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# SOLITON DYNAMICS IN BOSE-EINSTEIN CONDENSATES

## DINÂMICA DO SÓLITON EM CONDENSADOS DE BOSE-EINSTEIN

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Tatiana de Picoli Ferreira

# Soliton dynamics in Bose-Einstein condensates

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I stand
on the sacrifices of a million woman before me
thinking
*what can I do*
*to make this mountain taller*
*so the women after me*
*can see farther*

*legacy* - rupi kaur

# Resumo

Neste trabalho, nós estudamos um método de quantização e os efeitos dissipativos de uma solução topológica em um Condensado de Bose-Einstein. Em particular, nosso trabalho focou no estudo de um condensado unidimensional na presença de uma solução do tipo bright soliton. Este tipo de sistema apresenta uma simetria translacional que permite ao sóliton se mover no espaço. No entanto, o método de quantização das Coordenadas Coletivas não é suficiente para explicar a dinâmica e dissipação deste tipo de sistema. Nós desenvolvemos um Método das Coordenadas Coletivas Estendido, com intuito de quantizar o campo de Schrödinger corretamente, descrevendo um sistema efetivo do tipo partícula acoplada a um banho térmico. Este método se mostrou adequado para tratar de Lagrangianas lineares na derivada temporal. O método foi aplicado a um sistema com uma solução do tipo bright soliton. Nós mostramos que tanto a função de decaimento quanto a de difusão, são dependentes da temperatura e Markovianas. A forma da função de decaimento nos permitiu mostrar que o processo de dissipação do sóliton se dá por uma força do tipo Abraham-Lorentz.

# Abstract

In this work, we studied the quantization and dissipation effects of a topological solution of a Bose-Einstein Condensate. In particular, we used the one-dimensional condensate with a bright soliton solution. This system presents a translational symmetry that allows the soliton to travel in space. However, the quantization method of the Collective Coordinate is not sufficient to explain either dynamics or dissipation that this soliton may present. We have developed an Extended Collective Coordinate Method in order to correctly quantize a Schrödinger field, describing an effective system of a particle coupled with a thermal bath. This method proved itself adequate to treat Lagrangians that are linear in the time derivative. We have applied our method in a system with a bright soliton. We found both damping and dissipative functions to be temperature-dependent and Markovian. The form of the damping function showed us that the dissipation process of the soliton is due to a Abraham-Lorentz force.

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# Introduction

The beginning of the 20th century was marked by countless works that shaped the foundations of a new subject of physics that today we know as quantum mechanics. At that time, quantum mechanics emerged as a mathematical artifact to explain radiation effects that the classical mechanics was not able to describe. Max Planck himself was reluctant to believe that a real physical system could present quantized levels of energy, contradicting his findings [1]. Not long after, physicists from all over the world started to reach an agreement about this new subject. The rising of quantum mechanics showed that other branches of physics could also benefit from the richness of this new theory. In fact, the decade of 1920 is abundant in papers that marked the beginning of quantum electrodynamics, quantum field theory, and quantum statistics. In the path of field theory, in 1928 W. Heisenberg and W. Pauli published a work [2] in which they developed a canonical quantization method to study quantum behavior from classical field theory. The canonical quantization method was the first method presented to go to a quantum approach from a classical description. Thenceforth, several different methods of quantization were developed.

The decade of 1920 was also marked with the rising of the Bose-Einstein statistics. In 1924, the Indian physicist Satyendra N. Bose was studying blackbody radiation, which he was able to describe with the photon spectrum obtained by the statistics of indistinguishable particles. At that time, he was unable to publish his findings as quantum mechanics was still not widely accepted in the scientific community. Convinced by the importance of his findings, he sent the article for one of that most prestigious physicists at the time – Albert Einstein. Recognizing the impact of this work, Einstein translated the paper himself and published in 1925 what is known nowadays as Bose-Einstein statistics [3, 4]. This system of indistinguishable particles presented a threshold in which the particles were expected to condensate in the lowest energy state, given rise to a Bose-Einstein Condensate. Although this system was predicted at the beginning of the century, the existent technology restricted its experimental achievement for 70 years [5].

The system predicted by Bose and Einstein presents a very interesting statistics, but it is quite different from real physical systems. Their description does not take into account interactions, confinement potential, and other constraints that may arise. In 1961, Gross [6] and Pitaevskii [7] derived, independently, a field equation to describe the dynamics of Bose-Einstein Condensates of interacting particles, known as the Gross-Pitaevskii equation. This equation is a non-linear Schrödinger-like equation that describes a classical mean-field $\Psi(\mathbf{r}, t)$. Although its derivation comes from quantum field theory, the macroscopic occupancy of the condensate allows us to study the mean-field behavior through the condensate wave function

One of the main features of an interacting Bose gas lies on its nonlinearity. A non-linear theory provides a very rich physics in which topological deformations such as solitons and vortex may arise. These topological effects are a special class of solutions of the non-linear equation of motion, as the Gross-Pitaevskii equation. Here we will use topological, localized, and solitonic solutions interchangeably. These solitonic solutions can be defined as a shape conserved packet of matter that can travel freely in space. For having a fixed shape and mass, these solitonic solutions resemble the behavior of an extended particle. Nevertheless, this resemblance is merely qualitative, since classical field theory does not define the concept of a particle.

To treat this localized solution as a particle, one must use a quantization method to go beyond the mean-field approximation. The most adequate quantization method to treat systems with a dynamical behavior is called Collective Coordinate Method [8]. This method promotes the coordinate of the center of the localized solution to a dynamical variable, called collective coordinate. A great advantage of using this method is to rewrite a complicated field system as an effective system of a single particle in a bath of harmonic oscillators. The particle is described by the center of the solitonic solution and the bath is composed of excited quasiparticles acting as a thermal reservoir.

Although both particle and bath share the same microscopic origin, this effective system resembles the one used within the system-plus-reservoir approach for treating the quantum Brownian motion. Therefore, it seems reasonable to connect the quantization method of dynamical systems with the quantum Brownian approach to study the dissipation and dynamical properties of localized solutions. In fact, a systematic methodology was developed by Castro Neto and Caldeira in which they unified those two theories for a one-dimensional scalar field [9].

Since the nonlinearity of the Bose-Einstein Condensation allows the appearance of topological solutions, in this work we want to see how this theory can be applied in the Schrödinger field. In particular, we want to apply the quantization method of the Collective Coordinate and study dissipation effects in the one-dimensional Schrödinger field. Both the Collective Coordinate Method and the study of quantum dissipation can be found in literature applied for scalar fields [8,10]. In this work, we want to investigate the applicability of those methods for a complex field with a Lagrangian that is linear in the time derivative. As the experiments suggest, solitons are not stable configurations in Bose-Einstein Condensates [11,12], and they suffer dissipative effects that are not well explained by the mean-field approximation. In the literature, we can find several approaches to explain dynamical and dissipative processes, which usually involves phenomenological methods. For instance, one can make use of the stochastic projected Gross-Pitaevskii equation (SPGPE), in which dissipative terms are added into the original equation [13,14]. Here we want to be able to explain dissipative behavior via a microscopic theory. We expect to be able to write an effective system of a particle in a bath of oscillators using the adequate quantization method. Then, we can study a theory for quantum dissipation in which we expect our dissipation parameters (as damping and diffusion) to be temperature-dependent.

We begin our discussion, in Chapter 1, by presenting differences and similarities between classical and quantum theories. The discussion follows the use of quantization methods, in which we develop a very useful method for quantizing translationally invariant systems. The method of Collective Coordinate is applied for a one-dimensional scalar field, which is shown to present an effective behavior of a particle coupled with a bath of non-interacting quasi-particles.

In Chapter 2, we introduce our object of study, which is a Bose-Einstein Condensate. We begin with a basic analysis for the ideal Bose gas, in which we introduce the Bose-Einstein distribution function $\bar{n}$ and show the possibility of high occupancy of its ground state. Our discussion proceeds to the quantum field approach to understanding non-ideal Bose gases. Some topological solutions for this system are shown and we close our literature review with some important results found when we treat these systems beyond the mean-field approximation.

Chapter 3 is dedicated to the development of our theory. We start defining the Schrödinger field and proceed to apply the Collective Coordinate Method showed in Chapter 1. After the discussion about the applicability of the method, we propose an Extended Collective Coordinate Method to treat Lagrangians that are linear in the time derivative of the fields. Our effective Lagrangian is used in the Feynman-Vernon approach of the influence functional for the study of quantum dissipation, showing that the effective system can be treated in the same way as we show in the first chapter.

In Chapter 4, we apply the method developed in the previous chapter to the bright soliton solution in a Bose-Einstein Condensate. We compare our results with the ones found in the literature, showing that our theory is suitable to describe quantum dissipation in the Schrödinger field. In this chapter, we also discuss the mechanism involved in the soliton dissipation.

We finish the discussion of this dissertation with our conclusions in Chapter 5. Here we review what we can find in the literature on this subject, the literature gaps we expect to have filled, our main results, and propose future discussions on subjects that we may have left behind.

# Chapter 1

# Dynamics of Quantum Fields

When we first begin to learn about quantum theory, it becomes clear that quantum and classical systems present very different approaches and results. In fact, each theory was developed to deal with different systems. Whilst classical mechanics deals with macroscopic systems with deterministic physical quantities, quantum theory was designed for treating small atomic systems at zero (or very low) temperature. The intriguing part begins when a system composed of many quantum particles (bosons or fermions) presents, among others, a collective macroscopic behavior at finite temperatures. In this case, quantum and classical theories coalesce, giving rise to several fascinating effects. In this work, we are particularly interested in the study of the quantum dissipation of a collective variable in a macroscopic system, such as a Bose-Einstein Condensate. This chapter is dedicated to the understanding of how to reconcile classical and quantum theories, so we can reach our goal. We will describe the method to quantize a scalar field and to study its dissipation properties.

## 1.1 From Classical to Quantum Theories

Before we discuss the method to investigate quantum properties in a scalar field, let us review some disparities and similarities between classical and quantum theories.

In classical mechanics, a typical starting point is a system of a non-relativistic particle in one-dimension. This particle can be described by a general Lagrangian of the form

$$L = \frac{m\dot{q}^2}{2} - V(q), \tag{1.1}$$

where $m$ is the particle mass, $\dot{q}$ is the velocity related to the $q$ coordinate and $V(q)$ is an arbitrary potential energy, that depends only on the space variable. From the Lagrangian, it is possible to write equations of motion and find solutions for $q(t)$. As we know, the solution depends strongly on the form of the potential in which the particle is placed. Suppose a bistable potential $V(q)$ showed in figure 1.1.

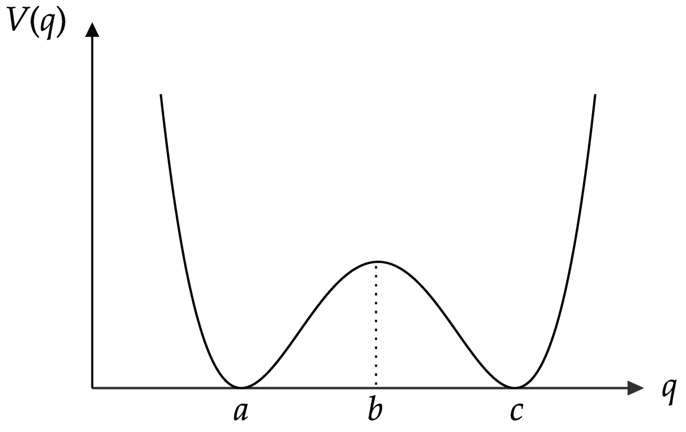

Figure 1.1: Illustrative bistable potential $V(q)$ in one dimension.

This potential allows three distinct static solutions: $q(t) = a$, $q(t) = b$ and $q(t) = c$. The first and the third are stable solutions with the lowest potential energy $V(a) = V(c)$, while $q(t) = b$ is an unstable one. Here we will focus only on the two stable solutions. A classical particle placed in one of these two solutions will remain static unless an amount of energy is given to the system. If a particle placed in $q(t) = a$ receives an energy $E$ smaller than the energy barrier $V(b)$, it starts to describe an oscillatory movement around $q(t) = a$. Only when a sufficient amount of energy is given, $E > V(b)$, the particle is allowed to move to the other side of the barrier. The same happens for the other stable solution, $q(t) = c$. However, if the Lagrangian 1.1 describes a quantum particle, the description would be quite different – the Heisenberg uncertainty principle does not allow the same set of solutions as in the classical case. The uncertainty principle states that, in quantum systems, solutions with both momentum and position cannot be determined with such a precision. A direct consequence of this principle is that the lowest energy state is no longer $V(a)$ or $V(c)$. In fact, if the barrier $V(b)$ is large enough, in the sense that tunneling effects could be neglected, the potential in the neighborhood of $q(t) = a$ and $q(t) = c$ can be approximated by a harmonic one. In this approximation, the lowest energy level is

$$E = V(a) + \frac{1}{2}\hbar\omega, \tag{1.2}$$

in which $\omega$ is the natural frequency of the harmonic oscillator. Another peculiarity is that even if the particle presents energy lower than $V(b)$, and moves about $q = a$, it could cross the potential barrier through a tunneling process.

From the single-particle case, the differences between classical and quantum systems become clear. However, the similarities between these two theories arise in the context of many-particle physics where there may be quantum systems with a collective behavior which act as a classical particle, even though quantum effects are not fully suppressed. There are several physical systems which lie on the edge between classical and quantum physics. Some of the quantum systems that may present macroscopic behavior are, for example, superfluids, superconductors, and laser light.

To understand how quantum properties appear in a macroscopic system, it is convenient to use the concept of a field to describe a large amount of particles. Suppose we have $N$ identical particles connected one-by-one by $N-1$ massless springs. To understand this system, we could write the discrete form of the Lagrangian as equation 1.1, describing the dynamics of each coordinate $q_1$, $q_2$, ..., $q_N$. This system is exactly solvable for harmonic springs, and many times it is convenient to describe it in the long wavelength or continuous limit. In this description, our system can be interpreted as a continuous elastic string for $N \to \infty$. This string is now represented by the field $\phi(x,t)$, instead of the coordinate $q(t)$. The Lagrangian that describes a classical field can be written as a functional of the field $\phi$,

$$L\left[\phi\right] = T\left[\phi\right] - V\left[\phi\right], \tag{1.3}$$

where the first term describes the kinetic energy and the second one the potential energy.

Suppose that this classical Lagrangian describes our elastic string in a similar bistable potential as before. The Lagrangian density for a particular bistable potential of the type $U(\phi) = (\phi^2 - 1)^2$ can be written as

$$\mathscr{L}\left[\phi\right] = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\phi^2 - 1\right)^2 . \tag{1.4}$$

This is the well-known $\phi^4$ theory [8]. The Euler-Lagrange equation of motion originated from the $\phi^4$ theory presents, among others, a new class of solutions called the kink solution [8]. The kink solution is a topological deformation in the field space with the form

$$\phi(x) = \pm\ \tanh\left(x - x_0\right), \tag{1.5}$$

in which the solution with the plus sign is called "kink" and the minus sign the "antikink", with both forms showed in figure 1.2.

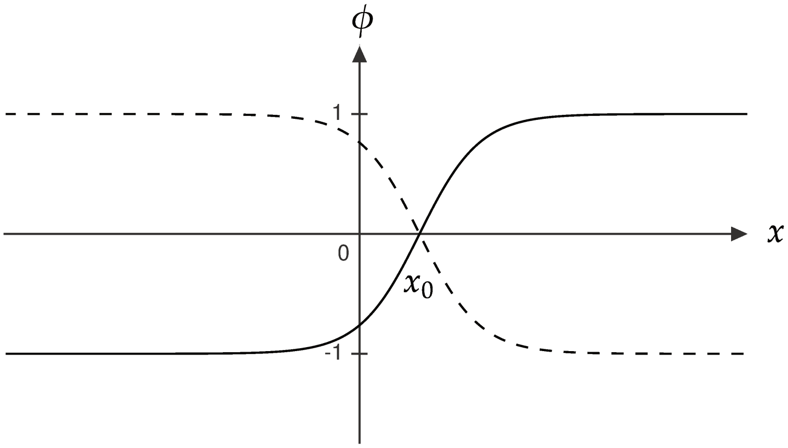

Figure 1.2: Plot of the kink solution 1.5, in which the "kink" is represented by the continuum line and the "antikink" by the dashed line.

In order to illustrate the kink solution for our toy model of the elastic string, we can imagine the generalization of the bistable potential of figure 1.1 to the potential energy density of the field $\phi$. In this system, a kink solution could represent this string interpolating from the minimum at $\phi(x) = -1$ to $\phi(x) = 1$ going over the potential energy barrier between the two minima, as figure 1.3 illustrates.

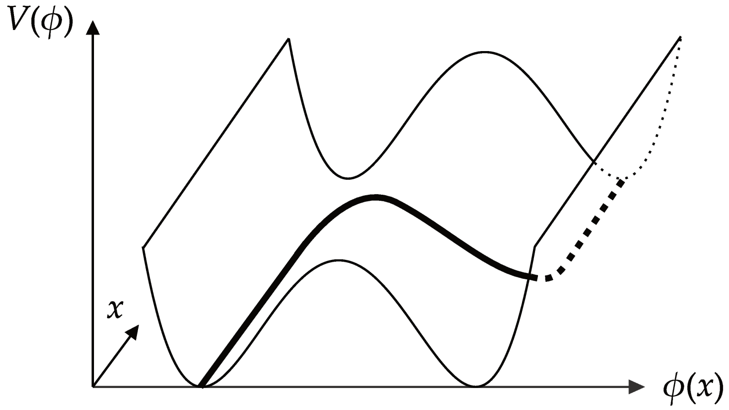

Figure 1.3: Illustrative form of the kink solution for an elastic string.

From this model, it is clear that this string presents a modulation around $x = x_0$, showing the topological nature of the kink solution. Leaving our toy model of a elastic string for a more general classical field, the kink solution usually resembles an extended particle, in the sense that it has a localized energy density about $x_0$. This means that there is a localized energy stored in the deformation of the field configuration when the solution interpolates from one classical energy minimum to the other. Although the behavior of this solution resembles an extended particle, the concept of a particle does not exist in the classical field theory.

A classical field can be defined as a scalar function of space and time $\phi(\mathbf{r}, t)$. The states are specified by the fields themselves and their dynamics are described by an equation of motion. In particular, for non-linear equations of motion, localized solutions are merely a class of topological solutions of these differential equations. However, in quantum field theory, the concept of a particle exists and can be understood as follows. Quantum fields are no longer functions, but operators of space and time. The dynamics of these fields obey the Heisenberg field equations of motion, but the solution of these equations are, again, operators in the Hilbert space. These operators can act on vectors in that space, creating and annihilating specific states in the Hilbert space. The concept of a particle arises from a very specific set of states which carry energy and momentum. In other words, they are the quanta of the classical normal modes which are labelled by frequency and wavevectors.

Since the concept of a particle can only exist in the context of quantum field theory, we can expect that some classical localized solutions, such as the kink solution, to have a quantum analogue. In fact, there are several systems with classical localized solutions that may exhibit quantum behavior, despite of describing in first approximation a macroscopic classical behavior of the system. This macroscopic behavior is often described by a mean-field. If the classical mean-field has a topological solution that resembles a particle, the system can be quantized so we can explore its genuine characteristic of a quantum particle. In the next few sections, we will show how to quantize a classical field and study, for example, quantum dissipation in the effective quantum system.

## 1.2 Quantization of Localized Solutions

In treating quantum many-body problems, a set of new mathematical tools is necessary. In this theory, the fields are operators in the Fock space that obey Heisenberg equations of motion. Although the interpretation of a classical field diverges from that of quantum fields in many ways, it is possible to neglect this difference at a first sight and make a semi-classical approximation to obtain the mean-field behavior for the system. In doing so, we neglect several quantum effects that must be recovered by approximation methods. Thus, the use of a semi-classical approximation requires a quantization method to reincorporate quantum fluctuations to the problem. In this section, we will show how to evaluate those quantum fluctuations around a static localized mean-field.

Suppose that a many-body system can be described by the general Lagrangian

$$L\left[\phi\right] = \int dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U\left(\phi\right)\right], \tag{1.6}$$

in which $\phi = \phi(x,t)$ is a scalar function of space and time. Since we want to study a general system, we set $U\left(\phi\right)$ as an arbitrary function of the field. Using the definition

$$L\left[\phi\right] = T\left[\phi\right] - V\left[\phi\right], \tag{1.7}$$

we can define the kinetic and potential energy functionals, respectively as,

$$T\left[\phi\right] = \int dx \frac{1}{2}\dot{\phi}^2, \tag{1.8}$$

and

$$V\left[\phi\right] = \int dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U\left(\phi\right)\right]. \tag{1.9}$$

From Lagrangian 1.6, we can find the Euler-Lagrange equation of motion for $\phi(x,t)$. If we choose to analyze a particular static solution, it can be obtained as an extreme of the potential energy. In our case, the static solution is given by a functional derivative of the potential 1.9,

$$\frac{\delta V}{\delta \phi} = 0. \tag{1.10}$$

Suppose one of the solutions of this equation is a localized static solution $\phi_0(x)$. This solution describes a topological deformation of the field. Since our solution is a static one, the classical energy is simply given by

$$E = V\left[\phi_0(x)\right]. \tag{1.11}$$

Any quantum fluctuation of this system will oscillate around the classical solution. To find high order corrections for the energy, we need to expand the potential 1.9 around the static solution. The functional Taylor expansion of the potential reads

$$V = V\left[\phi_0\right] + \frac{1}{2}\int dx \delta\phi \left[-\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{d^2U}{d\phi^2}\right)_{\phi_0}\right]\delta\phi. \tag{1.12}$$

Here we have expanded our field as a mean-field $\phi_0(x)$ plus fluctuations $\delta\phi(x,t)$ as

$$\phi(x,t) = \phi_0(x) + \delta\phi(x,t). \tag{1.13}$$

The first term of equation 1.12 is the potential evaluated at $\phi_0$. Terms that are of the first order in the fluctuations vanish because our solution is static and obeys the condition 1.10. Second order terms in the fluctuations obey an eigenvalue problem that can be solved by expanding $\delta\phi(x,t)$ as

$$\delta\phi(x,t) = \sum_k c_k(t)\delta\phi_k(x), \tag{1.14}$$

where $\{c_k(t)\}$ are the set of new coordinates of our problem and $\delta\phi_k(x)$ describe the normal modes of the fluctuations, which are solutions of

$$\left[-\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{d^2U}{d\phi^2}\right)_{\phi_0}\right]\delta\phi_k(x) = \omega_k^2\delta\phi_k(x). \tag{1.15}$$

Solving the eigenvalue problem 1.15, the effective potential can be written as

$$V = V\left[\phi_0\right] + \sum_{k=0}\frac{1}{2}c_k^2\omega_k^2. \tag{1.16}$$

To write an effective Lagrangian for this system, we must write the kinetic energy 1.8 in the new set of coordinates 1.14. In order to accomplish it, we first write

$$\begin{aligned}\dot{\phi} &= \frac{\partial}{\partial t}\left(\phi_0(x) + \sum_k c_k(t)\delta\phi_k(x)\right), \\ &= \sum_k \dot{c}_k(t)\delta\phi_k(x).\end{aligned} \tag{1.17}$$

Substituing 1.17 in the kinetic energy 1.8, we can write an effective Lagrangian of the form

$$L = \sum_{k=0}^{\infty}\left[\frac{1}{2}\dot{c}_k^2 - \frac{1}{2}c_k^2\omega_k^2\right] - V\left[\phi_0\right]. \tag{1.18}$$

To summarize, our starting point was a quantum field theory in which we used a classical approximation to find a static mean-field $\phi_0(x)$. To reincorporate the quantum fluctuations neglected by the classical approximation, we expanded our scalar field around the mean-field solution $\phi_0$ as done in 1.13. If we rewrite the Lagrangian using 1.13 and 1.14, we can show that the classical field has a behavior described by the effective Lagrangian 1.18. This

Lagrangian describes a set of infinite harmonic oscillators built upon the classical value $V[\phi_0]$. The quanta of these oscillators can be interpreted as the quasiparticles of the system with "wave functions" $\delta\phi_k(x)$ that are related to a wavenumber $k$ and frequency $\omega_k$.

Since the effective Lagrangian 1.18 describes a set of harmonic oscillators, the coordinates $c_k$ can be quantized to rewrite the energy of the system with quantum corrections as

$$E = V[\phi_0(x)] + \frac{1}{2}\sum_k \hbar\omega_k. \tag{1.19}$$

This means that, when we include quantum fluctuations into the problem, the energy 1.11 is corrected by a term that is a summation of the zero point energies of harmonic oscillators.

The quantization procedure described above is valid if, and only if, the second derivative of the potential 1.9 is different from zero for all values of $\phi$. The reason is that higher orders corrections to the energy 1.19 require the use of a perturbation method, in which the eigenvalues $\omega_k$ appear in the denominator to the perturbation theory. If the denominator is zero at any point, the correction for the energy diverges. Therefore, the zero eigenfrequency shows that the quantization method is not appropriate. In the case that the eigenvalue problem 1.15 allows a solution with zero eigenvalue, we must use another quantization procedure. The eigenvalue $\omega_0 = 0$ is known as a translational mode or zero-mode. In the next section, we will explore the conditions under which the translational mode appears, its physical meaning, and why the described procedure in this section is not appropriated to treat the zero-mode.

## 1.3 Translational Modes

In the evaluation of the quantum corrections of a classical static solution, we have seen that the procedure is highly dependent on the form of the potential $V[\phi]$. If the functional $V[\phi]$ has its second derivative equal to zero in any configuration, this means that our eigenvalue problem 1.15 allows a solution with eigenfrequency $\omega_0 = 0$,

$$\left[ -\left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{d^2 U}{d\phi^2} \right)_{\phi_0} \right] \delta\phi_0(x) = 0. \tag{1.20}$$

The zero-mode is also called the translational mode. That is because it occurs in the quantization of any static field $\phi_0(x)$ that is a solution of a translationally invariant potential $V[\phi]$.

Suppose $\phi_0(x)$ is a solution associated with the potential energy $V[\phi_0(x)]$. In a translationally invariant theory, a shifted solution $\phi_0(x - x_0)$ has the same energy $V[\phi_0(x)]$, independently of $x_0$. That is, in the space of functions $\phi$, every point $\phi_0(x - x_0)$ for any value of $x_0$ has the same energy, creating an equipotential curve. In the equipotential curve, we can always move from one point to another without changing the energy of the system. This is called neutral stability. Therefore, although $\phi_0(x)$ was found to be an extreme of the potential $V[\phi_0(x)]$, it is not a point of minimum in the field space, even locally.

To find the eigenfunction related to the zero-mode, we can write the equation of motion 1.10 in the form

$$-\frac{\partial^2 \phi(x)}{\partial x^2} + \frac{\partial U(\phi)}{\partial \phi} = 0. \tag{1.21}$$

Taking the derivative with respect to $x$, we obtain

$$\left( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2 U(\phi)}{\partial \phi^2} \right) \frac{\partial \phi_0}{\partial x} = 0. \tag{1.22}$$

Comparing this equation to the eigenvalue problem 1.20, we can see that the zero-mode eigenfunction is the local tangent in the field space, $\delta\phi_0 = \delta x (\partial\phi_0/\partial x)$, connecting all shifted functions along the equipotential curve.

Now that we have seen when the zero-mode appears, we can understand why the quantization procedure from the last section was not suitable. The reason is that when one of the modes has zero energy, the Lagrangian 1.18 misses one degree of freedom. The summation over the $k$ modes $\sum_{k=0}$ becomes $\sum_{k=1}$ if the first term of the summation is the zero-mode $\omega_0 = 0$. As it was shown, $x_0$ can change freely along the equipotential curve, and therefore, we expect the effective Lagrangian to have a kinetic term related to the motion of $x_0$. Therefore, the absent degree of freedom is related to the translational motion of the coordinate $x_0$.

Evidently, one could easily extend the presented approach for systems with more than one dimension. In one dimension, there is one zero-mode corresponding to the translation along the $x$ axes. Generalizing, in an $N$-dimensional problem, we should expect $N$ zero modes, one for each coordinate. Furthermore, there are other sources of zero modes, all related to symmetries that the system may present, but here we will focus on the translational symmetry only.

To treat the zero-mode without losing any degree of freedom of the system, we can make use of a quantization method called Collective Coordinate Method that was first presented by Christ and Lee [15] and is deeply discussed in [8].

## 1.4 Collective Coordinate Method

When the system presents some sort of symmetry that gives rise to zero modes, one needs a proper way to quantize this system. The literature shows several ways to treat the zero-mode [15–19], and many of them involve the use of collective coordinates. In this section, we will present the Collective Coordinate Method introduced by Christ and Lee [15]. This method can be used as a quantization method to write an effective Lagrangian and an effective Hamiltonian for translationally invariant theories.

Our starting point is a system described by a scalar field in $1+1$ dimensions (1 spatial and 1 temporal), containing one zero-mode due to the translational invariance. This system can be described by the general Lagrangian 1.6, written in terms of the scalar field $\phi(x,t)$, with a localized solution $\phi_0$ that obeys

$$-\frac{\partial^2\phi_0(x-x_0)}{\partial x^2}+\frac{\partial U(\phi_0)}{\partial\phi_0}=0. \tag{1.23}$$

To find quantum corrections to the the classical solution we follow the discussion, in the section 1.2 and expand our field as

$$\phi(x,t) = \phi_0(x - x_0) + \sum_{k=0} c_k(t)\delta\phi_k(x - x_0). \tag{1.24}$$

This expansion leads to the eigenvalue problem

$$\left[-\frac{\partial^2}{\partial x^2} + \left(\frac{d^2U}{d\phi^2}\right)_{\phi_0}\right]\delta\phi_k(x - x_0) = \omega_k^2\delta\phi_k(x - x_0), \tag{1.25}$$

that has a zero mode for $k = 0$. Since we have already seen that the zero modes are related to the translational freedom of the coordinate $x_0$, and the summation over $k$ in the Lagrangian 1.18 misses one degree of freedom if $\omega_0 = 0$, we must assume that the coordinate $x_0$ is a dynamical one. Therefore, a system with a zero-mode must have the center of the solution dependent of time, $x_0 = x_0(t)$. This dynamical coordinate is the so-called collective coordinate. Accordingly, the proper expansion 1.24 of the field $\phi(x,t)$ becomes

$$\phi(x,t) = \phi_0(x - x_0(t)) + \sum_{k=1} c_k(t)\delta\phi_k(x - x_0(t)). \tag{1.26}$$

Note that the zero mode, $k = 0$, was replaced by making $x_0 \to x_0(t)$ and now the summation begins at $k = 1$. This replacement is the key feature of the Collective Coordinate Method. Since a translationally invariant theory means that the potential $V[\phi_0]$ is independent of $x_0$, the effective potential 1.16 remains the same

$$V[\{c_k\}] = V[\phi_0] + \sum_{k=1} \frac{1}{2}c_k^2\omega_k^2. \tag{1.27}$$

On the other hand, the kinetic energy depends on the time derivative of the field 1.26,

$$\dot{\phi}(x,t) = -\left[\frac{\partial}{\partial x}\phi_0 + \sum_{k=1} c_k\frac{\partial}{\partial x}\delta\phi_k\right]\dot{x}_0 + \sum_{k=1}\dot{c}_k\delta\phi_k, \tag{1.28}$$

and, therefore, we should expect a different result for the effective kinetic energy.

Here it is convenient to make a change of variables and define $q_0(t) = x_0(t)$ and

$q_k(t) = c_k(t)$ for $k > 0$, which allows us to write the effective Lagrangian as

$$L = \frac{1}{2}\sum_{jk=0}^{\infty} \dot{q}_j M_{jk} \dot{q}_k - V[\{q_k\}], \tag{1.29}$$

where $M_{jk}$ is a matrix with elements

$$M_{00} = \int dx \left( \frac{\partial \phi_0}{\partial x}\phi_0 + \sum_{k=0} q_k \frac{\partial \delta\phi_k}{\partial x} \right)^2, \tag{1.30a}$$

$$M_{0k} = -\sum_{j=0} \int dx q_j \delta\phi_j \frac{\partial \delta\phi_k}{\partial x}, \qquad k \neq 0, \tag{1.30b}$$

$$M_{jk} = \delta_{jk}, \qquad j,k \neq 0. \tag{1.30c}$$

To write the matrix elements, we have used the orthogonality between the normal modes of the fluctuations $\delta\phi_k$. It is also worth remembering that the eigenfunction related to the zero-mode is $\delta\phi_0 = \delta x(\partial\phi_0/\partial x)$. Comparing the effective Lagrangian 1.29 with equation 1.18, it is already possible to perceive what difference the introduction of a collective coordinate makes.

It is also possible to write the effective Hamiltonian from the Legendre transformation

$$H = \sum_k \pi_k \dot{q}_k - L, \tag{1.31}$$

in which we have defined the canonical momenta as

$$\pi_k = \frac{\partial L}{\partial \dot{q}_k}. \tag{1.32}$$

Substituting the Lagrangian 1.29 in the Legendre transformation 1.31, a straightforward calculation leads us to the effective Hamiltonian

$$H = \frac{P^2}{2M} - \frac{P}{M}\sum_{k=1} M_{0k}\pi_k + \frac{1}{2}\sum_{j,k=1}\left( \delta_{jk} + \frac{M_{0k}M_{0j}}{M}\pi_k\pi_j \right) + V[\{c_k\}], \tag{1.33}$$

where $M$ is a number with unit of mass, calculated by $M = DetM_{jk} = M_{00} - \sum M_{0k}^2$. $P$ refers to the conjugate momentum of the coordinate $q_0 = x_0(t)$, that is $P = \pi_0$. This Hamiltonian describes a particle with momentum $P$ and mass $M$ interacting with a bath of harmonic oscillators, with the particularity that, in our case, it is the center of the static solution $x_0(t)$ which behaves as a particle. This particle (center of the solution), has an effective mass $M$, that refers to the mass of the topological excitation.

So far our discussion of the Collective Coordinate Method has been purely classical. The entire procedure of the method can be resumed in two steps: acknowledge the symmetries that produce the zero-modes and change the variables from $\phi$ to $\{q_k\}$ using the center of the classical solution as a dynamical variable. Although the Collective Coordinate Method can be easily found in the literature, it is limited to Lagrangians of the form 1.6 which are Lagrangians with a second-order time derivative for the kinetic energy. The literature fails to explain a systematic way to apply the method for Lagrangians with first-order time derivatives, such as the Schrödinger's field Lagrangian. Systems that are described by the Schrödinger equation can also present a collective coordinate relative to the center of a localized classical solution. The present method, as we are going to see, is not entirely suitable for dealing with these systems. Therefore, our work lies in the extension of the Collective Coordinate Method for Schrödinger fields.

The Hamiltonian 1.33 resembles the one of a Brownian particle, in which an external particle interacts with the particles of a given medium. The main difference between the Brownian motion described by the Hamiltonian 1.33 and the ordinary one is that the environment, responsible for making the particle with momentum $P$ to lose energy, have the same nature and microscopic origin as the Brownian particle itself. We will dedicate the rest of this chapter to review important features of quantum dissipation in this kind of system.

## 1.5 Quantum Dissipation

So far we have seen both differences and similarities between classical and quantum systems. In order to treat a quantum system with a semiclassical approximation, a quantization procedure is required to reincorporate quantum fluctuations into the system. Furthermore, we have seen that, in the process of quantization, classical fields with topological solutions can be described by an effective Hamiltonian of a particle coupled to a bath of oscillators, equation 1.33. Physically, these oscillators can be interpreted as quasiparticles, which collide with the quantum particle, thus behaving as a thermal bath to the latter. In other words, a paradigm of quantum Brownian motion.

Important features of the quantum Brownian motion were developed by Caldeira and Leggett [10]. In their work, they studied a system of a single particle coupled with a reservoir using the Feynman-Vernon formalism of the influence functional [20]. Among several important results, their work showed that the mobility properties of a quantum system can be described by damping and diffusion coefficients, in the same way as classical systems. Although their work was not directly related to topological excitations in field theory, we have seen that a field Lagrangian with localized solutions resembles an effective behavior of a particle-plus-reservoir system, in which both particle and bath share the same microscopic origin. A work done by Castro Neto and Caldeira [21] unified the quantization of a localized solution with the dissipative aspects of the quantum Brownian motion. In their work, the damping and diffusion functions of a scalar field in 1D were calculated successfully, showing that the dissipation properties of a scalar field are temperature dependent. Later, a similar approach was implemented for a scalar field in 2D [22]. Since we aim to investigate dissipative aspects of a localized solution in the Schrödinger picture, we will highlight important features and steps to understand quantum dissipation in these systems.

For the purpose of this dissertation, we will only focus on the main results of their approach, since the whole discussion of quantum dissipation involves extensive mathematical work. In appendices A and B, we develop the intermediate steps to obtain the main results we will show in this section. For further details, an elucidative review of this subject can be found in Ref. [23].

We begin with a Hamiltonian of a system-plus-reservoir in the form

$$H = H_S + H_I + H_R, \tag{1.34}$$

in which $H_S$, $H_I$ and $H_R$ refers to the Hamiltonian of the system, interaction and reservoir, respectively. To find average values of any observable $\hat{O}$ at time $t$ (Heisenberg picture), the operation

$$\left\langle \hat{O}(t) \right\rangle = \langle \psi(0) | \, \hat{O}(t) \, | \psi(0) \rangle \,, \tag{1.35}$$

must be performed for the system at an initially pure state $|\psi(0)\rangle$. Usually, for a system-plus-reservoir, the choice of a pure state is not the most appropriate one. Since the bath acts as a thermal reservoir for the system, and, as we know, a system at finite temperature is better described by a mixed state, we should represent it by the density operator

$$\hat{\rho}(0) = \sum_{\psi} p_{\psi} \, |\psi(0)\rangle \, \langle \psi(0)| \,, \tag{1.36}$$

where $p_{\psi}$ is the probability associated with the state $|\psi(0)\rangle$. The average of any observable is now given by

$$\left\langle \hat{O}(t) \right\rangle = Tr_{RS} \left\{ \hat{\rho}(t) \hat{O}(0) \right\} , \tag{1.37}$$

where $RS$ refers to the basis of the composite system formed by the reservoir ($R$) and the system ($S$). Since we are usually interested in the system $S$, we can use the properties of the trace (A.6) and write 1.37 as

$$\left\langle \hat{O} \right\rangle = Tr_{RS} \left\{ \hat{\rho}(t) \hat{O} \right\} = Tr_{S} \left\{ \left[ Tr_{R} \hat{\rho}(t) \right] \hat{O} \right\} = Tr_{S} \left\{ \widetilde{\rho}(t) \hat{O} \right\} , \tag{1.38}$$

where we have defined

$$\widetilde{\rho}(t) = Tr_{R} \hat{\rho}(t), \tag{1.39}$$

as the reduced density operator for the system. When we trace out the coordinates of the reservoir $R$, the only subspace remaining is that of the system of interest $S$. In the coordinate representation, it can be shown (in the appendix A) that the reduced density operator of a system-plus-reservoir can be written using the Feynman-Vernon formalism as

$$\widetilde{\rho}(x,y,t) = \int\int dx'dy' \; \widetilde{\rho}(x',y',0) \; \mathcal{J}(x,y,t;x',y',0), \tag{1.40}$$

in which $\mathcal{J}$ is the well-known superpropagator of the system, defined as

$$\mathcal{J}(x,y,t;x',y',0) = \int_{x'}^{x}\int_{y'}^{y} \mathcal{D}x\mathcal{D}y \; exp\left\{\frac{i}{\hbar}(\mathcal{S}_S[x]-\mathcal{S}_S[y])\right\} \; \mathcal{F}[x,y]. \tag{1.41}$$

Here, $\mathcal{S}_S$ is the action corresponding to the Hamiltonian $H_S$ and $\mathcal{F}$ is the so-called influence functional, defined in equation A.17.

Summarizing, a system in contact with a thermal reservoir cannot be represented by a pure state, because the reservoir itself mediates a coupling between the forward and backward time evolutions required to describe the dynamics of the density operator. Therefore, to understand the behavior of the subsystem $S$ (by evaluating $\widetilde{\rho}(t)$) one must firstly understand how the system $S$ acts on the reservoir $R$, and then how the latter reacts on the former. This sort of feedback effect represents the influence of the reservoir $R$ upon the system $S$, and can be understood by evaluating the influence functional $\mathcal{F}$.

Until now, our discussion was entirely general and could be applied to any system written as 1.34. However, the influence functional can only be analytically solvable for some specific systems. Therefore, we are going to show how this formalism can be applied for a Hamiltonian similar to 1.33 in order to obtain the damping and diffusion functions. Our model Hamiltonian in second quantization form reads

$$H = \frac{1}{2M}\left(P - \sum_{jk}\hbar g_{jk}a_j^{\dagger}a_k\right)^2 + \sum_k(\hbar\omega_k - \mu)a_k^{\dagger}a_k, \tag{1.42}$$

in which $g_{jk} = \langle j|\,\hat{p}\,|k\rangle$ is the coupling matrix between the quantum particle with momentum $P$ and the set of quasiparticles with eigenstates $|k\rangle$, now represented by the creation $a_k^{\dagger}$ and annihilation $a_k$ operators. The operator $\hat{p}$ is the one-body momentum operator that in one dimension reads $\hat{p} = (-i\hbar)\partial/\partial x$, $\omega_k$ are the eigenfrequencies of the $k$ state and $\mu$ is the chemical potential. This Hamiltonian represents a particle with total momentum $P$ surrounded by a set of indistinguishable particles that could be fermions or bosons. Since we want to apply this method for a Bose-Einstein Condensate later, the next results are for the particular bosonic case.

To evaluate the influence functional, that is, the influence the bath produces on the particle, one must define the appropriate set of coordinates for the bath. In dealing with creation and annihilation operators, it is convenient to evaluate the influence functional in the coherent state representation. A coherent state is defined as an eigenstate of the annihilation operator [23]

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1.43}$$

In this representation, it can be shown (appendix B) that the influence functional can be written as

$$\mathcal{F}[x,y] = \int d\mu(\alpha)d\mu(\alpha')d\mu(\gamma')\rho_R(\alpha'^*,\gamma')\int_{\alpha'}^{\alpha^*}\mathcal{D}\mu(\alpha)\int_{\gamma'^*}^{\alpha}\mathcal{D}\mu(\gamma)\;e^{\frac{i}{\hbar}(\mathcal{S}_{RI}[x,\alpha]-S^*_{RI}[y,\gamma^*])}, \tag{1.44}$$

where $d\mu$ and $\mathcal{D}\mu$ are defined in B.16 and B.13, respectively. Here, $\mathcal{S}_{RI}$ is the action associated with the reservoir $R$ and the particle-reservoir interaction that can be obtained from $H_I$, and is written as

$$\mathcal{S}_{RI}[x,\alpha] = \int dt'\left[\sum_k(\alpha_k^*\dot{\alpha}_k-\alpha_k\dot{\alpha}_k^*)+\dot{x}\sum_{jk}\hbar g_{jk}\alpha_j^*\alpha_k-\sum_k(\hbar\omega_k-\mu)\alpha_k^*\alpha_k\right]. \tag{1.45}$$

The influence functional is the average of the product of two time evolutions over the initial state of the bath. As we have mentioned above, one of the time evolutions refers to the evolution of the reservoir under the action of the system $S$, whereas the other is its time-reversed partner. In this model, in which the bath is a harmonic one, the influence functional can be exactly solved because the integrand of 1.44 is a Gaussian. The integral 1.44 has been evaluated elsewhere in the context of polaron [21] and skyrmion dynamics [22], resulting in

$$\mathcal{F} = e^{-\frac{i}{\hbar}\Phi_I[x,y]-\frac{1}{\hbar}\Phi_R[x,y]}, \tag{1.46}$$

where the functions $\Phi_I$ and $\Phi_R$ are defined as

$$\Phi_I = \int_0^t \int_0^\tau dt' d\tau' \left[x(\tau') - y(t')\right] M\gamma(\tau' - t') \left[\dot{x}(\tau') - \dot{y}(t')\right], \tag{1.47a}$$

$$\Phi_R = \int_0^t \int_0^\tau dt' d\tau' \left[x(\tau') - y(t')\right] D(\tau' - t') \left[x(\tau') - y(t')\right]. \tag{1.47b}$$

Here $\gamma(\tau' - t')$ and $D(\tau' - t')$ are known as the damping and diffusion functions respectively, being defined as

$$\begin{aligned}\gamma(t) &\equiv -\frac{\hbar}{M}\sum_{jk} |g_{jk}|^2 (\bar{n}_k - \bar{n}_j)(\omega_k - \omega_j)\cos(\omega_k - \omega_j)t \\ &= -\frac{\hbar}{2M}\theta(t)\int d\omega \int d\omega' \; S(\omega,\omega') \left[\overline{n}(\omega) - \overline{n}(\omega')\right](\omega - \omega')\cos(\omega - \omega')t,\end{aligned} \tag{1.48}$$

and

$$\begin{aligned}D(t) &\equiv \hbar^2 \sum_{jk} |g_{jk}|^2 (\bar{n}_k + \bar{n}_j + 2\bar{n}_j\bar{n}_k)(\omega_k - \omega_j)^2 \cos(\omega_k - \omega_j)t \\ &= \frac{\hbar^2}{2}\theta(t)\int d\omega \int d\omega' \; S(\omega,\omega') \left[\overline{n}(\omega) + \overline{n}(\omega') + 2\overline{n}(\omega)\overline{n}(\omega')\right](\omega - \omega')^2 \cos(\omega - \omega')t.\end{aligned} \tag{1.49}$$

respectively. Here $\bar{n}$ is the bosonic distribution function, and $\theta(t)$ is the Heaviside function. In the damping 1.48 and diffusion functions 1.49 we have introduced the scattering function $S(\omega, \omega')$, that is defined as

$$S(\omega,\omega') = \sum_{jk} |g_{jk}|^2 \delta(\omega - \omega_j)\delta(\omega' - \omega_k), \tag{1.50}$$

in the sense that all the transport coefficients (damping and diffusion) depend on the scattering of the quasiparticles by the particle. This means that, when the whole system is described by a Hamiltonian of the type 1.42 or an action written as 1.45, the only thing needed to understand the dissipative behavior of the particle is the coupling matrix $g_{jk}$ and the eigenfrequencies $\omega_k$.

With the influence functional calculated, we can proceed to understand the evolution of the system $S$. Substituting 1.46 in 1.41, the corresponding superpropagator can be written as

$$\mathcal{J}(x,y,t;x',y',0)=\int_{x'}^{x}\int_{y'}^{y}\mathcal{D}x\mathcal{D}y\;e^{\frac{i}{\hbar}\left(\mathcal{S}_S[x]-\mathcal{S}_S[y]-\Phi_I[x,y]\right)}\;e^{-\frac{1}{\hbar}\Phi_R}. \tag{1.51}$$

This expression show us that the quantum particle has its dynamics described by the effective action

$$\bar{\mathcal{S}}[x,y]=\mathcal{S}_S[x]-\mathcal{S}_S[y]-\Phi_I\left[x,y\right], \tag{1.52}$$

where $\mathcal{S}_S$ is the action of the system $S$. We can make a change of variables to

$$\begin{aligned} R &= \frac{x+y}{2} \\ r &= x-y, \end{aligned} \tag{1.53}$$

and derive equations of motion from 1.52 as

$$\ddot{R}+2\int_0^t\gamma(t-t')\dot{R}dt'=0, \tag{1.54a}$$

$$\ddot{r}-2\int_0^t\gamma(t-t')\dot{r}dt'=0, \tag{1.54b}$$

in which the function $\gamma$ is the damping function defined in 1.48. These equations are the equations of motion for the "center of mass" (equation 1.54a) and "relative" coordinate (equation 1.54b), associated to the coordinate representation of the reduced density operator of the system. Whereas the first equation can be intuitively interpreted as describing the dissipative motion of the particle represented by the density operator 1.40, the interpretation of the second equation is not so straightforward. In reality, it is related to the diffusion process the particle suffers, which can only be seen analyzing 1.54b in the Wigner representation of that equation.

Summarizing, we have started from a quantum field theory in which a set of bosons (or fermions) presented an organized behavior that could be described by a classical mean-field. The semiclassical quantization procedure showed that a localized mean-field solution presents an effective behavior that can be interpreted as that of an extended-particle (a soliton-like solution of the classical field equation) surrounded by a set of harmonic oscillators (quasiparticles). If the system was composed only by the extended particle, it would move with no resistance due to the translation invariance of the field equation. But, due to the collective coordinate quantization method, particle and normal modes become coupled, and the latter act as a thermal bath for the former. The particle's dissipation and diffusion coefficients are completely described by the action of the bath upon the particle, which can be quantified by the influence functional $\mathcal{F}$, and uniquely expressed in terms of the coupling matrix $g_{jk}$.

Therefore, our job in the next few chapters is to define the system of interest and apply the method presented here to find the coupling matrix and, consequently, the damping and diffusion functions.

# Chapter 2

# Bose-Einstein Condensate

The Bose-Einstein Condensate was first predicted theoretically at the beginning of the twentieth century and first achieved experimentally at its end. The complexity of both theory and experiment reflects the intriguing features that a Bose gas can present. This is a many-particle system that lies on the edge between quantum and classical mechanics. The Bose-Einstein Condensate is defined by a macroscopic occupation of its ground state, which provides a classically collective aspect for this system. However, at low temperatures, its properties are described by a single particle Schrödinger equation.

In this chapter, we will show the most important features of the Bose-Einstein Condensation, from the ideal gas to interacting and nonuniform systems. Our main goal is to provide the tools to understand our object of study, since this system will be used later to develop a theory similar to the one developed in the last chapter.

## 2.1 Ideal Bose gas

An ideal Bose gas is defined by a set of $N$ bosons at very low temperature without interparticle interactions or external potential. Although this ideal system differs considerably from condensates produced experimentally, its statistics provides the main basic concepts that appear in more realistic systems. Here we will briefly review the description of an ideal Bose gas and highlight the main concepts and results that arise in bosonic systems.

A gas of $N$ indistinguishable non-interacting bosonic particles has its thermodynamics quantities originated by the partition function

$$Z = \sum_{E_k} e^{-\beta(E_k-\mu)}, \tag{2.1}$$

in the grand-canonical ensemble. The parameter $\beta$ is defined as $\beta = 1/k_B T$, in which $k_B$ is the Boltzmann constant and $T$ the temperature of the gas. In order to describe this bosonic gas, we must find the eigenenergies $E_k$ of the system. For a system of non-interacting particles, the total Hamiltonian can be written as a sum of single-particle Hamiltonians

$$\hat{H} = \sum_i \hat{H}_i^{(1)}, \tag{2.2}$$

since each particle behaves independently. This separable Hamiltonian, allow us to write the energy $E_k$ as

$$E_k = \sum_i \bar{n}_i \epsilon_i, \tag{2.3}$$

in which, $\bar{n}_i$ is the occupation number of the $i^{th}$ single-particle energy state $\epsilon_i$, $i.e$, the number of bosons with the same set of quantum numbers $i$. If our particles were fermions, $\bar{n}_i$ could only assume the values 0 or 1, due to Pauli exclusion principle. But since we are interested in Bose systems, $\bar{n}_i$ can assume any integer value in the interval $[0, N]$. For such a system, the grand canonical potential reads

$$\begin{aligned} \Omega &= -\frac{1}{\beta} \ln Z, \\ &= \frac{1}{\beta} \sum_i \ln\left(1 - e^{\beta(\mu-\epsilon_i)}\right). \end{aligned} \tag{2.4}$$

With equation 2.4, we can evaluate the thermodynamic behavior of the system. For instance, the number of particles is given by

$$N = -\frac{\partial \Omega}{\partial \mu} = \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)} - 1} = \sum_i \bar{n}_i. \tag{2.5}$$

It is convenient to make a distinction between the number of particles in the lowest energy state, $i = 0$, and those in the excited states, $i \neq 0$,

$$\begin{aligned} N &= \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} + \sum_{i \neq 0} \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}, \\ &= N_0 + N'. \end{aligned} \tag{2.6}$$

Besides the definition of the number of particles in each state, this quantity determines that the chemical potential must be $\mu < \epsilon_0$ in order to have only positive values of $N_0$. The thermal component of the gas, $N'$, accounts for every particle that is not in the condensate. It is a summation over the excited states. For a given temperature and chemical potential, the number of particles in the excited states increases smoothly with the size of the system, as expected. However, the behavior of the condensate $N_0$ is quite intriguing. When the eigenenergy $\epsilon_0$ approximates the chemical potential $\mu$, the portion $N_0$ increases drastically and displays a divergence for $\mu = \epsilon_0$. This is the main mechanism for the Bose-Einstein condensation. In fact, $N_0$ is usually called the condensate component of the gas, since it shows the macroscopic occupation of the lowest level of energy as $\epsilon_0 \to \mu$.

For a bosonic gas obeying the condensation conditions, we have most of the particles found to be in the ground state, behaving as a whole. So, the many-body field operator

$$\hat{\Psi}(\mathbf{r}) = \sum_{i=0} \varphi_i \hat{a}_i, \tag{2.7}$$

that represents the set of all particles in the system, can be simplified for condensates. Here, $\hat{a}_i$ is the annihilation operator for bosons. The macroscopic nature of the particles in the ground state, allows us to neglect the commutation relations between the annihilation $\hat{a}_0$ and creation $\hat{a}_0^\dagger$ operators. Therefore, the condensate term $\varphi_0 \hat{a}_0$ can be approximated by a classical function

$$\begin{aligned} \hat{\Psi}(\mathbf{r}) &= \Psi_0(\mathbf{r}) + \sum_{i=1} \varphi_i \hat{a}_i, \\ &= \Psi_0(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r}). \end{aligned} \tag{2.8}$$

This is known as the Bogoliubov approximation. Thus, in diluted Bose gases at very low temperature, it is possible to treat the particles in the ground state as a macroscopic classical field. This result will become extremely useful in dealing with non-ideal Bose gas.

## 2.2 Non-ideal Bose gas

The difference between an ideal Bose gas and a non-ideal one begins with the nature of the Hamiltonian that describes the system. For the ideal case, it was possible to find a separable Hamiltonian, as equation 2.2, due to the absence of interparticle interactions. However, in a real Bose gas, particles do interact with each other. A general Hamiltonian for interacting systems can be written as

$$\hat{H} = \sum_{i} \hat{H}_i^{(1)} + \frac{1}{2} \sum_{i \neq j} \hat{H}_{ij}^{(2)} + \ldots \,. \tag{2.9}$$

Even if we consider only two-body interactions, this non-separable Hamiltonian makes the process of writing the grand canonical potential and finding thermodynamic quantities of the system an excessively hard task. So, for non-ideal Bose gas, it is convenient to find an alternative approach to study the system.

Since the majority of the particles are found to be in the ground state, we can make use of a microscopic theory developed by Bogoliubov to describe an interacting Bose gas [24]. His theory was based on a principle that a system with a large number of atoms in the same quantum state can be approximated by a classical function. This classical approximation provides the correct macroscopic properties of the quantum system, if the right parameters are chosen. To show how nonuniform Bose-Einstein condensate can be studied, we will apply the Bogoliubov approximation, as in equation 2.8, for the system of interest. In terms of the field operator $\hat{\Psi}$, our many-body system can be described by the Hamiltonian

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^{\dagger}(\mathbf{r}) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \hat{\Psi}(\mathbf{r}) + \frac{1}{2} \int \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^{\dagger}(\mathbf{r}) \hat{\Psi}^{\dagger}(\mathbf{r}') U(|\mathbf{r} - \mathbf{r}'|) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}'), \tag{2.10}$$

where the field operators, $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^{\dagger}(\mathbf{r})$, obey the commutation relations

$$\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^{\dagger}(\mathbf{r}')\right] = \delta(\mathbf{r} - \mathbf{r}'), \tag{2.11a}$$

$$\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')\right] = 0. \tag{2.11b}$$

The first and second terms of 2.10 are the one-body operators. The third term is the two-body operator that describes the interaction between the particles. In a dilute Bose gas, we can neglect higher orders of interactions, as three-body operators.

Considering that this Hamiltonian describes a bosonic system at very low temperature and most of the particles are found to be in the lowest energy state, we can proceed to apply the Bogoliubov approach. It would be careless to replace the field operator $\hat{\Psi}(\mathbf{r})$ for a classical function $\Psi(\mathbf{r})$ directly into the Hamiltonian 2.10 for a realistic potential. However, the dilute condition allows us to admit that the interparticle distance is large enough, so that the Born approximation can be applied. Our realistic potential $U$ can be replaced by a short-range pseudopotential $g\delta(\mathbf{r} - \mathbf{r}')$, where $g$ is a coupling constant [25]

$$g = \frac{4\pi\hbar^2 a}{m}, \tag{2.12}$$

with a fixed s-wave scattering length $a$. The approximation of the potential by a delta function means that the bosons only interact by contact.

Now that we have replaced the interparticle interaction $U$ by an effective potential that reproduces the correct scattering properties of the system, we can replace the field operator 2.8 into the Hamiltonian 2.10. Considering only first order terms in the fluctuations, the field equation of motion can be obtained in the Heisenberg picture by

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r}, t) = \left[\hat{\Psi}(\mathbf{r}, t), \hat{H}\right]. \tag{2.13}$$

Thus, the time dependent equation of motion for the condensate wave function can be written as

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g\,|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t). \tag{2.14}$$

This equation is known as the Gross-Pitaevskii equation (GPE) [25]. This equation is very similar to the Schrödinger equation. The main difference here is that the Gross-Pitaevskii equation describes a field, not a single particle. Besides, GPE usually has a non-linear term that is originated by the interparticle interaction.

For a field $\Psi$ to be well described by the GPE 2.14, there are some requirements that the system may obey. For instance, the number of bosons must be large enough to allow us to use the concept of Bose-Einstein Condensation, the temperature must be close enough to zero and our gas must be diluted. Under these conditions, we can assume that the interparticle interactions are given only by direct contact and that our field can be approximated by a macroscopic classical function. Moreover, the mean-field must be normalized to the total number of particles $\int |\Psi|^2\, d\mathbf{r} = N$. Therefore, non-ideal dilute Bose gas at low temperature can be described by a mean-field equation that comes from the second quantized Hamiltonian 2.10.

For a stationary field, the solution of the equation 2.14 can be separated in two parts, a spatial and a temporal one,

$$\Psi(\mathbf{r},t) = \Psi_0(\mathbf{r})\; e^{-\frac{i}{\hbar}\mu t} = \sqrt{n(\mathbf{r})}\; e^{-\frac{i}{\hbar}\mu t}, \tag{2.15}$$

where $\mu$ is the chemical potential and $n(\mathbf{r})$ is the density of the Bose-gas. With the separation 2.15, the time-independent GPE can be written as

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g\,|\Psi_0(\mathbf{r})|^2 - \mu\right)\Psi_0(\mathbf{r}) = 0. \tag{2.16}$$

This non-linear differential equation has a variety of interesting solutions. A simple solution of the GPE equation is when we consider the Thomas-Fermi limit. This limit occurs when the density of the condensate $n(\mathbf{r})$ changes very slowly in space. The first term in 2.16, which depends on position, is proportional to $\hbar^2$, and is known as quantum pressure. Therefore, the limit in which the quantum pressure can be neglected is called the Thomas-Fermi limit. If the spatial derivative can be neglected, equation 2.16 is no longer a differential equation, and its solution can be simplified. The configuration of the field for the Thomas-Fermi limit takes a particularly simple form [25],

$$n(\mathbf{r}) = \frac{\mu - V(\mathbf{r})}{g}. \tag{2.17}$$

Further developments on the form of the function $\Psi_0(\mathbf{r}) = \sqrt{n(\mathbf{r})}$ requires the definition of the external potential $V(\mathbf{r})$. The simplest case is the uniform gas, which occurs in the absence of an external potential. In this case the uniform gas presents a density $n_0 = \mu/g$. Both uniform gas and Thomas-Fermi limit results are very important solutions for the Gross-Pitaevskii equation. However, neither of them are localized solutions, in the sense of presenting a non-trivial topological deformation. As we have seen in the previous chapter, localized solutions present very interesting physics, in which the solution could behave as an extended-particle. The Gross-Pitaevskii equation allows numerous different localized solutions which are widely discussed both theoretically and experimentally in the literature. In the next few sections, we will present some important localized solutions for the time-independent Gross-Pitaevskii equation 2.16. These solutions are topological deformations of the particle density and they describe the mean-field behavior of a superfluid.

## 2.3 Soliton Solution

Starting with the one-dimensional case, the class of topological solutions is called soliton solutions [26]. The soliton solution is defined by a shape conserved wave-packet moving with a constant velocity. The one-dimensional time independent Gross-Pitaevskii equation (GPE) 2.16 can be written as

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + g\left|\Psi_0(x)\right|^2 - \mu\right)\Psi_0(x) = 0. \tag{2.18}$$

The GPE 2.18 allows three different types of solitonic solutions. The type of solution depends on the nature of the interaction between the bosons. For a system with repulsive interaction, the constant g is positive, $g > 0$. In this case, a localized suppression of the density appears in the density distribution of the superfluid. This density pattern is called a grey soliton. In the particular case in which the density completely vanishes at one point, this solution can be called a dark soliton. However, if the system has an attractive interaction between the bosons, $g < 0$, then the solution is called a bright soliton. The bright soliton is characterized by a localized wave-packet in the condensate. Particularly, the bright soliton has a solution of the form

$$\Psi(x,t) = |\Psi(0)| \ \operatorname{sech}\left(\frac{x}{\xi}\right) \ e^{-i\mu t/\hbar}, \tag{2.19}$$

where the parameter $\xi$ is the healing length, defined as

$$\xi = \frac{\hbar}{\sqrt{2mgn_0}}. \tag{2.20}$$

This quantity determines the minimum distance in which the superfluid order parameter can change and naturally appears in spatially deformed solutions [27].

Both grey soliton and bright soliton solutions arise from a translationally invariant theory, so the center of the solution can move freely in the space, acting as an ordinary particle. However, in realistic environments, they can also collide with thermal fluctuations as they move. These collisions are responsible for the soliton to lose energy and cease its motion. The bright soliton solution will be used in Chapter 4 to study these dissipation properties microscopically.

## 2.4 Vortex Solution

For a two-dimensional system, an important localized solution is a vortex. This type of solution is a topological deformation in which the superfluid revolves around an axis. To describe this system properly, let us define the polar coordinates $r$ and $\phi$. In these coordinates, the vortex solution has the form

$$\Psi(\mathbf{r},t) = |\Psi_0(r)| \ e^{iq\phi} \ e^{-i\mu t/\hbar}, \tag{2.21}$$

where the parameter $q$ is a integer to ensure that the GPE is single valued. The density can be defined as $\sqrt{n} = |\Psi_0(r)|$. Note that due to the symmetry of the problem, the density depends only on the radial coordinate $r$. Substituting 2.21 in the equation 2.16, the GPE can be written as

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\left(r\frac{d}{dr}\right) + \frac{\hbar^2}{2m}\frac{q^2}{r^2} + V(r) + g|\Psi_0|^2 - \mu\right]|\Psi_0| = 0. \tag{2.22}$$

The exact solution of this equation, considering the quantum pressure term, confinement potential, and interparticle interaction requires a numerical method [28]. To find an analytical solution for the equation 2.22, we can take the Thomas Fermi limit and consider that the first term of 2.22 vanishes. In this limit, the superfluid density can be written as

$$n_q(r) = n_0\left(1 - \frac{q^2\xi^2}{r^2} - \frac{V(r)}{\mu}\right), \tag{2.23}$$

where $\xi$ is the healing length defined as in equation 2.20.

The main difference between the density profile of a superfluid with a single vortex 2.23 and the vortex-free condensate in the TF limit 2.17 is the presence of the term $q^2\xi^2/r^2$. This term act as a centrifugal barrier, forcing the density to vanish at the center of the solution. In fact, note that for a single quantized vortex, $q = 1$. As the coordinate $r$ approximates the healing length $\xi$, the density tends to zero. This behavior characterizes a singularity at the core region of a quantized vortex.

## 2.5 Beyond the Mean-Field Approximation

Although the Bose-Einstein Condensate presents a macroscopic behavior described by a classical mean-field solution, some important quantum features have been neglected until now. To fully understand the behavior of this system, we must take into account quantum fluctuations. These fluctuations are responsible for describing any emergent, dissipative or dynamical phenomena the system may present. However, there is a variety of ways to include those quantum fluctuations into the problem.

One of the possible procedures to include these fluctuations is via numerical calculations. Numerical methods can provide comprehensive results of the system of interest, being a convenient choice to deal with non-linear theories [29, 30]. This is a very useful approach, but it has some limitations. An important constraint of this method is the computational cost of the application of an algorithm in a system with many degrees of freedom. Another aspect to take into account is that these methods are very sensitive to the specific values of the parameters of the system. Therefore, not all dynamical or transient systems can be fully understood by numerical methods alone. Still, there are several ways in which numerical calculations can help to understand quantum fluctuations of a Bose gas with specific configurations.

An alternative method to study these fluctuations emerge from systems in which the experimental behavior is well known, but the microscopic theory is not exactly solvable. This is the phenomenological approach, in which a theory is proposed to explain an experimental result that an *ab initio* theory was not capable to describe so far. This method is widely used in the literature to explain the behavior of a number of complex systems. For instance, we could mention the work done by Rooney, Bradley, and Blakie [14] in which the dynamics of a vortex is studied phenomenologically. In their work, the authors separate the condensate dynamics in three components: a movement without loss of energy, a decay by damping, and a decay by diffusion. The damping and diffusion functions appear in the theory by the argument that the experiments show this behavior that is not well explained by the mean-field solution itself.

Although the phenomenological approach can provide us with some good results, it would be desirable to have a first-principle theory to explain these systems. The more accurate way to go beyond the mean-field is by a microscopic theory. This is, to solve an equation of motion that describes the excited particles interacting with the mean-field. In 1972, Alexander L. Fetter proposed a microscopic theory to include quantum fluctuations in a static condensate [31]. In his work, the Bogoliubov approximation 2.8 was used to expand the second quantized Hamiltonian 2.10 without an external potential, $V(\mathbf{r}) = 0$. Then, quantum fluctuations were written in terms of a sum of quasiparticle creation $\alpha_k^\dagger$ and annihilation $\alpha_k$ operators

$$\delta\hat{\Psi}(\mathbf{r}) = \sum_k \left[u_k(\mathbf{r})\alpha_k - v_k^*(\mathbf{r})\alpha_k^\dagger\right], \tag{2.24a}$$

$$\delta\hat{\Psi}^*(\mathbf{r}) = \sum_k \left[u_k^*(\mathbf{r})\alpha_k^\dagger - v_k(\mathbf{r})\alpha_k\right], \tag{2.24b}$$

where the c-numbers $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ are amplitudes of the quasiparticles. This expansion of the fields $\delta\Psi$ was essential to write the coupled differential equations that describe the behavior of the amplitudes

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - \mu + 2g|\Psi_0|^2\right)u_k - (\Psi_0)^2 v_k = E_k u_k, \tag{2.25a}$$

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - \mu + 2g|\Psi_0|^2\right)v_k - (\Psi_0^*)^2 u_k = -E_k v_k. \tag{2.25b}$$

These equations are known as Bogoliubov equations. The solutions of these equations describe the spectrum of excited quasiparticles of the system. In his work, the author shows the application of these equations to the uniform condensate and a condensate with a vortex. Although such an analysis provides comprehensive results for the energy spectrum and behavior of the quasiparticles, it does not explain the dynamics of some systems. To study dissipation properties and dynamics, a more accurate theory must be used.

In Chapter 1, we have shown a method to describe the dynamics of a localized solution in a scalar field. In 2016, Efimkin, Hofmann, and Galitski showed a similar application of this method to study a particular Schrödinger field in one-dimension [32]. In their work, they studied a condensate with a dynamical bright soliton with solution 2.19. They proposed that a Schrödinger field with this localized solution should have the form

$$\Psi(x,t) = \psi_0(x - x_0(t)) + i\frac{\xi\pi_0}{\hbar}u_x(x - x_0(t)) + \sum_k \left[u_k(x - x_0(t))\alpha_k - v_k^*(x - x_0(t))\alpha_k^*\right], \tag{2.26}$$

in which the first term is the bright soliton solution, the second term they introduce for later convenience and the last term describes the modified Bogoliubov excitations 2.24. The parameter $\pi_0$ is the momentum of the condensate without the Bogoliubov excitations and $u_x$ is an arbitrary function that was not defined explicitly. This type of solution, 2.26, made it possible to write the corresponding Bogoliubov equations 2.25 and an effective Lagrangian for the moving center $x_0$,

$$L = \frac{M\dot{x}_0}{2} + \pi_{kk'}\dot{x}_0 + \sum_k \alpha_k^* \left( i\hbar \frac{\partial}{\partial t} - E_k \right) \alpha_k, \tag{2.27}$$

where $M$ is the effective mass of the topological solution and $E_k$ are the eigenvalues found in the Bogoliubov equations 2.25. From the effective Lagrangian 2.27, they proceed to define a coupling parameter $\pi_{kk'}$ of the form

$$\pi_{kk'} = \frac{1}{2} \sum_{kk'} \begin{pmatrix} \alpha_k^* & \alpha_k \end{pmatrix} \begin{pmatrix} \langle k|\, \sigma_z\ \hat{p}\, |k'\rangle & -\langle k|\, \sigma_z\ \hat{p}\overline{|k'\rangle} \\ -\overline{\langle k|}\sigma_z\ \hat{p}\, |k'\rangle & \overline{\langle k|}\sigma_z\ \hat{p}\overline{|k'\rangle} \end{pmatrix} \begin{pmatrix} \alpha_{k'} \\ \alpha_{k'}^* \end{pmatrix}, \tag{2.28}$$

where $\hat{p} = -i\hbar\partial/\partial x$ is the momentum operator, $\sigma_z$ is the Pauli matrix in the $z$ direction and the vectors $|k\rangle$ and $\overline{|k\rangle}$ are defined as

$$|k\rangle = \begin{pmatrix} u_k \\ -v_k \end{pmatrix}, \qquad \overline{|k\rangle} = \begin{pmatrix} -v_k^* \\ u_k^* \end{pmatrix}. \tag{2.29}$$

This coupling parameter accounts for every interaction between quasiparticles and the moving center $x_0(t)$.

To study the quantum dissipation of this system, in [32], they used the Keldysh formalism to write a general form of the damping function and a scattering function in the same form as we showed for a scalar field. Therefore, they showed that a Schrödinger field with a localized solution can be described by a particle in a bath of oscillators. Although their work also covers the main topics of our object of study, their discussion had a focus quite different from ours, leaving some important features open for future analysis. For instance, the second term in 2.26 was not well explored in their work. The usual application of the Collective Coordinate Method does not provide this type of term in the process of writing the effective Lagrangian. Thus, we must explore the application of the method and the existence of the second term in 2.26. Another topic that was left for future discussions was the exact calculation of the damping function. The method used to study quantum dissipation in their work was equivalent to the Feynman-Vernon formalism showed in Chapter 1. Therefore we could show the equivalence between these methods and calculate the damping and diffusion functions exactly. Lastly, it would be desirable for prospective applications to have a microscopic theory for a general localized solution of a Schrödinger field. A general development would allow us to understand the uses and limitations of the method. In the next chapter, we are going to show how to study quantum dissipation in a general Schrödinger field, following the method developed in Chapter 1 and the microscopic theory showed in this section.

# Chapter 3

# A General Analysis of the Schrödinger Field

So far we have seen that several many-particle systems require both quantum and classical theories to understand its physical behavior. In the first chapter, we have developed a method to study quantum dissipation in a one-dimensional scalar field. In particular, the quantization method uses the collective behavior of the system to write an effective Lagrangian of a particle coupled with a bath of harmonic oscillators. The properties of quantum dissipation were based on the quantum Brownian motion, in which the behavior of the particle can be completely understood by tracing out the coordinates of the reservoir. In the second chapter, we presented tools to understand our system of interest. Our field is defined by a diluted set of bosons at very low temperature. This system is known as Bose-Einstein Condensate, and it is described by a field equation known as the Gross-Pitaeviskii equation. Some solutions of this equation are localized ones, such as the soliton 2.19 and vortex solution 2.23. In this chapter, we will apply the quantization method developed in Chapter 1 to the Schrödinger field. As we are going to see, the direct application of the Collective Coordinate Method for this field will show some complications. After a discussion about the applicability of the method, we are going to develop an extended Collective Coordinate Method for the Schrödinger field and study quantum dissipation in this system.

## 3.1 Defining the System of Interest

The system of interest is a set of $N$ bosons that can be described by the Schrödinger Lagrangian

$$L = \int d\mathbf{r} \left\{ \frac{i\hbar}{2} \left( \Psi^* \dot{\Psi} - \Psi \dot{\Psi}^* \right) - \left[ \frac{\hbar^2}{2m} \left( \nabla \Psi^* \right) \cdot \left( \nabla \Psi \right) + U(\Psi^*, \Psi) \right] \right\}, \tag{3.1}$$

in which $L$ is a functional of the field $\Psi = \Psi(\mathbf{r}, t)$ and its complex conjugate $\Psi^* = \Psi^*(\mathbf{r}, t)$. Here we have assumed that the bosons only interact by contact, in the same way as in section 2.2. The function $U(\Psi^*, \Psi)$ is an arbitrary function of the fields, that can describe the external potential, chemical potential, or any other constraint on the system. From the Lagrangian 3.1, we can define a kinetic energy functional

$$T\left[\Psi^*, \Psi\right] = \int d\mathbf{r} \frac{i\hbar}{2} \left( \Psi^* \dot{\Psi} - \Psi \dot{\Psi}^* \right), \tag{3.2}$$

and a potential energy functional

$$V\left[\Psi^*, \Psi\right] = \int d\mathbf{r} \left[ \frac{\hbar^2}{2m} \left( \nabla \Psi^* \right) \cdot \left( \nabla \Psi \right) + U(\Psi^*, \Psi) \right]. \tag{3.3}$$

Here it is important to compare both functionals of the Schrödinger field with the ones associated with the scalar field, studied in Chapter 1. Both potential energy functionals, 1.9 for the scalar field and 3.3 for the Schrödinger field, have a $\nabla^2$ term and an arbitrary function $U$. Although the potential energy functional from both theories are very similar, the kinetic term changes considerably. For a scalar field, the kinetic energy functional has a second-order time derivative of the field $\phi$, whereas the Schrödinger field presents the important feature of being proportional to a first-order time derivative of its fields.

From the Lagrangian 3.1, we can find equations of motion for the fields by evaluating

$$\frac{\partial L}{\partial \Psi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\Psi}} - \nabla \cdot \frac{\partial L}{\partial \nabla \Psi} = 0, \tag{3.4a}$$

$$\frac{\partial L}{\partial \Psi^*} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Psi}^*} - \nabla \cdot \frac{\partial L}{\partial \nabla \Psi^*} = 0. \tag{3.4b}$$

As we have seen, the Schrödinger field is described by the Gross-Pitaeviskii equation. Thus, the equations of motion can be written in the same way as equation 2.14 for the field $\Psi$. Consider that the system in question allows for static localized solutions $\Psi_0$ and $\Psi_0^*$, such that $\dot{\Psi}_0 = \dot{\Psi}_0^* = 0$. In this case, the equations of motion can be written as,

$$\left.\frac{\partial U}{\partial \Psi}\right|_{\Psi_0^*,\Psi_0} - \frac{\hbar^2}{2m}\nabla^2 \Psi_0^* = 0, \tag{3.5a}$$

$$\left.\frac{\partial U}{\partial \Psi^*}\right|_{\Psi_0^*,\Psi_0} - \frac{\hbar^2}{2m}\nabla^2 \Psi_0 = 0. \tag{3.5b}$$

A localized solution of these equations of motion describes the collective behavior of the Bose gas. To find the fluctuation corrections for this system, we can apply the same procedure used for the scalar field. Therefore, it is convenient to make a harmonic approximation in the neighborhood of this static solution. A functional Taylor expansion of the potential functional 3.3, up to second order around $\Psi_0$ and $\Psi_0^*$, reads

$$\begin{aligned} V\left[\Psi^*, \Psi\right] = & V\left[\Psi_0^*, \Psi_0\right] + \int d\mathbf{r} \left[ \frac{\hbar^2}{2m}\left(\nabla \delta\Psi^*\right)\cdot\left(\nabla\delta\Psi\right) \right. \\ & \left. + \frac{1}{2}\left( \frac{\partial^2 U}{\partial\Psi^*\partial\Psi}\delta\Psi^*\delta\Psi + \frac{\partial^2 U}{\partial\Psi\partial\Psi^*}\delta\Psi\delta\Psi^* + \frac{\partial^2 U}{\partial\Psi^2}\delta\Psi\delta\Psi + \left.\frac{\partial^2 U}{\partial\Psi^{*2}}\delta\Psi^*\delta\Psi^*\right|_{\Psi_0^*,\Psi_0} \right)\right]. \end{aligned} \tag{3.6}$$

The first term is the potential 3.3 evaluated at the static solutions, so it is a constant. Terms that are of first-order in the fluctuations vanish because we are considering only static solutions that obey the equations 3.5. The only remaining terms are the ones quadratic in the fluctuations. We will call these quadratic terms as $V^{(2)}\left[\Psi^*, \Psi\right]$.

The usual procedure to diagonalize $V^{(2)}\left[\Psi^*, \Psi\right]$ is by expanding the fluctuations as a set of quasiparticles [31]

$$\delta\Psi(\mathbf{r},t) = \sum_k \left[u_k(\mathbf{r})\alpha_k(t) - v_k^*(\mathbf{r})\alpha_k^*(t)\right], \tag{3.7a}$$

$$\delta\Psi^*(\mathbf{r},t) = \sum_k \left[u_k^*(\mathbf{r})\alpha_k^*(t) - v_k(\mathbf{r})\alpha_k(t)\right], \tag{3.7b}$$

with amplitudes $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ ($u_k^*(\mathbf{r})$ and $v_k^*(\mathbf{r})$). These amplitudes must obey orthogonality relations defined as

$$\int d\mathbf{r}\,\left(u_j^* u_k - v_j^* v_k\right) = \delta_{jk}, \tag{3.8a}$$

$$\int d\mathbf{r}\,\left(u_j v_k - v_j u_k\right) = 0. \tag{3.8b}$$

The set of c-numbers $\alpha_k(t)$ and $\alpha_k^*(t)$ are analogous to annihilation and creation operators of the harmonic oscillator problem. In this case, they are the c-number representation of operators responsible for the annihilation or the creation of excited bosonic particles around the localized solution (distorted vacuum). The expansion 3.7 establishes these functions as a new set of coordinates of our system.

Substituting the expansion 3.7 in the quadratic terms of the potential, the second variation $V^{(2)}$ of the potential reads

$$\begin{aligned} V^{(2)} = \int d\mathbf{r} \sum_{jk} \Bigg\{ & \alpha_j^*\alpha_k \left[u_j^*\mathcal{L}u_k - \frac{1}{2}\left(\frac{\partial^2 U}{\partial\Psi^{*2}}u_j^* v_k + \frac{\partial^2 U}{\partial\Psi^2}v_j^* u_k\right)\right] \\ & + \alpha_j\alpha_k^* \left[v_j\mathcal{L}v_k^* - \frac{1}{2}\left(\frac{\partial^2 U}{\partial\Psi^{*2}}v_j u_k^* + \frac{\partial^2 U}{\partial\Psi^2}u_j v_k^*\right)\right] \\ & - \alpha_j\alpha_k \left[v_j\mathcal{L}u_k - \frac{1}{2}\left(\frac{\partial^2 U}{\partial\Psi^{*2}}v_j v_k + \frac{\partial^2 U}{\partial\Psi^2}u_j u_k\right)\right] \\ & - \alpha_j^*\alpha_k^* \left[u_j^*\mathcal{L}v_k^* - \frac{1}{2}\left(\frac{\partial^2 U}{\partial\Psi^{*2}}u_j^* u_k^* + \frac{\partial^2 U}{\partial\Psi^2}v_j^* v_k^*\right)\right]\Bigg\}, \end{aligned} \tag{3.9}$$

where we defined the Hermitian operator

$$\mathcal{L} = \frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}\frac{\partial^2 U}{\partial\Psi\partial\Psi^*} + \frac{1}{2}\frac{\partial^2 U}{\partial\Psi^*\partial\Psi}. \tag{3.10}$$

With the substitution of the expansion 3.7 in the equations of motion 3.4, it is possible to show that the quasiparticle amplitudes $u_j$ and $v_j$ obey the following equations

$$\mathcal{L}u_j - \frac{\partial^2 U}{\partial \Psi^{*2}} v_j = E_j u_j, \tag{3.11a}$$

$$\mathcal{L}v_j - \frac{\partial^2 U}{\partial \Psi^{2}} u_j = -E_j v_j. \tag{3.11b}$$

These two coupled eingenvalue equations 3.11 are the Bogoliubov equations showed in the previous chapter. They diagonalize correctly the second variation of the potential.

A straightforward calculation shows that the substitution of 3.11 in the potential 3.9 gives us

$$V = V\left[\Psi_0^*, \Psi_0\right] + \sum_k E_k \alpha_k^* \alpha_k. \tag{3.12}$$

Therefore, in the same way as for the scalar field, the Schrödinger field presents a classical collective behavior with energy $V\left[\Psi_0^*, \Psi_0\right]$ and fluctuations with energy $E_k$. The exact value of the quasiparticle energy $E_k$ can be found by solving the equations 3.11. The Bogoliubov equations allow for both positive and negative eigenvalues. However, the spectrum of energy corrections must be bounded from below, thus we must choose only values of $E_k \geqslant 0$. Another important feature of the Bogoliubov equations is the existence of a zero-mode $E_0 = 0$ corresponding to the condensate without thermal fluctuations. As we have seen, the appearance of the zero-mode implies that our system presents a translational symmetry, which allows the solutions $\Psi_0^*$ and $\Psi_0$ to have a degree of freedom related to it.

Therefore, in order to solve the problem analytically, we have made the approximation that our field was a static mean-field. Although this approximation was necessary to find a localized solution, we have missed the translational degree of freedom. To deal with this extra degree of freedom, we will use the so-called Collective Coordinate Method for the Schrödinger field.

## 3.2 Collective Coordinate Method

In principle, in the procedure to quantize a classical field theory, we can assume that the solution of the field equation of motion is a static one. However, as we have seen above, the evaluation of the fluctuation energy spectrum shows that some systems may present a zero frequency mode. This mode provides us with the information that the classical solution has a collective behavior of a moving particle. Therefore, in the quantization process, we expect our effective Lagrangian to have a kinetic energy term for the dynamics of the center of the classical solution. The Collective Coordinate Method reincorporates the dynamics of this coordinate into the system by regarding it as a dynamical variable. Here we will apply the method for a unidimensional Schrödinger field, in the same way as it was done in Chapter 1 for the scalar field.

The Lagrangian of the last section, equation 3.1, can be written as

$$L = T - \left( V\left[\Psi_0^*, \Psi_0\right] + \sum_k E_k \alpha_k^* \alpha_k \right), \tag{3.13}$$

where the third term describes the amplitude of the fluctuations about the classical solution of its equation of motion. Because of the existence of a the zero-mode for this system, the kinetic term cannot be suppressed. In order to include its dynamics into the system, the center of the solution is considered as a dynamical variable, which means $x_0 = x_0(t)$. In this way, our field decomposition becomes

$$\Psi(x,t) = \Psi_0(x - x_0(t)) + \delta\Psi(x - x_0(t), t). \tag{3.14}$$

Note that the potential energy term in the Lagrangian 3.1 does not change by making this assumption. The reason is that the potential energy functional depends only on the field itself and its partial derivatives on space, not on time. The kinetic functional, on the other hand, has a first order time derivative dependence, that using the definition 3.7 reads

$$\dot{\Psi} = -\left[ \frac{\partial}{\partial x}\Psi_0 + \sum_k \left( \alpha_k \frac{\partial}{\partial x} u_k - \alpha_k^* \frac{\partial}{\partial x} v_k^* \right) \right] \dot{x}_0 + \sum_k \left( u_k \dot{\alpha}_k - v_k^* \dot{\alpha}_k^* \right). \tag{3.15}$$

Substituting 3.15 in the kinetic energy functional 3.2, we can write the kinetic energy as

$$\begin{aligned} T = \int dx \frac{i\hbar}{2} \Bigg\{ & \left( \Psi_0 \frac{\partial}{\partial x} \Psi_0^* - \Psi_0^* \frac{\partial}{\partial x} \Psi_0 \right) \dot{x}_0 \\ & + \sum_{jk} \Bigg[ \alpha_j \alpha_k \left( u_j \frac{\partial}{\partial x} v_k + v_j \frac{\partial}{\partial x} u_k \right) + \alpha_j^* \alpha_k^* \left( v_j^* \frac{\partial}{\partial x} u_k^* + u_j^* \frac{\partial}{\partial x} v_k^* \right) \\ & \qquad - \alpha_j^* \alpha_k \left( v_j^* \frac{\partial}{\partial x} v_k + u_j^* \frac{\partial}{\partial x} u_k \right) - \alpha_j \alpha_k^* \left( u_j \frac{\partial}{\partial x} u_k^* + v_j \frac{\partial}{\partial x} v_k^* \right) \Bigg] \dot{x}_0 \\ & + \sum_{jk} \Bigg[ \dot{\alpha}_j^* \alpha_k \left( v_j^* v_k + u_j^* u_k \right) + \dot{\alpha}_j \alpha_k^* \left( v_j v_k^* + u_j u_k^* \right) \\ & \qquad - \dot{\alpha}_j \alpha_k \left( u_j v_k + v_j u_k \right) - \dot{\alpha}_j^* \alpha_k^* \left( v_j^* u_k^* + u_j^* v_k^* \right) \Bigg] \Bigg\}. \end{aligned} \tag{3.16}$$

Considering the center of the solution as a dynamical variable, some new terms arise in the kinetic energy functional.

Pursuing the quantization procedure, we must regard the new set of coordinates, $\alpha_k(t)$ and $\alpha_k^*(t)$, as bosonic annihilation and creation operators, respectively. Given this definition, we can interpret physically each new term. Terms that are proportional to $\alpha_j \alpha_k$ and $\alpha_j^* \alpha_k^*$ are responsible for producing Cherenkov radiation. This means that these terms are responsible for the absorption and emission of quasiparticles due to the movement of the center $x_0$. If the center of the solution moves very slowly compared to the characteristic speed of excitations of the medium, $\dot{x}_0 \ll c_s$, these radiation terms can be neglected. Terms that are proportional to $\alpha_j^* \alpha_k$ and $\alpha_j \alpha_k^*$ are related to scattering processes of the quasiparticles. Lastly, terms with $\dot{\alpha}_j^* \alpha_k^*$ and $\dot{\alpha}_j \alpha_k$ vanish because of the orthogonality relations 3.8. Therefore, within this set of approximations, the kinetic energy 3.16 can be written as

$$T = \frac{\kappa}{2} \dot{x}_0 + \dot{x}_0 \sum_{jk} \hbar g_{jk} \alpha_j^* \alpha_k + \frac{i\hbar}{2} \sum_k \left( \dot{\alpha}_k \alpha_k^* - \dot{\alpha}_k^* \alpha_k \right), \tag{3.17}$$

where we have defined the constant

$$\kappa = \int dx \; i\hbar \left( \Psi_0 \frac{\partial}{\partial x} \Psi_0^* - \Psi_0^* \frac{\partial}{\partial x} \Psi_0 \right), \tag{3.18}$$

and the coupling matrix as

$$g_{jk} = \frac{1}{2}\int dx\ i\left(v_j^*\frac{\partial}{\partial x}v_k - u_j^*\frac{\partial}{\partial x}u_k - v_j\frac{\partial}{\partial x}v_k^* + u_j\frac{\partial}{\partial x}u_k^*\right). \tag{3.19}$$

Comparing the kinetic energy 3.17 found by the direct use of the Collective Coordinate Method to the Schrödinger field with equation 1.29 found for the scalar field, we can see some disparities. The first and most important difference from both developments is that, in the Schrödinger field, the kinetic energy does not have a quadratic term in the velocities, as in 1.29. The reason for this is related to the fact that the initial Lagrangian 3.1 was only first-order in the time derivative. Since we already know that the localized solution can move in space, the effective Lagrangian of the center of the solution is expected to have a kinetic energy of the form $M\dot{x}_0^2/2$. So, from the effective Lagrangian found by the direct application of the method for a Schrödinger field, it is possible to conclude that the method was not appropriate to reincorporate the translational mode into the problem. Therefore, we must develop an extended Collective Coordinate Method to apply to a Schrödinger field.

## 3.3 Extended Collective Coordinate Method

The Collective Coordinate Method is widely discussed in the literature for theories described by a Lagrangian of the type 1.6, in which the kinetic term is proportional to $\dot{\phi}^2$. However, as we have just mentioned above, its application to the Schrödinger field cannot account for the kinetic energy associated to the localized solution. In the direct application of the method, the field, initially assumed to be static, is rewritten as a mean field plus fluctuations all centered at a dynamical variable as in 3.14. A direct consequence of this substitution is the form of the kinetic energy, since it depends on the time derivative of the field. Another subtle consequence, that is usually overlooked in the discussion of the method, is the effect of the substitution 3.14 in the equations of motion 3.5. The initial static condition for the solution $\Psi_0$ allows us to write the equations of motion 3.5 with $\dot{\Psi}_0 = 0$. However, in the quantization method the solution depends indirectly on time, so the equations of motion should be rewritten as

$$i\hbar\frac{\partial}{\partial x}\Psi_0^*\dot{x}_0 = \left.\frac{\partial U}{\partial \Psi}\right|_{\Psi_0^*,\Psi_0} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_0^*, \tag{3.20a}$$

$$-i\hbar\frac{\partial}{\partial x}\Psi_0\dot{x}_0 = \left.\frac{\partial U}{\partial \Psi^*}\right|_{\Psi_0^*,\Psi_0} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_0, \tag{3.20b}$$

where the left hand side of the equations are the time derivatives of $\Psi_0(x - x_0(t))$. In the standard application of the method, we assume that the left hand sides of the equations 3.20a and 3.20b are close enough to zero, $\dot{\Psi}_0(x - x_0(t)) \approx 0$. The physical implication of this assumption is that the solution $\Psi_0$ is not deformed by the moving center $x_0(t)$ since the equations of motion remain unchanged.

To improve the precision of the method for the Schrödinger field, we will consider deformations of the solution due to the movement of its center. Therefore, to continue the quantization procedure, we must solve the new set of equations of motion 3.20. These are coupled partial differential equations with a forced term. Therefore, we must determine the homogeneous ($\psi_0$) and the particular ($\psi_P$) solutions

$$\Psi_0(x - x_0(t)) = \psi_0(x - x_0(t)) + \psi_P(x - x_0(t)), \tag{3.21}$$

separately. The solution of the homogeneous case $\psi_0$ obeys 3.5 as before. To determine the particular solution, we can make a substitution of 3.21 in the equations 3.20. The equations of motion for the particular solution can be written as

$$i\hbar\left(\frac{\partial}{\partial x}\psi_0^*\right)\dot{x}_0 = \left.\frac{\partial^2 U}{\partial\Psi\partial\Psi^*}\right|_{\psi_0^*,\psi_0}\psi_P + \left.\frac{\partial^2 U}{\partial\Psi^*\partial\Psi}\right|_{\psi_0^*,\psi_0}\psi_P^* - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_P^*, \tag{3.22a}$$

$$-i\hbar\left(\frac{\partial}{\partial x}\psi_0\right)\dot{x}_0 = \left.\frac{\partial^2 U}{\partial\Psi\partial\Psi^*}\right|_{\psi_0^*,\psi_0}\psi_P^* + \left.\frac{\partial^2 U}{\partial\Psi^*\partial\Psi}\right|_{\psi_0^*,\psi_0}\psi_P - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_P. \tag{3.22b}$$

To solve these equations correctly, the solution $\psi_P$ must have the form of the forced term. So, we can write the particular solutions as

$$\psi_P = -i\varphi(x)\dot{x}_0, \tag{3.23a}$$

$$\psi_P^* = i\varphi^*(x)\dot{x}_0, \tag{3.23b}$$

in which $\varphi$ is a function of space that will define the deformation of the localized solution. Substituing 3.23 in 3.22, the equations of motion become

$$\hbar\left(\frac{\partial}{\partial x}\psi_0^*\right) = \left.\frac{\partial^2 U}{\partial\Psi\partial\Psi^*}\right|_{\psi_0^*,\psi_0}\varphi - \left.\frac{\partial^2 U}{\partial\Psi^*\partial\Psi}\right|_{\psi_0^*,\psi_0}\varphi^* + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi^*, \tag{3.24a}$$

$$-\hbar\left(\frac{\partial}{\partial x}\psi_0\right) = \left.\frac{\partial^2 U}{\partial\Psi\partial\Psi^*}\right|_{\psi_0^*,\psi_0}\varphi - \left.\frac{\partial^2 U}{\partial\Psi^*\partial\Psi}\right|_{\psi_0^*,\psi_0}\varphi^* - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi. \tag{3.24b}$$

These are the equations of motion that determine the deformation of the solution due to the movement of the center $x_0$. Further developments on the form of the functions $\varphi$ and $\varphi^*$ require information about $U(\Psi, \Psi^*)$.

A direct consequence of considering the deformation of the static solution is in the form of the field. The former field decomposition, written as a mean field plus fluctuations, now has an additional deformation term

$$\Psi(x,t) = \psi_0(x - x_0(t)) - i\varphi(x)\dot{x}_0 + \delta\Psi(x - x_0(t), t), \tag{3.25}$$

where the first term is the localized solution of the homogeneous equation 3.5, the second term is the deformation of this solution, and the last one is the fluctuation 3.7. Note that, when we explicitly include the deformation contribution, the form of the field becomes the same as that proposed in [32], as we have written in 2.26.

Substituing 3.25 in the kinetic functional 3.2, we can write it as

$$\begin{aligned} T = {} & \frac{\hbar}{2}\int dx\ \left(\varphi^*\frac{\partial}{\partial x}\psi_0 + \varphi\frac{\partial}{\partial x}\psi_0^*\right)\ \dot{x}_0^2 \\ & + \frac{\kappa}{2}\ \dot{x}_0 + \dot{x}_0\sum_{jk}\hbar g_{jk}\alpha_j^*\alpha_k + \frac{i\hbar}{2}\sum_k\left(\dot{\alpha}_k\alpha_k^* - \dot{\alpha}_k^*\alpha_k\right), \end{aligned} \tag{3.26}$$

in which $\kappa$ and $g_{jk}$ are a constant and the coupling matrix defined by 3.18 and 3.19, respectively. When we consider the deformation of the mean-field solution, the kinetic energy functional gains a new term, dependent on $\dot{x}_0^2$. This term describes the movement of the center $x_0$. Since the kinetic energy is defined by $Mv^2/2$, the soliton mass must be defined as

$$M = \hbar \int dx \, \left( \varphi^* \frac{\partial}{\partial x} \psi_0 + \varphi \frac{\partial}{\partial x} \psi_0^* \right), \tag{3.27}$$

where a dimensional analysis confirms that $M$ has unity of mass. The resulting effective Lagrangian for the Schrödinger field can be written as

$$L = \frac{M\dot{x}_0^2}{2} + \frac{\kappa}{2}\dot{x}_0 + \dot{x}_0 \sum_{jk} \hbar g_{jk} \alpha_j^* \alpha_k + \frac{i\hbar}{2} \sum_k \left( \dot{\alpha}_k \alpha_k^* - \dot{\alpha}_k^* \alpha_k \right) - \sum_k E_k \alpha_k^* \alpha_k - V\left[\psi_0^*, \psi_0\right]. \tag{3.28}$$

The extended Collective Coordinate Method we have developed is different from the conventional one described in section 3.2, because it takes into account solitonic deformations. These deformations allow us to write an effective Lagrangian with a kinetic energy term for the moving coordinate $x_0$. Although we had to adjust the method, the effective Lagrangian 3.28 is very similar to equation 1.29 found for the scalar field case. This is, the effective behavior of a Bose gas is of a moving particle $x_0$ in a bath of harmonic oscillators. The main difference is the presence of an additional term, proportional to $\kappa$. In the next section, we will discuss how to apply the method developed in Chapter 1 to the study of quantum dissipation in the effective Lagrangian found here.

## 3.4 Quantum Dissipation for a Bose gas

In the last section, we have showed that a Bose gas with a localized solution has an effective behavior of a particle in a "bath of oscillators". Physically, the particle is represented by the center of the topological solution $\psi_0$ in $x_0$, and the reservoir is composed of excited quasiparticles. To understand quantum dissipation in this system, our starting point is the effective Lagrangian 3.28 found in the last section.

The total action of this system is defined by

$$\mathcal{S} = \int L dt. \tag{3.29}$$

Since our system is composed of two main components, the particle and the bath, the total action can be broken into two subsystems: a particle action $\mathcal{S}_S$ plus a bath and interaction action $\mathcal{S}_{RI}$,

$$\mathcal{S}\left[x_0, \alpha\right] = \mathcal{S}_S\left[x_0\right] + \mathcal{S}_{RI}\left[x_0, \alpha\right], \tag{3.30}$$

in which $\mathcal{S}_S$ and $\mathcal{S}_{RI}$ can be defined by the Lagrangian 3.28 as

$$\mathcal{S}_S\left[x_0\right] = \int \left(\frac{M\dot{x}_0^2}{2} + \frac{\kappa}{2}\dot{x}_0 - V\left[\psi_0^*, \psi_0\right]\right) dt, \tag{3.31a}$$

$$\mathcal{S}_{RI}\left[x_0, \alpha\right] = \int dt \left[\dot{x}_0 \sum_{jk} \hbar g_{jk} \alpha_j^* \alpha_k + \frac{i\hbar}{2} \sum_k \left(\dot{\alpha}_k \alpha_k^* - \dot{\alpha}_k^* \alpha_k\right) - \sum_k E_k \alpha_k^* \alpha_k\right]. \tag{3.31b}$$

The particle action $\mathcal{S}_S$ contains the kinetic energy for the dynamics of $x_0$ and an additional term proportional to $\dot{x}_0$. This additional term is the main difference between the scalar case developed in Chapter 1 and the Schrödinger case. Since this new term is linear in the velocity, the equation of motion for the center of the solution must remain unchanged. Furthermore, the dynamics and dissipation of the system depend only on the behavior of the reservoir, thus we must compare the bath action from both theories. Note that the action $\mathcal{S}_{RI}$ of the subsystem $R$ is the same as the one used in section 1.5, and therefore, the procedure for dealing with both cases must be the same.

To summarize, a Schrödinger field with a localized solution has an effective behavior of a particle coupled to a reservoir. To understand the behavior of the particle, we must evaluate the influence functional 1.44 and write the associated equations of motion for the system 1.54. This procedure results in two quantities that describe the dissipation of this particle: the damping and diffusion functions. Fortunately, the description of the Schrödinger field reflects exactly the results found for a scalar field. Therefore, to study the dissipative dynamics emerging from a Schrödinger field, we only need to evaluate the expressions for the damping 1.48 and diffusion 1.49 functions.

Both damping and diffusion functions depend on the form of the coupling constant $g_{jk}$, defined in 3.19. This constant is written in terms of the solutions of the Bogoliubov equations 3.11. Therefore, in order to calculate those functions, we need the solutions for the quasiparticles amplitudes. Some problems allow for an exact solution of the Bogoliubov equations. In this case, we can use the discrete description of the excitations to perform our calculations, in the same way it was done for the optical polaron case in [9]. However, if we only have the asymptotic form of the quasiparticle solutions, which usually is the case, we can use the continuum approximation as above to find both damping and diffusion functions in terms of the phase shifts of the scattered quasiparticles. This last scenario was used in the calculations of the acoustical polaron case in [9] and in the study of skyrmions in a quantum Hall ferromagnet [22].

## 3.5 Summary of the Method

As we have seen, the method described in Chapter 1 can be adapted to study quantum dissipation of localized solutions of a Schrödinger field. The Lagrangian that describes this field is first-order in time derivative. This feature requires an adaptation of the Collective Coordinate Method to write an effective Lagrangian for a static solution.

In section 3.3, we presented an extended Collective Coordinate Method, suitable to deal with the Schrödinger field. This method presents an additional term for the effective Lagrangian, which is not necessary for dealing with the scalar case. In section 3.4, we showed that the additional term does not interfere with the equations of motion for the moving center $x_0$. Therefore, the quantum dissipation analysis remains the same for both theories.

Summarizing, in this chapter we developed a general method to study quantum dissipation in a Schrödinger field. The application of this method for a particular case can be summarized in the following steps:

1. Define the initial Lagrangian 3.1;
2. Find a localized solution from the equation of motion 3.20;
3. Solve the Bogoliubov equations 3.11;
4. Make the coordinate $x_0$ a dynamical variable;

5. Solve the equations for the deformation 3.24;
6. Evaluate the damping 1.48 and diffusion 1.49 functions, using the results of steps 2, 3 and 5.

Note that the main difference between the adapted method for the Schrödinger field and the one developed in Chapter 1 is the step 5. For a scalar field, the deformation due to the movement of the solution is negligible. However, a system described by the Lagrangian 3.1 showed an effective behavior highly sensitive to the deformation of the localized solution.

# Chapter 4

# Bright Soliton Analysis

In the previous chapter, we have developed a one-dimensional theory to study quantum dissipation in the Schrödinger field. Until this point, our discussion has been entirely general and could be applied for any topological solution which may present dynamical behavior. In this chapter, we will apply our theory to the case of a one-dimensional condensate with a bright soliton solution.

## 4.1 Static Bright Soliton

Our analysis begins with the discussion of the static mean-field solution and its small amplitude fluctuations. In this section, we will develop the steps 1 to 3 of our method, that is, the discussion before promoting the center of the solution to a dynamical variable.

Our system is a one-dimensional Bose gas in the absence of an external potential. The corresponding Lagrangian for this system is a particular case of equation 3.1 and can be written as

$$L = \int dx \left[ \frac{i\hbar}{2} \left( \Psi^* \dot{\Psi} - \Psi \dot{\Psi}^* \right) - \frac{\hbar^2}{2m} \left| \frac{\partial \Psi}{\partial x} \right|^2 + \mu |\Psi|^2 - \frac{g}{2} |\Psi|^4 \right] . \tag{4.1}$$

The parameter $\mu$ is the chemical potential and $g$ is the interaction parameter. Assuming the condensation condition, we set the chemical potential to be negative. For this system, we will suppose a attractive interaction, which means $g < 0$.

The equation of motion for this system can be obtained by the extremization of the action related to the Lagrangian 4.1 resulting in the Gross-Pitaeviskii equation without the external potential,

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + g\left|\Psi_0(x)\right|^2 - \mu\right)\Psi_0(x) = 0. \tag{4.2}$$

As we have seen in Chapter 2, this equation allows a topological solution called the bright soliton. The normalized solution can be written in the same way as 2.19,

$$\Psi_0(x,t) = \sqrt{\frac{2\left|\mu\right|}{g}}\ \mathrm{sech}\left(\frac{x-x_0}{\xi}\right)\ e^{-i\mu t/\hbar}. \tag{4.3}$$

The solution 4.3 is the generalization of equation 2.19 showed in Chapter 2. Here, the center of the solution is placed at an arbitrary position $x_0$. In figure 4.1, we show the density distribution related to the bright soliton solution. Note that the density is centered around $x_0 = 0$ and it vanishes for $x \gg \xi$, characterizing the topological aspect of the solution.

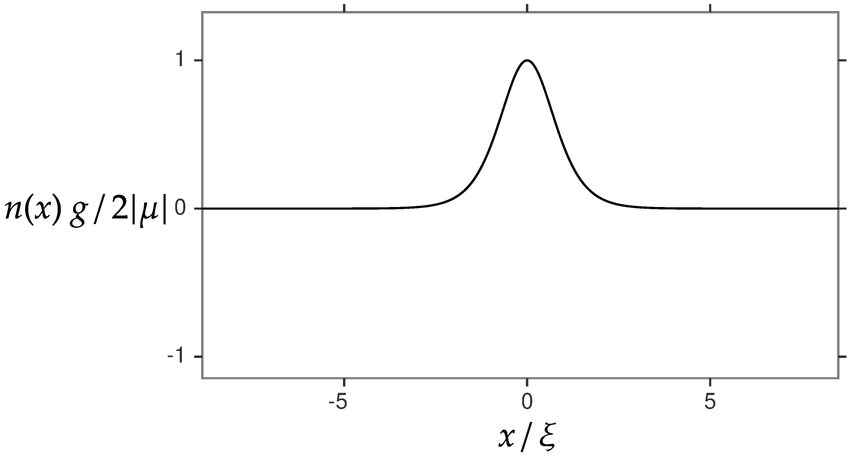

Figure 4.1: Density distribuition representation of the static bright soliton centered in $x_0 = 0$.

If we consider small amplitude oscillations around this topological solution, we can expand the Lagrangian 4.1 up to second order in the fluctuations. This expansion leads to the Bogoliubov eigenvalue equations 2.25 for the bright soliton

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu + 2g|\Psi_0(x)|^2\right)u_k - (\Psi_0(x))^2 v_k = \hbar\omega_k u_k, \tag{4.4a}$$

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu + 2g|\Psi_0(x)|^2\right) v_k - (\Psi_0^*(x))^2 u_k = -\hbar\omega_k v_k. \tag{4.4b}$$

Here the quasiparticle energy was written in terms of the frequencies as $E_k = \hbar\omega_k$. The corresponding frequency for this fluctuations have the form

$$\omega_k = \frac{\hbar k^2}{2m}, \tag{4.5}$$

where $k$ is the wave number and $[k] = L^{-1}$.

The eigenmodes of the Bogoliubov equations 4.4 for the bright soliton solution 4.3 have already been obtained in [33] and they have the form

$$u_k = \frac{e^{ikx}}{(k^2\xi^2+1)}\left[k\xi + i\tanh\left(x/\xi\right)\right]^2, \tag{4.6a}$$

$$v_k = \frac{e^{ikx}}{(k^2\xi^2+1)}\ \mathrm{sech}^2(x/\xi). \tag{4.6b}$$

Therefore, we have a static mean-field 4.3 with small amplitude fluctuations with the form 4.6. This completes the first three steps of our method in which it was possible to use the approximation of a static field.

## 4.2 Effective Lagrangian

To study the dissipation properties of the solitonic solution we must promote the center of the solution to a dynamical variable $x_0 = x_0(t)$. Our aim in this section is to define a dynamical collective coordinate for the center of the solution and write an effective Lagrangian for the localized solution.

As it was discussed in the last chapter, the Schrödinger field requires that we also modify the equations of motion for the moving soliton when defining the collective coordinate. This modified equation of motion describes the deformation of the solution due to its movement. Therefore, our field is composed of a mean-field term $\psi_0$, a deformation correction $-i\varphi(x)\dot{x}_0$ and fluctuations $\delta\Psi$:

$$\Psi(x,t) = \psi_0(x - x_0(t)) - i\varphi(x)\dot{x}_0 + \delta\Psi(x - x_0(t), t). \tag{4.7}$$

Substituing the expansion 4.7 in the original Lagrangian 4.1, we can write

$$\begin{aligned} L =& \frac{M\dot{x}_0^2}{2} + \dot{x}_0 \sum_{jk} \hbar g_{jk}\alpha_j^*\alpha_k + \frac{i\hbar}{2}\sum_k \left(\dot{\alpha}_k\alpha_k^* - \dot{\alpha}_k^*\alpha_k\right) \\ & - \sum_k (\hbar\omega_k - \mu)\alpha_k^*\alpha_k - V\left[\psi_0^*, \psi_0\right], \end{aligned} \tag{4.8}$$

in which the soliton mass $M$ and coupling constant $g_{jk}$ are defined in 3.27 and 3.19, respectively. For the particular case of the bright soliton, the term of 3.28 dependent of $\kappa$ does not appear. The reason for this is that $\kappa$ is calculated in terms of the mean-field solution and since the bright soliton is a real solution, it turns out to be zero, by definition.

At this point, we have shown that our system behaves as a particle with coordinate $x_0$ in a thermal bath. We can separate the Lagrangian 4.8 in three main contributions,

$$L\left[x_0, \alpha\right] = L_S\left[x_0\right] + L_R\left[\alpha\right] + L_I\left[x_0, \alpha\right], \tag{4.9}$$

where $L_S$ is the Lagrangian of the particle, $L_R$ the Lagrangian of the reservoir, and $L_I$ is the interaction between the particle and the bath. Each contribution is defined as

$$L_S\left[x_0\right] = \frac{M\dot{x}_0^2}{2} - V\left[\psi_0^*, \psi_0\right], \tag{4.10a}$$

$$L_R\left[\alpha\right] = \frac{i\hbar}{2}\sum_k \left(\dot{\alpha}_k\alpha_k^* - \dot{\alpha}_k^*\alpha_k\right) - \sum_k (\hbar\omega_k - \mu)\alpha_k^*\alpha_k, \tag{4.10b}$$

$$L_I\left[x_0, \alpha\right] = \dot{x}_0 \sum_{jk} \hbar g_{jk}\alpha_j^*\alpha_k. \tag{4.10c}$$

Now we can proceed to study the quantum dissipation of this effective particle.

## 4.3 Coupling Constant

Both damping 1.48 and diffusion 1.49 functions are dependent on the coupling constant. This constant is responsible for describing the interaction between the particle and the reservoir. Therefore, in order to understand dissipative properties of this system, we must calculate exactly the constant defined by 3.19

$$g_{jk} = \frac{1}{2}\int dx\ i\left(v_j^*\frac{\partial}{\partial x}v_k - u_j^*\frac{\partial}{\partial x}u_k - v_j\frac{\partial}{\partial x}v_k^* + u_j\frac{\partial}{\partial x}u_k^*\right). \tag{4.11}$$

It is important to compare the form of our coupling $g_{jk}$ and the constant $\pi_{jk}$ defined in 2.28. The interaction term found in [32] has two main components, the scattering terms which are proportional to $\alpha\alpha^*$ and radiation terms proportional to $\alpha\alpha$ and $\alpha^*\alpha^*$. In Chapter 3, we have used the approximation that the center of the solution $x_0$ travels with a small speed $v \ll c_s$ and any interaction with the quasiparticles is by scattering process. Therefore, if we neglect the radiation terms of $\pi_{jk}$, the interaction term we find is in accordance with the one found in [32]. In fact, the substitution of the fluctuations amplitudes 4.6 in the coupling constant 4.11 gives the same result as the scattered contribution of $\pi_{jk}$. A straightforward calculation of 4.11 gives

$$g_{jk} = \frac{\pi}{3L}\frac{\left(k^2 - j^2\right)\left(k^2 + j^2 + kj + 1/\xi^2\right)}{\sinh\left[\frac{\pi}{2}\xi\left(j-k\right)\right]\left(k^2 + 1/\xi^2\right)\left(j^2 + 1/\xi^2\right)}. \tag{4.12}$$

This constant describes a quasiparticle in the $j^{th}$ state interacting with the center of the solution $x_0$ and being scattered to the $k^{th}$ state. The coupling constant is symmetric in the interchange of $j$ and $k$, that is, $g_{jk} = g_{kj}$. In the limit when $j = k$ the coupling takes the form

$$g_{kk} = \frac{4}{3L\xi}\frac{k\left(3k^2 + 1/\xi^2\right)}{\left(k^2 + 1/\xi^2\right)^2}. \tag{4.13}$$

To visualize the behavior of $g_{jk}$, we can define the dimensionless quantities $k\xi$, $j\xi$ and $Lg_{jk}$ and plot [1] 4.12 for $j \neq k$ and 4.13 for $j = k$. Figure 4.2 shows the behavior of the coupling

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[1] The plot of figure 4.2 was made using the Python libriries pandas [34] and Matplotlib [35].

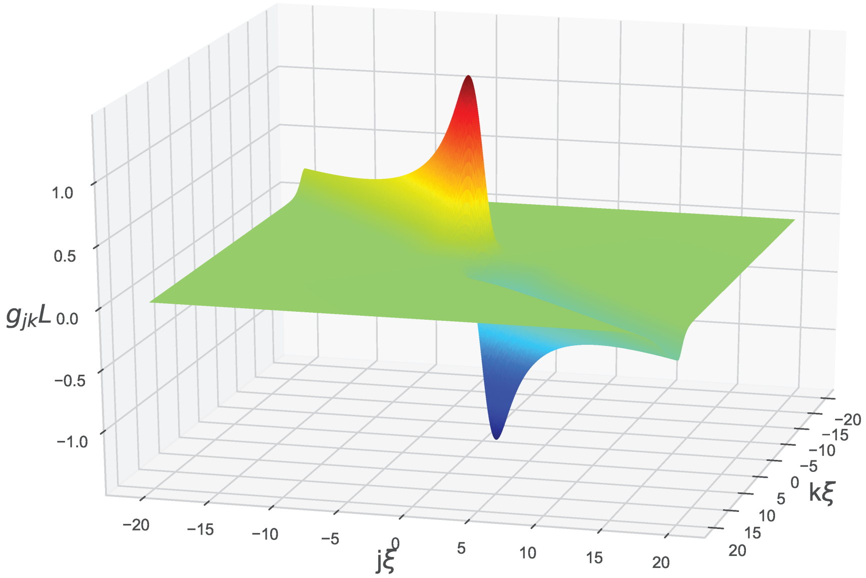

Figure 4.2: Behavior of the coupling constant for values of $k\xi$ and $j\xi$ in the interval $[-20, 20]$.

constant for different values of $j$ and $k$. One notices that the diagonal along which $j \approx k$ turns out to provide the most relevant values of the coupling $g_{jk}$.

## 4.4 Damping and Diffusion

Here we will develop the procedure to calculate both damping and diffusion functions.

Starting with the continuum approximation for the quasiparticle spectrum as defined in 1.48, we have

$$\gamma(t) = -\frac{\hbar}{2M}\theta(t)\int d\omega \int d\omega'\ S(\omega,\omega')\left[\overline{n}(\omega) - \overline{n}(\omega')\right](\omega-\omega')\cos(\omega-\omega')t. \tag{4.14}$$

As we have seen in the last section, the scattered quasiparticles present a peak in the coupling for values of $k \approx j$. For this reason, it is convenient to define a new set of variables in the form

$$\eta = \omega - \omega', \tag{4.15a}$$

$$\varphi = \frac{\omega + \omega'}{2}, \tag{4.15b}$$

and rewrite the damping 1.48 as

$$\gamma(t) = -\frac{\hbar}{2M}\theta(t)\int_0^\infty d\varphi \int_{-\infty}^\infty d\eta \; S(\varphi,\eta) \; \frac{[\overline{n}(\varphi+\eta/2) - \overline{n}(\varphi-\eta/2)]}{\eta} \; \eta^2 \cos(\eta t). \tag{4.16}$$

This change of variables allows us to calculate this function in the peak region of the coupling, since its main contribution occurs when $\eta \rightarrow 0$. In the appendix C, we show the calculation of the scattering function $S(\varphi,\eta)$ in the limit of $\eta \rightarrow 0$,

$$\lim_{\eta \longrightarrow 0} S(\varphi,\eta) = \frac{4\hbar^2}{9\xi^2\pi^2} \; f(\varphi), \tag{4.17}$$

where the function $f(\varphi)$ is defined as

$$f(\varphi) = \frac{(2\hbar\varphi + |\mu|)^2}{(\hbar\varphi + |\mu|)^4}. \tag{4.18}$$

Furthermore, the term dependent on the bosonic distribution $\bar{n}$ can be approximated by a derivative with respect to the variable $\varphi$, for $\eta$ close to zero

$$\lim_{\eta \longrightarrow 0} \frac{\overline{n}(\varphi+\eta/2) - \overline{n}(\varphi-\eta/2)}{\eta} = \frac{d\bar{n}(\varphi)}{d\varphi}, \tag{4.19}$$

which allows us to write the damping function as

$$\gamma(t) = \frac{4\hbar^3}{9M\pi^2\xi^2}\theta(t)\int_0^\infty d\varphi \; f(\varphi)\left(-\frac{d\bar{n}(\varphi)}{d\varphi}\right)\int_0^\infty d\eta\eta^2\cos(\eta t). \tag{4.20}$$

The term $\cos(\eta t)$ oscillates rapidly for long periods of time. Thus, we would expect that the integral in $d\eta$ to vanish, unless $\eta$ is so small that it counterbalances large values of $t$. Since we have shown that $\eta$ is in fact close to zero, the integral in $d\eta$ can be evaluated exactly. For long periods of time $t$, we can write the damping function as

$$\gamma(t) = -\bar{\gamma}(T)\delta''(t), \tag{4.21}$$

where the temperature dependent $\bar{\gamma}(T)$ is defined as

$$\bar{\gamma}(T) = \frac{4\hbar^4}{9M\pi\xi^2}\left(\frac{1}{k_BT}\right)\int_0^\infty d\varphi\ f(\varphi)\frac{e^{(\hbar\varphi+|\mu|)/k_BT}}{\left(e^{(\hbar\varphi+|\mu|)/k_BT}-1\right)^2}, \tag{4.22}$$

and $\delta''(t)$ is the second derivative of the Dirac delta function.

The diffusion function can be handled analogously, and reads

$$D(t) = \bar{D}(T)\delta''(t), \tag{4.23}$$

where $\bar{D}(T)$ is given by

$$\bar{D}(T) = \frac{4\hbar^4}{9\pi\xi^2}\int_0^\infty d\varphi\ f(\varphi)\left[\frac{1}{\left(e^{(\hbar\varphi+|\mu|)/k_BT}-1\right)}+\frac{1}{\left(e^{(\hbar\varphi+|\mu|)/k_BT}-1\right)^2}\right]. \tag{4.24}$$

Since the Bose-Einstein Condensation is expected to happen at low temperatures, we can analyze both $\bar{\gamma}(T)$ and $\bar{D}(T)$ in this temperature limit. Accordingly, our damping and diffusion coefficients can be written as

$$\bar{\gamma}(T) = \frac{4\hbar^4}{9M\pi\xi^2}\left(\frac{1}{k_BT}\right)\int_0^\infty d\varphi\ e^{-(\hbar\varphi+|\mu|)/k_BT}f(\varphi), \tag{4.25a}$$

$$\bar{D}(T) = \frac{4\hbar^4}{9\pi\xi^2}\int_0^\infty d\varphi\ e^{-(\hbar\varphi+|\mu|)/k_BT}f(\varphi), \qquad \text{for}\ \ k_BT \ll (\hbar\varphi-\mu). \tag{4.25b}$$

In this limit, as the temperature decreases, both damping and diffusion coefficients go to zero very rapidly. This is expected since, for $T = 0$, there are no quasiparticles being scattered. With this last result, we can also calculate the ratio between $\bar{D}(T)$ and $\bar{\gamma}(T)$ which reads

$$\frac{\bar{D}(T)}{\bar{\gamma}(T)} = 2Mk_BT. \tag{4.26}$$

Notice that, even at low temperatures, this ratio is the same as the classical result for the fluctuation-dissipation theorem for the Brownian motion [36].

Lastly, one of the main reasons to calculate explicitly the form of the damping function is to write the equation of motion for our particle. The equation of motion for the coordinate of the center of the solution is given by

$$\ddot{R} + 2\int_0^t \gamma(t-t')\dot{R}dt' = 0. \tag{4.27}$$

Substituing 4.21 in 4.27 and integrating by parts we can write the equation of motion for the center of the solution as

$$\ddot{R} - 2\bar{\gamma}(T)\dddot{R} = 0. \tag{4.28}$$

This equation is called the Abraham-Lorentz equation which is characterized by the time derivative in the acceleration. Thus, the equation of motion for the center of the soliton shows us a dissipative behavior, which we will discuss in more details in the next section.

## 4.5 Discussion

In this chapter, we have used the theory developed in Chapter 3 for the study of quantum dissipation in the particular case of a bright soliton solution. As we have shown, a system that can be described by the effective Lagrangian of the form 3.28 has its dissipative properties completely described by the evaluation of the coupling constant $g_{jk}$. The coupling accounts for any interaction between the excited quasiparticles and the center of the solution $x_0$. Since we are only interested in solutions that move with speeds much smaller than the characteristic speed of excitations in the condensate, we can assume that only scattering processes play a significant role in the dissipative mechanism of the soliton.

For the bright soliton, the coupling constant was found to be a symmetric function peaked in the neighborhood of the diagonal $j \approx k$. The exact form of the coupling allow us to evaluate both damping 4.21 and diffusion 4.23 functions. They are both temperature-dependent, showing that the dissipation of the soliton is highly related to the behavior of the excited bosons. The time dependence is given by the second derivative of the Dirac delta function. This characterizes a Markovian process, which is described by a delta function or its derivatives. Physically, this means that our dissipation has no memory. Therefore, the dynamics of the particle does not depend on its previous history but is local in time. In the limit of low temperatures, we have shown that both damping and diffusion coefficients decrease exponentially as the temperature goes to zero. In fact, for zero temperature, we expect no excited quasiparticles in our system. Thus, in this limit, the soliton has its mobility increased, since there is no dissipative process occurring. The particular case of high temperatures was not investigated, since the Bose-Einstein condensation is expected to happen only for temperatures below a critical value.

Using the exact form of the damping 4.21 in the equation of motion for the center of the solution 1.54, we were able to write the corresponding equation of motion for the center of the soliton, as equation 4.28. This equation of motion has a Abraham-Lorentz type of force, that is proportional to $\dddot{R}$. In electrodynamics, this kind of force can be interpreted as a radiation damping, since it is related with the emission of photons by an accelerating charge. Here, the interpretation of this force is quite different. As discussed in Chapter 3, our analysis considered the low speed limit, in which the radiation terms $\alpha\alpha$ and $\alpha^*\alpha^*$ were neglected. Since our effective Lagrangian 4.8 takes into account only scattering process of the quasiparticles with the soliton, there is no radiation term involved in the exchange of energy from the soliton to the excited bosons. Therefore, although our equation of motion gives rise to an Abraham-Lorentz force, here the correct interpretation is that the soliton will experience dissipation due to inelastic scattering of the quasiparticles. This result agrees with the one found in [32], where the authors also find an equation of motion with an Abraham-Lorentz type of force.

In order to compare our results with [32], we can use their approximation of the low frequency limit in 4.25a to evaluate the integral exactly. In this limit, the parameter $f(\varphi)$ in 4.18 can be approximated by

$$\lim_{\varphi \longrightarrow 0} f(\varphi) = \frac{1}{|\mu|^2}. \tag{4.29}$$

Substituting 4.29 in the expression 4.25a, we can perform the integral to rewrite 4.27 as

$$\ddot{R} - \frac{16 m \hbar}{9 M \pi \, |\mu|} \, e^{-|\mu|/k_B T} \, \dddot{R} = 0, \tag{4.30}$$

which agrees exactly with the equation of motion found in [32].

# Chapter 5

# Conclusions

In this work, we have seen the importance of considering quantum effects in macroscopic systems such as the Bose-Einstein Condensate. A Bose gas has the intriguing feature of condensing the majority of its particles in a single particle ground state. This is the main reason that allows us to analyze a quantum gas by a macroscopic wave function $\Psi$. This function is a classical mean-field that obeys a non-linear Schrödinger-like equation called the Gross-Pitaevskii equation. This equation is suitable to understand condensates when the temperature is close enough to zero. For higher values of the temperature, the system will present excited bosons that we may call quasiparticles, which result from the quatization of fluctuations about the classical solution. Those quasiparticles are responsible for any dissipative effect the condensate may present.

The nonlinearity of the Gross-Pitaevskii equation allows this system to present topological solutions that are of great experimental interest. To treat this system properly, our approach considered two main steps: a quantization method and the study of dissipation. Our idea was to apply a quantization method called Collective Coordinate Method in order to show that this system is equivalent to the model of a particle coupled to a reservoir, where the center of the topological solution is represented by the particle and the excited quasiparticles play the role of the thermal reservoir. This reminds us of a semi-phenomenological approach to deal with the quantum Brownian motion, in which a quantum particle has its dynamics determined by its coupling to a properly chosen bath. In our case, both particle and bath have the same microscopic origin, and therefore, we expected dissipative properties to depend on the temperature of the excited quasi-particles.

The Collective Coordinate Method is presented in the literature in several contexts, but it usually involves the quantization of scalar fields that obey a Lagrangian of the type

$$L\left[\phi\right] = \int dx \left[ \left(\frac{\partial\phi}{\partial t}\right)^2 - \mathcal{V}\left(\phi\right) \right], \tag{5.1}$$

in which the kinetic energy is quadratic in the time derivative of the field $\phi$. As we saw, the direct application of this method led us to an effective behavior without a kinetic term $Mv^2/2$ for the center of the solitonic solution. This result contradicts both experiments and theory, showing that the Collective Coordinate method is not suitable to quantize localized solutions of the equations of motion resulting from a Lagrangian of the type

$$L = \int dx \left[ \frac{i\hbar}{2}\left(\Psi^*\dot{\Psi} - \Psi\dot{\Psi}^*\right) - \mathcal{V}(\Psi^*, \Psi) \right]. \tag{5.2}$$

The Schrödinger Lagrangian presents a kinetic energy that is written in terms of a first-order time derivative. Since the quantization method is highly dependent on the form of the kinetic term, in order to quantize a topological solution described by 5.2 we must make adjustments to the method.

In the usual application of the method, the field is expanded as a mean-field plus fluctuations as

$$\phi(x,t) = \phi_0(x - x_0(t)) + \delta\phi(x - x_0(t), t). \tag{5.3}$$

The substitution of this expansion in the Lagrangian 5.1, neglecting any changes on the equations of motion, results in an effective kinetic term for the coordinate $x_0$. We have shown that, for the Schrödinger field, the effective behavior of the dynamics of the coordinate $x_0$ is highly dependent on the changes in the equations of motion. In other words, in order to obtain the correct effective behavior, our expansion must include deformations of the static solution. Therefore, the correct expansion in the Extended Collective Coordinate Method has the form

$$\Psi(x,t) = \psi_0(x - x_0(t)) - i\varphi(x)\dot{x}_0 + \delta\Psi(x - x_0(t), t), \tag{5.4}$$

where the first term has the form of the static solution, and the second term describes the deformation of $\psi_0$ as it moves in the space. The last term describes the excited quasiparticles of our bosonic field. The substitution of the expansion 5.4 in the Lagrangian 5.2 gives the expected equivalent system of a particle in a bath of harmonic oscillators. Then, we have shown that this effective behavior can be described in the same way as it was done in the case of scalar fields.

With a general theory developed for Schrödinger fields, we were able to apply our method for the case of a bright soliton in one-dimension. In Chapter 4, we have recovered some results that were found in the literature by other methods and have obtained the precise form of the damping and diffusion coefficients. Both of those functions are temperature-dependent, showing that the soliton mobility highly depends on the temperature of the excited quasiparticles. In the low-temperature limit, we have shown that our mobility increases drastically, as the damping and diffusion coefficients vanish. The time dependence of the damping and diffusion functions is given by a second time derivative of the Dirac delta function, showing that the dissipation is Markovian. This reminds us of the quantum Brownian motion, although the damping force now depends on the first time derivative of the acceleration of the particle. This results agrees with the one found in the literature by a similar formalism. The term with the time derivative of the acceleration resembles the Abraham-Lorentz force responsible for the phenomenon of radiation damping in electromagnetism. However, in our case, the method we have developed does not take into account any contribution for emission of excitations (quasi-particles) by the soliton. Therefore, our Abraham-Lorentz type of force must be interpreted as a dissipative effect due to inelastic scattering by the quasiparticles of the system. In other words, the coupling between the soliton and quasiparticles describes a scattering process in which the latter can absorb energy from the soliton, leading to the eventual dissipation of the topological solution.

We should remark that a viscous force proportional to the velocity of the soliton would result from the backscattering of quasiparticles [9,21,22,32], a process ($j = -k$) which is absent from the coupling constant 4.12.

In this work, we were able to study the quantum dissipation of a soliton-like solution of a Schrödinger field by a semi-classical quantization procedure. We have proposed an appropriate way to apply the quantization method for the dynamical solitonic solutions and shown that the effective behavior can be studied within a system-plus-reservoir approach.

Although we have applied our method only for a one-dimensional system, the procedure can be easily extended for other systems. For instance, one could choose a Bose gas with a external potential in a "pancake" shape, this is a quasi two-dimensional condensate. As we know, a vortex solution 2.23 can be obtained from the Gross-Pitaevskii equation. Studies have shown that the center of the vortex obeys an equation of motion with an Ohmic friction term [37], different from what we have found for the solitonic case. Thus, a Schrödinger field can present different types of dissipation mechanisms that are worthy investigating. Our method could also improve the understanding of the dynamics of other topological solutions such as systems with vortex-antivortex pair in a "pancake" shaped potential and vortex and current dissipation in a toroidal condensate. Therefore, future work on this subject will involve the application of the method we have developed in different topological solutions of the Schrödinger field to study dissipation properties by first-principles theory. It would also be desirable to apply our Extended Collective Coordinate Method to other types of Lagrangians that are linear in the field time derivative.

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# Appendices

# Appendix A

# Feynman-Vernon Formalism

In this appendix our aim is to highlight the main steps to understand how the reduced density matrix can be written in the Feynman-Vernon Formalism of the influence functional.

As we did in Chapter 1, we begin with a Hamiltonian of a system-plus-reservoir in the form

$$H = H_S + H_I + H_R, \tag{A.1}$$

in which $H_S$, $H_I$ and $H_R$ refers to the Hamiltonian of the system, interaction and reservoir respectively. To find average values of any observable $\hat{O}$ at time $t$ (Heisenberg picture), the choice of a pure state is not the most appropriate one in the system-plus-reservoir approach. Since the system $R$ act as a thermal reservoir for the system $S$, and, as we know, a system at finite temperature is described by a mixed state, we should represent it by the density operator

$$\hat{\rho}(0) = \sum_{\psi} p_{\psi} \left|\psi(0)\right\rangle \left\langle\psi(0)\right|, \tag{A.2}$$

where $p_{\psi}$ is the probability associated with the the pure state $|\psi(0)\rangle$. Since the time evolution of the physical state $|\psi(0)\rangle$ is given by

$$\left|\psi(t)\right\rangle = e^{-iHt/\hbar} \left|\psi(0)\right\rangle, \tag{A.3}$$

we can write the time evolution of the density operator as

$$\hat{\rho}(t) = \sum_{\psi} p_{\psi} \left|\psi(t)\right\rangle \left\langle\psi(t)\right| = e^{-iHt/\hbar}\hat{\rho}(0)e^{iHt/\hbar}. \tag{A.4}$$

In this representation, the average of any observable is now given by

$$\left\langle \hat{O}(t) \right\rangle = Tr_{RS}\left\{\hat{\rho}(t)\hat{O}(0)\right\}, \tag{A.5}$$

where $RS$ refers to the basis of the composite system $R \otimes S$. One important property of the trace is that given a linear operator $\hat{O}$ that lives in the space $A \otimes B$, the trace of this operator is equal to the partial traces over each subspace

$$Tr_{AB}\left\{\hat{O}\right\} = Tr_{A}\left\{Tr_{B}\hat{O}\right\} = Tr_{B}\left\{Tr_{A}\hat{O}\right\}. \tag{A.6}$$

Since we are usually interested in the behavior of the system $S$, we can use this property in A.5 to write

$$\left\langle \hat{O} \right\rangle = Tr_{RS}\left\{\hat{\rho}(t)\hat{O}\right\} = Tr_{S}\left\{\left[Tr_{R}\hat{\rho}(t)\right]\hat{O}\right\} = Tr_{S}\left\{\widetilde{\rho}(t)\hat{O}\right\}, \tag{A.7}$$

where we have defined

$$\widetilde{\rho}(t) = Tr_{R}\hat{\rho}(t), \tag{A.8}$$

as the reduced density operator of the system. The reduced density operator is the trace of the full density operator over the coordinates of the reservoir. Therefore, once we have the knowledge of this operator, we know how the reservoir acts on the subsystem $S$ of interest.

In order to write explicitly the density operator, we will define $x(t)$ as the coordinate of the subsystem $S$ and $\mathbf{R}(t)$ as the vector that represents the coordinates of the reservoir $R$. In the coordinate representation, the total density operator is given by

$$\hat{\rho}(x,\mathbf{R},y,\mathbf{Q},t)=\int\int\int\int dx'dy'd\mathbf{R}'d\mathbf{Q}'\;K(x,\mathbf{R},t;x',\mathbf{R}',0)\;K^{*}(y,\mathbf{Q},t;y',\mathbf{Q}',0)\\ \times\hat{\rho}(x',\mathbf{R}',y',\mathbf{Q}',0), \tag{A.9}$$

where $K(x,\mathbf{R},t;x',\mathbf{R}',0)$ and $K^{*}(y,\mathbf{Q},t;y',\mathbf{Q}',0)$ are the Feynman propagators defined as

$$K(x,\mathbf{R},t;x',\mathbf{R}',0)=\langle x\mathbf{R}|\,exp\left(-\frac{i}{\hbar}\hat{H}t\right)|x'\mathbf{R}'\rangle\,,\\ K^{*}(y,\mathbf{Q},t;y',\mathbf{Q}',0)=\langle y\mathbf{Q}|\,exp\left(\frac{i}{\hbar}\hat{H}t\right)|y'\mathbf{Q}'\rangle\,. \tag{A.10}$$

Therefore, to write the reduced density matrix, we must trace out the coordinates of the bath, that is, to evaluate

$$\widetilde{\rho}(x,y,t)=\int d\mathbf{R}\,\langle x\mathbf{R}|\,\hat{\rho}(t)\,|y\mathbf{R}\rangle\,, \tag{A.11}$$

in which $\hat{\rho}(t)$ is defined in A.9. We can make a simplification of A.11 and assume that our system is initially separable. This means that the composite system RS, at $t=0$, can be written as

$$\hat{\rho}(x',\mathbf{R}',y',\mathbf{Q}',0)=\hat{\rho}_{S}(x',y',0)\hat{\rho}_{R}(\mathbf{R}',\mathbf{Q}',0). \tag{A.12}$$

Substituting this specific choice in A.9, we can write

$$\widetilde{\rho}(x,y,t)=\int\int dx'dy'\;\mathcal{J}(x,y,t;x',y',0)\widetilde{\rho}(x',y',0), \tag{A.13}$$

in which $\mathcal{J}$ is the well-known superpropagator of the system defined as

$$\mathcal{J}(x,y,t;x',y',0)=\int\int\int d\mathbf{R}'d\mathbf{Q}'d\mathbf{R}K(x,\mathbf{R},t;x',\mathbf{R}',0)\;K^{*}(y,\mathbf{R},t;y',\mathbf{Q}',0)\\ \times\hat{\rho}_{R}(\mathbf{R}',\mathbf{Q}',0). \tag{A.14}$$

This superpropagator $\mathcal{J}$ is responsible for the time evolution of the reduced density operator,

$\rho_S(x', y', 0)$, describing the system $S$ at time $t' = 0$ to a final state $\rho_S(x, y, t)$ at the time $t' = t$.

The time evolution of the system of interest $S$ is related to the behavior of the bath and the interaction between them. To evaluate the total propagator $K(x, \mathbf{R}, t; x', \mathbf{R}', 0)$ can be quite laborious, since it involves an enormous number of variables. In order to ease this procedure, we can make use of the Feynman path integral formulation. In the functional integral representation, it can be shown [38] that the total propagator can be written as

$$K(x, \mathbf{R}, t; x', \mathbf{R}', 0) = \int_{x'}^{x} \int_{\mathbf{R}'}^{\mathbf{R}} \mathcal{D}x(t')\mathcal{D}\mathbf{R}(t') exp \left\{ \frac{i}{\hbar} S \left[ x(t'), \mathbf{R}(t') \right] \right\}, \tag{A.15}$$

where $S[x, \mathbf{R}]$ is the action of the composite system $RS$. Substituing A.15 in the superpropagator A.14 we can write

$$\mathcal{J}(x, y, t; x', y', 0) = \int_{x'}^{x} \int_{y'}^{y} \mathcal{D}x\mathcal{D}y \; exp \left\{ \frac{i}{\hbar}(S_S[x] - S_S[y]) \right\} \; \mathcal{F}[x, y], \tag{A.16}$$

in which $S_S$ is the action corresponding to the Hamiltonian $H_S$ and $\mathcal{F}$ is the so-called influence functional. The influence functional in the coordinate representation is defined as

$$\mathcal{F}[x, y] = \int d\mathbf{R}' d\mathbf{Q}' d\mathbf{R} \rho_R(\mathbf{R}', \mathbf{Q}', 0) \int_{\mathbf{R}'}^{\mathbf{R}} \mathcal{D}\mathbf{R} \int_{\mathbf{Q}'}^{\mathbf{R}} \mathcal{D}\mathbf{Q} \; exp \left\{ \frac{i}{\hbar}(S_{RI}[x, \mathbf{R}] - S_{RI}[y, \mathbf{Q}]) \right\}, \tag{A.17}$$

where $S_{RI}$ is the action of the reservoir $R$ and interaction $I$.

With the superpropagator written in the Feynman-Vernon formalism, we can finally write the reduced density matrix as

$$\begin{aligned} \widetilde{\rho}(x, y, t) &= \int\int dx' dy' \; \widetilde{\rho}(x', y', 0) \; \mathcal{J}(x, y, t; x', y', 0) \\ &= \int\int dx' dy' \; \widetilde{\rho}(x', y', 0) \int_{x'}^{x} \int_{y'}^{y} \mathcal{D}x\mathcal{D}y \; e^{\frac{i}{\hbar}(S_S[x] - S_S[y])} \; \mathcal{F}[x, y]. \end{aligned} \tag{A.18}$$

# Appendix B

# Influence Functional in the Coherent State Representation

For a system with a Hamiltonian written in terms of creation and annihilation operators for the quasiparticles interacting with a particle as equation 1.42, it is convenient to write the influence functional in the coherent state representation. For the particular case of bosons, these operators obey the following commutation relations

$$[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0 \qquad \text{and} \qquad \left[a_k, a_{k'}^\dagger\right] = \delta_{kk'}. \tag{B.1}$$

A coherent state is defined as an eigenstate of the annihilation operator

$$a \left|\alpha\right\rangle = \alpha \left|\alpha\right\rangle . \tag{B.2}$$

In order to write the influence functional in the coherent state representation for the reservoir, we will rewrite the propagator of expression A.14 as

$$K(\alpha^*, t; \alpha', 0) = \left\langle\alpha\right| \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt' \hat{H}(t')\right) \left|\alpha'\right\rangle , \tag{B.3}$$

where $\mathcal{T}$ is the time ordering operator. We can split our time interval in equally spaced parts defined by $\epsilon = t/(N-1)$. Using this definition we can write

$$K(\alpha^*, t; \alpha', 0) = \langle\alpha| \left[\exp\left(-\frac{i}{\hbar}\hat{H}\epsilon\right)\right] \left[\exp\left(-\frac{i}{\hbar}\hat{H}\epsilon\right)\right] ... \left[\exp\left(-\frac{i}{\hbar}\hat{H}\epsilon\right)\right] |\alpha'\rangle\,. \tag{B.4}$$

Using the completeness relation

$$\int |\alpha\rangle\,\langle\alpha|\,\frac{d\alpha}{\pi} = 1, \tag{B.5}$$

we can rewrite the propagator as

$$K(\alpha^*, t; \alpha', 0) = \left(\prod_{j=1}^{N-1}\int\frac{d^2\alpha_j}{\pi}\right)\prod_{j=0}^{N-1}\langle\alpha_{j+1}|\exp\left(-\frac{i}{\hbar}\hat{H}\epsilon\right)|\alpha_j\rangle\,. \tag{B.6}$$

Now, we can expand the exponentials up to first order in the parameter $\epsilon$ as

$$K(\alpha^*, t; \alpha', 0) = \left(\prod_{j=1}^{N-1}\int\frac{d^2\alpha_j}{\pi}\right)\prod_{j=0}^{N-1}\langle\alpha_{j+1}|\alpha_j\rangle\left[1 - \frac{i\epsilon}{\hbar}\frac{\langle\alpha_{j+1}|\,H(t_j)\,|\alpha_j\rangle}{\langle\alpha_{j+1}|\alpha_j\rangle}\right]. \tag{B.7}$$

The term of B.7 in parenthesis can be re-exponentiated resulting in

$$K(\alpha^*, t; \alpha', 0) = \left(\prod_{j=1}^{N-1}\int\frac{d^2\alpha_j}{\pi}\right)\prod_{j=0}^{N-1}\langle\alpha_{j+1}|\alpha_j\rangle\exp\left(\sum_{j=0}^{N-1}\frac{i}{\hbar}H_{j+1,j}\right), \tag{B.8}$$

where we have defined the matrix element $H_{j+1,j}$ as

$$H_{j+1,j} = \frac{\langle\alpha_{j+1}|\,H(t_j)\,|\alpha_j\rangle}{\langle\alpha_{j+1}|\alpha_j\rangle}. \tag{B.9}$$

It can be shown that the scalar product of two coherent states $\langle\alpha_{j+1}|\alpha_j\rangle$ reads

$$\langle\alpha_{j+1}|\alpha_j\rangle = \exp\left\{-\frac{1}{2}\left[\alpha_j^*\left(\alpha_j - \alpha_{j-1}\right) - \left(\alpha_j^* - \alpha_{j-1}^*\right)\alpha_{j-1}\right]\right\}. \tag{B.10}$$

Substituing B.10 in B.8, the propagator becomes

$$K(\alpha^*,t;\alpha',0)=\left(\prod_{j=1}^{N-1}\int\frac{d^2\alpha_j}{\pi}\right)\exp\frac{i}{\hbar}\sum_{j=0}^{N-1}\epsilon\left\{\frac{i\hbar}{2}\alpha^*_{j+1}\frac{(\alpha_j-\alpha_{j-1})}{\epsilon}\right.$$
$$\left.-\frac{i\hbar}{2}\frac{\left(\alpha^*_j-\alpha^*_{j-1}\right)}{\epsilon}\alpha_{j-1}-H_{j+1,j}\right\}. \tag{B.11}$$

In the limit that $\epsilon\to 0$, we can further write B.11 as

$$K(\alpha^*,t;\alpha',0)=\int\mathcal{D}\mu(\alpha(t'))\exp\left\{S\left[\alpha(t'),\alpha^*(t')\right]\right\}, \tag{B.12}$$

where we have defined

$$D\mu(\alpha)=\lim_{N\to\infty}\left\{\prod_{j=1}^{N-1}\int\frac{d\alpha_j d\alpha^*_j}{2\pi i}\right\}, \tag{B.13}$$

and

$$S\left[\alpha(t'),\alpha^*(t')\right]=\int_0^t dt'\left\{\frac{i\hbar}{2}\left[\alpha^*(t')\dot{\alpha}(t')-\alpha(t')\dot{\alpha}^*(t')\right]-H(\alpha^*,\alpha)\right\}. \tag{B.14}$$

Substituting the propagator B.12 in the superpropagator A.14, the influence functional in the coherent state representation can be finally written as

$$\mathcal{F}[x,y]=\int d\mu(\alpha)d\mu(\alpha')d\mu(\gamma')\rho_R(\alpha'^*,\gamma')\int_{\alpha'}^{\alpha^*}\mathcal{D}\mu(\alpha)\int_{\gamma'^*}^{\alpha}\mathcal{D}\mu(\gamma)\ e^{\frac{i}{\hbar}(S_{RI}[x,\alpha]-S^*_{RI}[y,\gamma^*])}, \tag{B.15}$$

in which $d\mu(\alpha)$ is defined as

$$d\mu(\alpha)=\prod_{j=1}\frac{d\alpha_j d\alpha^*_j}{2\pi i}. \tag{B.16}$$

# Appendix C

# Scattering Function

For the evaluation of the scattering function

$$S(\omega,\omega') = \sum_{jk} |g_{jk}|^2 \delta(\omega-\omega_j)\delta(\omega'-\omega_k), \tag{C.1}$$

we will first suppose the wave numbers $k$ and $j$ can be written as

$$k = \frac{n\pi}{L}, \tag{C.2}$$

where $n$ is an integer. Taking the continuum limit of C.1, the scattering function can be written as

$$S(\omega,\omega') = \left(\frac{L}{\pi}\right)^2 \int dkdj\ |g_{jk}|^2 \delta(\omega-\omega_j)\delta(\omega'-\omega_k). \tag{C.3}$$

In order to take advantage of the integrals of Dirac delta functions, let us remind the relationship between the wave number $k$ and the eigenfrequencies $\omega_k$

$$\omega_k = \frac{\hbar k^2}{2m}. \tag{C.4}$$

Now, we can make a change of variables and rewrite our integral as

$$S(\omega,\omega') = \left(\frac{L}{\pi}\right)^2 \frac{m}{2\hbar} \int d\omega_k d\omega_j\ \frac{|g_{jk}|^2}{\sqrt{\omega_k\omega_j}}\ \delta(\omega-\omega_j)\delta(\omega'-\omega_k). \tag{C.5}$$

Performing the integrals in $d\omega_k$ and $d\omega_j$, our scattering function $S(\omega, \omega')$ obtain its final form

$$S(\omega, \omega') = \left(\frac{L}{\pi}\right)^2 \frac{m}{2\hbar} \frac{|g(\omega, \omega')|^2}{\sqrt{\omega\omega'}}. \tag{C.6}$$

Here we have used the definition C.4 to rewrite the coupling function defined in 4.12 as

$$g(\omega, \omega') = \frac{\pi}{3L} \frac{(\omega - \omega')\left(\omega + \omega' + \sqrt{\omega\omega'} + |\mu|/\hbar\right)}{\sinh\left[\frac{\pi}{2}\xi\sqrt{\frac{2m}{\hbar}}\left(\sqrt{\omega'} - \sqrt{\omega}\right)\right](\omega + |\mu|/\hbar)(\omega' + |\mu|/\hbar)}. \tag{C.7}$$

We can now perform the substitution of the variables $\omega, \omega'$ for $\varphi, \eta$ defined in 4.15, and take the limit when $\eta \to 0$

$$\lim_{\eta \longrightarrow 0} S(\varphi, \eta) = \frac{4\hbar^2}{9\xi^2\pi^2} f(\varphi). \tag{C.8}$$

where the function $f(\varphi)$ is defined as

$$f(\varphi) = \frac{(2\hbar\varphi + |\mu|)^2}{(\hbar\varphi + |\mu|)^4}. \tag{C.9}$$