lambda L} + e^{i\beta})}{\sqrt{(1 - e^{-2\lambda L})}}, \tag{2.16}$$

$$\phi_\beta(L) = \cancelto{0}{\phi_0(L)} + \frac{\sqrt{2\lambda}e^{-\lambda L}}{\sqrt{(1 - e^{-2\lambda L})}} + e^{i\beta}\frac{\sqrt{2\lambda}e^{\lambda L}}{\sqrt{(e^{2\lambda L} - 1)}} = \frac{\sqrt{2\lambda}e^{-\lambda L + i\beta}(e^{-i\beta} + e^{\lambda L})}{\sqrt{(1 - e^{-2\lambda L})}}; \tag{2.17}$$

from what one can readily verify that

$$\phi_\beta(L) = e^{i\beta}\frac{(e^{-i\beta} + e^{\lambda L})}{(e^{\lambda L} + e^{i\beta})}\phi_\beta(0) \equiv e^{i\alpha}\phi_\beta(0). \tag{2.18}$$

Thus, the extensions $p_\alpha : \mathcal{D}(p_\alpha) \to \mathcal{L}^2(0, L)$ are defined on[3]

$$\mathcal{D}(p_\alpha) := \{\psi \in \mathcal{L}^2(0, L) | \psi(L) = e^{i\alpha}\psi(0), \ \alpha \in [0, 2\pi]\}.$$

Notice that $\alpha = 0$ or $\alpha = 2\pi$ recover the usual periodic condition.

Recalling definitions 3 and 5, consider $\psi \in \mathcal{D}(p_\alpha^\dagger)$, then there is $\psi^\dagger \in \mathcal{H}$ such that:

$$\langle \psi, p_\alpha\phi \rangle = \langle \psi^\dagger, \phi \rangle, \quad \forall\phi \in \mathcal{D}(p_\alpha), \tag{2.19}$$

[3] We exchanged the index β with α, but it is worth pointing out that $\alpha \equiv \alpha(\beta)$, as defined by Eq. (2.18)

with $\psi^\dagger = p_\alpha^\dagger \psi$. In the integral form, the equation above becomes

$$\int_0^L dx\ \psi^* \left(-i\frac{d\phi}{dx}\right) = \int_0^L dx\ (p_\alpha^\dagger \psi)^* \phi. \tag{2.20}$$

Conversely, we must obtain the same result through integration by parts of the left-hand side of the equation above, as follows

$$\int_0^L dx\ \psi^* \left(-i\frac{d\phi}{dx}\right) = -i[\psi^* \phi]_0^L + \int_0^L dx\ (-i\frac{d\phi}{dx}\psi)^* \phi = \int_0^L dx\ (p_\alpha^\dagger \psi)^* \phi. \tag{2.21}$$

We must then impose $\psi^*(L)\phi(L) - \psi^*(0)\phi(0) = 0$, which, considering the boundary condition of $\mathcal{D}(p_\alpha)$ for ϕ, reduces to $\psi^*(L)e^{i\alpha}\phi(0) - \psi^*(0)\phi(0) = 0$. Hence, $\psi(L) = e^{i\alpha}\psi(0)$, which implies that $\psi \in \mathcal{D}(p_\alpha)$. Indeed, $\mathcal{D}(p_\alpha^\dagger) = \mathcal{D}(p_\alpha)$ and p_α is self-adjoint.

In this example, we were able to see the close relationship between the extended domain of the operator and the imposition of boundary conditions at the boundaries of the interval. It is worth pointing out that the standard approach in QM textbooks is to consider periodic, $\alpha = 0$, vanishing boundary conditions for the wave function at the walls of the box. Naturally, such choice influences the eigenfunctions of the momentum operator and, consequently, the expansion of wave functions in terms of them (for a complete discussion, see Ref. [17]).

2.3.2 Laplace operator in three-dimensional Euclidean space with a point removed

Let us consider the three-dimensional Euclidean space with a point removed, $\mathbb{R}^3 - \{0\}$, provided with metric tensor in the usual spherical coordinates, (r, θ, φ), given by[4]

$$h_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta). \tag{2.22}$$

The Laplacian operator ∇^2 is an elliptic operator of the form (2.6) with h_{ij} given by Eq. (2.22), and its explicit form is

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2}. \tag{2.23}$$

where $\phi \in \mathcal{L}^2(\mathbb{R}^3 - \{0\}, r^2 dr d\Omega_2)$.

The last two terms in Eq. (2.23) may be identified as the usual squared angular momentum operator of QM, $\mathbf{L}^2/r^2$. The elliptic operator $\mathbf{L}^2$ corresponds to the Laplace-Beltrami operator on a unit 2-sphere and its eigenfunctions are the Spherical Harmonic functions, $Y_l^m(\theta, \varphi)$, with eigenvalues $l(l+1)$. We may then separate the functions ϕ as

[4] We chose to center the origin of the spherical coordinates at the removed point, $r = 0$.

$\phi(r,\theta,\varphi) = R(r)Y_l^m(\theta,\varphi)$, and reduce the eigenvalue equation $\nabla^2\phi = -k^2\phi$ to its radial component, as follows

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{l(l+1)}{r^2}R(r) = -k^2R(r). \tag{2.24}$$

The left-hand side of the equation above is an elliptic operator defined on $\mathcal{L}^2(\mathbb{R}_{++}, r^2dr)$. However, it will be more convenient to work with an operator defined on the real semi-axis with measure dr, for what we consider the transformation $R(r) \to R(r) = G(r)/r$, which reduces the equation above to

$$-A_lG \equiv \frac{d^2G(r)}{dr^2} - \frac{l(l+1)}{r^2}G(r) = -k^2G(r). \tag{2.25}$$

We reduced the problem of an elliptic operator ∇^2 defined on $\mathcal{L}^2(\mathbb{R}^3 - \{0\}, r^2drd\Omega_2)$ to another elliptic operator A_l but defined on $\mathcal{L}^2((0,\infty), dr)$ for each angular momentum mode, labeled by $l \in \mathbb{N}_0$. Let $A_l : \mathcal{D}(A_l) \to \mathcal{L}^2((0,\infty), dr)$ be defined over the domain $\mathcal{D}(A_l) = \{G \in C_0^\infty(0,\infty) | G(0) = 0\}$. In these conditions, the operators A_l are densely defined on $\mathcal{L}^2((0,\infty), dr)$, closed and unbounded. We can readily check that A_l are symmetric on the domain $\mathcal{D}(A_l)$, indeed, for $G_1, G_2 \in \mathcal{D}(A_l)$, it follows that

$$\begin{aligned}\langle G_1, A_lG_2\rangle &= \int_0^\infty dr\ G_1^*\left(-\frac{d^2G_2}{dr^2} + \frac{l(l+1)}{r^2}G_2\right) \\ &= -\cancelto{0}{\left[G_1^*\frac{dG_2}{dr}\right]_0^\infty} + \cancelto{0}{\left[\frac{dG_1^*}{dr}G_2\right]_0^\infty} + \int_0^\infty dr\left(-\frac{d^2G_1}{dr^2} + \frac{l(l+1)}{r^2}G_1\right)^* G_2 \\ &= \langle A_lG_1, G_2\rangle,\end{aligned} \tag{2.26}$$

where the boundary terms vanish because of the conditions on G_1 and G_2 since they belong to $\mathcal{D}(A_l)$. With little effort, one an check that the operator is positive as well. Thus, it admits at least one positive self-adjoint extension, which could be the operator itself, in case it is essentially self-adjoint.

Now, we follow the procedure of Sec. 2.2 to find out whether the operators A_l are self-ajoint or not. According to theorem 2 we must find the deficiency subspaces, i.e., $\ker(A^\dagger \mp 2i)$ (taking $\lambda = 2$ with no loss of generality), then it follows that we need solutions to the equation

$$A_l^\dagger\Psi_{l\pm} = -\frac{d^2\Psi_{l\pm}}{dr^2} + \frac{l(l+1)}{r^2}\Psi_{l\pm} = \pm 2i\Psi_{l\pm}, \tag{2.27}$$

which are given by

$$\Psi_{l\pm}(r) = a_{l\pm}\sqrt{r}H^{(1)}_{l+\frac{1}{2}}\big((\mp 1 - i)r\big) + b_{l\pm}\sqrt{r}H^{(2)}_{l+\frac{1}{2}}\big((\mp 1 - i)r\big), \tag{2.28}$$

where $H^{(1)}$ and $H^{(2)}$ are Hankel functions of the first and second kinds, respectively. One can readily verify that, requiring $\Psi_{l\pm} \in \mathcal{L}^2(0,\infty)$, we need $a_{l\pm} \equiv 0$, for all l, since $H^{(1)}$ diverges at the origin and at infinity. Even though $H^{(2)}$ vanishes rapidly enough at infinity, it diverges at the origin for all $l > 0$, hence $b_{l\pm} \equiv 0$, for $l > 0$. Thus, the only non-vanishing coefficients are $b_{0\pm}$. These results indicate that, for $l > 0$, Eq. (2.27) has no square-integrable solutions on the real semi-axis, hence $\mathcal{N}_{l\pm} = \varnothing$ and $n_\pm = 0$. Thus, theorem 2 implies that A_l is essentially self-adjoint for all $l > 0$. Conversely, for $l = 0$, $\mathcal{N}_{0\pm}$ are spanned by

$$\Psi_{0\pm}(r) = \sqrt{r} H^{(2)}_{1/2}\big((\mp 1 - i)r\big) \propto e^{(-1\pm i)r}, \tag{2.29}$$

so the deficiency indices are $(n_+, n_-) = (1,1)$ and, by theorem 2, there are infinitely many self-adjoint extensions of A_0.

Still according to theorem 2, the extensions of A_0 are parametrized by $U(1)$ isometries between $\mathcal{N}_+ \to \mathcal{N}_-$. Let U be such an isometry, then $U\Psi_{0+} = e^{i\theta}\Psi_{0-}$, for $\theta \in [0, 2\pi]$, and the most generic form of the domain of the self-adjoint extensions $A_{0\theta}$ is

$$\mathcal{D}(A_{0\theta}) = \{\Psi_\theta(r) = \Psi_0(r) + \Psi_+(r) + e^{i\theta}\Psi_-(r) | \Psi_0 \in \mathcal{D}(A_0),\ \Psi_\pm \in \mathcal{N}_{0\pm}; \theta \in [0, 2\pi]\}. \tag{2.30}$$

Like we did for the momentum operator in the box, we can infer a boundary condition for $l = 0$ modes of the Laplacian operator in order to make it self-adjoint, as follows[5]

$$\begin{aligned}\frac{\Psi'_\theta(0)}{\Psi_\theta(0)} &= \beta_0 \frac{-(1-i) - e^{i\theta}(1+i)}{1 + e^{i\theta}} \\ &= -\beta_0 \frac{(1-i)e^{-i\theta/2} + (1+i)e^{i\theta/2}}{(e^{-i\theta/2} + e^{i\theta/2})} \\ &= -\beta_0\left(1 + \tan(\theta/2)\right) := -\frac{\beta_0}{\beta}.\end{aligned} \tag{2.31}$$

Finally, the domain of the self-adjoint extension $A_{0\beta}$ is

$$\mathcal{D}(A_{0\beta}) = \{G_0 \in C_0^\infty(0, \infty)|\ \beta_0 G_0(0) + \beta G_0'(0),\ \beta \in \mathbb{R}\}. \tag{2.32}$$

These are known as Robin (or mixed) boundary conditions.

In this last example, we saw that the spherical decomposition of the eigenfunctions of the Laplace operator leads to the appearance of other elliptic differential operators A_l. They exclusively act on the radial components of each l-mode, $G_l(r)$. For all non-spherically symmetric modes, such operators were essentially self-adjoint on the original domain. We can interpret this result by noting that the term $l(l+1)/r^2$ in A_l acts as a potential barrier around the origin, forcing all modes with $l > 0$ to vanish at $r = 0$, just as the original domain $\mathcal{D}(A_l)$ prescribes. On the other hand, the spherically symmetric mode $l = 0$ 'feels' the origin and might behave in not-so-obvious ways at that point. This situation translates into having an operator A_0 that is not self-adjoint but whose self-adjoint extensions introduce a set of boundary conditions at the origin. These conditions dictate a more general behavior of the function around $r = 0$.

5 Here, we introduce a dimensional factor β_0 that balances the dimension of the derivative, hence it has dimension of energy.

Finally, having seen both examples, it should be clear that the process of finding self-adjoint extensions to differential operators is closely related to the prescription of boundary conditions at adequate regions of space. The relevance of this discussion to our work will become more evident in the next chapter. We will see that self-adjointness plays a central role in the dynamics of quantum fields. Naturally, the boundary conditions emerging from self-adjoint extensions of the relevant operators shall influence their eigenfunctions directly.

3 Scalar fields in non-globally hyperbolic spacetimes I: a prescription for dynamics

In Chapter 1, we presented a brief review on Quantum Field Theory in curved spacetimes for free scalar fields. Without much reasoning, we assumed that our spacetime $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic. It turns out that such an assumption is rather crucial in the development of the theory. Indeed, this causal property of the spacetime guarantees that hyperbolic partial differential equations - such as the wave equation - yield unique solutions to the Cauchy problem anywhere and at any time of the spacetime. However, an extensive collection of spacetimes does not satisfy global hyperbolicity, which jeopardizes the quantization procedure in them. Then, how does one develop a QFT in the absence of global hyperbolicity? By far, this is not a simple question. Nonetheless, Wald proposes that one might be able to do so by following a set of reasonable assumptions, that we shall present later in this Chapter. We begin by following Ref. [23] to give a brief overview of the definition of global hyperbolicity in static spacetimes[1]. Next, we will discuss how to quantize scalar fields in non-globally hyperbolic spacetimes.

3.1 Non-global hyperbolicity

Let us consider a *static* spacetime $(\mathcal{M}, g)$. A curve in $\mathcal{M}$ whose tangent vector is either null or timelike and always future-directed is denominated a *future-directed causal curve*, and a similar definition follows for past-directed causal curves. For an event $p \in \mathcal{M}$, we define its *causal future*, $\mathbf{J}^+(p)$, as the set of all events that can be reached by *future-directed causal curves*. The *causal past* of p, $\mathbf{J}^-(p)$, is defined analogously.

Now consider a spacelike hypersurface H corresponding to a static slice of $\mathcal{M}$. We can define the causal future and past of the entire hypersurface as follows:

$$\mathbf{J}^{\pm}(H) := \bigcup_{p\in H} \mathbf{J}^{\pm}(p). \tag{3.1}$$

We provide a graphical concept of $\mathbf{J}^+(H)$ and $\mathbf{J}^-(H)$ in Fig. 1 as the entire shaded regions bounded by inclined lines (light cones) above and below the surface H, respectively. It is worth pointing out that not all events in $\mathbf{J}^{\pm}(H)$ will be completely causally related to H. For instance, there might be past-directed causal curves culminating with an event $q \in \mathbf{J}^+(H)$ that does not intersect the hypersurface H, i.e., $\mathbf{J}^-(q)$ is not entirely enclosed by $\mathbf{J}^+(H) \cup \mathbf{J}^-(H)$.

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[1] Here, we chose to use a discursive approach in order to use as fewer definitions as possible, like in [1]. For a complete and formal description, see Chap. 8 in Ref. [23]

Figure 1 – Causal future (on the left) and past (on the right) of the spacelike hypersurface H.

We can restrict our definitions in order to define a causal region *entirely* related to H, which will be regarded as *domain of dependence* of H. For an event $q^+ \in \mathbf{J}^+(H)$, this means that, if *all* past causal curves emanating from q^+ intersect the hypersurface H, then q is inside the *future domain of dependence* of H, denoted $\mathbf{D}^+(H)$. Thus, $\mathbf{D}^+(H)$ is a set of all events whose causal pasts are necessarily affected by H. On the other hand, the *past domain of dependence* of H, denoted $\mathbf{D}^-(H)$, is formed by events $q^- \in \mathbf{J}^-(H)$ whose future causal curves all intersect the hypersurface H. Accordingly, $\mathbf{D}^-(H)$ is the set of all events whose causal futures will necessarily affect H. Naturally, the *full domain of dependence* of H, $\mathbf{D}(H)$, is obtained joining past and future domains, i.e.,

$$\mathbf{D}(H) = \mathbf{D}^+(H) \cup \mathbf{D}^-(H). \tag{3.2}$$

All events taking place in $\mathbf{D}(H)$ will somehow affect or be affected by the hypersurface. In Fig. 2, we can see $\mathbf{D}^+(H)$ and $\mathbf{D}^-(H)$ as dark-shaded regions above and bellow H, respectively, and their union, $\mathbf{D}(H)$, is the dark-shaded, diamond-shaped region. It is to notice that $H \subset \mathbf{D}^{\pm}(H) \subset \mathbf{J}^{\pm}(H)$ hence $H \subset \mathbf{D}(H) \subset \mathbf{J}(H)$.

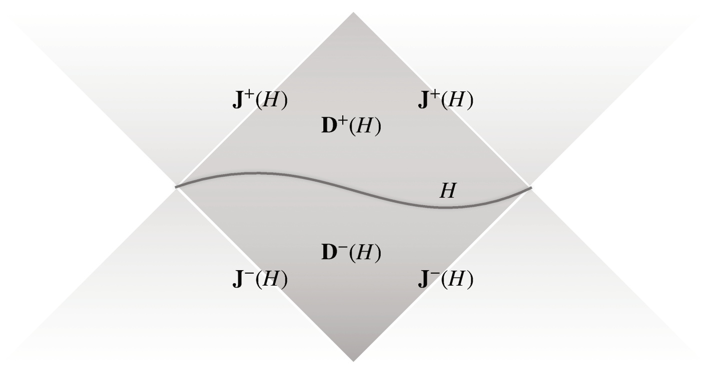

Figure 2 – Causal regions of spacetime related to the spacelike hypersurface H.[1]

The spacelike hypersurface H will be called a *Cauchy surface* if $\mathbf{D}(H) = \mathcal{M}$, i.e., all events in the spacetime manifold $\mathcal{M}$ are causally related to H. It follows readily that the entire history of the spacetime can be determined by specifying its conditions at some moment in time. For instance, equations dictating the dynamics of fields propagating throughout the manifold can be solved deterministically, given appropriate initial conditions. Such equations are predominantly hyperbolic and, for that, if a spacetime admits a Cauchy surface, then it is called *globally hyperbolic*.

A notable example of a globally hyperbolic spacetime is Minkowski spacetime, represented on the left of Fig. 3 by its Penrose diagram (see Sec. 3.1 of Ref. [5]). For any dimension higher than $(1+1)$, the left and right sides of the diagram are redundant due to spherical symmetry. It is straightforward to notice that all spacelike hypersurfaces (represented by dashed horizontal curves) are Cauchy surfaces. Indeed, we observe that the domain of dependence of any of the dashed slices is the entire manifold by direct comparison with Fig. 2. Moreover, timelike curves, like λ in Fig. 3, can be predicted or retrodicted only by specifying their state at some static spacelike slice.

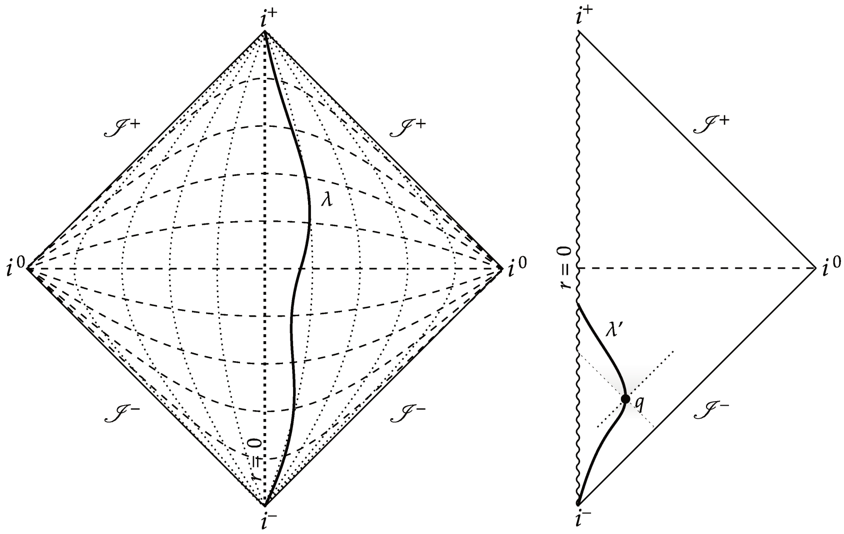

Figure 3 – Penrose (or Conformal) diagrams of Minkowski spacetime (left) and Minkowski with a point removed (right). All timelike curves start at the past timelike infinity i^- and end at the future one i^+. While, all null curves begin (end) at the past (future) null infinity $\mathscr{I}^-$ ($\mathscr{I}^+$). The spacelike infinity i^0 is an infinitely distant endpoint for spacelike curves.

If we consider now Minkowski spacetime with a point removed, the conformal diagram will be as shown on the right of Fig. 3 - the origin of spherical coordinates $r = 0$ was centered at the point removed for convenience. Consider the spacelike hypersurface Σ denoted by a dashed horizontal line in Fig. 3, and an event $q \in \mathbf{J}^-(\Sigma)$, shown in Fig. 3 with its light cone (dotted lines) attached to it. As we are not certain of what happens at the point removed, there might be causal curves passing through q that abruptly end at

$r = 0$. An example of one such curve is shown in Fig. 3 as λ'. Because of λ' and all other causal curves that might not intersect Σ, $q \notin \mathbf{D}^{-}(\Sigma)$ hence the full domain of dependence of Σ cannot include q, so it is not the entire manifold, i.e., $\mathbf{D}(\Sigma) \neq \mathcal{M}$. Thus, Σ 'would like' to be a Cauchy surface as in the previous case, but it is not one. These considerations can be extended analogously to any static slice of Minkowski, for any point q in the causal past or future of the hypersurface. The conclusion will always be the same: no spacelike hypersurface of the spacetime can be a Cauchy surface. Therefore, Minkowski with a point removed is said to be a *non-globally hyperbolic spacetime.*

Let us keep our discussion for a bit longer on Minkowski spacetime with a point removed. We just saw that global hyperbolicity is absent; hence the Cauchy problem is not well-posed. According to our previous discussion, one would find impracticable to obtain a complete set of eigenfunctions of the wave equation and build the field operator from them. We understand this impossibility as a consequence of information leaving or entering the spacetime at any time of its history through the removed point. Nevertheless, It is to notice that, if the behavior of the field was specified at the removed point, one could find a complete set of positive-energy modes and proceed with the quantization of the field. It is needless to say, however, that the spacetime would still be non-globally hyperbolic and there could be infinitely-many possible behaviors of the field at the removed point. In an attempt to shed light on this issue, we will present in the next section a possible solution to it.

3.2 Klein-Gordon equation in non-globally hyperbolic spacetimes

It is well known that if the spacetime is globally hyperbolic, there will be a unique solution $\phi \in C_0^{\infty}(\mathcal{M})$ to the Klein-Gordon equation (1.2) for each initial values $(\phi_0, \dot{\phi}_0) \in C_0^{\infty}(\Sigma) \times C_0^{\infty}(\Sigma)$ prescribed on a Cauchy surface Σ. The solution ϕ is such that $\phi|_{\Sigma} = \phi_0$ and $\tau^{\mu}\nabla_{\mu}\phi|_{\Sigma} = \dot{\phi}_0$. Conversely, as discussed in the last section, in non-globally hyperbolic spacetimes, Cauchy surfaces shall not exist hence the dynamical prescription explained above must not be valid. Let us now focus on determining an appropriate dynamical evolution in such causally pathological spacetimes.

Since we have been considering static spacetimes $(\mathcal{M}, g_{\mu\nu})$, their metric tensor admit the following decomposition[24]

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -V^2dt^2 + h_{ij}dx^i dx^j. \tag{3.3}$$

In Eq. (3.3), for a given timelike Killing field τ^{μ} of the metric, we define $V = \sqrt{-\tau^{\mu}\tau_{\mu}}$, and h_{ij} is the metric induced on a hypersurface Σ orthogonal to τ^{μ}. In this particular case, the Laplace-Beltrami operator (1.3) reduces to

$$\Box\phi = \nabla_\mu \nabla^\mu \phi = -\frac{1}{V^2}\partial_t^2\phi + \frac{1}{V}D^i(VD_i\phi), \tag{3.4}$$

in which D_i is the covariant derivative operator on a hypersurface Σ. Thus, we can write Klein-Gordon equation (1.2) as follows

$$\partial_t^2\phi = -A\phi, \tag{3.5}$$

in which

$$A := -Vh^{ij}D_j(VD_i\phi) + m^2V^2 + \xi \mathrm{R} V^2 \tag{3.6}$$

is the spatial component of the wave operator.

We begin noting that A is a second-order, linear elliptic operator of the form (2.6) plus a function $c(x) = m^2V^2 + \xi \mathrm{R}V^2$. It is densely defined on a Hilbert space $\mathcal{H} = \mathcal{L}^2(\Sigma, \mu)$, where μ is the measure in Σ, with domain $\mathcal{D}(A) = C_0^\infty(\Sigma)$. Now, let A be a positive, symmetric operator on $\mathcal{H}$. Thus, as shown in Chap. 2, there exists at least one positive self-adjoint extension of A (Friedrichs extension), and there might be infinitely many according to Theorem 2.

Let A_E be one of the self-adjoint extensions of A, then given well-posed initial conditions to the Cauchy problem - i.e., $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$, for all $t \in \mathbb{R}$ - we define

$$\phi_t = \cos(A_E^{1/2}t)\phi_0 + A_E^{-1/2}\sin(A_E^{1/2}t)\dot{\phi}_0, \tag{3.7}$$

where the operators $\cos(A_E^{1/2}t)$ and $A_E^{-1/2}\sin(A_E^{1/2}t)$ are defined via the spectral theorem (see Ref. [16]). In [25], Wald shows that ϕ_t exists for all t and there exists a unique solution $\phi \in C_0^\infty(\mathcal{M})$ to Eq. (1.2) over the entire $\mathcal{M}$, such that $\phi|_{\Sigma_t} = \phi_t$ and $\tau^\mu \nabla_\mu \phi|_{\Sigma_t} = \dot{\phi}_t$, for all t.

Additionally, Ishibashi and Wald, in [26], demonstrate that Eq. (3.7) is the only one that prescribes the dynamics of scalar fields in non-globally hyperbolic static spacetimes in a physically sensible way. By comparison with the globally hyperbolic case, they establish a set of conditions that determine whether a time evolution prescribed by the before-mentioned ϕ is acceptable or not, namely[2]

1. solutions of the form ϕ must be causally compatible with initial data, i.e., the support of ϕ lies within the causal future and causal past of the supports of $(\phi_0, \dot{\phi}_0)$;

2. the prescription for dynamics must be invariant under time translation and time reflection, i.e., if the initial data $(\phi_0, \dot{\phi}_0)$ is translated or reflected in time, then its corresponding solution ϕ should undergo the same transformations;

[2] See Ref. [26] for detailed proofs and definitions of the conditions listed here.

3. there exists a conserved energy also respecting time translation and reflection invariance. It must also be in agreement with the globally hyperbolic case, so it shall be given by an inner product $\mathcal{E}(\cdot,\cdot)$ defined as

$$\mathcal{E}(\psi,\phi) = \langle\dot{\psi}_0,\dot{\phi}_0\rangle_{\mathcal{L}^2} + \langle\psi_0, A\phi_0\rangle_{\mathcal{L}^2}\,, \tag{3.8}$$

where ϕ is a solution with initial data $(\phi_0,\dot{\phi}_0)$ in $C_0^\infty(\Sigma)\times C_0^\infty(\Sigma)$. While, ψ is a general solution of the Klein-Gordon equation that can be expressed as a finite linear combination of time translations of functions of the form ϕ.;

4. solutions satisfy a convergence condition. Such a condition requires that a sequence ψ_n converges in the norm given by $\mathcal{E}$ to a solution ψ of the same form as in condition 3, i.e.,

$$\lim_{n\to\infty}\mathcal{E}(\psi_n-\psi,\psi_n-\psi). \tag{3.9}$$

According to Ishibashi and Wald, the conditions listed above guarantee that their prescription for dynamics associated with Eq. (3.7) is the only physically sensible one. However, they point out that for each extension A_E there will be an associated dynamical evolution of Eq. (3.7). Although initial conditions will uniquely define the Cauchy evolution, the solutions will not all have the same outcomes. Indeed, we recall our discussion in Chapter 2 and notice that self-adjoint extensions are closely related to specific boundary conditions. We identify those non-equivalent solutions as outcomes of various boundary conditions that one can impose at a region in space, such as a singularity or a boundary[25]. We now have all the tools necessary to study the dynamics of scalar fields in non-globally hyperbolic spacetimes, as we will show in Chapter 4.

4 Scalar fields in non-globally hyperbolic spacetimes II: applications

In this Chapter, we aim to employ the prescription for dynamics developed in Chapter 3 to three cases of non-globally hyperbolic spacetimes. As discussed previously, we will require the self-adjointness of the spatial component A of the Klein-Gordon equation and, when necessary, will find its self-adjoint extensions. The latter process will provide us with a set of boundary conditions that appropriately extend the domain of A. We will examine the effects that these conditions will carry to physically relevant quantities, especially expectation values of field dependent objects.

In order to proceed with our applications, we shall establish a systematic prescription for studying scalar fields in non-globally hyperbolic spacetimes, as follows:

1. Given a metric tensor, write down the Klein-Gordon equation (1.2) in the appropriate coordinate system and reduce it to the form

$$\partial_t^2\phi = -A\phi.$$

2. Check whether A is symmetric and positive. If so, employ Theorem 2 to determine the self-adjointness of A.
3. If A is not essentially self-adjoint, find its self-adjoint extensions according to Theorem 3 and the suitable boundary conditions.
4. Given initial values, solve the wave equation to find a unique dynamical evolution to each boundary condition, i.e., to each self-adjoint extension of A.
5. Expand the scalar field in terms of a complete set of eigenfunctions obtained in step 3.
6. Calculate all relevant quantities, such as scattering cross section, Green's functions and other expectation values.

We shall apply the prescription above to two types of static, non-globally hyperbolic spacetimes, namely solutions of Einstein equation with naked singularity and with conformal boundary. The former includes two of our examples: the $(2+1)$-dimensional cone and the Global Monopole. While, the latter includes the anti-de Sitter spacetime.

4.1 $(2+1)$-dimensional Cone

Three dimensional gravity is known to always be locally flat in the absence of energy and matter. In fact, Eintein equations (1.5) may be rewritten in the form

$$\mathrm{R}_{\mu\nu} = \kappa[T_{\mu\nu} - g_{\mu\nu}T^{\lambda}{}_{\lambda}], \tag{4.1}$$

and the Riemann curvature tensor may be decomposed as follows[27]

$$\mathrm{R}_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}\mathrm{R}_{\beta\delta} - g_{\alpha\delta}\mathrm{R}_{\beta\gamma} - g_{\beta\gamma}\mathrm{R}_{\alpha\delta} + g_{\beta\delta}\mathrm{R}_{\alpha\gamma} - \frac{1}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)\mathrm{R}. \tag{4.2}$$

From both equations it is straightforward to notice that $T_{\mu\nu} = 0$ (absence of energy and matter) implies $\mathrm{R}_{\mu\nu} = 0$ and $\mathrm{R}_{\alpha\beta\gamma\delta} = 0$, i.e., the spacetime is locally flat.

Now, if we consider a point-like source of mass M placed at the origin of the coordinate system, the energy-momentum tensor would then be

$$T_{\mu\nu} = \mathrm{diag}(M\delta^2(\vec{r}), 0, 0). \tag{4.3}$$

We can plug this energy-matter distribution into Einstein equations and find the resulting metric tensor, which will still be locally flat since is a localized source. However, the resulting spacetime will have non-trivial topology and its line element will be[28]

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2, \tag{4.4}$$

with $0 < r < \infty$, $0 \leqslant \theta \leqslant 2\pi$ and $0 < \alpha < 1$. The mass of the souce and the parameter α are related by $M = 2\pi(1-\alpha)/\kappa$.

A static slice (t = const.) returns a line element dl^2 that can be embedded into 3-dimensional Euclidean space upon the following relation

$$dl^2 = dr^2 + \alpha^2 r^2 d\theta^2 = dR^2 + R^2 d\Theta^2 + dZ^2, \tag{4.5}$$

where (R, Θ, Z) are the usual cylindrical coordinates of $\mathbb{R}^3$. We may identify $R \equiv \alpha r$ and $\Theta = \theta$ to eliminate angular dependence, which reduces the equation to

$$dZ^2 = \left(\frac{1}{\alpha^2} - 1\right) dR^2, \tag{4.6}$$

from where we find

$$Z(R) = \sqrt{\frac{1-\alpha^2}{\alpha^2}}R + Z_0. \tag{4.7}$$

This equation parametrizes a 2-dimensional cone in $\mathbb{R}^3$ (see Fig. 4 for schematic representation) with its vertex located at $R = 0$ and $Z = Z_0$, and total angular opening $\varphi = 2\sin^{-1}\alpha$. The angular deficit of the cone will depend on the mass of the source through α, and it is given by $\Delta = 2\pi(1-\alpha)$, as shown on the right of Fig. 4.

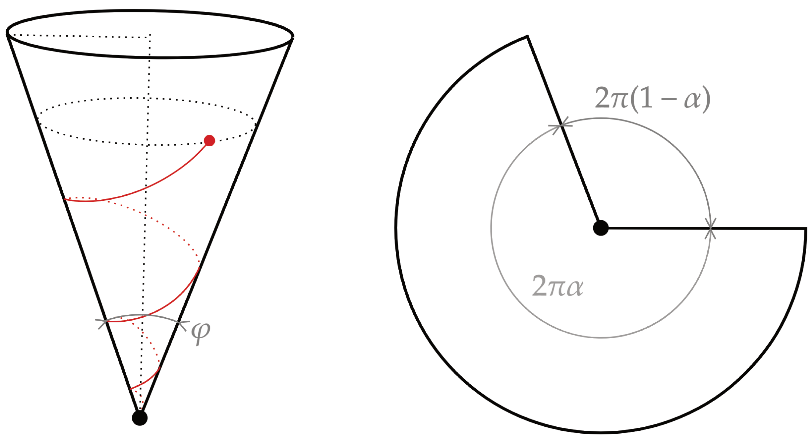

Figure 4 – Left: drawing of a 2-dimensional cone. The red line indicates what would be the path of a point-like classical particle spiralling towards the tip of the cone. Right: unwrapped cone.

The spatial component h_{ij} of the metric tensor (4.4) can be written in Cartesian coordinates (x^1, x^2) as follows

$$h_{ij} = \delta_{ij} + \frac{1-\alpha^2}{\alpha^2[(x^1)^2 + (x^2)^2]} x^i x^j, \tag{4.8}$$

which reveals the singular nature of the vertex ($r = 0$ or $x^1 = x^2 = 0$). This singularity of the metric is not of the curvature type; instead, the tip of the cone is regarded as a conical singularity, which appears as well in Cosmic Strings[29],

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2 + dz^2, \tag{4.9}$$

whose surfaces $z =$ const. are all $(2+1)$-dimensional cones.

Recalling our discussion from Chapter 3, in this case, no static slice of the spacetime, i.e., a 2-dimensional cone, can be a Cauchy surface. Indeed, there is nothing preventing a particle from suddenly disappearing at or emerging from the vertex of the cone. Accordingly, the $(2+1)$-dimensional cone is not deterministic for quantum fields dynamics; hence it is non-globally hyperbolic. For its technical simplicity, we chose this spacetime to perform our first analysis of the dynamics of scalar fields. The entire procedure and results can be found in Annex A, in which our article [30] examining these topics is attached.

In [30], we considered a massive scalar field propagating on the cone. Given the non-globally hyperbolicity of the spacetime, we considered Wald's and Ishibashi's prescription for dynamics, as described in Chap. 3. After the usual separation of variables, we were able to find the spatial component of the Klein-Gordon equation, denoted $-\Delta_m$ for each angular mode m, and examine its self-adjointness. For that, we followed Kay and Studer, whose work in [31] was to obtain the self-adjoint extensions of $-\Delta_m$. They concluded that the operator is essentially self-adjoint for all $m \neq 0$ in the domain $C_0^\infty(\mathbb{R}_{++})$. On the other hand, for $m = 0$, $-\Delta_0$ has infinitely many self-adjoint extensions, which are associated with a set of boundary conditions prescribed at the singularity $r = 0$ (see Eqs. (11) and (12) in Annex A[30]). This case is analogous to that described in Sec. 2.3.2, in which the Laplace operator in a space with a point removed was essentially self-adjoint for all modes except the spherically symmetric one ($l = 0$). Here, modes with $m > 0$ 'feel' a potential barrier around the conical singularity that shields them. Conversely, that barrier is absent for the $m = 0$ mode; hence the behavior of the wave function must be specified at $r = 0$.

Our procedure continued with solving the Klein-Gordon equation considering the appropriate boundary conditions for the circular mode ($m = 0$) and obtaining its normal modes. We then extended to our case the work of Deser and Jackiw in [32], in which they study classical and quantum scattering in the conical spacetime. Similarly, we took the positive-energy modes of the KG equation and scattered them off the cone but looking for the effects of the boundary conditions on the process. Our results in Sec. IV of [30] (p. 3 of Annex A) for the scattering amplitude show an analytic contribution to that of Deser and Jackiw, as shown in Fig. 2 of [30] (p. 5 of Annex A). This contribution depends on the boundary condition parameter and the remaing components of the scattering amplitude are exclusively related to topological properties of the conical spacetime.

4.2 Global Monopoles

Global Monopoles are topological defects that would have appeared as results of phase transitions in the early universe, and would have arose from global symmetry breaking[29]. We consider the simplest model that yields global monopoles given by the Lagrangian[33]

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - \eta_0^2)^2, \tag{4.10}$$

where ϕ^a, for $a = 1, 2, 3$. The global $O(3)$ symmetry of the Lagrangian is spontaneously broken to $U(1)$. In these conditions, monopoles appear when the field assumes a hedgehog configuration, i.e.,

$$\phi^a = \eta_0 f(r)\frac{x^a}{r}, \tag{4.11}$$

for $x^a x^a = r^2$. The function $f(r)$ is such that it vanishes as $r \to 0$ and tends to one for $r \gg r_\text{c}$, where $r_\text{c} \sim (\sqrt{\lambda}\eta_0)^{-1}$ is the size of the monopole's core.

We can compute the energy-momentum distribution of the field outside the monopole's core using the tensor in Eq. (1.4), which reduces to

$$T_{\mu\nu} = \partial_\mu\phi^a\partial_\nu\phi^a - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^a, \tag{4.12}$$

where $\eta_{\mu\nu}$ is Minkowski metric written in spherical coordinates (r, θ, φ). The non-vanishing components are

$$T_{tt} = -T_{rr} = \frac{\eta_0^2}{r^2}, \tag{4.13}$$

which can be plugged into Einstein equations assuming a spherically symmetric metric tensors that is given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4.14}$$

where A and B are functions to be found. Solving the field equations one finds the Schwarzschild-like coefficients

$$B(r) = A^{-1}(r) = 1 - \eta^2 - \frac{2GM_\mathrm{c}}{r}, \tag{4.15}$$

where $\eta \equiv \sqrt{8\pi G}\eta_0$ and $M_\mathrm{c} \approx -6\pi\eta_0\lambda^{-1/2}$ [34].

Since the mass of the monopole's core M_c is negative, the metric tensor does not allow for a Schwarzschild black hole to exist. Vilenkin and Shellard argue in [29] that the mass term is completely negligible in astrophysical scales, and the metric of the global monopole far from its core can be approximated to

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2}dr^2 + r^2 d\Omega_2, \tag{4.16}$$

where $\alpha \equiv 1 - \eta^2$ and $0 < \alpha < 1$. In these coordinates, the contraction of the Ricci tensor R (scalar curvature) and of the Riemann tensor $\mathrm{R}_{\alpha\beta\gamma\delta}\mathrm{R}^{\alpha\beta\gamma\delta}$ (Kretschmann scalar) are

$$\mathrm{R} = 2\frac{\eta^2}{r^2}, \tag{4.17}$$

$$\mathrm{R}_{\alpha\beta\gamma\delta}\mathrm{R}^{\alpha\beta\gamma\delta} = 4\frac{\eta^4}{r^4}, \tag{4.18}$$

respectively. Clearly, both quantities diverge at $r = 0$, which indicates a true curvature singularity at $r = 0$. Indeed, the global monopole has a *naked singularity* lying inside its massive core.

We can also rescale t and r to write the metric as follows

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{4.19}$$

This form of the metric shows that a static hypersurface with constant r of the line element (4.19) reveals a 2-sphere with solid angle smaller than 4π. Actually, it is readily verified that the total solid angle is $4\pi\alpha$, i.e., the global monopole spacetime has a deficit solid angle of $\delta\Omega = 4\pi(1-\alpha) = 8\pi^2 G\eta_0^2$, as shown in Fig. 5. In fact, at the equator $\theta = \pi/2$, instead of reducing to a flat $(2+1)$-dimensional geometry, the metric reduces to that of a cone, like in Eq. (4.4). Also, suppressing angular dependence in the metric tensor (4.19) shows that the Penrose diagram of the global monopole is similar to that of Minkowski with a point removed, as shown on the right of Fig. 5.

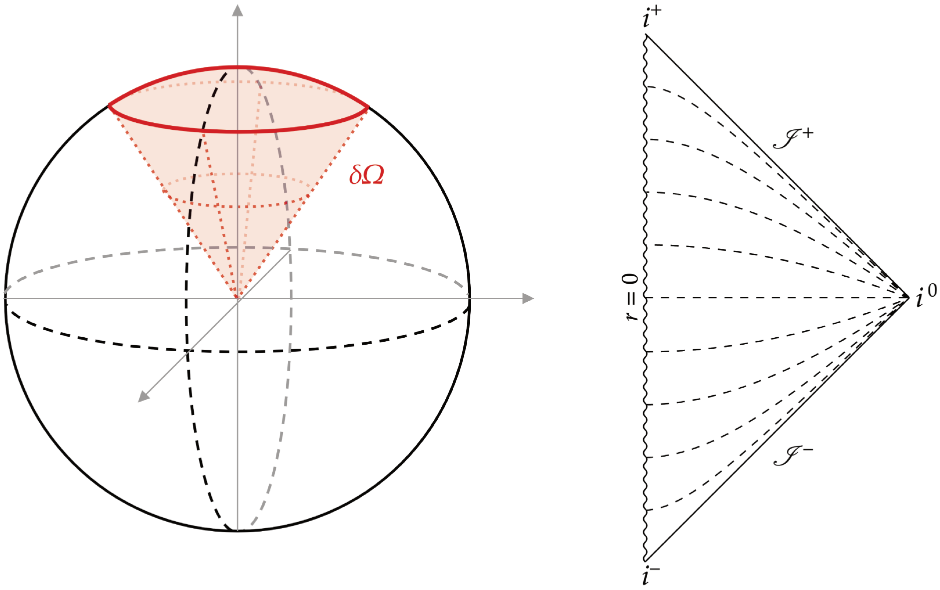

Figure 5 – Left: Scheme showing a sphere with deficit solid angle $\delta\Omega$ in light red. Right: Penrose diagram of the Global monopole. The vertical wavy line represents the naked singularity at $r = 0$. Static hypersurfaces are dashed, curved horizontal lines starting at $r = 0$ and finishing at spatial infinity i^0. These slices are all wannabe Cauchy surfaces, but the singularity prevents them from being so.

The same discussion of the causal structure of Minkowski with a point removed stands for the global monopole. Because of the naked singularity inside the monopole's core, no static slice of the spacetime can be a Cauchy surface. Thus, global monopole fails to be globally hyperbolic. Similarly to the cone, the naked singularity in the spacetime leads to the breakdown of predictability of the wave equation. News can emerge from or fall into the singularity, which alters unprecedentedly the history of a particle or field. Once again, boundary conditions at the singularity are necessary to regulate the behavior of the fields that we are treating here. We examined the physical consequences of Wald's and Ishibashi's prescription for sensible dynamics, and displayed our results in two articles, available in Annex B and Annex C.

In [35], shown in Annex B, we considered a massless, minimally coupled scalar field in the global monopole. We followed Ref. [36], in which the authors employ Wald's prescription for a dynamical evolution and find the appropriate boundary conditions at the naked singularity. As a matter of fact, in the case that we have considered in [35], the problem reduces to studying the Laplace operator in a space with a point removed as we did in the last section of Chapter 2. Let us recall that only spherically symmetric modes need boundary conditions, which are of the Robin-type (see Eq. (2.32)). All other modes satisfy Dirichlet conditions.

We then selected positive-energy modes of the KG equation and scattered them off the global monopole, as described in Sec. III of [35] (p.3 of Annex B). Our results for the scattering amplitude and the differential cross section show an intimate relationship between the scattering behavior and the boundary conditions (see Fig. 1 in [35], p.5 of Annex B). In particular, if one were to measure, for instance, the backscattering, they could determine which is nature's choice for the boundary condition of the field. Additionally, we briefly analyzed the spacetime's stability when tested by scalar fields (Sec. IV of [35], p.5 of Annex B). We concluded that, for a range of the boundary condition parameter, the spacetime is mode unstable. However, a more accurate discussion would be necessary to describe stability adequately.

Our next step in the global monopole was to examine the proper quantization of scalar fields. In [37], displayed in Annex C, we calculated the expectation values of some field-dependent quantities. This time, however, we considered massless scalar fields with an arbitrary coupling to curvature, which we have restricted later to minimal and conformal couplings. In these conditions, the Klein-Gordon operator could be reduced to a spatial operator depending on the radial component solely. This operator is of the Calogero-type[38, 39, 40], i.e.,

$$A_{\mathrm{Cal}} = -\frac{d^2}{dr^2} + \frac{a}{r^2}, \tag{4.20}$$

where a is a parameter that varies for each problem but determines the self-adjointness of A_{Cal}. The authors in Ref. [41] discuss all cases for which A_{Cal} is (or not) self-adjoint and find its eigenfunctions in each case.

Using the results from [41], we built a set of positive-energy eigenfunctions of the wave equation and proceeded with the quantization of fields according to our discussion of Chapter 1, both shown in Sec. II.B of [37] (p.3 of Annex C). For minimally (Sec. III.A of [37]) and conformally (Sec. III.B of [37]) coupled scalar fields ($\xi = 0$ and $\xi = 1/6$, respectively), we obtained analytical expressions to the proper Green's functions using Schwinger's representation (1.32) up to order η^2. As a basis of comparison, we kept the work by Mazzitelli and Lousto in Ref. [42], where they compute the quadratic fluctuations of the expectation value of the field and indicate a systematic procedure to find the stress-energy tensor. The authors adopt Dirichlet boundary conditions in all modes. Our computations for the stress-energy tensor are in Sec. IV of [37], p.6 in Annex C.

Our results showed that the interaction of the spherical mode with the naked singularity in a non-trivial way brings contributions to those of Mazzitelli and Lousto (see Eqs. (38) and (40) in [37]). In leading order of the small deficit η^2, our fluctuations are the same as those in Minkowski spacetime with a point removed. As one takes the deficit to zero ($\eta^2 = 0$), the metric of the global monopole recovers the flat Minkowski metric in spherical coordinates. So it is expected that our results do the same. We point out as well that the topological structure of the global monopole ultimately did not interfere in our computations.

4.3 Anti-de Sitter spacetime

The set of hyperbolic spaces comprises the surfaces of constant negative curvature in geometry. Examples include saddle surfaces, such as the shape of the Pringles® potato chip, or Gabriel's horn, a trumpet-shaped surface with finite volume but an infinite surface area. In gravitational settings, the n-dimensional anti-de Sitter space is an equivalent of such spacetimes. It appears as a solution to Einstein equations with a negative cosmological constant ($\Lambda < 0$) in the absence of matter and energy. Setting $\Lambda := -\dfrac{(n-1)(n-2)}{2H^2}$, the Einstein equations become[43]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{(n-1)(n-2)}{2H^2} g_{\mu\nu} = 0. \tag{4.21}$$

The outcome is an n-dimensional maximally symmetric pseudo-Riemmanian metric defined over a Lorentzian manifold with constant negative curvature, namely the adS_n spacetime.

We can understand adS_n as an isometric embedding of a single sheeted n-dimensional hyperboloid in an $(n+1)$-dimensional flat spacetime. Take the embedding space to be $\mathbb{R}^{n-1,2}$ provided with the set of coordinates $\{y^b\}$, for $b = 0, 1..., n$, and the metric

$$\eta_{ab} = \mathrm{diag}(-1, 1, ..., 1, -1). \tag{4.22}$$

We can identify adS_n as the timelike hypersurface

$$\eta_{ab} y^a y^b = -(y^0)^2 + \sum_{j=1}^{n-1} (y^j)^2 - (y^n)^2 = -H^2, \tag{4.23}$$

which reveals the hyperbolic structure of the spacetime. Suppressing all coordinates y^b, for $2 \leqslant b \leqslant n-1$, one obtain a visual representation of the spacetime as a 2-surface in $\mathbb{R}^3$, as shown in Fig. 6.

We can parametrize the hyperboloid in Eq. (4.23) using a set of n coordinates $x^\mu = (\tau, \rho, \theta_j, \varphi)$. We may identify: ρ as a radial coordinate defined over the interval $[0, \pi/2)$; θ_j $(j = 1, \ldots, n-3)$ and $\varphi := \theta_{n-2}$ as the polar and azimuthal coordinates on the unit $(n-2)$-sphere, respectively, each satisfying $0 \leqslant \theta_j \leqslant \pi$ and $0 \leqslant \varphi < 2\pi$; finally, τ is identified as a timelike coordinate ranging from $-\pi$ to π.

In the set of suitable parametrized coordinates, the induced metric on AdS_n is

$$ds^2 = g_{\mu\nu} x^\mu x^\nu = H^2 (\sec\rho)^2 [-d\tau^2 + d\rho^2 + (\sin\rho)^2 d\Omega_{n-2}^2], \tag{4.24}$$

where $d\Omega_{n-2}^2$ is the line element on a unit $(n-2)$-sphere. With the metric tensor in hands, we can find some relevant geometrical quantities, namely the Riemann curvature tensor:

$$\mathrm{R}_{\mu\nu\alpha\beta} = -H^{-2} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}); \tag{4.25}$$

the Ricci tensor:

$$\mathrm{R}_{\mu\nu} = -H^{-2} (n-1) g_{\mu\nu}; \tag{4.26}$$

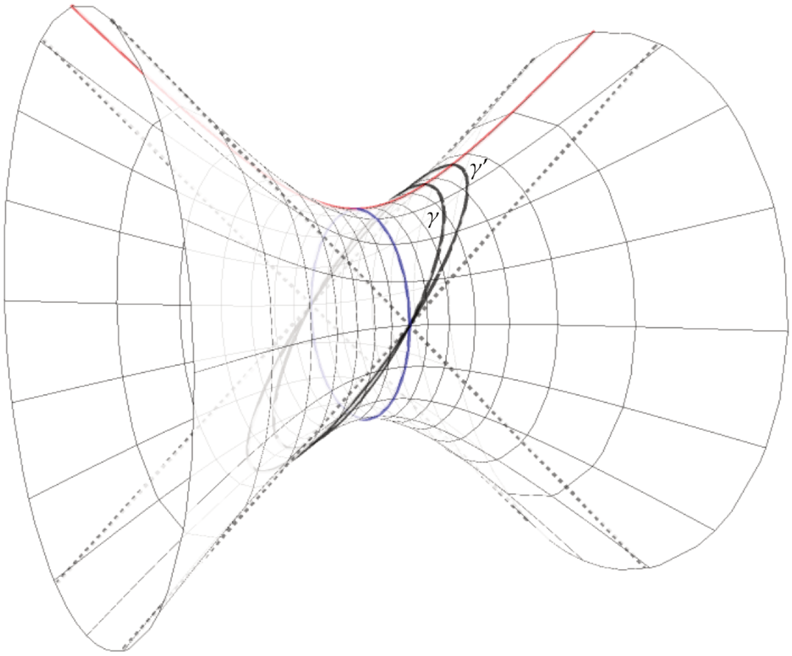

Figure 6 – Representation of AdS_n as a single-sheeted 2-hyperboloid[1].

and the scalar curvature:

$$\mathrm{R} = -H^{-2}n(n-1). \tag{4.27}$$

As expected, the curvature R is constant and negative, given the hyperbolic geometry of the spacetime.

Suppressing all angular coordinates θ_j and φ, we are left with the hyperboloid shown in Fig. 6. For a fixed ρ, the blue curve in Fig. 6 is generated by varying τ, like all other transverse section circles - i.e., purely timelike geodesics. On the other hand, when fixing τ, the radial coordinate ρ generates longitudinal curves (red hyperbola in Fig. 6) - i.e., spacelike geodesics. Besides, the radius of the blue circle centered at the origin of the coordinate system - the waist of the hyperboloid - is H, the curvature parameter of adS.

As the timelike coordinate τ varies within a periodic range, given a point in spacetime, we can return to it by traveling on a timelike geodesic of length 2π in τ. At this point, the topology of AdS_n becomes apparent, namely $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, revealing that the spacetime has closed timelike curves. It is straightforward to notice this feature from the circular geometry of the hyperboloid, as seen in Fig. 6.

Let us take a look on the conformal structure of AdS and how causality takes place in this background. The topology $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ of AdS_n allows the existence of closed timelike curves. Accordingly, unphysical events can take place at the spacetime, such as a particle returning to the same position $(\tau^{(O)}, \rho^{(O)}, \theta_k^{(O)}, \phi^{(O)})$ in space after moving along a timelike path by a periodic motion of 2π in time, i.e., $(\tau, \rho, \theta_k, \phi) \equiv (\tau^{(O)} + 2\pi m, \rho^{(O)}, \theta_k^{(O)}, \phi^{(O)})$, for $m \in \mathbb{Z}$. In Fig. 6, we can see two of these closed curves, γ and γ', in black, surrounding the waist of the hyperboloid. In such conditions, Wald remarks in [23] that observers following these geodesics would have no difficulty altering past events hence breaking causality.

In an attempt to solve this primary causality issue of AdS, we can 'unwrap' the hyperboloid along the time coordinate direction and patch one unwrapped hyperboloid after the other. In other words, we construct a spacetime spatially identical to AdS but extended in time, i.e., the temporal coordinate no longer ranges from $-\pi$ to π but from $-\infty$ to ∞. We refer to such procedure as the universal covering of AdS, and the resulting spacetime as CAdS. To visualize the scheme described above, we shall use a conformal diagram (Penrose diagram) of these spacetimes.

In the particular case of AdS_3, the conformal metric is as follows

$$g^c_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + d\rho^2 + (\sin\rho)^2 d\varphi^2, \tag{4.28}$$

which describes a cylindrical spacetime with radius $\pi/2$. At the walls of the cylinder ($\rho = \pi/2$) there is a spatial boundary that we recognize as the conformal infinity of AdS. Also, the axis of the cylinder is a timelike straight line centered at $\rho = 0$ with height 2π. In higher dimensions, the circular cap on top of the cylinder will be replaced by an $(n-2)$-dimensional spherical hemisphere. Moreover, suppressing all angular coordinates θ and φ, we get to the conformally compactified version of AdS_n in two dimensions, as shown on the left-hand side of Fig. 7.

Also, on the right-hand side of Fig. 7, there is the conformal diagram for the universal cover of AdS_n, CAdS_n. The closed timelike curves we discussed previously, γ and γ', appear in Fig. 7 as curvilinear paths arising from $\rho = 0$ and heading back there after $\Delta\tau = 2\pi$. However, in CAdS these curves evolve throughout the spacetime without ever returning to their initial point. Even though unwrapping AdS prevents the existence of closed timelike curves, other causality issues remain.

If we employ the same causal analysis that we discussed in Chapter 3 to AdS, we conclude that no static slice will be a Cauchy surface. For instance, taking a spacelike hypersurface at $\tau = \pi/2$ (which would appear as a straight horizontal line in Fig. 7), we verify that the white diamond-shaped regions give its domain of dependence. The remaining filled triangles are still part of the manifold but not entirely causally related to the hypersurface. We can understand this situation in the same way we did for Minkowski with a point removed: there is an entire region of events in the spacetime whose causal curves might end or begin at the spatial boundary $\mathscr{I}$ without ever intersecting a chosen spacelike hypersurface. Since these conclusions can be extended to all static slices of AdS, we can affirm that none of them are Cauchy surfaces and AdS is a non-globally hyperbolic spacetime.

Because of its causal structure, AdS displays a lack of predictability associated with fields propagating on this curved background. The Cauchy problem will not be

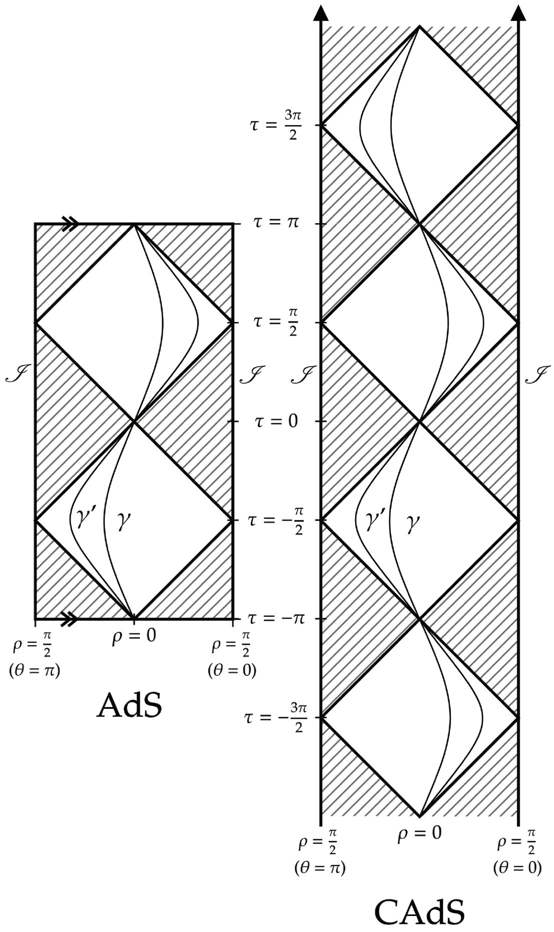

Figure 7 – Penrose diagram showing AdS (left) and CAdS (right) with all angular coordinates suppressed. In AdS, the hypersurfaces $\tau = \pi$ and $\tau = -\pi$ are identified.[2]

well-posed, yielding non-unique dynamics for a given set of initial conditions. We can understand this scenario as a result of information leaking through the spatial infinity of the spacetime, i.e., flowing in from, or out to the boundary. In an attempt to solve such a pathological behavior, in [2], the authors develop a quantum field theory in AdS_4 by regulating information leaving or entering the spacetime *by hand*. Their approach is based on imposing boundary conditions at the spatial infinity, hence controlling whether information crosses or is reflected by the boundary.

In our approach to AdS, we considered the boundary conditions found in [44] by Ishibashi and Wald. For simplicity, we treated the case of a massless, conformally coupled scalar field in four-dimensional AdS, for what the boundary conditions reduced to the Robin-type (see Eq.(2.32)). Our results are available in the Annex D, where the Ref. [45] is shown. As reference to further comparison, we kept the results of Kent and Winstanley in Ref. [43], in which they compute the fluctuations of the expectation values of the field squared and the energy-momentum tensor. In AdS, the boundary conditions must be specified for all modes of the wave equation (see Sec. 3 of [45], p.6 of Annex D). For operational reasons, we chose to prescribe Robin boundary conditions at spherically symmetric modes only. Accordingly, we detailed a modified procedure of obtaining the Green's functions in Sec. 4 of [45] (p.9 of Annex D). As direct consequences of our procedure, the quantities that we have computed do not respect the spacetime invariances.

In Sec. 5 of [45] (p.11 of Annex D), we present the reader with an extensive description of the numerical routine employed in the computations of the renormalized quantities, namely the fluctuations of the field squared and the stress-energy tensor. Such results are plotted in Fig. 2 and 3 of [45] (p.16 and p.21 of Annex D), in which the influence of the boundary condition parameter on the physical quantities is clear. Furthermore, Fig. 4 in [45] reveal the violation of the weak condition as a straightforward result of the imposition of non-Dirichlet boundary conditions at the boundary. Additionally, the break of the spacetime invariance becomes even more evident and we argue that a backreaction process would yield a spacetime with less symmetries than AdS, so non-maximally symmetric, naturally.

5 Final Remarks

The lack of global hyperbolicity in a variety of spacetimes poses a severe issue to the quantization of fields on these backgrounds. Naked singularities or conformal boundaries represent points or regions in space where information can leak through. Hence, knowing the configuration of a quantum field at an instant of time - but everywhere in space - would not be enough to determine its entire history. In other words, the Cauchy problem is not well-posed, which leads to a loss of predictability in the equations of motion governing the dynamics of quantum fields.

Wald and Ishibashi propose a physically sensible prescription for the dynamics of quantum fields in non-globally hyperbolic spacetimes. We have seen that their procedure consists in determining the self-adjoint extensions of the spatial component of the differential wave operator. To each extension, we have found boundary conditions at appropriate regions in space which one should employ when solving the wave equation. We expected that the parameter of the b.c. would appear in quantities obtained from the fields, i.e., our computations would not be able to remove the dependence on the b.c. Indeed, that was what happened in the cases treated in this dissertation.

In the first two examples, the Cone and the Global Monopole, we have observed that the parameter of the Robin boundary conditions in each case led to new scattering patterns. These results gave us our first glimpse on how non-trivial (non-Dirichlet) interactions between the fields and the singularities could yield equally non-trivial physical outcomes. Our next step was then to extend the analysis to the quantization of scalar fields, for what we used the Global Monopole as background to our toy-model.

In the Global Monopole, the naked singularity - placed at the origin of the coordinate system for convenience - was perceived by spherically symmetric modes of the wave equation only. Naturally, the boundary condition parameter appeared solely in the $(l = 0)$-mode. Due to operational difficulties, we adopted a perturbative approach in terms of the small curvature of the spacetime and then calculated the Euclidean Green's function. Using the work of Mazzitelli and Lousto as our guide, we derived some expectation values from the Green's function, namely the field squared and the Energy-Momentum Tensor. Both results had terms in zeroth order of the curvature parameter, which indicates that the relevant contribution was due to the Robin-like interaction of one mode of the field with the naked singularity exclusively. In particular, our contribution to fluctuations of the energy density carried an explicit dependence on the boundary condition parameter.

Once again, we followed Wald and Ishibashi through their developments in the anti-de Sitter spacetime. In this particular case, they showed that the imposition of mixed boundary conditions at the spatial infinity is sufficient to determine the evolution of quantum fields uniquely. Independently of the maximal symmetry of AdS, we have found that the imposition of different boundary conditions for each mode of the wave equation results in Green's functions that are not AdS-invariant. For simplicity, we restricted the upcoming computations to conformally invariant scalar fields in AdS_4.By the quadratic field fluctuation, we had our first evidence of a violation of AdS invariance. As expected, this endured and appeared in the fluctuations of the energy-momentum tensor as well.

Our results reveal that if one takes a close look at the energy density distributions of the fields in the Global Monopole and AdS, they should be able to specify the boundary conditions that scalar fields satisfy at the appropriate regions in space. As we are not aware of any constraints in Nature that indicate a specific choice of boundary condition, we argue that Wald's prescription should be taken into account when studying QFTCS in the absence of global hyperbolicity.

Annexes

ANNEX A – Article 1

Quantum scattering on a cone revisited[30]

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Quantum scattering on a cone revisited

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We revisit the scattering of quantum test particles on the conical $(2+1)$-dimensional spacetime and find the scattering amplitude as a function of the boundary conditions imposed at the apex of the cone. We show that the boundary condition is responsible for a purely analytical term in the scattering amplitude, in addition to those coming from purely topological effects. Since it is possible to have nonequivalent physical evolutions for the wave packet (each one corresponding to a different boundary condition), it seems crucial to have an observable quantity specifying which evolution has been prescribed.

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I. INTRODUCTION

In classically singular spacetimes, the evolution of wave packets may not be uniquely determined by the initial conditions. It is possible to have an infinite number of boundary conditions at the classical singular point, each one giving a reasonable physical evolution. To predict physical effects in such spacetimes, we need to know which evolution has been prescribed. In this way, it is essential to have an observable quantity depending on the possible boundary conditions. This is the main goal of this paper. We will show that the differential cross section of wave scattering on the cone carries the information we need.

In globally hyperbolic spacetimes, the propagation of particles and waves are uniquely determined. Given the initial position and velocity of a particle, the classical trajectory can be extended for all times. In a similar way, given the initial wave packet $\Psi(0)$ [and possibly $\dot{\Psi}(0)$] on a Cauchy surface, there is a well defined evolution $\Psi(t)$. However, in a singular spacetime, a classical trajectory which reaches the singularity cannot be extended and the future of the particle becomes unknown. In a similar way, since no Cauchy surface exists, the evolution of waves may be ambiguous.

In static singular spacetimes [1], a boundary condition must be imposed at the singular point in order to find the evolution of the wave packet. These boundary conditions are the ones which turn into self-adjoint the spatial part of the wave operator, giving rise to a sensible dynamics [2]. If there is only one boundary condition, there is no ambiguity, and we say that the spacetime is quantum mechanically (QM) nonsingular [3] and that the singularity has been "healed" by quantum mechanics. On the other hand, if there is an infinite number of possible boundary conditions, the evolution is uncertain. Since there is no privileged evolution, we say that the spacetime in this case is QM singular [3]. The $(2+1)$-dimensional cone is an example of a QM singular spacetime and will be used as a toy model to show that it is possible to find observational parameters related to boundary conditions in nature.

It is well known that solutions of Einstein field equations

$$R_{\mu\nu} = \kappa[T_{\mu\nu} - g_{\mu\nu}T^{\lambda}{}_{\lambda}] \tag{1}$$

in $(2+1)$ dimensions are locally flat in the absence of matter. This happens essentially because the Riemann curvature tensor may be written as

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta} + g_{\beta\delta}R_{\alpha\gamma} - \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R, \tag{2}$$

and $T_{\mu\nu} = 0$ implies $R_{\mu\nu} = 0$ through Einstein equations [4]. In addition to the $(2+1)$-dimensional Minkowski solution, there is a solution with nontrivial topology, which represents a massive point object and is identified as the product of a timelike straight line and a two-dimensional cone. It has the following metric [5]:

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2, \tag{3}$$

with $0 < r < \infty$, $0 \leq \theta \leq 2\pi$, and $0 < \alpha < 1$. The mass M of the object located at $r = 0$ is related to α by $M = 2\pi(1-\alpha)/\kappa$. The cone generated by the spatial part of metric (3) has the opening angle given by $\varphi = 2\sin^{-1}\alpha$ and in three dimensions is parametrized by $z(r) = (\alpha^{-2} - 1)^{1/2} r$. It has its vertex at $r = 0$ which is a classical spacetime singularity.

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Following the definition of quantum singularities due to Horowitz and Marolf [3] (see also [6]), it is known that the cone is also quantum mechanically singular; i.e., there is an infinite number of possible boundary conditions at the apex of the cone with sensible dynamics. In Ref. [7], these boundary conditions have been found. However, the scattering of waves on the cone has only been studied using a particular boundary condition [8], namely, the Friedrichs boundary condition [9].

This gives rise to the following question: how is the dynamics of a quantum test particle affected by the singularity at the cone vertex if we consider an arbitrary boundary condition? We attempt to answer this question by analyzing the scattering behavior of a scalar field on the cone. As we will see, the contribution of a general boundary condition to the differential cross section is purely analytical, in the sense that it is always present and is independent of the angular deficit $\Delta = 2\pi(1-\alpha^2)$. This contribution also adds up to the purely topological contribution which depends directly on α.

This paper is organized as follows: in Sec. II we give a brief review on the theory describing the dynamics of quantum test particles in classically singular spacetimes. Then, in Sec. III, we present a solution for the Klein-Gordon equation on the cone with the appropriate boundary conditions at $r=0$. We revisit the scattering of a quantum test particle on a conical spatial background in Sec. IV. In Sec. V we conclude by discussing the differences between our results and previous ones.

II. QUANTUM SINGULARITIES AND SELF-ADJOINT OPERATORS

A classical singularity is indicated by incomplete geodesics or incomplete curves of bounded acceleration [10]. Accordingly, the evolution of classical particles following these geodesics may not be defined for all values of its affine parameter [11]. At the center of a spherically symmetric black hole, for example, we have a very strong singularity, with infinite tidal forces. However, it is also possible to have milder singularities as solutions of Einstein equations. This is the case of the cosmic string background [12], given by the metric

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2 + dz^2, \tag{4}$$

which is locally flat (each section $z = \text{const}$ is a cone). In this way, there are geodesics which approach the singularity at $r=0$ feeling zero tidal forces. This is an example of a naked singularity.

In some cases, a naked singularity can be healed when the spacetime is probed by waves. As an example, we have the hydrogen atom, in which the position of the proton ($r=0$) is a classical singularity. However, when solving the Schrödinger equation, we must only impose square integrability to find a complete set of orthonormal solutions. In this way, the evolution $\Psi(t)$ of any wave packet is uniquely determined by the initial condition $\Psi(0)$. Since there is no ambiguity in the solution of the wave equation for the hydrogen atom, we say that it is QM nonsingular. As the evolution of waves is unique in QM nonsingular spacetimes, physical predictions are then uniquely determined.

In a QM singular spacetime, since the evolution of waves is no longer unique, the physical system does not give unique physical predictions. Each physical evolution is attached with a specific boundary condition at the singularity. We present the general theory of QM singularities due to Horowitz and Marolf [3] in what follows.

Let us restrict to static spacetimes with timelike Killing vector ξ^μ, where t denotes its parameter. The Klein-Gordon equation

$$(\nabla_\mu \nabla^\mu - \mu^2)\psi = 0 \tag{5}$$

can be rewritten as

$$\frac{\partial^2 \psi}{\partial t^2} = -A\psi, \tag{6}$$

where $A \equiv -VD^i(VD_i) + \mu^2$, $V^2 = -\xi^\mu \xi_\mu$, and D_j is the spatial covariant derivative on a static slice of space Σ.

Since we do not know exactly what happens at the singularity, consider the domain of operator A as being $C_0^\infty(\Sigma)$. Since the singular points are not part of Σ, the singularity is not being considered. With this choice, it is easy to see that the operator $(A, C_0^\infty(\Sigma))$ is symmetric and positive definite but not self-adjoint. However, it has at least one self-adjoint extension (Friedrichs extension [9]).

A general solution for Eq. (6) has the form

$$\psi_E(t) = \psi(0)\cos(A_E^{1/2} t) + \dot{\psi}(0) A_E^{-1/2} \sin(A_E^{1/2} t), \tag{7}$$

where A_E is a self-adjoint extension of the operator A. If this extension is unique, A_E represents a single operator (A is essentially self-adjoint) and, since there is no ambiguity in the evolution of a wave packet, the spacetime is said to be QM nonsingular. If there is an infinite number of self-adjoint extensions, E represents a parameter and the spacetime is QM singular. To each self-adjoint extension, there corresponds a boundary condition at the singularity. In the next section, we give an example of this procedure for the Klein-Gordon equation on the cone. Following Ref. [7], we present each boundary condition which turns into self-adjoint the spatial part of the wave operator.

III. KLEIN-GORDON EQUATION ON THE CONE

The Klein-Gordon equation on the conical $(2+1)$-dimensional spacetime given by the metric (3) is written as

$$\frac{\partial^2\phi(t,r,\theta)}{\partial t^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\alpha^2 r^2}\frac{\partial^2}{\partial\theta^2} - \mu^2\right)\phi(t,r,\theta)$$
$$\equiv (\Delta - \mu^2)\phi(t,r,\theta). \tag{8}$$

Solutions may be separated as $\phi(t,\vec{r}) = e^{-i\omega t}\Psi(r,\theta)$, and Eq. (8) reduces to

$$-\Delta\Psi(r,\theta) = (\omega^2 - \mu^2)\Psi(r,\theta). \tag{9}$$

In Sec. II, we discussed the importance of having A as essentially self-adjoint to ensure the uniqueness of the solution. However, in the conical spacetime, this is not the case. As discussed in [7], the operator $-\Delta$ on the domain $C_0^\infty(\mathbb{R}^+\times\mathbb{S}) \subset \mathcal{L}^2(\mathbb{R}^+\times\mathbb{S}, rdrd\theta)$ has a family of self-adjoint extensions $-\Delta^R$ parametrized by one parameter $R\in[0,\infty)$. If we consider another separation of variables to the solution, namely $\Psi(r,\theta) = \sum_m f_m(r)e^{im\theta}$, our operator $-\Delta$ reduces to

$$-\Delta_m = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{m^2}{\alpha^2 r^2}. \tag{10}$$

These $-\Delta_m$ on the domain $C_0^\infty(\mathbb{R}^+) \subset \mathcal{L}^2(\mathbb{R}^+, rdr)$ are essentially self-adjoint for $m\neq 0$. Nevertheless, for $m=0$, $-\Delta_0$ has infinitely many self-adjoint extensions, $\{-\Delta_0^R, R\in[0,\infty)\}$. As previously presented, with every extension $-\Delta_0^R$ there must be an associated boundary condition at the singularity ($r=0$), as follows [7]:

$$\lim_{r\to 0}\left[\ln\left(\frac{r}{R}\right)r\frac{d}{dr} - 1\right]f_0(r) = 0, \quad \text{for } R\in(0,\infty), \tag{11}$$

$$\lim_{r\to 0} r\frac{d}{dr}f_0(r) = 0, \quad \text{for } R = 0. \tag{12}$$

As one solves the eigenvalue problem $-\Delta_m f_m(r) = \lambda f_m(r)$ with the appropriate boundary conditions, we find that, for $m\neq 0$ and $m=0$ with $R=0$, there is a complete set of eigenfunctions with positive eigenvalues k^2. For $m=0$ and $R\neq 0$, $-\Delta_0^R$ may be negative. If we redefine the boundary condition parameter as

$$q = 2e^{-\gamma}R^{-1} (\gamma = \text{Euler-Mascheroni constant}), \tag{13}$$

the operator $-\Delta_0^R + \mu^2$ has a negative eigenvalue $-\omega_q^2 = -q^2 + \mu^2$ as long as $q > \mu$. In this case, it is possible to have a solution of the form

$$\Psi(t,r) = K_0(\omega_q r)e^{\pm\omega_q t}. \tag{14}$$

The positive exponential leads to an unstable configuration if the wave equation appears as a linear perturbation of the spacetime. Since physical predictions in unstable spacetimes are meaningless, we will consider the case $0\leq q\leq\mu$.

Then we have only positive eigenvalues for the operator $-\Delta_0^R + \mu^2$, and the complete set of solutions of the Klein-Gordon equation is given by

$$\left\{\frac{1}{\sqrt{2\pi}}\frac{J_0(kr) + \beta(k)N_0(kr)}{\sqrt{1+\beta^2(k)}}\right\} \cup \left\{\bigcup_{m\neq 0}\frac{1}{\sqrt{2\pi}}J_{\frac{|m|}{\alpha}}(kr)\right\}, \tag{15}$$

where J_n is the nth order Bessel function, N_0 is the 0th order Neumann function and $\omega_k^2 = k^2 + \mu^2$, $m\in\mathbb{Z} - \{0\}$, and

$$\beta(k) = \frac{\pi}{2}\left[\ln\left(\frac{q}{k}\right)\right]^{-1}. \tag{16}$$

Note that $q=0$ corresponds to $\beta=0$, so that in this case only regular solutions at $r=0$ are considered (Friedrichs boundary condition).

We point out that this development is completely applicable to the nonrelativistic case by simply setting $k = \sqrt{2\mu\omega_k}/\alpha$. It is now clear that solution (15) is not unique, for it depends on the chosen boundary condition. Therefore, the conical $(2+1)$-dimensional spacetime is quantum mechanically singular when tested by a Klein-Gordon field.

IV. QUANTUM SCATTERING REVISITED

In [8], Deser and Jackiw studied quantum scattering on the cone. We revisit their work in a relativistic version, considering now solution (15), which takes into account the appropriate boundary conditions at the vertex. Bound states are not relevant in scattering, so the spatial part is considered as

$$\Psi(r,\theta) = \frac{a_0}{\sqrt{2\pi}}\frac{J_0(kr) + \beta(k)N_0(kr)}{\sqrt{1+\beta^2(k)}} + \sum_{m\neq 0}\frac{a_m}{\sqrt{2\pi}}J_{\frac{|m|}{\alpha}}(kr)e^{im\theta}. \tag{17}$$

We follow the procedure in [8], so the total wave is

$$\psi(r,\theta) = \psi_{\rm in}(r,\theta) + \psi_{\rm sc}(r,\theta), \tag{18}$$

where $\psi_{\rm in}$ and $\psi_{\rm sc}$ are the incident and the scattered waves, respectively. Both satisfy the following asymptotic behavior as $r\to\infty$:

$$\psi_{\rm in}(r,\theta) \sim e^{ikr\cos\theta}, \tag{19}$$

$$\psi_{\rm sc}(r,\theta) \sim \sqrt{\frac{i}{r}}f(\theta)e^{ikr}. \tag{20}$$

In order to compare both asymptotic forms of Ψ and ψ, we must use the following relation:

$$e^{ikr\cos\theta} = \sum_{m=-\infty}^{\infty} e^{im\frac{\pi}{2}} J_m(kr) e^{im\theta}, \tag{21}$$

and the asymptotic forms of Bessel and Neumann functions

$$J_m(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr - m\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{22}$$

$$N_m(kr) \sim \sqrt{\frac{2}{\pi kr}} \sin\left(kr - m\frac{\pi}{2} - \frac{\pi}{4}\right). \tag{23}$$

When comparing the asymptotic forms of Eqs. (17) and (18), from incident modes e^{-ikr} we have

$$a_0 = \sqrt{2\pi} \frac{1 - i\beta(k)}{\sqrt{1 + \beta(k)^2}}, \tag{24}$$

$$a_m = \sqrt{2\pi} e^{-i\frac{|m|}{2}(\omega_c - \pi)}. \tag{25}$$

From scattered modes e^{ikr} we get the scattering amplitude

$$f(\theta) = \frac{1}{\sqrt{2\pi k}} \left\{ \frac{-2\beta(k)[1 - i\beta(k)]}{1 + \beta(k)^2} - i \sum_{m=-\infty}^{\infty} (e^{-i|m|\omega_c} - 1) e^{im\theta} \right\}, \tag{26}$$

where $\omega_c \equiv \pi(\alpha^{-1} - 1)$ is the angle between the projections of the asymptotic paths of a classical particle onto the x-y plane (see [8]).

Now the total wave becomes

$$\Psi(r,\theta) = \frac{-\beta(k)[1 - i\beta(k)]}{1 + \beta(k)^2} \{iJ_0(kr) - N_0(kr)\} + \sum_{m=-\infty}^{\infty} e^{-i\frac{|m|}{2}(\omega_c - \pi)} J_{\frac{|m|}{\alpha}}(kr) e^{im\theta}. \tag{27}$$

Note the appeareance of β, which is related to the choice of the boundary condition. It does not depend on α and adds up to the purely topological terms. This term represents a point interaction between the incoming wave and the apex of the cone.

Equation (26) may be rewritten after a few regularizations as

$$\sqrt{2\pi k} f(\theta) = \frac{-2\beta(k)[1 - i\beta(k)]}{1 + \beta(k)^2} + \frac{\sin\omega_c}{\cos\omega_c - \cos\theta} - i\pi \sum_n (\delta(\theta + \omega_c - 2\pi n) + \delta(\theta - \omega_c - 2\pi n) - 2\delta(\theta - 2\pi n)). \tag{28}$$

This scattering amplitude cannot satisfy the optical theorem, since its delta functions and divergences at $\theta = \pm\omega_c$ invalidate integration over all angles between 0 and 2π. However, one can check that the Klein-Gordon probability current remains divergenceless and, then, solution (27) holds probability conservation.

As proposed by Deser and Jackiw, the part of the scattered wave that asymptotically gives rise to deltas in $f(\theta)$ may be replaced into the incoming wave. We separate the total wave function ψ arbitrarily as

$$\psi(r,\theta) = \tilde{\psi}_{\text{in}}(r,\theta) + \tilde{\psi}_{\text{sc}}(r,\theta), \tag{29}$$

with the following asymptotic conditions:

$$\tilde{\psi}_{\text{sc}}(r,\theta) \sim \sqrt{\frac{i}{r}} \tilde{f}(\theta) e^{ikr}, \tag{30}$$

$$\tilde{f}(\theta) = \frac{1}{\sqrt{2\pi k}} \left\{ \frac{-2\beta(k)[1 - i\beta(k)]}{1 + \beta(k)^2} + \frac{\sin\omega_c}{\cos\omega_c - \cos\theta} \right\}. \tag{31}$$

Now we must find a new incident wave resulting from this redefinition. In [8], only the second term in Eq. (27) corresponds to the total wave, and the authors find a contour integral representation for it. To do so, they use Schläfli's representation for Bessel functions, given by

$$J_\nu(x) = \frac{1}{2\pi} \int_\Gamma dz e^{-i(k\sin z - \nu z)}, \tag{32}$$

where Γ is a complex contour coming from $-\pi + i\infty$ to $-\pi$, passing by the real axis to π, and then returning to $\pi + i\infty$. Following this procedure, the sum in Eq. (27) becomes

$$\sum_{m=-\infty}^{\infty} e^{-i\frac{|m|}{2}(\omega - \pi)} J_{\frac{|m|}{\alpha}}(kr) e^{im\theta} = \frac{1}{4\pi i} \int_C dz \tan\left(\frac{z}{2\alpha}\right) e^{-ikr\cos(z - \alpha\theta)} \equiv \frac{1}{4\pi i} I_C, \tag{33}$$

where C is the contour given in Fig. 1.

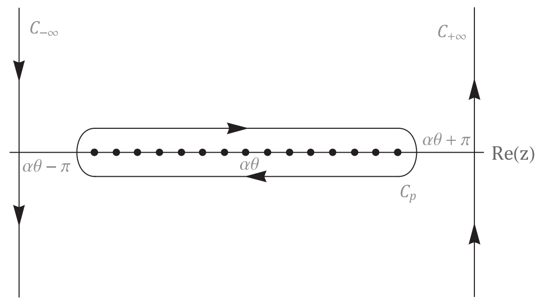

FIG. 1. Integration contour C for Eq. (33) separated in the other three: $C_{-\infty}$, $C_{+\infty}$, and C_p. Contour C_p is a clockwise Cauchy contour around the real poles of $\tan(z/2\alpha)$ between $\alpha\theta - \pi$ and $\alpha\theta + \pi$.

Integration over $C_{+\infty}$ and $C_{-\infty}$ can be rewritten using the function

$$\chi(r,\xi) = \int_{-\infty}^{\infty} dy e^{ikr\cosh y} \tan\left(\xi + i\frac{y}{2\alpha}\right). \quad (34)$$

Cauchy's residue theorem allows us to express integration over C_p as the sum of all residues of the integrated function at the poles. Finally, integral I_C can be separated in three others over the contours presented in Fig. 1, as follows:

$$\begin{aligned} I_C &= \left(\int_{C_{+\infty}} + \int_{C_{-\infty}} + \int_{C_p}\right) dz \tan\left(\frac{z}{2\alpha}\right) e^{-ikr\cos(z-\alpha\theta)} \\ &= i\left[\chi\left(r,\frac{\theta}{2}+\frac{\pi}{2\alpha}\right) - \chi\left(r,\frac{\theta}{2}-\frac{\pi}{2\alpha}\right)\right] \\ &\quad + 4\pi i\alpha \sum_{\substack{n \\ \alpha|\theta_n|<\pi}} e^{-ikr\cos(\alpha\theta_n)}, \end{aligned} \quad (35)$$

where $\theta_n \equiv \theta - (2n+1)\pi$.

We compared the asymptotic forms of (27) and (29), as in [8], and by their asymptotic contributions identified our new incoming and scattered waves

$$\Psi(r,\theta) = \underbrace{\alpha \sum_{\substack{n \\ \alpha|\theta_n|<\pi}} e^{-ikr\cos(\alpha\theta_n)}}_{\tilde{\psi}_{\rm in}(r,\theta)} + \underbrace{\frac{-\beta(k)[1-i\beta(k)]}{1+\beta(k)^2}\{iJ_0(kr) - N_0(kr)\} + \frac{1}{4\pi}\left[\chi\left(r,\frac{\theta}{2}+\frac{\pi}{2\alpha}\right) - \chi\left(r,\frac{\theta}{2}-\frac{\pi}{2\alpha}\right)\right]}_{\tilde{\psi}_{\rm sc}(r,\theta)}. \quad (36)$$

As a cone is not an asymptotically flat spacetime, topological scattering may produce those undesirable deltas at the amplitude (28). All these delta functions had their contributions placed into the incident wave from Eq. (36). This new incident wave can be seen as a composition of plane waves in variate directions depending on the deficit angle α and carrying the topological characteristics of space. We treat it as a redefinition of plane waves on the cone, for the original definition led to the appearance of deltas.

The incident wave we found in (36) is the same as Deser and Jackiw found in [8], and this shows the prescription of a boundary condition at the vertex affects only the scattered wave. Furthermore, the topological scattering is responsible for the second term of $\tilde{\psi}_{\rm sc}$ in (36) and was already found in [8].

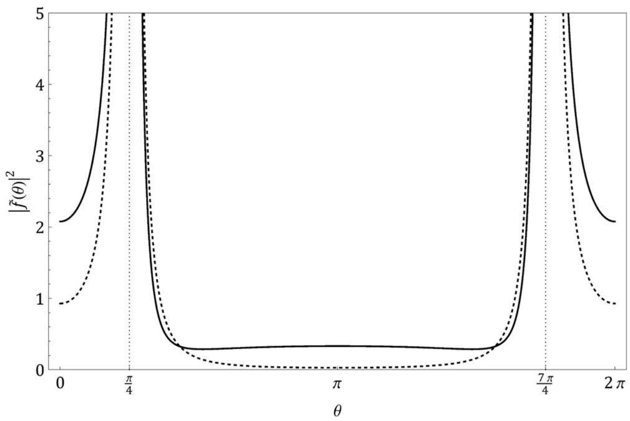

FIG. 2. Plot of $|\tilde{f}(\theta)|^2$ for $\beta = 0$ (dashed line) and $\beta \neq 0$ (filled line). The parameters α, related to the deficit angle, and q, which sets the boundary condition, are $\alpha = 0.8$ and $q = 10$. The frequency k is set equal to one. Dotted horizontal lines indicate the divergence angles $\omega_c = \pi/4$ and $\theta = 2\pi - \omega_c = 7\pi/4$ for the amplitude.

We point out that the term in $\tilde{\psi}_{\rm sc}$ results from a pointlike interaction of the wave and the localized massive object at the vertex.

In [8], the incident wave $\tilde{\psi}_{\rm in}$ would be scattered by the spacetime topology, generating part of our scattered wave. In our picture, a spherically symmetric term appears at $\tilde{\psi}_{\rm sc}$ as the incident wave perceives the boundary condition at $r = 0$, showing it is a purely analytic interaction.

In Fig. 2 we show the behavior of $|\tilde{f}(\theta)|^2$, when $\beta = 0$ and there are only topological effects, and when $\beta \neq 0$ and a purely analytical term arises. There are divergences at the classical scattering angle $\theta = \omega_c$, as well as at $\theta = 2\pi - \omega_c$. These divergences appear due to topological effects as in Ref [8] and are the signatures of the cone. The main effect of a non-null β appears at $\theta = 0$. By looking at the scattering at this angle we can infer the choice of the boundary condition.

V. FINAL REMARKS

The $(2+1)$-dimensional cone was used as a toy model to illustrate the effects of an arbitrary choice of boundary conditions in QM singular spacetimes. Studying the scattering of waves, we showed that the differential cross section depends explicitly on the boundary condition, so that, through the observation of scattered waves, it may be possible to infer which evolution has been prescribed. If we want to construct quantum field theory (QFT) in non-globally hyperbolic spacetimes (see [13] for QFT in AdS spacetime with general boundary conditions) and predict physical observables, we need to know which evolution has been preferred by nature. Our result gives us a hint of how to solve this question. We also argue that this simple model can be extended to more significant spacetimes, such as the spacetime of a cosmic string and the spacetime of a global monopole [14]. These spacetimes are QM singular (see

Refs. [15,16]) so that physical observables will depend on the boundary conditions.

To the best of our knowledge, for the first time an observable has been related to the prescribed evolution. In general, predictions in naked singular spacetimes are meaningless, since there is always an unknown parameter (the exact interaction between classical test particles and the singularity) which are not predicted by general relativity. We showed that with the introduction of quantum mechanics we are able to find the analytical interaction between waves and the singularity by means of a single observation. Now that we can find the prescribed evolution, the next step would be the search for other observables (the expectation value of the renormalized stress tensor, for example [17]) to see how different are the predictions compared to the usually chosen Friedrichs boundary condition.

Since the perturbation of the spacetime leads to the wave equation, it is also possible that the stability of QM singular spacetimes depends on the physical prescription [see discussion below Eq. (14)]. If perturbations with a wide range of possible boundary conditions are present, the spacetime will certainly be unstable. This can explain why such spacetimes have never been observed.

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ANNEX B – Article 2

Scattering cross section and stability of global monopoles[35]

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Scattering cross section and stability of global monopoles

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We study the scattering of scalar waves propagating on the global monopole background. Since the scalar wave operator in this topological defect is not essentially self-adjoint, its solutions are not uniquely determined until a boundary condition at the origin is specified. As we show, this boundary condition manifests itself in the differential cross section and can be inferred by measuring the amplitude of the backscattered wave. We further demonstrate that whether or not the spacetime is stable under scalar perturbations also depends on the chosen boundary condition. In particular, we identify a class of such boundary conditions that significantly affects the differential cross section without introducing an instability.

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I. INTRODUCTION

Topological defects are formed during phase transitions in the early Universe. They originate from the breakdown of gauge symmetries and are believed to seed the formation of large-scale structure in the Universe [1,2]. Depending on which symmetry is broken, they are classified as domain walls, cosmic strings or global monopoles [2]. Global monopoles, in particular, arise when the global $O(3)$ symmetry of the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2, \qquad (1)$$

where ϕ^a $(a = 1,\ 2,\ 3)$ is a triplet of scalar fields, is spontaneously broken to $U(1)$ [3].

The metric around a global monopole, once its core size has been neglected, can be approximated by

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2)$$

where the parameter $\alpha = 1\text{–}8\pi G\eta^2$ depends on the symmetry breaking energy scale (typically $8\pi G\eta^2 \approx 10^{-5}$ in grand unified theories). This metric describes a spacetime with a deficit solid angle (the section $\theta = \pi/2$ coresponds to a cone with deficit angle $\Delta = 8\pi^2 G\eta^2$). The spacetime is not flat, being characterized by the curvature scalar $R = 2(\alpha^{-2} - 1)r^{-2}$ [4]. The energy density, determined by the 00th component of the stress-energy-momentum tensor $T_{\mu\nu}$, is given by $T_{00} \sim G\eta^2/r^2$ so that the total energy $E(r) \sim 4\pi G\eta^2 r$ is linearly divergent for large r. Despite the fact that the Ricci scalar goes to zero when $r \to \infty$, the global monopole is not asymptotically flat since there are nonzero components of the Riemann curvature tensor $R_{\rho\sigma\mu\nu}$ for arbitrarily large r. In particular, the $R_{\theta\phi\theta\phi} = (1 - \alpha^2)\sin^2\theta$ component is nonzero if $\alpha \neq 1$. In this paper, we will consider scattering on the global monopole spacetime. We argue that our results are valid in the $\alpha \lessapprox 1$ (i.e. $\Delta \ll 1$) regime, which is the realistic one predicted by grand unified theories.

The propagation of a massless scalar field Ψ around the global monopole background is governed by the Klein Gordon equation, $(\nabla_\nu\nabla^\nu - \mu^2)\Psi = 0$. Its solutions, however, are not uniquely determined by the initial data. In fact, if the spatial part of the wave equation is seen as an operator A acting on a certain L^2 Hilbert space, an infinite number of sensible dynamical prescriptions may be defined, each one corresponding to a different choice of a self-adjoint extension for A [5]. These various extensions are encoded in the arbitrary specification of a boundary condition at $r = 0$.

According to Horowitz and Marolf [6], a classically singular spacetime is said to be quantum mechanically singular when the evolution of a wave packet on the spacetime background depends on extra information not

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predicted by the theory. In this sense, the evolution of a wave packet in the global monopole spacetime is as uncertain as the evolution of a classical particle due to the presence of the classical singularity at $r = 0$. Even though the chosen boundary condition cannot be directly observed, we should expect that some physical observable quantities will depend on it. The phase difference between incident and scattered waves is an example of that [7], but in this paper we will focus on the differential scattering cross section. As we show, this cross section is not determined until we specify a boundary condition. Stated in another way, one could use observable information obtained from a scattering experiment (i.e. the cross section) to determine the boundary condition favored by nature.

In Ref. [8], the scattering of scalar waves by a global monopole was analyzed for a Dirichlet boundary condition and an approximation for the total cross section was obtained. More recently, in Ref. [9], Anacleto *et al.* considered the scattering of scalar waves by a black hole with a global monopole and showed that the differential cross section for small angles contains explicitly the α parameter of the global monopole. Here, on the other hand, we consider not only a Dirichlet boundary condition (which is usually assumed since it leads to regular solutions at the origin), but all possible boundary conditions allowed by self-adjointness. We investigate how much the differential cross section of the global monopole for scalar waves depends on the choice of the boundary condition.

To be physically relevant, a spacetime should be stable (or, if unstable, should have an instability time scale small enough compared to the time scales of the effects under investigation). Because of that, we also study the stability of the global monopole. Similar work was done in Refs. [10,11]. In Ref. [10], for instance, it was demonstrated that the global monopole is stable under a radial rescaling $r \to \kappa r$. That is, if we impose a cutoff $r = R_c$ in a cosmological setup, the energy $E(R_c)$ has a minimum at $\kappa = 1$. In Ref [11], it was shown that the global monopole is stable under axisymmetric perturbations of the triplet ϕ^a. In this paper, on the other hand, we follow a different approach by considering perturbations of a scalar test field. Encoding the arbitrary boundary condition as a free parameter, we search for solutions of the scalar wave equation which correspond to unstable modes, i.e. purely outgoing modes at spatial infinity that grow exponentially in time.

Our work is organized as follows: in Sec. II, we briefly review the necessity of choosing a boundary condition to solve the wave equation in the global monopole spacetime. We follow Ref. [12], where the singular nature of the global monopole spacetime was analyzed and the authors found that a Robin boundary condition is necessary to make the spatial part of the wave operator self-adjoint. Next, in Sec. III, we use the method of partial waves to find the differential cross section for scalar waves and investigate its relation to the boundary condition. In Sec. IV, we analyze the stability of the global monopole under scalar perturbations of a test field and show that the spacetime is stable for a class of boundary conditions. The last part, Sec. V, is reserved for our final considerations.

II. BOUNDARY CONDITIONS FOR THE KLEIN-GORDON EQUATION

Consider a massless scalar field Ψ propagating on the global monopole background. (The massive case, discussed in Ref. [12], only brings unnecessary complications). The associated Klein-Gordon equation, in view of (2), can be cast as

$$\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{r^2}\left(r^2\frac{\partial^2\Psi}{\partial r^2}\right) + \frac{1}{\alpha^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\alpha^2 r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}. \tag{3}$$

Due to the spherical symmetry, the above equation is separable under the *ansatz* $\Psi(t,r,\theta,\varphi) = R_{\omega\ell}(r) Y_\ell^m(\theta,\varphi)e^{-i\omega t}$, where $Y_\ell^m(\theta,\varphi)$ are the usual spherical harmonics, $\ell \in \mathbb{N}$ is the orbital quantum number, $m \in \mathbb{Z}$ $(-\ell \le m \le \ell)$ is the azimuthal number, and $\omega \in \mathbb{C}$ is the complex wave frequency. A straightforward calculation transforms Eq. (3) into an equation for the radial function $R_{\omega\ell}$,

$$R''_{\omega\ell}(r) + \frac{2}{r}R'_{\omega\ell}(r) + \left[\omega^2 - \frac{\ell(\ell+1)}{\alpha^2 r^2}\right]R_{\omega\ell}(r) = 0. \tag{4}$$

Note that scalar waves are only affected by the α parameter of the global monopole through the inverse square potential $V_\ell(r) = \ell(\ell+1)/(\alpha^2 r^2)$. In other words, only nonzero $(\ell \ne 0)$ angular momentum waves will perceive the angular deficit. Spherical waves $(\ell = 0)$, on the other hand, are unaffected by the parameter and will propagate as in Minkowski spacetime. Therefore, the true classical singularity at $r = 0$ will be perceived only by $\ell = 0$ waves since, for $\ell \ne 0$ waves, it becomes "invisible" due to the strong repulsive potential.

Let us now understand how the remark above translates into the necessity of a boundary condition for spherical waves. The general solution of Eq. (4) is simply

$$R_{\omega\ell}(r) = A_{\omega\ell}\frac{J_{\nu_\ell}(\omega r)}{\sqrt{\omega r}} + B_{\omega\ell}\frac{N_{\nu_\ell}(\omega r)}{\sqrt{\omega r}}, \tag{5}$$

where $A_{\omega\ell}$, $B_{\omega\ell}$, and $\nu_\ell = \frac{1}{2}\sqrt{1+\frac{4\ell(\ell+1)}{\alpha^2}}$ are constants, and $J_\nu(\omega r)$ and $N_\nu(\omega r)$ are the νth-order Bessel and Neumann functions, respectively. Note that if we restrict the frequency to be real, i.e. $\omega \in \mathbb{R}$, the function $J_{\nu_\ell}(\omega r)/\sqrt{\omega r}$ is square-integrable near the origin for all $\ell \in \mathbb{N}$:

$$\int_0^c \left|\frac{J_{\nu_\ell}(\omega r)}{\sqrt{\omega r}}\right|^2 r^2 dr < \infty, \tag{6}$$

where c is an arbitrary positive constant. On the other hand, the function $N_{\nu_\ell}(\omega r)/\sqrt{\omega r}$, with $\omega \in \mathbb{R}$, is square-integrable near the origin only for $\ell = 0$. In view of that, to avoid non-square-integrable solutions, the boundary condition $B_{\omega\ell} = 0$ naturally arises for $\ell \neq 0$ waves. For $\ell = 0$, since both solutions are square-integrable, an arbitrary boundary condition at $r = 0$ must be chosen.

It is important to remark here that, even though the wave equation is the same for the Minkowski spacetime and for the global monopole (when $\ell = 0$), there is a crucial difference between the two cases. In the first one, the origin is not a singularity of the spacetime. Consequently, the coefficient $B_{\omega 0}$ must also vanish since the Laplacian of $N_{\nu_0}(\omega r)/\sqrt{\omega r}$ is proportional to the Dirac delta function $\delta^3(r, \theta, \phi)$, which fails to be square-integrable [13]. The global monopole, however, has a singularity at the origin $r = 0$, which is not part of the manifold. As a result, $N_{\nu_0}(\omega r)/\sqrt{\omega r}$ is square-integrable and the mode $\ell = 0$ is allowed.

It is convenient to define a new radial function $G_{\omega\ell}(r) = rR_{\omega\ell}(r)$ so that, in terms of $G_{\omega\ell}$, Eq. (4) becomes

$$\frac{d^2 G_{\omega\ell}(r)}{dr^2} + [\omega^2 - V_\ell(r)]G_{\omega\ell}(r) = 0. \tag{7}$$

Another way to understand why a boundary condition is needed when $\ell \neq 0$ is that the inverse square potential $V_\ell(r)$ falls off faster than $3/4r^2$, which is a well-known requirement for having a function whichf is not square-integrable [14]. When $\ell = 0$, the repulsive potential is absent, and the equation above reads

$$\frac{d^2 G_{\omega 0}(r)}{dr^2} + \omega^2 G_{\omega 0}(r) = 0. \tag{8}$$

The most general boundary condition for $G(r)$ is the Robin mixed boundary conditions (see Refs. [7,13]),

$$G_{\omega 0}(0) + \beta G'_{\omega 0}(0) = 0, \tag{9}$$

where $\beta \in \mathbb{R} \cup \{-\infty, +\infty\}$ is an arbitrary parameter. When this boundary condition is taken into account, the solution of (8), written in terms of the parameter β, becomes

$$G^\beta_{\omega 0}(r) \sim \begin{cases} \sin(\omega r) - \beta\omega\cos(\omega r), & \text{for } \beta = \pm\infty, \\ \cos(\omega r), & \text{for } \beta \neq \pm\infty. \end{cases} \tag{10}$$

To the best of our knowledge, all previous work on scattering by the global monopole spacetime assumed a Dirichlet boundary condition ($\beta = 0$), which does not allow for the existence of bound states. For some other values of the boundary condition parameter β, however, bound states do exist. In fact, if we let the frequency ω be imaginary so that $\omega^2 = -\lambda^2 < 0$, the general solution of Eq. (8) becomes

$$G_{\lambda 0}(r) = C_{\lambda 0} e^{-\lambda r} + D_{\lambda 0} e^{\lambda r}, \tag{11}$$

where $C_{\lambda 0}$ and $D_{\lambda 0}$ are constants (without loss of generality we can assume $\lambda > 0$). Since we are looking for square-integrable solutions, we must have $D_{\lambda 0} = 0$. In such a case, the boundary condition (9) transforms into

$$C_{\lambda 0}(1 - \lambda\beta) = 0. \tag{12}$$

In order to have nontrivial solutions, the parameters must be related by $\lambda = 1/\beta$, which only makes sense when $\beta > 0$ (otherwise λ would be negative). The associated solution is then the bound state

$$R_{\text{bound}}(r) \sim \frac{e^{-r/\beta}}{r}. \tag{13}$$

When $\beta = 0$ or $\beta = \pm\infty$, it is straightforward to show that no bound states are allowed.

III. WAVE SCATTERING

In this section, we study the scattering of incident scalar waves on the global monopole. Using the method of partial waves, our main goal is to determine the differential cross section of the global monopole as a function of the boundary condition parameter β. We reemphasize that our results must be applied to the realistic case considered in grand unified theories $8\pi G\eta^2 \approx 10^{-5}$ ($\alpha \lessapprox 1$), where $R_{\theta\phi\theta\phi} \approx 10^{-5}\sin^2\theta$. However, even for such small angular defects, the equatorial plane corresponding to a cone with a very small deficit angle. Scattering in conical spacetimes was discussed in Refs. [15,16]. In [15] the authors showed that even though the partial wave analysis in conical spacetimes leads to divergences, it is possible to redefine the incident wave in order to absorb the divergent terms. In [16], the authors showed that the same procedure can be done when an arbitrary boundary condition is chosen at the origin. Since the global monopole is plagued with the same problem (the spacetime is not asymptotically Minkowski), we also expect divergences in the partial wave analysis of our work. As we will see, these divergences can be handled by smoothing the singularities of the scattering amplitude [see Eq. (22) below]. Despite the fact that the spacetime is topologically nontrivial, we are able to analyze, at least qualitatively, the scattering of waves satisfying different boundary conditions.

To accomplish that, we consider an incident wave $\Psi_{\text{in}} = e^{ikz}e^{-i\omega t}$, with wave number $k = \omega$, propagating along the z axis. It is convenient to expand it into spherical waves using the standard plane wave decomposition,

$$e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell j_\ell(\omega r) P_\ell(\cos\theta), \tag{14}$$

where $j_\ell(\omega r)$ is the ℓth-order spherical Bessel function and $P_\ell(\cos\theta)$ is the ℓth-order Legendre polynomial.

This incident plane wave is scattered by the global monopole, so that the total wave can be written as

$$\Psi = \Psi_{\text{in}} + \Psi_{sc}, \tag{15}$$

where Ψ_{sc} corresponds to the scattered part. Far away from the singularity (as $r \to \infty$), this scattered part is an outgoing wave of the form

$$\Psi_{sc} = \frac{f(\theta)}{r} e^{ikr}. \tag{16}$$

Similarly, the large-r behavior of the incident part can be easily determined from Eq. (14) with the help of the asymptotic expression for the spherical bessel function.

To determine the scattering amplitude $f(\theta)$, we need the asymptotic behavior of the solutions we found in the previous section. From Eqs. (5) and (10), we find that the general radial solution of the wave equation for an arbitrary parameter β is given by

$$R_{\omega\ell}(r) \sim \begin{cases} \frac{J_{1/2}(\omega r)}{\sqrt{\omega r}} + \beta\omega \frac{N_{1/2}(\omega r)}{\sqrt{\omega r}}, & \text{for } \ell = 0, \\ \frac{J_{\nu_\ell}(\omega r)}{\sqrt{\omega r}}, & \text{for } \ell \neq 0. \end{cases} \tag{17}$$

Therefore, the full solution (15), when decomposed into the mode solutions, becomes

$$\Psi(t,r,\theta,\phi) = a_{00}\left(\frac{J_{1/2}(\omega r)}{\sqrt{\omega r}} + \beta\omega\frac{N_{1/2}(\omega r)}{\sqrt{\omega r}}\right)e^{-i\omega t} + \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\frac{J_{\nu_\ell}(\omega r)}{\sqrt{\omega r}} P_\ell(\cos\theta) e^{im\phi} e^{-i\omega t}, \tag{18}$$

where $a_{\ell m}$ are constants to be determined.

Due to the spherical symmetry, all $m \neq 0$ modes are irrelevant for the scattering, so that $a_{\ell m} = 0$ for them. By comparing the asymptotic behavior of solution (18) with the asymptotic behavior of (15), we can determine the coefficients $a_{\ell 0}$ to be

$$a_{\ell 0} = \begin{cases} \sqrt{\frac{\pi}{2}}\frac{i}{i-\beta\omega}, & \text{for } \ell = 0, \\ \sqrt{\frac{\pi}{2}}(2\ell+1) i^\ell e^{i\delta_\ell}, & \text{for } \ell \neq 0, \end{cases} \tag{19}$$

where the phase shifts are given by $\delta_\ell = \frac{\pi}{2}(\ell + \frac{1}{2} - \nu_\ell)$. Similarly, the scattering amplitude $f(\theta)$ can be written as

$$f(\theta) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{2i\omega} P_\ell(\cos\theta), \tag{20}$$

where

$$b_\ell = \begin{cases} \frac{2\beta\omega}{i-\beta\omega}, & \text{for } \ell = 0, \\ (2\ell+1)(e^{2i\delta_\ell} - 1), & \text{for } \ell \neq 0. \end{cases} \tag{21}$$

Now, we would like to use the expression above for the scattering amplitude to calculate the differential cross section $d\sigma/d\Omega = |f(\theta)|^2$. However, as in Refs. [8,9], the infinite sum in (20) is, depending on the angle θ, either poorly convergent or divergent [9,17,18]. While nothing can be done for divergent sums, slow convergence can be dealt with the method described below [18].

The first step is to multiply the scattering amplitude by $(1-\cos\theta)^n$, where $n \in \mathbb{N}$, and expand the obtained function in terms of the Legendre polynomials,

$$(1-\cos\theta)^n f(\theta) = \frac{1}{2i\omega}\sum_{\ell=0}^{\infty} b_\ell^{(n)} P_\ell(\cos\theta), \tag{22}$$

where $b_\ell^{(n)}$ are constant coefficients. By resorting to Bonnet's recursion formula for the Legendre polynomials, it is possible to show that the new coefficients are related to the old ones through $b_\ell^{(n)} = b_\ell$, if $n = 0$, and through the recursive relation

$$b_\ell^{(n)} = b_\ell^{(n-1)} - \frac{\ell+1}{2\ell+3} b_{\ell+1}^{(n-1)} - \frac{\ell}{2\ell-1} b_{\ell-1}^{(n-1)}, \tag{23}$$

if $n \geq 1$. In the end, the scattering amplitude can be written as a so-called reduced series,

$$f(\theta) = \frac{1}{2i\omega}\sum_{\ell=0}^{\infty} b_\ell^{(n)} \frac{P_\ell(\cos\theta)}{(1-\cos\theta)^n}, \tag{24}$$

which converges faster than the series appearing in Eq. (20).

The last step consists in the numerical implementation of the of the recursive relation (23), followed by the calculation of the differential cross section. Using Mathematica, we were able to show that taking $n = 6$ is enough to guarantee enough precision when computing the partial sum of the first few terms of the reduced series for $\pi/4 \leq \theta \leq \pi$, $\omega = \beta = 1$. This precision does not seem to change much when different values of β and ω are used. To understand the effect of the boundary conditions on the scattering, we choose $\omega = 1$ and plot in Fig. 1 the differential cross section as a function of the scattering angle θ for different boundary condition parameters β. Even though we use an exaggerated α parameter ($\alpha = 0.95$) for better visualization of the effects of the boundary condition, the qualitative behavior is the same for the more realistic value $\alpha \approx 1 - 10^{-5}$.

The most evident effect of the boundary condition appears to be on the backscattered part ($\pi/2 \leq \theta \leq \pi$) of the wave. The dirichlet boundary condition, which is usually considered in the literature, produces no backscattering, while the Neumann boundary condition produces the most. We have tested different values of the

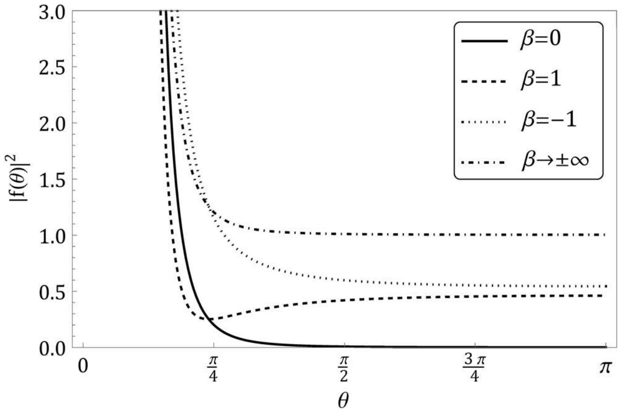

FIG. 1. Plot of the reduced series of the differential cross section for $\omega = 1$, $\alpha = 0.95$, and several values of β.

frequency ω and different values of the parameter α, and this behavior seems to be universal. As seen in Fig. 1, the amount of backscattering is uniquely related to the chosen boundary condition. By measuring the amplitude of the backscattered wave, one is able in principle to determine which boundary condition has been specified by nature.

IV. STABILITY

Everything we did so far would be less relevant if the spacetime happens to be highly unstable. In view of that, our final task is to analyze how the stability of the system depends on the boundary condition. To do so, we recall that global monopoles allow for the existence of bound states only when $\beta > 0$. For $\beta \leq 0$ and $\beta = \pm\infty$, only scattering modes $\omega^2 > 0$ are allowed and, therefore, the system is mode stable. If $\beta > 0$, on the other hand, the $\ell = 0$ case admits a bound state of the form (13), so that an arbitrary solution of the Klein-Gordon equation has to include not only the scattering states but also these bound states.

More precisely, the most general scalar wave can be decomposed as

$$\begin{aligned} &\Psi_\beta(t, r, \theta, \varphi) \\ &= A\frac{e^{-r/\beta}}{r} e^{-t/\beta} + B\frac{e^{-r/\beta}}{r} e^{t/\beta} \\ &\quad + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} C_{\omega\ell m} Y_\ell^m(\theta, \varphi) R_{\omega\ell}^\beta(r), \end{aligned} \tag{25}$$

where A, B, and $C_{\omega\ell m}$ are constants. The first term in the expansion above decays exponentially in time, becoming irrelevant after a sufficient long time (of order β). The second term of (25), however, grows exponentially in time. Nonetheless it still corresponds to a square-integrable solution because, for a fixed time t, the integral

$$\int_0^\infty \left| \frac{e^{-(r-t)/\beta}}{r} \right|^2 r^2 dr = \frac{\beta}{2} e^{2t/\beta} \tag{26}$$

is finite. Since this mode represents a growing perturbation, after a sufficient long time (of order β), nonlinear effects will become important. While these effects may be able to control the exponential growth and restore the stability of the system, a full nonlinear treatment of the Einstein-Klein-Gordon equations would be required to investigate that. For now, what we can say is that test scalar fields on the global monopole are mode unstable for $\beta > 0$.

V. FINAL REMARKS

We have seen that the propagation of scalar waves around a global monopole is not determined until a boundary condition at the origin is prescribed. This characterizes the global monopole as a quantum mechanically singular spacetime. The propagation of waves is as uncertain as the evolution of point particles reaching the classical singularity at $r = 0$. Assuming nature has a preferred physical evolution scheme, this could be, as we discuss, inferred phenomenologically, allowing us to identify the boundary condition, for theory by itself is unable to predict it.

It is important to mention that the necessity of a boundary condition is due to the idealization of the global monopole's core. If we do not neglect its finite size, the boundary condition can, in principle, be related to the way the core is modelled (see, for instance, Ref. [19]). Thus, perhaps a more physical and less mathematical way to interpret the main results of our analysis is that the differential cross section, instead of being related to the boundary condition chosen by nature, is related to the specific details of the monopole's core.

In more detail, our analysis shows that the scattering amplitude and the differential cross section encode the arbitrariness of the boundary condition. In particular, the amount of backscattering is intimately related to the boundary condition parameter β. Consequently, in principle, by measuring the amplitude of a wave which is scattered by the global monopole (specially the backscattered part), one could determine the boundary condition in a given experiment (and, according to the reasoning above, extract information about the monopole core).

Another important question we address in this paper concerns the stability of the global monopole. As we have shown, its stability under scalar perturbations depend on the boundary condition parameter. For $\beta > 0$, the spacetime is unstable while for $\beta \leq 0$ and $\beta = \pm\infty$, the spacetime is mode stable under perturbations of a test scalar field. Note, however, that the final word on the stability of the global monopole background requires a fully nonlinear treatment of the problem. We also remark that the scalar test field under consideration here is not the same as the scalar fields ϕ^a that determine the global monopole background through (1). The analysis developed in Ref. [11] involving perturbations of such fields has shown no instabilities.

Finally, the ideas presented here can, in theory, be extended to other naked singularity spacetimes, like the cosmic string, the negative-mass Schwarzschild, the overcharged Reissner-Nordström, and the overspinning Kerr spacetimes. The major technical difficulty in these cases, however, is to determine the self-adjoint extensions for the spatial part of the wave operator.

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ANNEX C – Article 3

Vacuum fluctuations and boundary conditions in a global monopole[37]

Vacuum fluctuations and boundary conditions in a global monopole

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We study the vacuum fluctuations of a massless scalar field $\hat{\Psi}$ on the background of a global monopole. Due to the nontrivial topology of the global monopole spacetime characterized by a solid deficit angle parametrized by η^2, we expect that $\langle\hat{\Psi}^2\rangle_{\text{ren}}$ and $\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}}$ are nonzero and proportional to η^2, so that they annul in the Minkowski limit $\eta \to 0$. However, due to the naked singularity at the monopole core, the evolution of the scalar field is not unique. In fact, they are in one-to-one correspondence with the boundary conditions which turn into self-adjoint the spatial part of the wave operator. We show that only the Dirichlet boundary condition corresponds to our expectations and gives zero contribution to the vacuum fluctuations in the Minkowski limit. All other boundary conditions give nonzero contributions in this limit due to the nontrivial interaction between the field and the singularity.

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I. INTRODUCTION

Grand unified theories (GUTs) predict spontaneous symmetry breaking of matter fields during phase transitions in the early Universe. As a result, topological defects might appear in the spacetime manifold, namely cosmic strings, domains walls and global monopoles [1]. Global monopoles arise when a global $O(3)$ symmetry of a triplet scalar field ϕ^a ($a = 1, 2, 3$) is spontaneously broken to $U(1)$ in the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - \eta_0^2)^2. \tag{1}$$

The fields assume a "hedgehog" configuration,

$$\phi^a = \eta_0 f(r)\frac{x^a}{r}, \tag{2}$$

for $x^a x^a = r^2$, with $f(r)$ vanishing as r approaches 0 and tending to 1 for r much bigger than a typical value $r_c \sim (\sqrt{\lambda}\eta_0)^{-1}$. For $r \gg r_c$, the only nonzero components of the energy-momentum tensor are

$$T_t{}^t = T_r{}^r = \eta_0^2/r^2, \tag{3}$$

which leads to a linearly divergent energy.

Furthermore, for typical GUT scales ($\eta_0 \sim 10^{16}$ GeV), the energy density is extremely high; thus one might expect gravitational effects around the monopole. Accordingly, Einstein's and field equations for a spherically symmetric solution yield the global monopole spacetime metric given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_2, \tag{4}$$

with Schwarzschild-like coefficients [2]

$$B(r) = A^{-1}(r) = 1 - \eta^2 - \frac{2GM_c}{r}, \tag{5}$$

where $\eta \equiv \sqrt{8\pi G}\eta_0$ and $M_c \approx -6\pi\eta_0\lambda^{-1/2}$ [3].

We can see from the metric described by (4) that the global monopole spacetime can be treated as a highly massive core centered at the origin with characteristic radius r_c, plus a spherically symmetric spacetime with deficit solid angle. Far from the core, the metric (4) can be approximated by

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2}dr^2 + r^2 d\Omega_2, \tag{6}$$

where $\alpha^2 \equiv 1 - \eta^2$ and $0 < \alpha < 1$. The spacetime described by (6) is curved and has scalar curvature $\mathcal{R}(x) = 2\eta^2/r^2$. Physically acceptable values for η^2 lie around the GUT scale, which predicts $\eta^2 \sim 10^{-5}$, thus, a very small curvature. Despite the Schwarzschild-like coefficients in the metric element (4), there is no event horizon since the monopole's

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mass M_C is negative. Not only the metric coefficient $B(r)$ diverges at $r = 0$ but also other geometrical quantities, e.g., the scalar curvature (which scales as r^{-2}). Therefore, another relevant feature of the global monopole spacetime is the presence of a naked singularity at $r = 0$.

We expect that fluctuations will appear on the expectation values of quantum fields due to the curvature of the spacetime. Indeed, Mazzitelli and Lousto found them in [4], where they obtained that the vacuum expectation values of the field squared $\langle\hat{\Psi}^2\rangle$ and the energy-momentum tensor $\langle\hat{T}_{\mu\nu}\rangle$ both fluctuate in order η^2. However, care must be taken when studying semiclassical effects in spacetimes having naked singularities, which is the case for the global monopole. In such spacetimes, the dynamics of the fields is not uniquely defined by initial conditions, and this represents a serious difficulty in the field quantization. In a practical sense, one may not be able to solve the Klein-Gordon equation,

$$(-\Box_x + M^2 + \xi\mathcal{R}(x))\Psi = 0, \tag{7}$$

and find a complete set of positive energy eigenfunctions defined through the whole history of the spacetime, which jeopardizes any attempt of quantizing the field operator at all times in terms of positive energy modes and annihilation/creation operators.

However, Wald [5] and Ishibashi [6] proposed a sensible dynamical description of the field by means of the imposition of boundary conditions at the spacetime singularity. This procedure allows us to consistently quantize scalar fields in the global monopole spacetimes, which, as already discussed, has a naked singularity. In this paper we will study quantum fields on the idealized global monopole spacetime given by the metric (6). The boundary conditions will model, somehow, the interaction of the field with the monopole core, which will be responsible for the appearance of interesting physical effects.

We organized this article as follows. In Sec. II, we will show how to consistently prescribe the dynamics of massless scalar fields in spacetimes with naked singularities. In particular, we intend to briefly discuss the procedure behind it and how the prescription of boundary conditions at the spacetime singularity will help us on our goal. Considering the appropriate boundary conditions, we will find a complete set of eigenfunctions for the Klein-Gordon operator in two cases of coupling with the curvature, namely minimal and conformal. Our choice for the coupling constant was merely for simplicity. As will be discussed later, the results found here remain valid within a certain range of positive values of the constant. For those outside this range, a more detailed discussion is required, and we leave it for future studies. With the eigenfunctions in hand, we will find the Euclidean Green's function and, from it, the quadratic vacuum expectation value of the field in Sec. III. We point out the contributions coming strictly from the boundary conditions, and we found its effects on the 00th component of the energy-momentum tensor in Sec. IV. Section V is devoted to our final remarks.

II. KLEIN-GORDON EQUATION AND NAKED SINGULARITIES

As first proposed by Penrose in 1969 [7], gravitational collapse may always produce "covered" singularities; i.e., no naked singularities are allowed in nature. This statement became known as cosmic censorship conjecture and was extensively developed and improved (see [8]). Whether naked singularities exist or not is still an open question, but one might ask if quantum effects in general relativity could sustain the conjecture, or even provide some kind of "cosmic censor." In the absence of a well-established theory of quantum gravity, one may refer to the foundations of semiclassical gravity in order to shed some light on those topics [9]. On the other hand, in spacetimes where naked singularities exist, it might not always be possible to consistently describe the evolution of quantum fields, which would inhibit any semiclassical approach. This pitfall may be overcome by the prescription of boundary conditions for the equation of motion at the spacetime singularity, i.e., imposing "by hand" the interaction between the fields and the classical singularity.

A. Nonglobally hyperbolic static spacetimes

In globally hyperbolic spacetimes, the dynamics arising from the Cauchy problem for well-posed initial conditions is uniquely defined at all times. Conversely, this is not true in nonglobally hyperbolic spacetimes, such as those containing a naked singularity. Classical test particles following geodesics may have their trajectory interrupted at the singularity, and the future of these particles may be compromised. Thus, one may find difficulties in studying the propagation of scalar fields in the background of nonglobally hyperbolic spacetimes.

If the spacetime is also static, it can be shown that Eq. (7) reduces to

$$\frac{\partial^2\Psi}{\partial t^2} = -A\Psi, \tag{8}$$

where $A \coloneqq -VD^i(VD_i\phi) + M^2V^2 + \xi RV^2$ and $V^2 = -\chi_\mu\chi^\mu$, where χ^μ are the timelike Killing fields of the spacetime. Wald argues in [5] that A can be seen as a strictly spatial differential operator acting on a Hilbert space defined over a static spatial slice Σ. Our ignorance on what happens at the singularity may be solved if we consider the domain of A to be $C_0^\infty(\Sigma)$. But even though the operator $[A, C_0^\infty(\Sigma)]$ is symmetric, it might not be self-adjoint. It can be shown, however, that it might admit a unique self-adjoint extension A_E or an infinite set of them [10].

As shown in [5], given initial conditions $(\Psi_0, \dot{\Psi}_0) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$, the solution to Eq. (8) for any $t \in \mathbb{R}$ is

$$\Psi_t = \cos(A_E^{1/2} t)\Psi_0 + A_E^{-1/2} \sin(A_E^{1/2} t)\dot{\Psi}_0. \tag{9}$$

The dynamics associated to Eq. (9) is unique if A is essentially self-adjoint, i.e., only one extension exists (Friedrichs extension). Nonequivalent dynamics arise from Eq. (9) when the operator has infinitely many self-adjoint extensions A_E. To each extension, there can be associated a boundary condition at the classical singularity labeled by the parameter E. Nevertheless, Ishibashi and Wald showed in [6] that it is possible to construct a physically sensible evolution of the fields in nonglobally hyperbolic spacetimes using Eq. (9). We will discuss in Sec. II B how these considerations can be adopted in the case of the global monopole spacetime.

B. Boundary conditions and solutions to the Klein-Gordon equation

In the process of extending the domain of the operator A to make it self-adjoint, one will find out the need of prescribing boundary conditions at the spacetime singularity, corresponding to each extension found. Indeed, many authors have considered these boundary conditions on the study of quantum fields propagating in nonglobally hyperbolic spacetimes (see [11–14]). In [15], the authors treated the particular case of the global monopole using Robin boundary conditions found in [16]. We calculated the scattering pattern of massless scalar waves and, as expected, the parameter of the boundary conditions endured throughout the whole process. Direct effects appeared in the differential cross section so that one could predict which boundary condition is the one chosen by nature comparing it to experimental scattering data, if available. The parameter also influenced the stability of the spacetime, since for some particular values of it there exist bound states with divergent growth in time.

In [15], we only studied minimally coupled massless scalar fields ($\xi = 0$) in the global monopole. Now, we intend to extend the analysis to conformally coupled massless scalar fields ($\xi = 1/6$) as well. To consistently do it, we must analyze the Klein-Gordon equation to find the appropriate boundary conditions. Equation (7) can be solved in spherical coordinates under separation of variables, i.e., $\Psi(x) = e^{-i\omega t} Y_l^m(\theta, \varphi) R(r)$, and it reduces to

$$-\alpha^2\left(-\frac{d^2}{dr^2} + \frac{\nu_\ell^2 - 1/4}{r^2}\right)u(r) = -\frac{\omega^2}{\alpha^2}u(r), \tag{10}$$

for $u(r) = rR(r)$ and $\nu_\ell = \frac{1}{2}\sqrt{1 + \frac{4\ell(\ell+1)}{\alpha^2} + 8\xi\frac{\eta^2}{\alpha^2}}$. Separation of variables simplifies the three-dimensional spatial differential operator to one depending on the radial coordinate. Thus, our problem reduces to an analog study of the well-known Calogero problem on the semiaxis in quantum mechanics [17–19]. In [20], the authors study in detail the self-adjoint extensions of the Calogero operator $[A_C = -d^2/dr^2 + ar^{-2}$ in $\mathcal{L}^2(\mathbb{R}^+)]$ and develop its spectral analysis. They discuss the conditions on a so that A_C is self-adjoint or not. In our case, we identify $a \equiv \nu_\ell^2 - 1/4$ and two of the cases treated in [20] will be relevant, namely $a \geq 3/4$ and $-1/4 < a < 3/4$. We will briefly discuss each of these cases and then apply them for minimal and conformal fields to find the solutions to Eq. (10).

(a) For $a \geq 3/4$ the operator A_C defined over the domain $C_0^\infty(\mathbb{R}^+)$ is essentially self-adjoint, i.e., $A_C = A_C^\dagger$ and $\mathcal{D}(A_C) = \mathcal{D}(A_C^\dagger)$, and no boundary conditions are necessary. This condition requires $\nu_\ell^2 \geq 1$, which implies in $\ell \geq 1$ for both cases: $\xi = 0$ and $\xi = 1/6$. Thus, all nonspherically symmetric modes ($\ell \neq 0$) will interact trivially with the singularity at $r = 0$. The eigenfunctions of the operator A_C with eigenvalues $p^2 > 0$ will then be

$$u_{\ell,p}(r) = \sqrt{\frac{pr}{2}}J_{\nu_\ell}(pr), \tag{11}$$

where J_ν are Bessel functions of order ν.

(b) For $-1/4 < a < 3/4$ the operator A_C defined over the domain $C_0^\infty(\mathbb{R}^+)$ is not self-adjoint but it admits a one-parameter $U(1)$ family of self-adjoint extensions $A_{C\beta}$ labeled by a real parameter β. This case requires the prescription of boundary conditions at $r = 0$ and that $0 < \nu_\ell^2 < 1$. Minimally coupled fields, as well as conformally coupled ones, will only feel the effects of the boundary conditions through their spherically symmetric modes (since $0 < \nu_\ell^2 < 1 \Rightarrow \ell = 0$, for $\xi = 0$ and $\xi = 1/6$). This conclusion endures as long as ξ is positive and much smaller than $1/\eta^2$. For some negative values of ξ, the inequality may not hold only for $\ell = 0$; thus other modes ($\ell = 1, 2, \ldots$) might need boundary conditions as well. The solutions must satisfy the following asymptotic boundary conditions near $r = 0$:

$$u_\beta(r) \sim \begin{cases} \mathcal{N}[r^{1/2+\nu_0} + \beta r^{1/2-\nu_0}], & |\beta| < \infty; \\ \mathcal{N}[r^{1/2-\nu_0}], & |\beta| = \infty. \end{cases} \tag{12}$$

The eigenvalue p^2 will only be positive for $\beta \geq 0$, and the associated eigenfunctions will be

$$u_{0,p,\beta}(r) = \sqrt{\frac{pr}{2}}\frac{J_{\nu_0}(pr) + \gamma(\beta,p)J_{-\nu_0}(pr)}{\sqrt{1 + 2\cos(\pi\nu_0)\gamma(\beta,p) + \gamma(\beta,p)^2}}, \tag{13}$$

where[1]

$$\gamma(\beta,p)=\beta\frac{\Gamma(1-\nu_0)}{\Gamma(1+\nu_0)}\left(\frac{p}{2p_0}\right)^{2\nu_0}. \quad (14)$$

Bound states appear as one considers negatives values of β. We are interested in the continuous spectrum of the fields; hence only non-negative values of β will be considered on what follows.[2] In the case $\xi=0$, conditions (12) are equivalent to Robin boundary conditions and, in accordance with [16], the eigenfunction reduces to

$$u^{\xi=0}_{0,p,\beta}(r)=\sqrt{\frac{pr}{2}}\frac{J_{1/2}(pr)+\beta pJ_{-1/2}(pr)}{\sqrt{1+(\beta p)^2}}. \quad (15)$$

For $\xi=1/6$, it suffices to apply $\nu_0=\frac{1}{2}\sqrt{1+\frac{4\eta^2}{3\alpha^2}}$ on (13).

Upon the discussion done so far, we can establish a complete set of positive energy eigenfunctions $f^{\xi}_{\lambda,\ell,m}(x)$ to the Klein-Gordon operator $(-\Box+\xi\mathcal{R})$ with eigenvalues $\lambda^2=-\omega^2/\alpha^2+p^2\alpha^2$, namely

$$f^{\xi}_{\lambda,\ell,m}(x)=\frac{e^{-i\omega t}}{\sqrt{2\pi r}}\begin{cases}\frac{1}{\sqrt{4\pi}}u^{\xi}_{0,p,\beta}(r) & \text{for } \ell=0;\\ Y^m_l(\theta,\varphi)u^{\xi}_{\ell,p}(r) & \text{for } \ell>0.\end{cases} \quad (16)$$

A completeness relation is naturally available to them, i.e.,

$$\frac{\delta^4(x,x')}{\sqrt{-g}}=\int_\lambda\sum_{\ell,m}f_{\lambda,\ell,m}(x)f^*_{\lambda,\ell,m}(x'). \quad (17)$$

With these in hand, we can follow the procedure of quantization of fields. The scalar field Ψ will be promoted to an operator $\hat{\Psi}$ defined as an expansion in terms of annihilation $[a(p)]$ and creation $[a^\dagger(p)]$ operators satisfying the canonical commutation relations. These operators will be weighted by the positive energy modes of the Klein-Gordon equation. In the next section, we will find the vacuum expectation value of some quantities depending on the field operator.

III. GREEN'S FUNCTIONS AND $\langle\hat{\Psi}^2\rangle$

Green's functions are fundamental to the computations of many quantities in quantum field theory, such as scattering cross section, decay rates, and the vacuum expectation value (VEV) of the energy-momentum tensor. The latter is of particular interest for us, since it plays a relevant role in general relativity and, notably, gives rise to semiclassical effects in gravity. As already indicated by QFT in flat spacetime results, computations involving quadratic mean values of the field operator are expected to have divergences all over the way. In curved spacetimes, the situation is no different, and it can be even worse. That is why renormalization procedures are necessary to obtain quantities such as the VEV of the energy-momentum tensor. In particular, a widely applied method is the point-splitting renormalization scheme (see [9]). In [4], the authors use this technique to obtain the renormalized quadratic mean value $\langle\hat{\Psi}^2\rangle$. To do so, they take a coincidence limit on the positions of the renormalized Green's functions as follows:

$$\langle\hat{\Psi}^2(x)\rangle=\frac{1}{2}G^{(1)}(x,x)=iG_F(x,x)=G_E(x,x), \quad (18)$$

where $G(x,x)$ indicates a formal limit $x'\to x$ on $G(x,x')$.

For convenience, Mazzitelli and Lousto follow the computation using the Euclidean Green's function, $G_E(x,x')$. For that they use the Euclideanized metric of the global monopole, which is

$$ds^2=\alpha^2d\tau^2+\alpha^{-2}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \quad (19)$$

The Klein-Gordon operator will now have eigenvalues $\lambda^2=\omega^2/\alpha^2+p^2\alpha^2$ associated with the same eigenfunctions $f^{\xi}_{\lambda,\ell,m}(x_E)$ (under the interchange $t\to\tau$).

To evaluate the Green's function, we can use Schwinger's integral representation

$$G_E(x,x')=\int_0^\infty ds\exp[-s(-\Box+\xi R)]\frac{\delta^4(x,x')}{\sqrt{-g}}, \quad (20)$$

which, using Eq. (17), can be expressed in terms of the eigenfunctions, as follows:

$$G_E(x,x')=\int_0^\infty ds\int_{-\infty}^{+\infty}d\omega\int_0^\infty d\,pe^{-s\lambda^2}\times\sum_{\ell,m}f_{\lambda,\ell,m}(x)f^*_{\lambda,\ell,m}(x'). \quad (21)$$

Because the boundary conditions we considered affect only spherically symmetric modes, we expect that, except by the term $\ell=0$ in the sum, the Green's function remains the same as the one found in [4]. Thus, it seems reasonable to separate it into two parts, one containing contributions from the boundary condition parameter (G^β_E), and the other equal to the one found by Mazzitelli and Lousto (G^{ML}_E). We then have

$$G_E(x,x')=G^\beta_E(x,x')+G^{ML}_E(x,x'), \quad (22)$$

where

[1]The momentum parameter p_0 was introduced to make $\gamma(\beta,p)$ dimensionless. From now on, we will set it to 1.

[2]We are also interested in the stable case, so that $\beta\geq 0$ (see [15]).

$$G_E^{ML}(x,x') = \int_0^\infty ds \int_{-\infty}^{+\infty} d\omega \int_0^\infty dp p e^{-s\lambda^2} \frac{e^{-i\omega(\tau-\tau')}}{2\pi\sqrt{rr'}} \sum_{\ell,m} Y_\ell^m(\theta,\varphi) Y_\ell^{m*}(\theta',\varphi') J_{\nu_\ell}(pr) J_{\nu_\ell}(pr'), \tag{23}$$

and[3]

$$G_E^\beta(x,x') = \int_0^\infty ds \int_{-\infty}^{+\infty} d\omega \int_0^\infty dp p \frac{e^{-i\omega(\tau-\tau')}}{2\pi\sqrt{rr'}} \frac{e^{-s\lambda^2}}{4\pi} \times \left[\frac{(J_{\nu_0}(pr) + \gamma(\beta,p) J_{-\nu_0}(pr))(J_{\nu_0}(pr') + \gamma(\beta,p) J_{-\nu_0}(pr'))}{1 + 2\gamma(\beta,p)\cos\pi\nu_0 + \gamma^2(\beta,p)} - J_{\nu_0}(pr) J_{\nu_0}(pr') \right]. \tag{24}$$

It is easy to see that the contribution $G_E^\beta(x,x')$ vanishes as β goes to zero, and our Green's function recovers identically the one from [4]. We will then need to compute only $G_E^\beta(x,x')$ in both cases of the coupling constant. Integration over ω and s can be directly performed to give

$$G_E^\beta(x,x') = \int_0^\infty dp \frac{e^{-\alpha^2 p(\tau-\tau')}}{8\pi\sqrt{rr'}} \left[\frac{(J_{\nu_0}(pr) + \gamma(\beta,p) J_{-\nu_0}(pr))(J_{\nu_0}(pr') + \gamma(\beta,p) J_{-\nu_0}(pr'))}{1 + 2\gamma(\beta,p)\cos\pi\nu_0 + \gamma^2(\beta,p)} - J_{\nu_0}(pr) J_{\nu_0}(pr') \right]. \tag{25}$$

We are interested in the effects appearing as a consequence of the spacetime curvature, which, as discussed previously, scales with a factor $\eta^2 \ll 1$. As found in [4], the fluctuations begin to appear only in first order of η^2 or higher. In comparison, we will expand our Green's function in powers of η^2 as follows:

$$G_E^\beta(x,x') = G_{E,0}^\beta(x,x') + \eta^2 G_{E,2}^\beta(x,x') + \mathcal{O}(\eta^4), \tag{26}$$

so we will be able to separate which contributions appear in each order.

A. Minimally coupled field

For $\xi = 0$, we have $\nu_0 = 1/2$ and $\gamma(\beta,p) = \beta p$, and after integrating the expression above in p, we get to the following result,

$$G_E^\beta(\tau,\tau',r,r') = -\frac{i}{16\pi^2 rr'} \times \{ e^{\tilde{\aleph}_\beta}[-2i\mathrm{Ci}(-i\tilde{\aleph}_\beta) - 2\mathrm{Si}(-i\tilde{\aleph}_\beta) + \pi] - e^{\tilde{\aleph}_\beta^*}[2i\mathrm{Ci}(i\tilde{\aleph}_\beta^*) - 2\mathrm{Si}(i\tilde{\aleph}_\beta^*) + \pi] \}, \tag{27}$$

where

$$\tilde{\aleph}_\beta \equiv \tilde{\aleph}_\beta(\tau,\tau',r,r') = \frac{(r+r') + i(1-\eta^2)(\tau-\tau')}{\beta}, \tag{28}$$

and Ci (Si) is the cosine (sine) integral function. The expansion in powers of η^2 yields

$$G_{E,0}^\beta(\tau,\tau',r,r') = -\frac{i}{16\pi^2 rr'} \times \{ e^{\aleph_\beta}[2i\mathrm{Ci}(i\aleph_\beta) + 2\mathrm{Si}(-i\aleph_\beta) + \pi] - e^{\aleph_\beta^*}[2i\mathrm{Ci}(i\aleph_\beta^*) - 2\mathrm{Si}(i\aleph_\beta^*) + \pi] \}, \tag{29}$$

and

$$G_{E,2}^\beta(\tau,\tau',r,r') = -\frac{(\tau-\tau')}{16\pi^2 rr'\beta} \times \left\{ e^{\aleph_\beta}[2i\mathrm{Ci}(i\aleph_\beta) + 2\mathrm{Si}(i\aleph_\beta) + \pi] - e^{\aleph_\beta^*}[2i\mathrm{Ci}(i\aleph_\beta^*) - 2\mathrm{Si}(i\aleph_\beta^*) + \pi] - 2i\frac{\aleph_\beta - \aleph_\beta^*}{\aleph_\beta \aleph_\beta^*} \right\}, \tag{30}$$

where

$$\aleph_\beta \equiv \aleph_\beta(\tau,\tau',r,r') = \frac{(r+r') + i(\tau-\tau')}{\beta}. \tag{31}$$

Taking the coincidence limit of the coordinates on $G_E^\beta(x,x')$ we have the contribution to the quadratic mean value of the field, which added to the one found in [4] returns the total value up to first order in η^2, i.e.,

$$\langle\hat{\Psi}^2\rangle^{\xi=0} \equiv \langle\hat{\Psi}^2\rangle^{ML} + \langle\hat{\Psi}^2\rangle^\beta = -\frac{1}{4\pi^2 r^2} \left\{ \eta^2 \left[\frac{p}{2\sqrt{2}} - \frac{1}{6}\log\mu r \right] + e^{\frac{2r}{\beta}} \mathrm{Ei}\left(-\frac{2r}{\beta} \right) \right\}, \tag{32}$$

[3]The last term subtracted in Eq. (24) was added to Eq. (23) in order to complete the sum in ℓ from 0 to ∞.

where $p = -0.39$, μ is a mass scale introduced by Mazzitelli and Lousto in the renormalization procedure, and Ei is the exponential integral function. What is interesting about the mean value (32) is the appearance of a contribution in zeroth order of η^2 due exclusively to the boundary conditions at the singularity. In fact, it is direct to check that the new term we obtained vanishes as we take the limit $\beta \to 0$. Moreover, the first order term of our Green's function $G^{\beta}_{E,2}$ vanishes in the coincidence limit, so no contributions from the boundary conditions on the quadratic mean value can appear in order η^2.

B. Conformally coupled field

In the case $\xi = 1/6$, we must take $\nu_0 = \frac{1}{2}\sqrt{1 + \frac{4\eta^2}{3\alpha^2}}$ and apply it in Eq. (25). Integration, however, becomes impractical to be done analytically; hence we will directly compute the series coefficients in (26). After expanding the argument inside the integral in Eq. (25) in powers of η^2 and evaluating the integral to the first coefficient, we have

$$\begin{aligned} G^{\beta}_{E,0}&(\tau,\tau',r,r') \\ &= -\frac{i}{16\pi^2 rr'} \times \{e^{\aleph_\beta}[-2i\mathrm{Ci}(-i\aleph_\beta) + 2i\mathrm{Shi}(\aleph_\beta) + \pi] \\ &\quad - e^{\aleph^*_\beta}[2i\mathrm{Ci}(i\aleph^*_\beta) - 2i\mathrm{Shi}(i\aleph^*_\beta) + \pi]\}, \end{aligned} \tag{33}$$

where Shi is the hyperbolic sine integral function. We were not able to evaluate analytically the first order term $G^{\beta}_{E,2}$.

Again the coincidence limit of the coordinates can be taken to obtain the quadratic mean value of the field, which to the lowest order of η^2 will be

$$\langle\hat{\Psi}^2\rangle^{\xi=1/6} = -\frac{1}{4\pi^2 r^2} e^{\frac{2r}{\beta}}\mathrm{Ei}\left(-\frac{2r}{\beta}\right) + \mathcal{O}(\eta^2). \tag{34}$$

Like in the minimally coupled case, a contribution appears in order $\mathcal{O}(\eta^0)$. In particular, $\langle\hat{\Psi}^2\rangle^{\xi=1/6}$ and $\langle\hat{\Psi}^2\rangle^{\xi=0}$ are identical in the lowest order of η^2, which indicates a fluctuation independent of the curvature. Furthermore, for all values of the coupling constant within the range $0 \le \xi \ll 1/\eta^2$, the contribution in order $\mathcal{O}(\eta^0)$ remains the same as in Eq. (34).

IV. CONTRIBUTIONS TO $\langle\hat{T}_{\mu\nu}\rangle$

One of the most relevant quantities in semiclassical approaches to gravity is the energy-momentum tensor of the quantum fields propagating over the classical background of general relativity. In this context, the computation of the energy-momentum tensor demands renormalization procedures to be followed. As discussed in [9], the renormalized VEV of the energy-momentum tensor, $\langle\hat{T}_{\mu\nu}\rangle$, can be calculated in terms of the renormalized Green's function, as follows:

$$\langle\hat{T}_\mu{}^\nu(x)\rangle_{\mathrm{ren}} = \lim_{x'\to x}(\mathcal{D}_\mu{}^\nu(x,x')G_{\mathrm{ren}}(x,x')), \tag{35}$$

where $\mathcal{D}_{\mu\nu}(x,x')$ is a nonlocal differential operator. One can find it by constructing the operator $\hat{T}_{\mu\nu}$ from the Lagrangian density of a scalar field and taking its expectation value with respect to a vacuum state [9].

The computation of this quantity can be highly nontrivial since it is locally defined and we are operating over nonlocal objects (Green's functions) to obtain it. For instance, what Mazzitalli and Lousto do in [4] is to not explicitly calculate $\langle\hat{T}_{\mu\nu}\rangle$, but to use symmetry arguments and intrinsic properties of the definition of the tensor to at least find its form, which is

$$\langle\hat{T}_\mu{}^\nu(x)\rangle^{ML}_{\mathrm{ren}} = \frac{1}{16\pi^2 r^4}[A_\mu{}^\nu(\xi,\eta^2) + B_\mu{}^\nu(\xi,\eta^2)\log(\mu r)], \tag{36}$$

where $A_\mu{}^\nu$ and $B_\mu{}^\nu$ are both tensors with numerical components depending on the coupling constant ξ in order $\mathcal{O}(\eta^2)$. The authors obtain explicitly only $B_\mu{}^\nu$ as a consequence of the renormalization procedure. To find $A_\mu{}^\nu$ it would be necessary to compute at least one of the components of $\langle\hat{T}_\mu{}^\nu\rangle^{ML}_{\mathrm{ren}}$ and to know its trace.

We are interested in the contributions to $\langle\hat{T}_\mu{}^\nu\rangle_{\mathrm{ren}}$ arising from the boundary conditions we have prescribed. In particular, we will compute the contributions to $\langle\hat{T}_t{}^t\rangle$ which represents the energy density of the fields gravitating around the monopole. To do so, we followed direct differentiation of the Green's function using Eq. (35). The functional form of the operator $\mathcal{D}_t{}^t(x,x')$ depends on the coupling parameter ξ; hence we will consider the minimally and conformally coupled cases separately. Only $\ell = 0$ modes are influenced by the boundary conditions, its spherical symmetry implies the Green's function and, accordingly, the differential operator will depend only on $\tau(\tau')$ and $r(r')$. As we showed in Sec. III, in both cases the Green's functions are finite so no further renormalization procedures must be followed.

A. Minimally coupled field

The differential operator in this case will be

$$\mathcal{D}_t{}^t(x,x') = -\frac{1}{2}\left(\alpha^{-2}\frac{\partial^2}{\partial\tau^2} + \alpha^2\frac{\partial^2}{\partial r\partial r'}\right) \tag{37}$$

and we must apply it on the Euclidean Green's function in Eq. (27). After that, we take the coincidence limit, which leads to the 00th component of the energy-momentum tensor (energy density)

$$\langle\hat{T}_t{}^t(x)\rangle^{\xi=0}_\beta = -\frac{1-\eta^2}{8\pi^2 r^4}\left[1 + e^{\frac{2r}{\beta}}\left(\frac{2r}{\beta} - 1\right)\mathrm{Ei}\left(-\frac{2r}{\beta}\right)\right]. \tag{38}$$

Again, in the limit $\beta \to 0$ the energy density vanishes, and the whole contribution will come from the Dirichlet boundary condition term, $\langle \hat{T}_\mu{}^\nu(x)\rangle_{\text{ren}}^{ML}$. On the other hand, for the Neumann boundary condition ($\beta \to \infty$), the energy density diverges negatively to infinity, i.e., $\langle \hat{T}_t{}^t(x)\rangle_\beta^{\xi=0} \to -\infty$.

B. Conformally coupled field

In this case, the differential operator will be as follows:

$$\mathcal{D}_t{}^t(x,x') = -\frac{1}{6}\left(5\alpha^{-2}\frac{\partial^2}{\partial\tau^2} + \alpha^2\frac{\partial^2}{\partial r\partial r'} + \frac{\eta^2}{3r^2}\right). \quad (39)$$

As we only have available the Green's function in zeroth order for the conformally coupled field, we will follow the same procedure as before, i.e., to expand the energy-momentum tensor in a series of η^2. We will compute here only the contribution to the energy-momentum tensor in the lowest order of η^2 applying the differential operator over the Euclidean Green's function in Eq. (33), which yields

$$\langle \hat{T}_t{}^t(x)\rangle_\beta^{\xi=1/6} = -\frac{1}{8\pi^2 r^4}\left[\frac{2r}{3\beta} + \frac{e^{\frac{2r}{\beta}}}{3}\left(\frac{4r^2}{\beta^2} + \frac{2r}{\beta} - 1\right)\mathrm{Ei}\left(-\frac{2r}{\beta}\right)\right] + \mathcal{O}(\eta^2). \quad (40)$$

Similar to the minimally coupled case, the Dirichlet boundary condition leads to a vanishing contribution, and the Neumann one contributes with a negatively divergent energy density.

V. FINAL REMARKS

The global monopole spacetime is characterized by a solid deficit angle proportional to η^2. This is responsible for the emergence of a naked singularity and the appearance of a strong curvature which is also proportional to η^2. As a matter of fact, we should expect only physical effects proportional to η^2 to appear.

In this paper, however, we found that the nontrivial interaction between the quantum field and the classical singularity may bring contributions to the vacuum fluctuations which do not disappear in the limit $\eta \to 0$. Only in the case where the field does not effectively realize the presence of the naked singularity (Dirichlet boundary condition) do we have zero contribution in Minkowski limit. In this case, the fluctuations are purely topological. When nontrivial interactions are taken into account, an analytic contribution arises. Such results stand for a range of positive values of ξ much smaller than a characteristic scale $1/\eta^2$. For negative values of ξ, the singularity may be perceived by nonspherically symmetric modes ($\ell = 1, 2, \ldots$) as well, and boundary conditions might be necessary for them. For instance, such is the circumstance of the Bañados-Teitelboim-Zanelli (BTZ) spacetime, which requires boundary conditions for all modes. However, we leave this case for further analysis in a future work.

The analytic contribution to the vacuum fluctuations seems to be originated by the local interaction between the field and the singularity, having nothing to do with the topology of spacetime. If nature abhors this behavior somehow, we have found out a preferred choice for the boundary conditions, namely the Dirichlet one. Otherwise, we have to accept such pathological behavior and try to find evidence in more realistic models, as the extreme Reissner-Nordström spacetime.

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ANNEX D – Article 4

Boundary Conditions and Vacuum Fluctuations in AdS_4[45]

Boundary Conditions and Vacuum Fluctuations in AdS$_4$.

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Abstract. Initial conditions given on a spacelike, static slice of a non-globally hyperbolic spacetime may not define the fates of classical and quantum fields uniquely. Such lack of global hyperbolicity is a well-known property of the anti-de Sitter solution and led many authors to question how is it possible to develop a quantum field theory on this spacetime. Wald and Ishibashi took a step towards the healing of that causal issue when considering the propagation of scalar fields on AdS. They proposed a systematic procedure to obtain a physically consistent dynamical evolution. Their prescription relies on determining the self-adjoint extensions of the spatial component of the differential wave operator. Such a requirement leads to the imposition of a specific set of boundary conditions at infinity. We employ their scheme in the particular case of the four-dimensional AdS spacetime and compute the expectation values of the field squared and the energy-momentum tensor, which will then bear the effects of those boundary conditions. We are not aware of any laws of nature constraining us to prescribe the same boundary conditions to all modes of the wave equation. Thus, we formulate a physical setup in which one of those modes satisfy a Robin boundary condition, while all others satisfy the Dirichlet condition. Due to our unusual settings, the resulting contributions to the fluctuations of the expectation values will not respect AdS invariance. As a consequence, a back-reaction procedure would yield a non-maximally symmetric spacetime. Furthermore, we verify the violation of weak energy condition as a direct consequence of our prescription for dynamics.

Introduction

One of the most remarkable outcomes of string theory was the proposition of the AdS/CFT correspondence [1]. It is conjectured that a theory of quantum gravity on n-dimensional AdS displays an underlying equivalent conformal quantum field theory without gravity, taking place at the $(n-1)$-dimensional conformal boundary of AdS. Accordingly, applications to high energy and condensed matter physics appeared within the efforts to test the limits of this new conjecture, placing the anti-de Sitter spacetime under the scientific spotlight.

Although most of the developments in AdS rely on string theory techniques, on a recent work [2], the authors have focused on studying semiclassical properties of the spacetime. Using the mathematical apparatus of Quantum Field Theory (QFT) in curved spaces, they have found the fluctuations of the expectation values of the energy-momentum tensor and the field squared in AdS_n. However, they did not discuss in depth the implications of the causal structure of the spacetime, i.e., the effects of non-globally hyperbolicity.

Since AdS has a conformal boundary, we may not be able to determine much about the history of a physical quantity without specifying its behavior at infinity. Such a circumstance poses a fundamental issue on the quantization procedure, namely the solutions of the wave equation will not be uniquely defined by initial conditions in AdS, i.e., the Cauchy problem is not well-posed. Thus, unless we give extra information at the conformal boundary, the lack of predictability makes it impracticable to build a quantized field whose dynamical evolution comprises the entire history of the spacetime.

Avis, Isham, and Storey [3] were the first ones to address the causal pathology of AdS when solving field equations. They developed QFT on AdS_4 by regulating information leaving or entering the spacetime *by hand*. Their approach proposes the imposition of boundary conditions at the spatial infinity in order to control whether information flows through (or is reflected by) the conformal boundary. Even though Avis et al. provide us with physically consistent solutions to the wave equation, works by Wald [4] and Ishibashi [5, 6] reveal that a broader category of boundary conditions might be employed to obtain a physical dynamical evolution.

In [5], the authors present a prescription for dynamics of fields in general non-globally hyperbolic spacetimes based on the grounds of physical consistency. In order to fulfill some reasonable physical requirements (to be explained later), they argue that the spatial component of the differential wave operator must be self-adjoint. Besides, in [6], they show that the prescription for dynamics in AdS translates into specifying boundary conditions at the conformal boundary. While Kent and Winstanley, in [2], impose the Dirichlet boundary condition at infinity, perhaps without realizing, they are neglecting an entire set of non-equivalent dynamical outcomes. According to Ishibashi and Wald [6], those outcomes would correspond to the various boundary conditions that one could have specified at infinity.

In this paper, we study physical effects that may arise due to non-Dirichlet boundary conditions at the conformal boundary. We investigate those effects by computing the vacuum fluctuations of the expectation values of the quadratic field and the energy-momentum tensor for conformally coupled scalar fields in AdS_4. Also, we will keep Ref. [2] as a basis for our results and shall return to it for further comparison.

We have organized this article as follows. In Sec. 1, we briefly review some of the fundamental aspects of the anti-de Sitter solution. Then, in Sec. 2, we display the systematic procedure that describes the dynamics of scalar fields in non-globally hyperbolic spacetimes - such as AdS - first presented by Wald and Ishibashi. With that scheme in hands, we show the implications their prescription has on scalar fields propagating on AdS, in Sec. 3. Our next step is to build the proper Green's functions in Sec. 4, and employ them in the computations of the renormalized quantities of interest, namely the fluctuations of the expectation values of the field squared and the energy-momentum tensor, both shown in Sec. 5. Finally, we discuss our results in Sec. 6.

1. Anti-de Sitter spacetime

Surfaces of constant negative curvature are well-known in geometry and comprise the set of hyperbolic spaces. In the context of General Relativity, the equivalent to those spaces is the n-dimensional anti-de Sitter space, which appears as a solution to Einstein equations when choosing a negative cosmological constant ($\Lambda < 0$) in the absence of matter and energy. Setting $\Lambda := -\frac{(n-1)(n-2)}{2H^2}$, we may write the Einstein equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{(n-1)(n-2)}{2H^2}g_{\mu\nu} = 0. \tag{1}$$

The outcome is an n-dimensional maximally symmetric pseudo-Riemmanian metric defined over a Lorentzian manifold with constant negative curvature, i.e., the AdS_n spacetime. In a suitable set of parametrized coordinates $\{x^\mu\}$*, the line element for the induced metric $g_{\mu\nu}$ on AdS_n is

$$ds^2 = H^2(\sec\rho)^2[-d\tau^2 + d\rho^2 + (\sin\rho)^2 d\Omega^2_{n-2}], \tag{2}$$

where $d\Omega^2_{n-2}$ is the line element on a unit $(n-2)$-sphere.

1.1. Topology

We may understand AdS_n as an isometric embedding of a single sheeted n-dimensional hyperboloid in an $(n+1)$-dimensional flat space provided with metric $\mathrm{diag}(-1, 1, \cdots 1, -1)$. Timelike curves in AdS are transverse sections of the hyperboloid, and they are always closed. The periodicity of the timelike coordinate, τ, suggests that given a point in spacetime, we can return to it by only traveling along a timelike geodesic of length 2π in τ. Accordingly, the topology of AdS_n becomes apparent, namely $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, which is compatible with the existence of closed timelike curves. Thus,

$*$ The radial coordinate, ρ, is defined over the interval $[0, \pi/2)$. The polar and azimuthal coordinates on the unit $(n-2)$-sphere are θ_j ($j = 1, \ldots, n-3$) and $\varphi := \theta_{n-2}$, respectively, each satisfying $0 \leq \theta_j \leq \pi$ and $0 \leq \varphi < 2\pi$. The timelike coordinate, τ, ranges from $-\pi$ to π.

unphysical events can take place in the spacetime, such as a particle returning to the same position through a periodic motion in time.

1.2. Causal structure

Wald remarks in [7] that observers following closed timelike geodesics would have no difficulty altering past events hence breaking causality. In an attempt to solve this primary issue, we can 'unwrap' the hyperboloid along the timelike direction, and patch together unwrapped hyperboloids one after the other. In other words, we construct a spacetime spatially identical to AdS but extended in time, i.e., the temporal coordinate no longer ranges from $-\pi$ to π but from $-\infty$ to ∞. We refer to such procedure as the universal covering of AdS, and the resulting spacetime as CAdS.

Even though the unwrapping of AdS prevents the existence of closed timelike curves, another fundamental causality issue remains, namely the lack of predictability associated with fields propagating on the spacetime. Indeed, no Cauchy hypersurfaces exist in AdS (and CAdS) hence portraying it as a non-globally hyperbolic spacetime. The Cauchy problem will not be well-posed, yielding non-unique dynamics for a given set of initial conditions. We can understand this scenario as a result of information leaking through the spatial infinity of the spacetime, i.e., flowing in (out) from (through) the boundary. In order to solve such a pathological behavior, we shall discuss in the next sections how to adequately address causality issues associated with field equations in non-globally hyperbolic spacetimes.

2. Scalar fields in non-globally hyperbolic static spacetimes

An extensive literature (see, for instance, [8] and references therein) provides a complete guide on QFT in curved spaces, and conduct us through a generalized quantization procedure based on that of QFT in Minkowski spacetime. Nevertheless, several researchers developed most of it in a category of spacetimes whose causal structure is thoroughly well-defined, namely globally hyperbolic spacetimes. Indeed, as we discussed previously if a spacetime does not feature global hyperbolicity, then basic field equations might not have causal solutions, which jeopardizes the quantization of fields. On what follows, we use works by Wald [4] and Ishibashi [5, 6] to prescribe the appropriate dynamics of scalar fields in non-globally hyperbolic spacetimes.

Let us consider a static spacetime $(\mathcal{M}, g_{\mu\nu})$, which admits the following decomposition of its metric [9]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -V^2dt^2 + h_{ij}dx^i dx^j. \tag{3}$$

In Eq. 3, h_{ij} is the metric induced on a hypersurface Σ orthogonal to a given timelike Killing field τ^μ of the metric, and we define $V^2 = -\tau^\mu\tau_\mu$. In this particular case, Klein-Gordon equation,

$$\nabla_\mu\nabla^\mu\phi - m^2\phi - \xi R\phi = 0, \tag{4}$$

reduces to

$$\partial_t^2 \phi = -A\phi, \tag{5}$$

in which $A := -VD^i(VD_i\phi) + m^2V^2 + \xi RV^2$ is the spatial component of the wave operator, and D_i is the covariant derivative in a spatial slice of Σ.

Wald points out in [4] that A is an operator defined on a Hilbert space $\mathcal{H} = \mathcal{L}^2(\Sigma)$ with domain $\mathcal{D}(A) = C_0^\infty(\Sigma)$, and whose self-adjointness properties are relevant to examine the dynamical evolution appropriately. An extensive literature on Functional Analysis (e.g., see [10, 11]) discusses the properties of such operators and present a systematic procedure for obtaining their self-adjoint extensions, accredited to Weyl and von Neumann.

It can be easily checked that $(A, \mathcal{D}(A))$ defined above is symmetric. For such a symmetric operator, we denote by $(A^\dagger, \mathcal{D}(A^\dagger))$ its adjoint operator. Symmetry of A implies that $A = A^\dagger$. However, we may have $\mathcal{D}(A) \neq \mathcal{D}(A^\dagger)$ - when A is not self-adjoint. In this case it may be possible to find the self-adjoint extensions of A. In order to find these extensions, let us define the *deficiency subspaces* of A, denoted $\mathcal{N}_\pm \subset \mathcal{H}$, by

$$\mathcal{N}_\pm = \{\psi_\pm \in \mathcal{D}(A^\dagger) |\ A^\dagger \psi_\pm = \pm i\lambda\psi_\pm, \lambda \in \mathbb{R}^+\}, \tag{6}$$

and the *deficiency indices* as $n_\pm = \dim(\mathcal{N}_\pm)$. There are three cases to be considered:

(i) If $n_+ \neq n_-$, then A has no self-adjoint extension.

(ii) If $n_+ = n_- = 0$, then A is essentially self-adjoint, and we obtain it by taking the closure, $\bar{A}$, of A.

(iii) If $n_+ = n_- = n \geq 1$, then infinitely many self-adjoint extensions of A may exist. They are in one-to-one correspondence to the isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$ parametrized by an $n \times n$ unitary matrix, U.

Certainly, the third case is more complex than the others, and we must follow a method for obtaining the self-adjoint extensions (see [11] for a proper description of it). They are given by A_E, with E being a parameter labeling the extension, defined by

$$\mathcal{D}(A_E) = \{\Phi_0 + \Phi_+ + U\Phi_+ |\ \Phi_0 \in \mathcal{D}(A), \Phi_+ \in \mathcal{N}_+\}, \tag{7}$$

and

$$A_E\Phi = A\Phi_0 + i\Phi_+ - iU\Phi_+, \tag{8}$$

for all $\Phi \in \mathcal{D}(A_E)$. This procedure can always be followed to find whether an operator has self-adjoint extensions and identify them, in case they exist.

In particular, Wald [4] proposes that there might exist a set of solutions of the wave equation 5 associated with each self-adjoint extension, i.e.,

$$\phi_t = \cos(A_E^{1/2}t)\phi_0 + A_E^{-1/2}\sin(A_E^{1/2}t)\dot{\phi}_0, \tag{9}$$

given well-posed initial conditions to the Cauchy problem, namely $(\phi_0, \dot{\phi}_0) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$, for all $t \in \mathbb{R}$.

It is straightforward to notice that for each extension A_E there will be an associated dynamical evolution of Eq. 9. Consequently, the dynamics of the field is not uniquely determined by initial conditions. We identify those non-equivalent solutions as a result of various boundary conditions that one can impose at a region in space, such as a singularity or a boundary [4]. Ishibashi and Wald, in [5], argue that Eq. 9 is the only one that prescribes a physically sensible dynamics of scalar fields in non-globally hyperbolic static spacetimes. By comparison with the globally hyperbolic case, they establish a set of conditions that determine whether a time evolution is consistent or not, namely:

(i) solutions of the wave equation must be causal;

(ii) the prescription for dynamics must be invariant under time translation and reflection;

(iii) there exists a conserved energy functional also respecting time translation and reflection invariance, in agreement with the globally-hyperbolic case;

(iv) solutions satisfy a convergence condition, as proposed in [4].

3. Boundary conditions at infinity of anti-de Sitter

Let us now consider Klein-Gordon equation 5 in AdS$_n$, as follows

$$\partial_t^2\phi = -(\sec\rho)^2\Big\{(\cot\rho)^2\Big[-(n-2)\tan\rho\partial_\rho^2 - \Delta_S\Big] - H^2 m_\xi^2\Big\}\phi, \tag{10}$$

where m_ξ is the effective mass of the field defined by $m_\xi^2 = m^2 - \xi n(n-1)H^{-2}$, and

$$\Delta_S = \sum_{j=1}^{n-3}\Big[(n-2)\cot\theta_j\partial_{\theta_j} + \prod_{k=1}^{j-1}(\csc\theta_k)^2\partial_{\theta_j}^2\Big] + \prod_{j=1}^{n-3}(\csc\theta_j)^2\partial_\varphi^2 \tag{11}$$

is the Laplace-Beltrami operator on the unit $(n-2)$-sphere whose eigenfunctions are Generalized Spherical Harmonic functions, $Y_l(\theta_j,\phi)$, with eigenvalues $l(l+n-3)$. We may recall that a static slice of AdS$_n$ can be decomposed into a real interval $[0,\pi/2)$, labeled by the radial coordinate ρ, and an $(n-2)$-dimensional unit sphere $\mathbb{S}^{n-2}$, parametrized by the angular coordinates θ_j and φ. It is also worth pointing out that, as the spacetime is static, there exists a timelike Killing field ∂_t, whose eigenfunctions $e^{-i\omega t}$ with positive energy, $\omega > 0$, can be used to expand the solution ϕ. Thus, ϕ will be an eigenfunction of the quadratic operator ∂_t^2 with eigenvalue $-\omega^2$. With those considerations in hand, let us write the solution as

$$\phi(t,\rho,\theta_j,\varphi) = \sum_{\omega,l} e^{-i\omega t}\tilde{f}_{\omega,l}(\rho)Y_l(\theta_j,\phi). \tag{12}$$

Under the transformation

$$\tilde{f}_{\omega,l}(\rho) = (\cot\rho)^{\frac{n-2}{2}} f_{\omega,l}(\rho), \tag{13}$$

and omitting temporal and angular dependence, Eq. 10 reduces to

$$Af_{\omega,l}(\rho) = \omega^2 f_{\omega,l}(\rho), \tag{14}$$

upon the identification [6]*

$$A \equiv -\frac{d^2}{d\rho^2} + \frac{\nu^2 - 1/4}{(\cos\rho)^2} + \frac{\sigma^2 - 1/4}{(\sin\rho)^2}, \tag{15}$$

which is a differential operator whose domain is $C_0^\infty(0, \pi/2)$ defined over a Hilbert space $\mathcal{H} = \mathcal{L}^2([0, \pi/2], d\rho)$, and the coefficients of the equation are defined as

$$\nu^2 - 1/4 = \frac{n(n-2)}{4} + H^2 m^2 - n(n-1)\xi, \tag{16}$$

and

$$\sigma^2 - 1/4 = \frac{(n-2)(n-4)}{4} + l(l+n-3). \tag{17}$$

From Eq. 17, it is straightforward to check that

$$\sigma = l + \frac{n-3}{2}. \tag{18}$$

The coefficient ν is taken to be the positive square root of ν^2 and will depend on the mass and coupling factor of the field. In such conditions, there are four relevant cases to be analyzed, namely

(i) $\nu^2 \geq 1$: in this case, the effective mass of the field satisfies the relation $H^2 m_\xi^2 \geq -(n+1)(n-3)/4$, which comprise the minimally coupled, massless scalar field for $n \geq 3$.

(ii) $0 < \nu^2 < 1$: this case occurs for $-(n-1)^2/4 < H^2 m_\xi^2 < -(n+1)(n-3)/4$, and includes conformally invariant scalar fields in all dimensions.

(iii) $\nu^2 = 0$: this is the case when the effective mass squared reaches a critical value, namely $H^2 m_\xi^2 \equiv -(n-1)^2/4$.

(iv) $\nu^2 < 0$: in this case, the effective mass squared is lower than the critical mass, i.e., $H^2 m_\xi^2 < -(n-1)^2/4$.

In [6], the authors examine the positivity of the operator A in terms of ν. They demonstrate that, in all cases in which $\nu^2 \geq 0$ - i.e., in (*i*), (*ii*) and (*iii*) - A is a positive operator. Meanwhile, in case (*iv*), the operator is unbounded bellow. Consequently, A has no positive, self-adjoint extensions in case (*iv*). On the other hand, at least one self-adjoint extension to A exists - that is, the Friedrichs extension[10] - in all other cases: (*i*), (*ii*) and (*iii*).

The solutions to Eq. 14 are given by

$$f_{\omega,l}(\rho) = \mathbf{C} \cdot (\cos\rho)^{\nu+1/2} \cdot (\sin\rho)^{\sigma+1/2} \times {}_2F_1\left(\frac{\nu+\sigma+\omega+1}{2}, \frac{\nu+\sigma-\omega+1}{2}; 1+\sigma, (\sin\rho)^2\right). \tag{19}$$

The other linear independent solution is never square-integrable, so we neglect it here. According to Eq. 6, to construct the deficiency subspaces $\mathcal{N}_\pm$, we must take $\omega^2 = \pm\lambda i$,

*Ishibashi and Wald define the radial coordinate x for the spatial infinity to be located at $x = 0$[6]. It relates to our radial coordinate ρ by $x = \pi/2 - \rho$.

so $\omega \in \mathbb{C}$. In such conditions, as shown in [6], solution 19 fails to be square integrable in case (i), i.e., $\nu \geq 1$. However, for $0 \leq \nu < 1$, which corresponds to cases (ii) and (iii), f is square integrable for all $\omega \in \mathbb{C}$.

In case (i), the deficiency subspaces are trivial, so $n_+ = n_- = 0$, and the operator admits a unique self-adjoint extension. In other words, the repulsive effective potential in A, i.e., $(\cos\rho)^{-2}$, prevents the fields from reaching spatial infinity. Hence, they vanish there, and no additional boundary conditions are required. Conversely, in cases (ii) and (iii), the deficiency subspaces $\mathcal{N}_\pm$ are each spanned by an eigenfunction $f_\pm$ of A with eigenvalue $\omega^2 = \pm 2i$. Thus, the deficiency indices in these cases are $n_+ = n_- = 1$, so infinitely many positive self-adjoint extensions of A exist. Now, the effective potential is not as strong as in case (i); hence we may associate the extensions to boundary conditions prescribed at infinity.

A one-parameter family of self-adjoint extensions, A_β, of A exists for $0 \leq \nu^2 < 1$ (cases (ii) and (iii)). Equation 7 provides us with the appropriate domain of A_β. Since the domain of A consists of functions in C_0^∞, all additional information needed to prescribe a physically consistent dynamical evolution must come from the asymptotic behavior of f_+ and Uf_+, for all isometries U.

Let U_β denote the isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$, given by

$$U_\beta f_+ = e^{i\beta} f_-, \tag{20}$$

for $\beta \in (-\pi, \pi]$. Let us consider the function

$$f_\beta := f_+ + U_\beta f_+ \equiv f_+ + e^{i\beta} f_-, \tag{21}$$

whose behavior near infinity ($\rho = \pi/2$) dictates the boundary conditions satisfied by all solutions ϕ_t of the form 9. For $0 < \nu < 1$, the asymptotic behavior at $\rho = \pi/2$ is

$$f_\beta \propto (\sin\rho)^{\sigma+1/2} \cdot (\cos\rho)^{-\nu+1/2} \times (a_\nu + b_\nu (\cos\rho)^{2\nu} + c_\nu (\cos\rho)^2 + \ldots), \tag{22}$$

where the coefficients of the leading terms, a_ν and b_ν, are functions of ν, σ, the spacetime dimension n and the parameter β. The leading powers in ρ of f_+ are

$$f_\beta \approx b_\nu \left(\frac{\pi}{2} - \rho\right)^{\nu+1/2} \left\{1 + \frac{a_\nu}{b_\nu}\left(\frac{\pi}{2} - \rho\right)^{-2\nu}\right\}, \tag{23}$$

from which we can see that the asymptotic boundary condition depends on the ratio a_ν/b_ν, which may take any real value. For $\nu = 0$, we have

$$f_\beta \propto (\sin\rho)^{\sigma+1/2} \cdot (\cos\rho)^{1/2} \times (a_0 \log(\cos^2\rho) + b_0 + c_0 (\cos\rho)^2 \log(\cos^2\rho) + \ldots), \tag{24}$$

and an analogous procedure reveals that the asymptotic boundary condition depends on a_0/b_0 also in this case. However, the function $(\sin\rho)^{-\sigma-1/2} \cdot (\cos\rho)^{-1/2} \cdot f_\beta$ and its first derivative in ρ both scale with a_0 when approaching infinity $\rho = \pi/2$. Setting $a_0 = 0$, we recover Dirichlet and Neumann boundary condition imposed simultaneously, which is precisely Friedrichs extension.

On what follows, we shall denote the ratio a_ν/b_ν by α_ν, hence all self-adjoint extensions of the operator will be parametrized by α instead of β, although $\alpha \equiv \alpha(\beta)$. From Eq. 23, we can check that∗

$$\left.\frac{\frac{d}{d\rho}[(\sin\rho)^{-\sigma-1/2}\cdot(\cos\rho)^{\nu-1/2}\cdot f_\alpha]}{[(\sin\rho)^{-\sigma-1/2}\cdot(\cos\rho)^{3\nu-3/2}\cdot f_\alpha]}\right|_{\rho=\pi/2} = -2\nu\frac{1}{\alpha_\nu}, \tag{25}$$

which we identify as generalized Robin boundary conditions for $0 < \nu < 1$. One recovers generalized Dirichlet or Neumann boundary conditions by setting α_ν equals to 0 and $\pm\infty$, respectively. In the particular case $\nu = 1/2$, Eq. 23 reduces to an even simpler form of the boundary conditions given by†

$$\left[\frac{df_\alpha}{d\rho}\Big/f_\alpha\right]_{\rho=\pi/2} = -\frac{1}{\alpha}, \tag{26}$$

which is the usual Robin boundary condition, hence mixing Dirichlet ($\alpha = 0$) and Neumann ($\alpha = \pm\infty$) conditions.

Even though the extensions A_α are now parametrized by a real parameter α_ν, not all of them are positive. Except for $\nu^2 \geq 1$, whose unique self-adjoint extension is already positive, the remaining cases satisfy the positivity conditions shown in [6]:

For $0 < \nu^2 < 1$, we have

$$\frac{b_\nu}{a_\nu} \equiv \frac{1}{\alpha_\nu} \geq -\left|\frac{\Gamma(-\nu)}{\Gamma(\nu)}\right| \frac{\Gamma(\frac{\sigma+\nu+1}{2})^2}{\Gamma(\frac{\sigma-\nu+1}{2})^2}. \tag{27}$$

For $\nu^2 = 0$, we have

$$\frac{b_0}{a_0} \leq 2\gamma + 2\psi\left(\frac{\sigma+1}{2}\right), \tag{28}$$

where γ is the Euler gamma and ψ is the digamma function.

It is worth pointing out that equations 25 and 26 must be satisfied mode by mode, i.e., for each spherical label l - and for each σ, indirectly (see Eq. 17) -, the conditions are satisfied by $f_{\beta,\omega,l}$. Accordingly, there are infinitely many parameters $\alpha_{\nu,l}$ associated to each $f_{\beta,\omega,l}$, and they all satisfy different positivity conditions, given in equations 27 and 28.

4. Green's functions in AdS

In [12], Allen and Jacobson show that, in a maximally symmetric spacetime, two-point functions such as $G_F(x,x') = -i\langle\psi|T\{\phi(x)\phi(x')\}|\psi\rangle$, where $|\psi\rangle$ is a maximally symmetric state, may be written in terms of the geodetic interval $s(x,x')$‡, i.e.,

$$G_F(x,x') := G_F(s(x,x')) \equiv G_F^{(AJ)}(s). \tag{29}$$

∗We exchanged all indices β for α.

†In case $\nu = 1/2$, we drop the index of $\alpha_{1/2}$ and replace it simply by α.

‡In AdS, s is constructed so that it goes to zero as $x' \to x$ and goes to infinity as we approach the boundary

Their proposition simplifies the computations considerably since the wave equation becomes an ODE of the variable s. They also require that the Green's function falls off as fast as possible at spatial infinity, which in AdS translates into: $G_F \to 0$ as $s \to \infty$. In other words, they are choosing Dirichlet boundary condition for the field ϕ. Kent and Winstanley, in [2], exploit this simplicity to find the fluctuation of the field squared and the energy-momentum tensor in all spacetime dimensions of AdS. They also verify that their results are compatible with the ones of Burgess and Lütken, whose approach in [13] was to perform a summation of modes of the wave solutions.

We are not aware of any law of nature that restricts the boundary conditions of all modes to Dirichlet ones. Indeed, Ishibashi and Wald showed in [5] that there is an entire category of boundary conditions that prescribe a physically consistent dynamical evolution. Additionally, there is no guarantee that all modes must satisfy the same boundary condition.

Let us then consider a setup in which one of the modes of the wave equation, $u_{\omega_\alpha,l_\alpha}$, is chosen so that its radial component $f_{\omega_\alpha,l_\alpha}(\rho)$ satisfies a generalized Robin boundary condition with parameter α. Meanwhile, the components $f_{\omega,l}(\rho)$ of all other modes $u_{\omega,l}(x)$ $(l \neq l_\alpha)$ satisfy Dirichlet boundary conditions.

The Green's function in this case is given by mode sum (from now on, we consider $\tau > \tau'$)

$$\begin{aligned} G_F(x,x') = -i(\cot\rho\cot\rho')^{\frac{n-2}{2}} & \\ \times\Big\{ \sum_{\omega_\alpha} |\mathcal{N}_{\omega_\alpha,l_\alpha}|^2 Y_{l_\alpha}(\theta_j,\varphi) Y^*_{l_\alpha}(\theta'_j,\varphi') f_{\omega_\alpha,l_\alpha}(\rho) f_{\omega_\alpha,l_\alpha}(\rho') e^{-i\omega_\alpha(\tau-\tau')} & \\ + \sum_{\substack{l\geq 0\\ l\neq l_\alpha}} \sum_{\omega} |\mathcal{N}_{\omega,l}|^2 Y_l(\theta_j,\varphi) Y^*_l(\theta'_j,\varphi') f_{\omega,l}(\rho) f_{\omega,l}(\rho') e^{-i\omega(\tau-\tau')} \Big\}, & \quad (30) \end{aligned}$$

where $\mathcal{N}_{\omega,l}$ are normalization constants. We may complete the last term in the summation for all Dirichlet modes by adding them to and subtracting them off Eq. 30, i.e.,

$$\begin{aligned} G_F(x,x') = -i(\cot\rho\cot\rho')^{\frac{n-2}{2}} & \\ \times\Big\{ \sum_{\omega_\alpha} |\mathcal{N}_{\omega_\alpha,l_\alpha}|^2 Y_{l_\alpha}(\theta_j,\varphi) Y^*_{l_\alpha}(\theta'_j,\varphi') f_{\omega_\alpha,l_\alpha}(\rho) f_{\omega_\alpha,l_\alpha}(\rho') e^{-i\omega_\alpha(\tau-\tau')} & \\ - \sum_{\omega} |\mathcal{N}_{\omega,l_\alpha}|^2 Y_{l_\alpha}(\theta_j,\varphi) Y^*_{l_\alpha}(\theta'_j,\varphi') f_{\omega,l_\alpha}(\rho) f_{\omega,l_\alpha}(\rho') e^{-i\omega(\tau-\tau')} & \\ + \sum_{l,\omega} |\mathcal{N}_{\omega,l}|^2 Y_l(\theta_j,\varphi) Y^*_l(\theta'_j,\varphi') f_{\omega,l}(\rho) f_{\omega,l}(\rho') e^{-i\omega(\tau-\tau')} \Big\}. & \quad (31) \end{aligned}$$

Let us denote the last term in equation 31 by $G_F^{(D)}$, and the first two terms by $G_F^{(\alpha)}$. The Green's function $G^{(D)}$ is obtained by the summation of Dirichlet modes purely. Thus, in the coincidence limit, it recovers the same results as $G^{(BL)}$, by Burgess and

Lütken, and $G_F^{(AJ)}$, by Allen and Jacobson. On the other hand, $G_F^{(\alpha)}$ lacks contributions from all spherical components, since it is not summed over all angular modes l. Hence, $G_F^{(\alpha)}$ may not be a maximally symmetric function. It seems reasonable for us to write that

$$G_F(x,x') \equiv G_F^{(\alpha)}(x,x') + G_F^{(D)}(s(x,x')). \tag{32}$$

Equation 32 illustrates the break of AdS invariance of the Green's function, as it may not depend on the geodetic interval s entirely anymore. We attribute the break on the maximal symmetry of G_F to the imposition of different boundary conditions for each angular mode.

5. Renormalized quantities for a conformal massless scalar field in AdS$_4$

In order to shed light on what we have discussed so far, we shall specialize to four spacetime dimensions, AdS$_4$. For simplicity on the computation of quantities of interest, let us restrict ourselves to a conformally invariant, massless scalar field, ϕ, i.e., $m = 0$ and $\xi = \frac{1}{6}$. In this case, from Eq. 16, we get $\nu = 1/2$, and from Eq. 17, we find that $\sigma = (2l+1)/2$. Equation 14 becomes

$$\left(-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\sin^2\rho}\right) f = \omega^2 f, \tag{33}$$

and its solutions are

$$f = \sqrt{\sin\rho}\left(\mathbf{C}_1 \cdot P^{l+1/2}_{\omega-1/2}(\cos\rho) + \mathbf{C}_2 \cdot Q^{l+1/2}_{\omega-1/2}(\cos\rho)\right), \tag{34}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are constants to be determined, and P and Q are the associated Legendre functions of the First and Second kinds, respectively. Square integrability requires f to fall off at the origin $\rho = 0$, hence $\mathbf{C}_1 \to 0*$. A complete set of eigenfunctions is then

$$f_{\omega,l}(\rho) = \mathcal{N}_{\omega,l} \cdot \sqrt{\sin\rho} \cdot Q^{l+1/2}_{\omega-1/2}(\cos\rho), \tag{35}$$

for normalization constants $\mathcal{N}_{\omega,l}$ to be determined.

As discussed in Sec. 3, boundary conditions at infinity are necessary to prescribe the dynamical evolution of the field in AdS$_n$. In case $\nu = 1/2$, Robin boundary conditions 26 are the appropriate ones. We aim to provide an example of the setups discussed in the last section. For that, we will consider that all non-spherically symmetric modes respect Dirichlet boundary conditions. However, the $l = 0$ mode will be chosen to satisfy Robin condition with a parameter α. As discussed above, the vacuum will not be AdS invariant in this case. However, since the non-trivial boundary condition is on $l = 0$ mode, we still preserve spherical symmetry.

*Formula 14.8.1 of Ref. [14] shows the divergence of P at $\rho = 0$.

Formulas 14.5.3 and 14.5.4 in Ref. [14] allow us to describe the behavior of $f_{\omega,l}$ and its derivative at the boundary, as follows

$$f_{\omega,l}(\rho \to \pi/2) \sim -\mathcal{N}_{\omega,l} \frac{2^{l-1/2}\sqrt{\pi}\sin\left(\frac{(l+\omega)\pi}{2}\right)\Gamma\left(\frac{l+\omega+1}{2}\right)}{\Gamma\left(\frac{-l+\omega+1}{2}\right)}, \tag{36}$$

$$\left.\frac{df_{\omega,l}}{d\rho}\right|_{\rho\to\pi/2} \sim -\mathcal{N}_{\omega,l} \frac{2^{l+1/2}\sqrt{\pi}\cos\left(\frac{(l+\omega)\pi}{2}\right)\Gamma\left(\frac{l+\omega+2}{2}\right)}{\Gamma\left(\frac{-l+\omega}{2}\right)} \tag{37}$$

For $l > 0$, all modes satisfy $f_{\omega,l}(\rho \to \pi/2) = 0$ (Dirichlet boundary condition), thus its positive quantized frequencies are

$$\omega = 2r + l,\ r \in \mathbb{N} \cup \{0\}. \tag{38}$$

For $l = 0$, we calculate the ratio between derivative 37 and function 36 to use it in 26, i.e.,

$$\begin{aligned}\left[\frac{df_{\omega,0}}{d\rho}\Big/ f_{\omega,0}\right]_{\rho=\pi/2} &= 2\cot\left(\omega\frac{\pi}{2}\right)\frac{\Gamma\left(1+\frac{\omega}{2}\right)}{\Gamma\left(\frac{\omega}{2}\right)} \\ &= \omega\cot\left(\omega\frac{\pi}{2}\right) = -\frac{1}{\alpha}.\end{aligned} \tag{39}$$

Positivity condition 27 requires that

$$\frac{1}{\alpha} \geq -\left|\frac{\Gamma(-1/2)}{\Gamma(1/2)}\right| \frac{\Gamma(1)^2}{\Gamma(1/2)^2} = -\frac{2}{\pi} \Rightarrow \alpha \leq -\frac{\pi}{2} \text{ or } \alpha \geq 0. \tag{40}$$

In our analysis, we consider $\alpha \geq 0$, which includes Dirichlet, $\alpha = 0$, and Neumann, $\alpha \to \infty$, cases.

Equation 39 imposes a quantization condition for the frequencies ω in terms of the parameter α. Except for $\alpha = 0$ and $\alpha = \infty$, it cannot be solved analytically for an arbitrary value α. One can readily verify that, in the Neumann case ($\alpha \to \infty$), the frequencies are odd integers. Meanwhile, for Dirichlet, they are even integers, which is consistent with Eq. 38.

In our procedure, we employed the software Mathematica [15] to solve equation 39 numerically in a determined range of ω for several values of α. As shown in Fig. 1, the solutions of 39 are given by the intersection points between the two functions. We can see that ω values for arbitrary α always lie between an odd number and its next even integer, which are precisely the frequencies for Neumann and Dirichlet conditions, respectively. Thus, given a Neumann frequency, $\omega_{N,r} = 2r - 1$, and a Dirichlet one, $\omega_{D,r} = 2r$, for $r > 0$, we may denote an α frequency between them as $\omega_{\alpha,r}$, even though it is not an integer number.

5.1. Quadratic field fluctuations $\langle\phi^2\rangle$

Before computing the Green's function, it is useful to write solution $f_{\omega_{\alpha,r},0}$ in a more convenient form and normalize it accordingly. Using Ref. [14], we find∗

$$f^{(\alpha)}_{r,0}(\rho) = H^{-1}\sqrt{\frac{2}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}}\sin(\omega_{\alpha,r}\rho). \tag{41}$$

Now, we recall our discussion from last section to construct the appropriate Green's function. We can decompose our Green's functions in two parts, i.e.,

$$\begin{aligned} G^{(\alpha)}_F(x,x') = {}& -i\frac{\cot\rho\cot\rho'}{4\pi H^2} \\ &\times\sum_{r>0}\Bigg(\frac{2}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}\sin(\omega_{\alpha,r}\rho)\sin(\omega_{\alpha,r}\rho')e^{-i\omega_{\alpha,r}(\tau-\tau')} \\ &\qquad - \frac{2}{2r\pi}\sin(2r\rho)\sin(2r\rho')e^{-i2r(\tau-\tau')}\Bigg), \end{aligned} \tag{42}$$

and

$$\begin{aligned} G^{(D)}_F(x,x') = {}& -iH^{-2}\cot\rho\cot\rho' \\ &\times\sum_{r\geq 0}\sum_{l\geq 0}\sum_{m=-l}^{l}|\mathcal{N}_{r,l}|^2 Y^m_l(\theta_j,\varphi)[Y^m_l(\theta'_j,\varphi')]^* f_{r,l}(\rho)f_{r,l}(\rho')e^{-i2r(\tau-\tau')}. \end{aligned} \tag{43}$$

∗For convenience, we change the lower label in $f_{\omega,0}$ from $\omega_{\alpha,r}$ to r simply, and add an upper index α to denote our choice of boundary condition.

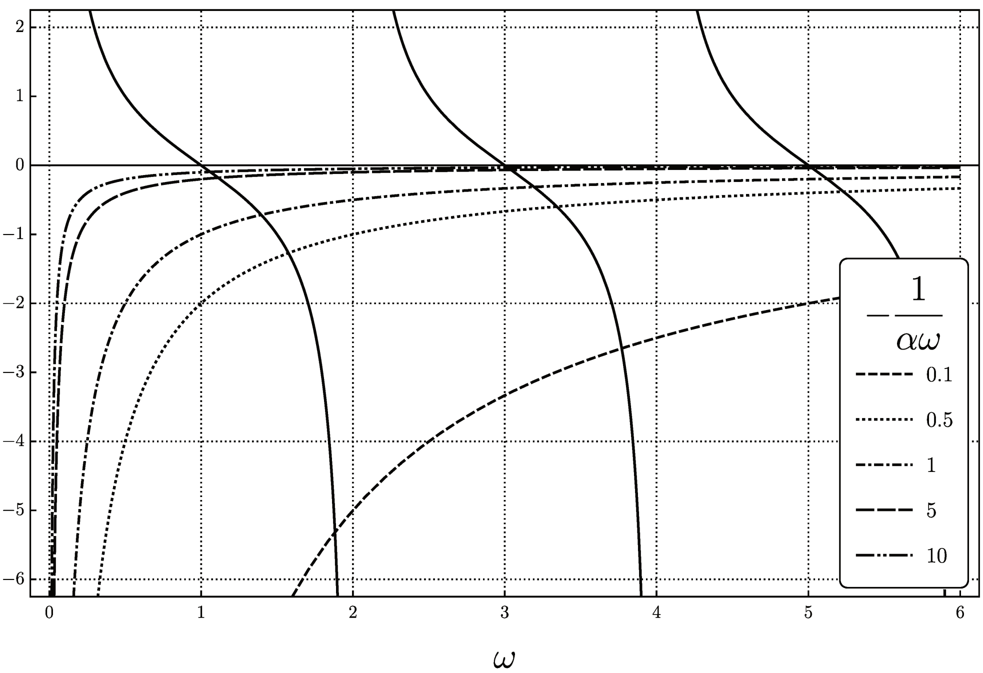

Figure 1. Quantization condition for ω imposed by 39. The solid lines show the function $\cot(\omega\pi/2)$ and all other curves are $-1/(\alpha\omega)$, for a few values of α.

Our 'Dirichlet' Green's function 43 is obtained from a summation of AdS invariant modes of the wave equation. Hence, it respects maximal symmetry and recovers the results of Burgess and Lütken, $G_F^{(BL)}$, and Allen and Jacobson, $G_F^{(AJ)}$, i.e., $G_F^{(D)}(x,x') \equiv G_F^{(D)}(s(x,x'))$. As Kent and Winstanley show in [2], approaching the coincidence limit $s \to 0$, the function $G_F^{(D)}$ diverges according to the Hadamard form. Thus, point-splitting renormalization can be employed to compute finite quantities. Furthermore, they obtain the Hadamard forms in AdS for any spacetime dimension through a systematic method, based on [16].

In the particular case of AdS$_4$, for a conformally invariant field, the Green's function $G_F^{(D)}$ has the Hadamard form given by

$$G_F^{(D)}(s) \sim -\frac{i}{4\pi^2 s^2}, \quad s \to 0. \tag{44}$$

After renormalization, it may be written as [2]

$$\left[G_F^{(D)}\right]_{\text{ren}}(s) = -\frac{i}{8\pi^2 H^2}\left\{-\frac{1}{6} + \frac{13}{240}\frac{s^2}{H^2} + \mathcal{O}(s^4)\right\}. \tag{45}$$

We may find the expectation value of the quadratic field fluctuations as follows

$$\langle \phi^2 \rangle^{(D)} = i \lim_{s \to 0}\left[G_F^{(D)}\right]_{\text{ren}}(s) = -\frac{1}{48\pi^2 H^2}, \tag{46}$$

which is naturally in accordance with the results in Ref. [2]. Analogously, the effect of our Green's function 42 on $\langle \phi^2 \rangle$ appears when taking the coincidence limit $x' \to x$. However, calculating $G_F^{(\alpha)}$ analytically is impossible, since the summation is taken over numerical values of frequencies. Hence, we adopt a numerical approach to find our results.

We expect $G_F^{(\alpha)}$ to be finite, since the Hadamard form took care of the divergences in $G_F^{(D)}$. On the other hand, we cannot perform the infinite sum in 42 numerically, so a residual divergent behavior might appear. Through our computations, we noted it was convenient to take the coincidence limit in the radial coordinate first, i.e., $\rho' \to \rho$, and then in the time coordinate. Thus, our final step would be to take the limit of $\tau' \to \tau$. It is more convenient though, to analytically extend the function on the complex plane and take the limit through the imaginary axis, i.e., $\tau' \to \tau + i\epsilon$, hence $\tau - \tau' \to -i\epsilon$. Finally, by multiplying $G_F^{(\alpha)}$ by i, we will have an entirely real-valued function that, in the limit $\epsilon \to 0$, yields directly the quadratic fluctuations of the field, and it is much simpler for us to handle it numerically.

Before implementing the numerical routine, we considered the only case that can be treated analytically, which is the Neumann condition, $\alpha \to \infty$. In this situation, the frequencies are $\omega_{\infty,r} = 2r - 1$, for $r > 0$, and the Green's function reduces to the following summation

$$iG_F^{(\infty)}(\epsilon,\rho,\rho) = \frac{\cot^2\rho}{2\pi^2 H^2} \times \sum_{r>0}\Big(\frac{\sin^2((2r-1)\rho)}{2r-1}e^{-(2r-1)\epsilon} - \frac{\sin^2(2r\rho)}{2r}e^{-2r\epsilon}\Big), \tag{47}$$

which we calculated using Mathematica [15], resulting

$$iG_F^{(\infty)}(\epsilon,\rho,\rho) = \frac{\cot^2\rho}{16\pi^2 H^2} \times \log\left[\cosh\left(\frac{\epsilon}{2}\right)^4 \sec\left(\rho - i\frac{\epsilon}{2}\right)^2 \sec\left(\rho + i\frac{\epsilon}{2}\right)^2\right]. \tag{48}$$

It is straightforward to find the expectation value $\langle\phi^2\rangle^{(N)}$ by simply taking $\epsilon \to 0$, i.e.,

$$\langle\phi^2\rangle^{(N)}(\rho) = \frac{\cot^2\rho}{4\pi^2 H^2}\log\left[\sec\left(\rho\right)\right]. \tag{49}$$

The function $\langle\phi^2\rangle^{(N)}$ is finite because both terms inside the sum in Eq. 47 diverge with same strength. Naturally, their subtraction eliminates the infinities. In particular, the last term in Eq. 47, the Dirichlet counterpart of $G_F^{(\alpha)}$, denoted $G_F^{(\alpha,D)}$, appears for all values of α and dictates the divergent behavior at $\epsilon \to 0$. We find its form by calculating the infinite summation and expanding it in powers of ϵ, i.e.,

$$iG_F^{(\alpha,D)}(\epsilon,\rho) = \frac{\cot^2\rho}{8\pi^2 H^2}\left\{-\log\epsilon + \log[\sin(2\rho)] + \mathcal{O}(\epsilon^2)\right\}. \tag{50}$$

Our numerical approach to find the expectation value $\langle\phi^2\rangle^{(\alpha)}$ proceeded as follows:

(i) Given a value for α, solve Eq. 39 to find the frequencies $\omega_{\alpha,r}$ up to $r_{\max} = 5000$;

(ii) Given a value of ρ between 0 and $\pi/2$, compute numerically the truncated summation

$$iG_F^{(\alpha)}(\epsilon,\rho,\rho) \approx \frac{\cot^2\rho}{2\pi}\sum_{r=1}^{r_{\max}}\left(\frac{\sin^2(\omega_{\alpha,r}\rho)}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}e^{-\omega_{\alpha,r}\epsilon} - \frac{\sin^2(2r\rho)}{2r\pi}e^{-2r\epsilon}\right) =: \mathtt{f}_\rho^{(\alpha)}[\epsilon], \tag{51}$$

for 50 values of ϵ equally spaced in the range 0.002 to 0.1.*

(iii) Fit the function $\mathtt{f}_\rho^{(\alpha)}[\epsilon]$ using a model that reproduces the divergent behavior in Eq. 50 followed by a Taylor expansion up to order ϵ^2, i.e.,

$$\mathtt{f}[\epsilon] = \mathtt{a} + \mathtt{b}\log[\epsilon] + \mathtt{c}\cdot\epsilon + \mathtt{d}\cdot\epsilon^2. \tag{52}$$

As $G_F^{(\alpha)}$ is a finite quantity, we expect the divergent behavior of $\mathtt{f}_\rho^{(\alpha)}[\epsilon]$ to be extremely attenuated. We have found coefficients $\mathtt{b}$ ranging between 10^{-9} and 10^{-12}, recovering the expected *almost-finite* behavior. The coefficients $\mathtt{c}$ and $\mathtt{d}$ were effective on reducing the residuals of the fit. Finally, $\mathtt{a}$ gives the approximated finite numerical value of $\langle\phi^2\rangle^{(\alpha)}$ at the point ρ.

(iv) Repeat steps 2 and 3 for as many values of ρ between 0 and $\pi/2$ as desired.

(v) Repeat the entire procedure for another value of α.

We followed the scheme described above for 14 values for the parameter α. We chose 80 equally spaced points in the range $(0,\pi/2)$ to obtain a good resolution of the behavior of $\langle\phi^2\rangle^{(\alpha)}(\rho)$. Our results are plotted in Fig. 2. The curve corresponding to $\alpha = 1000$ reproduces almost perfectly the analytic Neumann result 49. Accordingly, as we approach the other extreme, $\alpha = 0$ - corresponding to Dirichlet conditions - we can see the curves getting closer to zero. Consistently, if $\alpha = 0$, then $G_F^{(\alpha)}$ indeed vanishes, as one can see from Eq. 42.

*Our choice for $r_{\max}$ and the range of ϵ was made so the last term of the sum would be negligible with respect to the first one. Indeed, the first term is of order $e^{-2\cdot 1\cdot 0.002} \sim 10^{-1}$, while the last is $e^{-2\cdot 5000\cdot 0.002} \sim 10^{-9}$. Also, we needed ϵ small enough so the divergent behavior would appear.

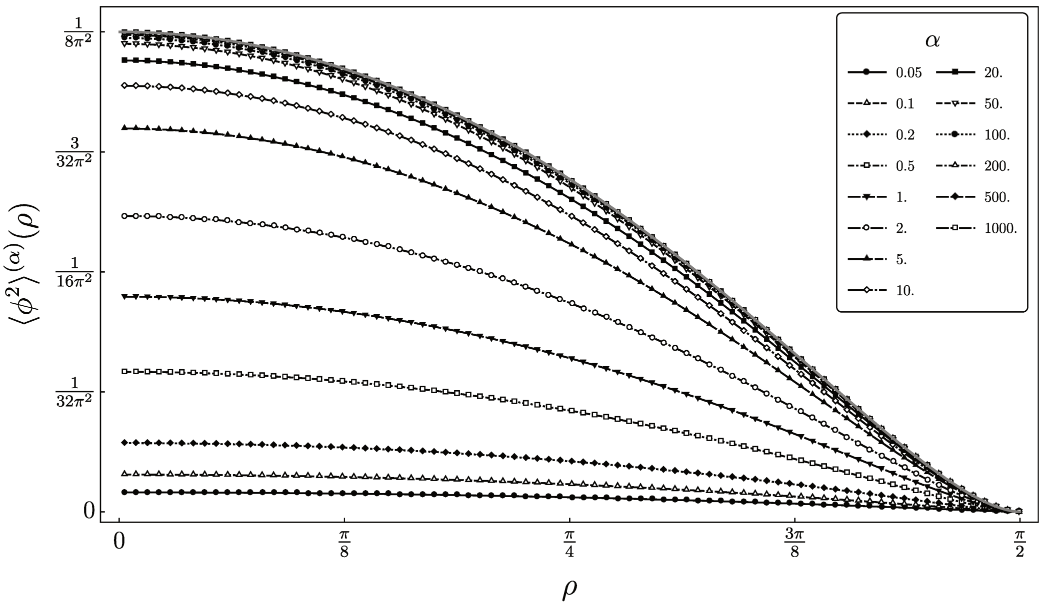

Figure 2. Contribution to the expectation value of the quadratic field fluctuations due to Robin boundary conditions at infinity for the spherically symmetric mode. The solid gray curve shows the analytical solution of the Neumann case, $\langle\phi^2\rangle^{(N)}$. H is set to one.

5.2. Energy-momentum tensor fluctuations $\langle T^{\mu}_{\ \nu}\rangle^{(\alpha)}$

In [2], the authors obtain the renormalized energy-momentum tensor $\langle T^{\mu}_{\ \nu}\rangle_{\text{ren}}$ in AdS_n. They use the formula from Ref. [16]

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = -[G]_{\mu\nu} + \frac{1}{2}(1-2\xi)[G]_{;\mu\nu} + \frac{1}{2}\left(2\xi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\nabla^\sigma[G] + \xi R_{\mu\nu}[G]\right) + \Theta_{\mu\nu}, \quad (53)$$

where

$$[G](x) \quad := \lim_{x'\to x} i\,[G_F]_{\text{ren}}(x,x'), \quad (54)$$

$$[G]_{\mu\nu}(x) := \lim_{x'\to x} i\,[G_F]_{\text{ren}}(x,x')_{;\mu\nu}, \quad (55)$$

and $\Theta_{\mu\nu}$ is a purely geometric tensor constructed to be conserved. Kent and Winstanley find that the non-geometrical component of the tensor is proportional to the metric tensor, which is completely consistent with the maximal symmetry of AdS. In our particular case of a conformally invariant field in four spacetime dimensions, we have

$$\langle T_{\mu\nu}\rangle^{(D)}_{\text{ren}} = -\frac{1}{960\pi^2 H^4} g_{\mu\nu}, \quad (56)$$

and the geometric tensor $\Theta_{\mu\nu}$ is identically zero. We may obtain this renormalized expectation value from Green's function $\left[G_F^{(D)}\right]_{\text{ren}}$, hence is associated with Dirichlet conditions in all modes of the wave equation.

Here, we want the contributions to the energy-momentum tensor coming from $G_F^{(\alpha)}$. Our approach will be analogous to that of the Green's functions: we decompose

$\langle T^{\mu}{}_{\nu}\rangle_{\text{ren}}$ into two parts, one carrying the boundary condition, denoted $\langle T^{\mu}{}_{\nu}\rangle^{(\alpha)}_{\text{ren}}$ with 16 components $\mathrm{T}^{\mu}{}_{\nu}$, and another one reproducing the Dirichlet results as in Eq. 56.

In our case, equations 54 and 55 may be written as

$$[G](\rho) = \lim_{\epsilon\to 0} iG_F^{(\alpha)}(\epsilon, \rho, \rho), \tag{57}$$

and

$$\begin{aligned}[G]_{\mu\nu}(\rho) &= \lim_{\epsilon\to 0} i(G_F^{(\alpha)})_{;\mu\nu}(\epsilon, \rho, \rho) \\ &= \lim_{\epsilon\to 0} i\left[(G_F^{(\alpha)})_{,\mu\nu} - \Gamma^{\lambda}_{\mu\nu}(G_F^{(\alpha)})_{,\lambda}\right],\end{aligned} \tag{58}$$

from which it follows that $[G](\rho) \equiv \langle\phi^2\rangle^{(\alpha)}(\rho)$. According to formula 53, we have here

$$\langle T_{\mu\nu}\rangle^{(\alpha)}_{\text{ren}} = -[G]_{\mu\nu} + \frac{1}{3}[G]_{;\mu\nu} - \frac{1}{12}\nabla_\kappa\nabla^\kappa[G]g_{\mu\nu} - \frac{1}{2H^2}[G]g_{\mu\nu}. \tag{59}$$

Considering all non-vanishing Christoffel symbols, the definitions for $[G]$ and $[G]_{\mu\nu}$, and the symmetric condition $\langle T_{\mu\nu}\rangle^{(\alpha)}_{\text{ren}} = \langle T_{\nu\mu}\rangle^{(\alpha)}_{\text{ren}}$, we readily verify that the only non-vanishing components are diagonal terms and the term $\mathrm{T}_{\tau\rho}$ ($= \mathrm{T}_{\rho\tau}$). Let us recall the temporal inversion ($\tau \to -\tau$) symmetry of AdS, denoted I, given in four dimensions by the transformation matrix $\mathrm{I}^{\mu'}_{\mu} = \mathrm{diag}(-1, 1, 1, 1)$. As none of our quantities depend explicitly on τ, we expect this discrete symmetry to be preserved. In particular, we expect $\mathrm{T}_{\tau \mathrm{x}^{\mathrm{j}}} = \mathrm{T}_{-\tau \mathrm{x}^{\mathrm{j}}} = \mathrm{T}_{\tau' \mathrm{x}'^{\mathrm{j}}}$, for $x^j = (\rho, \theta, \varphi)$. On the other hand, $\langle T^{\mu}{}_{\nu}\rangle^{(\alpha)}_{\text{ren}}$ transforms as a tensor, so we have

$$\langle T_{\mu'\nu'}\rangle^{(\alpha)}_{\text{ren}} = \mathrm{I}^{\mu}_{\mu'}\mathrm{I}^{\nu}_{\nu'}\langle T_{\mu\nu}\rangle^{(\alpha)}_{\text{ren}} \Rightarrow \mathrm{T}_{-\tau\rho} = \mathrm{T}_{\tau'\rho'} = \mathrm{I}^{\tau}_{\tau'}\mathrm{I}^{\rho}_{\rho'}\mathrm{T}_{\tau\rho} = -\mathrm{T}_{\tau\rho}. \tag{60}$$

That yields $\mathrm{T}_{\tau\rho} = -\mathrm{T}_{\tau\rho}$, which then implies $\mathrm{T}_{\tau\rho} = \mathrm{T}_{\rho\tau} \equiv 0$.

At this point, we have a diagonal tensor, whose remaining components may be calculated using Eq. 59. Our computational efforts were not successful when trying to compute the numerical expressions directly. However, we came up with a solution based on some properties that $\langle T^{\mu}{}_{\nu}\rangle^{(\alpha)}_{\text{ren}}$ must satisfy, based on the definition of $\langle T^{\mu}{}_{\nu}\rangle_{\text{ren}}$.

Let us first consider the effect of the trace anomaly. One can readily verify that it is respected by $\langle T^{\mu}{}_{\nu}\rangle^{(D)}_{\text{ren}}$ [2, 16], i.e.,

$$\langle T^{\mu}{}_{\mu}\rangle_{\text{ren}} = \langle T^{\mu}{}_{\mu}\rangle^{(D)}_{\text{ren}} = -\frac{1}{240\pi^2 H^4}, \tag{61}$$

so our tensor $\langle T^{\mu}{}_{\nu}\rangle^{(\alpha)}_{\text{ren}}$ must be traceless,

$$\langle T^{\mu}{}_{\mu}\rangle^{(\alpha)}_{\text{ren}} \equiv 0 = \mathrm{T}^{\tau}{}_{\tau} + \mathrm{T}^{\rho}{}_{\rho} + \mathrm{T}^{\theta}{}_{\theta} + \mathrm{T}^{\varphi}{}_{\varphi}, \tag{62}$$

which is our first constrain on the remaining diagonal components. We may use the symmetries of AdS as well. Although our Green's function breaks AdS invariance of the radial coordinate ρ, all other symmetries should remain valid. In AdS$_4$ there exist 10 Killing fields corresponding to the following isometries: one temporal translation, three rotations, four boosts and four spatial translations. From which, we only expect the first two to be preserved after imposing Robin boundary conditions in only one of the modes.

The temporal Killing field, $t = \partial_\tau$, yields a conservation equation along with its flow, given by the Lie derivative of the tensor with respect to t, i.e.,

$$\mathcal{L}_t \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow t^\sigma \partial_\sigma \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\tau \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0, \tag{63}$$

which shows that all components of $\langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}}$ are independent of τ. Additionally, we have the generators of spherical symmetry, given by the following Killing fields

$$\chi_1 = \partial_\varphi, \tag{64}$$

$$\chi_2 = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi, \tag{65}$$

$$\chi_3 = -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi. \tag{66}$$

Since a combination of them is still a Killing field, we may use χ_2 and χ_3 to obtain $\chi_4 = \partial_\theta$. We can use χ_1 and χ_4 to find other two conservation equations similar to that of t, as follows

$$\mathcal{L}_{\chi_1} \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow \chi_1^\sigma \partial_\sigma \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\varphi \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0, \tag{67}$$

$$\mathcal{L}_{\chi_4} \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow \chi_4^\sigma \partial_\sigma \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\theta \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} = 0. \tag{68}$$

These equations show us that $\langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}}$ can be a function of ρ only, i.e., $\langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}} \equiv \langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}}(\rho)$.

Finally, the conservation equation,

$$\nabla^\nu \langle T^\mu{}_\nu \rangle_{\text{ren}} = 0, \tag{69}$$

provide us with the last set of constrains. As $\langle T^\mu{}_\nu \rangle^{(D)}_{\text{ren}}$ is proportional to the metric, it is automatically conserved, since $\nabla^\mu g_{\mu\nu} = 0$. Hence, for $\langle T^\mu{}_\nu \rangle_{\text{ren}}$ to be entirely conserved, we must impose Eq. 69 on $\langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}}$ as well, which, using the properties we have found for $\langle T^\mu{}_\nu \rangle^{(\alpha)}_{\text{ren}}$ so far, reduces to

$$\begin{aligned}\partial_\rho \mathrm{T}^\rho{}_\rho - \tan\rho \mathrm{T}^\tau{}_\tau + (4\csc(2\rho) + \tan\rho)\mathrm{T}^\rho{}_\rho \\ -2\csc(2\rho)(\mathrm{T}^\theta{}_\theta + \mathrm{T}^\varphi{}_\varphi) = 0,\end{aligned} \tag{70}$$

$$\cot\theta(\mathrm{T}^\theta{}_\theta - \mathrm{T}^\varphi{}_\varphi) = 0 \Rightarrow \mathrm{T}^\theta{}_\theta = \mathrm{T}^\varphi{}_\varphi. \tag{71}$$

Before discussing our numerical approach for the expectation value of the energy-momentum tensor, we treat the case $\alpha \to \infty$, i.e., Neumann boundary condition. Again, we were able to find an analytic result only in this situation. We used equation 59 to find the formulas for components,∗

$$\begin{aligned}\mathrm{T}^\tau{}_\tau = \cos^2\rho\left([G]_{\tau\tau} - \tan\rho[G]_r\right) + \frac{1}{3}\cos^2\rho\tan\rho[G]_{,\rho} \\ -\frac{1}{12}\left(\cos^2\rho[G]_{,\rho\rho} + 2\cot\rho[G]_{,\rho}\right) - \frac{1}{2}[G],\end{aligned} \tag{72}$$

$$\begin{aligned}\mathrm{T}^\rho{}_\rho = -\cos^2\rho\left([G]_{\rho\rho} - \tan\rho[G]_r\right) + \frac{1}{3}\cos^2\rho\left([G]_{,\rho\rho} - \tan\rho[G]_{,r}\right) \\ -\frac{1}{12}\left(\cos^2\rho[G]_{,\rho\rho} + 2\cot\rho[G]_{,\rho}\right) - \frac{1}{2}[G],\end{aligned} \tag{73}$$

∗We are setting $H = 1$ to clear the expressions, later on we reinsert it.

$$\mathrm{T}^{\theta}{}_{\theta} = \mathrm{T}^{\varphi}{}_{\varphi} = -\cot^2\rho\tan\rho[G]_r + \frac{1}{3}\cot^2\rho\tan\rho[G],_\rho \\ -\frac{1}{12}\left(\cos^2\rho[G],_{\rho\rho} + 2\cot\rho[G],_\rho\right) - \frac{1}{2}[G]. \tag{74}$$

Our attempts to compute $[G]_{\rho\rho}$ and $[G]_\rho$ analytically and numerically were not successful. Hence, we adopted another approach that combined the explicit formulas above and the constrains given by equations 62 and 70.

Let us conveniently define a function $\mathrm{F}(\rho)$ depending exclusively on the quantities we were able to compute, namely $[G]_{\tau\tau}(\rho)$ and $[G](\rho)$, as follows

$$\mathrm{F}(\rho) := \csc^2\rho\mathrm{T}^{\tau}{}_{\tau}(\rho) - \mathrm{T}^{\theta}{}_{\theta}(\rho) \\ = \cot^2\rho\left\{[G]_{\tau\tau} - \frac{1}{12}\left(\cos^2\rho[G],_{\rho\rho} + 2\cot\rho[G],_\rho\right) - \frac{1}{2}[G]\right\}. \tag{75}$$

Using Eq. 62 and recalling that $\mathrm{T}^{\theta}{}_{\theta} = \mathrm{T}^{\varphi}{}_{\varphi}$, we find that

$$\mathrm{T}^{\rho}{}_{\rho}(\rho) = 2\mathrm{F}(\rho) - (1 + 2\csc^2\rho)\mathrm{T}^{\tau}{}_{\tau}(\rho), \tag{76}$$

and applying it to 70, we have

$$\partial_\rho\mathrm{T}^{\tau}{}_{\tau} + 2\frac{9 - \cos(2\rho) + 2\csc^2\rho}{(5 - \cos(2\rho))\cot\rho}\mathrm{T}^{\tau}{}_{\tau} = 2\frac{(\sin(2\rho)\mathrm{F}' + (7 - \cos(2\rho))\mathrm{F})}{(5 - \cos(2\rho))\cot\rho}. \tag{77}$$

The equation above is of the form

$$u'(\rho) + p(\rho)u(\rho) = q(\rho), \tag{78}$$

upon the identifications $u \equiv \mathrm{T}^{\tau}{}_{\tau}$,

$$p(\rho) = 2\frac{9 - \cos(2\rho) + 2\csc^2\rho}{(5 - \cos(2\rho))\cot\rho} \tag{79}$$

and

$$q(\rho) = 2\frac{(\sin(2\rho)\mathrm{F}'(\rho) + (7 - \cos(2\rho))\mathrm{F}(\rho))}{(5 - \cos(2\rho))\cot\rho}. \tag{80}$$

One can verify that

$$u(\rho) = \exp\left[-\int d\rho\ p\right]\left(\int_0^\rho d\rho'\ \exp\left[\int d\rho' p\right]\cdot q + \mathbf{C}\right) \tag{81}$$

solves the equation. In our case, we have

$$\exp\left[\int d\rho\ p\right] = \tan\rho\sec^3\rho\sqrt{5 - \cos(2\rho)}, \tag{82}$$

which vanishes at $\rho = 0$ and diverges at $\rho = \pi/2$. Naturally, the inverse function $\exp\left[-\int d\rho\ p\right]$ vanishes at the boundary, but diverges at $\rho = 0$ with strength $1/\rho$. As it is physically reasonable to ask for a finite $\mathrm{T}^{\tau}{}_{\tau}$ at $\rho = 0$, we set $\mathbf{C}$ to zero. Finally, we compute $\mathrm{T}^{\tau}{}_{\tau}$ using the following expression

$$\mathrm{T}^{\tau}{}_{\tau}(\rho) = 2\frac{\cot\rho\cos^3\rho}{\sqrt{5 - \cos(2\rho)}}\int_0^\rho d\rho'\ \frac{\tan^2\rho'\Big(\sin(2\rho')\mathrm{F}'(\rho') + (7 - \cos(2\rho'))\mathrm{F}(\rho')\Big)}{\cos^3\rho'\sqrt{5 - \cos(2\rho')}}. \tag{83}$$

For the Neumann case, we used our previous analytic results and found F to be

$$\mathrm{F}(\rho) = \frac{\cot^2\rho}{48\pi^2}\Big(\csc^2\rho + 2 + 2(\csc^4\rho - 1)\log(\sec\rho)\Big). \tag{84}$$

Applying it in Eq. 83, and then using 76 and 75, we find

$$\langle T^{\mu}{}_{\nu}\rangle^{(N)}_{\text{ren}}(\rho) = \frac{\cot^2\rho}{48\pi^2 H^4}\Big\{\Big(\sin^2\rho\Big)\mathrm{diag}(1,-1,0,0) + \Big(1 - 2\cot^2\rho\log(\sec\rho)\Big)\mathrm{diag}(1,1,-1,-1)\Big\}. \tag{85}$$

Now, we have a result to compare our numerical ones with.

To compute the function F numerically, we used our previous results of $\langle\phi^2\rangle^{(\alpha)}(= [G])$, but we also need $[G]_{\tau\tau}$. According to 55, we find it by taking the second derivative of $G_F^{(\alpha)}(\tau,\tau',\rho,\rho)$ with respect to τ and, then, taking the coincidence limit. In the convention we adopted, $\partial_{\tau\tau} = -\partial_{\epsilon\epsilon}$. Its then expected that the divergent behavior of the Dirichlet counterpart $G_F^{(\alpha,D)}$ is not that of Eq. 43 anymore. Indeed, we find it to be

$$-\partial_{\epsilon\epsilon}G_F^{(\alpha,D)} = \frac{\cot^2\rho}{8\pi^2}\times\Big\{-\frac{1}{\epsilon^2} - \frac{1}{24}(5+\cos(4\rho))\csc^2\rho\sec^2\rho + \mathcal{O}(\epsilon^2)\Big\}. \tag{86}$$

Our numerical procedure to find the expectation value of the energy-momentum tensor fluctuations was:

(i) Given a value for α, use the frequencies $\omega_{\alpha,r}$ found before;

(ii) Given a value of ρ between 0 and $\pi/2$, compute numerically the truncated summation

$$-i\partial_{\epsilon\epsilon}G_F^{(\alpha)} \approx -\frac{\cot^2\rho}{2\pi}\sum_{r=1}^{r_{\max}}\Big(\frac{\omega_{\alpha,r}^2\sin^2(\omega_{\alpha,r}\rho)}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}e^{-\omega_{\alpha,r}\epsilon} - \frac{(2r)^2\sin^2(2r\rho)}{2r\pi}e^{-2r\epsilon}\Big), \tag{87}$$

denoted $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$, for 50 values of ϵ equally spaced in the range 0.002 to 0.1.

(iii) Fit the function $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$ using a model that reproduces the divergent behavior followed by a Taylor expansion up to order ϵ^2, i.e.,

$$\mathtt{h}[\epsilon] = \mathtt{a} + \frac{\mathtt{b}}{\epsilon^2} + \mathtt{c}\cdot\epsilon + \mathtt{d}\cdot\epsilon^2. \tag{88}$$

As expected, the divergent behavior of $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$ is extremely attenuated, and the coefficient `b` is negligible compared to the others. Again, the coefficients `c` and `d` were effective on reducing the residuals of the fit. Finally, `a` gives the finite approximated numerical value of $[G]_{\tau\tau}$ at the point ρ.

(iv) Repeat steps 2 and 3 for as many values of ρ between 0 and $\pi/2$ as desired to obtain the complete $[G]_{\tau\tau}(\rho)$.

(v) Use our previous results for $[G]$ together with $[G]_{\tau\tau}$ in Eq. 75 to find a numerical interpolation of $\mathrm{F}(\rho)$, denoted $\mathtt{F}[\rho]$.

(vi) Given a value of ρ between 0 and $\pi/2$, use $\mathtt{F}[\rho]$ in Eq. 83 and perform a numerical integration to obtain an approximate value of $\mathrm{T}^{\tau}{}_{\tau}$ at that specific ρ.

(vii) Repeat step 6 for several values of ρ to find a complete numerical function $\mathrm{T}^{\tau}{}_{\tau}$. With that in hands, compute $\mathrm{T}^{\rho}{}_{\rho}$ and $\mathrm{T}^{\theta}{}_{\theta}$ using equations 76 and 75.

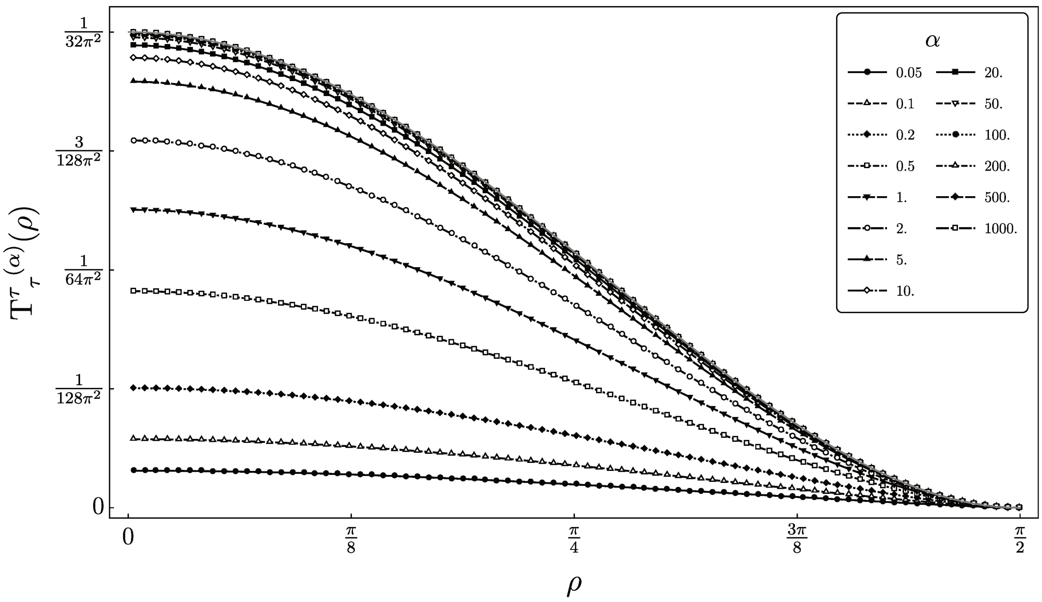

Figure 3. Contribution to the expectation value $\mathrm{T}^{\tau}{}_{\tau}$ due to Robin boundary conditions at infinity for the spherically symmetric mode. H is set to one.

(viii) Repeat the entire procedure for a different value of α.

Similarly to our results for the expectation value of the field squared, we followed the numerical procedure for 14 values of α. We have found all components of $\langle T^{\mu}{}_{\nu}\rangle^{(\alpha)}_{\mathrm{ren}}$. In Fig. 3, we can see $\mathrm{T}^{\tau}{}_{\tau}$ for several values of α, it is clear that the form of the function follows the analytic result for the Neumann condition (plotted in gray).

6. Discussion and further remarks

Avis, Isham, and Storey took a first-step, in Ref. [3], towards the development of a quantum field theory in anti-de Sitter spacetime. They acknowledged that the conformal infinity poses a serious causality issue to the wave equation but solve it by regulating the information flow through the boundary 'by hand.' They imposed the so-called 'transparent' and 'reflective' boundary conditions at infinity in analogy to a box in Minkowski spacetime. In this way, they quantized the fields in the Einstein Static Universe and restricted it to the AdS later.

Conversely, in this article, we considered the developments made by Ishibashi and Wald in [5], where they propose a physically consistent prescription for the dynamical evolution of fields. In the particular case that we have considered, they show that the imposition of mixed boundary conditions at the spatial infinity is sufficient to determine the evolution of quantum fields uniquely.

In the setup studied by Kent and Winstanley, in Ref. [2], all angular modes of the wave equation satisfy the same Dirichlet boundary condition at infinity. Their results are consistent with the maximal symmetry of AdS. Hence, the expectation values of field-dependent quantities fluctuate in the same way throughout spacetime, i.e., they are coordinate-independent. In the light of Wald's and Ishibashi's developments, we presented a setup here that puts up to question how necessary it is to impose the same boundary conditions to all modes of the wave equation. Indeed, we are not aware of any requirement of nature that precludes us from considering various setups in terms of boundary conditions.

Our analysis indicated a violation of AdS invariance in the Green's functions, which carried out implications on the related quantities: the quadratic fluctuations of the field and the energy-momentum tensor. Both of them are now dependent on the radial coordinate for any values of the parameter α, as shown in Fig. 2 and 3. At this stage, any attempt of obtaining a back-reacted metric using Einstein's semi-classical equations,

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle_{\mathrm{ren}} \equiv 8\pi G\left(\langle T_{\mu\nu}\rangle^{(D)}_{\mathrm{ren}} + \langle T_{\mu\nu}\rangle^{(\alpha)}_{\mathrm{ren}}\right), \tag{89}$$

would not yield a maximally symmetric metric anymore, but a spherically symmetric one. In these conditions, the coordinate system used to define the angular modes of the wave equation will be privileged. In particular, in this system, the energy density reaches its minimum at the origin $\rho = 0$, as shown in Fig. 4.

In Fig. 4, we can see a clear violation of the weak energy condition in most of the spacetime, except close to the boundary, where the Dirichlet contribution, $-\langle T^{\tau}{}_{\tau}\rangle^{(D)}_{\mathrm{ren}}$, pushes the energy density back up over zero. Even though such violation is no stranger to us - as can be observed in the Casimir effect - it appeared as a consequence of the contribution from the Robin boundary condition exclusively. Indeed, the Dirichlet term, $-\langle T^{\tau}{}_{\tau}\rangle^{(D)}_{\mathrm{ren}}$, of the energy density is positive throughout the entire spacetime. Thus, it is safe to assert that the violation of the weak energy condition is a direct consequence of the imposition of non-Dirichlet boundary conditions at infinity.

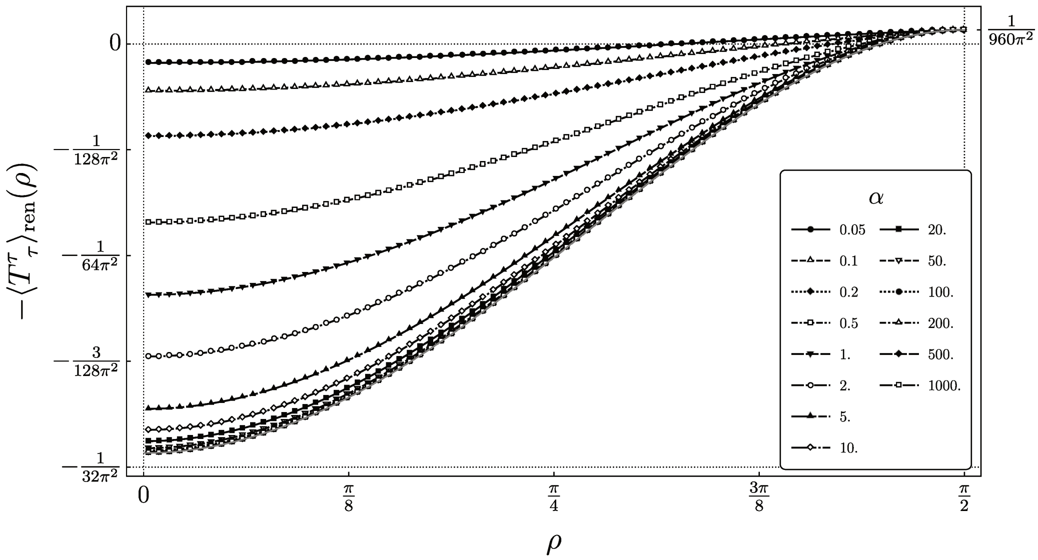

Figure 4. Energy densitiy $-\langle T^{\tau}{}_{\tau}\rangle_{\mathrm{ren}}$ of a massless scalar field conformally coupled to AdS_4 for several Robin boundary conditions. H is set to one.

Acknowledgments

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