Universidade Estadual de Campinas
Instituto de Filosofia e Ciências Humanas

João Vitor Schmidt

On Frege's definition of the Ancestral Relation: logical and philosophical considerations

Sobre a definição fregeana da Relação Ancestral: considerações lógicas e filosóficas

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Orientador: Marco Antonio Caron Ruffino

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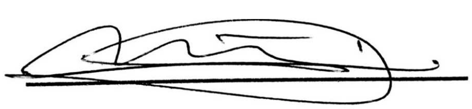

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A Ata de Defesa, assinada pelos membros da Comissão Examinadora, consta no processo de vida acadêmica do aluno.

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Resumo

Neste trabalho, examinamos a famosa definição fregeana da Relação Ancestral em seus aspectos lógicos e filosóficos. O logicismo de Frege é o tema central: demonstrar as bases lógicas da aritmética. Isto é feito ao mostrar-se a natureza lógica dos números e a natureza lógica do raciocínio matemático. Ao fazê-lo, faz-se importante mostrar como os números naturais constituem uma série ordenada. Esta é a tarefa da definição do Ancestral. Com isso em mente, focaremos nos seguintes tópicos. Na introdução, o problema sobre séries será discutido, seguido de uma introdução do oponente mais famoso de Frege sobre isso: Kant. Isto será feito no Capítulo 1. Entender a filosofia de Kant é o passo inicial para as motivações filosóficas de Frege. Elas serão então discutidas no Capítulo 2, juntamente às suas reinterpretações da terminologia kantiana. Terminaremos este capítulo discutindo a lógica conceitográfica de Frege. No capítulo 3, introduzimos um sistema simples para lógica de segunda ordem, necessário para avaliar a definição de Frege e fiel o bastante às suas motivações filosóficas. Então, o Ancestral será finalmente e detalhadamente introduzido e discutido. Além disso, provamos os teoremas necessários para o logicismo de Frege, similarmente ao modo como o próprio antecipou na Begriffsschrift e Die Grundlagen der Arithmetik. Isso culminará no que hoje é conhecido como o Teorema de Frege. Finalmente, no capítulo 4, discutimos um dos problemas da definição de Frege: sua suposta circularidade. Mais precisamente, argumentamos contra essa conclusão, acrescentando que o Ancestral, embora impredicativo, não é prejudicial como esperado, dadas as motivações filosóficas e lógicas de Frege perante ela.

Palavras-chave: Frege, Definição Ancestral, Logicismo, Kant, Teorema de Frege.

Abstract

In this work, we examine Frege's famous definition of the Ancestral Relation both in its philosophical and logical aspects. Frege's logicism is the main theme on both: to show the logical grounds of arithmetic. This is done by showing the logical nature of numbers and the logical nature of mathematical reasoning. In doing so, it's important to show how the natural numbers constitute an ordered series. This is the task of the Ancestral. With that in mind, we focus on the following topics. In the Introduction, the problem about series is discussed, followed by an assessment of Frege's most famous opponent regarding it: Kant. This is done in chapter 1. Understanding Kant's philosophy is the starting point for Frege's own philosophical motivations. In Chapter 2, we avaluate them, alongside Frege's own interpretations of the kantian notions. We finish the chapter by introducing Frege's concept-script logic. In Chapter 3 we introduce a simple system for second-order logic, necessary to evaluate Frege's definition and faithful enough to his philosophical motivations. Then, the Ancestral is finally and thoroughly introduced and discussed. Most importantly, we prove the theorems necessary for Frege's logicism, similarly as himself envisaged in the Begriffsschrift and Die Grundlagen der Arithmetik. This culminates in what is nowadays known as Frege's Theorem. Finally, in chapter 4, we discuss one of the problems of Frege's definition: its alleged circularity. More precisely, we argue against this conclusion, adding that the Ancestral, although impredicative, is not as harmful as supposed, given Frege's philosophical and logical motivations for it.

Keywords: Frege, Ancestral definition, Logicism, Kant, Frege's Theorem.

List of Abbreviations

1. Frege's Works

   BS - Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens, translated by William T. Bynum as *Conceptual notation: A formula language of pure thought modelled upon the formula language of arithmetic*, (FREGE, 1992)

   GLA - Die Grundlagen der Arithmetik, translated by John Austin as *The Foundations of Arithmetic*, (FREGE, 1953)

   GGA - Grundgesetze der Arithmetik, translated by Philip Ebert and Marcus Rossberg as *The Basic Laws of Arithmetic*, (FREGE, 2013)

   PMC - Wissenschaftlicher Britfwechsel, translated by Felix Klein as *Philosophical and Mathematical Correspondence*, (FREGE, 1980)

   PW - Nachgelassene Schriften und Wissenschaftlicher Briefwechsel, translated by Peter Long and Roger White as *Posthumous Writings*, (FREGE, 1979)

   CP - Kleine Schriften, translated by Max Black, V. H. Dudman *et. al.*, edited by Brian McGuinness as *Collected Papers on Mathematics, Logic, and Philosophy*, (FREGE, 1984)

2. Kant's Works

   CPR - Kritik der Reinen Vernunft, translated and edited by Paul Guyer and Allen Wood as *Critique of Pure Reason*, (KANT, 1998)

   JL - Immanuel Kant's Logik, ein Handbuch zu Vorlesungen, translated by J. Michael Young as *Immanuel Kant's Logic - A manual for lectures* in (KANT, 1992).

Contents

Introduction

This work have one major theme: Frege's famous definition of the Ancestral of a Relation. But the approach here taken is twofold: to discuss such definition in the philosophical perspective, and to show its accomplishments logicwise. Both perspectives are mutually related. The quest for the foundations of arithmetic is the starting point for both, and the problem of series is one of the main fields of discussion. It's inevitable in discussing Frege's philosophy to introduce some kantian terminology, and this is certainly the case regarding the Ancestral definition. This is included in the philosophical discussion. As a renowned logician, the main place for Frege's philosophy to be held is in logic, and so, we take the discussion in logical terms as well. We start, then, with series.

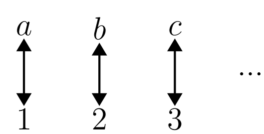
The notion of a Series

The capacity for establishing order in a finite set of objects can be seen as a basic human capability. An order may be seen as a series or sequence $< x_1, ..., x_n >$ of n objects such that a path is given from one element to another through a finite number of steps. Such a path is said to be reversible, that is, for every reached step, one could go back to the starting point with the same number of steps. Simply put, a series is a collection of objects in which one come after the other.

Such an elementary notion has many intuitive or empirical applications: the arrangement of books in a shelf, the final order of pilots in a race, or even the order of births in a group of siblings. In all these cases, a specific criteria for ordering is given: the author's name, the arrival time and date of birth, respectivelly. In fact, we can formulate an order even without a specific ordering rule: given a finite set of elements, we can choose a first element, a second, third and so on randomly.

The variety of ways in which a series can emerge is also noteworthy. It can be linear, circular, contain nodes, branches, or even be endless. It can also have one especific member in two different positions. They are, for that matter, something different than sets, or simple groupings of things, since the order of apperance of its elements are also an important factor. To give more examples: the hours in a day, the victory conditions for the rock-paper-scissors game, a family-tree, the digits in the decimal expansion of the number π, the series of winners of the world cup of football, all show how general such notion is.

As we stated above, an order can be arbitrarily given following any criteria or even with a random selection. But, in such cases, a connection with the series of natural numbers, taken as an ordered set, is immediately suggested. From a given set of elements, we can establish an order through a bijection between such set and a subset of natural numbers: for each element of the set, we associate a unique natural number n:



Another way to understand this is by given an index to each element of a set, therefore giving it an order. Most ordered series are, in fact, ordered due to such indexing, or bijection on the natural numbers. Take the examples above: the alphabetic order of the set of letters, which can be stated using ordinals, used to determine the series of books in a shelf. The duration of time intervals determines the sequence of hours in a day, or the final order of pilots in a race. The series of consecutive years determines the age difference between persons. Even if we pick randomly, it is this association with the natural number series that allows the ordering of each element picked. This is akin to the process of counting: if we want to know the size or the quantity of a finite set of objects, we can associate each element with a natural number. If all objects are associated, the size or cardinality of such set is the number associated with the last object that we pick in such process. But for that to be successful, it is required the set of natural numbers to be ordered prior, or, that to any successive selection of numbers, the greater number to be the last selected.

All these seems to be dependent on the series of natural numbers. For Russell (1901), these are considered *series by correlation*, that we can generate by correlating each element to an element of another ordered series. They are dependent on what he calls intrinsic or *independent series*, like the natural numbers. The question, then, is how the series of natural numbers acquire such order in the first place. One might answer this by the way numbers are constructed, since their size, or magnitude, are ordered through the relation of 'being greater than'. But this still demands an account on what such relation is.

These questions are something taken very seriously by philosophers of mathematics throughout the history. Two classical problems in the philosophy of mathematics can be stated through them. On the intrinsic order, or the independent order of the natural number series, lies a *foundational* problem: that of giving foundations for the number series; on correlational series lies an *applicability* problem: that of giving an account on the fact that the number series is applicable to any set of objects in the world, empirical or not. For that reason, starting in the nineteenth century, works on formal logic start treating series and the notion of order from the perspective of logical relations. The examples above also shows how relations can order objects.

Formal Treatment

There are many ways in which we can generate a series or an order from a given set of elements. But from a formal point of view, this is done basically with relations. A relation R can be seen a binary property or concept connecting two elements in the form $R(x, y)$. It can even be extended for any $n \geq 2$ in the general form $R^n(x_1, ..., x_n)$, where R is said to be an n-ary relation, even though only binary relations are used in orderings. In modern set-theoretic approaches, a relation R is a set of ordered pairs $< x, y >$, *e.g.*, $\{< x, y >, < y, z >\}$ is a relation R in which $R(x, y)$ and $R(y, z)$ holds[1]. Either way, an usual reading for $R(x, y)$ is that "x is related with y through R".

For a relation R in a domain D, and for every x, y, z in D, a common set of properties of relations is the following:

- If $R(x, x)$, then R is said to be reflexive;

[1]Take Enderton (1977) as a good example.

- If $R(x, y)$ holds and can be inferred that $R(y, x)$ also holds, then R is said to be symmetric;
- If $R(x, y)$ and $R(y, z)$ holds, and it can be inferred that $R(x, z)$ also holds, then R is said to be transitive.

Any R that fails such conditions are said to be not reflexive, not symmetric and not transitive, respectively. Moreover, if this holds for every x, y, z in D, then we have the following set of opposite properties:

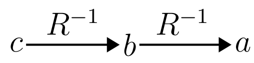
- If, for every x in D, $R(x, x)$ does not hold, then R is said to be irreflexive;
- If, for every x and y in D, $R(x, y)$ holds and $R(y, x)$ doesn't, then R is said to be assymmetric;
- If for any x and y in D, both $R(x, y)$ and $R(y, x)$ holds, and it can be inferred that $x = y$, then R is said to be antisymmetric.
- If, for every x, y and z in D, $R(x, y)$ and $R(y, z)$ holds, but $R(x, z)$ does not, then R is said to be antitransitive.

The most basic condition for any order is to the relation in question be transitive. This provides a basic path that connects the objects correlated by the relation. Russell (1901, 1996), for example, states that any ordering requires a relation that is both transitive and asymmetrical. This is a stronger condition. For a given set $A = \{a, b, c\}$ of three elements, a simple example would yield something like:

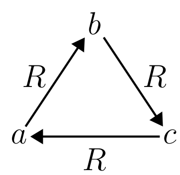
$$a \xrightarrow{R} b \xrightarrow{R} c$$

In which, clearly, there is a path from one given element of the set to another[2]. The reverse path would result a different order, and we can provide one assuming the inverse of R, R^{-1}:

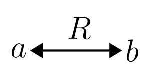
[2] More precisely, there is a path from a to c through b. But since R is taken to be transitive, there should be a line connecting a to c directly, as is normally depicted. Nevertheless, the important point is the existence of a path connecting both elements (a and c), which the above drawing clearly shows. An antitransitive relation R, such that $R(a, b)$ and $R(b, c)$, would require a distinct representation, and for that reason, we opt to represent transitive relations visually even without a direct line between every element, as long as the connection is given.



Such inverse relation is what Russell takes to be different senses of a relation, something that we could restate as different directions for R. The requirement is that R and R^{-1} must not be simultaniously true for the same pair of arguments, *viz.* $R(a,b)$ and $R^{-1}(a,b)$ cannot be both true. Series could also be considered open or closed, depending on the existence of a path from the last to the first element. But in such cases, Russell's restriction for order fails, if the transitivity is assumed: take $R(a,b)$, $R(b,c)$ and $R(c,a)$ (*i.e.* the series is closed) and assume that R is transitive. Then R is also symmetric, against the restriction. This exclude cases such as:



and



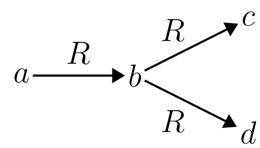
Provided that R is transitive in the first case. From $R(a,b)$ and $R(b,c)$, then $R(a,c)$ (from transitivity), but since $R(c,a)$, R is then not assymmetric. The same holds for the second case, since $R(a,b)$ and $R(b,a)$ holds[3].

The usual relation described by such restrictions is the $<$ relation on natural numbers. Such relation is transitive, but not symmetric. It cannot be the case that $n < m$ and $m < n$. It cannot be reflexive either, since this would contradicts the asymmetry, given that $R(x,x) \rightarrow R(x,x)$ trivially. Also, if a relation is both transitive and irreflexive, it cannot be symmetric, otherwise it would be reflexive. Therefore, the relation $<$ and Russell's definition of order describes the following:

Definition (Partial Ordering)**.** *A relation R is a partial ordering on a domain D if R is both transitive and irreflexive on D.*

[3]For that reason, Russell assumes only with antitransitive relations that such orders are possible in closed series.

Partial orders can also be used to generate trees or series with branches, not only simple linear series, allowing a greater variety of cases. Particulary, consider the following:



From that, we have that $R(a, b)$, $R(b, c)$ and $R(b, d)$. R is a partial order: it's irreflexive, meaning that there is no path from each element back to itself, and it's transitive, meaning that there is a path that goes from a to c and to d. But c and d are not connected through R. Such connectedness can be provided if we add the following condition:

- If exactly one of the following holds: $R(x, y)$, $R(y, x)$ or $x = y$, then R is said to satisfy trichotomy.

When R satisfies trichotomy, then all objects correlated by R are in fact connected. This is the expected case for the natural numbers. If a relation is a partial order and satisfies this, then such relation is called a linear or total ordering:

Definition (Linear Ordering). *A relation R is a linear ordering on a domain D if R is a partial order and satisfies trichotomy on D.*

Linear orderings are more restricted. They are the ones that matter for natural numbers, since they form a line with only one direction, without branches. If one is concerned with the order of them from a foundational point of view, the series must be linear. With numbers in mind, the relations $<$ and $\leq$ are usually what is used to describe orderings, but with some minor differences. For what we have exposed, $<$ is a partial order but also linear, or total, order. It is also described as a strict total order. By contrast, the relation $\leq$ is a weak total order. The major difference is that the irreflexive condition must be dropped, since $x \leq x$ must hold. To overcome this, $\leq$ describes a partial order if is (1) transitive, (2) antisymmetric and (3) reflexive. If trichotomy is added, then $\leq$ is a weak total order as well.

Summary of the chapters

From all the variety of series, linear or total orders are what properly describe the series of natural numbers. For that reason, the search for the foundations of mathematics, mainly the arithmetic, usually involves inquiries about series in such terms. Our focus here is to evaluate a classic dispute in philosophy of mathematics that takes place in this context: between Frege and Kant.

Frege held the famous logicist thesis, that arithmetic has logical foundations, being an analytic and ampliative science. His main declared opponent was Kant, who defended that arithmetic, as an ampliative science, cannot be purely logical, having to borrow its foundations from human cognitive capabilities, above all, intuitions. As it is well known, they differ on the ontological and epistemological status of numbers, but this also includes a difference between the way ordered series are conceived. So our task is to address this main difference.

The main theme is Frege's philosophy and logic. The focus on such dispute is not just a matter of choice. Frege describes his own goals in kantian terms, and from the beginning, in 1879, logicism already emerged as an anti-kantian approach to logic and mathematics, where the important Ancestral Definition is introduced. So, understanding Kant's position is crucial to better understand Frege's. Moreover, there is a great number of important topics to be considered: the analytic-synthetic distinction, the views on logic, the role of intuitions, the ampliative/informative character of mathematics and logic, all figure as important themes. Frege's importance for modern logic is undisputable, but his philosophical motivations were also notable. Then, both aspects of his work should be considered.

Keeping that in mind, in the first chapter, we'll assess Kant's critical philosophy in order to understand his positions about mathematics, numbers and ordered series. It's Kant's critical philosophy, mainly the one exposed in the *Critique of Pure Reason*[4] and subsequent logical texts, mostly in the *Jäsche Logic*[5], that will be analysed. Kant's famous conclusion that mathematical knowledge is synthetic *a priori* was the focal point of criticisms in the nineteenth and twentieth century philosophy of mathematics[6]. Starting with a theory of logic that regards it as fruitless, with no subject matter of his own serving only as a set of rules for reasoning, Kant saw in intuitions, in its pure form, the best explanation for mathematics success in science. In doing so, he saw a connection between the way human thought is organized in time with the way arithmetic is constructed: both being linearly ordered, in

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[4]Henceforth *CPR*.
[5]Henceforth *JL*.
[6]See, *e.g.* (COFFA, 1982) and the first chapters in (POTTER, 2000).

accordance to what is necessary to describe the series of natural numbers properly. For that, we start with Kant's terminology and important notions (1.1). Next, the role of intuitions, and especially, time, is discussed (1.2). This is then contrasted with Kant's two notions on logic (1.3). Finally, the importance of time in human thinking (1.4) and for the arithmetic (1.5) is addressed.

But the cost for such solution was considered high, and Frege was one of the dissatisfied customers. Already in his *Begriffsschrift*[7], this rejection can be found. In his *Die Grundlagen der Arithmetik*[8], his most philosophical book, Kant's thesis is constantly quoted as something to be overcome[9]. His motto was to show that arithmetical knowledge is analytic and *a priori*, while keeping its ampliative character. This is a philosophical thesis, one with mathematical purposes. The demands to accomplish this are what we now regard as Frege's major achievement: his logical calculus. But from a more modern standpoint, Frege's early logic must be seen with caution. All these topics are discussed in chapter 2. Precisely, Frege's philosophical motivations will be discussed (2.1), including his denial of kantian intuitions (2.1.1), his notion of analyticity (2.1.2), definitions (2.1.3) and his thesis that logic can be informative (2.1.4). Finally, the logic of *BS* is described (2.2), since it's in logic that Frege's thesis are to be proved.

It's in *GLA* that most of this philosophical discussion appears. But to prove his claims much had to be done, and the first step was already taken in the *BS*: in section III of the book, Frege derives important theorems that follows from a few definitions, all regarding series. It's here that we found the important Ancestral definition. The theorems are the ones expected to describe linear-orderings. From this, Frege's argument is simple: to show that such ordering principles can be stated without appeal to intuitions, as Kant did. Added to

[7]Published in 1879. Henceforth quoted as *BS*. In english, the term received multiple translations: Stefan Bauer-Mengelberg uses "ideography" in (FREGE, 1967), since this was the term addopted by Phillip Jourdain assessments of Frege's logic, which was readed and accepted by Frege. In Austin's translation of *GLA* in (FREGE, 1953), "concept writing" is the opted translation of the term, while in William Bynum's translation in (FREGE, 1992) it's "conceptual notation". In the new *GGA* translation by Philip Ebert and Marcus Rossberg, (FREGE, 2013), the more usual "concept-script" is used. We opt to use "concept-script" as well, when reffering to Frege's logic, keeping the german term, or the abbreviation *BS*, when reffering to the 1879 book.

[8]Published in 1884. Henceforth *GLA*.

[9]As Coffa lightly put it: "There are many ways of looking at Frege's marvelous book; I prefer to think of it as a gigantic fly-swatter, ominously surveying the whole filed of arithmetic, ready to squash pure intuitions as soon as it comes in" (COFFA, 1982, p.38).

that, such principles can also derive important mathematical modes of inference, and they are then used to prove the basic laws of arithmetic. This is what Frege hinted in *GLA*, and finally showed in *Grundgesetze der Arithmetik*[10]. In chapter 3, this accomplishments will be addressed. First, we will restate Frege's logic into a more modern correlate (3.1). Then, all important definitions of section III, including the Ancestral definition, will be presented and discussed (3.2). To show the importance of such definitions, important theorems will be derived in (3.2). They show how Frege's definition derives linear-orderings, suitable enough for arithmetical considerations. This is done, finally, in (3.3), as Frege's Theorem will be discussed. A complete proof of all theorems declared is available in Appendix A.

Finally, Frege's definition was subject of many criticisms. Specially, given his unrestricted second-order quantification, it was quickly recognized as impredicative already in 1887 by Benno Kerry, and latter by Ignacio Angelelli in 2012. They both accused Frege's Ancestral as circular. In chapter 4 we discuss this problem and evaluate their arguments. First, in (4.1), we restate them. Then, we offer objections to their conclusions in (4.2). More precisely, we argue that given Frege's philosophical motivation, showed in (4.2.1), the alleged circularity is a confusion about Frege's logical system. This is discussed in (4.2.2). We also offer some additional problems regarding their premises in (4.2.3), to finally conclude in (4.3) that Frege's definition of the Ancestral, although impredicative, is not circular as they declared.

[10]Published in two volumes in 1893 and 1903. Henceforth *GGA*.

Chapter 1

Kant's Proposal

In this chapter, our goal is to address Kant's thesis that arithmetical judgements are synthetic *a priori*. We aim to restate Kant's thesis that the notion of an ordered series is dependent on pure intuitions, since this is Frege's main opponent in his logical definition for the same notion. The notion of an ordered series plays a similar role in arithmetic for Kant as it for Frege. Since for Kant intuitions are necessary for ordered series, the foundations of arithmetic must rely on it as well. Kant's thesis is better exposed in the *Critique of Pure Reason*, a product of his mature critical philosophy. We shall first restate the kantian taxonomy of notions and terminology in *CPR* in order to better address his main thesis about arithmetic. For such, the difference between concepts and intuitions, and the role of logic will be discussed. Then, we shall see how time is important for human thinking, and for numbers and arithmetic as well. As a brief conclusion, we shall see how Kant also offers a solution for the applicability and foundational problems.

1.1 Kant's Critical Philosophy

Most of the kantian distinctions and terminology comes in a dual form: intuitions and concepts, *a priori* and *a posteriori*, and so on. One of the reasons for this binary division of notions is that the Kantian mature philosophy is a treatise of a conciliation between two faculties: understanding and sensibility. No knowledge is possible without sensible experience. But no sensible object can be known without our own cognitive capabilities. As he puts it, "there are two stems of human cognition, [...] namely sensibility and understanding, through the first of which objects are given to us, but through the second of which they are thought" (*CPR*, B29). There should be, and that's the business of the *Critique*, a science that determines the conditions under which a object is given to us, and a science that determines the conditions under which a object is thought by us. The transcendental *aesthetic* and *logic* are such sciences.

The main notion in Kantian terminology is that of *representations*: "[...] inner determinations of our mind in this or that temporal relation" (*CPR*, B242). Moreover, there are two kinds of representations, with our without consciousness. Only the former is what matters to Kant. In (*CPR*, B376) he offers a taxonomy, such that under the notion of a representation:

> stands the representation with consciousness (*perceptio*). A perception that refers to the subject as a modification of its state is a sensation (*sensatio*); an objective perception is a cognition (*cognitio*). The latter is either an intuition or a concept (*intuitus vel conceptus*).

Therefore, a conscious representation is a perception, and those objective representations are intuitions or concepts. The same distinction is made in (*JL*, p.589): "All cognitions, that is, all representations related with consciousness to an object, are either intuitions or concepts". Their main difference is the way each representation can refer to an object: intuitions are said to refer immediately *i.e.*, they designate a singular object and relate immediately to it, as a concept refers only mediately and can relate to many objects through marks [*merkmals'*]. As Kant puts it, "An intuition is a singular representation (*repraesentatio singularis*), a concept a universal (*repraesentatio per notas communes*) or reflected representation (*repraesentatio discursiva*)" (*JL*, p.589).

It's no secret that intuitions and concepts are the two main elements of Kant's philosophy. The relation between reason and sensibility is then explicable through the relation between them. Since, for Kant, human intuitions are only of the sensible kind, aesthetic is the science of intuitions. Logic, as the science on how we think an object given through an intuition, is the science of concepts, of how they can relate to an intuition. This conection should explain the most crucial question in the *Critique*, on how synthetic *a priori* judgements are possible.

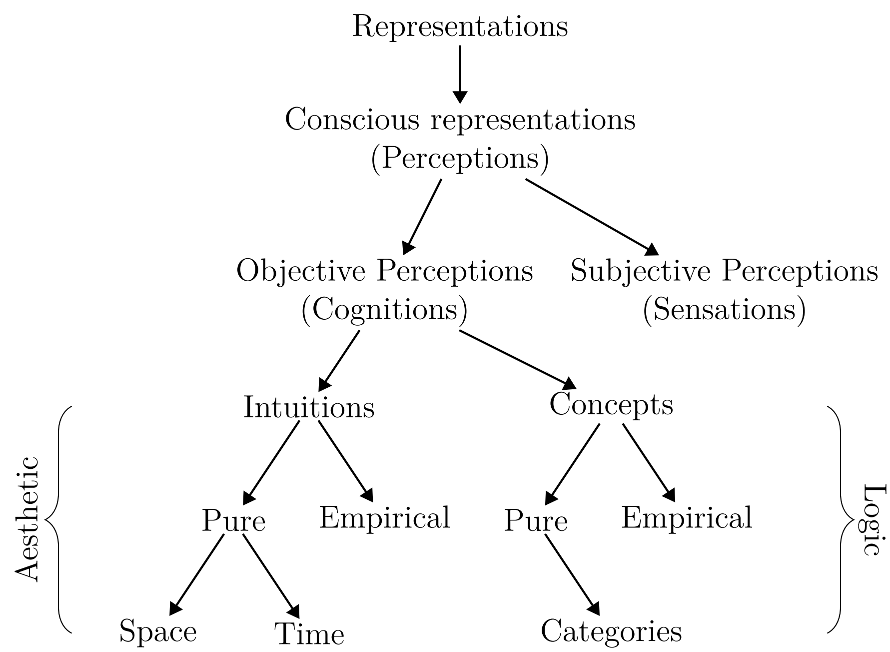


Figure 1.1: Kant's Taxonomy

Kant has, then, two major distinctions that need to be addressed: between pure (*a priori*) and empirical (*a posteriori*) cognitions and between analytic and synthetic judgements. A cognition is said to be *a priori* if it is "independent of all experience and even of all impressions of the senses" (*CPR*,B2), and a cognition is said to be *a posteriori* if it has its sources in experience, *viz.* it is empirical. Kant defines a judgement to be *a priori* if its justification is independent of experience. Hence, this distinction is not a matter of origin, but justification. However, if a cognition is *a priori* also in its origin, then it's said do be a pure cognition. The core of Kant's critical philosophy is to unveil such pure parts: pure intuitions and pure concepts (see Figure 1.1).

The relation between the aesthetic and logic is central for explaining the possibility of synthetic *a priori* judgements. A judgement is, for Kant, a relation between concepts. Furthermore, judgements are "[...] functions of unity among our representations" (*CPR*,B94). Concepts relate only indirectly to an object, being "the unity of the action of ordering different representations under a common one" (*CPR*,B93)[1]. They are the basis for our discursive reasoning and "the understanding can make no other use of these concepts than that of judg-

[1]In (*JL*,p.597), Kant likewise defines judgements to be "the representation of the unity of the consciousness of various representations, or the representation of their relation".

ing by means of them" (*CPR*,B93), that is, forming judgements. Like concepts, they do not refer imediately to an object. In fact, a concept refers only to other representations, either to other concepts or an intuition. Thus, the question whether a judgement is analytic or synthetic rests upon what kind of relation exists between the concepts (the representations) being judged.

The main kind of a judgement that Kant is dealing with is that of the subject-predicate relation, like "A is B". To say that "A is B" is the same as saying that the complex representation A has the representation B as subcomponent. Hence, this is a sort of membership relation applied to representations: "$B \in_R A$". The relation between the subject and the predicate can be of two kinds, according to Kant in (*CPR*, B10):

1. Either the predicate B is contained in the subject A: $B \in_R A$, or
2. The predicate B is not contained in the subject A. $B \notin_R A$.

Since a judgement is the representation of a unity, the two previous conditions above determines what kind of unity is being performed. The unity is trivial, if case (1) holds, or ampliative, if (2) holds prior to the unity. We would be saying something like $A \cap B \subseteq_R A$ and $B \in_R A$ as analogous for each case. Analysis is what it takes to justify judgements of the first kind, since in performing it, we discover that A and $A \cap \{B\}$ are names for the same representation. They are, then, analytic. For the second kind, since the predicate representation is not a previous subcomponent of the subject representation, their unity is something ampliative and synthetis is what it takes to justify it. They are, then, called synthetic judgements[2]. We can now offer more simple definitions:

1. A cognition is *a priori* if it is experience independent;
2. A cognition is *a posteriori* if it is experience dependent;
3. A judgement is **analytic** if the representation of its unity is trivial;
4. A judgement is **synthetic** if the representation of its unity is not trivial.

[2]Kant states that "One could also call the former **judgements of clarification**, and the latter **judgements of amplification** since through the predicate the former do not add anything to the concept of the subject [...] while the latter, on the contrary, add to the concept of the subject a predicate that was not thought in it at all" (*CPR*, B11).

Since judgements are cognitions as well, the first distinction is also applicable to them. Therefore, every *a posteriori* judgement is synthetic and every analytic judgement is *a priori*. But the converse is not true, since not every synthetic judgement is *a posteriori* and not every *a priori* judgement is analytic. The existence and posibility of a third kind, the synthetic *a priori* judgements, is the main theme in the *Critique*. How are they possible? How ampliative reasoning independent of all experience is possible? Kant's answer depends on how our intuitions are structured.

1.2 Aesthetic and Intuitions

Kant is emphatic in saying that all our cognitions must have an intuition to relate to: that "[...] which all thought as a means is directed as an end, is intuition", and "all thought [...] must ultimately be related to intuitions" (*CPR*, B33). To our human case, sensibility is the only source for intuitions, being the "capacity (receptivity) to acquire representations through the way in which we are affected by objects". Those representations that results from a sensible object are called sensations, and the intuitions it provides are always empirical.

The important point is that Kant distinguishes the *matter* from the *form* of an intuition: while the former is what corresponds to the sensation related to the intuition, the latter is "that which allows the manifold of appearance to be intuited as ordered in certain relations" (*CPR*, B34). The form of an intuition is a *pure intuition*, since it is not itself a sensation. Moreover, pure intuitions are the conditions under which every empirical object can be received, and hence, are the form of the sensibility as well. The study of such pure forms of the sensibility is what Kant designates by Transcendental Aesthetic[3].

This distinction is one of the foundations for the kantian philosophy of mathematics. Pure intuitions, not being sensations themselves, are always *a priori*, since they are a "mere form of sensibility in the mind" (*CPR*, B35). Since mathematics is a *a priori* science as well, and since mathematical propositions are about mathematical objects (and not mere concepts),

[3]'Transcendental', in the kantian sense, means the conditions of a possible experience, or the "the possibility of cognition or its use *a priori*" (*CPR*,B81). The Transcendental Aesthetic studies the conditions under an object is intuitable. The Trasncendental Logic, likewise, studies the conditions under which an object is thinkable, or applicable to concepts.

pure intuitions must provide, somehow, their basis. There are two kinds of pure intuitions: space and time. Kant considers Geometry as the science of space, and Arithmetic as the science of time[4]. But he takes both geometry and arithmetic for granted as informative disciplines, that is, where synthetic *a priori* judgements are possible. They must be the key to explain how such reasoning is possible. Therefore, space and time are the central points for mathematics.

From the two forms of intuitions, two properties of the mind are then determined: *outer sense*, in which we "represent to ourselves objects as outside us, and all as in space" and the *inner sense*, "by means of which the mind intuits itself, or its inner state, [...] so that everything that belongs to the inner determinations is represented in relations of time" (*CPR*,B37). Space then "is nothing other than merely the form of all appearances of outer sense" (*CPR*,B42), and time "nothing other than the form of inner sense" [*CPR*,B49]. It follows that the inner sense is more inclusive, since every possible intuition, either in space or not, must be perceived in time. Kant's argument[5] shows that both space and time are (1) pure *a priori*, *i.e.* non-empirical, intuitions; (2) the conditions under which every external and internal objects of our sensibility must be received and (3) non-discursive or non-conceptual representations. Specifically, every representation of a series must take time as its foundation, and this explains not just the possibility of the ordenation of representations, but the synthetic *a priori* character of arithmetic judgements. Therefore, time is our main focus.

Time

Kant's argument, exposed in the Metaphysical Exposition in (*CPR*,§4), shows that only taking time as a pure *a priori* intuition we can explain its unidimensionality and infinitude. The unidimensionality of time states that "different times are not simultaneous, but successive (just as different spaces are not successive, but simultaneous)" (*CPR*,B47), that is, different points or intervals in time are successive and are points in the same countinuouss time line[6]. Time is, in fact, represented in this exact way:

[4]Even though in *CPR* it's the Kinematics that is assumed as the science of time.

[5]Namely, the metaphysical exposition of such concepts.

[6]This is described as apodictic principles, or axioms, about time, namely, that time has only one dimension and that different times are successive.

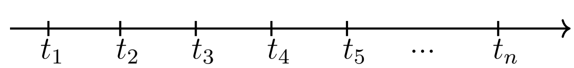
> time, although is not itself an object of outer intuition at all, cannot be made representable to us except under the image of a line, insofar as we draw it, without which sort of presentation we could not know the unity of its measure at all (*CPR*,B156).

The unity of such measure is the realization that the successive points in time form a series. Kant argues that time

> determines the relation of representations in our inner state. And just because this inner intuition yields no shape we also attempt to remedy this lack through analogies, and represent the temporal sequence through a line progressing to infinity, in which the manifold constitutes a series that is of only one dimension, and infer from the properties of this line to all the properties of time, with the sole difference that the parts of the former are simultaneous but those of the latter always exist successively (*CPR*, B50).

That is, points in time are successive, but in order to represent it we draw a series of points through a line that is simultaneous in space. This line represents the passage of time through a series of successive intervals. But drawing such line takes time, and hence, time is still the condition for grasping such series.

The passage of time, that we recognize from the three main points of past, present and future, happens through intervals where each one is the successor of another, and no interval have more than one successor, as the following:



The infinitude of time is the possibility to expand as long as we want the line above, or putting differently, the fact that every time interval represented in the line has a successor also in the line. Hence, for Kant, "the original representation **time** must therefore be given as unlimited" (*CPR*,B48). In order to maintain that time is unidimensional and unlimited, Kant argues that time must be an *a priori* intuition. He provides five arguments that we summarize as the following two points:

1. We cannot extract simultaneity or succession from experience. If we could, they would not be necessary principles. Moreover, the fact that "one cannot remove time, though one can very well take the appearances away from time" (*CPR*,B46) indicates that time precedes experience. Therefore, Time must be *a priori*.

2. Since time is unidimensional, different intervals are said to be part of the same time line and thus time must be a singular representation[7]. Also, for time to be unlimited, its representation must be originally unlimited as well, something that we cannot achieve from concepts[8]. Therefore, Time must be an intuition.

Kant's conclusion is that time is a pure *a priori* intuition, the condition for our representation of objects in the inner sense, and the condition for the possibilitity of every (empirical) intuition. This fact about time should explain how synthetic *a priori* judgements are possible in the field of kinematics and arithmetic as well: "Our concept of time therefore explains the possibility of as much synthetic *a priori* cognition as is presented by the general theory of motion, which is no less fruitful" (*CPR*,B49).

The way Kant depicts time as a line in space is consistent with our initial considerations about series. As stated in the introduction, a partial ordered series is any series that is both transitive and asymmetric, and a linear order is any that is both partial and satisfies trichotomy. Since, for Kant, time is unidimensional, it can only be represented with a line segment that has only one direction. Define, then, a successor relation on time intervals as $<_t$, meaning by $x <_t y$ that x is a time interval the appears before y in the time line. Suppose that $<_t$ is symmetric. Then, $t <_t t'$ means that $t' <_t t$ as well. In that scenario, after reaching t, we reach t' and t again. Since no time interval occupies two places in the segment, that means that time is not represented with a straight line, since there is a path that leads from t back to t again. Hence, the successor relation must be asymmetric if time is taken to be unidimensional. Likewise, if $t_1 <_t t_2$ and $t_2 <_t t_3$, it's safe to say that there is a finite amout of time intervals separating t_1 from t_3, *i.e.*, t_3 is reachable from t_1. This is possible because time, taken as unidimensional, must be transitive as well. Therefore, the relation $<_t$ is both

[7] Where Kant says that "[...] every determinate magnitude of time is only possible through limitations of a single time grounding it" (*CPR*,B48).

[8] "[...] for they contain only partial representations" (*CPR*,B48).

asymmetric and transitive, being in this case a partial order. But we can added the other desired condition as well. We already saw that $t <_t t'$ and $t' <_t t$ cannot both be the case. Now, suppose that $t <_t t'$ and $t = t'$. Then we have that $t <_t t$, which is the same problem stated above: the existence of a path from t back to t again. This is against Kant's notion of time, hence the supposition is false. Finally, from the same reasoning, $t' <_t t$ and $t = t'$ cannot both hold as well. Therefore, exactly one of them, that is $t <_t t'$, $t' <_t t$ or $t = t'$, holds. Hence, $<_t$ also satisfies trichotomy, and for that reason, describes a linear, or total, order as well.

This is so because time is a condition for all possible experience, one that ordenates the representation of objects aprehended in intuition. But as a pure intuition itself, time is a form of our sensibility, so the fact that we can ordenate our sensible experience and derive the cognition of a series is not solely determined by intuition, and requires the act of the understanding. More precisely, it requires a synthesis. As Kant puts it:

> inner sense [...] contains the mere form of intuition, but without combination of the manifold in it, and thus it does not yet contain any determinate intuition at all, which is possible only through the consciousness of the determination of the manifold through the transcendental action of the imagination (synthetic influence of the understanding on the inner sense), which I have named the figurative synthesis. (*CPR*, B154)

Intuition provides a manifold, one that is given successively in time but that requires a synthesis of the understanding in order to be thinkable. Both the object being perceived and the notion of succession itself, as Kant also explains, are products of understanding by means on how our sensibility is affected by objects:

> We cannot think of a line without **drawing** it in thought, [...] and we cannot even represent time without, in **drawing** a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold through which we successively determine the inner sense, and thereby attending to the succession of this determination in inner sense. [...] the synthesis of the manifold in space, if we abstract from this manifold in space and

> attend solely to the action in accordance with which we determine the form of **inner sense**, first produces the concept of succession at all. The understanding therefore does not find some sort of combination of the manifold already in inner sense, but **produces it**, by **affecting** inner sense. (*CPR*, B154-155)

First of all, the only way time is representable to us is through a straight line since that's the way our understanding synthesize the manifold of an intuition: successively. That's also how we manage to think about succession as well: by abstracting the content of a synthesis and focusing only on *how* we combine the manifold into a single representation. But that is according to the form of inner sense, time itself. Kant's argument, then, is something like the following:

1. When we synthesize the manifold of an intuition, we do so successively, according to the determinations of inner sense;

2. To have a representation of time, we must then abstract from the content of such synthesis and focus on the act;

3. We do that by drawing a figure that represents such act. And that figure is a straight line.

But why it is not possible to represent time as a circle or a triangle, or any other shape? Because our inner sense, the way we are affected internally by objects, is only possible if time is unidimensional, moving forward like a straight line being stretched indefinitely. Arithmetic would be affected if time weren't like that as well. But the role of time, although necessary, is not suficient. The same goes for ordered series. Only with the combination, or synthesis, of the manifold given by an intuition, we can determinate what a number or a series in fact is. To grasp such a series, that represent the manifold, a synthesis is required. Just as a line is representative of the passage of time, an ordered series would also be so. Not just because such series is a image of the passage of time, but because time itself is the condition under which a series is grasped, by the successive synthesis of the manifold given in intuition. But we should define what it is to synthesize a manifold for Kant, and this is the task for logic.

1.3 Logic and Concepts

Kant's aesthetic is conceived as the science of sensibility, about our receptivity of objects. Following the division that we already stated, Kant also takes that the science of our thinking, the active and spontaneous counterpart of the sensibility, is Logic. More precisely, logic is the science of rules of the understanding[9]. In Kant's time, logic was usually divided into three branches: a theory of concepts (of general representations), a theory of judgements (the relation between concepts) and a theory of inferences (the relation between judgements, *i.e.* that of derivability). His *JL* follows this schema. But the logic in *JL* is mainly a formal theory. Kant in fact recognizes two distinct logics: *general* and *transcendental*.

General Logic

In (*JL*, p.531), Kant defines general logic as follows:

> Logic is [...] a science *a priori* of the necessary laws of thought, not in regard to particular objects, however, but to all objects in general; hence a science of the correct use of the understanding and of reason in general, [...] according to principles a priori for how it ought to think.

It follows that general logic is a formal science on the laws of thought. For that matter, two points are important: (1) general logic has no relation to empirical or psychological conditions under which we first learn how to thought: its purpose is not to describe how humans happen to think, but how they ought to think. Therefore, its principles must be *a priori*. (2) For that reason, logic is regarded as formal because it abstracts entirelly from contents of the understanding, either empirical or not[10]. Another way to put it is that Kant's general logic has no interest in epistemology or metaphysics. It's only a *canon*, a method for examine the right use of reason, one that is not suitable for discoveries[11]. Furthermore, general logic

[9]The understanding is, as Kant defines it, "[...] the faculty for bringing forth representations itself, or the **spontaneity** of cognition" (*CPR*, B51).

[10]As Kant puts it in (*CPR*, B78), logic "abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking".

[11]Its actual counterpart is the applied logic, that Kant recognizes as a science that "[...] contains the rules for correctly thinking about a certain kind of objects" (*CPR*,B76), and for that reason, does not abstract entirely from the empirical conditions of our knowledge.

has nothing to say about the analytic-synthetic judgements or even the recognition of a true judgement. In fact, Kant considers that, as far as general logic is concerned, only a negative criteria of truth is possible: that the form of our thinking does not contradicts itself[12].

But how general (formal) logic ought to describe the mere form of thinking? It does so by studying the relation of its basic branches: the relation between concepts (whatever they may be) and the relation between judgements, that is, inferences, from a formal standpoint. Judgements are divided in respect to their form: quantity, quality, relation and modality. Inferences are organized according to the relation of such judgements, either immediate, also called inferences of the understanding, or mediate inferences, (inferences of reason for Kant), basically the aristotelic syllogistic suplemented with hipothetic and disjunctive inferences. In nowhere general logic is concerned with the content of concepts or judgements[13].

In sum, "Logic, having no specific subject matter, is *general*. Having nothing to do with human psychology, it is *pure*. Concerning only the form of thought, it is merely *formal*" (CAPOZZI; RONCAGLIA, 2009, p.144). This thesis about general logic yields some important conclusions from Kant's philosophy that puts him in the opposite side of the metaphysics of its time, like the Leibniz-Wolff fusion between logic and metaphysics, or even the later fregean position, something that we'll address later[14]. From the kantian perspective, no knowledge is possible through concepts alone: "[Logic] is far from sufficing to constitute the material (objective) truth of the cognition, [and] nobody can dare to judge of objects and to assert anything about them merely with logic" (*CPR*, B84). This conclusion also posits that logic and mathematics are two distinct disciplines. Kant is not a mathematical formalist, as he maintains that mathematical judgements are contentful. But, mathematics is a proper *organon* of reason, he declares: "[...] an organon presupposes exact acquaintance with the

[12]"But this criteria concerns only the form of truth, i.e., of thinking in general, and are to that extent entirely correct but not sufficient. For although a cognition may be in complete accord with logical form, i.e., not contradict itself, yet it can still always contradict the object" (*CPR*, B84).

[13]On concepts: "universal logic does not have to investigate the source of concepts, not how concepts arise as representations, but merely how given representations become concepts in thought" (*JL*,p.592). On judgements: "Since logic abstracts from all real or objective difference of cognition, it can occupy itself as little with the matter of judgements as with the content of concepts. Thus it has only the difference among judgements in regard to their mere form to take into consideration" (*JL*,p.598).

[14]MacFarlane (2002, p.28) argues that this isn't just about Kant changing the subject: accepting the generality of logic with the principles of his transcendental philosophy, yields the formality thesis as conclusion, *viz.* that logic is "completely indifferent to the semantic contents of concepts and judgements".

sciences, their objects and sources. Thus mathematics, for example, is an excellent organon" (*JL*,p.529), something that logic is not, since it has no objects or contents of its own[15].

Transcendental Logic

While the general logic is only concerned with the formal relation between concepts, the Transcendental Logic is the science where the relation between concepts and intuitions is properly investigated. This relation is central for a proper knowledge aquisition: "It is thus just as necessary to make the mind's concepts sensible (i.e., to add an object to them in intuition) as it is to make its intuitions understandable (i.e., to bring them under concepts). [...] Only from their unification can cognition arise." (*CPR*,B75). Kant maintains that concepts can only have content through the mediation of an object given from an intuition, and this is the only proper knowledge that we can obtain/produce. His hypothesis is that, just like the aesthetics shows the pure *a priori* conditions for our receptivity of objects, there must be pure *a priori* conditions for our thinking of such objects, or likewise, pure concepts of the understanding. General logic is not the science of such concepts, and for that reason, transcendental logic does not abstract entirely from the content of our cognitions.

Therefore, Kant postulates the existence of pure *a priori* parts of our understanding that mirror the pure *a priori* parts of our intuitions: the pure concepts. These concepts are the necessary conditions for our thinking, that are "[...] related to objects *a priori*, not as pure or sensible intuitions but rather merely as acts of pure thinking" (*CPR*,B81). Hence, the existence of such pure concepts determines the existence of a logic that is capable of determining the applicability of such pure concepts into objects, or "a science of pure understanding and of the pure cognition of reason, by means of which we think objects completely *a priori*" (*CPR*,B81), as Kant himself defines the transcendental logic.

As we discussed earlier, concepts and judgements are representations of the unity of other representations. They are functions of the understanding: the unity of different representations into one. This is what Kant designates by *synthesis*: "By synthesis [...] I understand

[15]The distinction between Canon and Organon is better clarified by Kemp Smith: "By a canon Kant means a system of *a priori* principles for the correct employment of a certain faculty of knowledge. By an organon Kant means instruction as to how knowledge may be extended, how new knowledge may be acquired. [...] A canon is therefore a discipline based on positive principles of correct use" (SMITH, 2003, p.170).

the action of putting different representations together with each other and comprehending their manifoldness in one cognition" (*CPR*,B103). Knowledge is the connection between concepts and intuitions: our understanding receives, by means of our intuitions, a manifold that must be synthetized and unified by some rules. The pure concepts are such rules. They provide the ways in which a synthesis can be brought upon, and this is so because "[...] the spontaneity of our thought requires that this manifold first be gone through, taken up, and combined in a certain way in order for a cognition to be made out of it" (*CPR*,B103).

The path to discover the pure concepts is given by general logic[16]: "The functions of the understanding can therefore all be found together if one can exhaustively exhibit the functions of unity in judgements" (*CPR*,B94). For that reason, Kant's discovery of the pure concepts, also known as categories, starts with the different forms of judgements of the general logic, and likewise, derives a table of categories similar to that[17]. Those are the "Clue to the Discovery of all Pure Concepts of the Understanding", as the title of the section alerts us[18]. Hence, Kant's table of categories is the following:

1. Of Quantity	**2. Of Quality**
Unity	Reality
Plurality	Negation
Totality	Limitation
3. Of Relation	**4. Of Modality**
Of Inherence and Subsistence	Possibility - Impossibility
Of Causality and Dependence	Existence - Non-existence
Of Community	Necessity - Contingency

Such categories are the functions under which our understanding can think and concep-

[16]It is not a mere enumeration.

[17]Although he assumes that this is a secure path for deriving the categories, Kant takes for granted that the table of judgements is suficient as given. More than that, Kant still believes that the classic form of judgements, the aristotelian subject-predicate form, is enough to express all the possible relations between concepts in a judgement. This point, among other problems in Kant's derivation, can be found in (YOUNG, 1992).

[18]As Kant declares in (*CPR*,B105): "The same understanding, therefore, and indeed by means of the very same actions through which it brings the logical form of a judgement into concepts by means of the analytical unity, also brings a transcendental content into its representations by means of the synthetic unity of the manifold in intuition in general, on account of which they are called pure concepts of the understanding that pertain to objects *a priori*; this can never be accomplished by universal logic".

tualize (*viz.* understand) an object given through an intuition and its manifold. This is how a concept obtain its content: by means of a synthesis[19].

Kant's transcendental logic, as so far presented, can be summarized as the following:

1. A judgement is the representation of the unity of other representations: a representation of the relation between concepts. General logic shows that there are four classes of judgements, that is, four classes of such unity.

2. Likewise, a concept is the representation of the unity of representations, more precisely, of other concepts or intuitions. This unity is what Kant calls synthesis.

3. There are different ways to synthesize diferent representations: the categories. They resembles the different forms of judgements, as logic teach us. It is in transcendental logic, the science of such pure concepts/categories, that the synthesis is performed.

4. Since Kant states that all concepts must have an intuition corresponding to it (either empirical or pure), at some level, the concept at hand, or its subcomponents, must relate to the synthesis of a manifold given by an intuition. That's how a concept can have an object.

In conclusion, Kant's transcendental philosophy postulates a twofold process in order for us to cognize a given object: (1) the perception of such object, given through the pure *a priori* forms of our sensations, space and time; (2) the tought of such object, given through the pure *a priori* concepts of the understanding. Those concepts are the main structure in forming both empirical concepts and judgements. Kant's next step is arguing how such connection is possible and how the categories are the necessary condition for all possible experience, that is, every possible cognition about objects. This is the main theme in the *The Deduction of the Pure Concepts of the Understanding.* As it turns out, time is one of the foundations for the process under which we acquire knowledge of an object through synthesis.

[19]Kant states that "[...] no concepts can arise analytically as far as the content is concerned. The synthesis of a manifold, however, (whether it be given empirically or *a priori*) first brings forth a cognition, which to be sure may initially still be raw and confused, and thus in need of analysis; yet the synthesis alone is that which properly collects the elements for cognitions and unifies them into a certain content; it is therefore the first thing to which we have to attend if we wish to judge about the first origin of our cognition." (*CPR*,B103). That's why any judgement in which the relation between the concepts are grounded in the synthesized manifold of an intuition must be synthetic. That doesn't mean that in analytic judgements the concepts involved have no sensible content. It's the justification of the judgement that determines it's classification.

1.4 The place of Time in human thinking

In the transcendental deduction of the categories Kant argues how the categories are the conditions for every possible experience. The basic tenet is that synthesis is what gives unity to our knowledge, and the categories must be the forms of such synthesis, the rules that prescribe how a unity can be acchieved. However, in this process we also learn that time, as the pure form of our intuitions, is a necessary condition as well. Likewise, every successive aprehension of objects, in which series can emerge, is subject of not only time, but a synthesis in order to us completely recognize it as an ordered whole. Therefore, the explanation provided by Kant on how categories works in putting together our representations is also an explanation on how series are products of our human understanding, since it's in ordering the manifold of our intuitions, *viz.*, putting them into concepts, that they are produced.

As Kant describes in the *Deduction*, there are three main synthesis to be performed: "that, namely, of the **apprehension** of the representations, as modifications of the mind in intuition; of the **reproduction** of them in the imagination; and of their **recognition** in the concept." (*CPR*, A97). For that matter, imagination, as a mixed faculty, plays the role of putting intuitions and concepts together in a cognition. This threefold process is, roughly, the following[20]:

1. Every representation, either pure or empirical, belongs to our inner sense (time), since it is a modification of the mind. It is required that every intuition in which a manifold is given be unified, through synthesis, into a new whole. But, as Kant says, they "would not be represented as such if the mind did not distinguish the time in the succession of impressions on one another" (*CPR*,A99). Moreover, "[...] we can become conscious of only one new item at a time. [...] Thus, to distinguish one impression from another, we must give them separate locations" (BROOK, 2016). That is, separate locations in time. This is the task of the synthesis of the aprehension.

2. The second synthesis is inseparably combined with the first. The synthesis of the aprehension is what transforms the raw material of the intuition, its manifold, into a

[20]The Transcedental Deduction is one of the most discussed and problematic sections of the *Critique*. Needless to say, I assume the brevity of my exposition here, that must avoid such problems and focus only in what sense time is a crucial factor for our human understanding.

whole perception. But since we can only perceive one representation at a time, we would not have a continuous perception if not synthesizing different successive representations into one. This is the role of the Synthesis of the reproduction in the imagination. Kant defines that "Imagination is the faculty for representing an object even without its presence in intuition" (*CPR*,B151), and it does so producing *images* of such representations. This ability to reproduce and access again our representations, is what enables the mind to perceive the same object in different moments[21].

3. Finally, what it's left is the application of concepts into the representations in question. What the synthesis of aprehension and of reproduction provides is an ordered whole, one that understanding must now cognize. The condition for that to happen is consciousness[22]. Without it, "all reproduction in the series of representations would be in vain", since "it is this one consciousness that unifies the manifold that has been successively intuited, and then also reproduced, into one representation" (*CPR*,A103)[23]. Hence, consciousness is what properly applies the categories to the objects provided by intuitions. It does so by relating such representations into different aspects in which they can be unified: the categories[24].

To sum up, each synthesis is related to three sources of cognitions: sense, imagination and apperception[25], and it's through them that we can have a proper cognition of an object.

[21]Otherwise, we would have, at each time interval, representations with total disconnection with each other. This seems to suggest that the synthesis of reproduction works similarly as memory, altought is just a way to associate different representations according to their position in the time line. As Brook (2016) argues, this synthesis provides a transition from one perception to another, where "Such transitions are the result of the setting up of associations (which, moreover, need not be conscious) and do not require memory". To give a simple example, if we are to perceive a sunset, the perception of the position of the sun in a time t_1 must be associated with the perception at t_2, otherwise, we would not be able to unify both perceptions into a whole, and conclude that it's the same sun in both.

[22]Kant considers it as *apperception*, *i.e.*, perception of the self. Kant's argument states the existence of a *a priori* condition for our apperception in its empirical use: the transcendental unity of apperception. Kant dedicates a great deal of the transcendental deduction discussing its importance, something that we avoid here.

[23]Guyer (2010, p.130) succintly explains it: "Kant argues that since the data for any cognition of an object are given over a period of time, the data must be severally and serially apprehended [...], earlier data in such a series of apprehensions must be able to be recalled when later data are experienced [...], and finally a connection among the data that constitutes them into representations of a single object must be recognized by means of a concept that links them as representations of states or properties of such an object".

[24]We shall exemplify this point later, in explaining how arithmetic produces its object in understanding.

[25]More precisely, as Kant explains: "Sense represents the appearances empirically in perception, the imagination in association (and reproduction), and apperception in the empirical consciousness of the identity of

This schema shows also how can knowledge of a series be achieved. In every aprehension of an object, time is presented as condition. As Kant himself assumes, "Time is in itself a series (and the formal condition of all series)" [*CPR*,B438], and "the synthesis of the manifold parts of space, through which we apprehend it, is nevertheless successive, and thus occurs in time and contains a series" [*CPR*,B439]. This is because, as we pointed out above, the synthesis of aprehension is subject of time.

We distinguished then two kinds of series, according to Kant's philosophy: the natural order of time as a pure form of our intuitions, and a empirical series of a given set of objects intuited[26]. The former is the condition of the latter, but it is itself only knowable to us through the latter. This is because, without the threefold process of synthesis, we cannot properly think or even be aware of an ordered series of objects whatsoever. That's why Kant assumes, as we showed earlier, that the straight line is the proper representation of time, since it's the figure that better represents its properties: unidimensionality and infinitude. Kant's argument goes beyond in saying that this is the *only* way we can represent time because this is the exact way, and the only way, that objects are presented and conceptualized in us, that is, successively.

Later, in the section *On the schematism of the pure concepts of the understanding*, Kant's gives details on how imagination intervenes in connecting concepts with intuitions and forming judgements through them[27]. The connection is only possible if something homogeneous to both concepts and intuitions is found, something that has both intelectual and sensible parts. This is what Kant designates by *schema*. An schema is a formal condition that determinates the applicability of a concept and differs from an image, which is only a particular representation of a singular intuition[28]. Both are products of the imagination, *i.e.* constructions of the mind. Schemas are central in arguing the possibility of synthetic *a priori* judgements, since they are responsible in connecting, synthetically and *a priori*, a concept with its corresponding object of intuition in a judgement. Once again, time is fundamental

these reproductive representations with the appearances through which they were given, hence in recognition" [*CPR*,A115].

[26]Kant gives a similar distintion at [*CPR*,B238].

[27]In other words, on how the synthesis of reproduction works to each particular category.

[28]While we can have an image of a particular triangle, a schema of a triangle is a rule to construct it in intuition. The difference, then, is that the latter is more general than the former.

to these schemas, since it's "the formal condition of the manifold of inner sense, thus of the connection of all representations" (*CPR*,B178). Hence, for each category, a correlate schema is produced according to each of the following corresponding relations in time:

> The schemata are therefore nothing but a priori time-determinations in accordance with rules, and these concern, according to the order of the categories, the time-series, the content of time, the order of time, and finally the sum total of time in regard to all possible objects. (*CPR*, B185)

In other words, the time-series defines quantity, the content of time defines quality, the order of time defines relations and the sum of times defines modality. It is in this perspective that numbers, and arithmetic, becomes a product of the time-series determinations. This is the root for the famous kantian thesis that arithmetic is synthetic *a priori*, something that finally we are in position to address.

1.5 Time, Numbers and Arithmetic

In a famous passage in the *Introduction* to *CPR*, Kant states his thesis about mathematics: that all mathematical judgements are synthetic *a priori*. In what matters to arithmetic, his example is that the proposition $7 + 5 = 12$ is synthetic, illustrating that, in order for us to proceede to such calculation, we have to construct the number by means of the numbers to be added. It's not a simple matter of analysing the concepts and discovering that the concept of 12 is a subcomponent of the concept of a sum between 7 and 5. Kant argues that, if we take larger numbers into the scene, it becomes obvious that the concept of a sum does not entails the concept of its result, since "[...] for it is then clear that, twist and turn our concepts as we will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts" (*CPR*,B16). Putting it differently, a sum takes time to be performed, and hence, takes time to be completed, that is, to get its result.

Numbers are, for that regard, constructions. In the preceding sum, one constructs the first number, using time and schemas, and then constructs the other, adding the first number as a starting point, just to finally reach the resulting number. Very late in the *CPR*, Kant states that "Philosophical cognition is rational cognition from concepts, mathematical cognition that from the construction of concepts. But to construct a concept means to exhibit a priori the intuition corresponding to it" (*CPR*,B741). In empirical situations, the ideia is that the manifold of intuition provides a whole, that is synthetized (according to the threefold process that we unveiled before) and where a schema is provided, according to the time-series, which in application to the category of quantity, provides a number to the phenomena in question. But the pure notion of a number must be empirically independent. In fact, Kant states that:

> The pure **schema of magnitude** (*quantitatis*), however, as a concept of the understanding, is **number**, which is a representation that summarizes the successive addition of one (homogeneous) unit to another. Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition (*CPR*,B182).

What Kant seems to suggest is this: in counting, we choose an homogeneous intuition, the things being counted, which we present repeatedly in apprehension (as synthesis of apprehension): fingers, strokes or even ideas[29]. After counting the first stroke, and successively the second, the time-series used in such apprehension determines that the first precedes the second, and so on for other strokes that are presented in the time-series. Then, the pure schema of quantity connects the manifold with the corresponding category, in which a discrete magnitude is produced[30]. As Kant states, "I think therein only the successive progress from one moment to another, where through all parts of time and their addition a determinate magnitude of time is finally generated" (*CPR*,B203). Therefore, in counting an empirical manifold, numbers are the schemas in which we apply the category of quantity in order to determine the answer to questions of the "how many?" type, that is, a cardinal number. If we take the succession of time as providing the ordinal numbers, we can state that Kant's

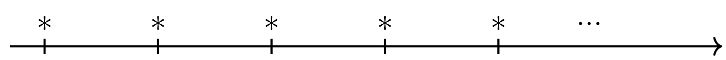
[29] As internal aprehensions of the mind.

[30] Friedman (1998, p.116) offers the same interpretation: "Lying at the basis of all operations with the concept of magnitude is the number series: the series of what we now call the natural numbers. And this series, for Kant, can in turn itself only be represented by means of a progression in time: the successive addition of unit to unit. In particular, it is only the necessarily temporal activity of progressive enumeration that allows us to find or determine the result of any calculation".

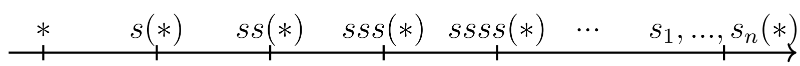
strategy is deriving the cardinals from the ordinals, from the order of the time-series to a schema for each interval.

The threefold process in perceiving, synthetizing and determining the quantity of a manifold, either pure or empirical, in such derivation can be analysed as the following[31].

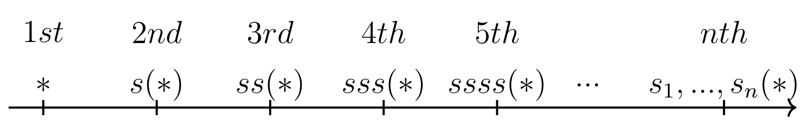
1. Consider that one is to count a series of n strokes: |||||. To determine it's number, or magnitude, we cannot apprehend them simultaneously, otherwise all quantities would have the same size, *i.e.* 1. We have to apprehend them successively, according to the time-series that we are subject to. Without any synthesis, intuition provides a time-series such as the following (where each $*$ is an interval):



2. The first step is to synthetize the manifold presented in each interval, through the synthesis of apprehension: we apprehend the units being counted, as they are exhibited at each interval. Since they are positioned successively in the time-series, denote $s(*)$ as the time-interval that is successive of $*$, then the representation of the time-series is:



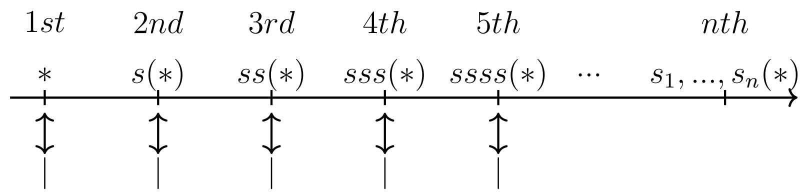
3. We still don't know the quantities that each $s_1, ...s_i(*)$[32] represents, but we can derive the ordinals from the successor in each interval:



4. Since we desire to find the quantity, or the cardinal number of the strokes, we first associate each stroke with the corresponding interval in the time-series, *i.e.* for each time interval $s_1, ..., s_i(*)$, we associate each of the strokes to be counted, that is, we count them one by one:

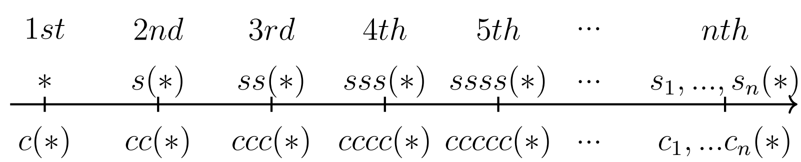
[31] Much of the following presentation is based on Wong's (1999) exposition about Kant's argument.

[32] Wheres if $i = 0$, the interval is $*$ itself. The notation $s_1, ..., s_n(*)$ denotes a series of n-applycations of the successor, that is: $\overbrace{s, ..., s}^{\text{n-times}}(*)$. For that reason, the index of each s can be discarded.



5. Numbers are schemas of quantity and hence, each duration of the time-series in the counting procedure determines a singular schema, or a singular number. In that scenario, each $s_1, ..., s_i(*)$ in the series is associated with a unique schema, that determines the procedure to construct a magnitude of that size. Accordingly, schemas can be regarded as iterations of the construction procedure, which follows the succesive structure of the time-series.

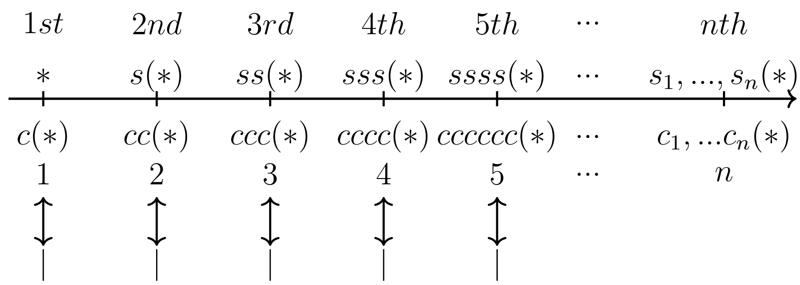
6. Denote by $c(s_1, ..., s_{i\leq n}(*))$ the construction performed at $s_1, ..., s_{i\leq n}(*)$, as regulated by a unique schema. If this construction procedure is just the addition of one more stroke, or one more unit to the counting being performed, then $c(s_1, ..., s_{i\leq n}(*))$ can be equally iterated in a way that $cc(s_1, ..., s_{i\leq n}(*))$ is the stroke, or the unit, being added to $c(s_1, ..., s_{i\leq n}(*))$. Likewise, it follows that $cc(*) = c(s(*))$ or that $c(ss(*)) = ccc(*)$, and so on. One simple way of reading something like $cc(*)$ is: add one more stroke to what $c(*)$ had originally yielded. Constructability is what licenses the transition from ordinals to cardinals. At this stage, what garantees the iteration of the constructability is the synthesis of reproduction. Accordingly we have:



7. Finally, each $c_1, ..., c_{i\leq n}(*)$ correspond to an individual number, since they are the link to the application of the category of quantity: taking each construction as a whole, we take the plurality expressed by it as a totality, reaching the final synthesis in the process, the synthesis of recognition. Adding the cardinal notation, we have that $c(*) = 1$, $cc(*) = 2$, $ccc(*) = 3$ and so on[33]. Needless to say, each $c_1, ..., c_{i\leq n}(*)$ correspond to

[33] We start with 1 because 0 denotes the number where no construction is performed. This can signify two things: either 0 is not a number at all, or it's a number where no intuition is required to be constructed. Avoiding the discussion, I take zero to be the number with an 'empty' construction.

an unique interval $s_1, ..., s_{i \leq n}(*)$, and for each of the strokes being counted, there'll be a unique construction that determines the actual number corresponding to it. Adding all together, the counting of the strokes in question can now be fully appreciated:



The upper side of the time-line represents the position of each interval, and the correspondent ordinal number, since this original order, which is inherent of our inner sense, has foundational priority over the cardinal numbers, represented in the bottom side of the line. As Wong (1999, p.363) also suggests, the time-series provides, from it's own, the structure for representing the ordinals, and hence, the order of time determines the order of numbers. Hence, "In order to determine *how many times* we need to apply the idea of *how many* to an ordinal sequence". In conclusion, the number of strokes enumerated is determined by the corresponding number of the last stroke counted.

Following this, we can define a recursive procedure for the addition operation, assuming such constructability as basic procedure (instead of the usual successor function). First, any number n is constructible if $n = c_1, ..., c_m(*)$, where $m = nth$ ordinal. This generates a list of cardinals as showed above, using the usual notation. We can abbreviate it as:

$$0 = * \tag{C1}$$

$$n + 1 = c(n) \tag{C2}$$

We're adding zero (0) as a number, even though it does not involve a construction. Nonetheless, this fits our purposes, since from this definition, $1 = c(0) = c(*)$, $2 = c(1) = cc(*)$ and so on. Then, recursive addition is:

$$n + 0 = n \tag{A1}$$

$$n + c(m) = c(n + m) \tag{A2}$$

Assuming that 0 is the number with an 'empty' construction. Using that, we can see how Kant's initial example fits our explanations, providing a complete proof of the proposition $7 + 5 = 12$:

$$\begin{aligned}
7 + 5 &= 7 + c(4) && \text{(C2)}\\
&= c(7 + 4) && \text{(A2)}\\
&= c(7 + c(3)) && \text{(C2)}\\
&= cc(7 + 3) && \text{(A2)}\\
&= cc(7 + c(2)) && \text{(C2)}\\
&= ccc(7 + 2) && \text{(A2)}\\
&= ccc(7 + c(1)) && \text{(C2)}\\
&= cccc(7 + 1) && \text{(A2)}\\
&= cccc(7 + c(0)) && \text{(C2)}\\
&= ccccc(7 + 0) && \text{(A2)}\\
&= ccccc(7) && \text{(A1)}\\
&= cccc(8) && \text{(C2)}\\
&= ccc(9) && \text{(C2)}\\
&= cc(10) && \text{(C2)}\\
&= c(11) && \text{(C2)}\\
&= 12 && \text{(C2)}
\end{aligned}$$

Now, this procedure tells us that, to perform the sum $7 + 5$, we have to construct 7 in the corresponding intuition, and procede to construct 5 on top of it. And we know that 7 is constructible, since $7 = c(6)$, from (C2). At the point where we get that $7 + 5 = ccccc(7)$ we could equally descend from (C2) to derive $cccccc(6)$ and so on, until we reach the base case and the answer: $cccccccccccc(*)$, *i.e.*, 12. This shows that to perform the sum, we have to successively add units to each other to reach the right answer, and this is only possible through time. The constructibility of 12 is a time-ordered procedure, which is synthetic by essence. Of course, one need not to start from $c(*)$ for every computation, otherwise, larger numbers would take an enormous ammount of time to be computed even for the most daily and mundane matter. The important thing, as far as Kant's philosophy is concerned, is that numbers must be constructible. As Wong (1999, p.370) points out, the empirical realization of the computation is not a necessary matter: "In order to stablish the representability of an arbitrarily large number, it suffices to construct an algorith that determines all the successive stages of its construction", just as we have showed above[34].

This constructions are called *symbolic* by Kant in (*CPR*,B745), and they are differentiated by the *ostensive* constructions of geometry, where's a particular object is constructed. The iteration of some procedure is present in both cases. But, following Friedman (1998), in geometry, that Kant held to be euclidian by nature, we take postulates as the starting points, such as to draw a lign segment from two given points, or to draw a circle from a fixed point as a center and a line as its radius, etc. A geometrical object is any object that we can construct in an iterative way using the postulates as rules. Since they are space-dependent, they are synthetic. This means that geometry, as far as constructions is concerned, is a temporal activity as well (even though its objects are spatial by nature). But arithmetic and algebra, as Kant states, "[...] entirely abstracts from the constitution of the object that is to be thought" (*CPR*,B745). This is because numbers are to be taken as rules, and not proper objects. Differently, geometry has objects (lines, circles, triangles, etc), while arithmetic has only schemas, which we use to define the discrete quantity of agregates, whatever they

[34]If this is right, Frege's criticism in (*GLA*,§89) is misleading in pointing out that a number like $1000^{1000^{1000}}$ would not be intuitable. Although, in Frege's defense, it's the foundational, and not the grasping of such number, that is the main problem with Kant's argument for such large quantities.

may be[35]. Arithmetic has no starting points as geometry has (the postulates): arithmetic is concerned solely with the notion of progressive iteration, *i.e.* time, and it's because of this, that the pure intuition of time is the *a priori* condition for the possibility of arithmetic.

1.6 Conclusion of Kant's proposal

It's hard to exaggerate the importance of time in Kant's philosophy, as well his considerations on arithmetic and numbers. Arithmetical judgements are necessarily synthetic, since the structure of numbers [36] is based in the structure of our pure intuition of time. Moreover, arithmetic is also *a priori*, because this structure is found in the pure part of our intuition. This explains why Kant rules out analysis as a sufficient procedure to derive arithmetical truths. The assumption, which defines his transcendental philosophy, is that only sensible objects are accessible for us, and hence, the *a priori* parts of our experience must be a product of our own minds, not an independent object that we only grasp. This is the case both for numbers as for series: we can only grasp an ordered series of objects insofar we "produce" such order in apprehending them, one by one, in time.

In that context, Kant's philosophy of mathematics is an attempt to solve both the applicability and foundational problems at once. Shabel (2005) describes it similarly as the problem of *applicability* and the problem of *apriority* of mathematics. Those were the main agenda for philosophers of mathematics in the modern period, spaning from Descartes to Kant: the apriority demand requires an explanation on the fact that mathematical features like universality, certainty and necessity are possible in an ontology that includes empirical objects. In the other hand, the applicability demand requires an explanation on the fact that, being universal, mathematics *is* applicable to empirical data.

This is the case for the applicability and apriority of the natural number series: how can such series be at the same time independent and applicable to experience. Kant's solution was a response against what he called the *mathematical investigators of nature* (especially Descartes and Newton) and the *metaphysicians of nature* (especially Leibniz). While the for-

[35] Algebra has, however, quantities that are not discrete, as the $\sqrt{2}$. In such cases, we can define $\sqrt{2}$ constructing the diagonal of a square, in geometry.

[36] And the operations defined on them.

mer, who assume space and time to be eternal and infinite subsisting real entities, fails on the applicability demand, the latter, who takes space to be relations of confused representations of real entities, fails on the apriority demand, according to Kant[37]. The superiority of Kant's proposal was the ability to solve both problems at once. That is, Kant's philosophy explains both (1) our ability to generate both series and numbers independent of all experience, that is, its foundational priority, and (2) our ability to apply such construction in the experience without loss of generality. Kant's argument, then, is "[...] about the connection between our way of intuiting, representing, and knowing the structure of space [and time] and our way of intuiting, representing, and knowing the features of the objects we experience to be in space [and time]: the former determines the latter" (SHABEL, 2005, p.48).

Finally, one could question the whole idea that numbers and series are time-dependent. As Friedman (1998, p.121) asks, even accepting that arithmetic involves progressive iterations, "[...] why should this idea, in turn, essentially involve time? Is not progressive iteration in fact a much more abstract concept than any temporal concept?". As it turns out, Frege was asking these exact questions.

[37]About the former, Kant argues that: "[...] they must assume two eternal and infinite self-subsisting non-entities (space and time), which exist (yet without there being anything real) only in order to comprehend everything real within themselves" (*CPR*,B56). Since in this account space and time are not appearances, their apriority is guaranteed. But their applicability fails because, by assuming such eternal and self-subsisting non-entities, they extend the realm of application to beyond possible experience. About the latter, Kant states that "[...] on this view the *a priori* concepts of space and time are only creatures of the imagination, the origin of which must really be sought in experience" (*CPR*,B56-57), which would fail in giving the *a priori* validity of mathematical knowledge, since such origin is only empirical.

Chapter 2

Frege's Proposal

Frege's proposal of a logical definition of an ordered series is based on his 1879 definition of the Ancestral of a relation, perhaps one of the most important achievements of his philosophy. In the *BS*, the ancestral of a relation is defined as an example of the capabilities of his logical notation. Moreover, such a definition has both logical and philosophical implications in Frege's program: it provides an example of the informativeness of his logic, which he believed to be absent from any form of intuition, and in the philosophy of mathematics, it aims to provide a logical derivation for a general principle of induction. It is also used in the definition of natural number and used extensively in proofs for the so-called basic laws of arithmetic. This was his first step toward his logicist program: the claim that arithmetic could be derived solely from logical principles. From a general principle of induction, one can derive the important principle of mathematical induction, provided that a logical definition for number is given, something he gave later in *GLA*. This results are also important in showing that Kant was wrong both in his notion of general logic and in his claim that arithmetic is synthetic *a priori* as a time-related procedure, precisely because no intuition is allegedly needed in Frege's proofs.

Our aim is to appreciate Frege's definition of the ancestral relation. For logicism to succeed, not only numbers has to be derived as logical objects, but it is important to also derive rules of reasoning in a logical matter as well. This is the case for the principle of mathematical induction. Since the Ancestral is the logical path for such, it must be possible to prove important facts about the desired relation. Furthermore, of much importance is to prove that the ancestral provides enough resources for linear-orderings. Proving those facts are also Frege's major point against intuition in his early philosophy. Frege takes the ancestral to be a logical notion, one that has maximum generality. Hence, the purpose was to provide a logical foundation for the ordinary understanding of the ancestral, and with that, a logical foundation for the notion of an linear-ordered series.

This chapter starts with Frege's philosophical background and motivations, specially his reinterpretation of the kantian terminology and disagreements. We also introduce Frege's conceptual-script logic, and discuss the philosophical topics regarding it. With that setup, the Ancestral definition will be postponed until the next chapter, alongside the important theorems that Frege proves about it.

2.1 Philosophical Motivations

Frege's philosophical influence is, today, undeniable. But since he was initially trained as a mathematician, this could be regarded as an accident of his more mathematical endeavour. But Frege's motivation was not only mathematical, but also philosophical. His interest in the foundational problem put him in the field of philosophy. This is the case for his three major works: *BS*, *GLA* and *GGA*. *GLA* is by far his most philosophical work and it shows that his motivations was highly philosophical[1]. His major interest is nowadays called *logicism*. In Fregean terms, logicism is the thesis that arithmetic has solely logical foundations. The details of such endeaveur was sugested in philosophical terms: to show the analyticity of arithmetical propositions. The first steps toward this thesis was already present in *BS*, in the definitions and proofs of part III regarding the ancestral of a relation. In *GLA*, the thesis

[1]It's regarded that Frege wrote *GLA* after Carl Stumpf suggested that he should write his proposals in ordinary language first, in order to be more accessible than *BS*. In a letter to Frege, dated from 1882, he wrote "I ask you whether it would not be appropriate to explain your line of thought first in ordinary language and then - perhaps separately on another occasion or in the very same book - in conceptual notation: I should think that this would make for a more favourable reception of both accounts" (*PMC*, p.172). I take that *GLA* is not more philosophical *because* Stumpf's suggestion, but because in it Frege is taking more time to elaborate the philosophical importance of his project. Benacerraf (1981) suggests differently that Frege's main purpose in *GLA* about analiticity is to give an account suitable enough for a mathematician's needs, which renders the conclusion that *GLA* is mainly mathematically motivated, and just acidentaly philosophical. As we see it, this doesn't take away the philosophical importance of Frege's book, neither the philosophical context of his endeaveurs.

is the main topic informally discussed and in *GGA* it is finally formally tested[2].

In *BS*, Frege raises the question about what should be the most secure foundation for a given proposition, answering that "The firmest method of proof is obviously the purely logical one [...] based solely upon the laws on which all knowledge rests" (*BS*, p.103). He also discusses his motivations: "to reduce the concept of ordering-in-a-sequence to the notion of *logical* ordering, in order to advance from here to the concept of number" (*BS*, p.104). Mainly, a secure foundation for arithmetic was already his goal. Two important philosophical topics are also present in *BS*: the rejection that intuitions have a foundational part in arithmetic, although in an obscure way, the analytic-synthetic distinction. There is no doubt that, in casting such terminology, Kant's philosophy was in Frege's mind.

This is further confirmed in *GLA*, where's logicism is then identified with the analyticity of arithmetic. This is Frege's philosophical motivation: "Philosophical motives too have prompted me to enquiries of this kind. The answers to the questions raised about the nature of arithmetical truths—are they a priori or a posteriori? synthetic or analytic?—must lie in this same direction" (*GLA*, §3). Latter, in the concluding remarks, Frege states his hypothesis:

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one. (*GLA*, §87)

In *GGA*, the most formal work and the epitome of Frege's career, this same pretension is in place, although not in the same extension as in *GLA*. There is no explicit discussion on philosophical topics in regard to the analyticity of the proofs. But Frege declares to be carrying out the project started in *BS* and *GLA*, namely, to "[...] make it plausible that arithmetic is a branch of logic and needs to rely neither on experience nor intuition as a basis for its proofs" (*GGA*, p.1). Moreover, Frege's purposes in logicism is also addressed in the letter to Anton Marty in 1882, where he declares to

> [...] demonstrate that the first principles of computation which up to now have

[2]With an unfortunate negative ending, following Russell's paradox.

> generally been regarded as unprovable axioms can be proved from definitions by means of logical laws alone, so that they may have to be regarded as analytic judgements in Kant's sense. (*PMC*, pp.99-100)

Also, in the 1885 paper *On Formal Theories of Arithmetic*, he more strongly suggest that "[...] there is no such thing as a peculiarly arithmetical mode of inference that cannot be reduced to the general inference-modes of logic" (*CP*, p.113), and that

> if arithmetic is to be independent of all particular properties of things, this must also hold true of its building blocks: they must be of a purely logical nature. From this there follows the requirement that everything arithmetical be reducible to logic by means of definitions. (*CP*, p. 114).

Hence, based in Frege's pretensions, we have the following assertions:

- The logical method of proofs does not rely on intuitions or experience, *i.e.* sensible data.
- This relates to kantian terminology: proofs that rely on intuitions are synthetic; proofs that rely on experience are *a posteriori*. Hence, a proof that does not rely on any of both is analytic and *a priori*.
- Arithmetic cannot rest his foundations in intuitions or experience, hence, a logical foundation must be found, *viz.* that arithmetic is analytic *a priori*.
- The way to show that arithmetic has logical grounds is:
  1. showing that arithmetical modes of inference are logical modes of inference;
  2. showing that definitions of important arithmetical concepts (the building blocks) can be reducible into a purely logical terminology.

Although Frege's project comes in kantian terms, there are major differences between the two. Frege's conception of logic differs from Kant's conception. As a consequence, Frege's intensions to prove the analytic *a priori* character of arithmetic deviates from Kant's famous thesis that arithmetical judgements are synthetic *a priori*. But despite such differences, they both agree that such judgements are contentful and that arithmetic is an informative science. We shall deal with these differences now.

2.1.1 The denial of Kantian Intuitions

According to our interest here, Frege's major philosophical motivation against the kantian thesis about mathematical judgements can be summarized in what may be called the *Content Principle* and the *Informativity Principle*:

> **Content Principle (CP)**: Logic has content of his own regardless of the subject matter (the domain that we assign to variables).

Since (CP) states that logic have a proper subject, it is to be regarded as a science. Therefore, logic can be informative about objects of its own:

> **Informativity Principle (IP)**: Logic is an ampliative science, that is, it can add content to our knowledge.

Fregean logicism, *viz.* the thesis that arithmetic has only logical grounds, can be regarded as a consequence of (CP) and (IP). The kantian thesis, as we already exposed, states that there is no proper logical domain regardless of the domain of sensible data (including both empirical and pure intuitions). For that reason, Kant denies (CP), and consequently, (IP). Such principles are the heart of the fregean-kantian dispute about arithmetic. But for the fregean argument to be sound, he has to give an account for the truth of both principles. As we shall see latter, the purpose of part III of *BS*, where the Ancestral relation appears, has this exact motivation.

Fregean defense of (CP) is completely in the domain of arithmetic. In *GLA*, the logical basis of arithmetic is closely tied with (CP), and Frege argues for the former also in terms of the second. The following summarizes the situation:

1. Concepts and objects are the most basic entities in Frege's Ontology[3], something that he might have borrowed from Kant's philosophy (that equaly devides all mental representations in General and Particular)

[3]The last of the three principles declared in *GLA*'s Introduction asserts: "never to lose sight of the distinction between concept and object".

2. Also following Kant, there would be only two ways in which an object can be given to us: through sensible intuition or logical intuition.

3. Kant denies the second, while Frege asserts the existence of objects which can be given whitout intuitions, *viz.* numbers.

4. Hence, for Frege, since numbers are not given through sensible intuition, they must be given to us through some kind of logical basis.

Therefore, numbers are the main case for Frege to assume the truth of (CP). This was, at least, Frege's belief. His appeal to extensions was central to maintain the thesis that numbers were logical objects. If it were not for the inconsistency of Basic Law V[4], (CP) would be true since numbers would be such logical contents. Hence, the above argument is only a hint: numerical words behave in such a way that numbers are to be regarded as objects.

Remember that Frege's position in *GLA* is built up in two parts: one *pars destruens* and one subsequent *pars construens*. The former is set up to stablished what numbers cannot be, where the latter to support what numbers in fact are, according to Frege's position. In (*GLA*, §45) he summarizes it:

> Number is not abstracted from things [...] nor is it a property of things [...]. Number is not anything physical, but nor is it anything subjective (an idea). Number does not result from the annexing of thing to thing. It makes no difference even if we assign a fresh name after each act of annexation.

What we see is Frege concluding his arguments against (1) some kind of abstractionism, (2) empiricism, (3) psychologism and (4) an extensional account of numbers as sets. Moreover, some features about the way we refer to numbers are taken as clues to the thesis that numbers are objects, *e.g.*, the use of definite article 'the' in number expressions like 'the number of cards is *n*'. In addition to that, numbers do not admit plurals: "When we speak of "the number one", we indicate by means of the definite article a definite and unique object of scientific study. There are not divers numbers one, but only one" (*GLA*, §38). We can

[4] Frege's axiom regarding extensions, or more precisely value-ranges, in *GGA*.

easily perceive the absurdities of any expression of the type 'The twos' or 'Some fives', Frege maintains.

All of these are indicatives that numbers behave like objects, since they cannot be properties (or concepts), cannot assume plural forms and always demands a definite article. But at the same time, they cannot be sensible entities. It is the kantian thesis about pure intuitions that remains as an option, which Frege promptly rejects as well. His rejection is based on the assumption that arithmetical claims has the most inclusive domain possible, *i.e.*, the domain of everything that is thinkable:

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. [...] The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. (*GLA*, §14)

There are two main topics that Frege is concerned here. First, it is to be taken as an evidence of the non-intuitional aspect of numbers that we can, according to him, enumerate or count anything. This is an applicability feature of arithmetic that we already pointed out earlier: every given set of elements, either empirical, imaginable, mental or intuitable can be enumerated, *i.e.* can be associated with a number. Second, this all-inclusive applicability relates to the fact that the denial of any arithmetical judgement is not imaginable. Not just a plain contradiction is derivable, but with a false arithmetical claim we could not even be able to think properly[5]. This is not the case with geometry, where intuition plays the foundational role:

> For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-

[5]We could prove that a statement like $1 = 0$ ensues a contradiction in a Fregean way. Since numerical statements are ascriptions about concepts, let 1 be the number of the concept F. Hence, there is one, and only one, object which falls under it (call it a). But since $1 = 0$, 0 is also the number of F's, hence, no object falls under it. From that, a is and is not an F. Contradiction. This is found in MacFarlane (2002), where he add that, since F is schematic, we could intanciate it to be 'circle' and derive an object that is and is not a circle, and hence, a contradiction in the field of geometry as well. Therefore, from an arithmetical falsehood, thinking becomes a confusion in all fields.

> contradictions when we proceed to our deductions, despite the conflict between our assumptions and our intuition [...] Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying any one of them, and complete confusion ensues. Even to think at all seems no longer possible. (*GLA*, §13)

The characteristics on how number-statements behave shows its logical grounds. Fregean *pars construens* starts with the observation that, following such numerical behavior, "[...] the content of a statement of number is an assertion about a concept" (*GLA*, §46). This is a fact that explains the aforementioned range of applicability of arithmetic, given that "[...] numbers are assigned only to the concepts, under which are brought both the physical and mental alike, both the spatial and temporal and the non-spatial and non-temporal" (*GLA*, §48). Conceptual words are often accompanied by indefinite articles or without articles in a plural form, like 'houses' or 'a house', and should not be mistakenly taken as proper names. The difference between conceptual words and proper names related to the prior distinction between concepts and objects. Anyhow, the applicability of a number to a given set of elements depends upon the concept choosen for it, and what Frege takes from such features is that numbers are regarded as objects, applycable to concepts: *e.g.* we can associate to a set of soldiers different numbers (as totality) depending on which concept we count, either a regiment, a platoon or soldiers.

But Frege takes such features for granted, and although this is not enough for a proper foundation, it shows Frege's position in contrast with Kant. MacFarlane (2002) gives an account of such distinctions[6], arguing that such features lead Frege to reject some basic kantian principles. According to him, in deducing the formality of logic from its generality, Kant made use of two principles which Frege would easily reject:

> (JO) Judgement is the mediate cognition of an object. (MACFARLANE, 2002, p.50)

> (OS) Objects can be given to us only in sensibility. That is, the only intuitions

[6] Our main descriptions of Frege's intensions in terms of (CP) and (IP) are mostly based in Macfarlane's paper.

> (singular representations) we are capable of having are sensible. (MACFARLANE, 2002, p.51)

Kant's thesis in (JO) asserts that in every judgement, there is a connection between a concept and an object. Where no object could be given, there is no possible judgement. Kant insists on the necessity of sensible intuitions in order for a judgement to grasp some content, but Frege rejects that in terms of the preceding assumption that statements of numbers are ascriptions about concepts: those are about concepts and not the objects numbered by the concept. From this possibility, as MacFarlane (2002, p.58), Frege can deny another kantian principle:

> (CO) For a concept to have content is for it to be applicable to some object that could be given in an intuition (singular representation). (MACFARLANE, 2002, p.52).

The denial of (CO) is what leads Frege to assert (CP)[7], and likewise, the acceptance of (CO) is what would make Kant reject (CP). Lastly, Frege's rejection of (OS) above is given in terms of the fact that numbers behave like objects, added with the thesis that numbers cannot be empirical or psychological entities. This was, nonetheless, a fregean leap of faith, since he maintains in *GLA* the belief that he could properly derive numbers as logical objects. As he declares in the (*GLA*, §89): "I must also protest against the generality of Kant's dictum: without sensibility no object would be given to us. Nought and one are objects which cannot be given to us in sensation". But since they can be given to us through ascriptions about concepts, Kant's dictum must be false.

This is Frege's defense of (CP). The truth of (CP) leads to the truth of (IP): since logic has its own objects, it is a science, and like any science, can be ampliative and informative in his proofs. And since we're dealing solely with logic, the analyticity of such proofs are also preserved. But Frege's position on the matter takes kantian terms into new interpretations. To better understand why Frege held (IP), we shall deal with this distinction first.

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[7]As long as logic is properly taked to be formal, *i.e.* in disregard to sensible data, (CO) is the opposing principle of (CP).

2.1.2 Fregean Analyticity

Recall that Kant's division into two logics (general and transcendental) are brought up to deal with (CO), as intuitions are concerned. A transcendental logic is built up as "[...] a science of pure understanding and of the pure cognition of reason, by means of which we think objects completely *a priori*", and that "[...] determine the origin, the domain, and the objective validity of such cognitions" (*CPR*, B81), *viz*, of pure *a priori* objects. Since Kant did not regard general logic capable of dealing with contents, it is incapable of explaining (JO), (OS) or (CO). Frege's rejection of such principles is, therefore, embedded in different considerations about what general logic is.

Frege wasn't kantian with regard to arithmetic mainly because his notion of general logic deviates from Kant's. With that, Frege held that Arithmetic is analytic, since he believed that no intuition was required to establish the truth of arithmetical statements. This involves at least three important topics: 1) although Frege states that he is assuming the notion in kantian terms, his definition for the analyticity of judgements are not quite the same; 2) his notion of definition has an important different aspect from Kant's notion, resulting from his logical inovations; 3) Frege held that arithmetic has maximum generality, being about everything thinkable. Since logic must hold the same generality, arithmetical truths must have a logical basis.

But there's some common ground between both, since Frege suggests that a truth is analytic if there is no need for special principles (i.e., non-logical) for justifying its truth. In (*GLA*, §3) he asserts that "Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement". Now, the way Frege regards such distinctions is specified in the following section:

> If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere

> of some special science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori. (*GLA*, §3)

Frege is talking about *truths* and hence worried about *how* a truth can be achieved, that is, its proofs. Thus, we have the following conditions:

1. A truth is analytic if its proof uses only logical principles;

2. A truth is synthetic if its proof uses, alongside logical, also non-logical or special principles, *e.g.* geometry requires principles that can be given only through some intuition, according both to Kant and Frege;

3. A truth is *a posteriori* if its proof use appeal to particular facts;

4. A truth is *a priori* if its proof is carried out without appeal to particular facts.

Following this, Frege and Kant agree that a judgement is analytic if its justification is based solely in logical principles. For Kant, the principle of non-contradiction is such a principle[8]: "we must also allow the *principle of contradiction* to count as the universal and completely sufficient *principle of all analytic cognition*" (*CPR*, B191). Then, if one is to analyze the concepts of mathematics in a judgement, according to Kant, one cannot realize its truth only by means of them: no analysis would be sufficient since mathematical objects are constructions (synthesis) of reason, and therefore, the principle of non-contradiction fails in those cases. As Kants puts it, in disregard of his objects, non-contradiction is a negative criterion for the truth of a judgement: if its negation is absurd, it must be regard as analytic. But whitout such a contradiction, "[...] it can still be either false or groundless" (*CPR*, B190). Although the principle of non-contradiction plays an important role (given that whitout it, one cannot reason at all), in regard of mathematical judgements, it's not a sufficient one. As

[8]His famous justification for the analytic/synthetic distinction as a matter of conceptual containment is based in such principle.

a consequence, a judgement like $7 + 5 \neq 12$ would not be a nonsense, even though it cannot be true, because to realize that, one must construct such objects and verify its falsehood, and by no means this envolves only analysis.

As it seems, Frege's definition of analyticity is not that different from the kantian. What changes is the way one can regard a truth to be analytic. First of all, Frege didn't agree with this kantian conclusion: one of the key features of arithmetic is that denying any arithmetical truth leads to contradictions, as exposed earlier. Even thought both regarded the question to be about the justification of a judgement, Kant seems to suggest that such justification lies in the content of his concepts, that is, we should look to the content of the concepts involved in a judgement in order to stablish what principle is involved in defining it to be true. If the principle of non-contradiction is such principle, the judgement is analytic, and synthetic if it's not.

The fregean rejection of such procedure is given in his anti-psychologism dictum: "always to separate sharply the psychological from the logical, the subjective from the objective" (*GLA*, xxii). Frege is rejecting not the kantian criteria for the justification of a judgement, but the necessity to look over what constitutes the concepts involved in the judgement in terms of contents (*i.e.* other concepts, his characteristic marks) in order to achieve such justification. Hence, the 'logical principles' claimed in the criteria for analyticity must be understood by two factors: (1) logical modes of inference and (2) definitions expressible using only logical terms. As we shall see, this two points, and how Frege takes them, are responsible for understanding the fregean acceptance of (IP).

2.1.3 Fregean Definitions, Conceptual-Contents

Definitions are, broadly speaking, concepts formation. In (*GLA*, §88) Frege states that Kant has underestimated the value of analytic judgements because his restricted notion of judgement[9] and a restricted notion of definition, where concepts are formed only by the union of different characteristic notes. Kantian definitions, as simple union of concepts, cannot extend knowledge whatsoever: if we define A to be a simple union of the concepts

[9]Where, e.g., existencial judgements are not expressable.

B and C[10], the judgements 'A is B' and 'A is C' are trivial analytic truths since nothing new is stated that was not given in the premises. Otherwise, judgements like 'A is D', which are in fact informative, are synthetic since the definition of A does not include D as one of its characteristic notes. In this sense, Kant's way of taking definitions perfectly fits with his notion of analiticity, beying in his case trivial.

Fregean definitions are stipulative signs, where a complex formula (*definiens*) fix the content of a simple sign (*definiendum*), which hasn't one, and where the complex formula can be variously analysable into the function-argument distinction. Fregean definitions preserve analyticity. As he states in *BS*, a definition is not a judgement, since it's a stipulation, but we can derive a judgement from it, one that simply "makes apparent again what was put into the new signs" (*BS*, §24), being in this case analytic. Moreover, we could derive the same theorems with or without the definitions, and this is a reason why definitions must preserve analyticity. From these considerations, we could easily believe that Frege regard definitions to be trivial, and hence, non-informative, just like Kant.

But Frege points out the importance of what he called *fruitful definitions*, from which we can obtain new concepts by the decomposition of complex judgements in a saturated and unsaturated part. Taking the example in *BS*, the proposition "hydrogen is lighter than carbon dioxide" can be decomposed to form new concepts, such as "being lighter than carbon dioxide" or "being heavier than hydrogen". This, of course, is only possible given Frege's substitution of the subject-predicate analysis by the function-argument one, another important feature of Frege's Logic. For fruitful definitions the matter rests upon what we can properly derive from them. As Frege points out: "The more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance" (*GLA*, §88). This is linked to (IP). The fruitfulness of a definition depends on which new concepts, or boundaries, could be achieved by means of the *definiendum*. As Frege puts it in *Boole's logical calculus and the Concept-Script*, "fruitfulness is the acid test of concepts" (*PW*, p.33). This means that definitions, albeit simple stipulations, can extend knowledge.

On the other hand, by taking a concept to be just the union of characteristic notes, which

[10]Which is actually the intersection between such concepts: $B \cap C$.

are also concepts, a definition for a new concept could be just stated as: $A =_{def} B \cap C$. That's just a way to say that something is an A if, and only if, it's both B and C, or, that the things defined to be A is all the things that belongs to the intersection between B and C. For example, the concept of 'husband' is just the intersection between 'maried' and 'man'. Not much could be achieved through such analysis of concepts. In *Boole's logical calculus and the Concept-script*, Frege characterizes such concept formation procedure using something like Venn diagrams, in which a concept is formed/defined by the intersection[11] of the extensions, represented by circles, of the original concepts (logical multiplication in Boole's logic). This is also, in Frege's mind, the kantian way of forming concepts, as he declares in (*GLA*,§88): "He seems to think of concepts as defined by giving a simple list of characteristics in no special order", and this is the same as the boolean representation of concepts through their logical operations[12]. The limitation of such is that:

> In this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot. (*PW*,p.34)

To overcome this, Frege's strategy is to focus on judgements. He declares to "start out from judgements and their contents, and not from concepts" (*PW*, p.16). That is, Frege does not consider concepts as given, to only then form judgements through them. He did not believe that concept formation was prior to the act of judging. This is how a fruitful definition, as concept formation, is possible.

[11] Or likewise with other logical operations, *e.g.* logical adition.

[12] Frege says: "If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do — in terms of our illustration — is to use the lines already given in a new way for the purpose of demarcating an area". (*GLA*,§88).

The question of whether definitions can extend knowledge is akin to the problem Frege begin his acclaimed *Über Sinn und Bedeutung*[13]: how can identity statements be informative? Frege's famous distinction between the sense and reference of a sign is given also to address this question, but things are different when such informativity is a result of some discovery, which would yield the identity synthetically informative. The question about the informativity *and* the analyticity of arithmetic, as present in *BS*, *GLA* and texts of this period, assumes a different aspect: no such sense-reference distinction is present. His earlier semantic notion is that of the 'conceptual-content'. The notion is of some considerable obscurity, since Frege was not clear what they really are, but the notion is nonetheless important in definitions, since they are performed with a sign for identity of conceptual-contents, and can be regarded of a relation between them. Albeit there are difficulties in understanding Frege's notion, we can recollect some of his thinking and shed some light on it.

Conceptual-Contents

Nowadays, with a model-theoretic approach to logic being the dominant trend, the relation between syntax and semantics becomes the centerpiece to any logical research. Looking at Frege's logic with this modern eyes, the relation between syntax and semantics is one of complete unity: in no place in Frege's logic an expression is taken as not interpreted, that is, every expression (which is well-formed) in the concept-script has a content (a thought, in the later semantics) associated with. As Shapiro (1991) puts it, Frege's logic was "fully interpreted"[14]. Another discussion relative to this question is the relation between a calculus and a language, that is, a deductive system and a semantics. This was, in fact, Frege's standpoint in regard to his notion of logic in contrast to the boolean algebrists. The dispute was declared into Leibnizian terms:

> Right from the start I had in mind the *expression of a content*. What I am striving after is a *lingua characterica* in the first instance for mathematics, not [just] a *calculus* restricted to pure logic. (*PW*, p.12)

[13]Translated as "On sense and meaning" in (*PMC*, p.157). We're opting for the usual 'sense and reference'.

[14]"[...] the logicists, including Frege, did not develop model-theoretic semantics, partly because their systems were fully interpreted. There was no non-logical terminology whose referents would vary from model to model" (SHAPIRO, 1991, p.11).

Frege's project was an attempt to complete Leibniz's project: the develpment of a logical system suitable for scientific purposes. In "Logic as Calculus and Logic as Language", Van Heijenoort famously called this feature as the *universality* of logic. In such, the introduction of quantifiers in *BS* features a great change and advance in comparison with boolean algebras[15]. Frege's ontology takes quantifiers as ranging over the whole universe of discourse, or as Heijenoort (1967, 325) says, "Frege's universe consists of all that there is, and it is fixed". In this universal conception of logic, sentences must have content and the formal system must be able to show its relation.

In Frege's conceptual-script, a judgement, which has logical priority in concept formation, expresses a content, *i.e.* something that is affirmed. In (*BS*,§2), this is regarded as "*a mere combination of ideas*", which in turn is composed of the logical relations between its components (functions and arguments). It's out of this that Frege defines the conceptual-content of an expression that is judgeable, and he does it in terms of logical consequence: "the only thing considered in a judgement is that which influences its *possible consequences*" (*BS*, §3). Moreover, he states that:

> I note that the contents of two judgements can differ in two ways: first, it may be the case that [all] consequences which can be derived from the first judgement combined with certain others can always be derived also from the second judgement combined with the same others; secondly, this may not be the case. [...] Now I call the part of the content which is the *same* in both the *conceptual content* (*BS*,§3).

More formally, this implies that the contents of two expressions φ and ψ are the same if, for Γ a set of sentences and S any individual sentence, the following holds[16]:

$$\Gamma, \varphi \vdash_F S \Leftrightarrow \Gamma, \psi \vdash_F S^{17}$$

[15]In Van Heijenoort's mind, boolean algebras represents only the propositional part of a logical calculus. This was, in fact, Frege's own criticism of Boole's logic, but this hardly fits into its later developments. For a good revision of Van Heijenoort's seminal distinction, see Peckhaus (2004).

[16]Taking $\vdash_F$ to be Frege's obscure consequence relation in *BS*.

[17]To give a simple example, consider the two judgements 'every single man is unmaried' and 'every maried man is not single', which we could express by the two different expressions $\forall x[S(x) \rightarrow \neg\exists y M(x,y)]$ and

This yields that the content of two judgeable sentences are the same just in case both are mutually inferable: since $\Gamma, \varphi \vdash_F \varphi$ and $\Gamma, \psi \vdash_F \psi$, and since φ and ψ have the same consequences, it follows that $\Gamma, \varphi \vdash_F \psi$ and $\Gamma, \psi \vdash_F \varphi$ (given that $\varphi \vdash_F \varphi$ and $\psi \vdash_F \psi$). Dropping Γ, we then have that φ and ψ have the same conceptual content just in case $\varphi \vdash_F \psi$ and $\psi \vdash_F \varphi$. But this can derive the undesired consequence that all theorems have the same conceptual content, as Kremer (2010) argues. This is actually the other direction of the consequence of the definition of conceptual content above. The argument (and the argument just showed above), however, pressuposes some logical tools that are unknown to Frege, specifically the *weakening* and *cut* rule for sequent-calculus[18]. The argument then runs as the following:

1. Assume that $\varphi \vdash_F \psi$ and $\psi \vdash_F \varphi$.
2. Assume that $\Gamma, \varphi \vdash_F S$. This yields, by weakening, $\Gamma, \varphi, \psi \vdash_F S$.
3. Using the same rule, $\psi \vdash_F \varphi$ yields $\Gamma, \psi \vdash_F \varphi$.
4. But then, using the cut rule, we derive from (2) and (3) $\Gamma, \psi \vdash_F S$.
5. Using the same reasoning, assuming $\Gamma, \psi \vdash_F S$ yields that $\Gamma, \varphi \vdash_F S$.
6. This satisfies the definition and hence, φ and ψ have the same conceptual contents.

Moreover, if this sequent-calculus notion of derivability is assumed, every logical truth is derivable from an empty set of premises, which implies that they are mutually inferable and hence, all logical truths would have the same conceptual content. This would be devastating for Frege's logicism and his thesis that analytic proofs can be nonetheless informative. However, as Kremer adverts correctly, Frege would not accept such notion of logical consequence: "For Frege, consequence, following from, is a *relation* between judgeable contents which enables one judgement to be *justified* on the *basis of others*" (KREMER, 2010, p.226). Frege's

$\forall x[\exists y M(x, y) \rightarrow \neg S(x)]$ respectively. Now, assuming as premise that 'x is single', we could derive that 'x is unmaried' from both judgements. Likewise, if we take that 'x is unmaried', both will derive that 'x is single'. Hence, they have the same conceptual content, even though expressed in different sentences.

[18]The *weakening* rule states that superfluous premises can be added to a proof without invalidating it. The *cut* rule states that, if $\varphi \vdash_F \psi$ and if $\varphi, \psi \vdash_F S$, that is, if a premise is an implication of the other, we might cut it and assume $\varphi \vdash_F S$.

system of logic is strange to the notion of logical laws as derivable from an empty set of premises, as it is to conditional (hypothetical) and *reductio ad absurdum* proofs. In Frege's view, logical laws express contents, and it's from them that new laws are derivable. In *BS*, theorems are proved from a set of axioms and rules, and so they're said to be "contained" in them. To a theorem to be "contained" solely in axioms and definitions means that it's analytic. The discovery of such relation yields information as well. As Kremer puts it,

> 'Consequence' is already for Frege a notion with epistemological import. Deducing consequences from basic logical laws is a process which generates content, insofar as the conclusions we deduce are contained in the basic laws collectively, but not individually. (KREMER, 2010, p.229)

This implies that in Frege's proofs of *BS*, there is an order of logical laws such that one is a result of the other, and this is made possible because the relation between the conceptual contents of them.

The notion of conceptual content is latter changed by the well-known distinction between sense and reference of sentences, and this is significant to understand what Frege thought conceptual contents suppose to be in the first place. He declares in a letter to Husserl that "What I used to call judgeable content is now divided into thought and truth value" (*PMC*,p.63). In *GGA*, Frege also notes the following:

> What was formerly the content-stroke reappears as the horizontal. These are consequences of a deep-reaching development in my logical views. Previously I distinguished two components in that whose external form is a declarative sentence: 1) acknowledgement of truth, 2) the content, which is acknoledged as true. The content I called judgleable content. This now splits for me into what I call thought and what I call truth-value. This is a consequence of the distinction between the sense and the reference of a sign. (*GGA*, *foreword X*)

So according to Frege himself, we should take the notion of conceptual-content[19] as a kind of combination of both sense and reference of a sign. To acknowledge a truth is to perform

[19]Frege actually talks about judgeable contents, but in the context of *BS* this are conceptual-contents that are of possible judgement. Their difference will be latter addressed.

a judgement over a conceptual (judgeable) content. For the mature distinction, a judgement "[...] in the narrower sense could be characterized as a transition from a thought [sense] to a truth value. [reference]" (*PMC*,p.63), that is, the sense of a expression somehow determines it's reference (or lack of). The same should be expected from conceptual-contents.

In *BS*, while speaking about what makes two distinct sentences have the same consequences, Frege uses the term 'sense' [*sinn*]. In explaining the need for a sign for identity of contents, he adds that a content can have "[...] two different *modes of determination* [*bestimmungsweisen*]" (*BS*, §8), that is, two different expressions can name the same content, or likewise, have the same logical consequences. This is very close to what he latter will call a "mode of designation [*Art des Gegebenseins*]" (*PMC*,p.157) of a sign in *Über Sinn und Bedeutung*, that is, it's sense. An expression has a 'mode of determination' or 'mode of designation' associated with, which in turns determines the object designated by the expression. Hence, we can conclude that in *BS*, the notion of conceptual-content includes an obscure notion of sense which is what determines the possible consequences of the thing being designated by the original expression. Accordingly, there should be a correlate part for the notion of reference of an expression as well. But in *BS* Frege had not yet identified the Truth and the False as references for expressions. He uses expressions like "the circunstance of [*der Umstand*]", "the case of [*der Fall*]" and "the proposition that [*der Satz, dass*]" (*BS*,§3), and he also often reads "is a fact [*ist eine Thatsache*]" when a judgement is confered to such content. This seems to suggest that, if there was a referent component in the conceptual-content notion, it plays a much closer role to something like russelian propositions than the latter identification of truth-values as objects[20].

Despite the differences, conceptual-contents are something like (but not exactly as) the latter distinction Frege has made in his logic. It was its imprecisions and problems especially with the identity of conceptual-contents signs that made Frege rethink this semantic notion.

[20]In a letter to Frege, Russell states against Frege's position about the designation of a proposition: "I believe that in spite of all its snowfields Mont Blanc itself is a component part of what is actually asserted in the proposition 'Mont Blanc is more than 4000 metres high'. We do not assert the thought, for this is a private psychological matter: we assert the object of the thought, and this is, to my mind, a certain complex (an objective proposition, one might say) in which Mont Blanc is itself a component part. If we do not admit this, then we get the conclusion that we know nothing at all about Mont Blanc. This is why for me the meaning of a proposition is not the true, but a certain complex which (in the given case) is true" (*PMC*, p.169). This is similar to what Frege once held in *BS*.

But such sign is crucial for definitions. Since the conceptual-content is not a syntatical notion, Frege's sign for *identity of contents* is introduced as a relation between sentences: it asserts that two expressions has the same conceptual-content, that is, have the same possible consequences[21]. With that in mind, definitions in *BS* are stipulations that fix a certain conceptual-content to a new introduced sign, through the sign for the identity of (conceptual) contents ($\equiv$). Definitions are also preceded by a double vertical stroke, in the form:

$$\Vdash \Gamma \equiv \Delta$$

Where Γ is the *definiens* and Δ is the *definiendum*, the defined sign. All expressions preceded by $\Vdash$ are not judgements. Otherwise, all definitions would be automatically synthetic. They are just conventions. We should, still, give an account for how fregean definitions, provided by the sign for identity of conceptual-contents, support (IP).

2.1.4 Logic and Informativity

There is a great deal of interpretations on how fregean account on definitions can support (IP)[22]. Gregory Landini (1996, 2012), offers one that regards such feature as an outcome of (1)

[21] The usage and problems of such sign will be addressed later.

[22] Here we just name a few.

- Danielle Macbeth (2005, 2012), argues that Frege's logical notation functions like an euclidian diagram: it shows the inferentially articulated contents of concepts as they matter to mathematical reasoning. She argues that Frege's concavity notation for generality is a sign for marking the subordination under higher-level functions. This, togheter with definitions, help articulating deductions about the defined concept, revealing diagramatically the subordination of such concepts under higher level concepts (or functions). This is what, in Macbeth's thesis, makes a fregean deduction informative.
- Ruffino (1991) explains such feature in terms of Frege's contextual principle in its epistemic role, that is, in concept-formation. In Ruffino's account, the construction of concepts through decomposition of judgeable contents (*i.e.* the Apriority Principle) is something governed by the context principle, where new and unexpected concepts are acquired. Hence, informativeness results.

As I take it, much of both accounts can be better explained taking as basis Frege's undeclared rule of substitution. In Macbeths account, Frege's proofs "[...] reveals logical relations among concepts and does so by combining, in joining inferences, parts of different wholes into new wholes" (MACBETH, 2012, p.307), which in turn yields the informativeness of logic. This is Frege's rule of substitution at work. In Ruffino's account, the Priority Principle is crucial, and this is something that Frege's certainly does in applying such rule. But there is no need for the context principle: definitions, together with such tool, helps Frege in instantiating new conceptual relations in which higher-order properties of the defined concepts are proved. Those are not components of the original defined expressions, hence, they are informative. Despite the differences, the common ground which I shall pursue is the Priority thesis and the decomposition of judgements into the function/argument analysis.

the sense/reference distinction added to (2) axioms schemas for comprehension in second-order logic. As we declared above, Frege's earlier semantic notion is that of conceptual-contents. Therefore, we shall assume Landini's interpretation in the context of this notion, applying a few amendments.

One of the key features of Frege's logic is the function-argument analysis. This inovation, added to the fregean principle that judgements precede concepts, made it possible to regard concept formation as an act of decomposition of expressions of the language. In the letter to Anton Marty, Frege asserts that:

> I do not believe that concept formation can precede judgement because this would presuppose the independent existence of concepts, but I think of a concept as having arisen by decomposition from a judgeable content. (*PMC*, p.101)

This is the *Priority Thesis*, that is, the thesis that "[...] complete judgements are undifferentiated into parts until precisified by concepts" (LANDINI, 1996, p.122). Moreover, it's the decomposition of expressions, which yields new functional expressions, that is the proper concept formation for Frege. This is a key feature for (IP):

> The power and importance of the priority thesis-understood as the possibility of alternative decompositions lies in its providing the philosophical justification for the comprehension of functions in Frege's formal system. (LANDINI, 1996, p.136)

Frege's logic extensively use one non-stated rule of inference nowadays called the *rule of substitution*. It allows Frege to, given a sentence, say $P \to (Q \to P)$, substitute uniformly any variable letter, *e.g.* P, for another simple or complex expression, say $(P \to P)$, deriving $(P \to P) \to (Q \to (P \to P))$. This is used both in the propositional level, such as this example[23], and for functional expressions. In this case, we use a parametric letter to fix the unsaturated part of the function to be substituted. From the sentence $\forall x(f(x) \to f(a))$, we could substitute $f(\Gamma)$ for $g(\Gamma) \to h(\Gamma)$, deriving $\forall x((g(x) \to h(x)) \to (g(a) \to h(a))$[24], using Γ as a parametric letter to fix the variable position. It's this rule of substitution that,

[23]This is a reason why Frege's logic did not have schematic axioms, nor did it need to.
[24]Frege makes this exact substitution to prove theorem 59 of *BS*.

following Boolos (1985), is equivalent to comprehension principles for second-order logic. In Landini's mind, this is what Frege needs to assert (IP):

> Looking back with the lenses of modern logic involving comprehension, it is quite clear that what Frege called "fruitful definitions" involves comprehension. Frege understands that definitions are stipulative conveniences of notation that are non-creative and wholly eliminiable. At the same time, the comprehension of new functions was effected in his systems by adopting principles for "defining" (or better forming) well-formed complex function terms of the language, and then using special parametric letters (not part of his object-language) which facilitate the rules for substituting complex function terms for primitive function terms such as fx, in the axioms of the formal system. (LANDINI, 2012, p.23)

This way of looking into Frege's logic pressuposes two kinds of "definitions": one stipulative and another decompositional. The first one is a convenience for manipulating proofs, but added to the decompositional one, it could help achieve new functions (or to form new concepts). Hence, a defined relation like $R(x, y)$ could be decomposed in the second variable $R(x, \Gamma)$, and using the substitution rule to derive a new complex judgement regarding the definition.

Landini's interpretation depends on another assumption, that the parametric letters used are *not* parts of the formal language, but used only to facilitate the decomposition and manipulation of the function newly acquired (LANDINI, 2012, pp.137-138). It's crucial in the sense/reference semantics that, for (IP) to be true, the sense of the new function decomposed is not part of the sense of the original expression. Otherwise, there would be no informativity. Therefore, he maintains the sense of a relation such as $R(x, y)$ is dependent solely on the senses of his basic components, *viz.* the sense of the binary relation $R(\Gamma, \Delta)$ and the senses of the two arguments x and y. Basically, the same usage of parametric letters is present in *BS*. Hence, when Frege decomposes from a judgeable content like $R(x, y)$ to acquire $R(x, \Gamma)$, this is not an original component of the first expression.

Landini's account applies better to Frege's mature semantics, but it can be extended to the conceptual-content semantics of *BS* and *GLA* as well. In that case, decomposition is applyed to judgements (judgeable contents), deriving new functional expressions using parametric letters. Since such decompositions are taken in the syntactic part of the language, one acquires new conceptual-contents, as long as they are judgeable contents as well. In that sense, the fruitful definitions and proofs are combined to express relations between contents. The informativity is due to the fact that certain contents are consequences of others, something that we could not trivially find out by just looking for the components of the judgeable content in hand. Hence, when Frege assumes (IP) to be true, he is answering Kant's thesis that analytic judgements are never ampliative. To Frege's mind, this is a deficiency of Kant's too narrow notions of judgement and definitions.

This is the case with the Fregean definition of the Ancestral of a Relation, a definition that was forged both to show Kant's limitation on logic and the capabilities of his own concept-script logic in terms of (IP), in showing that some important mathematical modes of inferences could be derived solely on logical grounds. Frege's Ancestral definition is a formalization of the basic notion of a series. In that sense, proving that such notion is logical in his essence is also a witness of (CP), as long as his definition is suitable to prove important theorems about it, which we could as well consider as properties of the definitions.

2.2 The Logic of the Concept-script

The importance of Frege's *BS* is well stablished in the history of logic as one of the firsts axiomatizations for propositional and quantificational logic. As we stated earlier, although Frege's logic is a 'concept-script', the basic starting point are judgements. The main task of the symbolism is to describe and exhibit the logical relations concerning such judgements, its contents. Precisely, what constitutes such contents are functions associated with the corresponding arguments.

The tradition before Frege takes judgements to be the classic aristotelic form of subject-predicate. Aware of the limitations of such form, Frege's way of dealing with judgements goes well beyond this. All features of judgements in the classical analysis are taken as features of the contents of judgements (not of judgements themselves), as exposed in (*BS*, §4): properties like universal, particular, negative, disjunctive, and others alike, are actually parts of contents of judgements. Now, the part of such content of judgements which are objective[25], Frege considers to be the aforementioned *conceptual-content*. To express this conceptual-content in the language, Frege introduces the content-stroke:

$$— \varphi$$

But not every conceptual-content is expressible. Those which can are called *judgeable contents*[26]. The content-stroke is only applycable to judgeable-contents. These are, simply put, contents from which a judgement can be made[27]. Frege is not clear in this matter, but as we take it, individual variables or constants alone and function variables without an argument are those contents regarded as unjudgeable. This might be a sign for specifying those formulas which are well-formed from those which are not, *viz.*, it is a syntactic notion[28] . In (*BS*,§2) Frege writes that "the idea 'house' cannot" be flanked with the content-stroke, but "the circunstance that there are houses would be an assertible [judgeable] content. (see §12) But the ideia 'house' is just a part of this". Here, §12 is where Frege explains existential judgements. Using modern symbolism, this is $\neg\forall x \neg H(x)$. There is evidence that concepts without sharp boundaries also are regarded as unjudgeable. In providing an example for theorem (81), Frege uses the concept 'heap', affirming that "there are certains z's for which, because of the indeterminateness of the concept 'heap', $F(z)$ is not an assertible [or deniable] content" [*BS*, §27]. In sum, the set of conceptual-contents is not the same as the set of judgeable contents, but in the concept-script language, only the overlap between them is considered.

Following a conteptual-content (that is also judgeable), the judgement of such content is expressible through the famous judgement-stroke:

$$\vdash \varphi$$

[25]That is, the part which does not rely on psychological grounds, or as Frege asserts, on "all aspects of ordinary language which result only from the intereaction of speaker and listener" (*BS*, §3).

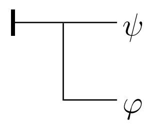
[26]This distinction is substituted in *GGA* by the horizontal function, which designates a truth-value to every term in the system.

[27]This is very much like contents that can have truth-values, but this was not already recognized in *BS*.

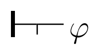
[28]There is no place in *BS* where Frege specifys the notion of well formed-formulas. For example, he's not clear if sentences like '$P \rightarrow$' are to be regarded as non-judgeable contents, but it's seems pretty clear that they are.

This asserts that 'φ is a fact', in fregean terms. The role and meaning of this sign is well discussed in the literature, something that we bypass here, only assuming that this asserts a judgement or the recognition of a formula as a fact[29]. In the context of *BS*, this is given in two ways: either the given sentence is an axiom, or it's provable from axioms, theorems and definitions.

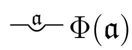
The *BS* notation is two-dimensional. In it, the logical operators considered are the conditional:



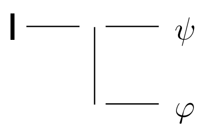
The negation:



And the universal quantifier:



Note that for every operator, we have content-strokes for every part to which it's applicable and the operator itself. Hence, for the conditional, we have the following constituents:



That is, from left to right, the judgement-stroke, a content-stroke for the conditional, the conditional-stroke, and the content-strokes of each subcomponent and the subcomponents themselves. The same form of compositionality holds for the negation, the universal quantifier, and every complex formula that can be composed through them.

Frege's sign for identity is the already mentioned identity of conceptual-contents:

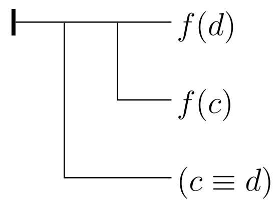
$$\vdash (\varphi \equiv \psi)$$

From this sign, according to Frege, "a bifurcation is necessarily introduced into the meaning

[29]Once again, we could say that this is a sign for the truth of a formula, but this was not the case already in *BS*.

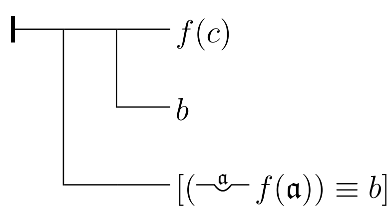
of every symbol" (*BS*, §8), given that it relates to names and not to contents. This sign is, however, the source of much trouble in Frege's early logic. They are the following:

1. The bifurcation is problematic, since in $\vdash\!\!-\!\!-(\varphi \equiv \psi)$, the names φ and ψ are being mentioned, and not used. Consider the axiom (52):

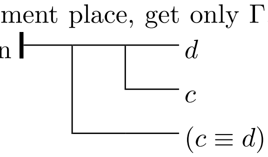
   

   The bifurcation implies that in $(c \equiv d)$ we have names being mentioned and in $f(c)$ and $f(d)$ the same names are being used.

2. Is not clear if the identity of conceptual-contents must be applicable only to judgeable contents or not. In fact, Frege seems to use it in both senses. A name in *BS* can be either of a judgeable expression, *i.e.* a well-formed formula in the system, or could denote an object, in which case it is an individual constant or variable. But Frege uses the identity sign without discrimination. In theorem (68):

   the variable b, flanking the identity, cannot denote an object, otherwise, the second antecedent, $-\!\!-\!\!-\, b$, would not be a judgeable content. Also, Frege often speaks about names as standing for objects, which is the case *e.g.* in definitions (99) and (115), where the terms flanking the identity seems to denote objects, and not judgeable contents[30]

[30]To make it more clear, very often Frege's rule of substitution allows him to substitute one function for its argument, that is, from $f(\Gamma)$, where Γ is used only to demarcate the argument place, get only Γ. From this, and from axiom (52) above, one could simply change $f(\Gamma)$ for Γ, to obtain  But in this case, c and d must denote judgeable contents, otherwise this would be an ill-formed expression: just let

Nowhere after *BS* Frege use this sign again. Even in the unpublished paper *Boole's logical Calculus and the Concept-script* of 1881, Frege already adopted the usual identity sign $=$. Following Landini (2012, p.44), this seems to suggests that after *BS*, Frege first starts to adopt both signs in order to avoid the bifurcation of its meaning[31]. In fact, Frege writes:

> I too have an identity sign, but I use it between contents of possible judgement almost exclusively to stipulate the sense of a new designation. Furthermore I now no longer regard it as a primitive sign but would define it by means of others. (*PW*,p.35-36).

It's not at all clear why Frege regarded his identity sign as definable from others[32], but what is noteworthy is his already present discontent with it. From that perspective, both signs could be assumed, taking the usual identity sign for identity between objects, as $a = b$, and the identity of conceptual-contents for only when the terms flanking the sign are judgeable contents, as $\varphi \equiv \psi$[33]. We'll adopt this alternative latter.

In sum, Frege's primitives of the concept-script logic already come with an implicit compositionality principle. The rules binding the formation of judgeable contents, together with their interpretations[34], can be expressed as the following:

1. *Content-Stroke*: If φ is a judgeable content, then —— φ denotes the conceptual content of φ.

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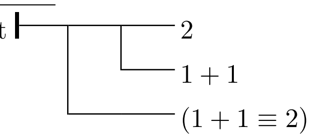
c to be $1+1$ and d to be 2, we then get 

which is clearly ill-formed, since —— $1+1$ or —— 2 are not judgeable contents.

[31]Until the complete abandonment of the $\equiv$ sign in the 1890's.

[32]He could have expected to define it as a biconditional, since this is basically the role it assumes in definitions.

[33]This is very similar to Frege's latter distinction between sense and reference, since $a = b$ seems to relate references and $\varphi \equiv \psi$ seems to relate to senses. I've already discussed this above, but once again, in introducing the horizontal function in *GGA*, Frege states that "Earlier I called it the *content-stroke*, when I combined under the expression 'judgeable content' that which I now have learnt to distinguish as truth-value and thought" (*GGA*, p.9, fn.2). The fact is that Frege's sign for identity of conceptual-contents is clearly problematic. To more problems on Frege's $\equiv$ sign, see Mendelsohn (1982).

[34]Once again, Frege's logic does not pressupposes a metatheory. There can be no 'outside' logic rather than the formal system of *BS*. This was one of the points made by Van Heijenoort: "Another important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasystematic question (consistency, independence of axioms, completeness). Frege is indeed fully aware that any formal system requires rules that are not expressed in the system; but this rules are void of any intuitive logic' (HEIJENOORT, 1967, p.326).

2. *Judgement-Stroke*: If $—\,\varphi$ is a recognized/provable true[35] content, then $\vdash\, \varphi$ denotes the judgement of $—\,\varphi$.

3. *Negation-Stroke*: If φ is judgeable, then $\neg\,\varphi$ denotes the content of the negation of the content of φ. If a judgement stroke is preceded, then the negation denotes the content of φ not being the case.

4. *Conditional-Stroke*: If φ and ψ are judgeable, then $$\begin{array}{l} —\!\!\top\!\!—\, \psi \\ \quad\ \, \llcorner\, \varphi \end{array}$$ denotes the content of φ being denied or ψ being affirmed[36].

5. *Generality*: If Φ is judgeable, then for some free a in Φ, $\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$ denotes the content of $\Phi(\mathfrak{a})$ being the case whatever $\mathfrak{a}$ denotes.

6. *Identity of Conceptual-contents*: For any φ and ψ, judgeable or not, $—\,(\varphi \equiv \psi)$ denotes that the signs 'φ' and 'ψ' have the same conceptual-content[37].

This rules specifies the set of possible judgeable expressions in Frege's logic. Moreover, Frege's notation was designed in a two-dimension way for the purpose of better visualizing deductions and to facilitate manipulations. As he argues against Schröder's review of *BS*, writing the formulas in different lines helps one to visualize the logical relations in question, avoiding the clumsy formulas when written in a single line[38].

His approach is also an axiomatic one. The axioms of the concept-script for propositional and quantificational logic are:

- Propositional axioms:

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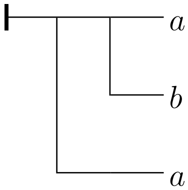
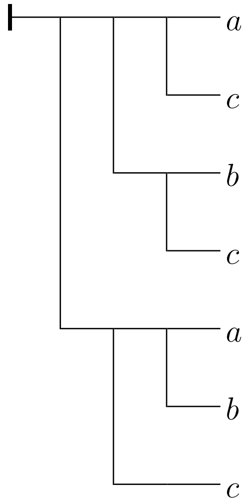
[35]In *BS*, Frege always speaks about "being the case" rather than "being true".

[36]Frege defines the conditional stroke rather from the condition where φ being affirmed and ψ being denied is denied.

[37]Here, φ and ψ can denote either individuals or complex judgeable contents in *BS*.

[38]In "Boole's logical Calculus and the Concept-script" Frege writes: "In fact I am in complete accord with usual practice; for in an arithmetical derivation too we put the individual equations in succession one beneath the other. [...] Now what I set beneath one another are also contents of possible judgement, or judgements. [...] We thus make use of the advantage that a formal language, laid out in two dimensions on the written page, has over spoken language, which unfolds in the one dimension of time". Moreover, the writting of formulas in a single line "[...] has the consequence that it would be extremely difficult to grasp what was going on" (*PW*, p.46).

(1)


(2)


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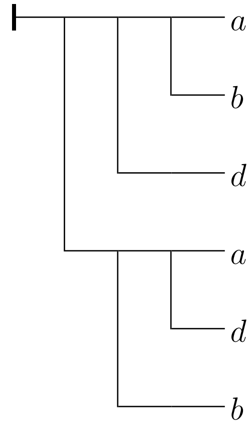

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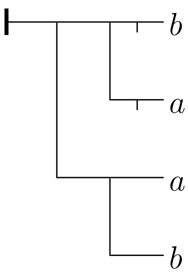

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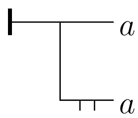

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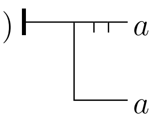
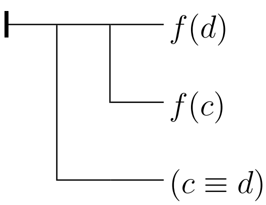


- Axioms governing the identity of conceptual-contents sign:

(52)


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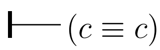

- And an axiom governing universal instantiation:

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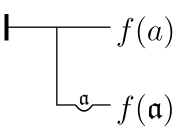
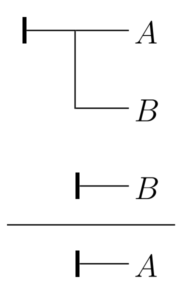


The only, declared, rule of inference was *modus ponens*:



But this is clearly not the only one, as we already stated. For some reason, Frege did not consider the rule of substitution as a rule. Since all variables in the concept-script logic denote generality, any judgeable formula can be instantiated to other specific judgeable contents as its variables. This rule is easier to state using comprehension axioms for second-order logic, but this approach makes use of schemas. This is not the approach followed by Frege. Schemas are formulated in the metalanguage of a logical system, and no metalanguage is avaiable in Frege's universal approach to logic.

Furthermore, the rule of substitution is more intuitive and easy to perform when applied to the propositional calculus. The difficulty arises with the quantification part of logic[39]. Special parametric letters are necessary to demarcate the argument position in every formula being substituted, and this are not part of the language, and seems not to be part of it in Frege's use of them. Both in *BS* and more precisely in *GGA*, uppercase greek letters are only used to fix the place of the argument in the function. Formulas containing them are not considered to be in the language. In (*BS*,§2) Frege calls them "abbreviations" for any sense that might be the case for them. In (*GGA*,§1) he then says that for parametric letters like 'ξ', "nothing is meant to be stipulated for the concept-script. Rather, 'ξ' itself will never occur in the concept-script developments".

To follow Frege, and the complexity of his notation and use of the rule of substitution, would be costly. The approach here will only assume the minimal system necessary to explain Frege's use of such rule, which is the main responsible for (IP), in the context of the theorems proved in *BS*[40]. Since, as it was already mentioned, Frege's approach to judgeable-contents could be understood as separating the well-formed expressions to other ill-formed ones, his logic will be present in the more familiar and usual way, starting with vocabulary, well-formed formulas, rules of inference and axioms. The more direct and easy, but also strange to Frege, proof method using hypothesis will be adopted.

We also shall use the modern symbolism for Frege main operators: $\rightarrow$ for conditionality, $\neg$ for negation and $\forall$ for the universal quantifier[41]. Meanwhile, the judgement-stroke will

[39] This is pointed out by Landini (2012, pp.19-21)

[40] Landini himself hints that "[...] hybrid systems are possible for axiomatization. That is, one can have a system that uses axioms and a rule of uniform substitution for the propositional calculus and then switch to using axiom schemas for quantification theory" (LANDINI, 2012, p.20)

[41] Assuming that this 'translation' do not make violence to Frege's perspective. Interpretations like those

be used for axioms and theorems and the double vertical stroke will be used for definitions. Also, we're assuming the bifurcation of the identity sign: the regular identity will be used as a relation between objects and Frege's sign $\equiv$ will be used for judgeable contents, only in definitions. This assumptions should facilitate the exposition of Frege's acchievements in *BS* and *GLA* without, we hope, altering Frege's philosophical motivation in (IP) and (CP).

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of Macbeth (2005) and Landini (2012) assumes that Frege's unusual notation has important features that cannot be translated into modern notation without some loss. But a middle ground is achievable. This should not alter the relevant philosophical side of Frege's informativity thesis.

Chapter 3

Frege's Ancestral

In this chapter, Frege's Ancestral definition will be finally introduced and properly discussed, following the philosophical discussion in chapter 2. We start unusually by departing from Frege's own logic, by describing the minimal second-order logic enough to appreciate the desired Ancestral results. For philosophical reasons already discussed, we avoid the semantic part of such calculus, assuming the usual interpretation of them, as pointed out in (2.2). After discussing Frege's Ancestral, his importance and logical aspects, we shall prove some theorems regarding it, and finally, how they may be seen as a anti-kantian argument regarding ordered-series. We'll close the chapter by discussing the intended proofs given in *GLA* for the basic laws of arithmetic, in which the ancestral plays a significant role.

3.1 Concept-script logic reconsidered

The following is the usual second-order logic with identity. But we assume functions to have foundational priority over concepts. In what follows, the vocabulary, formulas, rules and axioms are described[1].

[1]A note on the interpretation of Frege's own logic is necessary: (1) A judgeable content without further specification could be said to denote a propositional content. (2) although in *BS* Frege's theory of functions was not yet fully developed, it's safe to assume that those have priority over concepts, which are functions of a specific kind (In *BS*, Frege had not yet defined a concept as a function that has only truth-value as values. But, since the function-argument analysis is already present, we shall take, logically, concepts as unary functions). (3) In this reading, 'properties' or 'concepts' are unary functions yielding judgeable contents when saturated (this is the same as saying that they have truth-values), 'relations' are binary functions which are also judgeable. (4) First-order functions take objects in its domains and judgeable contents in its range (Save for the case when functions don't describe concepts, for in this case they have unjudgeable contents in its

1. **The vocabulary**:

   We shall use,

   (a) Lowercase roman letters for objects: variables $x_1, ..., x_n$ and constants: $a_1, ..., a_n$ for some denumerable n. (informally, x, y, z for variables and a, b, c for constants).

   (b) Lowercase roman letters $f_1^n, ..., f_m^n$ for first-order functions, for some denumerable m, n (informally, f, g, h).

   (c) Uppercase greek letters $M_1^n, ...M_m^n$ for higher-order functions, for some denumerable m, n (informally M, Ω).

   (d) Logical constants $\rightarrow$, $\neg$ and $\forall$.

   (e) Round and square brackets '(','\)','[','\]' for disambiguation.

   Definition (First-order Concepts)**.** *For $x_1, ..., x_n$ terms and f^n a function, if $f^n(x_1, ..., x_n)$ is a judgeable formula, that is, has a truth-value when saturated, then f^n is said to denote a Concept.*

   Observation 1: If $n = 1$, f^1 is a regular concept. We'll use F, G, H informally for such.

   Observation 2: For any $n \geq 2$, f^n is a n-ary relation. We'll use R, S, T informally for such.

   Definition (Second-order Concepts)**.** *For $f_1^n, ..., f_m^n$ first-order n-functions and M^m a second-order function, if $M_\beta^m(f_1^n(\beta), ..., f_m^n(\beta))$ is a judgeable content, that is, has a truth-value when saturated, then M^m is said to denote a second-order Concept.*

2. **Terms and Formulas**:

   (a) Individual variables and constants are terms;

   (b) If $\alpha_1, ..., \alpha_n$ are terms, for any denumerable n, and if f^n is a function that is not a concept, then $f^n(x_1, ..., x_n)$ is a term.

   (c) If α_1 is a term and F^1 is a concept, $F^1(\alpha_1)$ is an atomic formula;

range). Second-order functions can be taken to be from judgeable-contents to other judgeable-contents.

(d) If $\alpha_1, ..., \alpha_n$ are terms and R^n a n-ary relation, for any denumerable n, $R^n(\alpha_1, ..., \alpha_n)$ is an atomic formula;

(e) if $f_1^n, ..., f_m^n$ are first-order functions, and M^m a second-order function, then $M_\beta^m(f_1^n(\beta), ..., f_m^n(\beta))$ is a formula.

(f) If φ and ψ are formulas, then $\varphi \to \psi$, $\neg\varphi$ and $\forall x\varphi$ (for a free term x in φ) are formulas.

(g) There are no other formulas.

Definition (Derived operators)**.** *We add the following defined operators, for simplifying the derivations:*

(a) Conjunction*: $\varphi \wedge \psi$ holds if, and only if, $\neg(\varphi \to \neg\psi)$ holds;*

(b) Disjuntion*: $\varphi \vee \psi$ holds if, and only if, $\neg\varphi \to \psi$ holds;*

(c) Existencial quantifier*: $\exists x_1, ..., x_n\varphi$ holds if, and only if, $\neg\forall x_1, ...x_n\neg\varphi$ holds, for $x_1, ..., x_n$ free in φ.*

(d) Biconditional*: $\varphi \leftrightarrow \psi$ holds if, and only if, $(\varphi \to \psi) \wedge (\psi \to \varphi)$ holds.*

We then define the relations $=$ and $\equiv$:

Definition (Identity)**.** *Let R^2 be a binary relation, and x_1, x_2 terms. If*

$$R^2(x_1, x_2) \Leftrightarrow \forall f^1(f^1(x_1) \leftrightarrow f^1(x_2))$$

We say that R^2 is the identity relation $=(x_1, x_2)$ or informally $x_1 = x_2$.

Definition (Equivalency)**.** *Let M^2 be a binary second-order function, and F_1, F_2 concepts. If*

$$M_\beta^2(F_1(\beta), F_2(\beta)) \Rightarrow [(F_1(\beta) \to F_2(\beta)) \wedge (F_2(\beta) \to F_1(\beta))]$$

holds for M^2, then M^2 describes an equivalency relation between concepts. In this case, we say $\equiv(F_1, F_2)$ or informally $F_1 \equiv F_2$.

Observation: Since in Frege's original system of *BS*, the sign of identity $\equiv$ is not equivalent with $\leftrightarrow$, we omit the right-to-left direction in the condition above, keeping the left-to-right direction, which better describes Frege's original identity sign. This will be used only in the context of definitions. Furthermore, this should also encode Frege's definition of conceptual-content, which identity is described as mutual-inferentiability[2].

Observation: Definitions for higher-orders equivalency relation could be equally derived, but will be omited.

3. **Rules of inference**:

   (a) *Modus Ponens* **(MP)**: From φ and $\varphi \rightarrow \psi$, we may infer ψ.

   (b) *Universal Generalization* **(GEN)**: From φ we can infer $\forall x\varphi$, for a free x in φ.

   (c) *Universal Generalization - Second-Order* **(GEN^2)**: From $M^m_\beta(F(\beta))$ we can infer $\forall F[M_\beta(F(\beta))]$ provided that $F(\beta)$ is free in M^m_β.

   (d) *Existential Instantiation* **($\exists$E)**: From $\exists x\phi(x)$ we can infer $\phi(a)$, provided that a does not occur in ϕ or any previous step in a proof in which ϕ is used.

   (e) *Existential Generalization* **($\exists$GEN)**: From φ, one can infer $\exists x\varphi$, provided that x does not occur in φ.

Observation: Earlier, we mention the very important Fregean rule of substitution. Frege uses it applying for judgeable contents, either in the propositional or quantificational level, using parametric letters as a tool for fixing the argument places. Boolos (1985) showed that the rule of substitution is equivalent to axiom schemas for comprehension in second-order logic, in the form $\exists F\forall x(F(x) \leftrightarrow \varphi(x))$, that garantees the existence of a formula F if and only if $\varphi(x)$ holds, provided that F does not occurs free in φ.

But such axiom is a schema, one that have an infinite set of axioms (one for each formula) as instances. This is strange to Frege's approach. His axioms are not schemas, but real propositions of the object-language[3]. It is the rule of substitution that allows

[2]But, from mutual-inferentiability, one does not get an identity of conceptual-contents. That is, $((\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\varphi \equiv \psi)$ is not a theorem of *BS*. About this, see appendix in (DUARTE, 2009).

[3]Even though this distinction is not present in *BS*.

him to get multiple instances of an axiom in the object-language. Both approaches have a very important role in second-order logic: they provide the existence of a infinite set of concepts/properties/functions (depending which one has foundational priority in one's system) at hand.

Instead of using Frege's rule of substitution, it's simpler to use comprehension axioms. For example, the following instance states that the property *being self-identical* exists:

$$\exists I \forall x (I(x) \leftrightarrow (x = x))$$

Which is read: $I(x)$ holds if, and only if, x is self-identical. One way to make reference to such concepts is keeping track with the open variables naming it $[x : x = x]$, similarly as in lambda-calculus[4], and thus forming a term for such concepts or relations. Therefore, the name $[x : x = x]$ should be distinguished from the open formula $x = x$. This is a convenience for manipulating formulas such that comprehension provides the existence.

What matters, from Frege's perspective, is that comprehension axioms can be used to first get a new concept/function and then instatiate such formula in axioms or theorems. Frege does this with the rule of substitution and the decomposition of judgeable expressions, what is also called the Priority Thesis: that judgement precedes concepts formation[5]. What he does, indeed, is to take a possible judgement like $2^4 = 16$ and abstract/decompose one of his parts to achieve $\xi^4 = 16$, holding the argument place with ξ. Frege then can instantiate any axiom or theorem using such newly acquaired concept. Similarly, in the comprehension approach, such is possible giving the axiom instance

$$\exists F \forall x (F(x) \leftrightarrow (x^4 = 16))$$

Both are equivalent, and both can provide the name $[x : x^4 = 16]$ for such concept. From this perspective, the role of definitions in Frege's logic is not merely the one provided by the stipulation of the form $\Vdash \Delta \equiv \Gamma$, but is the condition under one is abble to, starting from it, decompose and instantiate such definition and acquire new

[4]We are assuming, then, a more informal version of such procedure, avoiding rules for conversion.
[5]"I only allow the formation of concepts to proceed from judgements" (*PW*, p.16).

contents (that is, judgements) about it[6]. This is an important point regarding the informativity of Frege's logic. Recalling Landini's interpretation:

> The power and importance of the priority thesis lies in its providing the philosophical justification for the comprehension of functions in Frege's formal system. [...] Accordingly, decomposition provides comprehension principles as rich as those of a standard second-order calculus. As is well-known, a second- order calculus is not decidable and not even semantically complete. There can be no question as to the semantic informativeness of its theses. Decomposition is, therefore, all that is required for informativity (LANDINI, 1996, p.136).

For that matter, comprehension is a easier and quicker way to represent Frege's achievements in second-order logic, without a full commitment to Frege's own notation.

4. **The axioms**: We take Frege's own axioms in *BS*, with a slightly modification:

   (a) $\varphi \to (\psi \to \varphi)$

   (b) $(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$

   (c) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

   (d) $\varphi \to \neg\neg\varphi$

   (e) $\neg\neg\varphi \to \varphi$

   (f) $\forall x_1, ..., x_n f^n(x_1, ..., x_n) \to f^1(y_1, ..., y_n)$, for any $y_1, ..., y_n$ free for f^1

   (g) $\forall f^n(M^m_\beta(f^n(\beta)) \to M^m_\beta(g^n(\beta))$, for any g^n free in M^m.

   (h) $(x_1 = x_2) \to (f^1(x_1) \to f^1(x_2))$

   (i) $(x_1 = x_1)$

   Observation: Here, (f) and (g) are axioms for universal intantiation in first and second-order versions. In (h) and (i), we substitute Frege's original axioms, using the identity

[6]Hence, I agree partially with Macbeth's (2005, 2012) thesis, although what she considers a feature of Frege's notation, taking diagramatically, I take it to be Frege's rule of substitution with decomposition, or comprehension schemas for second-order logic, as the main responsible for informativity.

sign for objects instead. The role of $\equiv$ in definitions is already supported by the definition provided above.

Finally, we add the important axiom schema for comprehension, both for first and second-order:

(j) $\exists f^n \forall x_1, ..., x_n (f^n(x_1, ..., x_n) \leftrightarrow \varphi(x_1, ..., x_n))$, where f^n is not free for φ.

(k) $\exists M^m \forall f_1^n, ..., f_m^n (M_\beta^m(f_m^n(\beta), ..., f_m^n(\beta)) \leftrightarrow \varphi(f_m^n(\beta), ..., f_m^n(\beta)))$, where M^m is not free for φ.

Observation: Axioms (j) and (k) are schemas, and hence, there will be one axiom for each φ, either first or second-order.

Finally, we take the following abbreviations: index and arity of variables and functions will be discarded when not resulting in ambiguity. Concepts and relations will be described with usual symbols. Functions signs will be used only when resulting in terms of the language.

This is a minimal description necessary for proving some results that Frege does in *BS*. The next step is to describe Frege's definitions. We'll take some time in them before proving facts and theorems. The aim is to show that Frege's Ancestral is an important step not only towards logicism, but also into Frege's account of (IP) and (CP).

3.2 The Ancestral of a relation

Frege's *Begriffsschrift* have three main parts: the presentation of the formal calculus of concepts, the derivation of theorems for propositional and quantificational calculus directly from the axioms of the system, and finally the derivation of theorems regarding some definitions. These definitions have a sole purpose: to represent, through logical terminology, the basic notions underlying ordered series. This is the task of the so-called Ancestral definition.

Frege's Ancestral was his first step towards a logicist account of arithmetic. The name 'ancestral' is commonly used today due to Russell's and Whitehead's treatment of the same relation in their *Principia Mathematica*. The reason is that, if we take the parent relation P, and apply Frege's definition, the resulting relation P^* is the usual ancestral relation. Frege's name for the same definition was 'following in a series'. The definition, and the results that follows from it, play an important role in Frege's project of showing that important notions can be logically defined and that arithmetical inferece-modes are logical by nature. This is a way of showing that (CP) holds, and consequently, (IP). Moreover, Frege's Ancestral has an important part in proving the Dedekind-Peano Axioms for arithmetic from second-order logic, and for that reason, is a central part in Frege's logicism.

In Frege's introduction for part III of *BS*, he states that:

> we see in this example how pure thought (regardless of any content given through the senses or even given *a priori* through an intuition) is able, all by itself, to produce from the content which arises from its own nature, judgements which at first glance seem to be possible only on the grounds of some intuition. (*BS*, §23)

This is a testimony for both (CP) and (IP): the results that the ancestral definition provides can be regarded as contents of the 'pure thought', confirming (CP), as the title of *BS* already anunciate. And since they are logical by nature and preserve analyticity, are also an affirmative case for (IP). This is also declared in the following section:

> I have, without borrowing any axiom from intuition, given a proof of a proposition which might at first sight be taken for synthetic [...]. From this proof it can be seen that propositions which extend our knowledge can have analytic judgements for their content. (*GLA*, §91)

The proposition in question is theorem **BS133** of *BS* concerning trichotomy, one that we shall present latter. Moreover, the same point is declared in the unpublished paper *Boole's logical Calculus and the Concept-script*:

> If we compare what we have here with the definitions contained in our examples, [...] that of following a series which I gave in §26 of my Begriffsschrift, we see that there's no question there of using the boundary lines of concepts we already have to form the boundaries of the new ones. Rather, totally new boundary lines are drawn by such definitions and these are the scientifically fruitful ones. Here too, we use old concepts to construct new ones, but in so doing we combine the old

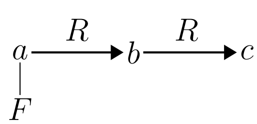
> ones together in a variety of ways by means of the signs for generality, negation and the conditional. (*PW*, p.34)

Hence, the definition of the ancestral (the 'following in a series') is central to show how decomposition is crucial for (IP). The definition is also aimed to give a logical account for the notion of a series, in accordance with (CP). Recall that, the notion of a linear ordering discussed in the introdution, requires a relation R that is both transitive and that satisfies trichotomy. This is what Frege's definition suppose to prove. Given any relation R, what we call the 'ancestral of R' is the series of objects that are connected through R. This is, then, a second order concept in which falls first order relations. Frege's ancestral is given through four definitions.

First, Frege defines what he calls the condition where a given property F is hereditary in a given relation R (*BS*, §24)[7]:

$$\Vdash Her(F, R) \equiv \forall x \forall y (F(x) \land R(x, y) \to F(y))$$

This means that F is a property that, if any given object x has it, x will pass it along R. In that case, '$Her(F, R)$' can be read as 'F is hereditary in the R-relation'. Properties like 'x is a human' or 'x is greater or equal to 0' are said to be hereditary in respect of some R in which they can relate, *viz.* the parent relation[8] or the successor relation on numbers. Frege reads this definition as the following: "If from the proposition that x has the property F, whatever x may be, it can always be inferred that each result of an application of the procedure R to x has the property F" (*BS*,§24)[9]. This defines a second-order relation: that of *being an hereditary property in respect to a relation*, in which first-order concepts and relations fall. We can visualize how this definition works in the following steps:

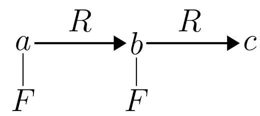


(1)

[7]Here we're using conjunctions $\land$ to simplify the definitions.

[8]Frege gives this exact example in (*BS*,§24).

[9]Here we transcribe Frege's own notation into ours: Frege's german letter for generality $\mathfrak{d}$ is the quantified x, and Frege's procedure f is the relation R.



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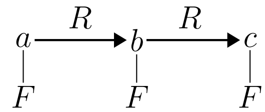
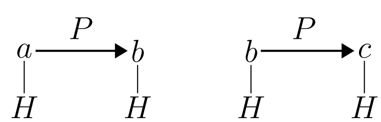


(3)

The above examples shows how an hereditary property F is passed along R: in (1), $F(a)$ holds, but since $R(a, b)$, $F(b)$ also must hold, as in (2). But, since $R(b, c)$, $F(c)$ holds as well, as in (3). In conclusion, F is a carried out through R, since, starting from a, every object that is related through R to some object that is itself related to a, also has F. Hence, $Her(F, R)$ holds. In Frege's own example, if some individual a is an human, and if b is son of a, then b is also an human. Hence, *being an human* is hereditary in respect to the parent relation[10].

The above definition, however, is not suficient for defining properly a series. It can be the case where $Her(F, R)$, $R(a, b)$ and $R(b, c)$ holds, even if there is no connection between them all. Consider the same example above: the parent relation. $P(x, y)$ states that "x is the father of y", or "y is a son of x". Consider the cases where $P(a, b)$ and $P(b, c)$. Even though b has the hereditary properties of a and c has the hereditary properties of b (like "x is an human", denoted by $H(x)$), is not the case that $P(a, c)$ holds, since clearly a is not the father of c. P is not a transitive relation. Since it is demanded that a series must contain a path from each element to another, we can visualize why the parent relation is not sufficient, as the following case:



Clearly, there's no path from a to c in this case, *i.e.*, the objects a, b and c are not connected. In order to overcome this, Frege offers the important *Ancestral Definition*. In his terms, this is the definition for the general notion of 'following in a series' (*BS*, §26):

$$\Vdash R^*(x, y) \equiv \forall F[Her(F, R) \land \forall z(R(x, z) \to F(z)) \to F(y)]$$

[10]But not, of course, in respect to every relation that a and b stands on.

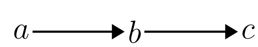
It renders the following condition: $R^*(x, y)$ holds in the case where y has all the hereditary properties that x has and which x pass through R[11]. This is to signify, according to Frege, that 'y follows x in the R-series', or 'x precedes y in the R-series'. This is a second-order concept or property in which first order relations falls.

The purpose of the definition is to provide a way to depict logically the conection between objects. We have stated that the hereditary property is not a sufficient condition for such task. Consider then that in $R^*(x, y)$, x is called the ancestral and y is the descendent. In order for R^* to hold, it is required to the descendent to have every hereditary property that the ancestral has and pass along through R. This is the clause $\forall z(R(x, z) \rightarrow F(z))$. The consequent of the definition completes the case, that $F(y)$ holds for every F that satisfies the clause. Frege considers that this definition is general enough to cover a variety of cases. In a footnote, he states that

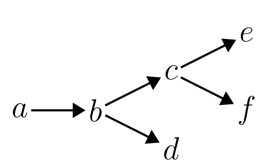
> To make clearer the generality of the concept of ordering-in-a-sequence given in this way, I remind the reader of some possibilities. Among these are not only a sequence such as beads on a string exhibit, but also branching like a family tree, a merging of several branches, as well as ringlike self-linking. (*BS*,§26).

Thus, the definition is inclusive enough to be instantiated into a variety of cases. The examples above can be visualized as:

- A linear order:

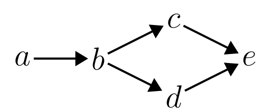


- A branching tree:

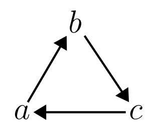


- A tree with closed branches:

[11]Frege's reading is: "If from the two propositions, that every result of an application of the procedure f to x has the property F, and that the property F is hereditary in the f-sequence, it can be inferred, whatever F may be, that y has the property F" (*BS*,§26). Where Frege writes f-sequence, we should read R-sequence.



- A cyclic order:



This examples can be better grasped considering the difference between many-to-one and one-to-many relations. A relation R is said to be one-to-many if it allows an object x to bear R to multiple objects, such that $R(x, y)$ and $R(x, z)$. A relation R is said to be many-to-one if it allows an object x to be reached by multiple objects, such as $R(x, y)$ and $R(z, y)$. A relation that is both many-to-one and one-to-many can be described as many-to-many. A relation R that is neither many-to-one nor one-to-many is said to be one-to-one. In such case each object can reach, or be reached, by a single object through R. In this case, a linear order like the above is such that the original relation R is one-to-one. A branching tree is one-to-many, while a tree with closed branches is many-to-many. A cyclic order, like the depicted above, is one-to-one, even though other cases could not be so[12].

We can still provide more intuitive examples. For the first case, consider the parent relation P as the starting point. If we consider it to be one-to-one *i.e.*, each parent can have only a single child, then P^*, Frege's definition applied to P, yields a linear-type series. If P is one-to-many, then P^* describes a series like a branching tree. For the third case, with the relation 'x is older than y', denoted by $O(x, y)$, O^* is a branching tree with closed branches. For the fourth case, consider the game of rock(r)-paper(p)-scissors(s), applyed to the relation 'x wins y' as $W(x, y)$. In this case, $W(p, r)$, $W(r, s)$ and $W(s, p)$ holds. Then, W^* describes a cyclic series. Frege's ancestral definition is general enough to include all such cases. What is required is that each object is somehow connected: that for every pair of objects $< x, y >$, a path from one to another must exist. This is the reason why Frege states that his definition "far surpass in generality all similar propositions which can be derived from any intuition of sequences" (*BS*,§23). This is in contrast to the kantian notion of a time-ordered series, which is more restrict.

[12]A branching tree with a cyclic closed branch, for example.

In all such examples, what is required is that R^* must be transitive. Considering again the parent relation P. If $P(a,b)$ and $P(b,c)$, it cannot be the case that $P(a,c)$, since the parent relation is not itself transitive, but $P^*(a,c)$ must hold. To visualize that, $P(a,b)$ implies $P^*(a,b)$, since a father is also the ancestral of his childs. Likewise, $R(b,c)$ implies $R^*(b,c)$. From that, transitivity must be the case in deriving $R^*(a,c)$, and so the connection between a, b and c is completed through the ancestral relation. Transitivity is an important theorem that Frege proves about his definition[13].

For arithmetical interests, only linear orderings are important, since that's what it takes to describe the natural numbers series. That's the case for Kant's approach for series as based in intuition. Since Frege's ancestral is more general, more restrictions are necessary for describing properties of linear orderings. Frege's definition of R^* is also known as Strict or Strong order, *i.e.* a relation that is both transitive and irreflexive[14]. For that reason, Frege's definition of "following in a series" is also called the Strong Ancestral.

The irreflexive and transitive properties implies that R^* is also asymmetric, since if it were, $R^*(x,y)$ and $R^*(y,x)$ would imply, using transitivity, $R^*(x,x)$, denying the irreflexivity. This is enough to describe a strict partial order. It is partial because not every two elements can be compared, like the examples showed ealier. Nonetheless, for linear orders like the required for arithmetic, more definitions are required. Frege's strategy is to convert the strict partial into what is called Weak Partial Orders. For that, he then defines what is called the Weak Ancestral of R (*BS*, §29):

$$\Vdash R^+(x,y) \equiv R^*(x,y) \lor (x = y)$$

That is, y is the weak ancestral of x in R just in case either x is the ancestral of y or x and y are equal. Frege did not have the disjunction sign '$\lor$' as a primitive. He used the equivalent form $(\neg P \to Q)$. This definition is called the weak ancestral simply because is a weakened version of the ancestral, where it follows that $R^+(x,x)$ holds. Frege reads it as 'y belongs to the R-series beginning with x'. Since R^+ is also reflexive, it is also antisymmetric. The weak

[13] We'll give this proof, among others, in the next section.

[14] Frege's definition is actually transitive and not necessarily reflexive, depending on the case of each R that we apply the definition on. The problematic case is the last example considered before: If R is cylic, R^* is reflexive, and hence, does not describes a strict order in this case.

ancestral is then a weak partial order[15].

The addition of such condition is clearly an antecipation of the less-or-equal-to relation. Providing the relation "m precedes n" for natural numbers, $P^{+}(m,n)$ is supposed to be a suitable definition for $m \leq n$, in the same sense as $P^{*}(m,n)$ is for $m < n$. But Frege still has to prove that the ancestral has the properties expected for those cases. For foundational purposes, given the necessary restrictions, R^{*} should necessarily describes a linear-ordering, that is, a total order. Hence, alongside the already known property of being transitive, the relation in question must satisfies trichotomy, wheres either $R^{*}(x,y)$, $R^{*}(y,x)$ or $x = y$ must hold exclusively.

For Kant, those properties are something easily derivable from series because time, as the *a priori* condition of our intuitions, is linear by essence. It's from this condition that numbers are not entities, but rules governing our description of quantities, that is only possible, according to Kant, from the pure intuition of time. Frege's argument is anti-kantian in a twofold manner: series are logically organized entities and do not required time to be ordered, and numbers are *sui generis* entities, which does not requires its position in a series in order to be about something[16].

Ultimately, as the examples showed earlier, the type of the series in question will depend on which relation R we assume. To derive the natural number series properly, the relation of predecession on numbers must be functional, that is, it cannot be the case for a number to be the predecessor of two distinct numbers, and according to the fregean foundational reasons, this cannot be a feature of our intuitions about numbers or series. Likewise, to derive a linear-ordered series from a relation, a functional condition is also required, and for that reason, the final definition that Frege gives in part III (*BS*, §31) is:

$$\Vdash Fun(R) \equiv \forall x \forall y \forall z (R(x,y) \land R(x,z) \rightarrow y = z)$$

[15]From a foundational perspective, it has no great difference on which relation to start. We could easily define a weak partial ordering first and acchieve a strict order by adding the condition $R^{*}(x,y) =_{def} R'(x,y) \land x \neq y$, providing that R' is a weak partial order. As we have seen, Frege opts to define a strict order first. Enderton (1977), for example, chooses the same approach, adding that the choice is just a matter of convenience. Frege's option is then treats the relation $<$ as more basic for its foundational purposes.

[16]This is just a way of saying that Frege does not derive numbers, or its cardinal character, from their ordinal position. Nonetheless, the ancestral plays a fundamental role in proving the existence of an infinite series of numbers.

This definition gives the notion of a relation being many-one. It's important to remind that the Fregean notion of a function is broader than what we nowadays take a function to be, since he assumes both properties and relations to be unary and binary functions. Nevertheless, this definition could be readed as 'R is a function', or, as Frege puts it, 'R is a many-one procedure'.

With those definitions at hand, Frege was able to define logically and generally how series or sequences works, and which kinds are properly ordered for arithmetical purposes, that is, what is minimally required to have a transitive and trichotomous relation. Another important topic is that of the principle of mathematical induction. The principle, a common proof method in mathematics, states that if a property holds for a given number n, and from that it could be derived that it holds for $n + 1$, then such property must holds for every n. In second-order logic, this is expressed as: $\forall F[F(n) \wedge \forall m(F(m) \rightarrow F(s(m))) \rightarrow \forall n F(n)]$, assuming that $s(n)$ is the successor function for numbers and n is a natural number. But this principle is dependent upon the well-ordering of the natural numbers, that is, that the series of natural numbers is a total order in which every subset has a least element[17]. Likewise, Frege's treatment of the ancestral should derive general principles for inductions.

To summarize, Frege's ancestral has a philosophical and a logical side. Philosophically, it provides an example on how analytic proofs can be informative, against the kantian argument of the syntheticity of arithmetic. Logically, by providing the logical analysis of ordered series and proving its basic properties, it's the starting point for Fregean logicism, something that Frege carried on in *GLA*.

3.3 Important Theorems

The definitions provided above are used to obtain important theorems regarding series, as Frege did in *BS*. As we stated already, logicism is clearly tied up with these definitions, since we can instatiate the relation R as the predecessor relation P on natural numbers, and define $x < y$ as $P^*(x, y)$ and $x \leq y$ as $P^+(x, y)$. But this identification would only appear in 1884 with *GLA*. Nonetheless, Frege's aim with these definitions is threefold, since there are

[17]Both are equivalent, meaning that from the well-ordering principle one can derive the principle of mathematical induction, and vice-versa.

three major theorems in *BS*:

BS81 $\vdash [F(x) \wedge Her(F,R) \wedge R^*(x,y) \rightarrow F(y)]$

BS98 $\vdash [R^*(x,y) \wedge R^*(y,z) \rightarrow R^*(x,z)]$

BS133 $\vdash [Fun(R) \wedge R^*(x,y) \wedge R^*(x,z) \rightarrow (R^*(y,z) \vee R^+(z,y))]$

Theorem **BS81**[18] is a form of the principle of induction. Theorems **BS98** and **BS133** are the important theorems for which a linear ordered series is defined: transitivity and trichotomy for R^*. Hence, if Frege's definition of the Ancestral is to be successful in expressing such notion into logical terms, these results are the most fundamental. Frege's axiomatic system does not use hypothetical proofs[19]. Hence, he requires proofs for every proposition, starting with axioms, rules of inferences and definitions. Many of the theorems are actually provable without the rule of substitution (or comprehension axioms) when using hypothesis, but the major theorem **BS133** still requires such tool.

3.3.1 Induction

Theorem **BS81** is a theorem for a general principle for induction. Frege declares that the Bernoullian induction, who is considered to be the first to explicitly uses such proof method, is based upon this theorem. This is not enough to derive the principle of mathematical induction *per se*, since no definition of natural number was given by Frege in *BS*. But Frege certainly believed that this was not a method peculiar to mathematics, having itself logical grounds, as he makes clear in [*PW*, p.31)]: "It follows from §§ 24 and 26 of my Begriffsschrift that this mode of inference is not, as one might suppose, one peculiar to mathematics, but rests on general laws of logic" and in (*GLA*, §79): "Only by means of this definition of following in a series is it possible to reduce the argument from n to $(n+1)$ which on the face of it is peculiar to mathematics, to the general laws of logic". The proof is the following:

Theorem BS81. $\vdash [F(x) \wedge Her(F,R) \wedge R^*(x,y) \rightarrow F(y)]$

[18]The numeration is the same that Frege uses in *BS*. We'll keep them in order to follow his strategy in deriving such theorems. Moreover, theorems presented in *BS* will be only referenced, without the corresponding proof or logical description.

[19]Something that would be better undestood only in the 1930's with the work of Gentzen.

Proof. We assume as premises $F(x)$, $Her(F,R)$ and $R^*(x,y)$. Now, assume that $R(x,w)$ holds for any given w. From that, and since $Her(F,R)$ and $F(x)$, we get $F(w)$ from **(MP)**. Applying **(GEN)** from $R(x,w)$ and $F(w)$, we derive $\forall w(R(x,w) \to F(w))$. Now, from the definition of $R^*(x,y)$, we get $\forall F[Her(F,R) \wedge \forall z(R(x,z) \to F(z)) \to F(y)]$. Given that we already have both premises, using **(MP)** we conclude $F(y)$ as desired. □

Frege's proof is far more complicated and uses the rule of substitution in several steps. Is not a surprising that both the definitions of the hereditary property and the Ancestral are used as important tools. Frege proves it from both definitions, applying theorem (68) and uniformly replacing parts of the judgeable contents by new terms and formulas. This is akin to comprehension axioms. The main path to theorem **BS81** is: (a) from (68) and (69) Frege gets (70), which with (19) he derives (71). (71) with (58) yeilds (72), which with axiom (8) derives (74). (b) from the definition of the ancestral (formula (76)) together with (68), Frege gets (77), and with (17) derives (78). Using axiom (2), Frege derive theorem (79) and with (5) gets (80). Theorems (80) and (74) finally results in the desired theorem **BS81**. This of course, demands proofs to all the theorems one by one without hypothesis.

Later, a principle for mathematical induction will be derived, but first the following similar principle is necessary from **BS81**[20]: since $R^*(x,y)$ implies $R^+(x,y)$, from the definition of the weak ancestral, we can get the equivalent formula $[F(x) \wedge Her(F,R) \wedge R^+(x,y) \to F(y)]$ and then $[F(x) \wedge Her(F,R) \to (R^+(x,y) \to F(y))]$. Finally, from **(GEN)** and changing $Her(F,R)$ by its *definiens*:

$$[F(x) \wedge \forall x \forall y(F(x) \wedge R(x,y) \to F(y)) \to \forall z(R^+(x,z) \to F(z))]$$

Frege didn't prove this theorem in *BS*, but it follows easily from **BS81**. It can be proved more directly as follows:

Theorem BS81a. $[F(x) \wedge \forall x \forall y(F(x) \wedge R(x,y) \to F(y)) \to \forall z(R^+(x,z) \to F(z))]$

Proof. Assume $F(x)$, $\forall x \forall y(F(x) \wedge R(x,y) \to F(y))$ and for a fixed z, $R^+(x,z)$. It suffices to show that $F(z)$ holds to then get the theorem by **(GEN)**. From this, either $x = z$ is true, in

[20]This is pointed out and proved by Landini (2012, p.70). The proof that follows is basically the same as his.

which $F(z)$ follows directly, or $x \neq z$. In this case, $R^*(x,z)$ holds. Applying the definition of the strong ancestral, and fixing a given F, we get $\forall z(R(x,z) \to F(z)) \wedge Her(F,R) \to F(z)$. Is sufficient then to show both antecedents. First, fix any y and assume $R(x,y)$. Then, from *definiens* of Her above, $F(y)$, and from **(GEN)** we get the first antecedent. The other one is just $Her(F,R)$, which is $\forall x \forall y(F(x) \wedge R(x,y) \to F(y))$ by definition. Hence, $F(z)$ holds, and using **(GEN)** again, we derive the theorem. □

3.3.2 Transitivity

Theorem **BS98** is an important theorem for arithmetic and the notion of an ordered series. It states the conditions under which a given relation is transitive. With such theorem Frege shows an important fact about his definition of the ancestral: the ancestral of any given relation R is transitive, i.e., if x is the ancestral of y and y is ancestral of z, certainly, x is ancestral of z. Once again, we give a simpler proof:

Theorem BS98. $\vdash [R^*(x,y) \wedge R^*(y,z) \to R^*(x,z)]$

Proof. We assume as premises $R^*(x,y)$ and $R^*(y,z)$. We want to prove $R^*(x,z)$, i.e., that $\forall F[\forall w(R(x,w) \to F(w)) \wedge Her(F,R) \to F(z))]$. We assume that $Her(F,R)$ and $\forall w(R(x,w) \to F(z))$. Since $R^*(x,y)$, $F(y)$ holds. We must show that $\forall z(R(y,z) \to F(z))$. For that, assume $R(y,t)$, for any given t. Since $Her(F,R)$, $F(y)$ and this assumption, we get $F(t)$. From **(GEN)**, we derive $\forall z(R(y,z) \to F(z))$. This is enough to get $F(z)$ from $R^*(y,z)$. Finally, from **(GEN^2)**, we derive the definition, and hence, $R^*(x,z)$ holds. □

Frege's own proof of **BS98** is, as one might suspect, longer and more complicated. And once again, multiple uses of the rule of substitution are made throughout the derivations. It follows from two basic points: the definition of the ancestral relation, which is numbered as formula 76 in *BS*, and theorem **BS81** above. From **BS81** Frege derive (84), which is an induction theorem just like **BS81**, but with switched subcomponents. The path from (76) to **BS98** is then the following:

$$(76)+(52)\to(90)+(5)\to(93)+(60)\to(94)+(7)\to(95)+(8)\to(96)+(75)\to(97).$$

Finally, (97) in addition to (84) gives the desired **BS98**. It is worth to take a closer look at the final step. (97) states the case where the 'the property of following x in the R-sequence is hereditary in the R-sequence'. Hence, we can use the original formula to express it: $\forall w \forall z(R^*(x,w) \wedge R(w,z) \rightarrow R^*(x,z))$. Let's denote this specific formula as $Her(R^*_x, R)$[21]. Taking theorem (84):

$$Her(F,R) \wedge F(x) \wedge R^*(x,y) \rightarrow F(y)$$

Frege applies the following substitutions: F(Γ) for $R^*(x,\Gamma)$, x for y and y for z. This results in the following formula:

$$Her(R^*_x, R) \rightarrow [R^*(x,y) \wedge R^*(y,z) \rightarrow R^*(x,z)]$$

Then, from (97) and **(MP)** he can derive **BS98**. The parametric letter Γ is used to demarcate the open variable in the formula $R^*(x,\Gamma)$. This is akin to the comprehension axioms stated earlier, since the following is an instance of it:

$$\exists F \forall z[F(z) \leftrightarrow R^*(x,z)]$$

Which allows us to use $[z : R^*(x,z)]$ as an open formula for z. This is analogous to Frege's procedure and his rule of substitution. Again, if we take the relation of predecession for natural numbers, then **BS98** results in an important theorem for arithmetic: $(x < y \wedge y < z) \rightarrow x < z$.

One could wonder that **BS98** was one of the two desired results that Frege was targeting with *BS*, since **BS98** is, togheter with **BS133**, the only theorem not used as a premise for other proofs. His intentions to provide a logical definition for ordered series, without intuitions, is clear enough.

[21]This is a clear deficiency of our way of depicting Frege's notation. This is because Frege's abreviation for the Hereditary property has variables bounded by quantifiers. This is a clear case why Frege chooses to keep them in the *definiens*.

3.3.3 Trichotomy

Theorem **BS133** is the last theorem proved by Frege. Its importance for mathematics is straightforward: if R is a function, and $R^*(x, y)$ and $R^*(x, z)$ hold, we have three possible cases. Either $R^*(y, z)$, $R^*(z, y)$ or $z = y$ exclusively holds. If the relation of predecession is such relation, then we have the law of trichotomy for arithmetic: for two given numbers m and n, either $m < n$, $n < m$ or $m = n$. This theorem is, together with **BS98**, what Frege needs for proving that the ancestral defines a linear-ordered series properly.

To prove theorem **BS133**, we need some preliminary results, all proved by Frege:

Theorem BS91. $\vdash R(x, y) \to R^*(x, y)$

Proof. Assume $R(x, y)$. We want to prove the definition:

$$\forall F[\forall z(R(x, z) \to F(z)) \wedge Her(F, R) \to F(y)]$$

For the assumption, we only need the first antecedent: $\forall z(R(x, z) \to F(z))$. Since $R(x, y)$, $F(y)$ holds, and using **(GEN)**, $R^*(x, y)$ holds. □

Theorem BS108. $\vdash R^+(x, z) \wedge R(z, w) \to R^+(x, w)$

Proof. Assume $R^+(x, z)$ and $R(z, w)$. We want to prove $R^+(x, w)$, i.e., $R^*(x, w) \vee x = w$. Since $R^+(x, z)$, we have two cases: 1) $R^*(x, z)$. Since $R(z, w)$, from theorem (91) above, $R^*(z, w)$. And from theorem (98) already proved, $R^*(x, w)$ holds. From the definition of weak ancestral, $R^+(x, w)$ as desired. 2) $x = z$. In this case, since Frege's sign for identity of content behave like an identity, we can substitute z for x in $R(z, w)$, and from theorem **BS91** achieve $R^*(x, w)$, which gives us $R^+(x, w)$ as desired. □

Theorem BS78. $\vdash [R^*(x, y) \wedge \forall z(R(x, z) \to F(z)) \wedge Her(F, R)] \to F(y)$

Proof. This is a straightforward proof: Assume the antecedentes, to show that $F(y)$. Since $R^*(x, y)$, and since $Her(F, R)$ and $(R(x, z) \to F(z))$, then $F(y)$ as desired. □

Theorem **BS78** is also an inductive theorem. To prove the next result, we shall use comprehension in a way very similar to Frege's rule of substitution, applying in theorem **BS78**.

Theorem BS124. $\vdash Fun(R) \wedge R(x,y) \wedge R^*(x,z) \rightarrow R^+(y,z)$

Proof. To prove such theorem, take the following instance of comprehension schema

$$\exists H \forall w(H(w) \leftrightarrow R^+(y,w))$$

Now, lets call the concept H as $[w : R^+(y,w)]$. Our main goal is to prove that $[w : R^+(y,w)]z$ holds true, which is the same as $R^+(y,z)$. For that, we use induction, i.e., Frege's theorem **BS78**[22]. Taking y to be z in **BS78** and F to be our concept $[w : R^+(y,w)]$, we arrive at:

$$R^*(x,z) \wedge \forall z(R(x,z) \rightarrow R^+(y,z)) \wedge Her \rightarrow R^+(y,z)$$

Here, Her is an abreviation for $\forall x \forall y(R^+(y,x) \wedge R(x,y) \rightarrow R^+(y,y))$. Our assumptions are $Fun(R)$, $R(x,y)$ and $R^*(x,z)$. Our goal is to prove each antecedent of the instantiated **BS78** above. 1) We first prove Her. Assume $R^+(y,x)$ and $R(x,y)$. From **BS108**, clearly, $R^+(y,y)$, hence, Her holds. 2) We want to prove that $\forall z(R(x,z) \rightarrow R^+(y,z))$. Assume $R(x,v)$ for any given v. Then, since $R(x,y)$ and $Fun(R)$, $v = y$, which gives us $R^+(y,v)$. From **(GEN)**, we arrive at the desired $\forall z(R(x,z) \rightarrow R^+(y,z))$. 3) Since $R^*(x,z)$ is one of our assumptions, we're done. This gives us the desired $R^+(y,z)$, completing the proof[23]. □

The proof of theorem **BS124** is central to prove **BS133**, which we can finally show:

Theorem BS133. $\vdash [Fun(R) \wedge R^*(x,y) \wedge R^*(x,z) \rightarrow (R^*(y,z) \vee R^+(z,y))]$

Proof. Our assumptions are $Fun(R)$, $R^*(x,y)$ and $R^*(x,z)$. Next, take the following instance of comprehension schema:

$$\exists H \forall w[H(w) \leftrightarrow R^*(y,w) \vee R^+(w,y)]$$

Once again, we could name H as $[w : R^*(y,w) \vee R^+(w,y)]$. We want to show that $H(z)$ holds. The proof is once again by induction, this time using theorem **BS81**. For that, we

[22] This should be a good example of Frege's induction theorem in use.
[23] The same proof, using theorem **BS78**, can be found in (LANDINI, 2012)

instantiate F to be $[w : R^*(y,w) \vee R^+(w,y)]$, and y to be z. Then, we arrive at:

$$[R^*(y,x) \vee R^+(x,y)] \wedge R^*(x,z) \wedge Her \rightarrow [R^*(y,z) \vee R^+(z,y)]$$

Now, Her is a abreviation for:

$$\forall v \forall w([R^*(y,v) \vee R^+(v,y)] \wedge R(v,w) \rightarrow [R^*(y,w) \vee R^+(w,y)])$$

To prove the theorem, it suffices to prove all three antecedents of the instantiated **BS81** above. 1) Since $R^*(x,y)$, we get $R^+(x,y)$, hence, $R^*(y,x) \vee R^+(x,y)$ holds. 2) $R^*(x,z)$ is one of our assumptions; 3) To prove Her, we assume both $R^*(y,v) \vee R^+(v,y)$ to some fixed v, and $R(v,w)$. From the first, we have three cases: (a) $R^*(y,v)$ holds. Since $R(v,w)$ also holds, from **BS91** we get $R^*(v,w)$, and from **BS98** we get $R^*(y,w)$, and finally, $R^*(y,w) \vee R^+(w,y)$ holds; (b) $R^*(v,y)$ holds. In this case, since $R(v,w)$ and $Fun(R)$, from theorem **BS124** we get $R^+(w,y)$, and hence, $R^*(y,w) \vee R^+(w,y)$ holds again; (c) $v = y$, and from $R(v,w)$, we obtain $R(y,w)$. Again, using **BS91** we get $R^*(y,w)$ and $R^*(y,w) \vee R^+(w,y)$. This complete the proof of Her, and in conclusion, we can use (MP) to obtain $R^*(y,z) \vee R^+(z,y)$ as desired. This complete the proof of **BS133**. □

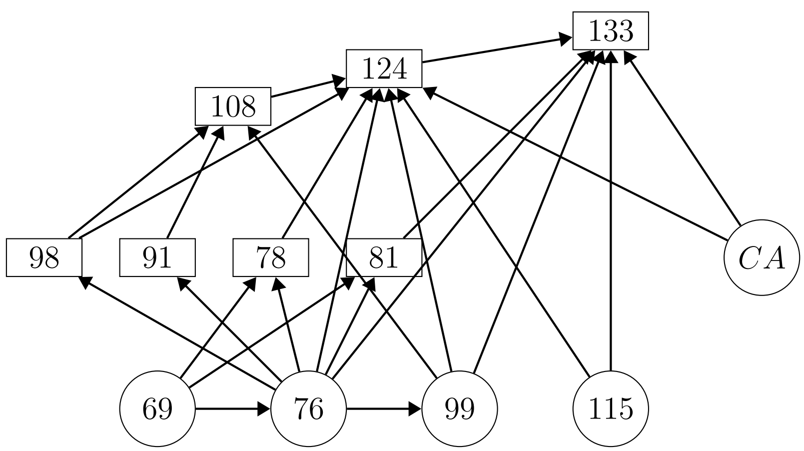
3.3.4 (IP) and (CP), once again

The use of comprehension in the proof above is essentially the same as Frege's rule of substitution. But this proof is much more direct, since it procedes with hypothesis. In Frege's axiomatic approach, since **BS133** is the last theorem of *BS*, the path leading to it is of considerable complexity. Frege proves it from (132) and (83). (83) can be achieved from theorem **81**, which we already exposed above. The proof of (132) starts from the definition of the many-one function, which is formula (115). Many of the proofs uses substitution for functions. The path can be break down as follows:

(115)+(68)→(116)+(9)→(117)+(58)→(118)+(19)→(119)+(58)→(120)
(120)+(20)→(121)+(112)→(122)+(19)→(123)+(110)→(124)+(20)→(125)
(125)+(114)→(126)+(12)→(127)+(51)→(128)+(111)→(129)

(129)+(9)→(130)+(75)→(131)+(9)→(132)

This long path is necessary in Frege's unawareness of hypothetical proofs for conditionals. But such unawareness is not unmotivated. Since every judgeable formulas of the language has a (conceptual) content, and since such content is what determines the possible consequences of each formula, the complete proof is a sort of map, that leads a proved proposition, in this case **BS133**, into all its foundations. Since everything in such proof is logically justifiable (propositonal and quantificational axioms and theorems that follows from them), the key for the information presented by **BS133** relies on all four definitions. Together, the propositions proved from this, form a kind of "proof-tree". For the simplified proof showed here, this "proof-tree" is something like:



Where numbers within circles are definitions, and numbers within boxes are theorems. The arrows indicate that the theorem made a directly use of a definition or theorem. The circle with "CA" denotes the use of comprehension axioms. No wonder that, in (*GLA*,§88), Frege states that the conclusions are contained in the definitions 'as plants are contained in their seeds, not as beams are contained in a house', a direct metaphor to the difference between his relation of consequence and the relation of consequence as a mere containment, something he thought to be the case for the Kantian and Boolean theory of concepts. This difference is only intelligible if we take into account the use of comprehension, or equivalently, his rule of substitution together with decomposition of judgeable contents.

If we look at Frege's definitions, they don't simply define a concept or relation from a list of given concepts. What they do is to define a concept through the expression of *the logical relations between concepts*. As Frege puts it, "we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation and the conditional" (*PW*,p.34). This are the fruitful kind of definitions.

From them, decompositions are performed. They can be decomposed in many ways, and Frege gives this hint every time he introduced a new one. To take an example, right after the definition the hereditary property, Frege says the following:

> The formula $\vdash f(\Gamma, \Delta)$ can be rendered 'Δ is a result of applying the procedure f to Γ', or 'Γ is the object of an application, with a result Δ, of the procedure f, or 'Δ bears the f-relation to Γ', or 'Γ bears the converse of the f-relation to Δ'. This expressions are to be taken as equivalent in meaning [*gleichbedeutend*] (*BS*, §24)

All those are different readings, which in turn yields different places where one can decompose such function. But still, they are the same function. By *equivalent in meaning* Frege might be assuming *equal in conceptual-contents*, since they all have the same logical consequences[24]. The fact that all those readings are possible is a glimpse for the multiple decompositions that we could get from it. The fact that here Frege uses parametric letters to mark the variable places is a hint that we can break down the same function in many diferent ways.

The proof of theorem **BS98** yields, in Frege's perspective, a judgeable content which is "a fact":

$$\vdash R^*(x, y) \wedge R^*(y, z) \rightarrow R^*(x, z)$$

By decomposition on R^* (or equivalently, by a second-order comprehension axiom) we can derive the following function/concept:

$$[M : M_{\beta,\gamma}(f(\beta, \gamma)) \wedge M_{\gamma,\delta}(f(\gamma, \delta)) \rightarrow M_{\beta,\delta}(f(\beta, \delta))]$$

This is actually a third-order unary function, that of *being a transitive binary second-order*

[24] Similarly, Bauer-Mengelberg translate this phrase as "[...] these expressions are to be taken as equivalent" (FREGE, 1967, p.56).

function[25]. Frege's definition of the ancestral made it possible to prove that

$$[M : M_{\beta,\gamma}(f(\beta,\gamma)) \wedge M_{\gamma,\delta}(f(\gamma,\delta)) \rightarrow M_{\beta,\delta}(f(\beta,\delta))]R^*$$

holds. This is not a component of the original definition, but is a property that holds for it. Hence, to say that R^* satisfies transitivity, for any R, is clearly something informative. The same can be said about **BS133**: any R^* that satisfies the condition of being many-one, also satisfies trichotomy. This is not a property of the original defined expression of R^*, but something that R^* made possible to prove. But all those proofs are only possible in second-order scenario, one that includes comprehension. In Frege's mind, and contrary to Quine's latter criticisms, this is logic. Such proofs, then, preserves analyticity in Frege's version of it. This is what (IP) is about, and this is how Frege thought that the ancestral was an example of fruitful definitions, one that allows informative proofs that maintains analyticity. Since the logical domain is all there is, such relation is also a proof of (CP): that the ancestral is a transitive relation does not depend upon the relation R in consideration. This is a property of the definition, one that has logical character.

It is for those reasons that the ancestral relation is clearly one of Frege's argument against a kantian foundation for arithmetic, since, supposedly, no intuition is present in any part of the proofs. What the ancestral also generates is, from **BS98** and **BS133**, a linear ordering, when applied to any many-one first-order relation. As we saw for Kant, numbers are rules to describe quantities, and the transition from the ordinals to the cardinals is possible because time, its foundation, is already linear by essence. It comprises both trasitivity and trichotomy by default, and hence, the series of natural numbers is exactly the series of quantities that we can construct from progressive time-intervals. In *BS*, Frege gave the first step to an alternative argument: to describe linear orderings logically, not intuitively. The second part is to show how *numbers*, or the things that describes quantities, are also of a logical character. This is annouced in *GLA* and later formally proved in *GGA*. In both, the ancestral reappears.

[25]This is so because R^* is a second-order concept, or function, in which first-order relations fall, with R being arbitrary.

3.4 The route to Frege's Theorem

Frege's Ancestral is, we saw, a definition in second-order logic for linear-ordered relations. This has an important application in the foundations of arithmetic. It was in *GLA* that such use was hinted, and in *GGA* that it was carried on more formally. One of them we've already antecipated: to derive a principle for mathematical induction. But this still asks for a formal definition of cardinal number, and once that is given, Frege's Ancestral is central in proving the existence of an infinite number of them. This achievements, present in *GLA* and *GGA*, were substantial. It's known that Frege provides a proof of Dedekind-Peano Axioms for arithmetic from second-order logic with comprehension and definitions.

Most of the *GLA* is devoted to offer a definition of cardinal number. This topic was already, but not sufficiently, discussed earlier. Recall that, as Frege concludes in (*GLA*,§46), "the content of a statement of number is an assertion about a concept". This is taken as an important aspect on number-words behavior that should clarify what, in fact, numbers are. This is the starting point in section §55, the beginning of the constructive part of *GLA*. And one of the first conclusions, rendered in (*GLA*,§57), is that numbers are not properties of concepts. This is to say that numbers are not something like an attribute[26], or higher-order properties of first-order concepts. In Frege's account, numbers are *self-subsistent objects*, since that is what identity statements provides. But this is not to be taken as signifying that numbers are independent of the propositions in which they appear, but only that they can be recognized again in different manners. In the equation $1 + 1 = 2$, we have an identity because both sign '$1+1$' and '2' denote the same number. This allows us to use the definite article "the number two". But the very notion of *number* is something that can be properly defined by taking into account the context in which they appear, and this context are identity statements. What the sentence "there are four apples in the basket" really express is an identity, one that is recognized when the original sentence is recasted as "the number that

[26]This is what the everyday use of numbers seems to imply. In sentences like "there are four apples in the basket", it seems that the number Four is a property, or adjective, that the concept "apples in the basket" possess, something that can be rephrased as "the number four belongs to the concept 'apples in the basket' ". But this assumption does not clarify what "the number Four" stands for. This is what the Julius Caesar problem in (*GLA*,§55) tries to show. Heck Jr. (2003) provides a simpler example on the inadequacy of this account on numbers: to a concept to have a number n, it must be the case that there are n object that falls into it. But in order to prove that there are infinitely many numbers, infinitely many objects must be likewise available. An assumption like that can hardly be accepted as logical, if can be shown to be true at all.

belongs to the concept 'apples in the basket' is four", where the term "is" clearly works as an identity sign[27].

This is Frege's task through sections §62 − 70: "[Number] is only due to be determined in the light of our definition of numerical identity. Our aim is to construct the content of a judgement which can be taken as an identity such that each side of it is a number" (*GLA*,§63). The first attempt is what nowadays is called *Hume's Principle*: that two numbers are equal if, and only if, there is an one-one correlation between the objects falling into one and the other, or putting in other terms, that F and G are equinumerous. Denote $\#F$ as the "the number of F's", and $F \approx G$ as "F and G are equinumerous", then Hume's Principle[28] is:

$$\#F = \#G \leftrightarrow F \approx G$$

Now, the condition for two concepts to be equinumerous is twofold: the existence of a relation R such that R correlates them both and R is a bijection (a one-one correspondence) between them. Frege delineates both these conditions in §71 and §72:

> If now every object which falls under the concept F stands in the relation R to an object falling under the concept G, and if to every object which falls under G there stands in the relation R an object falling under F, then the objects falling under F and under G are correlated with each other by the relation R. (*GLA*,§71)
>
> 1. If d stands in the relation R to a, and if d stands in the relation R to e, then generally, whatever d, a and e may be, a is the same as e. 2. If d stands in the relation R to a, and if b stands in the relation R to a, then generally, whatever d, b and a may be, d is the same as b. (*GLA*,§72)

Breaking down these conditions formally, the following definition for equinumerosity holds:

---

[27] This is pointed out by Frege in (*GLA*,§57).

[28] Henceforth (HP). The name was coined due to Frege's quotation of Hume's *Treatise* in *GLA*.

$$\Vdash F \approx G \equiv \exists R[\forall x \forall y \forall z(R(x,y) \wedge R(x,z) \rightarrow y = z) \wedge$$
$$\forall x \forall y \forall z(R(x,y) \wedge R(z,y) \rightarrow x = z) \wedge$$
$$\forall x[F(x) \rightarrow \exists y(R(x,y) \wedge G(y))] \wedge$$
$$\forall y[G(y) \rightarrow \exists x(R(x,y) \wedge F(x))]]$$

But Frege does not consider (HP) sufficient. Although this is an identity condition for numbers, it does not provide an explicit and necessary definition of them. This is the Julius Caesar problem: we have no means to decide what $\#F$, or the locution "the number that belongs to F", stands for in the left side of (HP), or at least, (HP) does not by itself provides such means. In fact, it seems to already presuppose that they denote numbers.

For that reason, Frege's strategy is to provide an explicit definition for numbers, and then, to derive (HP) from it, keeping the identity condition. In (*GLA*,§68) he provides the definition: "the Number which belongs to the concept F is the extension of the concept 'equinumerous[29] to the concept F' ". This is where extensions enters the scene in Frege's logic. They can be viewed as sets: the extension of a concept being the set of objects which has, or satisfies, such concept. In the definition, the extension of the concept "equinumerous to the concept F" is the set of all concepts that can be correlated with F by a one-one relation. From that perspective, a number is a set of equinumerous concepts. Extensions are also objects in Frege's ontology, and hence, they satisfies the criterion in describing numbers as self-subsistent objects that can be subject of the identity relation.

In *GLA* Frege does not bother in explaining what extensions of concepts are[30], but in *GGA*, they would be identified as an especific case of value-ranges. Since concepts are then taken as functions with truth-values as values, the value-range of a concept is the set of objects which, when saturating the concept, denotes the Truth as truth-value. Value-ranges are governed by Frege's famous axiom V in *GGA*:

$$\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = \forall x(f(x) = g(x))$$

[29] Austin's translation actually has "equal" instead for the german *gleichzahlig*.

[30] "I assume that it is known what the extension of a concept is", he declares in a footnote in (*GLA*,§69).

Where $\dot{\varepsilon}\Phi(\varepsilon)$ denotes the value-range of the function Φ. Saddly enough, this was an unfortunate move, since axiom V was proved inconsistent by Russell's Paradox. But although Frege's use of value-ranges in *GGA* is common, they are mostly eliminable but in one place: the derivation of (HP) as a theorem[31]. This discovery takes place amidst the revival of Frege's logicism started in the 80's by Wright (1983). The goal was to show that the Dedekind-Peano axioms for arithmetic follows from second-order logic added with (HP) as an axiom. This follows very closely what Frege does in *GLA*, skipping the reference to extensions in §68[32]. This is now known as Frege's Theorem.

After assuming extensions[33], and defining (HP) as we already quoted above, Frege starts to define numbers by choosing a representative concept, and then to obtain the set of equinumerous concepts[34]. First, in (*GLA*,§72) he defines "n is a (cardinal) number" to be "there exists a concept such that n is the Number which belongs to it". This is:

$$\Vdash Card(n) \equiv \exists F(n = \#F)$$

With that and suitable concepts, Frege is able to define numbers directly. He starts in [*GLA*,§74] with the concept "not identical to itself", and then defines the number 0:

$$\Vdash 0 \equiv \#[x : x \neq x]$$

The choice is based on a logical fact about the law of identity: since everything is self-identical, nothing can be not self-identical. We start from the fact that $\forall x(x = x)$, and then add double negation (one of Frege's axioms), to get $\neg\neg\forall x\neg\neg(x = x)$. From the definition of the existential quantifier, this is $\neg\exists x\neg(x = x)$, which is the same as $\neg\exists x(x \neq x)$. This shows Frege's strategy in "choosing one [concept] which can be proved to be such on purely logical

[31] See (HECK JR., 1993) about this.

[32] This revival started a great debate on whether (HP) can, or cannot, be taken as logical. Frege himself rejected it. Therefore, this fregeanism, identified as neo-logicism, has differences from Frege's own project, given that in Frege's version, the choice of taking extensions as logical objects plays a great role. See (RUFFINO, 2003) about this. On the difference between both logicisms, see (RUFFINO, 1998) and (HECK JR., 2003).

[33] We can ignore Frege's use of extensions, since the use of the ancestral is what matters here. The biggest difference is on how to interpret the numerical operator 'the number of F's', or $\#F$, since it's here that extensions are mentioned.

[34] Frege's strategy in *GLA* is also well summarized by Dummett (1991, pp.119-124).

grounds" (*GLA*,§74). That way, it is a logical fact that nothing falls under such concept.

Next is the definition of the Predecessor relation: "there exists a concept F, and an object falling under it x, such that the Number which belongs to the concept F is n and the Number which belongs to the concept 'falling under F but not identical with x' is m" (*GLA*,§76). Formally, this is:

$$\Vdash P(m,n) \equiv \exists F \exists x[F(x) \wedge n = \#F \wedge m = \#[z : F(z) \wedge z \neq x]]$$

This definition still demands a proof of the one-one character of the relation, that is, the uniqueness of the predecessor. But from it, Frege defines the number 1 and proves that $P(0,1)$ holds in (*GLA*,§77). The definion uses the concept "identical with 0":

$$\Vdash 1 \equiv \#[x : x = 0]$$

It's easy to see that $P(0,1)$ holds as Frege intended: let x be 0 and F be the concept $[x : x = 0]$, then the three conditions of the definition are met, since $[x : x = 0]0$ holds trivially, $1 = \#[x : x = 0]$ holds from the definition of 1, and $0 = \#[z : [x : x = 0]z \wedge z \neq 0]$, since nothing can be equal and not equal to zero. Following this, in (*GLA*,§78) Frege enumerates some propositions to be proved about the definitions presented so far:

GLA1: $\vdash P(0,a) \rightarrow a = 1$

GLA2: $\vdash 1 = \#F \rightarrow \exists x F(x)$

GLA3: $\vdash 1 = \#F \rightarrow (F(x) \wedge F(y) \rightarrow x = y)$

GLA4: $\vdash \exists x F(x) \wedge \forall x \forall y(F(x) \wedge F(y) \rightarrow x = y) \rightarrow 1 = \#F$

GLA5: $\vdash P(m,n) \wedge P(m',n') \rightarrow (m = m' \leftrightarrow n = n')$

GLA6: $\vdash \forall x[\exists F(x = \#F) \wedge x \neq 0 \rightarrow \exists y(\exists G(y = \#G) \wedge P(y,x))]$

The last theorem, in Frege's words, is "Every number except 0 follows in the series of natural numbers directly after a number". This could means that he was regarding it as the proposition $\neg P(n, 0)$. But in (*GGA*,§44), it is clear that is the former theorem that he has in mind. The proofs of these propositions can be found in Appendix A.

The other important theorem, resembling the last one, is that for every number there is another that follows it. That is, that there are infinitely many numbers. This is what Frege sketches starting in [*GLA*,§79], and it is here that the Ancestral shows its importance for logicism. To say that after every number there is another one that follows it is to say that the set of numbers forms an ordered series without a last member. For that reason, the proof of such proposition depends upon a given concept: "member of the series of natural numbers ending with n". The ideia underlying such proof is the following. If we take any number, say 5, and enumerate all n's that are less than or equal to 5, we get $[0, 1, 2, 3, 4, 5]$. As we can see, there are 6 numbers less than or equal to 5 in the number series. So, for any n, the number of the concept $[x : x \leq n]$ must be the successor of n. So, Frege must show that the number of the concept "member of the series of natural numbers ending with n" is the successor of n. In the example of the number 5, all objects are proved to already exists, and so, we only need to consider the set, or the number of them in order to prove the existence of the successor, and then give a name for it. Other way to put it is that, once we have the number 0 proved by logical means, we then can name $1 \equiv \#[x : x \leq 0]$, since only 0 satisfies $0 \leq 0$. Then, having 1 proved also by logical means, we can name $2 \equiv \#[x : x \leq 1]$. Both 0 and 1 satisfies this. If we accept, as Frege did, extensions as legitimate objects, all the series of natural numbers can be so generated taking nothing from intuition or empirical data.

The series of natural numbers mentioned is the series of objects correlated by the predecessor relation[35]. For that matter, Frege applies the ancestral, both the strong and weak. The readings are as follows: $P^*(x, y)$ is "y follows x in the series of natural numbers", and $P^+(x, y)$ reads "y follows x in the series of natural numbers beginning with x". Then, the very notion of natural or finite number is defined in (*GLA*,§83), where n is said to be a natural number if "n is a member of the series of natural numbers beginning with 0", that is

$$\Vdash\!\!\!-\ \mathbb{N}(n) \equiv P^+(0, n)$$

[35]Again, Frege believed that the only way those definitions captures the right objects, that is, numbers, is the inclusion of extensions.

It follows that $\mathbb{N}(0)$ is a natural number, since the weak ancestral is reflexive. What is noteworthy is that from Frege's ancestral it becomes easy to prove that any successor of a natural number is also a natural number. This is:

$$\vdash \mathbb{N}(m) \wedge P(m,n) \rightarrow \mathbb{N}(n)$$

This is the same as saying that the property of being a natural number is hereditary in the predecessor relation. It follows easily from theorem **BS108** of *BS*, one of the forms of transitivity of the weak ancestral.

Now that we know the definition of the natural number series, a simple corolary is the fact that such series is linearly ordered. Transitivity follows easily from **BS98**. Just let R be the relation P:

$$\vdash P^*(x,y) \wedge P^*(y,z) \rightarrow P^*(x,z)$$

It is also an easy corolary that trichotomy for natural numbers follows from **BS133**, the definition of $\mathbb{N}$, and the fact that P is a one-one relation, theorem **GLA5**:

$$\vdash \mathbb{N}(x) \wedge \mathbb{N}(y) \rightarrow (P^*(x,y) \vee P^*(y,x) \vee x = y)$$

We can now better understand how P^* comprises the meaning of the relation $<$ on natural numbers, and P^+ the sign $\leq$. This is how Frege had already in *BS* envisaged the use of the ancestral, a use that does not require intuitions in proving that the natural number series is linear-ordered, a clear argument against Kant's proposal.

Next, Frege's strategy described in (*GLA*,§82) to prove that for every natural number there is one that follows, have the following steps:

1. The task is to show that the number of the concept quoted above, "member of the series of natural numbers ending with n", now formally $[z : P^+(z,n)]$, follows n in the series of natural numbers. That is: $P(n, \#[z : P^+(z,n)])$.

2. To prove such theorem, Frege uses induction. Precisely, he alludes to an induction theorem derived from the ancestral definition: the **BS81a** that we showed earlier. This is what he explains in (*GLA*,§82): it must be proved (1) that $P(0, \#[z : P^+(z,0)])$, (2)

that the concept $[x : P(x, [z : P^{+}(z, x)]\}$ is hereditary in P. (1) is the usual base step, while (2) is the inductive step.

3. But in order to prove that, Frege enumerate three necessary lemmas in (*GLA*,§83):

   (a) $\forall x(\mathbb{N}(x) \rightarrow \neg P^{*}(x, x))$

   (b) $P(x, y) \rightarrow \forall z[P^{+}(z, y) \wedge z \neq y) \leftrightarrow P^{+}(z, x)]$

   (c) $n = \#[z : P^{+}(z, n) \wedge z \neq n]$

   That is, (a) no natural number follows after itself in the natural number series; (b) if x is the predecessor of y, then the concepts "member of the series of natural numbers ending with y, but not identical with y" and "member of the series of natural numbers ending with x" are coextensional; and (c) that n is the number of the concept "member of the series of natural numbers ending with n, but not identical with n".

4. Finally, from these lemmas and induction, it can be proved that

$$\forall n[\mathbb{N}(n) \rightarrow P(n, \#[z : P^{+}(z, n)])]$$

   Which yields that every natural number has a successor, yielding the infinity of the natural number series.

But things were not that simple as Frege supposed in *GLA*. In *GGA*, his proof for the same proposition does not followed this sketch precisely. The reason for this is showed by Boolos and Heck in (BOOLOS; HECK JR., 2011): these propositions, and the induction theorem as well, need a finitude restriction in order to be provable. This is because, taking lemma (c) as example, if $n = \aleph_0$, then we have $\aleph_0 = \#[z : P^{+}(z, \aleph_0) \wedge z \neq \aleph_0]$ which is false, given that $P(\aleph_0, \aleph_0)$[36], and from **GLA5**, $z \neq \aleph_0$ implies that $\#[z : P^{+}(z, \aleph_0) \wedge z \neq \aleph_0] = 0$. For that matter, following Boolos' and Heck's corrections, those lemmas become:

GLAL1 $\vdash \forall x(\mathbb{N}(x) \rightarrow \neg P^{*}(x, x))$

GLAL2 $\vdash \mathbb{N}(y) \wedge P(x, y) \rightarrow \forall z[P^{+}(z, y) \wedge z \neq y) \leftrightarrow P^{+}(z, x)]$

[36]Frege hints a similar result in (*GLA*,§84). We offer a proof in Appendix A.

GLAL3 $\vdash \mathbb{N}(y) \land P(x, y) \land P(x, \#[z : P^{+}(z, x)]) \to y = \#[z : P^{+}(z, y) \land z \neq y]$[37]

But this demands also a condition for the induction theorem, the finititude restriction. In applying it into theorem **BS81a**, we get one step from the principle of mathematical induction, yielding:

$$\vdash F(x) \land \forall v \forall w (R^{+}(x, v) \land F(v) \land R(v, w) \to F(w)) \to \forall z (R^{+}(x, z) \to F(z))$$

This is still a general form of induction, but is one suitable enough to derive the principle of mathematical induction in Fregean terms. It's enough to let x be 0 to get:

$$\vdash F(0) \land \forall v \forall w (\mathbb{N}(v) \land F(v) \land P(v, w) \to F(w)) \to \forall z (\mathbb{N}(z) \to F(z))$$

This is enough to prove the desired theorem from mathematical induction. We have to just let F be the concept $[x : P(x, \#[z : P^{+}(z, x)])\}$ and prove all antecedents. This requires all lemmas, together with an important lemma about the ancestral:

$$\vdash R^{*}(x, y) \to \exists z (R(z, y) \land R^{+}(x, z))$$

Finally, it follows the desired fact:

$$\vdash \forall n [\mathbb{N}(n) \to P(x, \#[z : P^{+}(z, x)])]$$

Which states that for all natural numbers there is another that follows it. From all this, one can derive easily the Dedekind-Peano Axioms for arithmetic. The complete prove of all facts mentioned and the so-called Frege's Theorem are showed in Appendix A.

What is more noteworthy is how important Frege's Ancestral definition is for such results. (1) it is the foundation of the notion of natural, or finite, number[38]; (2) It's what made possible

[37]Even though this does not correspond exactly with Frege's wordings in *GLA*, this is what, according to Boolos and Heck, he's probably intendeed to be proving. Nevertheless, those restrictions are in place when this lemma is used to prove the desired theorem, so this facilitate the proof of the lemma.

[38]But some digression is necessary. In (*GLA*,§84), in contrasting Cantor's work on infinite cardinals with his own conclusions, Frege states that "Finite Numbers, certainly, emerge as independent of sequence in series". This seem's odd, given his own definition in (*GLA*,§83) that "n is a finite number" is to be understood as "n is a member of the series of natural numbers beginning with 0". Two facts might help understanding this:

the derivation of a general principle of induction and then the principle of mathematical induction, (3) it shows how the series of natural numbers can be proved to be linearly ordered by pure logical means, and finally, (4) it's crucial in deriving the important fact that every natural number has a successor. As corollary, this yields that the natural number series is endless: there is no last number in such series. That is

$$\vdash\!\!\!\!\!\!\!-\!\!\!-\!\!\!-\, \neg\exists x(\mathbb{N}(x) \rightarrow \forall y(P(y,x))$$

This also shows that there are infinitely many numbers, something that Frege thought to be possible to prove by logical means alone, without any appeal to intuition or empirical facts. In contrast, Russell logicism or even Zermelo-Fraenkel's axiomatic set-theory requires axioms stating the existence of infinite objects. Such axioms would hardly be counted as logical by Frege. The existence of an infinite number of numbers should not be something pressuposed, but proved in logical terms. But of course, Frege's proof cannot be taken as fully successful, given the inconsistency of its logic. And from a neo-logicist standpoint, it's still uncertaing whether (HP) is, or isn't, a logical axiom.

first, Frege's word is the german *endliche Anzahl*. An *anzahl* is primary a number used for counting, one that answer questions of the "how many?" type. This is a cardinal notion, and this is his primary concern. Moreover, in the finite case, there is a correspondence between cardinals and ordinals, since one could get from one to another without any trouble. Only for infinite cases such relation is problematic, something that Cantor's work discovered. But Frege still considers the cardinal notion as basic, and he does not consider them as dependent upon the position in a series, otherwise, and this is the second point, they would not be self-subsistent objects as he intended. Frege's ancestral is more general than the series of natural numbers. It's only when (HP), or extensions, are brought upon that one is abble to prove the basic theorems necessary for arithmetic as Frege did. It's the ancestral definition increased with cardinal notions that results into the series of natural numbers. Therefore, even though the ancestral is an ordinal notion, Frege's theory of numbers is cardinal in essense, and the series of natural numbers is not a series of ordinals in Frege's sense.

Chapter 4

The impredicativity of Frege's Ancestral

The failure of Frege's axiom V was not just an isolated problem, but something that undermined mostly every attempt into the foundations of mathematics that uses an unrestricted notion of sets or classes. Russell too was affected by it, and in the years following the discovery of the paradox attempted his own amendment. The root of the problem, something that unrestricted comprehension axioms pressupose, was that any formula can define a property or class. This was something accepted by Frege from the beggining, already in 1879[1]. But Russell's Paradox shows otherwise. It cannot be the case that expressions like $x \notin x$ define a set or class. Russell (1907) calls such cases as *non-predicatives*, later called *impredicatives*. They are distinguished from those expressions that can define a set as *predicatives*. But what exactly prevents such impredicative expressions from defining sets was better argued latter by Poincaré[2]: the presence of a vicious circle, or, the attempt to define an object in terms of a domain that already contains it. Russell's Paradox is a prime example where such vicious circle leads to paradoxical situations. For that reason, Russell's solution was to ban such unrestricted quantification, which evolved into his Theory of Types. This was done based on what he calls the Vicious Circle Principle: that no object (or set) should be defined in terms of itself, or the aggregate in which it is an element.

[1] Although not in terms of sets or extensions, but concepts.
[2] About this, see (FEFERMAN, 2005).

But are all impredicative definitions harmful? Not everyone of them have paradoxical consequences, but they are not free from criticisms either. This is the case for Frege's Ancestral. Already in 1887, Benno Kerry pointed out that Frege's definition of the Ancestral was impredicative, given its circular nature. Since Frege's strategy was to define natural numbers in terms of the Ancestral, this definition was doomed to fail according to Kerry. More recently, Ignacio Angelelli argued the same, adding that such circularity also undermines Frege's reduction of the very notion of series in logical terminology. In this chapter, our aim is to assess Kerry's and Angelelli's objections and to argue that, although Frege's definition is impredicative, it is not harmful as they considered.

4.1 Kerry and Angelelli

Frege's definition of the Ancestral relation is, recalling it:

$$\Vdash R^*(x, y) \equiv \forall F[Her(F, R) \land \forall z(R(x, z) \to F(z)) \to F(y)]$$

Textually, this states that 'x is the ancestor of y' (i.e., y follows x in the R-series), just in case y has all F hereditary properties shared by all descendents of x (i.e., all objects that follows x in the R-sequence). This is a second order definition since it quantifies over the domain of concepts, which in Frege's case, is an unrestricted one. In addition to the more famous debate about Frege's theory of concepts, Benno Kerry also criticized Frege over this definition. This was done essentialy in 1887 in *Über Anschauung und ihre psychische Verarbeitung* (KERRY, 1887, p.295), where he states that:

> Now, this criterion is to begin with of dubious value because there is not a catalogue of such properties, hence one is never sure that one has examined the totality of them. Moreover, there is the crucial fact that, as the author himself has proved [in a footnote Kerry cites *BS*, p.71 Theorem 97], of the properties that are hereditary in the f-series is also the following: to follow x in the f-series. Thus, the determination of whether y follows x in the f-series, according to the definition given for this concept, depends on whether, in addition to a lot of other

> things on hereditary properties in general, one knows, in particular, about the hereditary property "being a descendant of x", that y has it or not. It is clear that this circle should totally prevent from saying, in Frege's sense, that any y follows x in an f-series[3]

In justifying if $R^*(x, y)$ holds or not, it is required that, for every hereditary property F, one can decide whether F is one of the properties that is passed along from x to y or not. Kerry's point is that, since Frege's definition quantifies over all hereditary properties, there could be one for which this task would be uncertain. The case in question, as Kerry quotes, is Frege's theorem (97) of *BS*:

$$\vdash \forall u \forall v (R^*(x, u) \wedge R(u, v) \rightarrow R^*(x, v))^4$$

This theorem states that the property $[z : R^*(x, z)]$ is hereditary in R. This is the same as saying that the property "being x descendent" is something that every descendent of x pass along the parent relation. Exemplifying it, if Gottlob Frege is Karl's descendent, then Alfred, Gottlob's son, is Karl's descendent as well. The problem that Kerry is warning is that, to determine whether Alfred is Karl's descendent, we have to find such hereditary properties and check if Alfred have them. But in this process, the above property would require to determine if Alfred is Karl's descendent in the first place. Hence the circularity.

This circularity was latter restated by Ignacio Angelelli in *Frege's Ancestral and his Circularities* (ANGELELLI, 2012). He actually presents two circularities, the first one being Kerry's, as stated above. The second one is aimed to show how Frege's intended reduction of the notion of a series failed as well. As we saw earlier, a series is simply a connection between elements, one that is at least transitive. This is also the notion that there is a 'chain' between one object to another given a finite number of steps. As we already alerted, Frege does not speak about the ancestral relation, but such notion is equivalent: a connection, or a path, between one person (the ancestor) to another (the descendent) that is also transitive. This is what Angelelli defines as the Ordinary Ancestral, which we now denote as (OA).

Frege's definition was a reduction of (OA) into logical terminology. This is what he

[3]The translation is Angelelli's (2012). Where he translates "being a descendent of x", Kerry actually writes simply "to follow x", since Kerry, as Frege, does not name such relation as Ancestral.

[4]We avoided here the *definiendum* of the Hereditary Property, showing only the *definiens*.

argues in (*BS*, p.104): "I sought first to *reduce* [*zurückführen*] the concept of ordering-in-a-sequence to the notion of *logical* ordering, in order to advance from here to the concept of number"[5]. This is also, according to Angelelli's interpretation, what Frege maintains in defining the Ancestral, in (*GLA*,§79), by saying that the *definiens* "is to mean the same as" [*sei gleichbedeutend*] the *definiendum*. For that reason, he concludes that

> It seems natural do interpret the product of the reduction as intended to replace the initial notion. In alternative terms, it seems natural to construe the Fregean ancestral as an *analysans* that replaces, in Frege's project, the *analysandum* (the common ancestral). Such would be the analysis interpretation. (ANGELELLI, 2012, p.478)

Since Frege was also interested in developing a logical system capable of replacing the natural language, his definition of the Ancestral, henceforth (FA), should be taken as a replacement of (OA). But is Frege's intended reduction successful? According to Angelelli, no.

4.1.1 The Circularity of (FA)

Both Kerry's and Angelelli's circularities follow the same basic principle: that (FA) is circular given it's unrestricted quantification over properties. We focus on Angelelli's version of both circularities, with the following fictional situation added by him. A certain Fritz is trying to convince a jury that he is Karl's descendent in order to inherit Karl's money. Since Fritz doesn't have the necessary documents, he is tempted to quote Frege's definition: that he is Karl's descendent if, and only if, he has all hereditary properties shared by Karl's descendents. The jury asks him to check whether Frege's definition could help him with two especific properties: 'being the fregean-descendent of Karl' and 'being the ordinary-descendent of Karl'. Let us recall that (FA) is $R^*(x, y)$, and consider now that $R^o(x, y)$ denotes that x is the ordinary ancestor (OA) of y. From that, the properties questioned by the jury are $R^*(a, x)$ and $R^o(a, x)$ respectively, where a is a name for Karl. Assuming that Fritz is tempted to check such properties, the first one generate Kerry's circularity, while the

[5]The first emphasis is ours, the second is Frege's. The phrase "logical ordering" is a translation for *logische Folge*, which Stefan Bauer-Mengelberg better translates as "logical consequence" in (FREGE, 1967, p.5).

second exemplifies Angelelli's argument against Frege's reduction of the Ancestral, what he calls the analysis interpretation. More precisely, the first is the following:

1. **Kerry's Circularity**

    (a) In order to prove that $R^*(a,b)$ holds, one has to show all F properties which are hereditary in R such that, if $\forall z(R(a,z) \rightarrow F(z))$ then $F(b)$;

    (b) $R^*(a,x)$ is such property;

    (c) Then, one has to show that b has the property $R^*(a,x)$, i.e., that $R^*(a,b)$ holds;

    (d) This is circular, hence, $R^*(a,b)$ cannot hold.

Assuming that b is a name for Fritz. The second circularity is an argument against the Analysis Interpretation. It goes as follows:

1. **Angelelli's Circularity**

    (a) Assume that (FA) is a reduction of (OA);

    (b) From (a), to show that $R^o(a,b)$ holds, one has to show that $R^*(a,b)$ holds;

    (c) From (FA), to prove that $R^*(a,b)$, one has to show all F properties which are hereditary in R such that, if $\forall z(R(a,z) \rightarrow F(z))$, then $F(b)$;

    (d) $R^o(a,x)$ is such property;

    (e) Hence, one has to prove that b has the property $R^o(a,x)$, that is, to prove that $R^o(a,b)$ holds;

    (f) This is circular. Therefore, (FA) is not reduction of (OA).

Assuming again that a and b are names for Karl and Fritz, respectively. Both circularities can be generalized for any x and y. The first circularity is what we already discussed above. The second one goes between (b) and (e). It simply states that, if one wants to prove that someone is his ordinary ancestor in terms of Frege's definition, at some point, he would have to prove that he is the ordinary ancestor of that person, hence, the proof would require the conclusion as one of his premises. The consequence of such circularity is that in order to (FA) successfully reduce (OA), (OA) could be stated in terms of (FA) and (FA) alone, but no such reduction is possible, since everytime we want to show that (OA) holds in terms of (FA), we are obbligated to prove (OA) in the first place.

Angelelli's conclusion, then, is that at best (FA) is an enrichment and a generalization of (OA), not an analysis or reduction. In his words

> The enrichment occurs through the "discovery" of the property of being hereditary that many properties have [and] includes the focusing on, and helps towards the demonstration of the formal properties of the ordinary ancestral, e.g., transitivity, which as Frege points out is what leads to the logical understanding of arithmetical induction. The generalization is accomplished in that the ancestral's underlying relation as such is conceived in a most abstract fashion. (ANGELELLI, 2012, p.498-499).

Since (FA) quantifies unrestrictedly over the domain of properties, the ordinary ancestral still appears in its scope. For that reason, (FA) cannot properly substitute (OA). The alleged enrichment of (FA) was already been extensively discussed above. Not just the transitivity, but given the suitable conditions, (FA) is also trichotomous and fundamental for the natural number series and the principle of mathematical induction. The generalization is due simply by the fact that the Ancestral is actually not a first-order relation *per se*, but a second-order property that can be applied to any first-order relation. Either way, Frege's definition is impredicative, as the first circularity clearly shows.

4.2 The circularity revisited

It's pretty clear that (FA) is impredicative. But yields the impredicative nature of (FA) a circular definition? Is it harmful as Kerry and Angelelli supposed? Apparently not. Both arguments have, at least, one problematic premise. This, added with Frege's philosophical motivations, might help understanding both circularities. In what follows, we argue against their conclusions. First, we recaptulate Frege's reduction of (OA) into (FA) and his philosophical justification for it. This will guide us into the second point: the fact that the circularity is due to some confusion about the role of quantification in Frege's concept-script. This help us understand why such circularities are not a problem for Frege, and perhaps also why he never answered Kerry's criticisms. Then, we argue that even granting Kerry's and Angelelli's interpretation as valid, the verification of (FA) is not possible as they imagined it to be. In arguing for such, we conclude that Frege's Ancestral is not circular, although still impredicative.

4.2.1 Frege's Reduction

We have already pointed out much of Frege's philosophical motivations. Logicism was, from the beggining, his main motivation, one that could very well be summarized as the attempt of freeing arithmetic from intuitions. This was done essentialy in two fronts: showing that arithmetical concepts are reducible into logical terminology and showing that arithmetical modes of inference are reducible to logical modes of inference. Angelelli is well aware of this motivations. He recalls two important points: 1) that for Frege any consideration about features of particular cases of (FA) are not essential and 2) that the proof of $R^*(x, y)$ should not be an enumeration of each point of the chain that starts from x and leads to y. These are important points that Angelelli rightfully mention, but failed to link to the circularity problem.

The first thing, argued in *BS*, is the generality of (FA):

> The propositions about sequences developed in what follows far surpass in generality all similar propositions which can be derived from any intuition of sequences. Therefore, if one wishes to consider it more appropriate to take as a basis an intuitive idea of sequences, then he must not forget that the propositions so obtained, which might have somewhat the same wording as the ones given here, would not state nearly so much as these because they would have validity only in the domain of the particular intuition upon which they were founded. (*BS*, §23)

From that, it follows that Frege's definition is one that generalizes the basic notion of a (transitive) series. Thus, the ordinary ancestral relation holding between human beings, the relation of one number following another in the natural number series, or any other partially ordered series is an instance of Frege's definition. Particular cases of (FA) can be about objects that are only aprehended from this or that intuition, or none at all. But (FA) does not need so, and this is confirmed by the fact that theorems regarding it are proved from "pure thought" alone (*BS*,§23). The generality of the definition is again mentioned in (*GLA*,§80): "since the relation R has been left indefinite, the series is not necessarily to be conceived in the form of a spatial and temporal arrangement, although these cases are not excluded".

Following this passage, Frege starts arguing against what we might consider a step-by-step proof procedure for the ancestral. This is: "if starting from x we transfer our attention continually from one object to another to which it stands in the relation Φ, and if by this procedure we can finally reach y, then we say that y follows in the Φ-series after x" (*GLA*,§80). From this procedure, if Fritz wants to prove that he is Karl's descendent, he just have to prove that he is the son of a son of ... a son of Karl. This is something like:

$$\exists x_1, ..., \exists x_n(P(a, x_1), ..., P(x_n, b))$$

Where P is the usual parent relation. Hence, a step-by-step proof like this requires that each link of the chain between Karl and Fritz to be proved. Frege does not rely on this kind of proofs. He continues by saying that "this describes a way of *discovering* that y follows, it does not define what is *meant* by y's following"[6] (*GLA*,§80). This discovery is not necessarily a definition, or at least is not a good one. And this is because Fritz being or not Karl's descendent does not depend upon his proof in front of the jury. As Frege argues:

> Whether y follows in the Φ-series after x has in general absolutely nothing to do with our attention and the circumstances in which we transfer it; on the contrary, it is a question of fact, just as much as it is a fact that a green leaf reflects light rays of certain wave-lengths whether or not these fall into my eye and give rise to a sensation, and a fact that a grain of salt is soluble in water whether or not I drop it into water and observe the result, and a further fact that it remains still soluble even when it is utterly impossible for me to make any experiment with it. (*GLA*,§80).

This is an aspect of Frege's realism: that there are facts independent of our way of aprehending them. This certainly is the case for arithmetic, since numbers are, as Frege regards

[6]The emphasis is ours.

them, self-subsisting objects and arithmetical propositions are true independent of our way of regarding them, which is also Frege's anti-psychologism. We cannot take the mental representation of an object as the object itself. At least as far as arithmetical propositions are concerned, the genesis of a representation has nothing to do with the justification for regarding it as true. This is argued in the very beginning of *BS*:

> on the one hand, we can ask by what path a proposition has been gradually established; or, on the other hand, in what way it is finally most firmly establishable. Perhaps the former question must be answered differently for different people. The latter [question] is more definite, and its answer is connected with the inner nature of the proposition under consideration. (*BS*, p.103).

He then concludes that it's "[...] not the psychological mode of origin, but the most perfect method of proof underlies the classification" (*BS*, p.103). The point was also made in (*GLA*, introduction) as a dictum: to "Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it". Still in the introduction, he also argues "[...] that a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes" (*GLA*, introduction). All this is to reminds us how Frege distinguished between the discovery that a proposition is true to the justification of it.

Returning to Fritz case, one could, and expect, to prove his heritages by showing each step of the chain between him and Karl. But Frege's realism towards truth assumes that this fact, whether Fritz is or isn't Karl's descendent, does not change according to Fritz defense for it. He is Karl's descendent or not regardless if the jury is convinced with his speech. Frege himself is saying that:

> What I have provided is a criterion which decides in every case the question Does it follow after?, wherever it can be put; and however much in particular cases we may be prevented by extraneous difficulties from actually reaching a decision, that is irrelevant to the fact itself. (*GLA*,§80)

Needless to say, Frege's definition is consistent with his own philosophical recomendations. $R^*(x, y)$ cannot rely upon a step-by-step proof, by showing each link of the connection between x and y. Theorem **BS98**, about the transitivity of the ancestral, could be equally proved in this intuitive way. If there is a path between x and y, and one from y and z, we could prove that there is another from x and z, by starting from x, reaching y and moving along until we finally reach z. But should the proof depends on such procedure? According to Frege, certainly not:

> We have no need always to run through all the members of a series intervening between the first member and some given object, in order to ascertain that the latter does follow after the former. Given, for example, that in the Φ-series b follows after a and c after b, then we can deduce from our definition that c follows after a, without even knowing the intervening members of the series. (*GLA*,§80).

As we saw, Frege is offering a logical definition that made possible to prove important facts about the ancestral in a pure logical fashion, with maximal generality and no need for intuitions in its proofs. This is (FA). (OA) in the other hand, is this exact intuitive notion that Frege wants to avoid, one that need a step-by-step proof, which is intuition-dependent and rely upon this or that particular case of application. Thus, and Angelelli is right about this, (FA) is a reduction of (OA), a generalization and certainly an enrichment, one that seeks to provide for a common notion, (OA), a formal and precise formulation that is purely logical. But Angelelli's and Kerry's argument for the circularity made use of an assumption that is very much akin to what Frege is trying to rule out in (FA).

4.2.2 Is there a circularity?

But then, why Fritz is incapable of proving his desired heritage of Karl by simply quoting (FA)? Because (FA) doesn't work that way. By asking Fritz to prove whether he has the property "being Karl's descendent", the jury is asking this exact step-by-step proof that Frege wants to shun, but instead of speaking of a proof of each link of the ancestor to the descendent, it's a proof of each hereditary property that is passed along from on to the other.

Moreover, both arguments for the circularity of (FA) uses the assumption that in order to verify the truth of a quantified sentence like $\forall x \varphi x$, one has to show first that φ holds for every object in the domain. That is, a quantified sentence requires a justification for each of it's instances in order to be true. More than that, this requires that the desired sentence is true *only* if we could, in principle, verify each instance in advance. This, of course, would get us into trouble if the quantified domain is not finite. Even in the case of (FA), it's not clear whether the domain of properties is finite or not. Either way, sentences as "every single man is unmarried" or "every natural number is either even or odd" would require more than is possible as justification in order to be true. This is not what quantification is about. But this is required if such circularity is to be derived from (FA): only in this scenario, the question whether b has the property $R^*(a, x)$ can be raised *prior* to the justification for $R^*(a, b)$ in the first case.

But this is what at least one premise of each argument pressuposes in order to be valid. They are premises (a) and (c) from the first and second arguments, respectively. They assert that:

> to prove that $R^*(a, b)$, one has to show all F properties which are hereditary in R such that, if $\forall z(R(a, z) \rightarrow F(z))$, then $F(b)$;

This is precisely the worry that Frege had answered in GLA, but now regarding properties. One need no checking of each step of the sequence to prove that $R^*(a, b)$. Analogously, one needs no checking of each hereditary property in order to prove that $R^*(a, b)$. In this case, both Kerry and Angelelli seems to be confusing a logical justification with an epistemological one. First, we have a step-by-step proof of the ancestral relation that Frege ruled out, i.e., the checking of each link of the series in order to verify the connection between the ancestor and the descendent. This is an empirical/epistemic proof. In the same way, Angelelli and Kerry are assuming a epistemic/empirical proof for the application of (FA): the checking of each hereditary property that the ancestor passes through R and the subsequent checking of the same properties in the descendent. Is safe to say that Frege would ruled that out too.

Thus, there is a confusion regarding the role of the universal quantifier and his logical justification. Frege never answered Kerry about the circularity problem, but Russell had in the appendix to Frege in his *Principles of Mathematics*:

> This argument, to my mind, radically misconceives the nature of deduction. In deduction a proposition is proved to hold concerning *every* member of a class,

> and may then be asserted of a particular member: but no proposition concerning *every* does not necessarily result from enumeration of the entries in a catalogue. (RUSSELL, 1996, p.522)

The same can be applied to Angelelli's argument. He even quote Russell but doesn't discuss his response in his paper. He also mention Carnap's response, but do not discuss it. Carnap himself argues in favour of Russell and Frege:

> [...] in order to demonstrate the truth of a universal sentence, it is not necessary to prove the sentences which result from it by the substitution of constants; rather, the truth of the universal sentence is established by a proof of that sentence itself. The demonstration of all individual cases is impossible from the start, because of their infinite number, and if such a test were necessary, all universal sentences and all indefinite predicates (not only the impredicative ones) would be irresoluble and therefore (by that argument) meaningless. (CARNAP, 1937, §44)

Then, not just this proof procedure is untenable, but is impossible in cases where the domain in question is infinite, since a proof is a finite operation.

As already mentioned, Angelelli is well aware of Frege's intentions, but he does not seem to find a connection between them and his own version of the circularity. In Angelelli's version, in contrast to Kerry's, the focus is on the proof that $R^o(a, x)$ is an hereditary property in R (premise (d)). But as we see it, this can be answered in the same way. That Fritz, here denoted by b, has the property $R^o(a, x)$ does not depends upon any checking. Either $R^o(a, b)$ holds or not. In the fictional story proposed by Angelelli, it's pretty clear that Fritz goes to the court to show his heritages, not to determine the truth of it.

Fulugonio (2008) gives an argument in favour of the circularity as well. Her arguments are very similar to Angelelli's[7], and reiterates that the circularity undermines Frege's project completely. She argues that:

> The only suposition in Kerry's criticisms to the Fregean definition of succession is that Frege's construction, and the definition of succession in particular, has a

[7]And in her defense, came first than Angelelli's paper.

> defined gnosiological motivation, one that Frege certainly explicits in many opotunities throughout his work. From that, if, as part of his project, Frege intends to clarify the notion of succession, his elucidation is - at least - insatisfactory if it demands, among other things, that it is known what it attemps to be clarifying. (FULUGONIO, 2008, p.9)[8]

As we discussed earlier, this is false. In many oportunities in *GLA* and already in *BS*, Frege argues in the oposite direction, regarding the justification for a proposition in contrast to the way one can regard it as true. As we take it, as far as the Ancestral is concerned, definitions and proof methods does not follow any epistemic worries, but only by assuming this that one can regard both circularity arguments as sound. Furthermore, Fulugonio argues that Russell's defence of Frege's definition of the Ancestral missed one important point, in not discussing the particular case where the property "[..] 'to follow x' is precisely one of the hereditary properties in which we find ourselves in a vicious circle" (FULUGONIO, 2008, p.8)[9]. But Russell's response goes to the heart of the problem: only by taking an incorret account on quantification that such property becomes problematic as Kerry's supposes. In his short response to Angelelli, Heck Jr. (2016, p.101), adds that "Russell is making an elementary logical *cum* epistemological point: a universal generalization does not have to be derived from the conjunction of its instances, and knowledge of a universal generalization need not rest upon knowledge of its instances".

It should also be noted that Fritz's case is a particular one: a particular application of (FA) to a particular individual. For that matter, it's hard to imagine how Fritz could prove $R^*(a, b)$ without a likely empirical (*viz.* intuitive) proof. And without some grounds, he cannot. It seems that (FA) is not suitable for helping his case, but again, this was not Frege's interest either. Heck also adds that "the power of Frege's definition shows itself not in particular cases but in results like [...] the *generalizations* that it allows to us to prove"

[8]In the original: "El único supuesto presente en la crítica de Kerry a la definición fregeana de sucesión es que toda la construcción fregeana, y la definición de sucesión en particular, tiene una aspiración gnoseológica definida, aspiración que por cierto Frege explicita en reiteradas oportunidades a lo largo de toda su obra. De modo que, si como parte de su proyecto lo que pretende Frege es elucidar la noción de sucesión, su elucidación es –cuanto menos– insatisfactoria si ella exige, entre otras cosas, que sea ya conocido aquello que se quiere elucidar". The translation is ours.

[9]In the original: "[...] que "seguir a x" es precisamente una de tales propiedades hereditarias, con lo cual nos encontramos ante un círculo vicioso". The translation is ours.

(HECK JR., 2016, p.101-102). Frege's intensions are with his logicism account of arithmetic, and in this particular case, no "checking" procedure is necessary if numbers are regarded as logical objects, otherwise, propositions about numbers would be empirical ones. As Russell (1996, p.522) completes, "Kerry's argument, therefore, is answered by a correct theory of deduction; and the logical theory of arithmetic is vindicated against its critics"[10], and here we add that Angelelli's argument is likewise answered by it.

4.2.3 Some further points

Let us consider for now that the problematic premise in both circularity arguments is true, thus asserting that (FA) is proved with a checking procedure, or in Kerry's words, that we check the catalogue of properties. More precisely, $R^*(a, b)$ holds if, and only if, for each F that is hereditary, it could be verified that all R-descendent of a have F, then $F(b)$ as well.

It's important to notice that a property can be hereditary in a given series even though it does not hold to all objects, that is, it could be the case that one property is hereditary in a given point of one series but not for the preceding objects of that point. In this case, consider the property 'being greater than 10'. This is an hereditary property for all numbers n such that $n > 10$, but is not for those n's such that $n \leq 10$. The same holds for the problematic property "being a's descendent", since a is not a descendent of itself, it does not have such property. But every R-descendent of a have it.

Keeping that in mind, Let R be a relation defined on a domain $D = \{a, b, c\}$, and let $R(a, b)$, $R(b, c)$ holds. We want to ask two questions: 1) whether $R^*(a, b)$ or $R^*(b, c)$ holds, and 2) if $R^*(a, c)$ holds as well.

1. To verify if $R^*(a, b)$, we must then look for all F-hereditary properties shared by R-descendents of a, and then look for these same properties on b. But since $R(a, b)$, there is no R-descendent of a prior to b, only b itself (for convienience, assume that R is one-to-one). So we must look into b's properties that are hereditary and check all of

[10]Russell also points out that this is very similar to Mill's objection to Barbara inferences, but can be likewise answered. He concludes that "general proposition can often be established where no means exist of cataloguing the terms of the class for which they hold; and even, as we have abundantly seen, general propositions fully stated hold of *all* terms, or, as in the above case, of *all* functions, of which no catalogue can be conceived" (RUSSELL, 1996, p.522).

them. But in this case, what would we find in such catalogue of b-properties? This is not clear at all. In fact, since b is the first imediate R-descendent of a, the only option is look directly over a's properties and then check if b has them. But this is not what (FA) states, and even if it was, the property $R^*(a, x)$ is not one of them, since (FA) is not reflexive.

2. Now, if we want to verify if $R^*(a, c)$, that is, the case where c is not an imediate successor of a in the R-series, we should check all hereditary properties of the R-descendents of a as well, in this case b. But again, there could be some property that b does not have and that c has exaclty because c follows b in the R-series, say, the property "being a's descendent that follows after b". If we consider Frege's definition of natural number in terms of the Ancestral, as exposed in chapter 3 above, we could very well ask about the proposition $P^+(0, 10)$. In this case, the number 10 has hereditary properties that no number before it in the natural number series has, and that 10 has *because* it is a natural number that is greater than all preceding numbers. For example, the property $(P^+(0, x) \wedge P^*(9, x))$, that is, "being a natural number greater than 9" is one of them.

These situations seems, and are, very confusing if we assume that every single property should be verified in order to prove any instantiation of (FA). Frege's response for this would be the following: one could, and should, reach the conclusion that "being a natural number greater than 9" is an hereditary property in the natural number series after the number 9 by logical means alone, i.e., whitout any intuition-based faculty, since we could not get this property by simply checking all preceding members of the series. Of course, this would be a direct response to Angelelli and Kerry, not for a kantian, since there is no need to check properties in the kantian sense. The problem is that one should be able to prove that, *e.g.* "being a natural number greater than 9" is a property that all numbers greater than 9 has, even though 9 itself does not have it.

This rather ordinary procedure is not the only way one can prove that $R^*(a, b)$, or to Fritz prove that he is Karl's descendent[11]. Fritz would easily prove that he has this property if he had grounds to show his heritages in the first place. This grounds, however, must be

[11]The same point is made by Heck Jr. (2016).

logical ones. But is hard to believe that this is what matters in Fritz case. If the question was to decide if $P^*(0,3)$, for example, there would be a logical way to check it, provided Frege's own results in *BS* and *GLA*[12]. But Fritz is not a logical object, so there would be no pure logical form to decide whether he is Karl's descendent or not. Angelelli argues that (FA) is a reduction to the sense that it's supposed to substitute (OA), to the extend that Frege's intention was to provide a more perfect language than the ordinary one. But Frege's intention, as we see it, certainly was a tentative to provide better definitions for ordinary concepts but only as they are necessary for mathematical purposes. In this sense, there is no reason to expect from Frege the intensions that Fritz had in the fictious example that Angelelli provided.

Finally, if we expect Fritz to prove his heritages *only* in the presence of a full proof of each instance of the parent relation from him to Karl, Fritz would have problems to show, for example, that if he is son of someone who is Karl descendent, he is Karl's descendent as well. But it is clear that, provided those cases, he would be, even whitout a demonstration of the complete path from him to Karl. If Fritz had those two premises proved, he could prove that he is Karl's descendent with the aid of Theorem (96) of *BS*:

$$\vdash R^*(x,y) \wedge R(y,z) \to R^*(x,z)$$

These are the kind of results that were important in Frege's point of view, i.e., that of arithmetic. Likewise, should one prove that $R^*(a,x)$ is an hereditary property by looking to every member of the R-series for such? Certainly not. It can be easily proved from the definition alone. And Theorem (97) of *BS* does exaclty that, not simply offering the problematic property as Kerry's argued.

4.3 Conclusion

In what we argued above, (FA) is not circular as Kerry and Angelelli intended. But is still an impredicative definition. The property $R^*(a,x)$ is still in the scope of the quantifier.

[12]And ignoring the inconsistency of his theory of extensions.

But this is not a real problem for Frege. At least not for the Ancestral. Frege's approach towards properties, or more precisely functions in his ontology, is a Realist one: the domain of functions is independent of the definition that one can give for them. There is no real problem in defining one function in terms of the totality in which is itself a member, since the function defined does not exists only *after* the definition.

But this problem is at the heart of logicism, since as we saw, the Ancestral is used to define the very notion of natural numbers. Recall that n is a natural number just in case $P^+(0, n)$ holds for n. But this is also an hereditary property: a successor of a natural number is itself a natural number[13]. But $P^+(0, x)$ holds if x has all hereditary properties of 0 that all P-successors of 0 have, where $P^+(0, x)$ is one of them. If we follow what we have discussed already, this is not itself a circularity, having no problem for a realist like Frege.

Hence, not every impredicative definition is harmful, or at least problematic from a fregean point of view. For that reason, not everyone was in full agreement with the complete ban of such definitions, as Russell's vicious circle principle recommended. As Ramsey (1931, p.41) famously put it, we can "[...] describe it in a certain way, by reference to a totality of which it may be itself a member, just as we can refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle". Likewise, Carnap (1937, p.164) just assume that accepting or not impredicative definitions was "[...] a question of choosing a form of language — that is, of the establishment of rules of syntax and of the investigation of the consequences of these".

The other place where impredicativity affects Frege was in his full comprehension for functions. This, added to Axiom V regarding value-ranges, yields Russell's paradox. It was this problem that motivates Russell in avoiding such impredicative cases. As we saw, most of the important theorems regarding the Ancestral made use of comprehension axioms as well. It was for this reason that, in declaring such axioms in the language, is necessary to provide some restrictions in order to avoid cases like

$$\exists F \forall x (F(x) \leftrightarrow \neg F(x))$$

[13] See theorems **BS108** and **Her(ℕ)** in Appendix A.

to be derivable. But even with this restriction, Frege's Ancestral works as intended.

Finally, accepting that Frege's definition of the Ancestral is not circular as both Kerry and Angelelli intended, we can restate his main argument of the paper as the following:

> P: Frege's ancestral is a reduction of the ordinary notion into logical terminology (The Analysis Interpretation);
>
> P: Frege's Ancestral is circular in regard to the ordinary notion;
>
> C: Therefore, it is not a proper analysis.

But having argued that there is no circularity in Frege's definition, we can attack Angelelli's conclusion by simply denying the second premise, keeping Frege's reduction safe.

Concluding Remarks

As we have seen, the very notion of an ordered series is crucial for the foundations of arithmetic. Frege's project, the reduction of arithmetic into logic, was assumed already in 1879, where the writing and developing of the concept-script logic had this in mind. Such project had philosophical motivations too, that includes, among others, a rejection of the kantian thesis about the same notion, a thesis that we saw in chapter 1. Where's Kant regarded intuition as necessary, both for explaining ordered series and numbers, Frege was convinced about the opposite. Numbers, assuming Frege's definition in terms of extensions, are not intuitive notions. Likewise, the very notion of ordered series is a logical procedure. This was his goal in the beggining with *BS*, which clearly shows how he was already committed to logicism. For that matter, his motivations were described in kantian terms: to prove the analyticity of arithmetical propositions, and to show how analytical proofs can be nonetheless informative. All this was discussed in chapter 2. We then finally showed, in chapter 3, the important fregean definition for the Ancestral. The theorems proved also show how his definition realizes his philophical motivations, and more precisely, its important role in proving the basic laws of arithmetic, which we described as the Dedekind-Peano axioms. But despite Frege's incredible achievements, problems and criticisms arised, and in chapter 4 we argued against one of them.

Frege's conception of logic as universal, ranging over the totality of objects and concepts, is brought up into question in much of those criticisms. And more precisely, quantification over the full domain of properties, or concepts, is what gives Frege's logic much trouble into modern eyes. There are other problems, or criticisms, about Frege's definition that worth being mentioned. To begin with, with the fact that (FA) is second-order, quantifying over the domain of properties without restriction, we have Quine's worries that second-order logic is "set theory in sheep's clothing", and not logic at all. This point is not of Frege's concern, since irrestrict second-order logic *is* logic by default for him. It is also prior to set-theory, given that extensions are, at least until 1924, logical objects.

Parsons (1995), quotes another problem that Frege's logical reduction in general is subject to[14]. Frege claims to have derived the principle of induction (both general and mathematical) from the ancestral definition and the definition of natural numbers. But, it can be argued, much of the work already seems to pressupose the very notion of induction. First, Frege might need such tool to define basic notions of the system[15], and second, to argue that such definitions captures the intended meanings[16].

More recently, Heck Jr. (2016) tackled this problem[17]. As we argued, Frege's Ancestral is his way of defining linear orders for foundational purposes. He does not name his definition as "Ancestral", but the question that Heck asks is whether his definition does behave like intended: as the usual ancestral relation, or equivalently, as a series that is at least transitive. But the problem is that, if we try to prove the extensional adequacy of Frege's definition with the desired behavior, one must proceed by induction. Then, "we have to use arithmetical induction to convince ourselves of the correctness of the definition we use to prove arithmetical induction" (HECK JR., 2016, p.102). Even if we skip the problem assuming that Frege was not at all worried about the ordinary ancestral, the same difficulty reapears in the definition of natural number. How can we be sure that Frege's natural numbers are in fact *the* natural numbers? We cannot rely on induction, otherwise, Frege's derivation of such rule of reasoning from the Ancestral would be utterly circular. Heck himself offers a solution: a different definition for the ancestral that captures the intensional meaning of the ordinary ancestral that, although extensionally equivalent to Frege's, does not use arithmetical induction. But to enter Heck's definition and proof would take us too far afield now.

Going back to the Frege-Kant differences, Boolos (1985), also adds the following questioning. Frege's proofs uses, as we saw, the rule of substitution, which is equivalent to

[14]This is actually credited to Seymour Papert's paper *Sur le réducctionnisme logique.*

[15]*e.g.* in defining what is a theorem, which usually is done by an inductive definitions.

[16]Parsons writes "Inductive definitions, especially, play an essential role both in setting up a system of set theory and in establishing the correspondence between it and the system of number theory." (PARSONS, 1995, p.202).

[17]It should be noted that much of this criticisms are discussed by Heck's paper. The notes here own's much to it.

comprehension axioms. In Frege's logic, this is provided by the fact that variables standing for concepts range over the domain of all properties. Hence, comprehension ensures us that, for a given predicative sentence, there exists a property/concept correlated to it. But as a kantian interlocutor might question, how can we be sure that there is such a property? The main case, as the interlocutor might argue, is that "[...] we cannot admit substitution as a logical rule unless we are prepared to admit that all instances of the comprehension schema $\exists X \forall x(Xx \leftrightarrow A(x))$ are logical truths" (BOOLOS, 1985, p.171). But this is hardly acceptable by a kantian, who concludes that it is intuitions that licenses comprehension, by showing the existence of such properties. Hence, the most important rule in Frege's proof might be argued to be some form of intuition. Boolos solution is actually to find a way to rephrase the formulas of the *BS* in order to avoid mentioning special items over which the second-order variables range. Once again, Boolos solution would take us too far.

Frege also believed that in deriving arithmetic as a "development of logic" (*GLA*,§87), the application of arithmetic should be understood as a corolary: since logic has the most inclusive domain and is applycable to everything[18], our use of numbers does not directly point to the world: "The laws of number are not really applicable to external things; [...] They are, however, applicable to judgements holding good of things in the external world: they are laws of the laws of nature" (*GLA*,§87).

The history of Frege's logicism is well know: Frege's axiom V is inconsistent, and the project was then, at the foundations, a failure. In 1924, Frege retracted from the view that arithmetic does not rely on intuitions. But this is, at least, the case for Frege's notion of cardinals numbers as dependent on extensions. The other part of Frege's project that long survived this disastrous faith is the Ancestral Definition. Russell's and Whitehead's analysis of the same notion follows Frege very closely, the same goes for Quine and even the neo-logicists today. But even this is not free from criticisms, and many problems are still open to debate, as we saw.

To sum up, in concluding *GLA*, Frege believed "to have made it probable that the laws of arithmetic are analytic judgements and consequently *a priori*" (*GLA*,§87). His criticisms

[18]Saving some restrictions: concepts without sharp boundaries cannot be counted, as Frege himself is aware of.

against Kant were by far the most common in *GLA*. But he is also careful, and did recognize Kant's merits, as a "genius to whom we must all look up with grateful awe", and moreover, declaring that "If Kant was wrong about arithmetic, that does not seriously detract, in my opinion, from the value of his work." (*GLA*,§89). We all know the fate of Frege's logicism, and for that matter, we owe him the same conclusion: if Frege was wrong about arithmetic, that does not detract from the value of his work. His achievements with the ancestral, both in *BS* and *GLA* are nothing but impressive and can still be regarded as successful.

Appendix A

Arithmetic in *GLA*

Here we prove the propositions in *BS* and *GLA* necessary for the proof of what nowadays is called Frege's Theorem: the derivation of the Dedekind/Peano axioms for arithmetic whithin second-order logic and definitions. Important definitions and theorems proved in Chapter 3 are mentioned, and should be considered as part of the general proof. It should also be mentioned that all the derivation that follows are not original. Not only the general strategy is taken directly from Frege's work, but minor details are also found already in the works of Boolos and Heck Jr. (2011), Boolos (1985) and Zalta (2017). For a complete account on this results, Almeida (2014) is recommended.

Alongside the definitions in Chapter 3, we add the following:

Definition. *Equinumerosity*

$$\begin{aligned} \Vdash F \approx G \equiv \exists R[&\forall x \forall y \forall z (R(x,y) \wedge R(x,z) \to y = z) \wedge \\ &\forall x \forall y \forall z (R(x,y) \wedge R(z,y) \to x = z) \wedge \\ &\forall x [F(x) \to \exists y (R(x,y) \wedge G(y))] \wedge \\ &\forall y [G(y) \to \exists x (R(x,y) \wedge F(x))]] \end{aligned}$$

Definition. *Hume's Principle (HP)*

$$\Vdash \#F = \#G \equiv F \approx G$$

Definition. *Cardinal Number*

$$\Vdash Card(n) \equiv \exists F(n = \#F)$$

Definition. *Zero*

$$\Vdash 0 \equiv \#[x : x \neq x]$$

Definition. *One*

$$\Vdash 1 \equiv \#[x : x = 0]$$

Definition. *Predecessor Relation*

$$\Vdash P(m,n) \equiv \exists F \exists x[F(x) \wedge n = \#F \wedge m = \#[z : F(z) \wedge z \neq x]]$$

First, we prove some results that Frege announced in (*GLA*,§78).

Theorem GLA1. $\vdash P(0,a) \to a = 1$

Proof. Imediate from the definition. □

Theorem GLA2. $\vdash 1 = \#F \to \exists x F(x)$

Proof. Assume $1 = \#F$. Then, from the definition of 1, $\#F = \#[x : x = 0]$. From **(HP)**, $F \approx [x : x = 0]$. Then, there is a R such that $\forall x([x : x = 0]x \to \exists y(R(x,y) \wedge F(y)))$ holds. Since we know that $[x : x = 0]0$ is the case, we get that $\exists y(R(0,y) \wedge F(y))$, which yields that $\exists y F(y)$, completing the proof. □

Theorem GLA3. $\vdash 1 = \#F \to (F(x) \wedge F(y) \to x = y)$

Proof. Assume $1 = \#F$, $F(x)$ and $F(y)$. From the definition of 1, $\#[x : x = 0] = \#F$. From **(HP)**, $[x : x = 0] \approx F$. Then, there is a R such that $\forall y[F(y) \to \exists x(R(x,y) \wedge [x : x = 0]x)]$. Since both $F(x)$ and $F(y)$ holds, we arrive at $\exists z(R(z,x) \wedge [x : x = 0]z)$ and $\exists z'(R(z',y) \wedge [x : x = 0]z')$. This shows that there is z, z' such that both $z = 0$ and $z' = 0$. Hence, $z = z'$. So, $R(z,x)$ and $R(z,y)$ holds as well. But from the equinumerosity definition, R also satisfies $\forall x \forall y \forall z(R(x,y) \wedge R(x,z) \to y = z)$. Finally, $x = y$ as intended. □

Theorem GLA4. $\vdash \exists x F(x) \wedge \forall x \forall y (F(x) \wedge F(y) \rightarrow x = y) \rightarrow 1 = \#F$

Proof. Following **GLA1** above, it suffices to show that $P(0, \#F)$. For that, we have to show all three conditions for the definition of P. We have as premises that $\exists x F(x)$ and $\forall x \forall y (F(x) \wedge F(y) \rightarrow x = y)$. (a) For the first one, since $\exists x F(x)$, fix x as a, then $F(a)$. (b) For the second, $\#F = \#F$ is enough. (c) Now, we must show that $0 = \#[z : F(z) \wedge z \neq a]$. Suppose there is a y such that $[z : F(z) \wedge z \neq a]y$, that is, $F(y) \wedge y \neq a$ holds. But from the premises, $F(a)$ and $F(y)$ yields that $a = y$, which contradicts the hypothesis. Therefore, nothing satisfies the concept $[z : F(z) \wedge z \neq a]$. Then, any R is sufficient to show that $[x : x \neq x] \approx [z : F(z) \wedge z \neq a]$, which yields from **(HP)** that $\#[x : x \neq x] = \#[z : F(z) \wedge z \neq a]$, and from the definition of 0, that $0 = \#[z : F(z) \wedge z \neq a]$. Finally, from existencial generalization, $P(0, \#F)$, and from **GLA1**, $1 = \#F$. □

Theorem GLA5. $\vdash P(m, n) \wedge P(m', n') \rightarrow (m = m' \leftrightarrow n = n')$

Proof. To prove such theorem, we need first to prove a lemma. One which states that, if F and G are equinumerous, and if $F(a)$ and $G(b)$ holds, the concepts 'being an F other than a' and 'being an G other than b' are also equinumerous, and vice versa.

Lemma GLA5b. $\vdash F(a) \wedge F(b) \wedge [(\#[x : F(x) \wedge x \neq a] = \#[x : G(x) \wedge x \neq b]) \leftrightarrow (\#F = \#G)]$

Proof. The proof has two parts:

1. Assume $F(a)$, $F(b)$ and that $\#[x : F(x) \wedge x \neq a] = \#[x : G(x) \wedge x \neq b]$. From **(HP)**, $[x : F(x) \wedge x \neq a] \approx [x : G(x) \wedge x \neq b]$, hence, there is a one-one R that correlate both concepts. We need a R' such that $F \approx G$ holds. Let $R' = [x, y : R(x, y) \vee (x = a \wedge y = b)]$. We need to prove that R' satisfies $F \approx G$

   (a) Assume $R'(x, y)$ and $R'(x, z)$. Then, we have two possible cases: $R(x, y)$ and $R(x, z)$ or $x = a$, $y = b$ and $z = b$. In both cases, $y = z$.

   (b) Assume $R'(x, y)$ and $R'(z, y)$. Two possible cases follows: $R(x, y)$ and $R(z, y)$ or $x = a$, $z = a$ and $y = b$. Either way, $x = z$ as expected.

(c) Since $F(a)$ and $G(b)$, $R'(a,b)$ holds easily since $a = a$ and $b = b$. Then from existential generalization, $F(a) \to \exists y(R'(a,y) \land G(y))$.

(d) The same holds for the last part of the definition: $G(b) \to \exists y(R'(b,y) \land F(y))$.

Applying **(GEN)** in both, we get that R' is one-one between the F's and the G's, and hence $F \approx G$. From **(HP)**, $\#F = \#G$.

2. Assume $F(a)$, $F(b)$ and $\#F = \#G$. From **(HP)**, $F \approx G$. Then, there is a R one-one between them. Define R' such that

$$R' = [x, y : F(x) \land x \neq a \land G(y) \land y \neq b \land [R(x,y) \lor (R(x,b) \land R(a,y))]]$$

We must show that R' is one-one between the concepts $[x : F(x) \land x \neq a]$ and $[x : G(x) \land x \neq b]$.

(a) Assume $R'(x,y)$ and $R'(x,z)$, then we have two possible cases to consider: $R(x,y)$ and $R(x,z)$ holds, or $(R(x,b) \land R(a,y))$ and $(R(x,b) \land R(a,z))$. Both cases yields $y = z$.

(b) Assume $R'(x,y)$ and $R'(z,y)$, then we have two possible cases: $R(x,y)$ and $R(z,y)$, or $(R(x,b) \land R(a,y))$ and $(R(z,b) \land R(a,y))$. Either way, $x = z$.

(c) Now, assuming $[x : F(x) \land x \neq a]x$, we have to pick a witness such that $[x : G(x) \land x \neq b]$ holds and bears R' to x. Since $F(x)$ and $x \neq a$, there some z such that $G(z)$ and $R(x,z)$. If $z \neq b$, than easily $R'(x,z)$ holds as required. If $z = b$, then $R(x,b)$ holds. More precisely, $\neg R(a,b)$ holds as well, since $x \neq a$. Since clearly there is some y such that $G(y)$ and $R(a,y)$, and since $y \neq b$, $R'(x,y)$ holds in such case. Either way, we derive that $[x : F(x) \land x \neq a]x \to \exists y(R'(x,y) \land [x : G(x) \land x \neq b]y)$ as expected.

(d) The same reasoning above holds for this case, hence, $[x : G(x) \land x \neq b]x \to \exists y(R'(x,y) \land [x : F(x) \land x \neq a]y)$

From that, R' is one-one, and hence, from **(HP)**: $\#[x : F(x) \land x \neq a] = \#[x : G(x) \land x \neq b]$.

This completes the proof of the lemma. □

We can now procede to prove the theorem.

1. Assume $P(m,n)$, $P(m',n')$ and $m = m'$. From that, $P(m,n)$ and $P(m,n')$ holds. From the definition of P and fixing F, G, a, b in both cases, we arrive at the following facts: $F(a)$, $G(b)$, $n = \#F$, $n' = \#G$, $m = \#[x : F(x) \wedge x \neq a] = \#[x : G(x) \wedge x \neq b]$. From the left-to-right direction of the Lemma **GLA5b**, we conclude that $\#F = \#G$ and hence $n = n'$.

2. Assume $P(m,n)$, $P(m',n')$ and $n = n'$. Then, $P(m,n)$ and $P(m',n)$ holds. Fixing F, G, a, b for both, we get the following facts: $F(a)$, $G(b)$, $n = \#F = \#G$, $m = [x : F(x) \wedge x \neq a]$ and $m' = [x : G(x) \wedge x \neq b]$. From the right-to-left direction of the Lemma **GLA5b**, we get that $\#[x : F(x) \wedge x \neq a] = \#[x : G(x) \wedge x \neq b]$, and hence, $m = m'$.

This concludes the proof of the theorem. □

Lemma GLA0a. $\vdash \#F = 0 \leftrightarrow \neg\exists x F(x)$

Proof.

1. Assume $\#F = 0$. Then, $\#F = \#[x : x \neq x]$. From **(HP)**, $F \approx [x : x \neq x]$. Then, there is a one-one R, where the following holds: $\forall x[F(x) \rightarrow \exists y([x : x \neq x]y \wedge R(x,y))]$. Assume that $F(a)$ holds for some a, then $\exists y([x : x \neq x]y \wedge R(x,y))$ and more precisely, $\exists y([x : x \neq x]y)$, and $\exists y(y \neq y)$, which is not possible. Hence, there can be no a, and hence $\neg\exists x F(x)$.

2. Assume $\neg\exists x F(x)$. There must be a R, one-one, such that $F \approx [x : x \neq x]$. Since nothing is an F, any one-one R is sufficient, since the correlation conditions holds trivially. Therefore, $\#F = 0$ as intended.

□

Corollary GLA0b. $\vdash \#F \neq 0 \leftrightarrow \exists x F(x)$

Proof. The proof is imediate by contraposition on lemma **GLA0a**. □

Theorem GLA6a. $\vdash \neg P(n, 0)$

Proof. Assume $P(n, 0)$, *viz.* that 0 is the successor of n. Then, from the definition of P, there is an F and x such that $F(x)$, $0 = \#F$ and $n = \#[z : F(z) \wedge R(z, y)]$. Since $0 = \#F$, we have that $\#[x : x \neq x]$, and from **(HP)**, $[x : x \neq x] \approx F$. Then, there is a one-one R such that the following hold: $\forall x[F(x) \to \exists y([x : x \neq x]y \wedge R(x, y))]$. Since $F(x)$ is the case, $\exists y([x : x \neq x]y)$ holds, that is, $\exists y(y \neq y)$ which is a contradiction. Hence, $\neg P(0, n)$, as intended. □

Theorem GLA6b. $\vdash \forall x[\exists F(x = \#F) \wedge x \neq 0 \to \exists y(\exists G(y = \#G) \wedge P(y, x))]$

Proof. Fix an x and assume $\exists F(x = \#F)$ and $x \neq 0$. Then, from corolary **GLA0b**, $\exists z F(z)$. Fix it as a, hence $F(a)$. We must show that there is a concept such that y is the number if it. From comprehension axiom, let such concept be $[z : F(z) \wedge z \neq a]$. From that, it follows trivially that $P(\#[z : F(z) \wedge z \neq a], \#F)$. Then, the theorem follows from existential generalization. □

The next theorems are proofs of some facts about the Ancestral, the natural numbers and the principle of mathematical induction.

Theorem BS108. $\vdash R^+(x, y) \wedge R(y, z) \to R^+(x, z)$

Proof. Assume $R^+(x, y)$ and $R(y, z)$. From the first premise, we have two cases:

1. $R^*(x, y)$ holds. Then, from $R(y, z)$ we get $R^*(y, z)$ from **BS91**. Then, from **BS98**, we arrive at $R^*(x, z)$. Finally, $R^+(x, z)$ holds from the definition of the weak ancestral.

2. $x = y$ holds. In that case, from the second premise, $R(x, z)$. From **BS91**, $R^*(x, z)$ and from the definition of the weak ancestral, $R^+(x, z)$.

This completes the proof. □

Theorem Her(ℕ). $\vdash \mathbb{N}(m) \wedge P(m, n) \to \mathbb{N}(n)$

Proof. Just let R be P and x be 0 in **BS108**, □

Theorem BS81b. $\vdash F(x) \wedge \forall v \forall w(R^+(x, v) \wedge F(v) \wedge R(v, w) \to F(w)) \to \forall z(R^+(x, z) \to F(z))$

Proof. The proof is from **BS81a**. Assume the antecedents: (a) $F(x)$, (b) $\forall v \forall w(R^{+}(x,v) \wedge F(v) \wedge R(v,w) \to F(w))$ and for some fixed z, (c) $R^{+}(x,z)$. The aim is to show $F(z)$. For that matter, we use **BS81a**, instantiating F as the concept $[x : F(z) \wedge R^{+}(z,x)]$, which is obtainable from comprehension axiom. This yields:

$$[F(x) \wedge R^{+}(x,x)] \wedge Her \to \forall z(R^{+}(x,z) \to [F(z) \wedge R^{+}(x,z)])]$$

Where's Her abreviattes:

$$\forall v \forall w([F(v) \wedge R^{+}(x,v)] \wedge R(v,w) \to [(F(w) \wedge R^{+}(x,w))])$$

So we have to show both antecedents.

1. $R^{+}(x,x)$ holds trivially, and $F(x)$ is assumption (a). Hence $[F(x) \wedge R^{+}(x,x)]$.

2. We must show Her. Fix v and w, and assume both antecedents. From $F(v)$, $R(v,w)$ and $R^{+}(x,v)$ and assumption (b), we get $F(w)$. Since $R^{+}(x,v)$ and $R(v,w)$ and from theorem **BS108**, $R^{+}(x,w)$. This proves Her.

Having proved all antecedents of the instantiated **BS81a**, we arrive at $[F(z) \wedge R^{+}(x,z)]$, which yields $F(z)$ as desired, proving the theorem with assumption (c)[1]. □

Theorem PIM. $\vdash F(0) \wedge \forall v \forall w(\mathbb{N}(v) \wedge F(v) \wedge P(v,w) \to F(w)) \to \forall z(\mathbb{N}(z) \to F(z))$

Proof. Just let x be 0 and R be the predecessor relation P in **BS81b** and apply the definition of natural number. □

Lemma BSL1. $\vdash R^{*}(x,y) \to \exists z(R(z,y) \wedge R^{+}(x,z))$

Proof. Assume $R^{*}(x,y)$. To prove it, we use the ancestral definition, assuming $R^{*}(x,y)$ and taking F to be the concept $[w : \exists z[R(z,w) \wedge R^{+}(x,z)]]$, obtaining:

$$\forall w(R(x,w) \to \exists z[R(z,w) \wedge R^{+}(x,z)]) \wedge Her \to \exists z[R(z,y) \wedge R^{+}(x,z)]$$

[1] A shortened version of this proof is originally found in (BOOLOS; HECK JR., 2011, p.79-80)

Where Her is an abbreviation for

$$\forall u \forall v(\exists z[R(z,u) \land R^{+}(x,z)] \land R(u,v) \to \exists z[R(z,v) \land R^{+}(x,z)])$$

We have to show both antecedents.

1. Fix w and assume $R(x,w)$. Since $R^{+}(x,x)$ holds trivially, we get $R(x,w) \land R^{+}(x,x)$, and from existential generalization $\exists z[R(z,w) \land R^{+}(x,z)]$. From (GEN), we prove the first antecedent.

2. Fix u and v, and assume $\exists z[R(z,u) \land R^{+}(x,z)]$ and $R(u,v)$. Fix z as a, then $R(a,u)$ and $R^{+}(x,a)$. From **BS108**, $R^{+}(x,u)$. Then, $R(u,v) \land R^{+}(x,u)$, we then arrive at $\exists z(R(z,v) \land R^{+}(x,z))$. From (GEN) we complete the proof, hence Her holds.

From both antecedents, then $\exists z[R(z,y) \land R^{+}(x,z)]$, completing the proof. □

Finally, we prove three lemmas, announced by Frege in (GLA,§§$82-83$), which are used in the proof of the Theorem **GLA7**, that every natural number has a successor.

Lemma GLAL1. $\vdash \forall x(\mathbb{N}(x) \to \neg P^{*}(x,x))$

Proof. Fix x and assume $\mathbb{N}(x)$. The proof is by induction, using a modified version of BS81[2]. Let F be $[x : \neg P^{*}(x,x)]$, x be 0, y be x, R be P and avoiding the *definiens* for the hereditary property, we get:

$$\neg P^{*}(0,0) \land P^{+}(0,x) \land \forall v \forall w(\neg P^{*}(v,v) \land P(v,w) \to \neg P^{*}(w,w)) \to \neg P^{*}(x,x))$$

We have to show all three antecedents.

1. Assume that $P^{*}(0,0)$. From lemma **BSL1**, $\exists z(P(z,0) \land P^{+}(0,z))$. Fixing z as n, we derive $P(n,0)$. This is against theorem **GLA6b**, hence, $\neg P^{*}(0,0)$.

2. Since $\mathbb{N}(x)$, we have $P^{+}(0,x)$.

[2]The difference is in the second conjunct: the former has the strong ancestral, whilst here the weak ancestral will be used instead. The derivation is easy and we omit it here.

3. Fix v and w, and assume $P^*(w,w)$ and $P(v,w)$. From **BS91**, $P^*(v,w)$. Using lemma **BSL1**, $\exists z(P(z,w) \wedge P^+(v,z))$. Fixing z as a, $P^+(v,a)$ has two cases to be considered:

   (a) $P^*(v,a)$ is the case. Then, from $P(a,w)$ and **BS91**, $P^*(a,w)$, and from **BS98** twice, $P^*(a,v)$ and then $P^*(a,a)$. Moreover, since P is one-one (**GLA5**), from $P(v,w)$ and $P(a,w)$ get that $v = a$. Hence, $P^*(v,v)$ holds.

   (b) $v = a$. Then, since $P^*(w,w)$, using **BSL1**, $\exists z(P(z,w) \wedge P^+(w,z))$. Fix z as b, then $P(b,w)$ and $P^+(w,b)$. From the first and **GLA5**, $a = v = b$. Now, from $P^+(w,b)$ we have to cases: Case 1) $w = b$. In that case, $a = v = b = w$ and hence $P^*(v,v)$. Case 2) $P^*(w,b)$. Then, also $P^*(w,v)$, and with $P^*(v,w)$ already acquired, $P^*(v,v)$ from **BS98**.

   $P^*(v,v)$ is obtained in both cases. Now, we derived $P^*(w,w) \wedge P(v,w) \to P^*(v,v)$. From contraposition, we conclude the desired $\neg P^*(v,v) \wedge P(v,w) \to \neg P^*(w,w)$

This proves all antecedents. Hence, $\neg P^*(x,x)$. □

Lemma GLAL2. $\vdash \mathbb{N}(y) \wedge P(x,y) \to \forall z[P^+(z,y) \wedge z \neq y) \leftrightarrow P^+(z,x)]$

Proof. Assume both $\mathbb{N}(y)$ and $P(x,y)$. We prove both directions of the biconditional.

1. Fix any z, and assume $P^+(z,y) \wedge z \neq y$. From the definition, then $P^*(z,y)$. Using **BSL1**, $\exists w(P(w,y) \wedge P^+(z,w))$. Now fix w as a, hence $P(a,y)$ and $P^+(z,a)$. From **GLA5**, $P(x,y)$ and $P(a,y)$, $x = a$. Then, from $P^+(z,x)$ as intended.

2. Fix z and assume $P^+(z,x)$. We have two cases to consider:

   (a) $x = z$ is the case. From that, $P(z,y)$. From **BS91**, $P^*(z,y)$ and $P^+(z,y)$ from the definition of the weak ancestral. Suppose now that $z = y$. Then $x = z = y$, which yields both $P(x,x)$ and $P^+(0,x)$. Now, from **BS91**, $P^*(x,x)$, and hence, $P^+(0,x) \to P^*(x,x)$ and $\mathbb{N}(x) \to P^*(x,x)$, which is against **GLAL1**. Hence, $z \neq y$, which yields $(P^+(z,y) \wedge z \neq y)$.

   (b) $P^*(z,x)$ is the case. From $P(x,y)$ and **BS91**, $P^*(x,y)$, and from **BS98**, we get $P^*(z,y)$, and then $P^+(z,y)$ from definition of the weak ancestral. Assume $z = y$, then $P^*(y,y)$ and $P^+(0,y)$, which contradicts **GLAL1**, hence $z \neq y$, and then $(P^+(z,y) \wedge z \neq y)$.

Finally, from (GEN), we complete the proof. □

Lemma GLAL3. $\vdash \mathbb{N}(y) \wedge P(x,y) \wedge P(x,\#[z : P^+(z,x)]) \rightarrow y = \#[z : P^+(z,y) \wedge z \neq y]$

Proof. Assume $\mathbb{N}(y)$, $P(x,y)$ and $P(x,\#[z : P^+(z,x)])$. From **GLA5**, $y = \#[z : P^+(z,x)]$. Now from **GLAL2** above, $\forall z(P^+(z,y) \wedge z \neq y \leftrightarrow P^+(z,x))$. Now, **(HP)** has as corolary $\forall x(F(x) \leftrightarrow G(x)) \rightarrow F \approx G$. From that, we arrive at $[z : P^+(z,y) \wedge z \neq y] \approx [z : P^+(z,x)]$, which from **(HP)** gives $\#[z : P^+(z,y) \wedge z \neq y] = \#[z : P^+(z,x)]$. Finally, this yields that $y = \#[z : P^+(z,y) \wedge z \neq y]$ □

Theorem GLA7. $\vdash \forall n[\mathbb{N}(n) \rightarrow P(x,\#[z : P^+(z,x)])]$

Proof. We assume **PIM**, taking F to be $[x : P(x,\#[z : P^+(z,x)])\}$ and rearranging the variables to arrive at:

$$P(0,\#[z : P^+(z,0)]) \wedge$$
$$\forall x \forall y(\mathbb{N}(x) \wedge P(x,\#[z : P^+(z,x)]) \wedge P(x,y) \rightarrow P(y,\#[z : P^+(z,y)])) \rightarrow$$
$$\forall n(\mathbb{N}(n) \rightarrow P(n,\#[z : P^+(z,n)]))$$

So it's enough to prove both antecedents.

1. To show that $P(0,\#[z : P^+(z,0)])$, we must show a concept F and an object x that satisfies the definition of P. Let F be $[z : P^+(z,0)]$ and x be 0. Then:
   (a) $P + (0,0)$ holds from reflexivity of P^+.
   (b) $\#[z : P^+(z,0)] = \#[z : P^+(z,0)]$ holds trivially.
   (c) We must now show that $0 = \#[z : [w : P^+(w,0)]z \wedge z \neq 0]$. By the definition of 0 and simplyfing a little, we must show that $\#[z : z \neq z] = \#[z : P^+(z,0) \wedge z \neq 0]$. For that, suppose that $\exists x(P^+(x,0) \wedge x \neq 0)$. Then, fixing x as a, $P^+(a,0)$ and $a \neq 0$ implies that $R^*(a,0)$. From that, using **BSL1**, $\exists z(R(z,0) \wedge R^+(a,z))$. Fixing z as b, then $P(z,0)$. This contradicts **GLA6b**, hence $\neg\exists x(P^+(x,0) \wedge x \neq 0)$. Then, from **GLA0a**, $0 = \#[z : P^+(z,0) \wedge z \neq 0]$.

This proves that $P(0, \#[z : P^+(z, 0)])$.

2. We must now show that the concept $[w : P(w, \#[z : P^+(z, w)]\}$ is hereditary in P. For that, fix x and y, assume $\mathbb{N}(x)$, $P(x, \#[z : P^+(z, x)])$ and $P(x, y)$. Then, from **GLA5**, $y = \#[z : P^+(z, x)]$. From **Her(ℕ)**, $\mathbb{N}(y)$, and from **GLAL3**, $y = \#[z : P^+(z, x) \wedge z \neq y]$. Now, it's suffice to show that $P(\#[z : P^+(z, y) \wedge z \neq y], \#[z : P^+(z, y)])$. For the definition P, let F be $[z : P^+(z, y)]$ and x be y. Then:

   (a) $P^+(y, y)$ is true given the reflexivity of P^+.

   (b) $\#z : P^+(z, y) = \#z : P^+(z, y)$ trivially.

   (c) Now since $[z : P^+(z, y) \wedge z \neq y]$ is the same as $[z : [w : P^+(w, y)]z \wedge z \neq y]$, then trivially $\#[z : P^+(z, y) \wedge z \neq y] = \#[z : P^+(z, y) \wedge z \neq y]$.

   Hence, $P(\#[z : P^+(z, y) \wedge z \neq y], \#[z : P^+(z, y)])$, and since $y = \#[z : P^+(z, y) \wedge z \neq y]$, $P(y, \#[z : P^+(z, y)])$ as intended.

Since both antecedents are proved, then $\forall n(\mathbb{N}(n) \to P(n, \#[z : P^+(z, n)]))$ as desired. □

Corollary GLA8. $\vdash \neg\exists x(\mathbb{N}(x) \to \forall y(P(y, x))$

Proof. Assume that there is such x. Name it a and assume $\mathbb{N}(a)$. Then, $\forall y(P(y, a))$. From **GLA7**, $P(a, \#[z : P^+(z, a)])$. Name $\#[z : P^+(z, a)]$ as b. From $\forall y(P(y, a))$, let y be b, and we derive $P(b, a)$. But then, $P(b, a)$ and $P(a, b)$ implies $a = b$ from **GLA5** and then $P(a, a)$ holds. From **BS91**, $P^*(a, a)$. This is against **GLAL1**. Hence, there are no such x. □

The next theorem states that the predecessor relation under natural numbers is trichotomous.

Theorem BS133b. $\vdash \mathbb{N}(x) \wedge \mathbb{N}(y) \to (P^*(x, y) \vee P^*(y, x) \vee x = y)$

Proof. Assume $\mathbb{N}(x)$ and $\mathbb{N}(y)$. From definition, $P^+(0, x)$ and $P^+(0, y)$. From definition of P^+, we have four cases to consider:

1. If both $0 = x$ and $0 = y$, then $x = y$ and the conclusion holds.

2. If $P^*(0, x)$ and $0 = y$, then $P^*(y, x)$, and the conclusion holds.

3. If $0 = x$ and $P^*(0, y)$, then $P^*(x, y)$, and the conclusion holds.

4. If $P^*(0, x)$ and $P^*(0, y)$, then from **BS133**, $(P^*(x, y) \vee P^*(y, x) \vee x = y)$ as intended.

In all cases the theorem follows. □

From all the facts proved above, we can now show how the Dedekind-Peano axioms are consequences. They are:

PA1 $\mathbb{N}(0)$, that is, zero is a natural number.

PA2 $\forall x \forall y (\mathbb{N}(x) \wedge P(x, y) \rightarrow \mathbb{N}(y))$, that is, the successor of any natural number is a natural number.

PA3 $\forall x \forall y \forall z (P(x, z) \wedge P(y, z) \rightarrow y = z)$, that is, no two numbers have the same successor.

PA4 $F(0) \wedge \forall v \forall w (\mathbb{N}(v) \wedge F(v) \wedge P(v, w) \rightarrow F(w)) \rightarrow \forall z (\mathbb{N}(z) \rightarrow F(z))$, that is, the principle of mathematical induction.

PA5 $\forall x (\mathbb{N}(x) \rightarrow \exists y (\mathbb{N}(y) \wedge P(x, y)))$, that is, every natural number has a successor.

PA1 follows directly from the definition of 0. **PA2** is theorem **Her($\mathbb{N}$)**. **PA3** follows from **GLA5**. **PA4** is **PIM**, and finally, **PA5** is derivable from **GLA7** just proved, adding the condition that $\mathbb{N}(y)$ from **Her($\mathbb{N}$)** and $\exists$**GEN**.

We offer now a result concerning infinite cardinals, as Frege discuss it in (*GLA*,§§$84-86$). Define $\aleph_0$ as $\#[z : \mathbb{N}(z)]$, that is, the number of the concept "being a natural number". Then the following holds:

Theorem GLA9. $\vdash P(\aleph_0, \aleph_0)$

Proof. Consider the concept $[z : \mathbb{N}(z)]$. This is by definition $[z : P^+(0, z)]$. Now, the following three facts holds:

1. $P^+(0, 0)$ from reflexivity of P^+.

2. $\#[z : P^+(0, z)] = \#[z : P^+(0, z)]$ trivially by identity.

3. Finally, it can be proved that $\#[z : P^{+}(0,z)] = \#[z : P^{+}(0,z) \wedge z \neq 0]$. Consider the relation P. P is one-one as **GLA5**. Now assume that $P^{+}(0,x)$ for any x. Then, from **GLA7**, $P(x, \#[z : P^{+}(z,x)])$. And from **GLA6a** it can be inferred that $\#[z : P^{+}(z,x)]) \neq 0$. Now, clearly from **Her(ℕ)**, $P^{+}(0, \#[z : P^{+}(z,x)])$. Hence, $P^{+}(0, \#[z : P^{+}(z,x)]) \wedge \#[z : P^{+}(z,x)] \neq 0$. From **(∃GEN)**, $\exists y(P^{+}(0,y) \wedge y \neq 0 \wedge P(x,y))$. From the assumption and (GEN), $\forall x[P^{+}(0,x) \rightarrow \exists y((P^{+}(0,y) \wedge y \neq 0)) \wedge P(x,y))]$. Now, from the same reasoning, it can be inferred that $\forall x[(P^{+}(0,x) \wedge x \neq 0) \rightarrow \exists y(P^{+}(0,y) \wedge P(x,y))]$. From definition of equinumerosity, $[z : P^{+}(0,z)] \approx [z : P^{+}(0,z) \wedge z \neq 0]$. The desired fact follows from (HP).

From those facts, $P(\aleph_0, \aleph_0)$ follows from the definition of P. □

As Frege hints in (*GLA*,§84), the following holds:

Theorem GLA10. $\vdash P^{*}(\aleph_0, \aleph_0)$

Proof. From **GLA9** and **BS91**. □

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