UNIVERSIDADE DE CAMPINAS
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BRUNO COSTA COSCARELLI

MODEL THEORY IN A PARACONSISTENT ENVIRONMENT

TEORIA DE MODELOS NUM AMBIENTE PARACONSISTENTE

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Supervisor/Orientador: Marcelo Esteban Coniglio

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Abstract

The purpose of this thesis is to develop a paraconsistent Model Theory from the basis launched by Walter Carnielli, Marcelo Esteban Coniglio, Rodrigo Podiack and Tarcísio Rodrigues in the article 'On the Way to a Wider Model Theory: Completeness Theorems for First-Order Logics of Formal Inconsistency' of 2014. The pursuit of a deeper understanding of the phenomenon of paraconsistency from an epistemological point of view leads to a reasoning system based on the Logics of Formal Inconsistency. Models are regarded as states of knowledge and the concept of isomorphism is reformulated so as to give raise to a new concept that preserves a portion of the whole knowledge of each state. Based on this, a notion of refinement is created which may occur from inside or from outside the state. In the sequel, two important classical results, namely the Omitting Types Theorem and Craig's Interpolation Theorem are shown to hold in the new system and it is also shown that, if classical results in general are to hold in a paraconsistent system, then such a system should be in essence how it was developed here. Finally, an essay of what a paraconsistent PROLOG should be is made in the light of the ideas developed so far.

Keywords: paraconsistency, model theory, logic programming, reasoning system.

Resumo

O propósito desta tese é desenvolver uma Teoria de Modelos paraconsistente a partir das bases lançadas por Walter Carnielli, Marcelo Esteban Coniglio, Rodrigo Podiack e Tarcísio Rodrigues no artigo 'On the Way to a Wider Model Theory: Completeness Theorems for First-Order Logics of Formal Inconsistency' de 2014. A busca por uma compreensão mais profunda do fenômeno da paraconsistência de um ponto de vista epistemológico leva a um sistema de raciocínio baseado nas "Lógicas de Inconsistência Formal" (LFI's). Os modelos são tratados como estados de conhecimento e o conceito de isomorfismo é reformulado de modo a gerar outro que preserva uma porção da totalidade do conhecimento de cada estado. Com base nisso, é criada uma noção de refinamento que pode acontecer de dentro ou de fora do estado. Na sequência, mostra-se que dois importantes resultados clássicos, a saber o Teorema da Omissão de Tipos e o Teorema da Interpolação de Craig, valem no novo sistema e se mostra ainda que, caso se queira que os resultados clássicos em geral valham em um sistema paraconsistente, é necessário que tal sistema seja essencialmente como o que foi desenvolvido aqui. Finalmente, é feito um ensaio sobre como deveria ser um PROLOG paraconsistente à luz das ideias que foram desenvolvidas até então.

Contents

Introduction

In 1963, Newton da Costa opened the promising branch of Paraconsistent Logic by presenting a hierarchy of such logics. For this new field to effectively blossom, however, an essential tool was lacking: the consistency operator, which provides a way for delimiting a portion of the logic that shall behave classically. In [15], Walter Carnielli, Marcelo Esteban Coniglio and João Marcos brought new breath to the new field of study by presenting the Logics of Formal Inconsistency (LFIs). In classical logic, a contradiction, that is, the assumption of an assertion and its negation at the same time, yields a theory where every assertion holds. Such a theory is said to be a trivial one. Paraconsistent logics are those that do not 'explode' into triviality when exposed to a contradiction. LFIs are those that not only can be exposed to contradictions but also have the necessary apparatus to accomplish the task of recovering classical behavior.

As the development of a new area progresses, links to other areas naturally arise. One of the most beautiful and prominent areas of classical logic is certainly that of Model Theory. The time to the inspiring adventure of plowing through this field came about in 2014, when Walter Carnielli, Marcelo Esteban Coniglio, Rodrigo Podiack and Tarcísio Rodrigues published the article 'On the Way to a Wider Model Theory: Completeness Theorems for First-Order Logics of Formal Inconsistency', cited as [13] in the references, launching the grounds to the development of a paraconsistent model theory.

In [15], a system named mbC is proposed as the paraconsistent propositional calculus. In [13], a system named QmbC is proposed as the predicate calculus derived from mbC. There, syntax and semantics were proposed and subtle technical difficulties had to be overcome in order to prove completeness, compactness and the Lowenhëim-Skolem Theorems. The purpose of the present work is to start from the point where that work stopped and go ahead.

This thesis is composed of four chapters. The first one presents the background to the research and introduces the propaedeutics for the sake of completeness whereas the last three ones present the original material. The contents of those chapters are correlate but have different aims. As a branch of Logic, Model Theory is a three-fold subject, having either philosophical, mathematical and computational interest. Keeping in mind that the aim of this work is Model Theory as a whole, each of the three first chapters aims one of the three fields of interest. A brief description of each chapter follows below.

Chapter 2 is dedicated to philosophical issues. The pursuit of a deeper understanding of the phenomenon of paraconsistency is based on the fundamental premise that the natural environment for the LFI's is that of epistemology, for this is the realm where contradictions inexorably impose their presence and do make sense as soundly instituted 'living beings'.

The plan for this chapter is to construct a reasoning system that shall comply with contradictory information. The idea is to consider models as states of knowledge. As expected, the consistency operator shall provide the means to split states in safe and unsafe -although maybe plausible- knowledge. In the sequel, there starts a search for ways of gaining new safe knowledge without compromising the previously safe portion.

In the struggle to get along with doubt, QmbC is doubtlessly the suitable tool for the task. However, the concept of isomorphism, which is perhaps the most essencial and ubiquitous one in Model Theory, fails to perform the task for which is was designed. For this reason, QmbC will be the starting point, but the result will be an enriched version of it.

The first two sections gradually delimit the safe portion of the whole, developing, at the same time, an extended QmbC that is intended to be a reasoning system. In the first section, the concept of isomorphism is reformulated into the concept of quasi-isomorphism, which plays a central role in the rest of the chapter. The second section concludes the job of developing a paraconsistent reasoning system. It is important to stress that the final result is just one among infinitely many possibilities and that choices have been made in order to make it fit into finitely many pages.

The third and the fourth sections present two distinct suggestions of ways to refine models in the search for new knowledge within a fixed universe of objects. They are horizontal refinements in some sense.

The fifth section does not suggest any new way of refinement. Instead, it treats the idea of a refinement that enlarges the universe of objects. This would be a vertical refinement in the same sense.

Finally, the sixth section draws on the results and methods from [13] in order to establish an axiomatization to the so far semantically developed reasoning system.

Chapter 3 is dedicated to mathematical issues. The aim now is to show that it is possible to transpose the main classical concepts and results to a paraconsistent environment. Basically, this chapter tackles three core problems. The first one is to determine what must be required from a logical paraconsistent system in order to render it possible to recover the classical concepts and results. The second one is to present such concepts and results. The third one is to give a sense to working with a mathematical paraconsistent system.

A great deal of work toward that task was done in Chapter 2. In fact, the problem of quasi-isomorphisms was treated there and so was the important concept of ultrafilter.

This chapter starts with the first problem. Section 1 determines what a system without a classical auxiliar negation should be like. Sections 2 and 3 present two important classical results, respectively, the Omitting Types Theorem and Craig's Interpolation Theorem. In both cases, the theorems are shown not only to hold in a theory based on QmbC, but also to be manageable in such a theory, in the sense that the results that follow from those theorems in the classical context also follow in the new context. In the case of Omitting Types, the theorem is shown to hold in a theory based on a weaker system that does not count with an auxiliar classical negation, or, what is the same, that does not count with a consistency operator, or, what is still the same, that is not an LFI. However, it turns out that the theorem is not manageable without a classical negation. In the case of Craig's Interpolation Theorem, severe technical difficulties arise when a classical negation is not available. The hindrances that loom are discussed.

The results from Sections 2 and 3 strongly suggest that QmbC is in fact a good starting point and that a system without a consistency operator is too weak for the task. Section 4 returns to the problem of elementary extensions in a different fashion as that of Chapter 2. Section 5 of Chapter 2 is concerned with the existence of an extension of a model that preserves all its knowledge. This time, the concern is on what extensions preserve from the models they extend. Once again, the standard form of QmbC turns out to be insufficient to support good results involving quasi-isomorphism. Nevertheless, good results are obtained when the structure of QmbC is enriched. The system proposed in Chapter 2 turns out to be suitable for the task.

The discussion developed in the first four sections proves that a paraconsistent model theory is plainly possible and that an enriched version of QmbC -not necessarily the one proposed in Chapter 2- shall be the bedrock for such a theory. At this point, the chapter's aim has been achieved.

Working with unsafe knowledge in Mathematics shall be the same as in science, that is, split what is supposedly known into safe and unsafe knowledge and treat the second part carefully. The most obvious concern of mathematical model theory is that of describing the class of models that satisfy a given theory and describing the set of sentences that are satisfied by a given class of models. In order to give a sense to working with a mathematical paraconsistent system, this point is analyzed carefully. This is what Section 5 is about.

Chapter 4 is dedicated to computational issues. It explores the links between paraconsistency and Prolog. Section 1 presents some preliminary definitions. Section 2 makes an essay on what a Paraconsistent Reasonig Prolog should be in the light of the ideas developed in Chapter 2. Section 3 identifies different kinds of paraconsistent negation that may be useful in Prolog from a procedural point of view. Section 4 briefly discusses normal programs.

How to Follow this Thesis

As mentioned in the Introduction, the last three chapters of this thesis contain the original results of the work and each of them treats the subject under consideration within a particular point of view. Being so, it is likely that some readers may be interested in some specific part of the work and not in some other. The purpose of these few lines is to serve as a guide for those readers.

Chapter 1 contains no original material and shortly covers the propaedeutics of what comes after. Being so, it can be skipped by the reader who is acquainted with those subjects.

Chapter 2 contains the philosophical material. Section 2.1 poses the problem to be tackled and points to the direction to be followed, settling the starting point to the construction of a system that shall account to the questions raised along the section. Section 2.2 actually constructs the system. Sections 2.1 and 2.2 contain the material that is necessary to the comprehention of the other parts of the text. Being so, a reader who is particularly interested in the mathematical or computational aspects treated in Chapters 3 and 4 can skip the rest of Chapter 2. A reader who wants to go still faster to the chapter of interest can also skip Section 2.1 at the cost of losing the intuition behind the construction. Section 2.2 is divided into two parts. In the first one, the construction is performed. In the second one, the definitions are presented. The reason for such a division is to concentrate in a single place the definitions that are necessary for a reader to understand any part of the work separately. Being so, the reader who wants to go directly to the point can skip even Subsection 2.2.1 at the cost of losing the process of constructing the system. Subsection 2.2.2 is the only part of the text that cannot be skipped at all.

Chapter 3 contains the mathematical material. Sections 3.1, 3.2 and 3.3 are independent of Chapter 2. Sections 3.4, 3.5 and 3.6 depend on Sections 2.1 and 2.2 as explained above.

Chapter 4 contains the computational material. It depends on Sections 2.1 and 2.2 and is independent of Chapter 3.

Chapter 1

Basic Concepts

1.1 Background to the Research

The proposal of this thesis is to create a system for reasoning in real life. What is intended as a result is a system that shall be able to accommodate theories in the process of construction. Indeed, every theory in real life practice is in the process of construction.

Model theory is the classical tool that have been designed to study theories. The perspective of this work is that this is in fact the most suitable tool to perform the task. However, classical model theory is based on classical logic and classical logic is not able to comply with the existence of contradictions, which is an unavoidable reality in theories that are in the process of construction. In fact, classical logic trivializes in the presence of a contradiction, in the sense that, assuming two contradictory premises, every conclusion follows. This is the logical principle dubbed *'Ex contradictione sequitor quodlibet'*. Such a behavior is a serious problem for the real life practice and it seems to be an unsurmountable one within the classical theory. The consequence of this feature in model theory is that classical models determine every aspect of reality. This is a good approach when models are intended to capture the reality as a whole. Such models, however, are not reachable from an epistemological point of view, that is, such an approach presupposes that the agents are omniscient. This is clearly not the case when scientists are exploring the world or when a detective is trying to elucidate a crime. In real life situations, it would be more appropriate to treat models as states of knowledge about the reality rather then as the reality itself.

In order to accomodate contradictions, some kind of non-classical logic is demanded. The classical feature the new logic must get rid of is not the principle of non-contradiction, but rather the principle of *ex contradictione sequitor quodlibet*.

In other words, what is desired in not a logic that deduces contradictions. On the contrary, it is highly desirable that it does not deduce contradictions. What is required is that it does not trivialize when exposed to a contradiction. In this sense, it is intended to be a careful logic that does not hastily take clonclusions from two pieces of contradictory information. Moreover, it would be desirable to have a way of separating consistent and inconsistent information.

The discussion above establishes the tool that will be used and the environment where this tool will be developed. The tool is model theory. The environment is the class of paraconsistent logics, which are the logics that do not trivilize when exposed to a contradiction. More specifically, the environment will be a class of paraconsistent logics that have a separate operator to split consistent and inconsistent information. Those are the Logics of Formal Inconsistency, from now on just LFI's.

After developing a system for reasoning, the natural step in the sequence is to look for applications of the system just developed. This is the task for the last chapter. The field where this application is made is logic programming. Logic is traditionally applied in computer science as the study of the underlying rules that base program construction. The idea behind logic programming is that logic may work as the program itself. The logic programming system that will be studied is PROLOG.

At this point, it is clear what the background to this work is: Model theory, paraconsistent logic, more specically the LFI's, and logic programming, more specifically PROLOG. The next sections of this chapter will be dedicated to present the notation and the basic concepts and results that will be directly used along the development of the work. This presentation shall be sufficient to render the exposition self-contained.

As already exposed in the Introduction, the starting point to this thesis is the article 'On the Way to a Wider Model Theory: Completeness Theorems for First-Order Logics of Formal Inconsistency', cited as [13]. It is clearly the main reference for this work. In order to guide the reader who wants a complete presentation of the subjects that compose the backgroung to the research or who wants to have a deep comprehention of some related topic, the rest of this section will be dedicated to provide references.

Model Theory: As a textbook to the subject, [18], [34], [30], [40], [29] are good references. For a historical view on the subject, [7] and [6]. For a philosophical treatment, [12] and [8]. In [31], the subject is presented with an emphasis in computer science.

Paraconsistent Logic: The most complete reference for the subject is [14], but [15] and [16] also contain the necessary content. In [20], a study on mbC is made. The works that gave born to the paraconsistent logics are [21] and [22]. For an essay on those seminal works, see [1]. For a historical view, see [25], [2], [3], [4], [26] or [28]. For a philolophical view, see [23], [24] or [17]. As exposed above, the view of this thesis is that paraconsistency is an epistemical phenomenon. The opposite view is that there are real contradictions in the world. A discussion on this view, which is called *dialetheism* can be found in [5] and [35]. Relevant logic is an example of a naturally non-explosive logic. For a study on this subject, see [33].

Logic Programming: The main reference for this subject is [32]. Alternative references are [10], [11] and [19]. For a general presentation of logic for computer science including PROLOG, see [39].

1.2 Basic Concepts of Logics of Formal Inconsistency

As the universe of the Logics of Formal Inconsistency (LFI's) is the environment where the whole work will be developed, it is natural that LFI's be the first topic to be presented. As discussed in the previous section, LFI's are the paraconsistent logics that are able to recover classical behavior through a paraconsistency operator. The possibility of recovering classical behavior opens the possibility of defining a classical auxiliar negation. As being able to recover classical behavior is the core feature of this class of logics and an auxiliar negation is a powerful tool, this section will be dedicated to discussing what a classical negation is, how non-classical negations should behave and how the existence of a consistency operator is linked to the existence of a classical auxiliar negation.

First of all, a negation is a unary connective. But this is not enough. There are two patterns of behavior that the negation of classical logic performs and that guarantee the core features of that system. Those patterns will be described bellow and negations satisfing one and the other will be distinguished. Finally, a negation will be called classical if it satisfies both of them.

Along the whole text, the symbol $\vdash$ will be systematically used to designate syntactical conclusion, $\vDash$ for semantical conclusion and $\Vdash$ if it is not being specified whether the reference is to a syntactical or to a semantical conclusion.

In the sequel, the negations characterized by the two mentioned patterns:

Definition 1.2.1 (Complementing Negation). *A unary connective $\neg$ is a* complementing negation *iff, for every sentence $\phi \in L_\Sigma$ and for every theory T, if $T \nVdash \phi$ and $T \cup \{\neg\phi\}$ is nontrivial, then $(T \cup \{\neg\phi\}) \nVdash \phi$ and, if $T \nVdash \neg\phi$ and $T \cup \{\phi\}$ is nontrivial, then $(T \cup \{\phi\}) \nVdash \neg\phi$.*

An equivalently formulation is: If $(T \cup \{\neg\phi\}) \Vdash \phi$*, then* $T \Vdash \phi$ *and if* $(T \cup \{\phi\}) \Vdash \neg\phi$*, then* $T \nVdash \neg\phi$*.*

Definition 1.2.2 (Supplementing Negation)**.** *A unary connective* $\urcorner$ *is a* supplementing negation *iff, given two arbitrary sentences* ϕ *and* ψ *in* L*,* $\{\phi, \urcorner\phi\} \Vdash \psi$*.*

Definition 1.2.3 (Classical Negation)**.** *A unary connective* $\sim$ *is a* classical negation *iff it is both a complementing and a supplementing negation.*

When a logical system is semantically determined, a stronger definition of complementing negation allows for an alternative characterization: A unary connective $\neg$ is a complementing negation iff, for every sentence $\phi \in L_\Sigma$, for every theory T and for every model $\mathfrak{A}$, if $(\mathfrak{A}, T \cup \{\neg\phi\}) \vDash \phi$, then $\mathfrak{A}, T \vDash \phi$ and if $(\mathfrak{A}, T \cup \{\phi\}) \vDash \neg\phi$, then $\mathfrak{A}, T \nvDash \neg\phi$. It is clear that a complementing negation in this sense is also a complementing negation in the original sense. In this case, the following proposition provides an alternative characterization for complementing negation.

Proposition 1.2.4. *Let* $\neg$ *be a negation in a semantically determined logical system. Then,* $\neg$ *is a complementing negation iff, for every sentence* ϕ *and for every model* $\mathfrak{A}$*,* $\mathfrak{A} \vDash \phi$ *or* $\mathfrak{A} \vDash \neg\phi$*.*

Proof. ($\Rightarrow$) Suppose $\neg$ is a complementing negation. Suppose, for the sake of contradiction, that there exists a model $\mathfrak{A}$ with valuation v such that $\mathfrak{A} \nvDash \phi$ and $\mathfrak{A} \nvDash \neg\phi$. As $v(\phi) = 0$, the condition $v(\psi) = 1$ for every $\psi \in \{\phi\}$ is not fulfilled. Thus, it is true that, if $v(\psi) = 1$ for every $\psi \in \{\phi\}$, then $v(\neg\phi) = 1$. Hence, it is the case that $\mathfrak{A} \nvDash \neg\phi$ while $\mathfrak{A}, \phi \vdash \neg\phi$. That is a contradiction against the fact that $\neg$ is complementing. Therefore, $\mathfrak{A} \vDash \phi$ or $\mathfrak{A} \vDash \neg\phi$ must hold.
($\Leftarrow$) Suppose that, for every sentence ϕ and for every model $\mathfrak{A}$, it holds that $\mathfrak{A} \vDash \phi$ or $\mathfrak{A} \vDash \neg\phi$. Now, let ϕ be an arbitrary sentence and $\mathfrak{A}$ an arbitrary model. Suppose that $\mathfrak{A} \nvDash \neg\phi$. Then, there is some $\vec{a} \in \bar{A}$ such that $v(\neg\phi(\vec{a})) = 0$. Moreover, $\mathfrak{A} \vDash \phi$, by hypothesis. In particular, $v(\phi(\vec{a})) = 1$. Hence, for such $\vec{a}$, $v(\psi(\vec{a})) = 1$, while $v(\neg\phi(\vec{a})) = 0$. Therefore, $\mathfrak{A}, \phi \nvDash \neg\phi$. In the same way, if $\mathfrak{A} \nvDash \phi$, then $\mathfrak{A}, \neg\phi \nvDash \phi$. Summing up, $\neg$ is in fact a complementing negation. ☐

An alternative characterization of supplementing negation is also available.

Proposition 1.2.5. *Let* $\urcorner$ *be a negation in a semantically determined logical system. Then,* $\urcorner$ *is a supplementing negation iff, for every sentence* ϕ *and for every nontrivial model* $\mathfrak{A}$*, at most* $\mathfrak{A} \vDash \phi$ *or* $\mathfrak{A} \vDash \urcorner\phi$*, that is,* $\mathfrak{A} \nvDash \phi$ *or* $\mathfrak{A} \nvDash \urcorner\phi$*.*

Proof. ($\Rightarrow$) Suppose $\urcorner$ is a supplementing negation. Suppose, for the sake of contradiction, that there exist a model $\mathfrak{A}$ and a sentence ϕ such that $\mathfrak{A} \vDash \phi$ and $\mathfrak{A} \vDash \neg\phi$. Then, $\mathfrak{A} \vDash \psi$ for every sentence ψ. Hence, $\mathfrak{A}$ is a trivial model.

($\Leftarrow$) Suppose that, for every nontrivial model $\mathfrak{A}$ and for every sentence ϕ, $\mathfrak{A} \nvDash \phi$ or $\mathfrak{A} \nvDash \urcorner \phi$. Let ϕ and ψ be two arbitrary sentences. It is the case that whatever model that satisfies both ϕ and $\urcorner \phi$ satisfies also ψ, for no nontrivial model satisfies ϕ and $\urcorner \phi$. Hence, $\mathfrak{A}, \phi, \urcorner \phi \vDash \psi$. As ϕ and ψ are arbitrary, $\urcorner$ is a supplementing negation. □

The corollary below is immediate from Proposition 1.2.4 and Proposition 1.2.5.

Corollary 1.2.6. *Let $\sim$ be a negation in a semantically determined logical system. Then, $\sim$ is a classical negation iff, for every sentence ϕ and for every nontrivial model $\mathfrak{A}$, exactly one of the following holds: $\mathfrak{A} \vDash \phi$ or $\mathfrak{A} \vDash \sim \phi$.*

In the previous section, the concept of a trivial theory was introduced on the fly. The following definition will formally present it together with a list of related concepts.

Definition 1.2.7. Trivial Theory *A theory T in a language L is* trivial iff, for every $\phi \in L$, $T \Vdash \phi$;

Nontrivial Theory A theory T in a language L is nontrivial iff, for some $\phi \in L$, $T \nVdash \phi$;

Maximal Nontrivial Theory A theory T in a language L is maximal nontrivial iff, for every $\phi, \psi \in L$, if $T \nVdash \phi$, then $T \cup \{\phi\} \Vdash \psi$;

Consistent Theory A theory T in a language L is consistent iff, for every $\phi \in L$, $T \Vdash \phi$ implies $T \nVdash \neg\phi$ and $T \Vdash \neg\phi$ implies $T \nVdash \phi$;

Inconsistent Theory A theory T in a language L is inconsistent iff, for some $\phi \in L$, $T \Vdash \phi$ and $T \Vdash \neg\phi$;

Maximal Consistent Theory A theory T in a language L is maximal consistent iff it is consistent and, for every $\phi \in L$, if $T \nVdash \phi$, then there is $\psi \in L$ such that $T \cup \{\phi\} \Vdash \psi$ and $T \cup \{\phi\} \Vdash \neg\psi$;

One of the most important techniques in classical model theory is that of enlarging a theory until a maximal consistent extension of it is reached. In the context of LFI's, the concept of maximal nontriviality does duty for that of maximal noninconsistency. For this reason, it is in order to explore how each kind of negation behaves with respect to that very concept.

Proposition 1.2.8. *Let $\neg$ be a complementing negation. If T is a maximal nontrivial theory, then, for every sentence $\phi \in L_\Sigma$, $T \Vdash \phi$ or $T \Vdash \neg\phi$.*

Proof. Let T be a maximal nontrivial theory. Suppose that, for some sentence ϕ, $T \nVdash \phi$ and $T \nVdash \neg\phi$. As T is a maximal nontrivial theory, it contains all the formulae it derives, whence T is properly contained in $T \cup \{\phi\}$. As $\neg$ is a complementing negation, $T \cup \{\phi\} \nVdash \neg\phi$. Hence, $T \cup \{\phi\}$ contains T properly and is not trivial, which is a contradiction against the fact that T is maximal nontrivial. □

On the other hand, there are maximal nontrivial theories that contain some sentence together with its negation. In fact, if $\neg$ is a complementing negation but not a supplementing one, then there are two sentences ϕ and ψ such that $\phi, \neg\phi \nVdash \psi$, which means that $T_0 = \{\phi, \neg\phi\}$ is a nontrivial theory. Taking an enumeration of the sentences in L_Σ and defining $T_{n+1} = T_n \cup \{\theta_{n+1}\}$ if $T_n \cup \{\theta_{n+1}\}$ is nontrivial and $T_{n+1} = T_n$ otherwise, a chain of nontrivial theories is constructed and their union $T = \cup_{n\in\mathbb{N}} T_n$ is a maximal nontrivial theory that contains the sentence ϕ and its negation $\neg\phi$ simultaneously.

The converse of Proposition 1.2.8 does not hold. That is, a theory T may contain ϕ or $\neg\phi$ for every sentence in L_Σ without being maximal nontrivial. Firstly, a result will be presented as a lemma, for it is interesting for its own.

Lemma 1.2.9. *Let $\neg$ be a complementing negation, T a nontrivial theory and ψ an arbitrary sentence. Then, $T \cup \{\psi\}$ or $T \cup \{\neg\psi\}$ is nontrivial.*

Proof. If $T \Vdash \psi$, then $T \cup \{\psi\}$ and T derive the same sentences (all the logical systems considered in this work are tarskian). As T is nontrivial, so is $T \cup \{\psi\}$. Analogously, if $T \Vdash \neg\psi$, then $T \cup \{\neg\psi\}$ is nontrivial.

If neither $T \Vdash \psi$ nor $T \Vdash \neg\psi$, then both $T \cup \{\psi\}$ and $T \cup \{\neg\psi\}$ are nontrivial. In fact, $T \cup \{\psi\} \nVdash \neg\psi$ and $T \cup \{\neg\psi\} \nVdash \psi$. □

Returning to the converse of Proposition 1.2.8, if $\neg$ is a complementing negation but not a supplementing one, then there is a sentence ϕ such that $T_0 = \{\phi, \neg\phi\}$ is a nontrivial theory. Taking again an enumeration of the sentences in L_Σ and defining $T_{n+1} = T_n \cup \{\psi_{n+1}\}$ if $T_n \cup \{\psi_{n+1}\}$ is nontrivial and $T_{n+1} = T_n \cup \{\neg\psi_{n+1}\}$ otherwise, a chain of theories is constructed. Each T_n is nontrivial, by Lemma 1.2.9. The union $T = \cup_{n\in\mathbb{N}} T_n$ is a nontrivial theory that contains, for each sentence ψ, ψ or its negation $\neg\psi$. This may be a maximal theory, but $T' = T \backslash \{\phi\}$ is still a nontrivial theory that contains, for each sentence ψ, ψ or its negation $\neg\psi$. Also, T' is not maximal, for T is a nontrivial theory that contains it properly.

The next proposition explores the relation between supplementing negations and maximal nontrivial theories.

Proposition 1.2.10. *Let $\urcorner$ be a supplementing negation. Let T be a nontrivial theory such that, for every sentence $\phi \in L$, $T \Vdash \phi$ or $T \Vdash \urcorner \phi$ (obviously, not both). Then, T is a maximal nontrivial theory.*

Proof. Let T be a theory as in the enunciation. Let $\phi \notin T$ be an arbitrary sentence not belonging to T.

By hypothesis, $\urcorner \phi \in T$. Thus, both ϕ and $\urcorner \phi$ belong to $T \cup \{\phi\}$, which turns out to be a trivial theory, for $\urcorner$ is a supplementing negation. As ϕ is an arbitrary sentence, T is maximal nontrivial. $\square$

The converse does not hold. Suppose $\urcorner$ is a supplementing negation but not a complementing one. In addition, suppose that there are a theory T and a sentence ϕ such that $T \nVdash \phi$, $T \nVdash \urcorner \phi$, $(T \cup \{\phi\}) \Vdash \urcorner \phi$ and $(T \cup \{\urcorner \phi\}) \Vdash \phi$. Take once again an enumeration of the sentences in L_Σ, defining:

- $T_0 = \emptyset$
- $T_{n+1} = T_n \cup \{\psi_{n+1}\}$ if $T_n \cup \{\psi_{n+1}\} \nVdash \phi$ and $T_n \cup \{\psi_{n+1}\} \nVdash \urcorner \phi$ and
- $T_{n+1} = T_n$, otherwise.

The union $T' = \cup_{n \in \mathbb{N}} T_n$ is maximal with respect to the property of not deriving either ϕ or $\urcorner \phi$ ($T' \nVdash \phi$ and $T' \nVdash \urcorner \phi$). It is also maximal with respect to the property of being nontrivial. In fact, let θ be an arbitrary sentence. If $\theta \notin T'$, then $T' \cup \{\theta\} \Vdash \phi$ or $T' \cup \{\theta\} \Vdash \urcorner \phi$. In any case, $T' \cup \{\theta\} \Vdash \{\phi, \urcorner \phi\}$, for $T' \cup \{\phi\} \Vdash \urcorner \phi$ and $T' \cup \{\urcorner \phi\} \Vdash \phi$. As $\urcorner$ is a supplementing negation, it follows that $T' \cup \{\theta\}$ is trivial. Therefore, T' is a maximal nontrivial theory that does not contain ϕ nor $\urcorner \phi$.

Looking carefully at the proof of Lemma 1.2.9, a finer result can be obtained: If $T \cup \{\psi\} \nVdash \neg\psi$, then $T \cup \{\psi\}$ is nontrivial and if $T \cup \{\neg\psi\} \nvdash \psi$, then $T \cup \{\neg\psi\}$ is nontrivial. Hence, it is not necessary that $\neg$ be a complementing negation in order to guarantee that maximal theories contain ψ or $\neg\psi$ for every ψ. It is enough that $\neg$ be so that, for every ψ, $T \cup \{\psi\} \nVdash \neg\psi$ or $T \cup \{\neg\psi\} \nVdash \psi$.

So much hairsplitting is to be avoided. Anyway, the observation above shows that it would not be possible to relax the requirement that $(T \cup \{\phi\}) \Vdash \urcorner \phi$ and $(T \cup \{\urcorner \phi\}) \Vdash \phi$ in the search for a counterexample to the converse of Proposition 1.2.10. The following corollary follows immediately from Proposition 1.2.8 and Proposition 1.2.10.

Corollary 1.2.11. *Let $\sim$ be a classical negation. A theory T is maximal nontrivial iff, for each sentence $\phi \in L$, it holds one and no more than one of the following:*

- $T \Vdash \phi$;
- $T \Vdash \sim \phi$.

In QmbC, besides the original negation $\neg$, at least two interesting negations can be defined. Let $\Lsh \alpha$ be an abbreviation of $\neg\alpha \wedge \circ\alpha$ and let $\sim \alpha$ be an abbreviation of $\alpha \rightarrow \Lsh \alpha$. In [15], it is proven that $\neg$ is a complementing but not a supplementing negation, that the auxiliar negation $\Lsh$ is a supplementing but not a complementing negation and that $\sim$ is a classical negation.

The existence of both a supplementing negation and a complementing one does not happen by accident. If, in a given logic, Modus Ponens is respected, the disjunction '$\wedge$' behaves classically and a supplementing negation $\Lsh$ is available, then a classical negation is available as well.

In fact, fix an arbitrary sentence θ. If T is a theory such that $T \Vdash (\theta \wedge \Lsh \theta)$, then T is obviously trivial. Let the symbol $\perp_\theta$ be an abbreviation of the formula $\theta \wedge \Lsh \theta$. If a formula $\perp$ is such that a theory T is trivial whenever it derives $\perp$, then it is called a bottom particle. It is clear that in a logic with a supplementing negation each sentence in the language provides a bottom particle. Hence, there are at least as many bottom particles as there are sentences.

Finally, if a bottom particle is available, then a classical negation is available as well. Define $\sim \alpha = \alpha \rightarrow \perp$. To prove that $\sim$ is a supplementing negation, just see that $\{\alpha, \sim \alpha\} \Vdash \perp$. To prove that $\sim$ is a complementing negation, let T be an arbitrary theory and ϕ an arbitrary sentence such that $T \nVdash \sim \phi$ and $T \cup \{\phi\}$ is not trivial. If $T \cup \{\phi\} \Vdash \sim \phi$, then $T \cup \{\phi\} \Vdash \perp$. Hence, $T \cup \{\phi\}$ would be trivial, which is a contradiction. Therefore, $T \cup \{\phi\} \nVdash \sim \phi$. Analogously, if $T \nVdash \phi$ and $T \cup \{\sim \phi\}$ is not trivial, then $T \cup \{\sim \phi\} \nVdash \phi$. As T and ϕ are arbitrary, $\sim$ is in fact a complementing negation.

Summing up, if a supplementing negation $\Lsh$ is available, then a bottom particle $\perp$ is available too; If a bottom particle is available, then a classical negation $\sim$ is available too; If a classical negation is available, then a supplementing negation is available too, for a classical negation is itself a supplementing negation. That is, having a supplementing negation, having a classical negation and having a bottom particle are equivalent conditions.

Naturally, classical negations are stronger than supplementing ones, in the sense that the former provide recourses to mimic classical logic in a broader way than the latter. For this reason, the fact that the availability of a supplementing negation avails the logic with a classical one suggests that it is not actually worth exploring supplementing negations' behavior so much deeper.

It must be stressed that supplementing negations are always extra entities, rather than basic ones, that they play an auxiliary role in whatever paraconsistent logic that happens to count with them. In fact, they respect the principle of explosion, whence the original paraconsistent negation cannot be a supplementing one.

On the other hand, paraconsistent negations must be complemeting. In fact, for a unary connective to deserve the name 'negation', it is supposed to behave somehow as a negation. As being complementing and supplementing are the basic features of a negation an paraconsistent negations cannot be supplementing, it is left for them to be complementing. From now on, a paraconsistent negation will be a one that is complementing but not supplementing.

Having presented a link between bottom particles and classical negations, this section will be concluded with the presentation of a link between consistency and bottom particles.

Proposition 1.2.12. *Consider a logic in a language L_Σ endowed with a classical conjunction and with a classical implication for which Modus Ponens and the Deduction Metatheorem (DMT) hold. Then, there exists a consistency connective $\circ$ (defined or basic in the signature) satisfying the schema $(\circ\alpha \to (\alpha \to (\neg\alpha \to \beta)))$ iff there exists a bottom particle (defined or basic in the signature).*

Proof. ($\Rightarrow$) Suppose there is a consistency connective $\circ$ satisfying the schema in the enunciation. Choose a sentence α in L_Σ and define the sentence $\bot = \circ\alpha \wedge (\alpha \wedge \neg\alpha)$. It will be proven that $\bot$ is a bottom particle, that is, $\bot \Vdash \beta$ for every arbitrary formula β. In fact,

1. $\bot \quad \Vdash \circ\alpha, \alpha, \neg\alpha$ (for '$\wedge$' is a classical conjunction);
2. $\Vdash \circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$
3. $\bot \quad \Vdash \beta$ (applying *Modus Ponens* three times in (1) and (2).

($\Leftarrow$) Suppose there exists a bottom particle $\bot$ and, for each formula α, let $\circ\alpha$ stand for the the formula $\alpha \to (\neg\alpha \to \bot)$. Let α and β be arbitrary formulae. Then,

1. $\circ\alpha, \alpha, \neg\alpha \Vdash \bot$ (applying *Modus Ponens* twice);
2. $\bot \quad \Vdash \beta$ (for $\bot$ is a bottom particle);
3. $\circ\alpha, \alpha, \neg\alpha \Vdash \beta$ (by (1) and (2), for the logic under consideration is a tarskian one);
4. $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$ (applying DMT three times).

□

The next corollary joins Proposition 1.2.12 and the discussion that precedes it.

Corollary 1.2.13. *If SYS is a paraconsistent logical system that behaves classically with respect to the positive connectives, then the following are equivalent:*

1. *SYS has an auxiliar supplementing negation;*

2. *SYS has an auxiliar classical negation;*

3. *SYS has a bottom particle;*

4. *SYS has a consistency operator.*

1.3 Basic Concepts of Classical Model Theory

The task of this work is to develop model theory in a paraconsistent environment. The main notions regarding the environment have just been discussed. Now it is time to discuss the main notions of the object to be studied in that envirionment, that is, model theory.

This section presents the basic concepts of classical model theory and introduces the notation that will be used along the text.

The first concept to be presented is that of signature. Constructing a theory about some physical or abstract world consists in determining three aspects: First, what the objects of that theory are. Second, what can be stated about those objects. Third, what assertions are true. A signature is an uninterpreted abstract object that serves as the basic framework to theorize about a class of possible realities. A signature fixes a set of objects (without giving any interpretation) that must be present in any theory of the class of theories it determines, symbols that gain new objects from already available objects and symbols that create basic assertions about the objects.

Definition 1.3.1 (Signature)**.** *A* signature Σ is a triple $\langle C, \bar{F}, \bar{P}, V\rangle$, where C is a set of symbols, $\bar{F} = \cup_{i=1}^{\infty} F_i$ is a union of sets of symbols F_i (which may be void), $\bar{P} = \cup_{i=1}^{\infty} P_i$ is a union of sets of symbols P_i (which may be void) and V is a set of symbols with cardinality $\aleph_0$. A symbol $c \in C$ is called a constant, a symbol $f \in F_n$ is called a function symbol of arity n, a symbol $P \in P_n$ is called a predicate symbol of arity n and a symbol in V is called a variable.

Each signature determines a set of terms, which are the entities that become the objects of a theory when interpreted. In other words, terms are uninterpretd objects.

Definition 1.3.2 (Term)**.** *Given a signature* Σ*, the* terms based on Σ are those recursively described as follows:

- A constant c is a closed term;

- A variable x is an open term that depends on x;
- If $f \in F_n$ is a function symbol of arity n and $\tau_1, \dots, \tau_n$ are terms that depend respectively on the variables in the sets $V_1, \dots, V_n$, then $f(\tau_1, \dots, \tau_n)$ is a term that depends on the variable in the set $V_0 = V_1 \cup \dots \cup V_n$;
- There are no other terms.

The notation $\tau[x_{i_1}, \dots, x_{i_m}]$ indicates that the term τ does not depend on any variable out of the set $\{x_{i_1}, \dots, x_{i_m}\}$. If the set of varibles of which the term τ depends is void, then it is said to be a closed term. Otherwise, it is said to be an open term.

Obs.: It shall be clear from Definition 1.3.2 that the notation $\tau[x_{i_1}, \dots, x_{i_m}]$ does not mean that τ depends on the variables $x_{i_1}, \dots, x_{i_m}$. It may be the case that τ does not depend on any of those variables.

Each signature determines a set of formulae from the set of terms. Those are the entities that become the assertions about the objects of a theory when interpreted. In other words, formulae are uninterpreted assertions about uninterpreted objects. The definition of formula recurs to the definition of term.

Definition 1.3.3 (Formula)**.** *Given a signature* Σ*, the* formulae based on Σ are those recursively described as follows:

- If $P \in P_n$ is a function symbol of arity n and $\tau_1, \dots, \tau_n$ are terms that depend respectively on the variables in the sets $V_1, \dots, V_n$, then $P(\tau_1, \dots, \tau_n)$ is a formula of complexity 0 that depends on the variables in the set $V_0 = V_1 \cup \dots \cup V_n$;
- If ϕ is a formula of complexity m that depends on the variables in the set V_ϕ, then $\neg\phi$ is a formula of complexity $m + 1$ that depends on the variables in V_ϕ;
- If ϕ is a formula of complexity m that depends on the variables in the set V_ϕ and ψ is a formula of complexity n that depends on the variables in the set V_ψ, then $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \to \psi$ are formulae of complexity $m + n + 1$ that depend on the variables in $V_\phi \cup V_\psi$;
- If ϕ is a formula of complexity m that depends on the variables in the set V_ϕ, then $\exists x\phi$ and $\forall x\phi$ are formulae of complexity $m + 1$ that depend on the variables in $V_\phi \backslash \{x\}$;
- There are no other formulae.

The notation $\phi(x_{i_1}, \ldots, x_{i_m})$ indicates that the formula ϕ does not depend on any variable out of the set $\{x_{i_1}, \ldots, x_{i_m}\}$. If the set of varibles on which the formula ϕ depends is void, then it is said to be a closed formula. Otherwise, it is said to be an open formula.

The complexity of a formula ϕ is designated by $comp(\phi)$.

A formula of complexity 0 is also called a basic formula or an atom.

Obs.: It shall be clear from Definition 1.3.3 that the notation $\phi(x_{i_1}, \ldots, x_{i_m})$ does not mean that ϕ depends on the variables $x_{i_1}, \ldots, x_{i_m}$. It may be the case that ϕ does not depend on any of those variables.

A set of terms together with a set of formulae constitute a language. When interpreted, a language is a universe of objects together with what can be asserted about those objects.

Definition 1.3.4 (Language)**.** *The* language based on a given signature Σ is the pair $\langle T_\Sigma, F_\Sigma \rangle$, denoted by L_Σ, where T_Σ is the set of terms based on Σ and F_Σ is the set of formulae based on Σ. If τ is a term, the fact that $\tau \in T_\Sigma$ is also expressed by saying that τ belongs to L_Σ and designated by $\tau \in L_\Sigma$. If ϕ is a formula, the fact that $\tau \in T_\Sigma$ is also expressed by saying that ϕ belongs to L_Σ and designated by $\phi \in L_\Sigma$.

A structure on (over, based on) a signature (or language) is a schema that allows the interpretation of the terms and formulae in the language based on that signature.

Definition 1.3.5 (Structure)**.** *A* structure *on a signature* Σ *(or on the language based on a signature* Σ*) is a pair* $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$*, where* A *is a nonempty set, which is the domain of interpretation, and* $I_{\mathfrak{A}}$ *is a function whose domain is* Σ*, called* interpretation function of $\mathfrak{A}$*, such that:*

- $I_{\mathfrak{A}}(c) \in A$
- $I_{\mathfrak{A}}(f) \in E_n$*, if* $f \in F_n$*, where*
  - E_n *is the space of functions from* A^n *to* A*.*
- $I_{\mathfrak{A}}(P) = A_P^{\mathfrak{A}}$ *is a subset of* A^n*, if* $P \in P_n$*.*

This is how a structure works.

Definition 1.3.6 (Interpretation of a Term). *Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ be a structure based on a signature Σ and let $\tau(x_{i_1}, \ldots, x_{i_m})$ be a term. The interpretation of τ in the sequence $(a_{i_1}, \ldots, a_{i_m})$ of elements in the domain of interpretation A is designated by $\tau^{\mathfrak{A}}(x_{i_1}, \ldots, x_{i_m})[a_{i_1}, \ldots, a_{i_m}]$ or simply by $\tau^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}]$ and is defined as follows:*

- *If τ is a constant c, then $\tau^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}] = I_{\mathfrak{A}}(c)$;*
- *If τ is the variable x_{i_k}, then $\tau^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}] = a_{i_k}$;*
- *If $f \in F_n$ is a function sysmbol of arity n, $\tau_1, \ldots, \tau_n$ are terms and $\tau = f(\tau_1, \ldots, \tau_n)$, then $\tau^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}] = I_{\mathfrak{A}}(f)(\tau_1^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}], \ldots$ $\ldots, \tau_n^{\mathfrak{A}}[a_{i_1}, \ldots, a_{i_m}])$.*

When a string of variables $(x_{i_1}, \ldots, x_{i_m})$ is substituted by a string of $(a_{i_1}, \ldots, a_{i_m})$ from the domain of interpretation in a term $\tau(x_{i_1}, \ldots, x_{i_m})$, the result is the intert-preted term $\tau(a_{i_1}, \ldots, a_{i_m})$. Likewise, when a string of variables $(x_{i_1}, \ldots, x_{i_m})$ is substituted by a strings $(a_{i_1}, \ldots, a_{i_m})$ of elements from the domain of interpretation in a formula $\phi(x_{i_1}, \ldots, x_{i_m})$, the result is the intertpreted formula $\phi(a_{i_1}, \ldots, a_{i_m})$. Philosophically speaking, uninterpreted terms are undetermined objects and uninterpreted formulae talk about those objects. In this line, it makes no sense to attribute a truth value to a formula that does not talk about real objects. For this reason, the function that atributes truth value to formulae is defined for interpreted formulae. Such a function is named ‘valuation’. Before presenting this important concept, however, some notation will be introduced.

- $\vec{x}$ is used to designate a sequence of variables and $\vec{a}$ is used to designate a sequence of elements in the domain of interpretation.
- The lenth of a sequence $\vec{x}$ or $\vec{a}$ is the number of elements it possesses.
- The notation $\mathfrak{A}, v \vDash \phi(\vec{a})$ is used to mean that the model $\mathfrak{A}$ is the structure $\mathfrak{A}$ endowed with the valuation v and that $v(\phi) = 1$. However, if confusion may not rise, the shorter notation $\mathfrak{A} \vDash \phi(\vec{a})$ can be used;
- The term ‘sentence’ is used as a synonym for closed formula. As a sentence does not depend on any variable, it holds that, if ϕ is a sentence, then, for every valuation v, given two sequences of elements is the doimain of interpretation $\vec{a_1}$ and $\vec{a_1}$, $v(\phi[\vec{a_1}]) = v(\phi[\vec{a_1}])$. For this reason, it can be written $v(\phi)$ and $\mathfrak{A}, v \vDash \phi$ instead of $v(\phi[\vec{a}])$ and $\mathfrak{A}, v \vDash \phi[\vec{a}]$
- Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ be a model over the signature Σ, whose set of constants is C. Let $X \subset A$ be a subset of A and $C_X = \{c_x | x \in X\}$ a set of new constants

(with the same cardinality of X, of course). Then, Σ_X is the signature with the same symbols of function and predicate and whose set of constants is $C \cup C_X$;

- In accordance with the item above, $I_{\langle \mathfrak{A}, X\rangle}$ is the interpretation function for the language L_{Σ_X}, defined so that $I_{\langle \mathfrak{A}, X\rangle}(P) = I_{\mathfrak{A}}(P)$, for every $P \in \bar{P}$; $I_{\langle \mathfrak{A}, X\rangle}(f) = I_{\mathfrak{A}}(f)$, for every $f \in \bar{F}$; $I_{\langle \mathfrak{A}, X\rangle}(c) = I_{\mathfrak{A}}(c)$, for every $c \in C$ and $I_{\langle \mathfrak{A}, X\rangle}(c_x) = x$, for every $x \in X$;
- The extended model $\langle \mathfrak{A}, X\rangle$ (or $\mathfrak{A}_X$) is the model over the language L_{Σ_X} with interpretation function $I_{\langle \mathfrak{A}, X\rangle}$ and whose valuation is the natural extension of $v_{\mathfrak{A}}$ to Σ_X;
- An interpreted formula $\phi(\vec{x})[\vec{a}]$ (or just $\phi[\vec{a}]$) is a formula $\phi(\vec{x})$ where each variable in $\vec{x}$ is interpreted by the correspondent element in $\vec{a}$.
- The set of all interpreted formulae in a model $\mathfrak{A}$ is denoted by $IF(\mathfrak{A})$.
- It will be written $\theta[x/\tau]$ or θ^x_τ to denote the formula obtained by uniformly substituting x by τ. If it is clear by the context what variable is being substituted, it may be written $\theta(\tau)$;
- A variant of a formula is a renaming of its variables.
- $L(\mathfrak{A})_{\vec{x},z}$ and $T(\mathfrak{A})_{\vec{x},z}$ stand for the set of formulae and terms, respectively, in the variables $\{\vec{x}, z\}$ that belong to the language where $\mathfrak{A}$ is based;
- An equality is a predicate symbol $\approx$ of arity 2 so that $(I(\approx) = \{(a,a)|a \in A\})$. Given two terms τ_1 and τ_2, it is written $\tau_1 \approx \tau_2$ instead of $\approx (\tau_1, \tau_2)$ to designate generated by $\approx$ from τ_1 and τ_2. It shall be clear that, given a structure $\mathfrak{A}$, $v(\tau_1 \approx \tau_2) = 1$ iff $\tau_1^{\mathfrak{A}} = \tau_2^{\mathfrak{A}}$ for whatever valuation $\mathfrak{A}$ may be endowed with.
- The symbol '=' always stands for equality in the metalanguage. The symbol $\approx$ stands for the equality in the language, when the referred language is endowed with an equality.

Finally, valuations can be defined. In the definition that follows, it is tacitly assumed that the lenths of the sequences cohere.

Definition 1.3.7 (Valuation)**.** *Let $\mathfrak{A}$ be a structure over the signature Σ with domain A. Let L_Σ, as usual, be the language constructed over Σ. The* valuation *over $\mathfrak{A}$ is the mapping $v : IF(\mathfrak{A}) \to \{0,1\}$ recursively defined by the following clauses:*

vPred $v(P(t_1 \ldots, t_n)[\vec{a}]) = 1$ *iff* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, *for* $P \in P_n$.

vOr $v((\alpha \vee \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *or* $v(\beta[\vec{a}]) = 1$.

vAnd $v((\alpha \wedge \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *and* $v(\beta[\vec{a}]) = 1$.

vImp $v((\alpha \to \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 0$ *or* $v(\beta[\vec{a}]) = 1$.

vNegClass $v(\neg\alpha[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 0$.

vEx $v(\exists x\alpha[\vec{a}]) = 1$ *iff* $v(\alpha(x, \vec{x})[a, \vec{a}]) = 1$ *for some* $a \in A$.

vUni $v((\forall x\alpha)[\vec{a}]) = 1$ *iff* $v(\alpha(x, \vec{x})[a, \vec{a}]) = 1$ *for every* $a \in A$.

This section will be closed with the concept of model, which joins the concepts of structure and valuation.

Definition 1.3.8. *The model based on a signature* Σ *(or on a language* L_Σ*) is the structure based on* Σ *together with the valuation based on the same signature (or language).*

As a model can be identified with its structure, the same notation can be used for the two objects.

1.4 Basic Concepts of QmbC Model Theory

Along Chapter 2, a paraconsistent system for resoning will be constructed and the starting point for this construction is the system QmbC.

In this section, the versions for QmbC of the classical concepts in Section 1.3 will be presented.

The notation in Section 1.3 remains valid not only for QmbC but also for the systems that will be constructed from it, although some concepts will be reformulated. When the context is clear, it may be written just ‘model’ to denote ‘QmbC-model’ or ‘reasoning model’, for instance. This section focus on what changes from classical to QmbC models.

The concepts of signature and term are the classical ones. The concept of formula is also the classical one augmented by the addition of a clause that introduces the consistency connective.

- If ϕ is a formula of complexity m that depends on the variables in the set V_ϕ, then $\circ\phi$ is a formula of complexity $m + 2$ that depends on the variables in V_ϕ.

Note that the consistency connective adds 2 points to the complexity of a formula, whereas the other connectives add 1 point. The convenience of such a definition will become clear as the proofs of the reults progress. In [13], complexity is defined so that the negation connective that adds 2 points and the consistency connective only 1.

The definition of structure is also the classical one. A small change in the notation will be introduced, namely, $A_{P1}^{\mathfrak{A}}$ will be written instead of $A_{P}^{\mathfrak{A}}$. The convenience of doing so will become clear as the text progresses.

Definition 1.4.1 (QmbC-Structure). *A* QmbC-structure *over a signature* Σ *is a pair* $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$*, where* A *is a nonempty set, which is the domain of interpretation, and* $I_{\mathfrak{A}}$ *is a function whose domain is* Σ*, called* interpretation function of $\mathfrak{A}$*, such that:*

- $I_{\mathfrak{A}}(c) \in A$
- $I_{\mathfrak{A}}(f) \in E_n$*, if* $f \in F_n$*, where*
  - E_n *is the space of functions from* A^n *to* A*.*
- $I_{\mathfrak{A}}(P) = A_{P1}^{\mathfrak{A}}$ *is a subset of* A^n*, if* $P \in P_n$*.*

The concept of interpretation of terms is the classical one. The concept of valuation, on its turn, must be reformulated. It shall be clear that the classical clause **vNegClass** must be substituted by a paraconsistent clause and that a new clause will be needed in order to rule the behavior of the consistency operator. Less obvious is the fact that the clauses for the paraconsistent negation and for the consistency operator do not respect uniform substitution and this calls for extra clauses that rule terms substitution in negated and paraconsistency formulae. The definition that follows is presented in [13].

Definition 1.4.2 (QmbC-Valuation). *Let* $\mathfrak{A}$ *be a structure over the signature* Σ *with domain* A*. Let* L_Σ*, as usual, be the language constructed over* Σ*. A mapping* $v : IF(\mathfrak{A}) \to \{0,1\}$ *is a* QmbC-valuation *over* $\mathfrak{A}$ *iff it satisfies the following clauses:*

vPred $v(P(t_1 \ldots, t_n)[\vec{a}]) = 1$ *iff* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$*, for* $P \in P_n$*.*

vOr $v((\alpha \vee \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *or* $v(\beta[\vec{a}]) = 1$*.*

vAnd $v((\alpha \wedge \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *and* $v(\beta[\vec{a}]) = 1$*.*

vImp $v((\alpha \to \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 0$ *or* $v(\beta[\vec{a}]) = 1$*.*

vNeg *If* $v(\alpha[\vec{a}]) = 0$*, then* $v(\neg\alpha[\vec{a}]) = 1$*.*

vCon *If* $v(\circ\alpha[\vec{a}]) = 1$*, then* $v(\alpha[\vec{a}]) = 0$ *or* $v(\neg\alpha[\vec{a}]) = 0$.

vVar $v(\alpha[\vec{a}]) = v(\beta[\vec{a}])$ *whenever* α *is a variant of* β.

vEx $v(\exists x\alpha[\vec{a}]) = 1$ *iff* $v(\alpha(x,\vec{x})[a,\vec{a}]) = 1$ *for some* $a \in A$.

vUni $v((\forall x\alpha)[\vec{a}]) = 1$ *iff* $v(\alpha(x,\vec{x})[a,\vec{a}]) = 1$ *for every* $a \in A$.

sNeg *For every context* $(\vec{x}, z)$ *and* $(\vec{x}, \vec{y})$*, for every sequence* $(\vec{a}, \vec{b})$ *in* A *interpreting* $(\vec{x}, \vec{y})$*, for every* $\phi \in L(\mathfrak{A})_{\vec{x},z}$ *and for every* $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ *such that* t *is free for* z *in* ϕ*, if* $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ *and* $b = (t[\vec{x},\vec{y}/\vec{a},\vec{b}])^{\widehat{\mathfrak{A}}}$*, then* $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b])$ *implies* $v((\neg\phi[z/t][\vec{x},\vec{y}/\vec{a},\vec{b}])) = v(\neg\phi[\vec{x},z/\vec{a},b])$.

sCon *For every context* $(\vec{x}, z)$ *and* $(\vec{x}, \vec{y})$*, for every sequence* $(\vec{a}, \vec{b})$ *in* A *interpreting* $(\vec{x}, \vec{y})$*, for every* $\phi \in L(\mathfrak{A})_{\vec{x},z}$ *and for every* $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ *such that* t *is free for* z *in* ϕ*, if* $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ *and* $b = (t[\vec{x},\vec{y}/\vec{a},\vec{b}])^{\widehat{\mathfrak{A}}}$*, then* $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b])$ *implies* $v((\circ\phi[z/t][\vec{x},\vec{y}/\vec{a},\vec{b}])) = v(\circ\phi[\vec{x},z/\vec{a},b])$.

In classical model theory, there is only one possible valuation for each structure. On the other hand, there are multiple possible valuations for a QmbC-structure. For this reason, there are multiple possible models as well.

Finally, the section will be closed with the concept of QmbC-model.

Definition 1.4.3. *A* QmbC-model based on a signature Σ (or on a language L_Σ) is a QmbC-structure endowed with a QmbC-valuation.

1.5 Basic Concepts of Logic Programming

Chapter 4 works out the system developed in Chapter 2 by developing the basis for a paraconsistent logic programming (PROLOG). This section presents some basic concepts of Logical Programming and introduces the notation and some results that will be used.

The first concept to be presented is that of substitution of variables.

Definition 1.5.1 (Substitution of Variables)**.** *A* substitution of variables or just *substitution* in a language is a function from the set of variables to the set of terms. Given a substitution θ and a term τ, $\tau\theta$ is the term recursively defined as follows:

- If τ is a constant, then $\tau\theta = \tau$;
- If τ is a variable x, then $\tau\theta = \theta(x)$;
- If $\tau = f(\tau_1, \ldots, \tau_n)$, then $\tau\theta = f(\tau_1\theta, \ldots, \tau_n\theta)$.

Given a substitution θ and a formula ϕ, $\phi\theta$ is the formula recursively defined as follows:

- If $\phi = P(\tau_1, \ldots, \tau_n)$, then $\phi\theta = P(\tau_1\theta, \ldots, \tau_n\theta)$;
- If $\phi = \neg\alpha$, then $\phi\theta = \neg(\alpha\theta)$;
- If $\phi = \circ\alpha$, then $\phi\theta = \circ(\alpha\theta)$;
- If $\phi = \alpha \vee \beta$, then $\phi\theta = (\alpha\theta) \vee (\beta\theta)$;
- If $\phi = \alpha \wedge \beta$, then $\phi\theta = (\alpha\theta) \wedge (\beta\theta)$;
- If $\phi = \alpha \to \beta$, then $\phi\theta = (\alpha\theta) \to (\beta\theta)$.

If S is a set of expressions (where an expression may be a term of a formula), then $S\theta = \{E\theta | E \in S\}$.

The convention for composition of substitutions is to use the inverse order as that of composition of functions.

Definition 1.5.2 (Composition of Substitutions)**.** *The composition of two substitutions θ and σ is the substitution $\theta\sigma = \sigma \circ \theta$.*

The concept below plays an important role in the theory of Logic Programming.

Definition 1.5.3. *A substitution θ is said to* unify a set of expressions S or to be a unifier for S if $S\theta$ is a singleton. The unifier θ is said to be a most general unifier for S if, for each unifier σ for S, there is a substitution γ such that $\sigma = \theta\gamma$.

The next concept to be introduced is that of program, which is a set of clauses.

Definition 1.5.4 (Clause)**.** *A clause is an arrangement of the form $B \leftarrow A_1, \ldots, A_n$, where $B, A_1, \ldots, A_n$ are atoms.*

- *B is called the head and $A_1, \ldots, A_n$ the body of the clause;*
- *The clause $B \leftarrow A_1, \ldots, A_n$ is logically equivalent to the implication $B \leftarrow A_1 \wedge \cdots \wedge A_n$.*

Definition 1.5.5 (Program)**.** *A program is a finite set of clauses.*

- *If P is a program, B_P is the set of atoms in the language of P;*
- *A bodyless clause $B \leftarrow$ in a program P determines that B is a consequence of P.*

A few extra definitions will be also needed.

- An atom $A \in B_P$ is a consequence of the program P iff A is a logical consequence of the clauses of P;
- A predicate is said to be defined by a program P is it is the head of some clause in P;
- An arrangement of the form $\leftarrow A_1, \ldots, A_n$ is called a goal;
  - A goal is said to be satisfied if there is a variable substitution θ such that $A_1\theta \wedge \cdots \wedge A_n\theta$ is a consequence of P.

The paraconsistent negations that will be defined in Chapter 4 are based on SLD-resolutions, which is a special case of SLD-derivation. This is the next concept to be introduced.

Definition 1.5.6 (SLD-derivation)**.** *Let G be the goal $\leftarrow A_1, \ldots, A_n$. If G' is a goal $\leftarrow A_1\theta, \ldots, A_{k-1}\theta, B_1\theta, \ldots, B_m\theta, A_{k+1}\theta, \ldots, A_n\theta$, C is the clause $A \leftarrow B_1, \ldots, B_m$ of the program P and θ is a mgu such that $A\theta = A_k\theta$, then G' is the goal derived from G using C and θ.*

Let P be a program and G a goal. An SLD-Derivation of $P \cup G$ consists of a (finite or infinite) sequence $G_0 = G_1, \ldots, G_n$ (or $G_0 = G_1, \ldots$) of goals, a (finite or infinite) sequence $C_0 = C_1, \ldots, C_n$ (or $C_0 = C_1, \ldots$) of variants of clauses in P and a (finite or infinite) sequence $\theta_0 = \theta_1, \ldots, \theta_n$ (or $\theta_0 = \theta_1, \ldots$) of mgu's.

Definition 1.5.7 (SLD-Resolution)**.** *An SLD-resolution of $P \cup G$ is a finite SLD-derivation such that the last goal has an empty body.*

This section will be ended with a brief presentation of some definitions and results concerning fixpoints.

Definition 1.5.8. *Let S be a set with a partial order $\leq$. Then, $a \in S$ is an upper bound of a subset X of S if $x \leq a$, for every $x \in X$. Similarly, b is a lower bound of a subset X of S if $b \leq x$, for every $x \in X$.*

Definition 1.5.9. *Let S be a set with a partial order $\leq$. Then, $a \in S$ is the least upper bound of a subset X of S if a is an upper bound of X and, for every upper bound a' of X, it is the case that $a \leq a'$. Similarly, $b \in S$ is the greatest lower bound of a subset X of S if b is an upper bound of X and, for every upper bound b' of X, it is the case that $b' \leq b$.*

When there exists an upper bound, it is unique and is denoted by $\mathrm{lub}(X)$. When there exists a lower bound, it is unique and is denoted by $\mathrm{glb}(X)$.

Definition 1.5.10. *A partially ordered set L is a complete lattice if lub(X) and glb(X) for every subset X of L.*

The lower upper bound of L (if it exists) is called the *top element* and is denoted by $\top$. The greatest lower bound of L (if it exists) is called the *bottom element* and is denoted by $\bot$.

Definition 1.5.11. *Let L be a complete lattice and $T : L \to L$ be a mapping. Then, T is said to be monotonic if $T(x) \leq T(y)$, whenever $x \leq y$.*

Definition 1.5.12. *Let L be a complete lattice and $X \subseteq L$. Then, X is said to be directed if every finite subset of X has an upper bound in X.*

Definition 1.5.13. *Let L be a complete lattice and $T : L \to L$ be a mapping. Then, T is said to be continuos if $T(lub(X)) = lub(t(X))$, for every directed subset X of L.*

Definition 1.5.14. *Let L be a complete lattice and $X \subseteq L$ be a mapping. Then, $a \in L$ is said to be the least fixpoint of T a is a fixpoint (that is, $T(a) = a$) and $a \leq b$, if b is a fixpoint. The greatest fixpoint is defined similarly.*

Proposition 1.5.15. *Let L be a complete lattice and let $T : L \to L$ be monotonic. Then, T has a least fixpoint lfp(T) and a greatest fixpoint gfp(T). Furthermore,*
lfp(T)=glb($\{x|T(x) = x\}$)=glb($\{x|T(x) \leq x\}$) and
gfp(T)=lub($\{x|T(x) = x\}$)=lub($\{x|x \leq T(x)\}$).

Definition 1.5.16. *Let L be a complete lattice and let $T : L \to L$ be monotonic. Define*

$T \uparrow 0 = \bot$

$T \uparrow \alpha = T(T \uparrow (\alpha - 1))$, *if α is a successor ordinal*

$T \uparrow \alpha =$lub($\{T \uparrow \beta|\beta \leq \alpha\}$), *if α is a limit ordinal*

$T \downarrow \top$

$T \downarrow = T(T \downarrow (\alpha - 1))$, *if α is a successor ordinal*

$T \downarrow \alpha =$glb($\{T \downarrow \beta|\beta \leq \alpha\}$), *if α is a limit ordinal*

Proposition 1.5.17. *Let L ba a complete lattice and let*
$T : L \to L$ be continuous. Then, lfp(T)= $T \uparrow \omega$.

Chapter 2

A Suitable System for Paraconsistent Reasoning

2.1 A Paraconsistent Account of Isomorphism

The notion of isomorphism is designed in each branch of mathematics as a tool for matching objects that are likely to be viewed as being the 'same thing', in the sense that they behave in the same manner with respect to some desired aspect. In classical theory of models, the concept of isomorphism is designed so as to preserve validity of sentences, that is, so as to match models that satisfy the same sentences.

Of course, isomorphism (or homomorphism) can be defined for QmbC just in the same way as for classical logic. The problem is that the classical-like definition does not preserve validity of sentences in QmbC, for it does not control the 'propagation' of negation and consistency. The goal of this section is to provide an account of isomorphism for QmbC which shall be able to preserve validity of sentences at least to some extent. It turns out that the concept of isomorphism is so much limited in QmbC, but it may be fruitful in some slight enrichments of it.

An important case of extension of a model $\mathfrak{A}_X$ is the one where $X = A$. In that regard, it holds the very useful result below. Its proof is straightfoward and identical to the proof for the classical case.

Lemma 2.1.1. *For every $\theta(x) \in L_\Sigma$ and $a \in A$, $\mathfrak{A} \vDash \theta[a]$ iff $\mathfrak{A}_A \vDash \theta(c_a)$.*

Obs.: In view of Lemma 2.1.1, working with sentences instead of interpreted formulae is not actually a loss of generality. In fact, if every element in the domain interprets some constant, then what is true about sentences is true about interpreted formulae. The developments in this section will be performed in terms of sentences because this is the usual way in the literature. However, the proofs are performed for extended models $\mathfrak{A}_A$. For this reason, they are valid for interpreted formulae as well. Further in the text, it will be convenient to work in terms of interpreted formulae.

The result below plays an important role in the proofs by induction.

Proposition 2.1.2. *Given a formula θ and a term τ in a language L_Σ, $comp(\theta^x_\tau) = comp(\theta)$.*

Proof. The proof is performed by induction on the complexity of θ.
If $comp(\theta) = 0$, then $\theta = P(\tau_1, \ldots, \tau_n)$, for some $P \in \bar{P}$ and some sequence of terms $(\tau_1, \ldots, \tau_n)$. Then, $\theta^x_\tau = P((\tau_1)^x_\tau, \ldots, (\tau_n)^x_\tau)$, which has complexity 0.
Suppose the property holds for $k \leq n$. If θ has complexity $n+1$, then

- If $\theta = \#\phi$, for $\# \in \{\neg, \exists, \forall\}$, then $comp(\phi) = n$. By the inductive hypothesis, $comp(\phi) = comp(\phi^x_\tau) = n$. Hence, $comp(\theta^x_\tau) = comp(\phi^x_\tau) + 1 = n + 1$.
- If $\theta = \phi\#\psi$, for $\# \in \{\vee, \wedge, \to\}$, then $comp(\phi) + comp(\psi) = n$, whence $comp(\phi), comp(\psi) \leq n$. By the inductive hypothesis,
  $comp(\phi) = comp(\phi^x_\tau)$ and $comp(\psi) = comp(\psi^x_\tau)$.
  Hence, $comp(\theta^x_\tau) = comp(\phi^x_\tau) + comp(\psi^x_\tau) + 1 = n + 1$.
- If $\theta = \circ\phi$, then $comp(\phi) = n - 1$. By the inductive hypothesis, $comp(\phi) = comp(\phi^x_\tau) = n - 1$. Hence, $comp(\theta^x_\tau) = comp(\phi^x_\tau) + 2 = n + 1$.

□

Finally, the introduction of really new concepts can begin. The first one will be that of quasi-isomorphism.

As the difference between classical and QmbC models lies in the level of valuation, rather than in the level of structures, the difference between isomorphism for classical and for QmbC models is likely to lie in the way they 'propagate' validity in the level of valuation. This consideration leads to a concept of isomorphism in QmbC that is to be set exactly in the same way as in the classical case. This concept, however, is to be enriched as the concept of structure gets enriched. The result is a kind of isomorphism that preserves validity for part of the formulae. Two isomorphic models in this sense are not structurally the same. A concept with such a behavior would hardly capture the spirit that mathematical tradition bestowed to the idea of isomorphism. For this reason, the new concept will be named *quasi-isomorphism* instead of isomorphism.
Here follows the concept:

Definition 2.1.3 (QmbC Quasi-Isomorphism). *Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle B, I_{\mathfrak{B}}\rangle$ be two QmbC-structures for the language L_Σ over the signature Σ. A* QmbC quasi-homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ *from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h : A \to B$ such that:*

1. *$(a_1, a_2, \ldots, a_n) \in A^{\mathfrak{A}}_{P1}$ implies $(h(a_1), h(a_2), \ldots, h(a_n)) \in B^{\mathfrak{B}}_{P1}$;*
2. *$h(f^{\mathfrak{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), h(a_2), \ldots, h(a_n))$ for every $f \in F_n$ and $(a_1, a_2, \ldots, a_n) \in A^n$;*
3. *$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for every $c \in C$.*

If h is a bijection and it holds 'iff' instead of 'implies' in (1), then h is a QmbC quasi-isomorphims from $\mathfrak{A}$ to $\mathfrak{B}$. If there exists a QmbC quasi-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ or a QmbC quasi-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are said to be quasi-isomorphic. This fact is denoted by $\mathfrak{A} \simeq \mathfrak{B}$ or $\mathfrak{B} \simeq \mathfrak{A}$.

In the pursuit of the goal of this section, which is to explore the possibilities of quasi-isomorphisms in preserving validity of sentences, a definition is in order:

Definition 2.1.4 (Preservation Kernel of a QmbC-structure). *Let $\mathfrak{A}$ be a QmbC-structure. The* preservation kernel *of $\mathfrak{A}$ is the set $Pk(\mathfrak{A}) = \{\theta | \forall \mathfrak{B}, (\mathfrak{A} \simeq \mathfrak{B}) \Rightarrow [(\mathfrak{A} \vDash \theta) \Leftrightarrow (\mathfrak{B} \vDash \theta)]$, that is, the set of all sentences from $\mathfrak{A}$ whose validity is preserved under whatever quasi-isomorphism. The preservation kernel of a family of quasi-isomorphic models is the preservation kernel of any of its models. This is well defined, for being quasi-isomorphic is an equivalence relation.*

A humble result can be stated here:

Proposition 2.1.5. *Let $\mathfrak{A}$ be a model and θ a sentence. If θ involves no connectives $\neg$ or $\circ$, then $\theta \in Pk(\mathfrak{A})$.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models and h a quasi-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
It must be proven that, for a given sentence θ involving no connectives $\neg$ and $\circ$, it happens that $\mathfrak{A} \vDash \theta$ iff $\mathfrak{B} \vDash \theta$. It will be proven the equivalent result that $\mathfrak{A}_A \vDash \theta$ iff $\mathfrak{B}_B \vDash \theta$. The equivalence of this result to the desired one follows from Lemma 2.1.1 and from the fact that the quasi-isomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$ is also a quasi-isomorphism from $\mathfrak{A}_A$ to $\mathfrak{B}_B$.
Firstly, it must be proven that if τ is a closed term, then $h(\tau^{\mathfrak{A}_A}) = \tau^{\mathfrak{B}_B}$. The proof is identical to that for classical models and will be omitted.
Having this fact as a lemma, the proof of the proposition can be performed by induction on the complexity of θ:
For atomic sentences:
$\mathfrak{A}_A \vDash P(\tau_1, \tau_2, \ldots, \tau_n)$ iff $(\tau_1^{\mathfrak{A}_A}, \tau_2^{\mathfrak{A}_A}, \ldots, \tau_n^{\mathfrak{A}_A}) \in P^{\mathfrak{A}_A}$ iff $(h(\tau_1^{\mathfrak{A}_A}), h(\tau_2^{\mathfrak{A}_A}), \ldots, h(\tau_n^{\mathfrak{A}_A})) \in$

$P^{\mathfrak{B}_B}$ iff $(\tau_1^{\mathfrak{B}_B}, \tau_2^{\mathfrak{B}_B}, \ldots, \tau_n^{\mathfrak{B}_B}) \in P^{\mathfrak{B}_B}$ iff $\mathfrak{B}_B \models P(\tau_1, \tau_2, \ldots, \tau_n)$.
Suppose the proposition is proven for sentences with complexity up to n and let θ be a sentence with complexity $n+1$.

- If $\theta = \phi \wedge \psi$, then $\mathfrak{A}_A \vDash \theta$ iff ($\mathfrak{A}_A \vDash \phi$ and $\mathfrak{A}_A \vDash \psi$) iff ($\mathfrak{B}_B \vDash \phi$ and $\mathfrak{B}_B \vDash \psi$) iff $\mathfrak{B}_B \vDash \theta$.

- If $\theta = \phi \vee \psi$, then $\mathfrak{A}_A \vDash \theta$ iff ($\mathfrak{A}_A \vDash \phi$ or $\mathfrak{A}_A \vDash \psi$) iff ($\mathfrak{B}_B \vDash \phi$ or $\mathfrak{B}_B \vDash \psi$) iff $\mathfrak{B} \vDash \theta$.

- If $\theta = \phi \to \psi$, then $\mathfrak{A}_A \vDash \theta$ iff ($\mathfrak{A}_A \nvDash \phi$ or $\mathfrak{A} \vDash \psi$) iff ($\mathfrak{B}_B \nvDash \phi$ or $\mathfrak{B}_B \vDash \psi$) iff $\mathfrak{B}_B \vDash \theta$.

- If $\theta = \exists x \phi(x)$, then $\mathfrak{A}_A \vDash \theta$ iff there is $a \in A$ such that $\mathfrak{A}_A \vDash \phi[a]$. But $\mathfrak{A}_A \vDash \phi[a]$ iff $\mathfrak{A}_A \vDash \phi(c_a)$ iff (by the inductive hypothesis)* $\mathfrak{B}_B \vDash \phi(c_a)$ iff $\mathfrak{B}_B \vDash \phi[c_a^{\mathfrak{B}_B}]$ iff $\mathfrak{B}_B \vDash \phi[h(a)]$. Therefore, there is $a \in A$ such that $\mathfrak{A}_A \vDash \phi[a]$ iff there is $h(a) \in B$ such that $\mathfrak{B}_B \vDash \phi[h(a)]$. Hence, $\mathfrak{A}_A \vDash \exists x \phi(x)$ iff $\mathfrak{B}_B \vDash \exists x \phi(x)$.

- If $\theta = \forall x \phi(x)$, then $\mathfrak{A}_A \vDash \theta$ iff, for all $a \in A$, $\mathfrak{A}_A \vDash \phi[a]$. As just proven above, $\mathfrak{A}_A \vDash \phi[a]$ iff $\mathfrak{B}_B \vDash \phi[h(a)]$. As $h : A \to B$ is a bijection, $\mathfrak{A}_A \vDash \phi[a]$ for all $a \in A$ iff $\mathfrak{B}_B \vDash \phi[b]$ for all $b \in B$. Hence, $\mathfrak{A}_A \vDash \forall x \phi(x)$ iff $\mathfrak{B}_B \vDash \forall x \phi(x)$.

*Note that Proposition 2.1.2 is being used at this point. □

In the search for some set of sentences to which some stronger result can be stated, it follows one more definition:

Definition 2.1.6 (Handable Sentence). *Given a model $\mathfrak{A}$ over the language L_Σ, a* handable sentence *in $\mathfrak{A}$ is a formula in L_Σ recursively formed by the following rules:*

- *If P is a predicate symbol of arity n and $\tau_1, \tau_2, \ldots, \tau_n$ are closed terms, then $P(\tau_1, \tau_2, \ldots, \tau_n)$ is a handable sentence.*

- *If ϕ and ψ are handable sentences, then $\phi \wedge \psi$ is a handable sentence.*

- *If ϕ and ψ are handable sentences, then $\phi \vee \psi$ is a handable sentence.*

- *If ϕ and ψ are handable sentences, then $\phi \to \psi$ is a handable sentence.*

- *If θ is a handable sentnece and $\mathfrak{A} \vDash \circ\theta$, then $\neg\theta$ is a handable sentence.*

- *If θ is a handable sentence, then $\circ\theta$ is a handable sentence.*

- *If, for every $a \in A$, $\theta(c_a)$ is a handable sentence, then $\exists x\theta(x)$ is a handable sentence.*
- *If, for every $a \in A$, $\theta(c_a)$ is a handable sentence, then $\forall x\theta(x)$ is a handable sentence.*
- *If, for some $a \in A$, $\theta(c_a)$ is a handable sentence and $\mathfrak{A} \vDash \theta(c_a)$, then $\exists x\theta$ is a handable sentence.*

Obs.: There is no problem in making recourse to the validity of formulae in $\mathfrak{A}$, for the set of formulae being defined is a subset of a previously defined set of formulae.

Now, a slightly stronger result can be stated for mbC-models.

Proposition 2.1.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two quasi-isomorphic models and θ a handable sentence (with respect to both $\mathfrak{A}$ and $\mathfrak{B}$) which involves no connective $\circ$. Then, $\mathfrak{A} \vDash \theta$ iff $\mathfrak{B} \vDash \theta$. Equivalently, given a model $\mathfrak{A}$, $\theta \in Pk(\mathfrak{A})$.*

Proof. The proof is made by induction on the complexity of the sentence and is identical to that of Proposition 2.1.5, just with an extra step of induction:

- If $\theta = \neg\phi$, then $\mathfrak{A}_A \vDash \neg\phi$ iff $\mathfrak{A}_A \nvDash \phi$ (for $\mathfrak{A} \vDash \circ\phi$, as θ is a handable sentence) iff $\mathfrak{B}_B \nvDash \phi$ (by the inductive hypothesis) iff $\mathfrak{B}_B \vDash \neg\phi$ (for $\mathfrak{B}_B \vDash \circ\phi$).

□

At this point, the possibilities for QmbC have been fully explored and some enrichment turns out to be necessary in order to gain some stronger result. A way of doing so is to assume the converse of **vCon**.

vConverseCon If $v(\alpha[\vec{a}]) = 0$ or $v(\neg\alpha[\vec{a}]) = 0$, then $v(\circ\alpha[\vec{a}]) = 1$.

An equivalent form of **vConverseCon** was studied in under the name **Ciw**.

Ciw $\circ\alpha \vee (\alpha \wedge \neg\alpha)$

Adding **Ciw** as a clause leads to the following definition:

Definition 2.1.8 (QmbCCiw-Valuation)**.** *A* QmbCCiw-valuation *v is a QmbC-valuation in which the extra clause below holds: If **Ciw** holds in a QmbC-valuation v restricted to a set of formulae K, then v is called a KQmbCCiw-valuation.*

Naturally, a QmbC-model based on an QmbCCiw-valuation will be called a *QmbCCiw-model*.

Fianally, it is possible to state a result for sentences involving also the connective $\circ$.

Proposition 2.1.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two quasi-isomorphic QmbCCiw-models and θ a handable in sentence both $\mathfrak{A}$ and $\mathfrak{B}$. Then, $\mathfrak{A} \vDash \theta$ iff $\mathfrak{B} \vDash \theta$.*

Proof. Again, by induction on the complexity of θ.
The proof is performed by the same steps of the proof of Proposition 2.1.5 with the additional step of the proof of Proposition 2.1.7 and yet with the additional step

- If $\theta = \circ\phi$ and $\mathfrak{A} \vDash \theta$, then $\mathfrak{A}_A \nvDash \phi$ or $\mathfrak{A}_A \nvDash \neg\phi$ (for **vCon**) and at most $\mathfrak{A}_A \nvDash \phi$ or $\mathfrak{A}_A \nvDash \neg\phi$ (for **vNeg**), which means that even $\mathfrak{A}_A \vDash \phi$ and $\mathfrak{A}_A \nvDash \neg\phi$ or $\mathfrak{A}_A \nvDash \phi$ and $\mathfrak{A}_A \vDash \neg\phi$.
  Hence, by the inductive hypothesis (remember that if $\circ\phi$ has complexity $n+1$, then ϕ has complexity $n-1$ and $\neg\phi$ has complexity n), even $\mathfrak{B}_B \vDash \phi$ and $\mathfrak{B}_B \nvDash \neg\phi$ or $\mathfrak{B}_B \nvDash \phi$ and $\mathfrak{B}_B \vDash \neg\phi$.
  In any case, $\mathfrak{B}_B \vDash \theta$, by **Ciw**.

□

This is an interesting result, for it does not exclude any connective and the set of handable sentences is not so restrictive.

As already mentioned, models endowed with **Ciw** have been actually studied in the literature and have turned out ot be a good solution for many purposes. In Chapter 4 of the present work, a situation in which **Ciw** appears naturally will be presented. However, the assumption of **Ciw** will not be always convenient and for the purposes that are being pursued here it will be not. The next lines show why.

There are three flaws that must be mended:

1. Sentences that should desirably be handable may not be so, for inconsistent sentences can derive from consistent ones.

2. The concept of handable sentence depends on the model under consideration, not on the notion of structure itself. This means that, in order to link the validity of a sentence in one model to its validy in another via quasi-isomorphism, it is necessary to make sure that it is handable in both models. So, little work has been done until now on behalf of the preservation kernel. As an illustration of the inconvenience of this fact, see that Proposition 2.1.9 does not allow one to conclude that if θ is a handable sentence for a model $\mathfrak{A}$, then $\theta \in Pk(\mathfrak{A})$.

3. There is a high philosophical price in assuming **Ciw**. Assuming **Ciw** is to assume that if an assertion has not been found to be inconsistent, then it is consistent. That may not be a problem in a dialeteistic approach, where inconsistency dwells in the world. But it is a great problem in a non-dialeteistic approach, where inconsistency is an epistemic phenomenon rather than an ontological one and knowledge is to be revised. In such a context, an inconsistent sentence is a one whose temporally epistemological inconsistency does not compromise the entire system of reasoning. Being so, assuming **Ciw** would be to assume that if a sentence has not been exposed to any contradiction so far, then it cannot be exposed to contradictions at all.

Regarding the first flaw, it would be desirable to find a set that should be closed under formation of more complex sentences by the use of connectives.

In this respect, not only does not **Ciw** solve the problem, but it also creates the odd situation where every false sentence is consistent. In fact, if $v(\phi) = 0$, then $v(\neg\phi) = 1$, by **vNeg**. By **Ciw**, $v(\circ\phi) = 1$.

A convenient solution may be to introduce clauses that shall be able to 'propagate' consistency. The eight clauses bellow will be called *Propagation Clauses*:

vPropOr If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $v((\circ(\alpha \vee \beta))[\vec{a}]) = 1$.

vPropAnd If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $v((\circ(\alpha \wedge \beta))[\vec{a}]) = 1$.

vPropImp If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $v((\circ(\alpha \to \beta))[\vec{a}]) = 1$.

vPropNeg If $v((\circ\alpha)[\vec{a}]) = 1$, then $v((\circ(\neg\alpha))[\vec{a}]) = 1$.

vPropCon For every α, $v((\circ(\circ\alpha))[\vec{a}]) = 1$ and $v((\circ(\neg \circ \alpha))[\vec{a}]) = 1$.

vPropUni If, for all $a \in A$, $v(\circ\alpha[a, \vec{a}]) = 1$, then $v(\circ(\forall x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

vPropEx If, for all $a \in A$, $v(\circ\alpha[a, \vec{a}]) = 1$, then $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

vPropEx' If, for some $a \in A$, $v(\alpha[a, \vec{a}]) = 1$ and $v(\circ\alpha[a, \vec{a}]) = 1$, then $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

With the clauses above, it is possible to delimit a less blur set.

Proposition 2.1.10. *Let v be a QmbC-valuation for which the Propagation Clauses hold and let K be a handable set of consistent formulae. The set K' of formulae that are recursively formed from K is a handable set. Moreover, if the formulae in K have complexity 0, then it is closed under subformulae and its formulae are consistent.*

Proof. The proof follows by induction on the complexity of formualae and is straightforward. □

In view of Proposition 2.1.10, assuming the Propagation Clauses seems to be an effective solution to the first flaw. In fact, let $K_{\mathfrak{A}}$ and $K_{\mathfrak{B}}$ be the sets of consistent sentences with complexity 0 in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $K_{\mathfrak{A},\mathfrak{B}} = K_{\mathfrak{A}} \cap K_{\mathfrak{B}}$ be the intersection of those sets and let $K'_{\mathfrak{A},\mathfrak{B}}$ be the set that is recursively generated by $K_{\mathfrak{A},\mathfrak{B}}$ in the canonical way with respect to the propagation clauses. That is, if α and β can be formed, then $\alpha \vee \beta$ can be formed; if $\alpha(x)$ is a formula and $\alpha(c_a)$ can be formed for every $a \in A$, then $\exists x\alpha(x)$ and $\forall x\alpha(x)$ can be formed (if necessary, define the sentences in $\mathfrak{A}_A$ and then take the restriction to the original signature); and so on.

Obviously, the sentences in $K_{\mathfrak{A},\mathfrak{B}}$ are consistent both in $\mathfrak{A}$ and $\mathfrak{B}$, and so are the sentences in the set $K'_{\mathfrak{A},\mathfrak{B}}$. Thus, a Ciw-valuation satisfying also the Propagation Clauses is able to preserve validity in $K'_{\mathfrak{A},\mathfrak{B}}$, which is a set that includes at least the main sentences that should be naturally expected to be preserved.

The focus now changes to the third flaw (there is a high price in assuming **Ciw**) and to the odd situation pointed out in the discussion that follows (all non-valid sentences would be consistent). The first flaw is mended. Nevertheless, for the reasons just exposed, it would be highly desirable a solution not involving **Ciw**, which is essencial in the induction step involving $\circ$.

The solution, however, is provided once again by the Propagation Clauses. The point is that, if a valuation v satisfies the Propagation Clauses, then it is true that, for every sentence $\theta \in K'_{\mathfrak{A},\mathfrak{B}}$, if $v(\theta) = 0$ or $v(\neg\theta) = 0$, then $v(\circ\theta) = 1$. In fact, it is true that $v(\circ\theta) = 1$ for every sentence $\theta \in K'_{\mathfrak{A},\mathfrak{B}}$. Summing up, **Ciw** can be discarded if the interest is driven to $K'_{\mathfrak{A},\mathfrak{B}}$.

The discussion above leads to the following definition:

Definition 2.1.11 (Propagating QmbC-Valuation)**.** *A* propagating QmbC-valuation *is a QmbC-valuation satisfying the Propagation Clauses.*
A model with a propagating QmbC-valuation is a propagating QmbC-model*, or just a propagating model.*

Proposition 2.1.12. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two quasi-isomorphic propagating models. Then, $\theta \in K'_{\mathfrak{A},\mathfrak{B}}$ implies $\mathfrak{A} \vDash \theta$ iff $\mathfrak{B} \vDash \theta$.*

The proof of the proposition above is the same as that of Proposition 2.1.9, with the difference that the induction step for $\circ$ is provided by the fact that the sentences in $K'_{\mathfrak{A},\mathfrak{B}}$ are consistent.

Proposition 2.1.12 provides a good way of disposing of **Ciw** and, consequently, getting rid of the third flaw and of the odd situation of non-valid sentences being consistent. The cost is to assume the Propagation Clauses. This assumption is quite acceptable, not to say natural, in a system designed for reasoning. In fact, the 'consistent portion' of a model shall behave classically. The 'syntactical matches' of **vPropOr**, **vPropAnd** and **vPropImp** (respectively, $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$, $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$ and $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$) actually hold in the logic C_1 of Da Costa. Clause **vPropNeg** would not be a problem, unless in an intuitionist context. For **vPropCon**, a discussion akin to the discussion in modal logic around axiom 4 ($\Box\alpha \to \Box\Box\alpha$) should take place. It is worth noticing that it appears in the literature in the logic mCi, described in [15]. Its assumption is harmless in the context aimed here, anyway.

There still remains the second flaw (The concept of handable sentence depends on the model under consideration). There is control of validity of basic sentences (with complexity 0) in the level of structure but there is only partial control of validity of complex sentences. For that reason, quasi-isomorphisms control the equivalence of basic sentences, but not the equivalence of complex sentences. So, the set of complex sentences that are valid for every model that is quasi-isomorphic to a given model is too small. In other words, very little can be said about the preservation kernel. In order to mend this flaw, it will be necessary to control somehow the validity of the consistency of complex sentences in the level of structure in the same fashion as this control occurs for basic sentences, at least to some extent. That can be done by enriching the concept.

Definition 2.1.13 ($\circ$-Structure)**.** *A* $\circ$-structure *over a signature Σ is a pair $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$, where A is a nonempty set, which is the domain of interpretation, and $I_{\mathfrak{A}}$ is a function, whose domain is Σ, which is the interpretation function, such that:*

- $I_{\mathfrak{A}}(c) \in A$
- $I_{\mathfrak{A}}(f) \in E_n$*, if* $f \in F_n$*, where*
    1. E_n *is the space of functions from* A^n *to* A*.*
- $I_{\mathfrak{A}}(P) = (A^{\mathfrak{A}}_{P1}, A^{\mathfrak{A}}_{P2})$*, if* $P \in P_n$*, where*
    1. $A^{\mathfrak{A}}_{P1} \subseteq A^n$
    2. $A^{\mathfrak{A}}_{P2} \subseteq A^n$

It is necessary to enrich the concept of valuation as well.

Definition 2.1.14 ($\circ$-Propagating QmbC-Valuation)**.** *A* $\circ$-propagating mbC-valuation *is a propagating mbC-valuation that satisfies*

vConPred $v(\circ P(t_1 \dots, t_n)) = 1$ *iff* $(t_1^{\mathfrak{A}}, \dots, t_n^{\mathfrak{A}}) \in A_{P2}^{\mathfrak{A}}$, *for* $P \in P_n$.

A model based on a $\circ$-structure and on a $\circ$-propagating QmbC-valuation is called a *$\circ$-propagating QmbC-model*, or just a *$\circ$-propagating model*.

As something has been added to the concept of structures, something alike must be added to the concept of quasi-(iso)homomorphism.

Definition 2.1.15 ($\circ$-Propagating Quasi-Isomorphism)**.** *Let* $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle B, I_{\mathfrak{B}} \rangle$ *be two $\circ$-structures for the language L_Σ over the signature Σ. A quasi-homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h : A \to B$ such that:*

1. $(a_1, a_2, \dots, a_n) \in A_{P1}^{\mathfrak{A}}$ *implies* $(h(a_1), h(a_2), \dots, h(a_n)) \in B_{P1}^{\mathfrak{B}}$*;*
2. $h(f^{\mathfrak{A}}(a_1, a_2, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), h(a_2), \dots, h(a_n))$ *for every* $f \in F_n$ *and* $(a_1, a_2, \dots, a_n) \in A^n$*;*
3. $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ *for every* $c \in C$.
4. $(a_1, a_2, \dots, a_n) \in A_{P2}^{\mathfrak{A}}$ *implies* $(h(a_1), h(a_2), \dots, h(a_n)) \in B_{P2}^{\mathfrak{B}}$*;*

If h is a bijection and it holds 'iff' instead of 'implies' in (1) and in (4), then h is a $\circ$-propagating quasi-isomorphism *from $\mathfrak{A}$ to $\mathfrak{B}$.*

Obs.: Again, it holds that if τ is a closed term, then $h(\tau^{\mathfrak{A}}) = \tau^{\mathfrak{B}}$ and the proof of this fact is identical to that for classical models, for it involves only clauses (2) and (3). The definition of $\circ$-propagating model has been designed so that the following result holds.

Proposition 2.1.16. *Let K' be the set of formulae recursively formed from a handable set K and let $\mathfrak{A}$ and $\mathfrak{B}$ be two quasi-isomorphic $\circ$-propagating models. Then,* $K'_{\mathfrak{A}} = K'_{\mathfrak{B}}$.

Proof. If θ has complexity 0, then $\theta = P(\tau_1, \tau_2, \dots, \tau_n)$ for some predicate function P and for some n-tuple of terms $(\tau_1, \tau_2, \dots, \tau_n)$. Therefore, $\theta \in K'_{\mathfrak{A}}$ iff $\mathfrak{A} \vDash \circ P(\tau_1, \tau_2, \dots, \tau_n)$ iff $(\tau_1^{\mathfrak{A}}, \tau_2^{\mathfrak{A}}, \dots, \tau_n^{\mathfrak{A}}) \in A_{P2}^{\mathfrak{A}}$ (by **vConPred**) iff $(h(\tau_1^{\mathfrak{A}}), h(\tau_2^{\mathfrak{A}}), \dots, h(\tau_n^{\mathfrak{A}})) \in A_{P2}^{\mathfrak{A}}$ (by (3) of Proposition 2.1.15) iff $(\tau_1^{\mathfrak{B}}, \tau_2^{\mathfrak{B}}, \dots, \tau_n^{\mathfrak{B}}) \in A_{P2}^{\mathfrak{B}}$ (by the observation just after the definition of quasi-isomorphism) iff $\mathfrak{B} \vDash \circ P(\tau_1, \tau_2, \dots, \tau_n)$ (by **vConPred**) iff $\theta \in K'_{\mathfrak{A}}$.
That is, $K_{\mathfrak{A}} = K_{\mathfrak{B}}$. As $K'_{\mathfrak{A}}$ is the set of sentences that are recursively formed from $K_{\mathfrak{A}}$ and $K'_{\mathfrak{B}}$ is the set of sentences that are recursively formed from $K_{\mathfrak{B}}$, it follows that $K'_{\mathfrak{A}} = K'_{\mathfrak{B}}$. □

Propositions 2.1.16 and 2.1.12 immediately yield

Proposition 2.1.17. *Let $\mathfrak{A}$ be a $\circ$-propagating model. Then,* $K'_{\mathfrak{A}} \subset Pk(\mathfrak{A})$.

Proposition 2.1.17 gives life to the concept of quasi-isomorphism, which is finally able to control equivalence in a discernible portion of the language.

The set $K'_{\mathfrak{A}}$ does not contain all the sentences in $Pk(\mathfrak{A})$.

As shown in Proposition 2.1.5, the set S of sentences that are free of $\neg$ and $\circ$ belong to $Pk(\mathfrak{A})$; Obviously, the sets T and F, respectively of sentences equivalent to $\top$ and $\bot$ in $\circ$-propagating models, are contained in $Pk(\mathfrak{A})$ too. So is the set $U = K'_{\mathfrak{A}} \cup S \cup T \cup F$.

Some results of partial closeness hold. If $D \subset Pk(\mathfrak{A})$, then $\theta \in Pk(\mathfrak{A})$ if one of the following conditions hold:

1. $\theta = \phi \wedge \psi$ and $\phi, \psi \in D$;
2. $\theta = \phi \wedge \psi$ and $\phi \in D$ and $v_{\mathfrak{A}}(\phi) = 0$. (likewise replacing ϕ by ψ);
3. $\theta = \phi \vee \psi$ and $\phi, \psi \in D$;
4. $\theta = \phi \vee \psi$ and $\phi \in D$ and $v_{\mathfrak{A}}(\phi) = 1$. (likewise replacing ϕ by ψ);
5. $\theta = \phi \to \psi$ and $\phi, \psi \in D$;
6. $\theta = \phi \to \psi$ and $\phi \in D$ and $v_{\mathfrak{A}}(\phi) = 0$;
7. $\theta = \phi \to \psi$ and $\psi \in D$ and $v_{\mathfrak{A}}(\psi) = 1$;
8. $\theta = \neg\phi$ and $\phi \in D$ and $v_{\mathfrak{A}}(\phi) = 0$;
9. $\theta = \neg\phi$ and $(\phi, \circ\phi \in D)$ and $v_{\mathfrak{A}}(\circ\phi) = 1$;

Let $\{U_n\}$ be the chain of sets recursively defined as follows:

- $U_0 = U$
- U_{n+1} is such that $\theta \in U_{n+1}$ iff $\theta \in U_n$ or θ satisfies some of the nine items above (where $D = U_n$).

It is straightforward to prove by induction that, for every n, $U_n \subset Pk(\mathfrak{A})$. Also, it is straightforward to prove that $U_\omega = \cup U_n \subset Pk(\mathfrak{A})$.

The set U_ω is not exhaustive yet. Completely delimiting $Pk(\mathfrak{A})$ is not a simple task at all. Nevertheless, it is possible to delimit, in the various contexts that may happen to present themselves, portions of $Pk(\mathfrak{A})$ that can be controlled and may be of great interest.

An analog search can be made for QmbCCiw-models or any other family of models.

The whole discussion in this section settles $\circ$-propagating QmbC-models as the starting point to the developments to be proposed in what follows.

2.2 Paraconsistent Reasoning Models

2.2.1 Construction

The point of view championed in this work is that of inconsistency as an epistemic phenomemon. In accordance with that line, this section will be dedicated for building paraconsistent models for reasoning. The consistent portion of models shall be the domain of safe knowledge and quasi-isomorphisms are expected to preserve validity within that domain.

In the previous section, the discussion on validity was made with the focus on sentences. For that, sometimes it was necessary to work with the extension $\mathfrak{A}_A$ instead of the model $\mathfrak{A}$ itself. This choice was made for the philosophical approach being adopted was that of regarding sentences as the cells that convey knowledge. This approach seems to have an epistemological flavor. The proposal for the rest of this chapter is to regard the domain of interpretation as the universe of objects in a given theory, which may be physical objects, for instance. Within this approach, predicate symbols predicate what can be stated about the objects of the universe and interpreted formulae are the cells that convey knowledge par excellence. In this sense, interpreted formulae are the statements that can be made in a model. For this reason, 'statement' will be used as a synomym for 'interpreted formula'. Both terms will be used along the text according to convenience. The term 'statement' will be preferred along the text and the term 'interpreted formula' will be preferred in the proofs. The terms 'basic (interpreted) formula' and 'basic statement' will denote a (interpreted) formula or a statement of complexity 0. Naturally, sentences are interpreted formulae as well.

Models will be regarded as states of knowledge and two different states shall agree with respect to safe knowledge. So, structures should control which statements are to be considered safe. Basic statements are thought of as the basic facts that can be stated about the world and it is not acceptable that every basic statement be controlled by the structure, unless the agent is omniscient. For this reason, it is convenient to adopt the clause **vPredPos** instead of **vPred** as described below:

vPredPos $v(P(t_1 \ldots, t_n)[\vec{a}]) = 1$ if $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, for $P \in P_n$

vPred $v(P(t_1 \ldots, t_n)[\vec{a}]) = 1$ iff $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, for $P \in P_n$.

At first sight, it seems mandatory that the consistency of statements be determined from the beginning. In fact, it is reasonable to suppose that the agent knows what statements can be exposed to a contradiction and what statements cannot. However, the next section will treat of refining a state of knowledge and two different models will be seen as quasi-isomorphic if one can be a refinement of the other. In other words, a statement that is determined from the beginning is to be regarded as a non-revisable statement. In this line, it is reasonable that the consistency of some statements shall be revisable. For this reason, it is convenient to adopt the clause **vConPredPart** instead of **vConPred** as described below:

vConPredPart $v(\circ P(t_1 \ldots, t_n)[\vec{a}]) = 1$ if $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P2}^{\mathfrak{A}}$, for $P \in P_n$.

vConPred $v(\circ P(t_1 \ldots, t_n)[\vec{a}]) = 1$ iff $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P2}^{\mathfrak{A}}$, for $P \in P_n$.

In QmbC or in $\circ$-propagation QmbC, **vPred** is able to delimit all basic statements with valuation 1 from the beginning and all basic statements with valuation 0 as well, for all statements are to have valuation 1 or 0 from the beginning. The assertion that a statement is delimited from the beginning is intended to mean that quasi-isomorphic models necessarily agree with respect to its valuation. Clause **vPredPos**, however, is only able to delimit the set of basic statements which are to have valuation 1 from the beginning. Thus, some clause is needed in order to delimit the set of basic statements with valuation 0. For that, an extra set A_{P3} will be added to the concept of structure together with a new clause **vPredNeg** in the valuation. In this new context, there are three kinds of basic statements: Those which have valuation 1 from the beginning, those which have valuation 0 from the beginning and those which have no valuation determined from the beginning.

As already advanced, models are states of knowledge, quasi-isomorphic models are compatible states of knowledge and statements that are preserved by quasi-isomorphism are non-revisable ones. If a statement is non-revisable, then it must be safe knowledge. So, in the search for an account of quasi-isomorphism in a reasoning context, it is a natural desideratum that if $\theta \in Pk(\mathfrak{A})$, then $\mathfrak{A} \vDash \circ\theta$ and $\circ\theta \in Pk(\mathfrak{A})$. For this desideratum to be fulfilled, it is necessary that $A_{P1} \cup A_{P3} \subset A_{P2}$. Obviously, $A_{P1} \cap A_{P3} = \emptyset$.

In a reasoning context, it makes sense to know that some statement implies some other even without knowing about the validity or about the consistency of them. In order to control this phenomenom, two sets A_{P4} and A_{P5} will be added to the structure together with the correspondent clause in the valuation rules. It also makes sense to know that at least one of two (or more) statement is true or that two (or more) staements cannot be true at the same time even without knowing anything about those statements. The same recourse will be used to deal with conjunction and disjunction. Separate sets and clauses will be needed in order to deal with conjunction and disjunction, for De Morgan does not hold here.

Once exposed the elements above, a kind of reasoning QmbC can be stated. Naturally, the first step is the definition of structure. Firstly, an enrichment of the concept of language is in order and also some notation is needed:

The sets A_{P4} to A_{P9} are designed with the purpose of linking predicate symbols. In order to render possible this linking, a way of referring to a predicate symbol without using its name is necessary, for a quasi-homomorphism is a function from one domain of interpretation to another. For that purpose, a codification of predicate symbols will be added to the concept of language. The next section will begin with the definition of En, which will serve as a codification of predicate symbols written in binary base.

2.2.2 Definition

A reasoning language Σ is a language endowed with a *codification function* $En : \bar{P} \to Bin$, where Bin is the set of tuples of any finite length whose entries are 1 or 0. Moreover, for every P, $En(P)$ ends in 111 and the sequence 111 appears nowhere else in $En(P)$. These requirements make it possible to identify where the representation of P ends and the sequence $\bar{a}$ begins in a sequence of the form $P \times A^n$. Without that, the notation would be ambiguous.
$En'(P, a, b)$ denotes the tuple obtained by the substituition of every occurrence of 1 in $En(P)$ by a and of every occurrence of 0 by b.

Yet some additional notation: $\overline{(P \times A)}_i(a, b)$ denotes the set of pairs of the form $En'(P, a, b) \times A^i$, where $P \in P_i$ and $\overline{P \times A}(a, b)$ denotes $\cup\overline{(P \times A)}_i(a, b)$.

Now, reasoning structures can be defined. In the definition, it is presupposed that A contains at least two elements.

Definition 2.2.1 (Reasoning Structure)**.** *A reasoning structure over a signature Σ is a pair $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$, where A is a nonempty set, which is the domain of interpretation, and $I_{\mathfrak{A}}$ is a function whose domain is Σ, which is the interpretation function, such that:*

- $I_{\mathfrak{A}}(c) \in A$, *if c is a constant.*
- $I_{\mathfrak{A}}(f) \in E_n$, *if $f \in F_n$, where*
  - E_n *is the space of functions from A^n to A.*
- $I_{\mathfrak{A}}(P) = (A^{\mathfrak{A}}_{P1}, A^{\mathfrak{A}}_{P2}, A^{\mathfrak{A}}_{P3}, A^{\mathfrak{A}}_{P4}, A^{\mathfrak{A}}_{P5}, A^{\mathfrak{A}}_{P6}, A^{\mathfrak{A}}_{P7}, A^{\mathfrak{A}}_{P8}, A^{\mathfrak{A}}_{P9}, a^{\mathfrak{A}}_1, a^{\mathfrak{A}}_2)$, *if $P \in P_n$, where*
  - $a^{\mathfrak{A}}_1, a^{\mathfrak{A}}_2 \in A$ *and* $a^{\mathfrak{A}}_1 \neq a^{\mathfrak{A}}_2$.
  - $A^{\mathfrak{A}}_{Pi} \subseteq A^n$, *for* $1 \leq i \leq 3$
  - $A^{\mathfrak{A}}_{Pj} \subseteq (\overline{P \times A})_n(a^{\mathfrak{A}}_1, a^{\mathfrak{A}}_2)$, *for* $4 \leq i \leq 9$
  - $a^{\mathfrak{A}}_1, a^{\mathfrak{A}}_2 \in A$ *and* $a^{\mathfrak{A}}_1 \neq a^{\mathfrak{A}}_2$.

As usual, the interpretation function $I_{\mathfrak{A}}$ induces an interpretation mapping $(\cdot)^{\mathfrak{A}} : CT_{\Sigma} \to A$ from the set CT_{Σ} of terms in Σ to the domain of interpretation. This map interprets each sentence recursively as follows:

- $c^{\mathfrak{A}} = I_{\mathfrak{A}}(c)$ if c is a constant.
- $f(\tau_1, \tau_2, \ldots, \tau_n)^{\mathfrak{A}} = I_{\mathfrak{A}}(f)(\tau_1^{\mathfrak{A}}, \tau_2^{\mathfrak{A}}, \ldots \tau_n^{\mathfrak{A}})$ if $f \in F_n$.

The next step is to define valuation in the new logic. It will be an extension of $\circ$-propagating valuations. In fact, propagating consistency is convenient for the reasons already discussed. This time, something else will be assured, that is, that every sentence of the kind $\circ\phi$ or of the kind $\neg \circ \phi$ be consistent. Also, it will be assured that every statement which has valuation 1 or 0 from the beginning be consistent. This requirement is fair in a context where knowledge stated from the beginning is safe knowledge, for safe knowledge shall be consistent. It is not a novel clause in the literature, for it holds in mCi (see again [15]).

Although some of the clauses were presented in the definition of mbC-valuation and in the discussion along the text, it is worth to present all of them in an organized way. For a better comprehension, they will be separated in four groups: The one of basic control, which comprises the clauses that state valuation for some basic statements and for some statements of complexity 1 'from the beginning'; The one of classical behavior, which comprises the clauses that control connectives that behave classically; The one of QmbC control, which comprises the clauses that rule paraconsistent behavior in QmbC; and the one of propagation control, which comprises the clauses that control propagation of consistency.

In the definition that follows $(P, t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$ will be used as a shorthand for $(En'(P, a_1^{\mathfrak{A}}, a_2^{\mathfrak{A}}), t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$.

Definition 2.2.2 (Reasoning Valuation)**.** *Let $\mathfrak{A}$ be a structure over the signature Σ with domain A and let $IF(\mathfrak{A})$ be the set of interpreted formulae over $\mathfrak{A}$. A mapping $v : IF(\mathfrak{A}) \to \{0, 1\}$ is a* reasoning valuation *over $\mathfrak{A}$ if it satisfies the following clauses:*

Basic Control Clauses:

vPredPos $v(P(t_1, \ldots, t_n)[\vec{a}]) = 1$ *if* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$*, for* $P \in P_n$*.*

vPredNeg $v(P(t_1, \ldots, t_n)[\vec{a}]) = 0$ *if* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P3}^{\mathfrak{A}}$*, for* $P \in P_n$*.*

vConPredPart $v((\circ(P(t_1, \ldots, t_n)))[\vec{a}]) = 1$ *if* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P2}^{\mathfrak{A}}$*, for* $P \in P_n$*.*

vConImpPred $v((P(t_1 \dots, t_n)) \to P'(t_1, \dots, t_m))[\vec{a}]) = 1$ *and* $v((\circ(P(t_1, \dots, t_n) \to P'(t_1, \dots, t_m)))[\vec{a}]) = 1$ *if* $(P', t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P4}^{\mathfrak{A}}$ *and* $(P, t_1^{\mathfrak{A}}[\vec{a}], \dots, t_m^{\mathfrak{A}}[\vec{a}]) \in A_{P'5}^{\mathfrak{A}}$, *for* $P \in P_n$ *and* $P' \in P_m$.

vConOrPred $v((P(t_1, \dots, t_m) \vee P'(t_1, \dots, t_n))[\vec{a}]) = 1$ *and* $v((\circ(P(t_1, \dots, t_m) \vee P'(t_1, \dots, t_n)))[\vec{a}]) = 1$ *if* $(P', t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P6}^{\mathfrak{A}}$ *and* $(P, t_1^{\mathfrak{A}}[\vec{a}], \dots, t_m^{\mathfrak{A}}[\vec{a}]) \in A_{P'7}^{\mathfrak{A}}$, *for* $P \in P_n$ *and* $P' \in P_m$.

vConAndPred $v((P(t_1, \dots, t_m) \wedge P'(t_1, \dots, t_n))[\vec{a}]) = 0$ *and* $v((\circ(P(t_1, \dots, t_m) \wedge P'(t_1, \dots, t_n)))[\vec{a}]) = 1$ *if* $(P', t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P8}^{\mathfrak{A}}$ *and* $(P, t_1^{\mathfrak{A}}[\vec{a}], \dots, t_m^{\mathfrak{A}}[\vec{a}]) \in A_{P'9}^{\mathfrak{A}}$, *for* $P \in P_n$ *and* $P' \in P_m$.

Classical Behavior Clauses:

vOr $v((\alpha \vee \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *or* $v(\beta[\vec{a}]) = 1$.

vAnd $v((\alpha \wedge \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 1$ *and* $v(\beta[\vec{a}]) = 1$.

vImp $v((\alpha \to \beta)[\vec{a}]) = 1$ *iff* $v(\alpha[\vec{a}]) = 0$ *or* $v(\beta[\vec{a}]) = 1$.

vEx $v((\exists x \alpha(x))[\vec{a}]) = 1$ *iff* $v(\alpha[a, \vec{a}]) = 1$ *for some* $a \in A$.

vUni $v((\forall x \alpha(x))[\vec{a}]) = 1$ *iff* $v(\alpha[a, \vec{a}]) = 1$ *for every* $a \in A$.

QmbC Clauses

vNeg *If* $v(\alpha[\vec{a}]) = 0$, *then* $v((\neg\alpha)[\vec{a}]) = 1$.

vCon *If* $v((\circ\alpha)[\vec{a}]) = 1$, *then* $v(\alpha[\vec{a}]) = 0$ *or* $v((\neg\alpha)[\vec{a}]) = 0$.

vVar $v(\alpha[\vec{a}]) = v(\beta[\vec{a}])$ *whenever* $\alpha[\vec{a}]$ *is a variant of* $\beta[\vec{a}]$.

sNeg *For every context* $(\vec{x}, z)$ *and* $(\vec{x}, \vec{y})$, *for every sequence* $(\vec{a}, \vec{b})$ *in* A *interpreting* $(\vec{x}, \vec{y})$, *for every* $\phi \in L(\mathfrak{A})_{\vec{x},z}$ *and for every* $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ *such that* t *is free for* z *in* ϕ, *if* $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ *and* $b = (t[\vec{x}, \vec{y}/\vec{a}, \vec{b}])^{\widehat{\mathfrak{A}}}$, *then* $v((\phi[z/t])[\vec{x}, \vec{y}/\vec{a}, \vec{b}]) = v(\phi[\vec{x}, z/\vec{a}, b])$ *implies* $v((\neg\phi[z/t][\vec{x}, \vec{y}/\vec{a}, \vec{b}])) = v(\neg\phi[\vec{x}, z/\vec{a}, b])$.

sCon *For every context* $(\vec{x}, z)$ *and* $(\vec{x}, \vec{y})$, *for every sequence* $(\vec{a}, \vec{b})$ *in* A *interpreting* $(\vec{x}, \vec{y})$, *for every* $\phi \in L(\mathfrak{A})_{\vec{x},z}$ *and for every* $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ *such that* t *is free for* z *in* ϕ, *if* $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ *and* $b = (t[\vec{x}, \vec{y}/\vec{a}, \vec{b}])^{\widehat{\mathfrak{A}}}$, *then* $v((\phi[z/t])[\vec{x}, \vec{y}/\vec{a}, \vec{b}]) = v(\phi[\vec{x}, z/\vec{a}, b])$ *implies* $v((\circ\phi[z/t][\vec{x}, \vec{y}/\vec{a}, \vec{b}])) = v(\circ\phi[\vec{x}, z/\vec{a}, b])$.

Propagation Clauses

vPropOr *If* $v((\circ\alpha)[\vec{a}]) = 1$ *and* $v((\circ\beta)[\vec{a}]) = 1$, *then* $v((\circ(\alpha \vee \beta))[\vec{a}]) = 1$.

vPropAnd *If* $v((\circ\alpha)[\vec{a}]) = 1$ *and* $v((\circ\beta)[\vec{a}]) = 1$*, then* $v((\circ(\alpha \wedge \beta))[\vec{a}]) = 1$.

vPropImp *If* $v((\circ\alpha)[\vec{a}]) = 1$ *and* $v((\circ\beta)[\vec{a}]) = 1$*, then* $v((\circ(\alpha \to \beta))[\vec{a}]) = 1$.

vPropNeg *If* $v((\circ\alpha)[\vec{a}]) = 1$*, then* $v((\circ(\neg\alpha))[\vec{a}]) = 1$.

vPropCon *For every* α*,* $v((\circ(\circ\alpha))[\vec{a}]) = 1$ *and* $v((\circ(\neg(\circ\alpha)))[\vec{a}]) = 1$.

vPropUni *If, for all* $a \in A$*,* $v(\circ\alpha[a, \vec{a}]) = 1$*, then* $v(\circ(\forall x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

vPropEx *If, for all* $a \in A$*,* $v(\circ\alpha[a, \vec{a}]) = 1$*, then* $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

vPropEx' *If, for some* $a \in A$*,* $v(\alpha[a, \vec{a}]) = 1$ *and* $v(\circ\alpha[a, \vec{a}]) = 1$*, then* $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

Definition 2.2.3. *Let* $\mathfrak{A} =< A, I_{\mathfrak{A}} >$ *be a reasoning structure. Then,* $\mathfrak{A}$ *satisfies a sentence* ϕ *if* $v(\phi) = 1$ *for every reasoning valuation* v *over* $\mathfrak{A}$*.*

Not every reasoning structure is compatible with a reasoning valuation, for the basic control clauses determine validity of some statements, that is, they determine part of the valuation function and this partial valuation function may fail to fulfill some of the classical behavior or QmbC clauses. So, in order to define a reasoning model, it is necessary to start from a compatible reasoning structure, that is, a reasoning structure whose partially determined valuation fulfills the classical behavior and the QmbC-valuation clauses. The logical system defined from the definitions of reasoning structures, reasoning valuations and reasoning models just given will be called *Paraconsistent Reasoning System* and will also be referred as PRS.

The concept of quasi-isomorphism will be the same as usual with the appropriate clauses added for matching the extra apparatus in the structure.

Definition 2.2.4 (Reasoning Quasi-Isomorphism)**.** *Let* $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle B, I_{\mathfrak{B}} \rangle$ *be two reasoning structures for the language* L_{Σ} *over the signature* Σ*. A reasoning quasi-homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ *from* $\mathfrak{A}$ *to* $\mathfrak{B}$ *is a function* $h : A \to B$ *such that:*

Clause i $(a_1, a_2, \ldots, a_n) \in A^{\mathfrak{A}}_{Pi}$ *implies* $(h(a_1), h(a_2), \ldots, h(a_n)) \in B^{\mathfrak{B}}_{Pi}$*; (for each* $i, 1 \leq i \leq 3$*)*

Clause j $(P, a_1, \ldots, a_n) \in A^{\mathfrak{A}}_{Pj}$ *implies* $(h(P), h(a_1), \ldots, h(a_n)) \in B^{\mathfrak{B}}_{Pj}$*; (for each* $j, 4 \leq j \leq 9$*)*

Clause 10 $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ *for every* $c \in C$*;*

Clause 11 $h(a^{\mathfrak{A}}_1) = b^{\mathfrak{B}}_1$ *and* $h(a^{\mathfrak{A}}_2) = b^{\mathfrak{A}}_2$*;*

Clause 12 $h(f^{\mathfrak{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), h(a_n), \ldots, h(a_n))$ *for every* $(a_1, a_2, \ldots, a_n) \in A^n$ *and* $f \in F_n$.

If h is a bijection and it holds 'iff' instead of 'implies' in clauses 1 to 9, then h is a reasoning quasi-isomorphim *from* $\mathfrak{A}$ *to* $\mathfrak{B}$.

If there is a reasoning quasi-isomorphis from $\mathfrak{A}$ *to* $\mathfrak{B}$ *or from* $\mathfrak{B}$ *to* $\mathfrak{A}$, *then* $\mathfrak{A}$ *and* $\mathfrak{A}$ *are said to be quasi-isomorphic. This fact is denoted by* $\mathfrak{A} \approxeq \mathfrak{B}$ *or* $\mathfrak{A} \approxeq \mathfrak{B}$.

The same definition of preservation kernel given to QmbC-structures can be given for reasoning models.

Definition 2.2.5 (Preservation Kernel of a Reasoning Model). *Let* $\mathfrak{A}$ *be a QmbC-structure. The* preservation kernel *of* $\mathfrak{A}$ *is the set* $Pk(\mathfrak{A}) = \{\theta | \forall \mathfrak{B}, (\mathfrak{A} \approxeq \mathfrak{B}) \Rightarrow [(\mathfrak{A} \vDash \theta) \Leftrightarrow (\mathfrak{B} \vDash \theta)]\}$.

Once defined reasoning quasi-(iso)homomorphism, the next step is to delimit, at least partially, the preservation kernel of a model, that is, the set of statements that it preserves. Unlike in the case of $\circ$-quasi-isomorphism, not every basic statement of a model belongs to its preservation kernel. This is not indeed a handicap. Rather, it is an advantage, for this is the feature that allows knowledge revision.

The only clauses that rule interpretation of terms are still the classical ones, that is, (10) and (12). For this reason, there remains valid that $h(\tau^{\mathfrak{A}}) = \tau^{\mathfrak{B}}$. Clause (11) guarantees that h matches the representation of a propositional function P in $\mathfrak{A}$ to its representation in $\mathfrak{B}$. Clauses (1) and (3) guarantee that if $(\tau_1^{\mathfrak{A}}, \tau_2^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}})$ belongs to $A_{P1}^{\mathfrak{A}}$ or to $A_{P3}^{\mathfrak{A}}$, then $P(\tau_1, \tau_2, \ldots, \tau_n)$ belongs to $Pk(\mathfrak{A})$ and so does $\circ P(\tau_1, \tau_2, \ldots, \tau_n)$, by clause (2), for $A_{P1}^{\mathfrak{A}} \cup A_{P3}^{\mathfrak{A}} \subset A_{P2}^{\mathfrak{A}}$. This all means that there is a set of basic interpreted formulae that belong to the preservation kernel together with their consistency. Likewise, clauses (4) and (5) determine a set of interpreted formuale of the form $\phi \rightarrow \psi$, where ϕ and ψ are basic statements, that is contained in the preservation kernel as well. Clauses (6) and (7) determine statements of the form $\phi \vee \psi$ and clauses (8) and (9), statements of the form $\phi \wedge \psi$.

Moreover, the propagation clauses of reasoning valuation guarantee that the set of statements derived from this basic set are consistent and, in the same way as in the case of $\circ$-models, this set is contained in the preservation kernel.

Summing up, the set of statements that are expected to constitute 'safe knowledge' is preserved by quasi-isomorphisms.

2.3 Refinement

The very purpose of a paraconsistent environment for reasoning is the possibility of dealing with unsafe knowledge while seeking new safe knowledge. This is where the convenience of regarding models as states lies. Being states of knowledge, models must be refinable. Following such a track, this section looks for possible ways of obtaining new and reasonable knowledge from what is already known.

There are two ways of gaining knowledge: The first one is from the world by some external source, like an experiment or whatever. The second one is from inside, that is, from what is already available.

There are two ways of gaining knowledge from inside: The first one is by deduction. The second one is by reasonability criteria, such as the existence of 'so much evidence' in favor of some fact.

The focus now will be in gaining knowledge from inside by reasonability criteria. Regarding a model as a state, a refinement of it is expected to be a new model which keeps what is safe in the original one and gains something else. The next definition makes this idea precise.

Definition 2.3.1. *Let L_Σ be a language over a signature Σ, A a set, M_A the set of reasoning models over L_Σ with domain A and $S(L_\Sigma)$ the power set of the set of interpreted formulae in L_Σ. A function $Ref : S(L_\Sigma) \times M_A \to M_A$ is a* refinement *in M_A iff, for every reasoning model $\mathfrak{A}$ and for every set $\Theta \in S(L_\Sigma)$,*

1. *$Ref(\Theta, \mathfrak{A}) \approxeq \mathfrak{A}$;*
2. *For every $\theta \in \Theta$, $v_{Ref(\Theta,\mathfrak{A})}(\theta) = 1$ iff $v_\mathfrak{A}(\theta) = 1$;*
3. *If $v_{Ref(\Theta,\mathfrak{A})}(\phi) \neq v_\mathfrak{A}(\phi)$, then $v_{Ref(\Theta,\mathfrak{A})}(\circ\phi) = 1$.*

As refinements are intended to gain new knowledge in a given state, it is natural that the domain of interpretation be fixed. This requirement means that the refinement of a model 'talks' about the same objects. It also guarantees that M_A, as stated in the definition above, be indeed a set.

Requirement (3) enforces the desirable property that knowledge gained through refinement be consistent.

As $Ref(\Theta, \mathfrak{A})$ is quasi-isomorphic to $\mathfrak{A}$, validity is preserved in the set $S_1^\mathfrak{A} = Pk(\mathfrak{A}) \cup \Theta$. This is the set of statements intended to be safe in the model to be refined. Two other sets are to be considered: The first one is $S_2^\mathfrak{A} = \{\phi | v_{Ref(\Theta,\mathfrak{A})}(\phi) \neq v_\mathfrak{A}(\phi)\}$, that is, the set of statements whose validity has been altered through refinement, which is to be the set of statements that are to be considered safe in the refinement. The second one is $S_3^\mathfrak{A} = S(L_\Sigma) \backslash (S_1^\mathfrak{A} \cup S_2^\mathfrak{A})$, which is intended to be the set of still unsafe interpreted formulae. The three sets are disjoint and cover $S(L_\Sigma)$.

As the refinement of a model has extra safe statements, it makes sense to reiterate the process in order to obtain new safe statements.

Definition 2.3.2 (n-Iterated Refinement)**.** *The n-iterated refinement $Ref_n(\mathfrak{A})$ of a model $\mathfrak{A}$ is defined recursively. It is necessary to define a set Θ_n simultaneously:*

- $\Theta_1 = Pk(\mathfrak{A})$
- $Ref_1(\mathfrak{A}) = Ref(\Theta_1, \mathfrak{A})$
- $\Theta_{n+1} = \Theta_n \cup S_2^{Ref_n(\mathfrak{A})}$
- $Ref_{n+1}(\mathfrak{A}) = Ref(\Theta_{n+1}, Ref_n(\mathfrak{A}))$

If, for some i, $\Theta_{i+1} = \Theta_i$, then $Ref_{i+1}(\mathfrak{A}) = Ref_i(\mathfrak{A})$. In this case, there exists a fixed point, which can be stated as a maximum refinement, and Θ_i would be the higher amount of safe knowledge that can be gathered. However, it may happen that there is no fixed point. In this case, the natural candidate to higher safe set is $\Theta_\omega = \cup_{i=1}^{\infty}\Theta_i$. Nevertheless, the process can be restarted taking $\Theta_1 = \Theta_\omega$ and Θ_α can be defined for higher ordinals. As languages are denumerable, the process ends until Θ_{ω_1}. A limit to the chain of refinements can eventually be stated for convenience. So, the maximum refinement of a model $\mathfrak{A}$, denoted by $MRef(\mathfrak{A})$, can be defined as $Ref(\Theta_{\omega_1}, \mathfrak{A})$ or as the refinement of $\mathfrak{A}$ up to some established limit.

Provided that Ref fulfills an extra condition, the assertion that the process ends until Θ_{ω_1} can be strenthened. Some definitions and results regarding fixpoints will be needed in the next few lines. For the reader that is not familiar with the theory of fixpoints, a brief presentation of it is located in Section 1.5. A detailed presentation can be found in [32].

Taking the power set of $S(L_\Sigma)$ as the 'Universe' U and inclusion ($\subseteq$) as the equivalence relation, it is straightforward that U is a complete lattice, where, for each subset X of U, $lub(X) = \{\cup C | C \in X\}$ and $glb(X) = \{\cap C | C \in X\}$. Moreover, $\top = S(L_\Sigma)$ and $\bot = \emptyset$.

Let Ref be a refinement and $\mathfrak{A}$ a fixed model. Consider the function $Ref_{\mathfrak{A}}(\Theta) = Ref(\Theta, \mathfrak{A})$. By the design of Ref, $Ref_{\mathfrak{A}}$ is a monotonic function from U to U, which may be continuous or not.

If Ref is such that $Ref_{\mathfrak{A}}$ is continuous, then Θ_ω, as described above, is a fixpoint, which that the process iterating the refinement ends until Θ_ω.

The idea of refining models can be joint to the idea of creating the family of quasi-isomorphic models to a given model to set a strong criterion to delimit safe knowledge. The proposal is to look at the set of quasi-isomorphic models (preserving domain of interpretation) and 'ask' each one about each interpreted formula. Such a criterion leads naturally to a modal approach. The traditional modal connectives can be defined. As they are supposed to look not at a single model, but at a set of models, they must be, actually, metaconnectives.

Definition 2.3.3 (Modal Metaconnectives)**.** *Given a statement* θ *and a model* $\mathfrak{A}$*, then*

- *The metastatement* $\diamond\theta$ *is valid in* $\mathfrak{A}$ *iff there exists a model* $\mathfrak{B}$ *such that* $\mathfrak{B} \approxeq \mathfrak{A}$ *and* $\theta \in MRef(\mathfrak{B})$*. In this case,* θ *is said to be* possible *in* $\mathfrak{A}$ *and the notation* $\mathfrak{A}\overline{\models} \diamond\theta$ *is used to designate this fact.*
- *The metastatement* $\Box\theta$ *is valid in* $\mathfrak{A}$ *iff for every model* $\mathfrak{B} \approxeq \mathfrak{A}$*,* $\theta \in MRef(\mathfrak{B})$*. In this case,* θ *is said to be* necessary *in* $\mathfrak{A}$ *and the notation* $\mathfrak{A}\overline{\models}\Box\theta$ *is used to designate this fact.*

The choice for using the symbol $\overline{\models}$ *instead of* $\models$ *was made for the obvious reason that* $\diamond$ *and* $\Box$ *are metaconnectives.*

Finally, the core concept for this section has been reached. Safe knowledge is to be identified with necessary knowledge. By design, every statement from the preservation kernel is necessary. Moreover, as being quasi-isomorphic is an equivalence relation, two quasi-isomorphic models $\mathfrak{A}$ and $\mathfrak{B}$ have the same quasi-isomorphic models. Consequently, for every statement θ, $\mathfrak{A}\overline{\models} \diamond\theta$ iff $\mathfrak{B}\overline{\models} \diamond\theta$ and $\mathfrak{A}\overline{\models}\Box\theta$ iff $\mathfrak{B}\overline{\models}\Box\theta$.

The definition of the set of necessary statements rises naturally. It will be called the *Necessitation Kernel*. It is clear from the discussion above that the preservation kernel of a model is contained in its necessitation kernel and that two quasi-isomorphic models have the same necessitation kernel.

Definition 2.3.4 (Necessitation Kernel)**.** *The* necessitation kernel *of a model* $\mathfrak{A}$ *is the set* $NKer(\mathfrak{A}) = \{\theta | \mathfrak{A}\overline{\models}\Box\theta\}$*.*

In the beginning of this section, it was said that there were two ways of gaining knowledge: The first one from outside and the seconde one from inside. The whole development after that focused on gaining knowledge from inside. This is time to ask about gaining knowledge from outside.

In the beginning of last section, it was suggested that basic statements should be the basic facts that can be stated about the world. In such a reasoning context, it makes sense to know with security about some basic facts and about some conjunctions, disjunctions and implications involving basic statements, that is, it makes sense to know that at least one of two facts must occur or that some basic fact forces some other basic fact, and so on. Summing up, it makes sense to know with security about the facts that are controlled by quasi-isomorphisms and about what can be inferred from those facts. In this line, the knowledge that can be gained from outside by observation, experimentation or whatever method is of the same kind. Thinking of models as states of knowledge, the discussion above leads to the conclusion that an enrichment of a state from outside is just a model that is able to control a higher amount of statements in the level of structure.

In other words, $\mathfrak{B}$ is a refinement of $\mathfrak{A}$ from outside iff $A_i^{\mathfrak{A}} \subset B_i^{\mathfrak{B}}$ for $1 \leq i \leq 9$. So, according to the definition of quasi-homomorphism, $\mathfrak{B}$ is a refinement of $\mathfrak{A}$ from outside iff there is a quasi-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. It is obvious that $Pk(\mathfrak{A}) \subset Pk(\mathfrak{B})$ and this implies $NKer(\mathfrak{A}) \subset NKer(\mathfrak{B})$, that is, the refinement from outside possesses a higher amount of safe knowledge.

Regarding a theory about whatever reality as a category, objects are states of knowledge (identified with models), morphisms are refinements of states of knowledge from outside (identified with quasi-homomorphisms) and isomorphisms (identified with quasi-isomorphisms) correlate states that agree about safe knowledge.

The imposition that refinements preserve domain of interpretation was made for philosophical reasons rather than for technical necessity. In fact, it makes sense that the search for knowledge be performed within a fixed universe of discourse, unless the objects under consideration are changed. Anyway, the same conclusions would be reached without this imposition under consideration and the category considered above needs not be restricted to models with a fixed domain.

Closing this section, two suggestions of how a refinement function may work will be given. They will be based on the classical ways to obtain knowledge in science: induction and abduction.

Both induction and abduction will be functions preserving both the domain of interpretation and the function I. In other words, Ref will be so that

1. $Ref(\mathfrak{A}, \Theta)$ has the same domain of interpretation as $\mathfrak{A}$;
2. $I_{Ref(\mathfrak{A},\Theta)}(c) = I_{\mathfrak{A}}(c)$, for every constant c;
3. $I_{Ref(\mathfrak{A},\Theta)}(f) = I_{\mathfrak{A}}(f)$, for every $f \in \bar{F}$;
4. $I_{Ref(\mathfrak{A},\Theta)}(P) = I_{\mathfrak{A}}(P)$, for every $P \in \bar{P}$, which means that $A_{Pi}^{Ref(\mathfrak{A},\Theta)} = A_{Pi}^{\mathfrak{A}}$, for $1 \leq i \leq 9$;
5. $a_1^{Ref(\mathfrak{A},\Theta)} = a_1^{\mathfrak{A}}$ and $a_2^{Ref(\mathfrak{A},\Theta)} = a_2^{\mathfrak{A}}$.

Induction is a process by which a fact is assumed to be true if it could not be refused so far and, besides that, it can be verified a sufficiently large number of times or in a sufficiently large number of situations.

Abduction is a process in which a fact is assumed to be true if it could not be refused so far and, besides that, there is a sufficiently large amount of evidence in favor of it, that is, it implies a large number of facts that happen to verify.

Basic facts are expressed by predicate functions. That is, $P(\tau_1, \ldots, \tau_n)$ means that the fact P is verified by the n-tuple of objects $(\tau_1, \ldots, \tau_n)$. Said this, induction and abduction are methods that look at basic statements.

Following this track, an auxiliar function will be needed in order to delimit the universe of objects to which each predicate function refers and a 'parameter of reliability'. The first coordinate function will delimit the universe of objects and the second coordinate function will determine the parameter of reliability.

Definition 2.3.5 (Frame Function)**.** *A* frame function *for a signature* Σ *is a function* $Fr : \bar{P} \to (S(L_\Sigma) \times \mathbb{N})$ *(*$S(L_\Sigma)$ *as in Definition 2.3.1) whose first coordinate function* Fr_1 *each* P*,* $Fr_1(P)$ *is a set of statements of the form* $P(\tau_1, \ldots, \tau_n)$*. The second coordinate function is designated by* Fr_2*.*

A predicate symbol P predicts something about sequences of objects $\vec{a}$. The first component Fr_1 delimits the set of sequences of objects to which that P is intended to hold after refinement. The idea is that $Fr_1(P)$ is the set of statements that predict P to a set of objects to which that predication makes sense and that have some similarity, so that such a predication makes sense to that set as a whole. The second component Fr_2 is intended to set a parameter. That is, the set of predications $Fr_1(P)$ is to be considered valid as a whole when some criterion is fulfilled. Such a criterion shall depend on a parameter that depends on its turn on P. Such a parameter is determined by Fr_2.

Refinements must gain new knowledge without contradicting already existent safe knowledge. For this reason, it is necessary to guarantee that the revision of $Fr_1(P)$ raises no contradiction against safe knowledge. So, it is necessary to define revisable predicate symbols.

Definition 2.3.6 (Revisable Predicate Symbol)**.** *A predicate* P *is revisable iff all the following requirements are fulfilled:*

1. *There is no* n*-tuple* $(\tau_1, \ldots, \tau_n)$ *such that* $P(\tau_1, \ldots, \tau_n) \in Fr_1(P)$ *and* $v(P(\tau_1, \ldots, \tau_n)) = 0$ *and* $v(\circ P(\tau_1, \ldots, \tau_n)) = 1$*;*

2. *There are not some formula* θ *and some* n*-tuple* $(\tau_1, \ldots, \tau_n)$ *such that* $P(\tau_1, \ldots, \tau_n) \in Fr_1(P)$ *and* $v(\theta) = 1$ *and* $v(\circ\theta) = 1$ *and* $v(P(\tau_1, \ldots, \tau_n) \wedge \theta) = 0$ *and* $v(\circ(P(\tau_1, \ldots, \tau_n) \wedge \theta)) = 1$*;*

3. *There are not some formula* θ *and some* n*-tuple* $(\tau_1, \ldots, \tau_n)$ *such that* $P(\tau_1, \ldots, \tau_n) \in Fr_1(P)$ *and* $v(\theta) = 1$ *and* $v(\circ\theta) = 1$ *and* $v((P(\tau_1, \ldots, \tau_n) \to \neg\theta) = 1$ *and* $v(\circ(P(\tau_1, \ldots, \tau_n) \to \neg\theta)) = 1$*.*

The concept of revisable predicate symbol is designed so that revising the set of basic sentences based on a given predicate symbol does not raise any conflict against any consistent statement. The next step is to enumerate predicate symbols and revise basic sentences from some criterion. So, given an enumeration of predicate symbols and a criterion for revision, a chain of revised valuations will be defined. It is necessary to define such a chain because the revision of a statement may turn a revisable predicate into a non revisable one. It is also important to observe that, despite revising basic statements does not raise contradiction against consistent statements, it may raise contradiction against non-consistent ones. So, the valuations in the chain may not be reasoning valuations. Said all that, the chain of revised valuations is recursively defined:

The first valuation in the chain is $v_0 = v$. For $i > 0$, v_i is defined in the following manner:

If θ is a statement with complexity greater than 0, then, for every i, $v_i(\theta) = v(\theta)$.

If θ has complexity 0, then $\theta = P_j(\tau_1, \ldots, \tau_n)$ for some j. Then,

- If $j < i$, then $v_i(\theta) = v_j(\theta)$;
- If $j > i$, then $v_i(\theta) = v(\theta)$;
- If $j = i$, then
  - If
    - P_i is a revisable predicate function in v_{i-1} and
    - P_i fulfills the revision requirement in v_{i-1},

    then $v_i(\theta) = 1$.
  - Otherwise, $v_i(\theta) = v(\theta)$.

From the chain of revised valuations, a limit valuation v_ω is defined:

If θ is a statement with complexity greater than 0, than, for every i, $v_i(\theta) = v(\theta)$. In this case, $v_\omega(\theta) = v(\theta)$

If θ has complexity 0, then $\theta = P_i(\tau_1, \ldots, \tau_n)$ for some i. In this case, $v_\omega(\theta) = v_i(\theta)$.

Now, it is time to fix the contradictions raised in non-consistent statements. Again, a recursive sequence of valuations will be defined:

- $v_\omega^0 = v_\omega$
- v_ω^{n+1} is such that

- If θ has complexity up to n or greater then $n+1$, then $v_\omega^{n+1}(\theta) = v_\omega^n(\theta)$;
- If θ has complexity $n+1$, then $v_\omega^{n+1}(\theta) = v_\omega^n(\theta)$ if it is compatible with the clauses of reasoning valuations and $v_\omega^{n+1}(\theta) \neq v_\omega^n(\theta)$ otherwise.

From the chain constructed above, $v_{Ref(\mathfrak{A})}$ is defined as a limit valuation:

If θ has complexity i, then $v_{Ref(\mathfrak{A})}(\theta) = v_\omega^i(\theta)$.

Finally, $Ref(\mathfrak{A})$ is the model having the same domain of interpretation, the same function I as $\mathfrak{A}$ and valuation $v_{Ref(\mathfrak{A})}$ as just described.

Both induction and abduction will be as described above, differing by the requirement criterion. So, induction and abduction are refinement functions as described, having as requirement criterion respectively:

Induction Requirement: The set $\{(\tau_1, \ldots, \tau_n) | P(\tau_1, \ldots, \tau_n) \in Fr_1(P)$ and $v(P(\tau_1, \ldots, \tau_n)) = 1$ and $v(\circ P(\tau_1, \ldots, \tau_n)) = 1\}$ has cardinality greater than $Fr_2(P)$.

Abduction Requirement: The set $\{(\theta, \tau_1, \ldots, \tau_n) | P(\tau_1, \ldots, \tau_n) \in Fr_1(P)$ and $v(\theta) = 1$ and $v(\circ\theta) = 1$ and $v(P(\tau_1, \ldots, \tau_n) \to \theta) = 1$ and $v(\circ(P(\tau_1, \ldots, \tau_n) \to \theta)) = 1\}$ has cardinality greater than $Fr_2(P)$.

The refinement criteria proposed above do not take non-consistent statements into account and are very strict in refusing contradictory information. A genuine reasoning paraconsistent context should not dismiss unsafe (but often reasonable) information. In order to comply with this kind of information, it is necessary to decide what to discard. A good way of doing so is to deal with information stochastically. In this line, a fact shall be accepted if there is much more evidence in favor of it that against it. This section will be closed with a suggestion of a stochastic induction requirement for refinement as an example of how to work out this proposal. Firstly, an extra definition is needed.

Definition 2.3.7 (Stochastic Refinement Schema)**.** *A stochastic refinement schema is a tuple $\langle E, S, V, p\rangle$, where E is an enumeration function for the set of statements, S is a σ-algebra over $\mathbb{N}$, V is a probability function over S and p is a real number in the closed interval* $[0, 1]$.

Finally, the requirement:

Stochastic Induction Refinement $\frac{V(B_{\alpha 1})}{V(B_\alpha)} > p$, where $B_{\alpha 1} = \{n | E^{-1}(n) \in Fr_1$ and $v(E^{-1}(n)) = 1\}$ and $B_\alpha = \{n | E^{-1}(n) \in Fr_1\}$ are, respectively, the set of labels of the true statements of the form $P(\tau_1, \ldots, \tau_n)$ and the set of labels of all statements of the true statements $P(\tau_1, \ldots, \tau_n)$.

2.4 Ultrafilters

In the last section, refinements were presented as a way of extracting new safe knowledge from a family of models, that is to say, the family of states of knowledge that are compatible with a given state.

In this section, the concept of reduced product of a family of models over a given ultrafilter will be recovered in order to provide an alternative way of extracting new knowledge from a given family of states of knowledge. Anyway, the content to be presented is worthy for its own, no matter how interesting it may be as an alternative to the refinement schemata suggested in the preceding chapter.

The concept of ultrafilter is the classical one. The concept of reduced product will be adapted to the context of reasonig models and the core result of this section is that the reduced product of a family of resoning models is a reasoning model itself and, moreover, if this family happens to be a one of isomrphic models, then safe knowledge is preserved.

Before starting the journey, some notation is in order. The models $\mathfrak{A}_i$ cited bellow are tacitly taken from a family $\{\mathfrak{A}_i\}_{i\in I}$ of reasoning models over the same language L_Σ.

- As usual, an ordered n-tuple $(a_1,\ldots,a_n)$ is also denoted by the abbreviated notation $\vec{a}$. If each a_i in the ordered n-tuple is a sequence, the notation $\vec{a_j}$ will stand for the n-tuple of the j^{th} entries from each a_i, that is, $\vec{a_j} = (a_{1,j},\ldots,a_{n,j})$;
- $\mathfrak{F}^{A^{\mathfrak{A}}_{Pk}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} = \{i \in I|(t_1^{\mathfrak{A}_i}[\vec{a_i}],\ldots,t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{P1}\}$, for $k \in \{1,2,3\}$;
- P_i stands for the sequence that corresponds to the predicate symbol P in $\mathfrak{A}_i$. Within the notation from Section 2.2, P_i stands for $En'(P, a_1^{\mathfrak{A}_i}, a_2^{\mathfrak{A}_i})$;
- $\mathfrak{F}^{A^{\mathfrak{A}}_{Pk}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} = \{i \in I|(P'_i, t_1^{\mathfrak{A}_i}[\vec{a_i}],\ldots,t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{Pk}\}$, for $k \in \{4,\ldots,9\}$;
- $\mathfrak{F}_{\theta[\vec{a}]} = \{i \in I|\mathfrak{A}_i \vDash \theta[\vec{a_i}]\}$;
- $\bar{\mathfrak{F}}_{\theta[\vec{a}]} = \{i \in I|\mathfrak{A}_i \nvDash \theta[\vec{a_i}]\}$.
- If the context is clear, the shorter forms $\mathfrak{F}_\theta$ and $\bar{\mathfrak{F}}_\theta$ will be used instead of $\mathfrak{F}_{\theta[\vec{a}]}$ and $\bar{\mathfrak{F}}_{\theta[\vec{a}]}$. This will largely contribute for visual cleanness.

Now, reduced structures and reduced products can be defined:

Definition 2.4.1 (Reduced Structure). *The* reduced structure *of a family* $\{\mathfrak{A}_i\}_{i\in I}$ *of reasoning models over an ultrafilter* $\mathfrak{F}$ *is the structure* $\mathfrak{A}_{\mathfrak{F}}$ *(or just* $\mathfrak{A}$*, if the context is clear) such that*

- *Each element from the domain of interpretation $A_{\mathfrak{F}}$ of $\mathfrak{A}_{\mathfrak{F}}$ is a sequence of elements from the domains of interpretation A_i of the models $\mathfrak{A}_i$. That is, for each $a \in A_{\mathfrak{F}}$, $a = \{a_i\}_{i \in I}$, with $a_i \in A_i$, for every $i \in I$;*

- *For each constant c, $I_{\mathfrak{A}_{\mathfrak{F}}}(c) = \{I_{\mathfrak{A}_i}(c)\}_{i \in I}$;*

- *For each $f \in F_n$, $I_{\mathfrak{A}_{\mathfrak{F}}}(f) = \{I_{\mathfrak{A}_i}(f)\}_{i \in I}$. That is, if $\vec{a} \in (A_{\mathfrak{F}})^n$, then $I_{\mathfrak{A}_{\mathfrak{F}}}(f)(\vec{a}) = \{I_{\mathfrak{A}_i}(f)(\vec{a_i})\}_{i \in I}$;*

- *$I_{\mathfrak{A}_{\mathfrak{F}}}(P) = (A^{\mathfrak{A}_{\mathfrak{F}}}_{P1}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P2}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P3}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P4}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P5}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P6}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P7}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P8}, A^{\mathfrak{A}_{\mathfrak{F}}}_{P9}, a_1^{\mathfrak{A}_{\mathfrak{F}}}, a_2^{\mathfrak{A}_{\mathfrak{F}}})$, if $P \in P_n$, where*
  - *$A^{\mathfrak{A}_{\mathfrak{F}}}_{Pk} \subseteq (A_{\mathfrak{F}})^n$ is such that $(t_1^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}], \ldots, t_n^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}]) \in A^{\mathfrak{A}_{\mathfrak{F}}}_{Pk}$ iff $\mathfrak{F}^{A^{\mathfrak{A}_{\mathfrak{F}}}_{Pk}}_{(t_1^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}],\ldots,t_n^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}])} \in \mathfrak{F}$, for $k \in \{1, 2, 3\}$;*
  - *$A^{\mathfrak{A}_{\mathfrak{F}}}_{Pk} \subseteq (\overline{P \times A})(a_1^{\mathfrak{A}_{\mathfrak{F}}}, a_2^{\mathfrak{A}_{\mathfrak{F}}})$ is such that $(P', t_1^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}], \ldots, t_m^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}]) \in A^{\mathfrak{A}_{\mathfrak{F}}}_{P'k}$, iff $\mathfrak{F}^{A^{\mathfrak{A}_{\mathfrak{F}}}_{Pk}}_{(P', t_1^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}],\ldots,t_m^{\mathfrak{A}_{\mathfrak{F}}}[\vec{a}])} \in \mathfrak{F}$, for $k \in \{4, \ldots, 9\}$;*
  - *$a_1^{\mathfrak{A}_{\mathfrak{F}}}, a_2^{\mathfrak{A}_{\mathfrak{F}}} \in A_{\mathfrak{F}}$ are the sequences $a_1^{\mathfrak{A}_{\mathfrak{F}}} = \{a_1^{\mathfrak{A}_i}\}_{i \in I}$ and $a_2^{\mathfrak{A}_{\mathfrak{F}}} = \{a_2^{\mathfrak{A}_i}\}_{i \in I}$;*

Definition 2.4.2 (Reduced Product)**.** *The* reduced product *of a family $\mathfrak{M} = \{\mathfrak{A}_i\}_{i \in I}$ of reasoning models over an ultrafilter $\mathfrak{F}$ is the reduced structure of $\mathfrak{M}$ endowed with the valuation $v_{\mathfrak{F}}$ as follows*

- *For each interpreted formula $\theta[\vec{a}]$, $v_{\mathfrak{F}}(\theta[\vec{a}]) = 1$ iff $\mathfrak{F}_{\theta[\vec{a}]} \in \mathfrak{F}$.*

Two lemmata will be needed for the demonstration of Reduced Product's Theorem:

Lemma 2.4.3. *Let $\mathfrak{F}$ be an ultrafilter over a set of indices I. If $A, B \subset I$ are such that $A \cup B \in \mathfrak{F}$, then $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.*

Proof. Suppose $A \notin \mathfrak{F}$. Then, $A \backslash (A \cap B) \notin \mathfrak{F}$. Otherwise, it would be the case that $A \in \mathfrak{F}$, for $(A \backslash (A \cap B)) \subset A$. Hence, $(A \cup B)^C \cup (A \backslash (A \cap B)) \notin \mathfrak{F}$. Otherwise, it would be the case that $(A \backslash (A \cap B)) = [((A \cup B)^C \cup (A \backslash (A \cap B))) \cap (A \cup B)] \in \mathfrak{F}$, for $(A \cup B) \in \mathfrak{F}$. Therefore, $B \in \mathfrak{F}$, for $B = [(A \cup B)^C \cup (A \backslash (A \cap B))]^C$, that is, B is the complement of that set.

By symmetry, if $B \notin \mathfrak{F}$, then $A \in \mathfrak{F}$. □

Lemma 2.4.4. *Let $\mathfrak{F}$ be an ultrafilter, $\{\mathfrak{A}_i\}_{i \in I}$ a family of models satisfying **vNeg** and $\theta[\vec{a}]$ an interpreted formula. Then, $(\mathfrak{F}_{\theta[\vec{a}]})^C \subset \mathfrak{F}_{\neg\theta[\vec{a}]}$.*

Proof. Let $i \in (\mathfrak{F}_{\theta[\vec{a}]})^C$, that is $i \notin \mathfrak{F}_{\theta[\vec{a}]}$. Then, $\mathfrak{A}_i \nvDash \theta[\vec{a_i}]$. By **vNeg**, $\mathfrak{A}_i \vDash \neg\theta[\vec{a_i}]$. Hence, $i \in \mathfrak{F}_{\neg\theta[\vec{a}]}$. □

The next theorem is the core result of the section.

Theorem 2.4.5 (Reduced Product's Theorem)**.** *The* reduced product $\mathfrak{A}_{\mathfrak{F}}$ *of a family of reasoning models* $\{\mathfrak{A}_i\}_i$ *over an ultrafilter* $\mathfrak{F}$ *is itself a reasoning model.*

Proof. The demonstration consists in proving that the reduced product's valuation satisfies each clause of reasoning valuations:

Basic Control Clauses:

vPredPos $v(P(t_1, \ldots, t_n)[\vec{a}]) = 1$ if $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P1}$, for $P \in P_n$.

If $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P1}$, then $\mathfrak{F}^{A^{\mathfrak{A}}_{P1}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$. For each $i \in \mathfrak{F}^{A^{\mathfrak{A}}_{P1}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])}$, $(t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{P1}$. By **vPredPos**, $\mathfrak{A}_i \vDash (P(t_1, \ldots, t_n)[\vec{a_i}])$. Hence, $\mathfrak{F}^{A^{\mathfrak{A}}_{P1}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \subset \mathfrak{F}_{(P(t_1,\ldots,t_n)[\vec{a}])}$. As $\mathfrak{F}^{A^{\mathfrak{A}}_{P1}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{(P(t_1,\ldots,t_n)[\vec{a}])} \in \mathfrak{F}$. Therefore, $v(P(t_1, \ldots, t_n)[\vec{a}]) = 1$, as desired.

vPredNeg $v(P(t_1, \ldots, t_n)[\vec{a}]) = 0$ if $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P3}$, for $P \in P_n$.

If $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P3}$, then $\mathfrak{F}^{A^{\mathfrak{A}}_{P3}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$. For each $i \in \mathfrak{F}^{A^{\mathfrak{A}}_{P3}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])}$, $(t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{P3}$. By **vPredNeg**, $\mathfrak{A}_i \nvDash (P(t_1, \ldots, t_n)[\vec{a_i}])$. Hence, $\mathfrak{F}^{A^{\mathfrak{A}}_{P3}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \subset \bar{\mathfrak{F}}_{(P(t_1,\ldots,t_n)[\vec{a}])}$. As $\mathfrak{F}^{A^{\mathfrak{A}}_{P3}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$, it holds that $\bar{\mathfrak{F}}_{(P(t_1\ldots,t_n)[\vec{a}])} \in \mathfrak{F}$. This implies that $\mathfrak{F}_{(P(t_1,\ldots,t_n)[\vec{a}])} = (\bar{\mathfrak{F}}_{(P(t_1,\ldots,t_n)[\vec{a}])})^C \notin \mathfrak{F}$. Therefore, $v(P(t_1, \ldots, t_n)[\vec{a}]) = 0$, as desired.

vConPredPart $v((\circ(P(t_1, \ldots, t_n)))[\vec{a}]) = 1$ if $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P2}$, for $P \in P_n$.

The proof of **vConPredPart** is the same as that of **vPredPos**, mutatis mutandis.

vConImpPred $v((P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a}]) = 1$ and $v(\circ(P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a}]) = 1$ if $(P', t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P4}$ and $(P, t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_m^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P'5}$, for $P \in P_n$ and $P' \in P_m$.

If $(P', t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P4}$ and $(P, t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_m^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P'5}$, then $\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$ and $\mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$.

So, $(\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \cap \mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])}) \in \mathfrak{F}$.
For each $i \in (\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \cap \mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])})$, $(P'_i, t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{P4}$ and $(P_i, t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_m^{\mathfrak{A}_i}[\vec{a_i}]) \in A^{\mathfrak{A}_i}_{P'5}$. By **vConImpPred**, $\mathfrak{A}_i \vDash (P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a_i}]$ and $\mathfrak{A}_i \vDash \circ(P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a_i}]$.
Hence, $(\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \cap \mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])}) \subset \mathfrak{F}_{(P(t_1,\ldots,t_n)\to P'(t_1,\ldots,t_m))[\vec{a}]}$ and
$(\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \cap \mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])}) \subset \mathfrak{F}_{\circ(P(t_1,\ldots,t_n)\to P'(t_1\ldots,t_m))[\vec{a}]}$.
As $(\mathfrak{F}^{A^{\mathfrak{A}}_{P4}}_{(P',t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \cap \mathfrak{F}^{A^{\mathfrak{A}}_{P'5}}_{(P,t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_m^{\mathfrak{A}}[\vec{a}])}) \in \mathfrak{F}$, it holds that
$\mathfrak{F}_{(P(t_1,\ldots,t_n)\to P'(t_1,\ldots,t_m))[\vec{a}]} \in \mathfrak{F}$ and $\mathfrak{F}_{\circ(P(t_1,\ldots,t_n)\to P'(t_1,\ldots,t_m))[\vec{a}]} \in \mathfrak{F}$.
Therefore, $v((P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a}]) = 1$ and $v(\circ(P(t_1, \ldots, t_n) \to P'(t_1, \ldots, t_m))[\vec{a}]) = 1$, as desired.

vConOrPred $v((P(t_1, \ldots, t_m) \vee P'(t_1, \ldots, t_n))[\vec{a}]) = 1$ and $v(\circ(P(t_1, \ldots, t_m) \vee P'(t_1, \ldots, t_n))[\vec{a}]) = 1$ if $(P', t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P6}$ and $(P, t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_m^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P'7}$, for $P \in P_n$ and $P' \in P_m$.

vConAndPred $v((P(t_1, \ldots, t_m) \wedge P'(t_1, \ldots, t_n))[\vec{a}]) = 1$ and $v(\circ(P(t_1, \ldots, t_m) \wedge P'(t_1, \ldots, t_n))[\vec{a}]) = 1$ if $(P', t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}], P') \in A^{\mathfrak{A}}_{P8}$ and
$(P, t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A^{\mathfrak{A}}_{P'9}$, for $P, P' \in P_n$ and $P' \in P_m$.

The proofs of **vConOrPred** and **vConAndPred** are the same as that of **vConImpPred**, mutatis mutandis.

Classical Behavior Clauses

vOr $v((\alpha \vee \beta)[\vec{a}]) = 1$ iff $v(\alpha[\vec{a}]) = 1$ or $v(\beta[\vec{a}]) = 1$.

$(\Rightarrow)$ If $v((\alpha \vee \beta)[\vec{a}]) = 1$, then $\mathfrak{F}_{\alpha\vee\beta} \in \mathfrak{F}$. For each $i \in I$, $i \in \mathfrak{F}_{\alpha\vee\beta}$ iff $\mathfrak{A}_i \vDash (\alpha \vee \beta)$. By **vOr**, that holds iff ($\mathfrak{A}_i \vDash \alpha$ or $\mathfrak{A}_i \vDash \beta$) iff ($i \in \mathfrak{F}_\alpha$ or $i \in \mathfrak{F}_\beta$) iff $i \in (\mathfrak{F}_\alpha \cup \mathfrak{F}_\beta)$. So, $\mathfrak{F}_{\alpha\vee\beta} = (\mathfrak{F}_\alpha \cup \mathfrak{F}_\beta)$. This means that $(\mathfrak{F}_\alpha \cup \mathfrak{F}_\beta) \in \mathfrak{F}$. By Lemma 2.4.3, $\mathfrak{F}_\alpha \in \mathfrak{F}$ or $\mathfrak{F}_\beta \in \mathfrak{F}$. Hence, $v(\alpha[\vec{a}]) = 1$ or $v(\beta[\vec{a}]) = 1$.
$(\Leftarrow)$ If $v(\alpha[\vec{a}]) = 1$, then $\mathfrak{F}_\alpha \in \mathfrak{F}$. For every $i \in \mathfrak{F}_\alpha$, $\mathfrak{A}_i \vDash \alpha$, which yields, by **vOr**, that $\mathfrak{A}_i \vDash (\alpha \vee \beta)$. Hence, if $i \in \mathfrak{F}_\alpha$, then $i \in \mathfrak{F}_{\alpha\vee\beta}$, that is, $\mathfrak{F}_\alpha \subset \mathfrak{F}_{\alpha\vee\beta}$. As $\mathfrak{F}_\alpha \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\alpha\vee\beta} \in \mathfrak{F}$. Therefore, $v((\alpha \vee \beta)[\vec{a}]) = 1$.
By symmetry, if $v((\beta)[\vec{a}]) = 1$, then $v((\alpha \vee \beta)[\vec{a}]) = 1$. Summing up, if $v((\alpha)[\vec{a}]) = 1$ or $v((\beta)[\vec{a}]) = 1$, then $v((\alpha \vee \beta)[\vec{a}]) = 1$, as desired.

vAnd $v((\alpha \wedge \beta)[\vec{a}]) = 1$ iff $v(\alpha[\vec{a}]) = 1$ and $v(\beta[\vec{a}]) = 1$.

($\Rightarrow$) If $v((\alpha \wedge \beta)[\vec{a}]) = 1$, then $\mathfrak{F}_{\alpha\wedge\beta} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{\alpha\wedge\beta}$, $\mathfrak{A}_i \vDash (\alpha \wedge \beta)$, which yelds, by **vAnd**, that $\mathfrak{A}_i \vDash \alpha$ and $\mathfrak{A}_i \vDash \beta$. Hence, $i \in \mathfrak{F}_\alpha$ and $i \in \mathfrak{F}_\beta$, that is, $\mathfrak{F}_{\alpha\wedge\beta} \subset \mathfrak{F}_\alpha$ and $\mathfrak{F}_{\alpha\wedge\beta} \subset \mathfrak{F}_\beta$. As $\mathfrak{F}_{\alpha\wedge\beta} \in \mathfrak{F}$, it holds that $\mathfrak{F}_\alpha \in \mathfrak{F}$ and $\mathfrak{F}_\beta \in \mathfrak{F}$. Therefore, $v((\alpha)[\vec{a}]) = 1$ and $v((\beta)[\vec{a}]) = 1$.
($\Leftarrow$) If $v((\alpha)[\vec{a}]) = 1$ and $v((\beta)[\vec{a}]) = 1$, then $\mathfrak{F}_\alpha \in \mathfrak{F}$ and $\mathfrak{F}_\beta \in \mathfrak{F}$, whence $(\mathfrak{F}_\alpha \cap \mathfrak{F}_\beta) \in \mathfrak{F}$. By **vAnd**, for every $i \in (\mathfrak{F}_\alpha \cap \mathfrak{F}_\beta)$, $\mathfrak{A}_i \vDash (\alpha \wedge \beta)$. Hence, if $i \in \mathfrak{F}_\alpha$ and $i \in \mathfrak{F}_\beta$, then $i \in \mathfrak{F}_{\alpha\wedge\beta}$, that is, $(\mathfrak{F}_\alpha \cap \mathfrak{F}_\beta) \subset \mathfrak{F}_{\alpha\wedge\beta}$. As $(\mathfrak{F}_\alpha \cap \mathfrak{F}_\beta) \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\alpha\wedge\beta} \in \mathfrak{F}$. Therefore, $v((\alpha \wedge \beta)[\vec{a}]) = 1$, as desired.

vImp $v((\alpha \to \beta)[\vec{a}]) = 1$ iff $v(\alpha[\vec{a}]) = 0$ or $v(\beta[\vec{a}]) = 1$.

$\Rightarrow$ If $v((\alpha \to \beta)[\vec{a}]) = 1$, then $\mathfrak{F}_{\alpha\to\beta} \in \mathfrak{F}$. By **vImp**, for every $i \in I$, $\mathfrak{A}_i \vDash (\alpha \to \beta)$ iff $\mathfrak{A}_i \nvDash \alpha$ or $\mathfrak{A}_i \vDash \beta$. Hence, $\mathfrak{F}_{\alpha\to\beta} = \bar{\mathfrak{F}}_\alpha \cup \mathfrak{F}_\beta$. As $\mathfrak{F}_{\alpha\to\beta} \in \mathfrak{F}$, it follows, by Lemma 2.4.3, that $\bar{\mathfrak{F}}_\alpha \in \mathfrak{F}$ or $\mathfrak{F}_\beta \in \mathfrak{F}$. If $\bar{\mathfrak{F}}_\alpha \in \mathfrak{F}$, then $\mathfrak{F}_\alpha = (\bar{\mathfrak{F}}_\alpha)^C \notin \mathfrak{F}$, which implies that $v((\alpha)[\vec{a}]) = 0$. If $\mathfrak{F}_\beta \in \mathfrak{F}$, then $v((\beta)[\vec{a}]) = 1$. Therefore, $v((\alpha)[\vec{a}]) = 0$ or $v((\beta)[\vec{a}]) = 1$.
($\Leftarrow$) Suppose that $v((\alpha)[\vec{a}]) = 0$ or $v((\beta)[\vec{a}]) = 1$. For every $i \in \bar{\mathfrak{F}}_\alpha$, $\mathfrak{A}_i \nvDash \alpha$, which implies, by **vImp**, that $\mathfrak{A}_i \vDash (\alpha \to \beta)$. For every $i \in \mathfrak{F}_\beta$, $\mathfrak{A}_i \vDash \beta$, which implies, again by **vImp**, that $\mathfrak{A}_i \vDash (\alpha \to \beta)$. So, $\bar{\mathfrak{F}}_\alpha \subset \mathfrak{F}_{\alpha\to\beta}$ and $\mathfrak{F}_\beta \subset \mathfrak{F}_{\alpha\to\beta}$. If $v((\alpha)[\vec{a}]) = 0$, then $\bar{\mathfrak{F}}_\alpha \in \mathfrak{F}$. If $v((\beta)[\vec{a}]) = 1$, then $\mathfrak{F}_\beta \in \mathfrak{F}$. In any case, $\mathfrak{F}_{\alpha\to\beta} \in \mathfrak{F}$. Therefore, $v((\alpha \to \beta)[\vec{a}]) = 1$.

vEx $v((\exists x\alpha(x))[\vec{a}]) = 1$ iff $v(\alpha[b, \vec{a}]) = 1$ for some $b \in A_{\mathfrak{F}}$.

$\Rightarrow$ Let $v((\exists x\alpha(x))[\vec{a}]) = 1$. Then, $\mathfrak{F}_{\exists x\alpha[\vec{a}]} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{\exists x\alpha[\vec{a}]}$, $\mathfrak{A}_i \vDash \exists x\alpha[\vec{a_i}]$. By **vEx**, there is some $e \in A_i$ such that $\mathfrak{A}_i \vDash \alpha[e, \vec{a_i}]$. So, for every $i \in \mathfrak{F}_{\exists x\alpha[\vec{a}]}$, choose $b_i \in A_i$ such that $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$ (at this point, Axiom of Choice is being used). Take $b = (b_i)_{i\in I}$ to be the sequence such that, for each $i \in \mathfrak{F}_{\exists x\alpha[\vec{a}]}$, b_i is the chosen b_i above and, for each $i \notin \mathfrak{F}_{\exists x\alpha[\vec{a}]}$, b_i is any fixed element of A_i. For the so constructed b, $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$ if $i \in \mathfrak{F}_{\exists x\alpha[\vec{a}]}$ and $\mathfrak{A}_i \nvDash \alpha[b_i, \vec{a_i}]$ if $i \notin \mathfrak{F}_{\exists x\alpha[\vec{a}]}$. Hence, $\mathfrak{F}_{\alpha[b,\vec{a}]} = \mathfrak{F}_{\exists x\alpha[\vec{a}]}$. As $\mathfrak{F}_{\exists x\alpha[\vec{a}]} \in \mathfrak{F}$, it holds that $v(\alpha[b, \vec{a}]) = 1$, Therefore, there is a $b \in A_{\mathfrak{F}}$ such that $v(\alpha[b, \vec{a}]) = 1$, as desired.
($\Leftarrow$) If $v(\alpha[b, \vec{a}]) = 1$ for some $b \in A$, then $\mathfrak{F}_{\alpha[b,\vec{a}]} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{\alpha[b,\vec{a}]}$, it holds that $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$. By **vEx**, $\mathfrak{A}_i \vDash \exists x\alpha[\vec{a}]$. Hence, $\mathfrak{F}_{\alpha[b,\vec{a}]} \subset \mathfrak{F}_{\exists x\alpha[\vec{a}]}$. As $\mathfrak{F}_{\alpha[b,\vec{a}]} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\exists x\alpha[\vec{a}]} \in \mathfrak{F}$. Therefore, $v(\exists x\alpha[\vec{a}]) = 1$, as desired.

vUni $v((\forall x\alpha)[\vec{a}]) = 1$ iff $v(\alpha[b, \vec{a}]) = 1$ for every $b \in A_{\mathfrak{F}}$.

$\Rightarrow$ If $v(\forall x\alpha(x)) = 1$, then $\mathfrak{F}_{\forall x\alpha[\vec{a}]} \in \mathfrak{F}$. Let $b = (b_i)_{i\in I}$ be an arbitrary element in $A_{\mathfrak{F}}$. For every $i \in \mathfrak{F}_{\forall x\alpha[\vec{a}]}$, it holds that $\mathfrak{A}_i \vDash \forall x\alpha[\vec{a_i}]$. By **vUni**, for every $e \in A_i$, it holds that $\mathfrak{A}_i \vDash \alpha[e, \vec{a_i}]$. In particular, $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$. Hence, $i \in \mathfrak{F}_{\alpha[b,\vec{a}]}$. So,

$\mathfrak{F}_{\forall x\alpha[\vec{a}]} \subset \mathfrak{F}_{\alpha[b,\vec{a}]}$. As $\mathfrak{F}_{\forall x\alpha[\vec{a}]} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\alpha[b,\vec{a}]} \in \mathfrak{F}$. Therefore, $v(\alpha[b,\vec{a}]) = 1$. As b is arbitrary, the desired result follows.
$\Leftarrow$ Let $v(\alpha[b,\vec{a}]) = 1$ for every $b \in A_{\mathfrak{F}}$. Suppose, for the sake of contradiction, that $v((\forall x\alpha)[\vec{a}]) = 0$. Then, $\mathfrak{F}_{\forall x\alpha[\vec{a}]} \notin \mathfrak{F}$. For every $i \notin \mathfrak{F}_{\forall x\alpha[\vec{a}]}$, it holds, by **vUni**, that $\mathfrak{A}_i \nvDash \forall x\alpha[\vec{a_i}]$. So, for every $i \notin \mathfrak{F}_{\forall x\alpha[\vec{a}]}$, choose $b_i \in A_i$ such that $\mathfrak{A}_i \nvDash \alpha[b_i, \vec{a_i}]$ (at this point, Axiom of Choice is being used). Take $b = (b_i)_{i\in I}$ to be the sequence such that, for each $i \notin \mathfrak{F}_{\forall x\alpha[\vec{a}]}$, b_i is the chosen b_i above and, for each $i \in \mathfrak{F}_{\forall x\alpha[\vec{a}]}$, b_i is any fixed element of A. For the so constructed b, $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$ if $i \in \mathfrak{F}_{\forall x\alpha[\vec{a}]}$ and $\mathfrak{A}_i \nvDash \alpha[b_i, \vec{a_i}]$ if $i \notin \mathfrak{F}_{\forall x\alpha[\vec{a}]}$. Hence, $\mathfrak{F}_{\alpha[b,\vec{a}]} = \mathfrak{F}_{\forall x\alpha[\vec{a}]}$. As $\mathfrak{F}_{\forall x\alpha[\vec{a}]} \notin \mathfrak{F}$, it holds that $v(\alpha[b,\vec{a}]) = 0$, which is a contradiction against that fact that $v(\alpha[b,\vec{a}]) = 1$ for every $b \in A_{\mathfrak{F}}$.

QmbC Clauses

vNeg If $v(\alpha[\vec{a}]) = 0$, then $v(\neg\alpha[\vec{a}]) = 1$.

If $v(\alpha[\vec{a}]) = 0$, then $\mathfrak{F}_\alpha \notin \mathfrak{F}$, which implies that $(\mathfrak{F}_\alpha)^C \in \mathfrak{F}$. By Lemma 2.4.4, $(\mathfrak{F}_\alpha)^C \subset \mathfrak{F}_{\neg\alpha}$. Hence, $\mathfrak{F}_{\neg\alpha} \in \mathfrak{F}$, which implies that $v(\neg\alpha[\vec{a}]) = 1$, as desired.

vCon If $v((\circ\alpha)[\vec{a}]) = 1$, then $v(\alpha[\vec{a}]) = 0$ or $v((\neg\alpha)[\vec{a}]) = 0$.

If $v((\circ\alpha)[\vec{a}]) = 1$, then $\mathfrak{F}_{\circ\alpha} \in \mathfrak{F}$. For each $i \in \mathfrak{F}_{\circ\alpha}$, $\mathfrak{A}_i \vDash \circ\alpha$. By **vCon**, $\mathfrak{A}_i \nvDash \alpha$ or $\mathfrak{A}_i \nvDash \neg\alpha$. Then, $\mathfrak{F}_{\circ\alpha} \cap (\mathfrak{F}_\alpha \cap \mathfrak{F}_{\neg\alpha}) = \emptyset$. So, $(\mathfrak{F}_\alpha \cap \mathfrak{F}_{\neg\alpha}) \subset (\mathfrak{F}_{\circ\alpha})^C$. Hence, $(\mathfrak{F}_\alpha \cap \mathfrak{F}_{\neg\alpha}) \notin \mathfrak{F}$, for, otherwise, it would be the case that $(\mathfrak{F}_{\circ\alpha})^C \in \mathfrak{F}$. But this is not the case, for $\mathfrak{F}_{\circ\alpha} \in \mathfrak{F}$. Finally, it is not the case that both $\mathfrak{F}_\alpha \in \mathfrak{F}$ and $\mathfrak{F}_{\neg\alpha} \in \mathfrak{F}$, for, otherwise, it would be the case that $(\mathfrak{F}_\alpha \cap \mathfrak{F}_{\neg\alpha}) \in \mathfrak{F}$. Therefore, it holds that $v(\alpha[\vec{a}]) = 0$ or $v((\neg\alpha)[\vec{a}]) = 0$, as desired.

vVar $v(\alpha[\vec{a}]) = v(\beta[\vec{a}])$ whenever $\alpha[\vec{a}]$ is a variant of $\beta[\vec{a}]$.

If $\alpha[\vec{a}]$ is a variant of $\beta[\vec{a}]$, then, for every $i \in I$, $\mathfrak{A}_i \vDash \alpha$ iff $\mathfrak{A}_i \vDash \beta$. Hence, $\mathfrak{F}_\alpha = \mathfrak{F}_\beta$. Therefore, $v(\alpha[\vec{a}]) = v(\beta[\vec{a}])$, as desired.

sNeg For every context $(\vec{x}, z)$ and $(\vec{x}, \vec{y})$, for every sequence $(\vec{a}, \vec{b})$ in A interpreting $(\vec{x}, \vec{y})$, for every $\phi \in L(\mathfrak{A})_{\vec{x},z}$ and for every $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ such that t is free for z in ϕ, if $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ and $b = (t[\vec{x},\vec{y}/\vec{a},\vec{b}])^{\widehat{\mathfrak{A}}}$, then $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b])$ implies $v((\neg\phi[z/t][\vec{x},\vec{y}/\vec{a},\vec{b}])) = v(\neg\phi[\vec{x},z/\vec{a},b])$.

If $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b])$, then even $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b]) = 1$ or $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b]) = 0$.
If $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b]) = 1$, then both $\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \in \mathfrak{F}$ and $\mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]} \in \mathfrak{F}$. So, $(\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]}) \in \mathfrak{F}$. For every $i \in (\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap$

$\mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]})$, it holds that $\mathfrak{A}_i \vDash (\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \vDash \phi[\vec{x_i},z_i/\vec{a_i},b_i]$. By **sNeg**, even ($\mathfrak{A}_i \vDash \neg\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \vDash \neg\phi[\vec{x_i},z_i/\vec{a_i},b_i]$) or ($\mathfrak{A}_i \nvDash \neg\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \nvDash \neg\phi[\vec{x_i},z_i/\vec{a_i},b_i]$). Therefore, $(\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]}) \subset [(\mathfrak{F}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\neg\phi[\vec{x},z/\vec{a},b]}) \cup (\bar{\mathfrak{F}}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \bar{\mathfrak{F}}_{\neg\phi[\vec{x},z/\vec{a},b]})]$. As $(\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]}) \in \mathfrak{F}$, it holds that $(\mathfrak{F}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\neg\phi[\vec{x},z/\vec{a},b]}) \cup (\bar{\mathfrak{F}}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \bar{\mathfrak{F}}_{\neg\phi[\vec{x},z/\vec{a},b]}) \in \mathfrak{F}$. By Lemma 2.4.3, even $\mathfrak{F}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \mathfrak{F}_{\neg\phi[\vec{x},z/\vec{a},b]} \in \mathfrak{F}$ or $\bar{\mathfrak{F}}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \bar{\mathfrak{F}}_{\neg\phi[\vec{x},z/\vec{a},b]} \in \mathfrak{F}$. In the first case, $\mathfrak{F}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]}, \mathfrak{F}_{\neg\phi[\vec{x},z/\vec{a},b]} \in \mathfrak{F}$, which yields $v((\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\neg\phi[\vec{x},z/\vec{a},b]) = 1$, In the second case, $\mathfrak{F}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} = (\bar{\mathfrak{F}}_{(\neg\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]})^C \notin \mathfrak{F}$ and $\mathfrak{F}_{\neg\phi[\vec{x},z/\vec{a},b]} = (\bar{\mathfrak{F}}_{\phi[\vec{x},z/\vec{a},b]})^C \notin \mathfrak{F}$, which yelds $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b]) = 0$.

If $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b]) = 0$, then both $\mathfrak{F}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \notin \mathfrak{F}$ and $\mathfrak{F}_{\phi[\vec{x},z/\vec{a},b]} \notin \mathfrak{F}$, which implies that both $\bar{\mathfrak{F}}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \in \mathfrak{F}$ and $\bar{\mathfrak{F}}_{\phi[\vec{x},z/\vec{a},b]} \in \mathfrak{F}$. So, $(\bar{\mathfrak{F}}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \bar{\mathfrak{F}}_{\phi[\vec{x},z/\vec{a},b]}) \in \mathfrak{F}$. For every $i \in (\bar{\mathfrak{F}}_{(\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]} \cap \bar{\mathfrak{F}}_{\phi[\vec{x},z/\vec{a},b]})$, it holds that $\mathfrak{A}_i \nvDash (\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \nvDash \phi[\vec{x_i},z_i/\vec{a_i},b_i]$. By **sNeg**, even $\mathfrak{A}_i \vDash (\neg\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \vDash \neg\phi[\vec{x_i},z_i/\vec{a_i},b_i]$ or $\mathfrak{A}_i \nvDash (\neg\phi[z_i/t_i])[\vec{x_i},\vec{y_i}/\vec{a_i},\vec{b_i}]$ and $\mathfrak{A}_i \nvDash \neg\phi[\vec{x_i},z_i/\vec{a_i},b_i]$. The rest of the proof follows as in the first part.

sCon For every context $(\vec{x},z)$ and $(\vec{x},\vec{y})$, for every sequence $(\vec{a},\vec{b})$ in A interpreting $(\vec{x},\vec{y})$, for every $\phi \in L(\mathfrak{A})_{\vec{x},z}$ and for every $t \in T(\mathfrak{A})_{\vec{x},\vec{y}}$ such that t is free for z in ϕ, if $\phi[z/t] \in L(\mathfrak{A})_{\vec{x},\vec{y}}$ and $b = (t[\vec{x},\vec{y}/\vec{a},\vec{b}])^{\widehat{\mathfrak{A}}}$, then $v((\phi[z/t])[\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\phi[\vec{x},z/\vec{a},b])$ implies $v(\circ\phi[z/t][\vec{x},\vec{y}/\vec{a},\vec{b}]) = v(\circ\phi[\vec{x},z/\vec{a},b])$.

The proof is identical to that of **sNeg**.

Propagation Clauses

vPropOr If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $v((\circ(\alpha \vee \beta))[\vec{a}]) = 1$.

vPropAnd If $v((\circ\alpha)[\vec{a}]) = 1$ and $v(\circ\beta) = 1$, then $v((\circ(\alpha \wedge \beta))[\vec{a}]) = 1$.

vPropImp If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $v((\circ(\alpha \to \beta))[\vec{a}]) = 1$.

If $v((\circ\alpha)[\vec{a}]) = 1$ and $v((\circ\beta)[\vec{a}]) = 1$, then $\mathfrak{F}_{\circ\alpha} \in \mathfrak{F}$ and $\mathfrak{F}_{\circ\beta} \in \mathfrak{F}$. So, $\mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta}$, it holds that $\mathfrak{A}_i \vDash \circ\alpha$ and $\mathfrak{A}_i \vDash \circ\beta$. By **vPropOr**, $\mathfrak{A}_i \vDash \circ(\alpha \vee \beta)$; By **vPropAnd**, $\mathfrak{A}_i \vDash \circ(\alpha \wedge \beta)$; By **vPropImp**, $\mathfrak{A}_i \vDash \circ(\alpha \to \beta)$. Hence, $(\mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta}) \subset \mathfrak{F}_{\circ(\alpha\vee\beta)}$, $(\mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta}) \subset \mathfrak{F}_{\circ(\alpha\wedge\beta)}$ and $(\mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta}) \subset \mathfrak{F}_{\circ(\alpha\to\beta)}$. As $\mathfrak{F}_{\circ\alpha} \cap \mathfrak{F}_{\circ\beta} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\circ(\alpha\vee\beta)} \in \mathfrak{F}$, $\mathfrak{F}_{\circ(\alpha\wedge\beta)} \in \mathfrak{F}$ and $\mathfrak{F}_{\circ(\alpha\to\beta)} \in \mathfrak{F}$. Therefore,

$v((\circ(\alpha \vee \beta))[\vec{a}]) = 1$, $v((\circ(\alpha \wedge \beta))[\vec{a}]) = 1$ and $v((\circ(\alpha \to \beta))[\vec{a}]) = 1$. In this way, **vPropOr**, **vPropAnd** and **vPropImp** are proven.

vPropNeg If $v((\circ\alpha)[\vec{a}]) = 1$, then $v((\circ(\neg\alpha))[\vec{a}]) = 1$.

If $v((\circ\alpha)[\vec{a}]) = 1$, then $\mathfrak{F}_{\circ\alpha} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{\circ\alpha}$, $\mathfrak{A}_i \vDash \circ\alpha$. By **vPropNeg**, $\mathfrak{A}_i \vDash \circ(\neg\alpha)$. Hence, $\mathfrak{F}_{\circ\alpha} \subset \mathfrak{F}_{\circ(\neg\alpha)}$. As $\mathfrak{F}_{\circ\alpha} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\circ(\neg\alpha)} \in \mathfrak{F}$. Therefore, $v((\circ(\neg\alpha))[\vec{a}]) = 1$, as desired.

vPropCon For every α, $v((\circ(\circ\alpha))[\vec{a}]) = 1$ and $v((\circ(\neg \circ \alpha))[\vec{a}]) = 1$.

For every $i \in I$, $\mathfrak{A}_i \vDash \circ(\circ\alpha)$ and $\mathfrak{A}_i \vDash \circ(\neg \circ \alpha)$. Hence, $\mathfrak{F}_{\circ(\circ\alpha)} = \mathfrak{F}_{\circ(\neg\circ\alpha)} = I$. Therefore, $v((\circ(\circ\alpha))[\vec{a}]) = 1$ and $v((\circ(\neg \circ \alpha))[\vec{a}]) = 1$, as desired.

vPropUni If, for all $b \in A_{\mathfrak{F}}$, $v(\circ\alpha[b, \vec{a}]) = 1$, then $v(\circ(\forall x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

vPropEx If, for all $b \in A_{\mathfrak{F}}$, $v(\circ\alpha[b, \vec{a}]) = 1$, then $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

Suppose that, for all $b \in A_{\mathfrak{F}}$, $v(\circ\alpha[b, \vec{a}]) = 1$. In the proof of **vUni** part $\Leftarrow$, it has been proven that, in this case, the set of indices labeling models that satisfy $\circ\alpha[e, \vec{a_i}]$ for every $e \in A$ belongs to $\mathfrak{F}$. That is, $J = \{i \in I | \text{for all } e \in A, \mathfrak{A}_i \vDash \circ\alpha[e, \vec{a_i}]\} \in \mathfrak{F}$. By **vPropUni**, $J \subset \mathfrak{F}_{\circ(\forall x\alpha(x,\vec{x}))[\vec{a}]}$. By **vPropEx**, $J \subset \mathfrak{F}_{\circ(\exists x\alpha(x,\vec{x}))[\vec{a}]}$. As $J \in \mathfrak{F}$, $\mathfrak{F}_{\circ(\forall x\alpha(x,\vec{x}))[\vec{a}]}$ and $\mathfrak{F}_{\circ(\exists x\alpha(x,\vec{x}))[\vec{a}]}$. Being so, if, for all $b \in A_{\mathfrak{F}}$, $v(\circ\alpha[b, \vec{a}]) = 1$, then $v(\circ(\forall x\alpha(x, \vec{x}))[\vec{a}]) = 1$ and $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$, as desired.

vPropEx' If, for some $b \in A$, $v(\alpha[b, \vec{a}]) = 1$ and $v(\circ\alpha[b, \vec{a}]) = 1$, then $v(\circ(\exists x\alpha(x, \vec{x}))[\vec{a}]) = 1$.

Suppose there is a $b \in A_{\mathfrak{F}}$ such that $v(\alpha[b, \vec{a}]) = 1$ and $v(\circ\alpha[b, \vec{a}]) = 1$. Now fix such a b. By the definition of validity in $\mathfrak{A}$, $v(\alpha[b, \vec{a}]) = 1$ and $v(\circ\alpha[b, \vec{a}]) = 1$ imply that $J' = \{i \in I | \mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]\} \in \mathfrak{F}$ and $J'' = \{i \in I | \mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]\} \in \mathfrak{F}$. So, $J = (J' \cap J'') \in \mathfrak{F}$. For each $i \in J$, $\mathfrak{A}_i \vDash \alpha[b_i, \vec{a_i}]$ and $\mathfrak{A}_i \vDash \circ\alpha[b_i, \vec{a_i}]$. By **vEx**, $\mathfrak{A}_i \vDash \circ(\exists x\alpha)[\vec{a}]$. Hence, $J \subset \mathfrak{F}_{\circ(\exists x\alpha)[\vec{a}]}$. As $J \in \mathfrak{F}$, it holds that $\mathfrak{F}_{\circ(\exists x\alpha)[\vec{a}]} \in \mathfrak{F}$. Therefore, $v(\circ(\exists x\alpha)[\vec{a}]) = 1$, as desired. □

What has been proven is in fact a broader result.

Corollary 2.4.6. *Let $\mathfrak{A}_{\mathfrak{F}}$ be the reduced product over an ultrafilter $\mathfrak{F}$ of a family of models $\{\mathfrak{A}_i\}_{i \in I}$ satisfying a given subset of the set of clauses satisfied by reasoning models denoted by SC. Then, $\mathfrak{A}_{\mathfrak{F}}$ is itself a model satisfying SC.*

Proof. This is rather a corollary of the proof of the theorem. In fact, the track was to prove, for each clause, individually and indepentently from the other clauses, that if the models from the family satisfy that clause, then the reduced product satisfies that clause as well. □

Reduced Product's Theorem holds also for QmbC-models.

Corollary 2.4.7. *The reduced product $\mathfrak{A}_{\mathfrak{F}}$ of a family of QmbC-models $\{\mathfrak{A}_i\}_{i\in I}$ over an ultrafilter $\mathfrak{F}$ is itself a QmbC-model.*

Proof. It is enough to prove independently that reduced products preserve **vPred** and then use Corollary 2.4.6.

vPred $v(P(t_1,\ldots,t_n)[\vec{a}]) = 1$ iff $(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, for $P \in P_n$.

$\Rightarrow$ If $v(P(t_1,\ldots,t_n)[\vec{a}]) = 1$, then $\mathfrak{F}_{P(t_1,\ldots,t_n)[\vec{a}]} \in \mathfrak{F}$. For every $i \in \mathfrak{F}_{P(t_1,\ldots,t_n)[\vec{a}]}$, it holds that $\mathfrak{A}_i \vDash P(t_1,\ldots,t_n)[\vec{a_i}]$. By **vPred**, $(t_1^{\mathfrak{A}_i}[\vec{a_i}],\ldots,t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A_{P1}^{\mathfrak{A}_i}$. Therefore, $\mathfrak{F}_{P(t_1,\ldots,t_n)[\vec{a}]} \subset \mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])}$. As $\mathfrak{F}_{P(t_1,\ldots,t_n)[\vec{a}]} \in \mathfrak{F}$, it holds that $\mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$. Therefore, $(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$.
$\Leftarrow$ If $(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, then $\mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$. For each $i \in \mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])}$, $(t_1^{\mathfrak{A}_i}[\vec{a_i}],\ldots,t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A_{P1}^{\mathfrak{A}_i}$. By **vPred**, $\mathfrak{A}_i \vDash (P(t_1,\ldots,t_n)[\vec{a_i}])$. Hence, $\mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \subset \mathfrak{F}_{(P(t_1,\ldots,t_n)[\vec{a}])}$. As $\mathfrak{F}^{A_{P1}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}],\ldots,t_n^{\mathfrak{A}}[\vec{a}])} \in \mathfrak{F}$, it holds that $\mathfrak{F}_{(P(t_1,\ldots,t_n)[\vec{a}])} \in \mathfrak{F}$. Therefore, $v(P(t_1,\ldots,t_n)[\vec{a}]) = 1$. ∎

The next corollary allows the use of reduced products as an alternative tool for refining models and renders that concept a great bit more interesting.

Corollary 2.4.8. *Let the family of models $\{\mathfrak{A}_i\}_{i\in I}$ be the equivalence class of a given reasoning model $\mathfrak{A}_0$ by the relation of quasi-isomorphism. Then, the reduced product of $\{\mathfrak{A}_i\}_{i\in I}$ over an ultrafilter $\mathfrak{F}$ is a reasoning model $\mathfrak{A}_{\mathfrak{F}}$ that satisfies the preservation kernel of $\mathfrak{A}_0$. That is, if $\theta \in Pk(\mathfrak{A}_0)$, then $\mathfrak{A}_{\mathfrak{F}} \vDash \theta$.*

Proof. If $\theta \in Pk(\mathfrak{A}_0)$, then, for every $i \in I$, $\mathfrak{A}_i \vDash \theta$. Hence, $\mathfrak{F}_\theta = I \in \mathfrak{F}$. Therefore, $\mathfrak{A}_{\mathfrak{F}} \vDash \theta$. ∎

In classical model theory, the interpretation function of a model completely defines its valuation. The interpretation function of the reduced product of a family of models with respect to an ultrafilter is defined in the same fashion of Definition 2.4.1 and the definition of a valuation is not necessary, for the defined interpretation function determines a valuation. What Łoś Theorem states is that the valuation determined by the interpretation function is exactly the one defined in Defintion 2.4.2, that is: For each interpreted formula $\theta[\vec{a}]$, $v_{\mathfrak{F}}(\theta[\vec{a}]) = 1$ iff $\mathfrak{F}_{\theta[\vec{a}]} \in \mathfrak{F}$, where $\mathfrak{F}_{\theta[\vec{a}]} = \{i \in I | \mathfrak{A}_i \vDash \theta[\vec{a_i}]\}$.

On the other hand, the interpretation function of a reasoning model does not define a valuation. Rather, it enforces the truth values of the interpreted formulae that belong to its preservation kernel. Summing up, it determines part of a valuation. Reduced Product Theorem states that the valuation defined in Definition 2.4.2 coincides with the part of valuation determined by the reduced structure. This discussion leads to a reformulation of Łoś Theorem that is an immediate corollary of Reduced Product Theorem (Theorem 2.4.5).

Theorem 2.4.9 (Łoś Theorem for Reasoning Models)**.** *Let $\mathfrak{A}_{\mathfrak{F}}$ be the reduced structure of a family $\{\mathfrak{A}_i\}_{i\in I}$ of reasoning models with respect to an ultrafilter $\mathfrak{F}$. Then, if a model $\mathfrak{A}'_{\mathfrak{F}}$ based on $\mathfrak{A}_{\mathfrak{F}}$ is endowed with a valuation v and $\phi(\vec{a}) \in Pker(\mathfrak{A}_{\mathfrak{F}})$, then $\mathfrak{A}'_{\mathfrak{F}} \vDash \phi$ iff $\mathfrak{F}_{\phi(\vec{a})} = \{i|\mathfrak{A}_i \vDash \phi(a_i)\} \in \mathfrak{F}$.*

2.5 Refinement through an Untrafilter

The use of reduced products as a tool for gaining new safe knowledge from a family of states makes sense within the light of the concept of 'relevant set of states'.

In a physical investigation, for instance, it is necessary to control variables or circumstances in order to decide about the validity of some property. So let Φ be a set of facts that depend on a set Ψ of circumstances and $\{\mathfrak{A}_i\}_{i\in I}$ a family of quasi-isomorphic states that preserve some kernel of safe knowledge. The way to come to conclusions about the facts in Φ is to compare states that agree with respect to some subset of circumstances in Ψ. If $J \subset I$ is the set of indices of states that control a determined subset of circumstances from Ψ, then J is likely to be relevant with respect to Φ, in the sense that, if the states delimited by J agree that a specific fact holds, then it actually holds. If J' is the set of indices of states that control another subset of circumstances, then it is likely to be relevant with respect to Φ too. If the sets of indices that delimit relevant states are gathered up, then the ultrafilter generated by them is a relevant one and the reduced product must lead to safe knowledge about facts in Φ.

An ultrafilter $\mathfrak{F}$ over a set of indices I is a subset of the power set of I satisfying three conditions:

1. If $J' \subset J''$ and $J' \in \mathfrak{F}$, then $J'' \in \mathfrak{F}$;
2. If $J', J'' \in \mathfrak{F}$, then $(J' \cap J'') \in \mathfrak{F}$;
3. If $J' \in \mathfrak{F}$, then $(J')^C \notin \mathfrak{F}$.

Thinking in terms of sets of states of knowledge, condition 1 means that, if the agreement of a set of states if enough to guarantee some given fact, then the agreement of a larger set of states is enough to guarantee that fact. This sounds as a mandatory condition from any philosophical point of view.

Condition 3 means that, if a given set of states is relevant with respect to a given set of facts, then its complement is not. This is not a mandatory condition, but still a very defendable one. In fact, if a set of states is relevant on regard of some given aspect, there remains outside this set what does not talk about that aspect in a relevant way.

Condition 2 means that, if each of two sets of states are individually able to guarantee some given fact, then their intersection is able to guarantee that fact. This sounds a little bit strange, but not really odd. On the one hand, the intersection of two sets of states is smaller then both sets. So, it is necessary to concede that a smaller set provides the same guarantee as a bigger one. On the other hand, the intersection is a more powerful set, for it fulfills two conditions, that is to say, the one that enables the first set to guarantee knowledge and the one that enables the second set to guarantee the same knowledge.

Another point to be discussed is that it is not acceptable that an ultrafilter be relevant with respect to the whole universe of discourse. In order to rely on the use of ultrafilters as a sound tool for gaining knowledge, it is necessary to delimit the range of relevance of each ultrafilter. This necessity calls for a schema of refinement that shall be able to enrich each state of a family with the new safe knowledge gained by a filtration and maintain the non-refined knowledge available for a further refinement through another ultrafilter that shall talk in a relevant manner about other aspects of the theory. A concept that is suitable to perform the task will be presented. Firstly, it will be necessary to introduce some elements:

- Let $\{\mathfrak{A}_i\}_{i\in I}$ be a family of quasi-isomorphic models, Φ a set of interpreted formulae closed under all the connectives ($\vee, \wedge, \rightarrow, \neg$ and $\circ$) and $\mathfrak{F}$ an ultrafilter.
- Let $\mathfrak{A}_{\mathfrak{F}}$ be the reduced product of $\{\mathfrak{A}_i\}_{i\in I}$ over $\mathfrak{F}$.
- Let Ψ be the set for interpreted formulae recursively generated by $\Phi \cup Pk(\{\mathfrak{A}_i\}_{i\in I})$, that is, its closure under all the connectives ($\vee, \wedge, \rightarrow, \neg$ and $\circ$).
- Let Ω be the set of interpreted formulae that have no formula in Ψ as a subformula.
- If Λ and Λ' are two sets of interpreted formulae, then $\Gamma(\Lambda, \Lambda')$ is defined so that, for every $\theta \in IF(\mathfrak{A})$, $\theta \in \Gamma(\Lambda, \Lambda')$ iff:
  - $\theta \in \Lambda$ or
  - There is a $\phi \in \Lambda$ such that
    - $\theta = \neg\phi$ or

    * $\theta = \circ\phi$ or
  - There are $\phi, \psi \in (\Lambda \cup \Lambda')$, not both in Λ' such that
    * $\theta = \phi \vee \psi$ or
    * $\theta = \phi \wedge \psi$ or
    * $\theta = \phi \rightarrow \psi$ or

- Let Ψ_j be the set of interpreted formulae recursively defined as follows:
  - $\Psi_0 = \Psi$
  - $\Psi_{j+1} = \Gamma(\Psi_j, \Omega)$
- Let $\Psi_\omega = \cup_{j\in\mathbb{N}}\Psi_j$
- For each index $i \in I$, let $\bar{\mathfrak{A}}_i$ be the model whose interpretation function is the same as that of $\mathfrak{A}_i$ and whose valuation $\bar{v}_i$ is defined from the valuation v_i of $\mathfrak{A}_i$ so that, for each interpreted formula θ:
  - If $\theta \in \Omega$, let $\bar{v}_i(\theta) = v_i(\theta)$;
  - If $\theta \in \Psi$, let $\bar{v}_i(\theta) = v_{\mathfrak{F}}(\theta)$;
  - If $\theta \in (\Psi_\omega \backslash \Psi)$, then there is a $j \in \mathbb{N}$ such that $\theta \in \Psi_{j+1}$ and $\theta \notin \Psi_j$. In this case, let
    * $\bar{v}_i(\theta) = v_i(\theta)$, if $v_i(\theta)$ is compatible with $\bar{v}_i$ restricted to Ψ_i;
    * $\bar{v}_i(\theta) = v_{\mathfrak{F}}(\theta)$, otherwise.

Finally, the promised concept:

Definition 2.5.1 (Refinement through an Ultrafilter)**.** *The* refinement *of the family* $\{\mathfrak{A}_i\}_{i\in I}$ *of models* through the ultrafilter $\mathfrak{F}$ *restricted to the set of interpreted formulae* Ψ *is the family of models* $\{\bar{\mathfrak{A}}_i\}_{i\in I}$.

The refinement of a family of models through an ultrafilter restricted to its range of relevance preserves the kernel of safe knowledge, refines the knowledge within the range of relevance of that ultrafilter and changes nothing more than the strictly necessary out of that range in order to keep a family of reasoning models. The knowledge out of the range of relevance is not changed, which allows a further process of refinement through some other ultrafilter with a different range.

This section will be closed with a result involving quasi-isomorphisms:

Proposition 2.5.2. *Let $\{\mathfrak{A}_i\}_{i\in I}$ and $\{\mathfrak{B}_i\}_{i\in I}$ be two families of reasoning models with the same set of indices I such that, for every $i \in I$, there is a bijection from the domain A_i of $\mathfrak{A}_i$ to the domain B_i of $\mathfrak{B}_i$. Let $\mathfrak{A}_{\mathfrak{F}}$ and $\mathfrak{B}_{\mathfrak{F}}$ be the ultraproducts of these families over the same ultrafilter $\mathfrak{F}$ and let $J = \{j \in I | \mathfrak{A}_j \approxeq_{h_i} \mathfrak{B}_j\}$. Being so, if $J \in \mathfrak{F}$, then $\mathfrak{A}_{\mathfrak{F}} \approxeq_h \mathfrak{B}_{\mathfrak{F}}$, where h takes an element $\{a_i\}_{i\in I} \in A_{\mathfrak{F}}$ to the element $\{h_i(a_i)\}_{i\in I} \in B_{\mathfrak{F}}$.*

Proof. The function h is a bijection by design. The work to be done is to prove each clause in Definition 2.2.4 from 1 to 9. Only clause 1 will be proven, for the other clauses follow analogously. For that, let $\mathfrak{A}_{\mathfrak{F}}$, $\mathfrak{B}_{\mathfrak{F}}$ and J be as in the enunciation.
Let $P \in P_n$, $J' = \{i | (t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A_{P1}^{\mathfrak{A}_i}\}$ and $J'' = \{i | (t_1^{\mathfrak{B}_i}[h_i(\vec{a_i})], \ldots, t_n^{\mathfrak{B}_i}[h_i(\vec{a_i})]) \in B_{P1}^{\mathfrak{B}_i}\}$.
Suppose that $\mathfrak{F}^{A_{Pi}^{\mathfrak{A}}}_{(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}])} \in A_{P1}^{\mathfrak{F}}$. Thus, $J' \in \mathfrak{F}$. If $i \in J \cap J'$, then $(t_1^{\mathfrak{A}_i}[\vec{a_i}], \ldots, t_n^{\mathfrak{A}_i}[\vec{a_i}]) \in A_{P1}^{\mathfrak{A}_i}$, whence $(t_1^{\mathfrak{B}_i}[h_i(\vec{a_i})], \ldots, t_n^{\mathfrak{B}_i}[h_i(\vec{a_i})]) \in B_{P1}^{\mathfrak{B}_i}$, for $\mathfrak{A}_j \approxeq \mathfrak{B}_j$. Therefore, $J \cap J' \subseteq J''$. As $J, J' \in \mathfrak{F}$, $J \cap J' \in \mathfrak{F}$, which yields $J'' \in \mathfrak{F}$, which finally yields $\mathfrak{F}^{A_{Pi}^{\mathfrak{B}}}_{(t_1^{\mathfrak{A}}[h(\vec{a})], \ldots, t_n^{\mathfrak{A}}[h(\vec{a})])} \in B_{P1}^{\mathfrak{F}}$, as desired. The converse is symmetric. □

2.6 Elementary Extensions

The concept of refinement proposed in the previous sections preserves the domain of interpretation. Regarding models as states of knowledge, this means that refinements gain new knowledge about a fixed universe of objects. However, the development of science is made by discovering new objects as well. The goal of this section is to present the extension of a state as a state that deals with a larger universe of objects. Naturally, extensions are expected to preserve what is already known about the objects dealt with by the original state. The task in this section is to capture this idea by the concepts of extension and elementary extension.

Definition 2.6.1 (Extension of a Model)**.** *A reasoning structure $\mathfrak{B} = \langle B, I_{\mathfrak{B}} \rangle$ is an extension of a reasoning structure $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ if*

1. *$A \subseteq B$;*
2. *$I_{\mathfrak{A}}(f) = I_{\mathfrak{B}}(f)|_{A^n}$, for $f \in F_n$ and*
3. *$I_{\mathfrak{A}}(P) = I_{\mathfrak{B}}(P) \cap A^n$, for $P \in P_n$, which means*
   - (a) *$(a_1, \ldots, a_n) \in A_{Pi}^{\mathfrak{A}}$ iff $(a_1, \ldots, a_n) \in (A_{Pi}^{\mathfrak{B}} \cap A^n)$, for $1 \leq i \leq 3$.*
   - (b) *$(P, a_1, \ldots, a_n) \in A_{Pi}^{\mathfrak{A}}$ iff $(P, a_1, \ldots, a_n) \in A_{Pi}^{\mathfrak{B}}$ and $(a_1, \ldots, a_n) \in A^n$, for $4 \leq i \leq 9$.*

A model $\mathfrak{B} = \langle B, I_{\mathfrak{B}}, v_{\mathfrak{B}} \rangle$ *is an* extension *of a given model* $\mathfrak{A} = \langle A, I_{\mathfrak{A}}, v_{\mathfrak{A}} \rangle$ *if* $\mathfrak{B} = \langle B, I_{\mathfrak{B}} \rangle$ *is an extension of* $\mathfrak{A} = \langle A, I_{\mathfrak{A}} \rangle$ *and* $v_A = v_B|_{IF(\mathfrak{A})}$.

It is quite natural to demand that extensions preserve knowledge not only about how the previous objects behave but also about the way in which those objects interact. In fact, they do preserve. So, in what sense are elementary extensions different? What kind of knowledge about previous objects may not be preserved by ordinary extensions? The only answer that makes sense is that ordinary extensions may not preserve the truth about the existence of objects that interact with the previous objects in some way. It may be the case that, among the universe of objects dealt with, there is no object that interacts with some given set of objects in some way but there is such an object in a larger universe of objects. In this line, it is expected that elementary extensions be those extensions that do not bring objects that interact with the previous ones in different ways. Refrasing the idea, if a new object interacts with a set of old ones in some way, then there is an old object that interacts with that set of objects in that same way.

The goal to be pursued from now on is to prove that things are as they are expected to be. This is the content of the Theorem of Elementary Extensions, which is the core result of this section. Before reaching the desired result, some work is required.

Lemma 2.6.2. *Let* $\mathfrak{B}$ *be an elementary extension of* $\mathfrak{A}$ *(*$v_{\mathfrak{A}} = v_{\mathfrak{B}}|_{IF(\mathfrak{A})}$*). Then, for every formula* $\theta(x, \vec{x})$ *and for every sequence* $\vec{a} \in A^n$ *(where* n *is the length of* $\vec{x}$*), it holds that, if* $v_{\mathfrak{B}}(\exists x \theta(x, \vec{a})) = 1$*, then there is* $a \in A$ *such that* $v_{\mathfrak{A}}(\theta(a, \vec{a})) = v_{\mathfrak{B}}(\theta(a, \vec{a})) = 1$.

Proof. Let $\vec{a} \in A^n$ be an arbitrary sequence. Then, $v_{\mathfrak{B}}(\exists x \theta(x, \vec{a})) = 1$ iff $v_{\mathfrak{A}}(\exists x \theta(x, \vec{a})) = 1$ iff $v_{\mathfrak{A}}(\theta(a, \vec{a})) = 1$ for some $a \in A$ iff $v_{\mathfrak{B}}(\theta(a, \vec{a})) = 1$ for such a. □

A list of definitions will be given in order to reach the definition of a sequence of sets S_n, which will be used in the proof of the Theorem of Elementary Extensions. So, let $A \subset B$. Then,

- $\bar{A}$ is the set of sequences $\vec{d} = (d_i)_{1 \leq i \leq n}$ of whatever length n where, for every $1 \leq i \leq n$, $d_i \in A$.

- $\bar{AB}$ is the set of sequences $\vec{d} = (d_i)_{1 \leq i \leq n}$ of whatever length n where, for some $1 \leq i \leq n$ (at least one index), $d_i \in B \backslash A$;

- $\bar{\bar{AB}}$ is the set of sequences $\vec{d} = (d_i)_{1 \leq i \leq n}$ of whatever length n where, for some $1 \leq i \leq n$ (at least one indice), $d_i \in A$ and, for some $1 \leq i \leq n$ (at least one index), $d_i \in B \backslash A$;

- F^0_{AB} (F^0_A) is the set of formulae with complexity 0 of the form $\theta(\vec{x}, \vec{c_d})$ with $\vec{d} \in \bar{AB}$ ($\vec{d} \in \bar{A}$). That is, basic formulae of the form $\theta(\vec{x}, \vec{y})$, where each variable y_i in $\vec{y}$ actually appears free in θ and is substituted by the constant $c_{d_i} \in \Sigma_B$;
- F_{AB} (F_A) is the set that is recursively generated from F^0_{AB} (F^0_A) by using the connectives from the signature in the canonical way;
- Given a set of formulae S, S' is the set of formulae formed from the formulae in S by using one connective.
- A sequence of sets of formulae in L_{Σ_B} is recursively defined as follows:
  - $F_0 = F_{AB} \cup F_A$;
  - $F_{n+1} = F_n \cup F'_n$.
- For each n, S^*_n is the subset of F_n formed by the closed formulae in F_n.
- For each formula $\theta \in S^*_n$, $\theta' = \theta(\vec{y})[\vec{d}]$ is the interpreted formula in L_Σ obtained by substituting each new constant c_b in Σ_B by a variable not occurring in θ and then interpreting this variable by b.
- For each n, $S_n = \{\theta' | \theta \in S^*_n\}$.

Reached the definition of $S_n = \{\theta' | \theta \in S^*_n\}$, some extra definitions will be presented:

- $IF|_{AB}$ is the set of interpreted formulae formed from F_{AB} in the same way as S_n is formed from F_n.
- $\sim \alpha$ is an abbreviation for $\alpha \to (\alpha \wedge (\neg\alpha \wedge \circ\alpha))$.
- An interpreted subformula of an interpreted formula $\theta(\vec{y})[\vec{d}]$ is an interpreted formula $\phi(\vec{x}, \vec{y})[\vec{e}, \vec{d}]$, where $\phi(\vec{x}, \vec{y})$ is a subformula of $\theta(\vec{y})$. When the context is clear, interpreted formulae and interpreted subformulae may be referred to just as formulae and subformulae, respectively.
- A partial valuation for a set of interpreted formulae S is a function $v : S \to \{0, 1\}$. It will be said that a partial valuation v is coherent for a set S or that a set S is coherent for a valuation v or that a valuation v with domain S is a reasoning partial valuation if no reasoning clause is violated by v for the interpreted formulae in S.

Some basic formulae and some formulae of the form $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \to \beta$ and $\circ\alpha$ are ruled by the Basic Control Clauses. The other ones are recursively defined. In this sense, the other clauses can be dubbed ‘recursive clauses’. The following lemma is straightforward and refrases this idea in more precise words.

Lemma 2.6.3. *Let S be a set of interpreted formulae, v a partial valuation that is coherent for S and θ an interpreted formula not contained in S. Suppose also that S contains every interpreted formula whose validity is ruled by the Basic Control Clauses. If S contains every interpreted subformula of θ and θ is not a subformula of any subformula in S that is not of the form $\circ\circ\alpha$ or $\circ\neg\alpha$, then there is a truth value $x \in \{0,1\}$ such that the extension $v' : S \cup \{\theta\} \to \{0,1\}$ of v defined so that, for every $\phi \in S \cup \{\theta\}$, $v'(\phi) = v(\phi)$ if $\phi \in S$ and $v(\theta) = x$ is a reasoning partial valuation.*

Finally, the core result of this section:

Theorem 2.6.4 (Elementary Extension Theorem)**.** *Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}}, v_{\mathfrak{A}}\rangle$ be a reasoning model, B an extension of the domain A ($A \subset B$) and $v_{\bar{A}B} : IF|_{\bar{A}B}(L_\Sigma) \to \{0,1\}$ a function that valuates formulae interpreted by sequences in $\bar{A}B$. Then, there exists a valuation $v_B : B \to \{0,1\}$ such that, for every sequence $\vec{a} \in A^n$ and for every formula $\theta(\vec{x})$, $v_B(\theta(\vec{a})) = v_{\mathfrak{A}}(\theta(\vec{a}))$ and, for every sequence $\vec{c} \in \bar{A}B$ and for every formula $\theta(\vec{x})$, $v_B(\theta(\vec{c})) = v_{\bar{A}B}(\theta(\vec{c}))$ iff, for every sequence $(b, \vec{a}) \in (B \times \bar{A})$ and for every formula $\theta(x, \vec{x})$, if $v_{\bar{A}B}(b, \vec{a}) = 1$, then there is $a \in A$ such that $v_{\mathfrak{A}}(a, \vec{a}) = 1$.*

Proof. ($\Rightarrow$) It follows from Lemma 2.6.2. Suppose there is a valuation v_B as described in the enunciation. If $v_{\bar{A}B}(\theta(b, \vec{a})) = 1$, then $v_B(\theta(b, \vec{a})) = 1$. As v_B is a reasoning valuation, $v_B(\exists x\theta(x, \vec{a})) = 1$. By Lemma 2.6.2, there is $a \in A$ such that $v_B(\theta(a, \vec{a})) = 1$, which implies that $v_{\mathfrak{A}}(\theta(a, \vec{a})) = 1$, as desired.
($\Leftarrow$) Suppose that, for every sequence $(b, \vec{a}) \in (B \times A^n)$ and for every formula $\theta(x, \vec{x})$, if $v_{\bar{A}B}(b, \vec{a}) = 1$, then there is $a \in A$ such that $v_{\mathfrak{A}}(a, \vec{a}) = 1$. It must be proven that there exists a valuation as described in the enunciation.
A sequence of valuations v_n will be recursively constructed. Each v_n has domain S_n.

- v_0
  - $v_0(\theta(\vec{d})) = v_{\mathfrak{A}}(\theta(\vec{d}))$, if $\theta(\vec{d}) \in S_0$ and $\vec{d} \in \bar{A}$;
  - $v_0(\theta(\vec{d})) = v_{\bar{A}B}(\theta(\vec{d}))$, if $\theta(\vec{d}) \in S_0$ and $\vec{d} \in \bar{A}B$.

  $v_0(\theta(\vec{d}))$ is not defined, otherwise.

- v_{n+1}
  - $v_{n+1}(\theta(\vec{d})) = v_n(\theta(\vec{d}))$, if $\theta(\vec{d}) \in S_n$;
  - Let $(\theta_i)_{i\in\mathbb{N}}$ be an enumeration of the interpreted formulae in $S_{n+1} \backslash S_n$. Then, $v_{n+1}(\theta_i)$ assumes some value that coheres with the truth values settled for S_n (if $i = 0$) or for $S_n \cup \{\theta_j\}_{j<i}$ (if $i > 0$).

- $v_{n+1}(\theta(\vec{d}))$ is not defined if $\theta \notin S_{n+1}$.

Each partial valuation v_n is coherent. The proof of this fact will be performed by induction.

First part ($n = 0$): Let S_0^m be the set of interpreted formulae with complexity m from S_0. This first part will be proven by induction on m. Observe that if an interpreted formula belongs to S_0, then all of its interpreted subformulae belong to S_0 as well.

It is immediate that v_0 is coherent for sets of formulae involving no quantifiers, for the domains of $v_{\mathfrak{A}}$ and $v_{\bar{AB}}$ (which are reasoning valuations) are disjoint and each one is closed for interpreted subformulae. If all the formulae in a given set have complexity 0, then all of them involve no connectives. In particular, they involve no quantifiers. Therefore, the result holds for S_0^0.

Suppose the result holds for S_0^m. To prove that S_0^{m+1} is coherent is the same as to prove that no reasoning clause is violated.

The fact follows easily for the clauses that do not involve quantifiers. In fact, if a clause that does not involve quantifiers is violated by a given set of formulae F, then there is a subset $F' \subset F$ with two or three formulae (according to the clause in question) that violates that clause. This is so because the connectives have arity 1 or 2, whence the recursive definition of valuation for formulae formed by connectives other than the quantifiers refer to at most two formulae already valuated and the one to be valuated. Moreover, one of the formulae (the one to be valuated) is of higher complexity than the other(s), by design of the clauses, which means that there can be at most one formula with complexity $m + 1$ in F'. Also, there must be a formula $\theta[\vec{d}]$ with complexity $m + 1$, for otherwise F' would be a subset of S_0^m, which is coherent, by the inductive hypothesis. Summing up, F' has exactly one formula of complexity $m + 1$. Now suppose, for the sake of contradiction, that there is such a set violating some clause not involving quantifiers and let $\theta[\vec{d}]$ be the only formula of complexity $m + 1$. The whole of possibilities for $\theta[\vec{d}]$ are analyzed below:

If $\theta[\vec{d}]$ is formed by a connective $\# \in \{\vee, \wedge, \rightarrow\}$, then $\theta[\vec{d}] = \phi[\vec{k}]\#\psi[\vec{l}]$ for some $\phi[\vec{k}]$ and $\psi[\vec{l}]$, were $\vec{k}$ and $\vec{l}$ are segments of $\vec{d}$. If $\vec{k}, \vec{l} \in \bar{A}$, then all the formulae in F' are valuated by $v_{\mathfrak{A}}$, which is a contradiction against the fact that $v_{\mathfrak{A}}$ is coherent. Likewise, if $\vec{k}, \vec{l} \in \bar{AB}$, then all the formulae in F' are valuated by $v_{\bar{AB}}$, which is a contradiction against the fact that $v_{\bar{AB}}$ is coherent. If $\vec{k} \in \bar{A}$ and $\vec{l} \in \bar{AB}$ (or $\vec{k} \in \bar{AB}$ and $\vec{l} \in \bar{A}$), then $\theta[\vec{d}] \notin S_0$, which contradicts the assumption that $F' \subset S_0^{m+1}$.

If $\theta[\vec{d}] = \#\phi[\vec{d}]$ for $\#\{\neg, \circ\}$, then all the formulae in F' are valuated by $v_{\mathfrak{A}}$, if $\vec{d} \in \bar{A}$ or by $v_{\bar{AB}}$, if $\vec{d} \in \bar{AB}$. In any case, there is a contradiction against the fact that $v_{\mathfrak{A}}$ and $v_{\bar{AB}}$ are coherent.

For the clauses involving quantifiers, things are a little bit delicate, for the truth value of an interpreted quantified formula may depend on the valuations of interpreted formulae valuated both by $v_{\mathfrak{A}}$ and $v_{\bar{AB}}$. Here, the condition imposed to $v_{\bar{AB}}$ and $v_{\mathfrak{A}}$ in the enunciation plays its role. The clauses involving quantifiers will be treated separately.

It is proven in [15] and in [14] that, for every mbC-model $\mathfrak{A}$ and for every formula α, $\mathfrak{A} \vDash \sim \alpha$ iff $\mathfrak{A} \nvDash \alpha$. The result holds for reasoning models as well.

vEx Let $\exists x\phi(x)[\vec{d}] \in S_0^{m+1}$. If $v_0(\exists x\phi(x)[\vec{d}]) = 1$, then either $v_{\mathfrak{A}}(\exists x\phi(x)[\vec{d}]) = 1$ or $v_{\bar{AB}}(\exists x\phi(x)[\vec{d}]) = 1$. In either case, there is a $b \in B$ such that $v_{\mathfrak{A}}(\phi[b, \vec{d}]) = 1$ (it may be the case that $b \in A$, despite the notation) or $v_{\bar{AB}}(\phi[b, \vec{d}]) = 1$, that is, such that $v_0(\phi[b, \vec{d}]) = 1$.
Conversely, if there is a $b \in B$ such that $v_0(\phi[b, \vec{d}])$, then there is a $b \in B$ such that $v_{\mathfrak{A}}(\phi[b, \vec{d}]) = 1$ or $v_{\bar{AB}}(\phi[b, \vec{d}]) = 1$. The possible cases for b and $\vec{d}$ will be analyzed separately:

- If $b \in A$ and $\vec{d} \in \bar{A}$, then $(b, \vec{d}) \in \bar{A}$. So, $v_{\mathfrak{A}}(\phi(b, \vec{d})) = 1$. Hence, $v_{\mathfrak{A}}(\exists x\phi(x)[\vec{d}]) = 1$, for $v_{\mathfrak{A}}$ is a reasoning valuation.
- If $\vec{d} \in \bar{AB}$, then $(b, \vec{d}) \in \bar{AB}$ (whether $b \in A$ or $b \in B \backslash A$). So, $v_{\bar{AB}}\phi(b, \vec{d}) = 1$. Hence, $v_{\bar{AB}}(\exists x\phi(x)[\vec{d}]) = 1$, for $v_{\bar{AB}}$ is a reasoning valuation.
- If $b \in B \backslash A$ and $\vec{d} \in A^k$, then $v_{\bar{AB}}(\phi[b, \vec{d}]) = 1$ and, by hypothesis, there is $a \in A$ such that $v_{\mathfrak{A}}(\phi(a, \vec{d})) = 1$. Hence, $v_{\mathfrak{A}}(\exists x\phi(x)[\vec{d}]) = 1$, for $v_{\mathfrak{A}}$ is a reasoning valuation.

In any case, $v_0(\exists x\phi(x)[\vec{d}]) = 1$.

vUni Let $\forall x\phi(x)[\vec{d}] \in S_0^{m+1}$. If $v_0(\forall x\phi(x)[\vec{d}]) = 1$, then either $v_{\mathfrak{A}}(\forall x\phi(x)[\vec{d}]) = 1$ or $v_{\bar{AB}}(\forall x\phi(x)[\vec{d}]) = 1$. In the first case, $\vec{d} \in \bar{A}$ and, for all $a \in A$, $v_0(\phi[a, \vec{d}]) = v_{\mathfrak{A}}(\phi[a, \vec{d}]) = 1$. For $b \in B \backslash A$, it holds that $v_0(\phi[b, \vec{d}]) = 1$ as well, for otherwise $v_{\bar{AB}}(\phi[b, \vec{d}]) = 0$, which would imply that $v_{\bar{AB}}(\sim \phi[b, \vec{d}]) = 1$. By hypothesis, there would be $a \in A$ such that $v_{\mathfrak{A}}(\sim \phi[a, \vec{d}]) = 1$, which would imply that $v_{\mathfrak{A}}(\phi[a, \vec{d}]) = 0$. Contradiction! Therefore, for every $b \in B$, $v_0(\phi[b, \vec{d}]) = 1$.
In the second case, for all $b \in B$ (whether $b \in A$ of $b \in B \backslash A$), $(b, \vec{d}) \in \bar{AB}$. Hence, for all $b \in B$, $v_0(\phi[b, \vec{d}]) = v_{\bar{AB}}(\phi[b, \vec{d}]) = 1$. Hence, $v_0(\phi[b, \vec{d}]) = 1$, as desired.
Conversely, suppose that $v_0(\phi[b, \vec{d}]) = 1$ for every $b \in B$. If $\vec{d} \in \bar{A}$, then, for every $a \in A$, $v_{\mathfrak{A}}(\phi[a, \vec{d}]) = v_0(\phi[a, \vec{d}]) = 1$. Therefore, $v_{\mathfrak{A}}(\forall x\phi(x)[\vec{d}]) = 1$. If $\vec{d} \in \bar{AB}$, then, for every $b \in B$, $v_{\bar{AB}}(\phi[b, \vec{d}]) = v_0(\phi[b, \vec{d}]) = 1$.Hence, $v_{\bar{AB}}(\forall x\phi(x)[\vec{d}]) = 1$. In any case, $v_0(\forall x\phi(x)[\vec{d}]) = 1$, as desired.

vPropUni and vPropEx Suppose that $v_0(\circ\phi[b,\vec{d}]) = 1$ for every $b \in B$. If $\vec{d} \in \bar{A}$, then, for every $a \in A$, $v_{\mathfrak{A}}(\circ\phi[a,\vec{d}]) = v_0(\circ\phi[a,\vec{d}]) = 1$. Therefore (as $v_{\mathfrak{A}}$ is a reasoning valuation and **vPropUni** and **vPropEx** hold in reasoning valuations), $v_{\mathfrak{A}}(\circ(\forall x\phi(x)[\vec{d}])) = 1$ and $v_{\mathfrak{A}}(\circ(\exists x\phi(x)[\vec{d}])) = 1$. Analogously, if $\vec{d} \in \bar{AB}$, then $v_{\bar{AB}}(\circ(\forall x\phi(x)[\vec{d}])) = 1$ and $v_{\bar{AB}}(\circ(\exists x\phi(x)[\vec{d}])) = 1$. In any case, $v_0(\circ(\forall x\phi(x)[\vec{d}])) = 1$ and $v_0(\circ(\exists x\phi(x)[\vec{d}])) = 1$, as desired.

vPropEx' Suppose there are a $b \in B$ and a $\vec{d} \in \bar{AB}$ such that $v_0(\phi[b,\vec{d}]) = v_0(\circ(\phi[b,\vec{d}])) = 1$. If $\vec{d} \in \bar{AB}$, then $v_{\bar{AB}}(\phi[b,\vec{d}]) = v_{\bar{AB}}(\circ(\phi[b,\vec{d}])) = 1$. Therefore, $v_{\bar{AB}}(\circ(\exists x\phi(x,\vec{x}))[\vec{d}]) = 1$. If $\vec{d} \in \bar{A}$ and $b \in A$, then $v_{\mathfrak{A}}(\phi[b,\vec{d}]) = v_{\mathfrak{A}}(\circ(\phi[b,\vec{d}])) = 1$. Therefore, $v_{\mathfrak{A}}(\circ(\exists x\phi(x,\vec{x}))[\vec{d}]) = 1$. If $\vec{d} \in \bar{A}$ and $b \in B\backslash A$, then $v_{\bar{AB}}(\phi[b,\vec{d}]) = v_{\bar{AB}}(\circ(\phi[b,\vec{d}])) = 1$. By hypothesis, there is $a \in A$ such that $v_{\mathfrak{A}}(\phi[a,\vec{d}]) = v_{\mathfrak{A}}(\circ(\phi[a,\vec{d}])) = 1$. Therefore, $v_{\mathfrak{A}}(\circ(\exists x\phi(x,\vec{x}))[\vec{d}]) = 1$. In any case, $v_0(\circ(\exists x\phi(x,\vec{x}))[\vec{d}]) = 1$, as desired.

Second part (Suppose the result holds for S_n and v_n): To prove that v_{n+1} is coherent is the same as to prove that it respects each of the reasoning clauses. The recursive definition of v_{n+1} states that each interpreted formula θ_i or σ_i is valuated so that the new set obtained by its addition is coherent. Being so, v_{n+1} is coherent for every set $S_n \cup \{\theta_j\}_{j\leq i}$, which implies that it is coherent for $S_{n+1} = S_n \cup \{\theta_j\}_{j\in\mathbb{N}}$. In fact, if v_{n+1} were not coherent for $S_n \cup \{\theta_j\}_{j\in\mathbb{N}}$, there would be some set of interpreted formulae violating some clause. As each clause involves finitely many formulae, there would be a finite set of formulae violating that clause. But every finite set is contained, for some i, in $S_n \cup \{\theta_j\}_{j\leq i}$. So, v_{n+1} would not be coherent for this set, leading to a contradiction.

There is a point, however: In the definition of v_{n+1}, it was taken for granted the existence of a coherent truth value for each θ_i with respect to the valuations already defined. This assumption must be proven. By Lemma 2.6.3, it is enough to prove that, for each $i \in \mathbb{N}$, v_{n+1} respects the Basic Control Clauses in $S_n \cup \{\theta_j\}_{j\leq i}$, that θ_{i+1} is not a subformula of any formula in that set, and that every subformula of θ_{i+1} belongs to the referred set.

It is clear that v_{n+1} respects the Basic Control Clauses, for all the interpreted formulae whose valuation is ruled by thoses clauses belong to S_0 and v_{n+1} is an extension of v_0. As the construction of the sequence $\{S_i\}$ is recursive, the subformulae of θ_{i+1} must be constructed before θ_{i+1} and the formulae that have θ_{i+1} as a subformula must be constructed after θ_{i+1}. In other words, all the subformulae of θ_{i+1} belong to S_n and all the formulae that have θ_{i+1} as a subformula are constructed only from S_{n+2} on and, therefore, do not belong to $S_n \cup \{\theta_j\}_{j\leq i}$.

At first sight, it may seem unnecessary to enumerate the formulae in $S_{n+1}\backslash S_n$ and then attribute valuation to one to one sequentially. But in fact it is. This is because **vCon**, **vVar**, **sNeg** and **sCon** may involve more then one formula in $\{\theta_j\}_{j\leq i}$ at one time.

The set $\cup_{i\in\mathbb{N}}S_i$ comprises the formulae in the language interpreted by the elements of B. The valuation v_B is finally defined so that $v_B(\theta[\vec{b}]) = v_n(\theta[\vec{b}])$, were n is the first index such that v_n valuates $\theta[\vec{b}]$.

The coherence of v_B follows from the fact that the coherence of a set of formulae with respect to a given clause involves a set of formulae that belong to S_n for some n and each S_n is coherent for every n. □

In [13], some very important results for classical logic are extended to mbC. In the next section, the extension of some of those results to reasoning logic is briefly discussed. Compacity and the fact that every consistent set of formulae have a model are among those results and, as a consequence of the validity of these results, the following proposition holds and its proof is identical to the proof for classical logic.

Proposition 2.6.5. *If $\mathfrak{F}$ is a family of elementary extensions of a model $\mathfrak{A}$, then there is a model that is an elementary extension of each model from $\mathfrak{F}$.*

The proposition above is not a complex result nor a central one, but it has an interesting meaning: Research can be made independently by several researchers, who may discover independent groups of objects, which can be lumped together in a single state of knowledge.

Obs.: In some sense, it would be appropriate to figure the refinements proposed in Sections 3 and Section 4 as horizontal gaining of knowledge and extensions as vertical gaining.

At this point, the task of giving an account of models as refinable states of knowledge is satisfactorily carried out. There remains to the final section the task of finding an axiomatization for the reasoning system developed along the previous sections.

2.7 Axiomatization

A QmbC-valuation is a one that satisfies exactly the Classical Behavior Clauses, the QmbC Clauses and **vPred** ($v(P(t_1\ldots,t_n)) = 1$ iff $(t_1^{\mathfrak{A}},\ldots,t_n^{\mathfrak{A}}) \in A_{P1}^{\mathfrak{A}}$, for $P \in P_n$). That is, reasoning valuations differ from QmbC-valuations for satisfying the Basic Control Clauses instead of **vPred** and also for satisfying the Propagation Clauses. Of course, reasoning models are endowed with the necessary apparatus to give sense to the Basic Control Clauses. A convenient kind of valuation to consider is

QmbC plus Propagation Clauses (for simplicity,PQmbC). Those valuations differ from reasoning valuations for satisfying **vPred** instead of the Basic Control Clauses and differ from QmbC-valuations for satisfying the Propagation Clauses as well as all the other QmbC Clauses.

In the search for an axiomatization to reasoning logic, it is convenient, before going on, to compare reasoning logic to PQmbC in what concerns semantical consequence.

Let $\mathfrak{A}$ be a reasoning model over a language L_Σ with domain of interpretation A, interpretation function $I_\mathfrak{A}$ and valuation function $v_\mathfrak{A}$. Let $I_\mathfrak{B}$ be a PQmbC interpretation function over the same language and with the same domain of interpretation such that

1. $I_\mathfrak{B}(c) = I_\mathfrak{A}(c)$, for every constant c;
2. $I_\mathfrak{B}(f) = I_\mathfrak{A}(f)$, for every function symbol f;
3. $(a_1, \ldots, a_n) \in I_\mathfrak{B}(P)$ iff $v_\mathfrak{A}(a_1, \ldots, a_n) = 1$, for every $P \in P_n$.

It is clear that $I_\mathfrak{A}$ and $I_\mathfrak{B}$ agree in the interpretation of each term. Also, for whatever valuation v based on $I_\mathfrak{B}$ and satisfying **vPred**, for every $P \in P_n$, and for every sequence of terms $\tau_1, \ldots, \tau_n$, it holds that
$v(P(\tau_1, \ldots, \tau_n)) = 1$ iff $(\tau_1^\mathfrak{B}, \ldots, \tau_n^\mathfrak{B}) \in I_\mathfrak{B}(P)$ iff $(\tau_1^\mathfrak{A}, \ldots, \tau_n^\mathfrak{A}) \in I_\mathfrak{B}(P)$ (for $I_\mathfrak{A}$ and $I_\mathfrak{A}$ agree in interpretations) iff $v_\mathfrak{A}(P(\tau_1, \ldots, \tau_n)) = 1$.

Conversely, if a valuation v agrees with $v_\mathfrak{A}$ about every basic statement, then v satisfies **vPred** according to $I_\mathfrak{B}$. In fact:
$v(P(\tau_1, \ldots, \tau_n)) = 1$ iff $v_\mathfrak{A}(P(\tau_1, \ldots, \tau_n)) = 1$ iff $(\tau_1^\mathfrak{A}, \ldots, \tau_n^\mathfrak{A}) \in I_\mathfrak{B}(P)$ iff $(\tau_1^\mathfrak{B}, \ldots, \tau_n^\mathfrak{B}) \in I_\mathfrak{B}(P)$.

Now, let $v_\mathfrak{B} = v_\mathfrak{A}$. As just proven, $v_\mathfrak{B}$ satisfies **vPred**. It also satisfies the Calssical Behavior Clauses, the mbC-Clauses and the Propagation Clauses, for $v_\mathfrak{A}$ satisfies those clauses. Summing up, $v_\mathfrak{B}$ is a PQmbC-valuation with respect to $I_\mathfrak{B}$.

Therefore, the model $\mathfrak{B}$ with domain of interpretation $B = A$, interpretation function $I_\mathfrak{B}$ as defined above and valuation function $v_\mathfrak{B} = v_\mathfrak{A}$ is a PQmbC model such that, for every formula θ, $\mathfrak{B} \vDash \theta$ iff $\mathfrak{A} \vDash \theta$. In other words, for each reasoning model over a language Σ, that is a PQmbC model over the same language that satisfies the same statements.

Conversely, let $\mathfrak{B}$ be a PQmbC model over a language L_Σ with domain of interpretation B, interpretation function $I_\mathfrak{B}$ and valuation function $v_\mathfrak{B}$. Let $I_\mathfrak{A}$ be a reasoning interpretation function over the same language and with the same domain of interpretation such that

1. $I_\mathfrak{A}(c) = I_\mathfrak{B}(c)$, for every constant c;
2. $I_\mathfrak{A}(f) = I_\mathfrak{B}(f)$, for every function symbol f;

3. For every predicate symbol P, $I_{\mathfrak{A}}(P)$ gives a sequence of void sets, that is, $A^{\mathfrak{A}}_{Pi} = \emptyset$, for $1 \leq i \leq 9$.

Again, it is clear that $I_{\mathfrak{B}}$ and $I_{\mathfrak{A}}$ agree about the interpretation of each term. Also, any valuation v (in particular, $v_{\mathfrak{B}}$) satisfies the Basic Control Clauses according to $I_{\mathfrak{A}}$ by vacuity. Being a PQmbC-valuation, $v_{\mathfrak{B}}$ also satisfies the Classical Behavior Clauses, the mbC-Clauses and the Propagation Clauses. Summing up, $v_{\mathfrak{B}}$ is also a reasoning valuation.

Therefore, the model $\mathfrak{A}$ with domain of interpretation $A = B$, interpretation function $I_{\mathfrak{A}}$ as defined above and valuation function $v_{\mathfrak{A}} = v_{\mathfrak{B}}$ is a reasoning model such that, for every formula θ, $\mathfrak{A} \vDash \theta$ iff $\mathfrak{B} \vDash \theta$. In other words, for each PQmbC model over a language Σ, there is a reasoning model over the same language that satisfies the same statements.

Joining the two parts, for every set of statements Θ, there is a reasoning model satisfying Θ iff there is a PQmbC model satisfying it too. Also, every reasoning model satisfies Θ iff every PQmbC model satisfies it too. In fact, if there is a PQmbC model that does not satisfy θ, then there is a reasoning model that does not satisfy it either.

This means that reasoning logic and PQmbC logic have the same semantical consequence relation. Consequently, one is sound or complete with respect to a given axiomatization iff the other one is respectively sound or complete with respect to the same axiomatization.

Profiting from the conclusion above, an axiomatization for reasoning logic will be pursued via PQmbC logic. This recourse facilitates so much the task, for the work has been carefully done for QmbC logic in [13]. In [13], QmbC has been proven to be sound and complete with respect to the following axiomatization:

Definition 2.7.1 (Hilbert Caulculus for QmbC)**.** *The list of axiom schemata and inference rules that below constitute the Hilbert Caulculus for QmbC.*

Axiom Schemata

Ax1 $\alpha \to (\beta \to \alpha)$

Ax2 $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

Ax3 $\alpha \to (\beta \to (\alpha \wedge \beta))$

Ax4 $(\alpha \wedge \beta) \to \alpha$

Ax5 $(\alpha \wedge \beta) \to \beta$

Ax6 $\alpha \to (\alpha \vee \beta)$

Ax7 $\beta \to (\alpha \vee \beta)$

Ax8 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Ax9 $\alpha \vee (\alpha \to \beta)$

Ax10 $\alpha \vee \neg\alpha$

Ax11 $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Ax12 $\phi[x/t] \to \exists x\phi$, *if* t *is a term that is free for* x *in* ϕ

Ax13 $\forall x\phi \to \phi[x/t]$, *if* t *is a term that is free for* x *in* ϕ

Ax14 $\alpha \to \beta$, *whenever* α *is a variant of* β

Rules of Inference

MP $\alpha, \alpha \to \beta/\beta$

$\forall$-In $\alpha \to \beta/\alpha \to \forall x\beta$, *if* x *is not free in* α

$\exists$-In $\alpha \to \beta/\exists x\alpha \to \beta$, *if* x *is not free in* β

It will be proven that PQmbC can be syntactically determined by the same axiom schamata and rules of inference plus some suitable axiom schemata that are able to rule propagation of consistency.

Definition 2.7.2 (Hilbert Calculus for PQmbC)**.** *PQmbC is syntactically determined by the same axiom schemata and rules of inference as QmbC plus the following axiom schemata:*

Ax15 $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$

Ax16 $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$

Ax17 $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$

Ax18 $\circ\alpha \to \circ\neg\alpha$

Ax19 $\circ(\circ\alpha)$

Ax20 $\forall x \circ \alpha \to \circ\forall x\alpha$

Ax21 $\forall x \circ \alpha \to \circ\exists x\alpha$

Ax22 $\exists x(\alpha \wedge \circ\alpha) \to \circ\exists x\alpha$

The next step is to prove that PQmbC is sound and complete with respect to the axiomatization described in Definition 2.7.2.

Proposition 2.7.3 (Soundness of PQmbC)**.** *PQmbC is sound with respect to the axiomatization described in 1.4.2 plus the **Propagation Clauses***

Proof. A PQmbC-valuation satisfies all clauses satisfied by QmbC plus something else. So, being a PQmbC model is more restrictive than being a QmbC model, which means that the class of PQmbC models is contained in the class of QmbC models. Therefore, every axiom that yields a tautology in the latter also yields a tautology in the former. Moreover, every rule of inference that preserves tautologies in the latter also preserves tautologies in the former. As it has been proven in [13] that QmbC is sound with respect to the axiomatization in Definition 2.7.1, all that is left to be proven is that axioms **Ax15** to **Ax22** yield tautologies in the class of PQmbC models.

Ax15, Ax16 and Ax17
- If $v(\circ\alpha) = 0$ or $v(\circ\beta) = 0$, then $v(\circ\alpha \wedge \circ\beta) = 0$, by **vAnd**. Hence, $v(Ax15) = v(Ax16) = v(Ax17) = 1$, by **vImp**.
- If both $v(\circ\alpha) = 1$ and $v(\circ\beta) = 1$, then
  - $v(\circ(\alpha \vee \beta)) = 1$, by **vPropOr**. Hence, $v(Ax15) = 1$, by **vImp**.
  - $v(\circ(\alpha \wedge \beta)) = 1$, by **vPropAnd**. Hence, $v(Ax16) = 1$, by **vImp**.
  - $v(\circ(\alpha \to \beta)) = 1$, by **vPropImp**. Hence, $v(Ax17) = 1$, by **vImp**.

Ax18 and Ax19 Immediate from **vPropNeg** and **vPropCon**.

Let $\alpha(x)$ be a formula depending at most on x.

Ax20 and Ax21
- If $v(\circ\alpha[a]) = 0$ for some $a \in A$, then $v(\forall x \circ \alpha) = 0$, by **vUni**. Hence, $v(Ax20) = v(Ax21) = 1$, by **vImp**.
- If $v(\circ\alpha[a]) = 1$ for all $a \in A$, then
  - $v(\circ\forall x\alpha) = 1$, by **vPropUni**. Hence, $v(Ax20) = 1$, by **vImp**.
  - $v(\circ\exists x\alpha) = 1$, by **vPropEx**. Hence, $v(Ax21) = 1$, by **vImp**.

Ax22
- If there is no $a \in A$ such that $v(\alpha[a]) = 1$ and $v(\circ\alpha[a]) = 1$, then there is no $a \in A$ such that $v((\alpha \wedge \circ\alpha)[a]) = 1$, by **vAnd**. By **vEx**, $v(\exists x(\alpha \wedge \circ\alpha)) = 0$. Hence, $v(Ax22) = 1$, by **vImp**.
- If there is $a \in A$ such that $v(\alpha[a]) = 1$ and $v(\circ\alpha[a]) = 1$, then $v(\circ\exists x\alpha) = 1$, by **vPropEx'**. Hence, $v(Ax22) = 1$, by **vImp**.

□

For completeness, differently from soundness, it is not possible to use the result for QmbC proven in [13], but it is possible to course the very same track.

For the sake of comprehensibility, here follows the step by step of the proof from [13]:

Theorem of constants: If Δ is a QmbC theory over a signature Σ and $\vdash_C$ is the consequence relation of QmbC over the signature Σ_C obtained by adding the set C of new constants, then, for every ϕ in the original language, $\Delta \vdash \phi$ iff $\Delta \vdash_C \phi$. That is, adding new constants provide conservative extensions.

Extension to a Henkin theory: Every theory can be conservatively extended to a theory with a set of witnesses C (a C-Henkin theory).

Extension of a Henkin theory: The extension of a Henking theory (within the same signature) is still a Henking theory.

Lindenbaum-Łoś́: Every theory that does not prove a given formula ϕ can be extended to a maximally non-trivial theory with respect to ϕ. Using this result together with the previous one, the theory can be extended to a Henkin maximally non-trivial theory with respect to ϕ.

Canonical interpretations: Given a Henkin maximally non-trivial theory Δ with respect to a statement ϕ, a model $\mathfrak{A}$ for Δ can be built with a canonical interpretation function and a canonical valuation v such that, for every statement θ, $\mathfrak{A}, v \vDash \theta$ iff $\Delta \vdash \theta$.

Conclusion: The canonical model so constructed is a QmbC model in an enriched signature that satisfies Δ and does not satisfy ϕ. Reducing this model, a model in the original signature is obtained which satisfies Δ and does not satisfy ϕ. This means that, if Δ does not syntactically entail ϕ, then it does not semantically entail ϕ, or, equivalently, if Δ semantically entails ϕ, then it syntactically entails ϕ.

The first four steps can be performed for PQmbC quite in the same way, in every detail. The fifth step, that is, the construction of a model by canonical interpretations can be performed in every detail for PQmbC too, but something else must be proven: That the model so constructed satisfies the propagation clauses as well. After that, the conclusion follows in the same way. So here follows the complement of the proof of the Canonical Interpretation's Theorem:

Proof. In the demonstration in [13], the canonical model $\mathfrak{A}$ for Δ is constructed over an extended signature Σ_C, obtained by the addition of a set C of new constants. The canonical interpretation function is defined so that its domain is the set of terms over the original signature Σ and each term over Σ is interpreted by one and just one new constant. A mapping $*$ is defined from the set of statements over Σ_C to the set of statements over Σ such that, for each statement ϕ, ϕ^* is exactly ϕ with the new constants substituted by the terms which interpret them. Naturally, if ϕ is a statement over Σ, then $\phi^* = \phi$.

Then, it is proven that, for each statement ϕ, $v(\phi) = 1$ iff $\Delta \vdash \phi^*$. Finally, it is proven that v satisfies each QmbC clause.

The whole sequence of steps briefly described above is carefully performed in [13] for QmbC and can be performed for PQmbC exactly in the same way. What remains to be proven is that the propagation clauses hold for v as well.

vPropOr, vPropAnd and vPropImp If $v(\circ\alpha) = 1$ and $v(\circ\beta) = 1$, then $\Delta \vdash \circ\alpha^*$ and $\Delta \vdash \circ\beta^*$. In this case, $\Delta \vdash (\circ\alpha^* \wedge \circ\beta^*)$, by **Ax3**. Then,

- $\Delta \vdash \circ(\alpha^* \vee \beta^*)$, by **Ax15** and MP. Hence, $v(\circ(\alpha \vee \beta)) = 1$.
- $\Delta \vdash \circ(\alpha^* \wedge \beta^*)$, by **Ax16** and MP. Hence, $v(\circ(\alpha \wedge \beta)) = 1$.
- $\Delta \vdash \circ(\alpha^* \to \beta^*)$, by **Ax17** and MP. Hence, $v(\circ(\alpha \to \beta)) = 1$.

vPropCon $\Delta \vdash \circ(\circ\alpha^*)$, by **Ax18** and $\Delta \vdash \circ(\neg\circ\alpha^*)$, by **Ax19**. Hence, $v(\circ(\circ\alpha)) = 1$ and $v(\circ(\neg \circ \alpha)) = 1$.

Let $\alpha(x)$ be a formula depending at most on x. For each $a \in A$, let c_a be a constant whose interpretation is a (which exists, for the proof is performed for a Henkin theory) and $\alpha(c_a)$ the substitution of x by c_a, as ususal.

vPropUni and vPropEx If $v(\alpha[a]) = 1$ fol all $a \in A$, then $\Delta \vdash \alpha^*(c_a)$ for all $a \in A$. In this case, it is provable in QmbC (and also in PQmbC) that $\Delta \vdash \forall x\alpha^*$. Then,

- $\Delta \vdash \circ\forall x\alpha$, by **Ax20** and MP. Hence, $v(\circ\forall x\alpha) = 1$.
- $\Delta \vdash \circ\exists x\alpha$, by **Ax21** and MP. Hence, $v(\circ\exists x\alpha) = 1$.

vPropEx' If, for some $a \in A$, $v(\alpha[a]) = 1$ and $v(\circ\alpha[a]) = 1$, then, for such a, $\Delta \vdash \alpha^*(c_a)$ and $\Delta \vdash \circ\alpha^*(c_a)$. In this case, $\Delta \vdash (\alpha^*(c_a) \wedge \circ\alpha^*(c_a))$, by **Ax3**. From this and from **Ax12**, it is provable in QmbC (and also in PQmbC) that $\Delta \vdash \exists x(\alpha^* \wedge \circ\alpha^*)$. Then, $\Delta \vdash \circ\exists x\alpha^*$, by **Ax22**. Hence, $v(\circ\exists x\alpha) = 1$.

$\square$

Final Considerations

As already commented, PRS is just one possibility of a system for paraconsistent reasoning among infinitely many, but it is a very suitable one. In PRS, the conjunction, the disjunction and the implication of a pair of basic statements can be determined from the beginning. It makes sense to expect that this control be possible to more complex conjunctions, disjunctions and implications. For that purpose, a further enrichment of the concept of structure can be made. However, the more the structure is enriched in order to keep control over larger sets of statements, the havier it becomes.

Also, some clauses could be added in order to reach a bigger portion of the preservation kernel. For instance, a clause could be added to work with respect to the universal quantifier as **vPropEx'** works with respect to the existential quantifier.

vPropUni': If, for some $a \in A$, $v(\alpha[a, \vec{a}]) = 0$ and $v(\circ\alpha[a, \vec{a}]) = 1$, then $v(\circ(\forall x \alpha(x, \vec{x}))[\vec{a}]) = 1$.

Again, the more the list of clauses is enlarged in order to keep control over a bigger portion of the preservation kernel, the havier it becomes. Being so, PRS has been delimited at this point in order to avoid an unwieldy system.

This thesis has [13] as its basis. At this point, it is in order to cite the other works that have had the same article as its basis. For far, two papers have been published from [13].

The first one is 'Fraïssè's theorem for logics of formal inconsistency', by Bruno Mendonça and Walter Carnielli, cited as [36] in the references.

The second one is 'The Keisler-Shelah Theorem for QmbC Through Semantical Atomization', cited as [27] in the references.

The two papers differ from the present thesis and from one another in pursuit and method, but all three works had to face the same question: How to work out the concept of isomorphism in a system that fails to determine truth value from the structure for a great deal of statements? Each one gave a response of its own to the challenge:

In the present thesis, the structures are enriched in order to become able to control a greater deal of statements.

In [36], the concept of isomorphism is reproposed in terms of a reformulated mechanism of back and forth. This reformulation relies on a new notion of quantifiers rank, which takes into account the fact that, whenever a statement is of the form $\neg\phi$ or $\circ\phi$, its valuation is not determined by the valuation of its subformulae.

In [27], the method that is employed is that of semantical atomization. This technique allows structures to control every statement at the cost of working with an unwieldy language that possesses a predicate symbol for each formula.

Chapter 3

The Power of Classical Negation

3.1 QmbC without a Consistency Operator

The purposes this chapter is to determine the strength of the classical auxiliar negation, that is, to verify what can be recovered from classical model theory and what cannot be recovered without it.

The task for the current section is to delimit the field. The system under consideration is QmbC. In order to study how essencial the classical auxiliar negation is in recovering classical results, it will be needed a system that shall behave like QmbC in all aspects except in that of having such a negation. That is what will be now studied.

By Proposition 1.2.12, this system cannot have a consistency connective. In view of this fact, the most natural idea is to work with a system obtained from QmbC by dropping $\circ$ from the signature and the clause **vCon** (if $v(\circ\alpha) = 1$, then $v(\alpha) = 0$ or $v(\neg\alpha) = 0$) from the list of clauses to be satisfied. Such a system would be as close to QmbC as possible, for it would only lack the forbidden connective $\circ$ (by Proposition 1.2.12) and would behave like QmbC with respect to the remaining connectives. However, it is still left to be answered whether an auxiliar classical negation can be defined even without the consistency connective or not. The answer is 'no'. It will be proven that no bottom particle $\perp$ can be defined. According to Corollary 1.2.13, this is equivalent to proving that no connective $\circ$ can be defined and to proving that no classical negation can be defined.

A nonquantified system without a consistency operator is studied by Batens in [9] under the name PI. It consists in the positive precicate calculus (PC+) with a paraconsistent negation. The system that will be defined can be regarded as QmbC minus consistency operator or by PI plus quantifiers. The former formulation will be the adopted.

This new system will be called QmbC$^-$ and will be presented in more precise terms below:

- A language L_Σ is a language for QmbC$^-$ iff it is a language for classical logic (without $\circ$).
- A structure S is a structure for QmbC$^-$ iff it is a structure for classical logic.
- A valuation v is a valuation for QmbC$^-$, or a QmbC$^-$-valuation, if it satisfies all the clauses for QmbC-valuations except for **vCon** (If $v(\circ\alpha) = 1$ then $v(\alpha) = 0$ or $v(\neg\alpha) = 0$).
- QmbC$^-$ is the system semantically defined with basis on QmbC$^-$– valuations in the canonical manner.

In the previous section, a bottom particle was defined to be a sentence '$\perp$' such that any model that derives it is trivial. In semantical terms, this is equivalent to being such that $v(\perp) = 0$ for every valuation of whatever model. In fact, a sentence $\perp$ that is always false derives whatever formula ϕ, for it is trivally the case that ϕ is true whenever $\perp$ is true. This is the characterization to be used in the proof of the next proposition, that is, it will be proven that there is no sentence that is always false.

Proposition 3.1.1. *Let L_Σ be a language for QmbC$^-$ and let Φ be an arbitrary finite and nonempty set of formulae in L_Σ. Then, there is a model $\mathfrak{A}$ with valuation $v_\mathfrak{A}$ where Φ is valid.*

Proof. Let Ψ_0 be the set of the formulae of complexity 0 that are subformula of some formula in Φ. For each formula $\theta \in \Psi_0$, there are a predicate symbol P^θ, a sequence of constants $(c_1^\theta, \ldots, c_i^\theta)$ and a sequence of variables $(x_{i+1}^\theta, \ldots, x_n^\theta)$ such that $\phi = P^\theta(c_1^\theta, \ldots, c_i^\theta, x_{i+1}^\theta, \ldots, x_n^\theta)$ or there are terms $\tau_1(\vec{x})$, $\tau_2(\vec{x})$ such that $\theta(\vec{x}) = \tau_1(\vec{x}) \approx \tau_2(\vec{x})$. Of course, these sequences may have length 0, despite the notation. In order to obtain a model where each $\theta \in \Psi_0$ is valid, take $\mathfrak{A} = \langle A, I_\mathfrak{A}\rangle$, where A is a unary set and $I_\mathfrak{A}(P^\theta) = A^n$ for all P that occurs in Ψ_0. Obviously, the valuation $v_\mathfrak{A}$ validates each θ in Ψ_0.
Let Ψ_{k+1} be the set of formulae ψ such that $\psi = \alpha\#\beta$ ($\# \in \{\vee, \wedge, \rightarrow\}$) or $\psi = \#x\alpha$ ($\# \in \{\exists, \forall\}$) or $\psi = \neg\alpha$, where $\alpha, \beta \in \cup_0^k \Psi_i$. If, for every $\gamma(\vec{x}) \in \cup_0^k \Psi$ and for every $a \in A^l$, $v_\mathfrak{A}(\gamma(\vec{a})) = 1$, then it must be defined $v_\mathfrak{A}(\psi) = 1$ if ψ is not $\neg\alpha$ for some α and it may be defined $v_\mathfrak{A}(\psi) = 1$ if ψ is $\neg\alpha$ for some α. Thus, v can be defined so that $v_\mathfrak{A}(\psi) = 1$ for every $\psi \in \Psi_{k+1}$. Defining v in this way until the maximum complexity among the formulae in Φ is reached and defining $v_\mathfrak{A}$ freely for the rest of the language, a model that validates all the formulae in Φ will be defined.

$\square$

The corollary below follows immediately:

Corollary 3.1.2. *There is no bottom particle in QmbC⁻.*

Corollary 3.1.2 together with the discussion that preceeds it settles that QmbC$^-$ is in fact the desired system, that is, the closest system to QmbC that does not have a classial negation, the one that does not have a consistency connective while still behaving like QmbC with respect to all other connectives. The discussion regarding this sort of question is satisfactorily concluded.

With regard to the proof of Proposition 3.1.1, it is possible to define a nonunary model that satisfies Φ if no variable that occurs in a subformula of the form $\tau_1(\vec{x}) \approx \tau_2(\vec{x})$ is free or is the scope of a universal quantifier. If the language L_Σ possesses closed terms, then there is at least one constant c. If $\mathfrak{A}$ is a nonunary model in L_Σ, then $\forall x(c \approx x)$ is a bottom particle in the theory of $\mathfrak{A}$. Thus, the class of nonunary models with QmbC$^-$ valuation possesses a bottom particle.

The task that imposes itself at this point is that of finding an aximatization for QmbC$^-$ and this is what will be pursued through the next lines.

As QmbC$^-$ was obtained from QmbC by dropping **vCon**, it is natural to search an axiomatization for it by dropping **Ax11** from the the axiomatization of QmbC. This axiomatization will be called **AX**, for simplicity. The proof of the fact that **AX** is sound with respect to QmbC$^-$ is a straightforward verification. The question of whether it is complete or not will not be explored here. If so, the proof of this fact shall be different. Instead, it will be presented an axiomatization that is sound (which is a straightforward verification) and that will be proven to be complete with respect to QmbC$^-$. Summing up, **AX** will be extended in order to obtain a sound and complete axiomatization of QmbC$^-$ and the question of whether it is really necessary to extend **AX** or **AX** is sufficient to axiomatize QmbC$^-$ will not be explored.

Plowing carefully through the steps of the proof of completeness for QmbC in [13], one can find that the strong negation was used twice. The first use of it was made for proving the Deduction Metatheorem. The second one was made for proving the schema that will be called **Ax23**.

Ax23 $(\alpha \to \exists x\beta) \to \exists x(\alpha \to \beta)$ if x does not occur free in α.

If one were able to prove DMT and **Ax23** prescinding from **Ax11** and, consequently, from using the strong negation (by Proposition 3.1.1), then completeness would proven. If one assumes DMT and **Ax23**, then completeness can be proven to an extension of **AX**.

The schema presented below, that will be called **Ax24**, enables one to prove DMT.

Ax24 $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$ if x does not occur free in α.

The Hilbert calculus obtained from **AX** by adding **Ax23** and **Ax24** will be called $\mathbf{AX}^+$. Next, it will be proven that $\mathbf{AX}^+$ is an axiomatization for QmbC$^-$. The first step is to prove DMT for $\mathbf{AX}^+$. Firstly, two lemmata are called for:

Lemma 3.1.3. *The schema $\alpha \vdash (\beta \to \alpha)$ holds in any Hilbert calculus that fulfills **Ax1** and **MP**.*

Proof. Hypothesis: α

1. $\alpha \to (\beta \to \alpha)$ [**Ax1**]
2. α [Hyp.]
3. $\beta \to \alpha$ [1, 2 by **MP**]

□

Lemma 3.1.4. *The schema $(\alpha \to \beta), (\beta \to \gamma) \vdash (\alpha \to \gamma)$ holds in any Hilbert calculus that fulfills **Ax1**, **Ax2** and **MP**.*

Proof. Hypothesis: $(\alpha \to \beta), (\beta \to \gamma)$

1. $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$ [**Ax2**]
2. $(\alpha \to \beta)$ [Hyp.]
3. $(\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma)$ [1 and 2 by **MP**]
4. $\beta \to \gamma$ [Hyp.]
5. $\alpha \to (\beta \to \gamma)$ [4 by Lemma 3.1.3]
6. $\alpha \to \gamma$ [3 and 5 by **MP**]

□

Now, DMT can be proven:

Theorem 3.1.5 (Deduction Metatheorem for $\mathbf{AX}^+$)**.** *Suppose that there exists in $\boldsymbol{AX}^+$ a derivation of ψ from $\Gamma \cup \{\phi\}$ such that no application of the rules $\exists$-**In** and $\forall$-**In** to formulae that depend on ϕ has as its quantified variables free variables of ϕ. Then, $\Gamma \vdash (\phi \to \psi)$.*

Proof. The proof is the same as that performed in [13] for QmbC, except for the part involving $\forall$**-In** in the second step of induction. So this is the only part to be presented here.

Suppose that, for some given $n \geq 2$, $\Gamma \vdash (\phi \to \phi_i)$ if ϕ_i is derived in i lines, whenever $1 \leq i < n$. Suppose ψ is derived in n lines.

The case that remains to be analyzed is: $\psi = \phi_i = \alpha \to \forall x\beta$ for some α and some β, with x not free in α and ψ is obtained by the application of $\forall$**-In** to $\phi_j = \alpha \to \beta$ $(j < i)$.

In this case, there are two possibilities:

1. ϕ_j does not depend on ϕ. In this case, $\Gamma \vdash \phi_j$, that is, $\Gamma \vdash (\alpha \to \beta)$, whence $\Gamma \vdash (\alpha \to \forall x\beta)$, by $\forall$**-In**. By Lemma 3.1.3, $\Gamma \vdash (\phi \to (\alpha \to \forall x\beta))$, that is, $\Gamma \vdash (\phi \to \psi)$, as desired.

2. x does not occur free in ϕ. By the inductive hypothesis, $\Gamma \vdash (\phi \to \phi_j)$, that is, $\Gamma \vdash (\phi \to (\alpha \to \beta))$. By $\forall$**-In**, $\Gamma \vdash (\phi \to \forall x(\alpha \to \beta))$ (x does not appear free in ϕ). By **Ax24**, $\Gamma \vdash (\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta))$. Applying Lemma 3.1.4, $\Gamma \vdash (\phi \to (\alpha \to \forall x\beta))$, that is, $\Gamma \vdash (\phi \to \psi)$.

□

The proof of the next result for $\mathbf{AX}^+$ is identical to its proof for QmbC. The same proof can be performed in **AX** as well.

Theorem 3.1.6 (Theorem of Constants)**.** *Let $\Delta \subseteq S_{L_\Sigma}$ be a theory in $\boldsymbol{AX}^+$ over a signature Σ and let $\vdash_C$ be the consequence relation of $\boldsymbol{AX}^+$ over the signature Σ_C, which is obtained from Σ by adding a set C of new individual constants. Then, for every $\phi \in S_{L_\Sigma}$,*

$$\Delta \vdash \phi \iff \Delta \vdash_C \phi$$

Again, the proof of the next result for $\mathbf{AX}^+$ is identical to its proof for QmbC. The same proof cannot be performed in **AX**, for it depends on **Ax23** and on DMT, which depends on **Ax24** on its turn.

Theorem 3.1.7. *Every theory $\Delta \subseteq S_{L_\Sigma}$ in $\boldsymbol{AX}^+$ over a signature Σ can be conservatively extended to a C-Henkin theory Δ^H in $\boldsymbol{AX}^+$ over a signature Σ_C, as in Theorem 3.1.6. That is, $\Delta \subseteq \Delta^H$ and, for every $\phi \in S_{L_\Sigma}$, then $\Delta \vdash \phi$ iff $\Delta^H \vdash_C \phi$. Additionally, any extension of Δ^H by sentences in the signature Σ_C is also a C-Henkin theory.*

The next theorem is presented in a stronger version than that presented in [13]. There, it is required that Δ be maximal nontrivial with respect to some sentence ϕ. Here, it is required that Δ be maximal nontrivial with respect to some set of sentences Φ. Actually, the property that is needed from Δ is that it be closed, that is, it must contain every formula that it derives. On the purpose of adapting the proof, the lemma below comes in help:

Lemma 3.1.8. *Let Δ be maximal nontrivial with respect to a set Φ. Then Δ is closed.*

Proof. Let $\psi \notin \Delta$ be an arbitrary formula not belonging to Δ. Suppose, for the sake of contradiction, that $\Delta \Vdash \psi$. As Δ is maximal nontrivial with respect to Φ, there is a sentence $\phi \in \Phi$ such that $(\Delta \cup \{\psi\}) \Vdash \phi$. Hence, $\Delta \Vdash \phi$, which is a contradiction. Therefore, if $\psi \notin \Delta$, then $\Delta \nVdash \psi$, as desired. □

Theorem 3.1.9 (Canonical Interpretation)**.** *Let Δ be a set of sentences over a signature Σ. Assume that Δ is a C-Henkin theory in $\boldsymbol{AX}^+$ for a nonempty set of constants C of Σ and that Δ is also maximal nontrivial with respect to some set of sentences Φ. Then, Δ induces a canonical structure $\mathfrak{A}$ and a canonical QmbC$^-$ valuation v such that, for every sentence ψ,*

$$\mathfrak{A} \vDash \psi \iff \Delta \vdash \psi$$

Proof. The proof is identical to that presented in [13] for QmbC, except for the part concerning the clause referring to the universal quantifier and this is the only step to be presented here. This time, there is no strong negation to help in the task.

Before going on, it is important to note that Δ is closed in view of Lemma 3.1.8 and this allows the steps in the proof for QmbC that depend on the maximal nontrivaility of Δ with respect to a sentence.

In order to prove **vUni**, firstly note that **Ax13** and **MP** yield that $\Delta \vdash \forall x\phi$ iff, for any term t, $\Delta \vdash \phi(x/t)$. Because of the fact that Δ is a Henkin theory, it is the case that $\Delta \vdash \forall x\phi$ iff, for any **closed** term t, $\Delta \vdash \phi(x/t)$. Consider now a sentence of the form $\forall x\psi$. Then, $v(\forall x\psi) = 1$ iff $\Delta \vdash (\forall x\psi)^*$ iff $\Delta \vdash \forall x(\psi)^*$, by the definition of *. From this and by the observation above, one infers that $v(\forall x\psi) = 1$ iff $v((\psi)^*(x/t)) = 1$ for every closed term $t \in L_\Sigma$. On the other hand, it can be proven by induction on the complexity of ψ that $((\psi)^*)(x/t) = (\psi(x/\bar{t}))^*$ for any closed term in L_Σ. Thus, $v(\forall x\psi) = 1$ iff $v(\psi(x/\bar{t})) = 1$ for every closed term t, that is, for every element t of CT_Σ (maintained the notation from the proof in [13]). □

Completeness follows exactly in the same way as in the classical case.

Proposition 3.1.10 (Completeness of QmbC$^-$)**.** *For every set of sentences $\Delta \cup \{\phi\}$ over a signature Σ, if $\Delta \vDash \phi$, then $\Delta \vdash \phi$.*

The task of finding the closest paraconsistent system from QmbC without a classical negation is accomplished. This section will be closed with the exploration of a few features of QmbC^- that make clear that it is significantly different from QmbC.

Proposition 3.1.1 derives a slightly stronger version of Corollary 3.1.2, namely, that, given a set of formulae Φ, and a formula ϕ, the validity of Φ does not entail the falsity of ϕ. In other words, not only there is no formula that is always false, but also there is no formula that is always false provided a certain set of formulae is valid. Still rephrasing the result, given a Theory T and a formula ϕ, ϕ is consistent with T, which renders the concept of consistency a trivial one. It may be, however, that every model for $\Phi \cup \{\phi\}$ is trivial. In fact, T may be a maximal nontrivial theory and ϕ a sentence not in T. In order to recover the concept of consistency in QmbC^- as an interesting one, a new formulation will be settled:

Definition 3.1.11 (Consistency of a Sentence with a Given Theory)**.** *A sentence ϕ is* consistent *with a theory T iff there is a nontrivial model of T that realizes ϕ.*

The definition above fits for QmbC and for classical logic as well, for there are no trivial models in those logical systems.

In the proof of Proposition 3.1.1, it was shown that, if the formulae of a set Φ have complexity up to n and $\neg\phi$ has complexity $n+1$, then, for every sequence $\vec{a}$ of elements in the domain of interpretation A, there is a valuation that validates every formula of Φ and such that $v(\neg\phi(\vec{a})) = 1$ and there is a valuation that validates every formulae of Φ and such that $v(\neg\phi(\vec{a})) = 0$. The facts below follow from the discussion just performed:

- If $(\Phi \cup \{\neg\phi\}) \vDash \psi$ and $\neg\phi$ is not a subformula of ψ, then $\Phi \vDash \psi$;
- If $\neg\phi$ is not a subformula of any formula in Φ, then $\Phi \nvDash \neg\phi$.

3.2 Omitting Types

In Section 2, QmbC^- was established as the closest system to QmbC that does not have an available auxiliar classical negation. This section starts the work of searching for results from classical model theory that remain valid in QmbC and/or in QmbC^- in order to find out how powerful the classical auxiliar negation is in recovering classical results and how far it is possible to go without it.

The first result to be pursued is the very one of omitting types and the answer is positive. Accordingly, not only is the Omitting Types Theorem valid in QmbC, but also it can be proven in the same way as the classical result in [18], provided that the negation be understood as the auxiliar classical negation whenever it occurs along the proof. The theorem is valid in QmbC$^-$ as well, but this time the classical proof cannot be transposed to the desired context. Fortunately, it can be adapted and this is what will be done in what follows.

In line with the developments of the first two sections, a type is to be understood as a maximal nontrivial theory. The concept of a theory to locally realize a set of formualae is the same as in classical logic, provided the concept of consistent formula be understood as in the end of the previous section.

Definition 3.2.1 (Local Realization). *Let $\Gamma(\vec{x})$ be a set of formulae in a language L_Σ. A theory T locally realizes Γ iff there is a formula $\phi(\vec{x})$ such that:*

1. *ϕ is consistent with T;*

2. *For every $\sigma \in \Gamma$, $T \vDash \phi \to \sigma$.*

A theory T is said to locally omit Γ if it does not locally realize Γ. It means that, for every consistent formula $\phi(\vec{x}) \in L_\Sigma$, there are a formula $\sigma(\vec{x}) \in \Gamma$, a model $\mathfrak{A} = \langle A, I, v\rangle$ for T and a string $\vec{a}$ of elements in A such that $v(\phi(\vec{a})) = 1$ and $v(\sigma(\vec{a})) = 0$. In QmbC and in classical logic, this is equivalent to state that $(\phi \wedge \sim \sigma)(\vec{x})$ is consistent, where $\sim$ is the classical negation.

In QmbC$^-$, where there is no available classical negation, this formulation is not possible. Fortunately, a satisfactorily manageable one can be given. Let $\mathfrak{A} = \langle A, I, v\rangle$ be a model as described above. Suppose there is a string of constants $\vec{c}$ (with the same length as that of $\vec{a}$) in the signature whose constants do not occur in $T \cup \{\phi, \sigma\}$. It is straightforward to prove by induction on the complexity of the formulae that a model $\mathfrak{A}' = \langle A, I', v'\rangle$ can be defined so that, for each indice i, $I'(c_i) = a_i$ where c_i is the i^{th} constant in $\vec{c}$ and a_i is the i^{th} element in $\vec{a}$; $I'(d) = I(d)$ if d is not in $\vec{c}$; and $v'(\psi(\vec{a})) = v(\psi(\vec{a}))$ if $\psi \in (T \cup \{\phi, \sigma\})$. In view of this fact, if the language L_Σ has infinitely many constants, then a theory T is said to locally omit Γ if, for each $\phi(\vec{x}) \in L_\Sigma$, there are a formula $\sigma(\vec{x}) \in \Gamma$, a string $\vec{c}$ of constants and a model $\mathfrak{A} = \langle A, I, v\rangle$ for T such that $v(\phi(\vec{c})) = 1$ and $v(\sigma(\vec{c})) = 0$.

The proposition below, like the Omitting Types Theorem, can be proven in QmbC in the same way as in the classical context. The proof above is performed with the suitable adaptations for making it fit to QmbC$^-$ as well.

Proposition 3.2.2. *Let T be a maximal nontrivial theory in a language L_Σ and let $\Gamma(\vec{x})$ be a set of formulae in L_Σ. If T has a model that omits Γ, then T locally omits Γ. That is, if T locally realizes Γ, then every model of T realizes Γ.*

Proof. Let T be a maximal nontrivial theory that locally realizes the set of formulae $\Gamma(\vec{x})$ and let $\phi(\vec{x})$ be a formula satisfying items (1) and (2) of Definition 3.2.1.

Suppose, for the sake of contradiction, that $T \not\models \exists\vec{x}\phi(\vec{x})$. Thus, $T \cup \{\exists\vec{x}\phi(\vec{x})\}$ is trivial, for T is maximal nontrivial. Hence, every model for $T \cup \{\exists\vec{x}\phi(\vec{x})\}$ is trival. Therefore, no nontrivial model of T validates $\exists\vec{x}\phi(\vec{x})$, which means that no nontrivial model of T realizes $\phi(\vec{x})$, which is a contradiction against the fact that $\phi(\vec{x})$ is consistent. This contradiction proves that $T \models \exists\vec{x}\phi(\vec{x})$, which yields that every model of T satisfies $\exists\vec{x}\phi(\vec{x})$, which means, on its turn, that every model of T realizes $\phi(\vec{x})$, which finally yields that every model of T realizes Γ, as desired.

□

The Omitting Types Theorem is the core result of this section. Proposition 3.2.2 is the converse of this celebrated theorem in the particular case that T is a maximal nontrivial theory. The proof below is performed for QmbC$^-$.

Theorem 3.2.3 (Omitting Types Theorem)**.** *Let T be a nontrivial theory in a denumerable language L_Σ and let $\Gamma(\vec{x})$ be a set of L_Σ-formulae. If T locally omits Γ, then T has a denumerable model that omits Γ.*

Proof. The proof will be presented for $\Gamma(x)$ instead of $\Gamma(\vec{x})$ just for the sake of simplicity of notation.

Let $C = \{c_i\}_{i\in\mathbb{N}}$ be a denumerable set of new constants and let L_{Σ_C} be the language obtained from L_Σ by the addition of those new constants.

Let $\phi_0, \phi_1, \phi_2, \ldots$ be an enumeration of the sentences in L_{Σ_C}. A chain of theories $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ in L_{Σ_C} and a chain of sets of sentences $\emptyset = S_0 \subseteq S_1 \subseteq S_2 \ldots$ will be built so that

1. Each T_m is nontrivial;
2. For every m, there are a formula $\sigma_m(x) \in \Gamma(x)$ and a constant $c_m \in C$ such that $\sigma_m(c_m) \in S_{m+1} \backslash S_m$;
3. If $1 \leq i \leq m$, then $T_m \nvdash (\sigma_i(c_i))$;
4. If $\phi_m = \exists x\psi(x)$ and $\phi_m \in T_m$, then $\psi(c_p) \in T_{m+1}$, where c_p is the first constant that does not occur either in T_m or in ϕ_m.
5. If $\phi_m = \forall x\psi(x)$ and $\phi_m \notin T_m$, then $\psi(c_p) \in S_{m+1}$, where c_q is the first constant that does not occur either in T_m or in ϕ_m.

Construction of T_m and S_m: Assume T_m and S_m already defined (with $m \geq 0$). Suppose $T_m = T \cup \{\theta_1, \ldots, \theta_r\}$.

Let n be the lower index such that the set $C_n = \{c_i\}_{0\leq i\leq n}$ contains all the constants in $\theta = \theta_1 \wedge \cdots \wedge \theta_r$. Let $\theta'(x_0, \ldots, x_n) = \theta^{c_0}_{x_0} \ldots^{c_n}_{x_n}$ be the formula (in L_Σ)

obtained by substituting each c_i by x_i after renaming the eventual occurrences of x_i. Finally, let

$$\theta''(x_m) = \begin{cases} \exists x_1 \ldots \exists x_{m-1} \exists x_{m+1} \ldots \exists x_n \theta' \; if \; m \leq n \\ \exists x_1 \ldots \exists x_n \theta' \; if \; m > n \end{cases}$$

It is easy to prove that $\theta''(x_m)$ is consistent with T. As T locally omits Γ and c_m does not occur in $T \cup \Gamma \cup \{\theta''\}$, there exist a formula $\sigma(x) \in \Gamma$ and a model $\mathfrak{A}$ with valuation v such that $v(\theta''(c_m)) = 1$ and $v(\sigma(c_m)) = 0$. Define

- $S'_{m+1} = S_m \cup \{\sigma(c_m)\}$.

Once defined S'_{m+1}, T_{m+1} is defined as follows:

- If $(T_m \cup \{\phi_m\}) \vdash \psi$ for some $\psi \in S'_{m+1}$, then $T_{m+1} = T_m$;
- If $(T_m \cup \{\phi_m\}) \nvdash \psi$ for all $\psi \in S'_{m+1}$, then
  - If ϕ is not of the form $\exists x \psi$, then $T_{m+1} = T_m \cup \{\phi_m\}$;
  - If $\phi = \exists x \psi$ for some ψ, then $T_{m+1} = T_m \cup \{\phi_m, \psi(c_p)\}$, where c_p is the first constant that does not occur in $T_m \cup \{\phi_m\}$.

Finally, define

- If $\phi_m = \forall x \psi(x)$ for some formula ψ and $\phi_m \notin T_{m+1}$, then $S_{m+1} = S'_{m+1} \cup \{\psi(c_p)\}$, where c_p is the first constant that does not occur in $T_m \cup \{\phi_m\}$;
- Otherwise, $S_{m+1} = S'_{m+1}$.

Now let $T_\omega = \cup_{i \in \mathbb{N}} T_i$ and $S_\omega = \cup_{i \in \mathbb{N}} S_i$.

If $\gamma \in S_\omega$ is an arbitrary formula in S_ω, then $T_\omega \nvdash \gamma$. In fact: if $T_\omega \vdash \gamma$ for some $\gamma \in S_\omega$, then $T_k \vdash \gamma$ for some k, by compacity, which yields that $T_r \vdash \gamma$ for every $r \geq k$. Suppose, for the sake of contradiction, that this is the case. For some indice l, $\gamma = \sigma_l(c_l)$ or $\gamma = \psi(c_p)$ and $\phi_l(x) = \forall x \psi(x)$. In any case, $\gamma \in S_r$ for every $r \geq l + 1$, which yields that $T_r \nvdash \gamma$ for every $r \geq l + 1$. Let $m = max\{k, l + 1\}$. Then, it is the case that $T_m \vdash \gamma$ on the one hand and that $T_m \nvdash \gamma$ on the other. That's a contradiction!

Moreover, if $\psi \notin T_\omega$, then $(T_\omega \cup \{\psi\}) \vdash \gamma$ for some $\gamma \in S_\omega$. In fact, let $\psi \notin T_\omega$. For some i, $\psi = \phi_i$. As $\psi \notin T_{i+1}$ (for, otherwise, it would be the case that $\psi \in T_\omega$), it happens that $(T_i \cup \{\phi_i\}) \vdash \gamma$ for some $\gamma \in S_{i+1}$. By monotonicity, $(T_\omega \cup \{\phi_i\}) \vdash \gamma$.

Summing up, T_ω is maximal nontrivial with respect S_ω.

By construction, T_ω is also a Henkin theory.

By Theorem 3.1.9, there is a denumerable model $\mathfrak{B}'$ that satisfies exactly the same sentences as those derived by T_ω, that is, $T_\omega \vDash \phi$ iff $\mathfrak{B}' \vDash \phi$ for every sentence

ϕ. This is a model in an extension for the extended language L_{Σ_C} of some model $\mathfrak{B}$ in the original language L_Σ. For each $i \in \mathbb{N}$, let $b_i = c_i^{\mathfrak{B}}$. So, $\mathfrak{B}' = \langle \mathfrak{B}, b_1, b_2, \dots \rangle$, where the elements b_i interpret the constants $c_i \in C$.

Let $\mathfrak{A}'$ be the model generated by $B^* = \{b_i\}_{i \in \mathbb{N}}$, that is, $\mathfrak{A}' = min\{\mathfrak{B}'' \subset \mathfrak{B}' | B^* \subset |\mathfrak{B}''|\}$. It will be proven that $|\mathfrak{A}'| = B^*$. For that, it is necessary and sufficient to prove that every function symbol interprets sequences of elements in B^* as elements in B^* (if $f \in F^n$, then $Im(f|_{B^*}) \subset B^*$) and that every constant is interpreted by some element in B^*.

In fact, let $f \in F^n$ and $(b_{i_1}, \dots, b_{i_n}) \in (B^*)^n$ and let $(c_{i_1}, \dots, c_{i_n})$ be the n-tuple of constants that are interpreted by the elements b_{i_j}. The sentence $\theta = \exists x (f(c_{i_1}, \dots, c_{i_n}) \approx x)$ is a theorem in QmbC$^-$, whence $T_\omega \vdash \theta$. As T_ω is maximal nontrivial with relation to S_ω, it follows by Lemma 3.1.8 that it is also closed. This implies that $\theta \in T_\omega$. As T_ω is also a Henking theory, for some constant $c \in \Sigma_C$, $(f(c_{i_1}, \dots, c_{i_n}) \approx c) \in T_\omega$. This means that $(f(c_{i_1}, \dots, c_{i_n}))^{\mathfrak{B}'} = c^{\mathfrak{B}'} = b$, for some $b \in B^*$, as desired. The proof of the fact that every constant in Σ is interpreted by some $b \in B_*$ is analogous. It uses the fact that the sentence $\exists x (c\ x)$ is a theorem in QmbC$^-$ and the rest follows the same track.

The next step is to prove that, for every ϕ, $\mathfrak{B}' \vDash \phi$ iff $\mathfrak{A}' \vDash \phi$. As a consequence, it will be proven that $T_\omega \vDash \phi$ iff $\mathfrak{A}' \vDash \phi$, that is, that $\mathfrak{B}'$ satisfies exactly the same sentences as those that T_ω derives.

The proof will be performed by induction on the complexity of ϕ. The basic case is immediate, for each interpretation function $I_{\mathfrak{A}'}(f)$ is a restriction of $I_{\mathfrak{B}'}(f)$ to the domain B^* and each set of tuples $I_{\mathfrak{A}'}(P)$ is a restriction of $I_{\mathfrak{B}'}(P)$ to the same domain. Now, suppose the result holds for sentences with complexity up to n and let ϕ be a sentence with complexity $n+1$. If ϕ is equal to $\neg\psi$, $\psi \wedge \gamma$, $\psi \vee \gamma$ or $\psi \to \gamma$, the proof is straightforward. The cases to be analized are $\phi = \exists x \psi$ and $\phi = \forall \psi$.

Let $\phi = \exists x \psi(x)$. If $\mathfrak{A}' \vDash \phi$, then there is an element $b \in B^*$ such that $\mathfrak{A}' \vDash \psi(b)$. As each element in B^* interprets some constant in $\mathfrak{B}'$, there is a constant c_b such that $c_b^{\mathfrak{A}'} = b$, whence $\mathfrak{A}' \vDash \psi(c_b)$. By the inductive hypothesis, $\mathfrak{B}' \vDash \psi(c_b)$. Therefore, $\mathfrak{B}' \vDash \exists x \psi(x)$, as desired.

Conversely, if $\mathfrak{B}' \vDash \phi$, then $\phi \in T_\omega$. In the enumeration of the formulae used in the construction of the sequence $\{T\}_i$, there is an index m such that $\phi = \phi_m$. By construction, there is a constant c_p such that $\psi(c_p) \in T_{m+1}$, which yields $\psi(c_p) \in T_\omega$, which yields, on its turn, $\mathfrak{B}' \vDash \psi(c_p)$. By the inductive hypothesis, $\mathfrak{A}' \vDash \psi(c_p)$. Therefore, $\mathfrak{A}' \vDash \exists x \psi(x)$, as desired.

Now, let $\phi = \forall x \psi(x)$. If $\mathfrak{B}' \vDash \phi$, then $\mathfrak{B}' \vDash \psi(b)$ for every $b \in |B|$. In particular, $\mathfrak{B}' \vDash \psi(b)$ for every $b \in B^*$. For each $b \in B^*$, there is a constant c_b such that $c_b^{\mathfrak{B}'} = c_b^{\mathfrak{A}'} = b$. As $\mathfrak{B}' \vDash \psi(b)$, it holds that $\mathfrak{B}' \vDash \psi(c_b)$. By the inductive hypothesis, $\mathfrak{A}' \vDash \psi(c_b)$, whence $\mathfrak{A}' \vDash \psi(b)$ for every $b \in B^* = |A'|$. Therefore, $\mathfrak{A}' \vDash \forall x \psi(x)$, as desired.

Conversely, if $\mathfrak{A}' \vDash \phi$, then $\mathfrak{A}' \vDash \psi(b)$ for every $b \in B^* = |A'|$. For each $b \in B^*$, there is a constant c_b such that $c_b^{\mathfrak{B}'} = c_b^{\mathfrak{A}'} = b$. Thus, for every $b \in B^*$, $\mathfrak{A}' \vDash \psi(c_b)$. By the inductive hypothesis, $\mathfrak{B}' \vDash \psi(c_b)$ for every $b \in B^*$. Hence, $\mathfrak{B}' \vDash \psi(b)$ for every $b \in B^*$. Suppose, for the sake of contradiction, that $\mathfrak{B}' \nvDash \forall x \psi(x)$. Then, $\forall x \psi(x) \notin T_\omega$. For some index k, $\phi_k = \forall x \psi(x)$. As $\forall x \psi(x) \notin T_\omega$, it is the case that $\forall x \psi(x) \notin T_k$ and that, for some c_p, $\psi(c_p) \in S_{k+1} \subseteq S_\omega$. Hence, $T_\omega \nvDash \psi(c_p)$. But $c_p^{\mathfrak{A}'} = c_p^{\mathfrak{B}'} = b_p \in B^*$. Therefore, $T_\omega \nvDash \psi(b_p)$, with $b_p \in B^*$. This is a contradiction against the fact that $\mathfrak{B}' \vDash \psi(b)$ for every $b \in B^*$.

Thus, the induction is concluded and the desired result ($T_\omega \vDash \phi$ iff $\mathfrak{A}' \vDash \phi$) is stated.

Moreover, $\mathfrak{A}'$ omits Γ. In fact, let $b \in B^* = |\mathfrak{A}'|$ be an arbitrary element in the domain of $\mathfrak{A}'$. For some indice m, b interprets the constant c_m and, for some $\sigma \in \Gamma$, $\sigma(c_m) \in S_{m+1} \subset S_\omega$. Being so, $\mathfrak{A}' \nvDash \sigma(c_m)$, whence $\mathfrak{A}' \nvDash \sigma[b]$. Therefore, b does not realize Γ. As b is an arbitrary element from the domain of $\mathfrak{A}'$, it is the case that no element of $|\mathfrak{A}'|$ realizes Γ, that is, $\mathfrak{A}'$ omits Γ.

Summing up, $\mathfrak{A}'$ is a model for T_ω (and, consequently, for T) that omits Γ.

The work is not done yet, for $\mathfrak{A}'$ is a model in the signature Σ_C. So take the reduct $\mathfrak{A}$ of $\mathfrak{A}'$ to the signature Σ.

The work is finally done, for $\mathfrak{A}$ is a model in the language L_Σ for the restriction of T_ω to L_Σ. Thus, $\mathfrak{A}$ is a model for T in L_Σ that omits Γ and the task of constructing such a model is accomplished. □

So far, so good. A very strong and celebrated classical result reveals itself to be valid in both QmbC and QmbC$^-$ and, in fact, it provides an important tool for gaining other results in QmC as much as in the classical context. Unfortunately, it turns out that such a powerful result is not plainly handable without the aid of a classical auxiliar negation. This section will be closed with two examples that illustrate this fact.

The first example presents the concept of ω-model, which is linked to the very interesting system of ω-logic.

Definition 3.2.4 (ω-Model)**.** *Let L_Σ be a language for arithmetic over the signature $\Sigma = \langle \bar{F}, \bar{P}, C \rangle$, where $F_1 = \{S\}$, $F_2 = \{+, \cdot\}$ and $C = \{0\}$. Let the term $\overbrace{S \ldots S}^{mtimes}(0)$ be denoted by $\bar{m}$ and let the constant* 0 *be denoted also by $\bar{0}$. A model $\mathfrak{A}$ over Σ with domain of interpretation A is defined to be an* ω-model *iff $A = \{\bar{m}^{\mathfrak{A}} : m \in \mathbb{N}\}$. A Σ-theory is said to be* ω-consistent *iff there is no formula $\phi(x)$ in L_Σ such that*

$$T \vDash \phi(\bar{0}), T \vDash \phi(\bar{1}), \ldots, T \vDash \phi(\bar{n}), \ldots$$

and $T \vDash \exists x \neg\phi(x)$*. Finally,* T *is said to be* ω-complete *iff, for every* $\phi(x) \in L_\Sigma$*,*

$$T \vDash \phi(\bar{0}), T \vDash \phi(\bar{1}), \ldots, T \vDash \phi(\bar{n}), \ldots$$

implies $T \vDash \forall x \phi(x)$*.*

The definition above is a classical one and fits for QmbC and QmbC$^-$ as well. In QmbC, it can be reformulated by substituting $\neg$ by $\sim$ and this reformulation shall be done. In QmbC$^-$, this very possibility does not exist.

In classical logic, as in QmbC, a Σ-model $\mathfrak{A}$ is an ω-model iff it omits the set

$$\Gamma(x) = \{\sim (x \approx \bar{0}), \sim (x \approx \bar{1}), \sim (x \approx \bar{2}), \ldots\}$$

In QmbC$^-$, all that can be stated is that if $\mathfrak{A}$ omits the set

$$\Gamma(x) = \{\neg(x \approx \bar{0}), \neg(x \approx \bar{1}), \neg(x \approx \bar{2}), \ldots\}$$

then it is an ω-model.

The following proposition is valid for classical logic as well as for QmbC with the same proof presented in [18] (provided $\neg$ is substituted by $\sim$, as always). A consistent theory is understood as a one that is satisfied in some nontrivial model.

Proposition 3.2.5. *Let* T *be a consistent theory in* L_Σ *(*Σ *as in Definition 3.2.4). Then,*

1. *If* T *is* ω*-complete, then* T *has an* ω*-model.*
2. *If* T *has an* ω*-model, then* T *is* ω*-consistent.*

In QmbC$^-$, item 2 does not hold. In fact, let $\mathfrak{A} = \langle A, I, v\rangle$ be a Σ-model where $A = \{0', 1', 2', \ldots\}$ and I works as in classical arithmetic. This means that $I(\bar{m}) = \bar{m}^{\mathfrak{A}} = m'$ for every $m \in \mathbb{N}$, $I(+)(a', b') = (a+b)'$ and $I(\cdot)(a', b') = (a \cdot b)'$, for every $a, b \in \mathbb{N}$. Naturally, $\mathfrak{A}$ is an ω-model and v works precisely in the same way as the valuation in a classical arithmetic model for formulae with complexity 0. Let $v(\neg\phi(\vec{a})) = 1$ for every $\phi(\vec{x})$. Finally, let v be defined as it must be in the other cases in order for it to be coherent with the mbC$^-$ clauses. The model $\mathfrak{A}$ just defined is clearly an ω-model for the theory $T = \{\phi | \mathfrak{A} \vDash \phi\}$. But T is not ω-consistent. In fact, take a negated formula $\neg\phi(\vec{x})$. Then, $T \vDash \neg\phi(\bar{n})$ for every n and $T \vDash \exists x(\neg\neg\phi)(\bar{n})$, as $T \vDash \neg\neg\phi(a)$ for every a in the domain of interpretation. It is not to be investigated here the question of whether item 1 holds in QmbC$^-$ or not. The point is that the strategy used in the classical/QmbC case does not work in this context. The strategy consists in proving that, if T is an ω-complete theory, then T omits $\Gamma(x) = \{\neg(x \approx \bar{0}), \neg(x \approx \bar{1}), \neg(x \approx \bar{2}), \ldots\}$. The point is that the strategy used to prove that T omits Γ does not work in QmbC$^-$. Just to begin with, the proof uses that, if $\theta(x)$ is consistent, then $\forall x \neg\theta(x)$ is a valid sentence. That is not true in QmbC$^-$ at all; For the proof to work, it is needed a way of finding a sentence that has valuation 0 whenever $\phi(x)$ has valuation 1 and that is just what there is not in QmbC$^-$. This is not the only point where the proof does not work, but it is the most striking one.

The discussion just developed calls for a very interesting definition:

Definition 3.2.6 (ω-Rule). *The* ω-rule *is the infinite inference rule*

$$\frac{T \vDash \phi(\bar{0}), T \vDash \phi(\bar{1}), \ldots, T \vDash \phi(\bar{n}), \ldots}{\forall x \phi(x)}$$

for every L_Σ-formula $\phi(x)$.

The ω-logic is the one obtained from the classical/QmbC/QmbC$^-$ system by adding the ω-rule and by allowing infinite proofs.

The next proposition closes the first example of a successful application of the Omitting Types Theorem in the classical/QmbC context that fails in the QmbC$^-$ context. Again, it is valid in QmbC with the same proof as that for the classical context. Again, the proof cannot be performed in QmbC$^-$. Again, it is not to be investigated the question of whether it holds for QmbC$^-$ or not. Again, the strategy consists in proving that a given theory T' omits $\Gamma(x) = \{\neg(x \approx \bar{0}), \neg(x \approx \bar{1}), \neg(x \approx \bar{2}), \ldots\}$ in order to allow the use of the Omitting Types Theorem in the sequel.

Proposition 3.2.7 (Completeness in the ω-logic). *A theory T in L_Σ is consistent in the ω-logic iff T has an ω-model.*

The second example presents similar difficulties.

Definition 3.2.8 (Complete Formula).

- *A formula $\phi(x_1, \ldots, x_n)$ is said to be* complete *in a theory T iff, for every formula $\psi(x_1, \ldots, x_n)$, exactly one of*

  $$T \vDash \phi \to \psi, T \vDash \phi \to \neg\psi$$

  holds;

- *A formula $\theta(x_1, \ldots, x_n)$ is said to be completable in a theory T iff there is a complete formula $\phi(x_1, \ldots, x_n)$ such that $T \vDash \phi \to \theta$. Otherwise, θ is said to be incompletable;*

- *A theory T is said to be atomic iff every consistent formula in T is completable in T;*

- *A model $\mathfrak{A}$ is said to be an atomic model iff every n-tuple $\vec{a}$ satisfies a complete formula in the theory determined by $\mathfrak{A}$.*

The theorem below is valid in QmbC with the same proof presented in [18] for the classical case, provided that the natural negation $\neg$ be substituted by the classical negation $\sim$.

Proposition 3.2.9. *Let T be a theory in a language L_Σ such that, for every sentence θ, exactly one of $T \vDash \theta$ or $T \vDash \neg\theta$ holds. Then, T has a countable atomic model iff T is atomic.*

The proof of the second half ($\Leftarrow$) of the proposition above, there is, of the fact that if T is atomic then T has an atomic model, makes use of a slightly stroger version of the Omitting Types Theorem. The proof of this stronger result is a simple adaptation of the original result and will not be treated here. The proof of the second half of Proposition 3.2.9 will be presented below in order to allow an analysis of why it does not work in the QmbC$^-$ context.

Proof. Assume T is atomic. For each $n \in \mathbb{N}$, let $\Gamma(x_1, \ldots, x_n)$ be the set of all negations of all complete formulae $\psi(x_1, \ldots, x_n)$ in T. For every formula $\phi(x_1, \ldots, x_n)$ that is consistent with T, there is a complete formula $\psi(x_1, \ldots, x_n)$ such that $\psi \to \phi$, whence there is some model $\mathfrak{A}$ with valuation v and some sequence $a_1, \ldots, a_n$ of elements in its domain such that $v(\phi(a_a, \ldots, a_n)) = 1$ and $v(\psi(a_1, \ldots, a_n)) = 1$. Being so, $v(\phi(\vec{a})) = 1$ and $v(\neg\psi(\vec{a})) = 0$. As $\neg\psi$ is an arbitrary formula in Γ_n, it follows that T locally omits Γ_n. As this is the case for every $n \in \mathbb{N}$, it is the case that T locally omits Γ_n for every $n \in \mathbb{N}$. By the Extended Omitting Types Theorem, T has a countable model $\mathfrak{B}$ that omits each Γ_n. Thus, each $\vec{a}$ satisfies a complete formula, whence $\mathfrak{B}$ is an atomic model. □

The problem in this example is basically the same as in the first one: From $v(\psi(\vec{a})) = 1$, it cannot be concluded that $v(\neg\psi(\vec{a})) = 0$. For this reason, it cannot be concluded that 'T omits each Γ_n'. Moreover, the concept of complete formulae itself is hardly likely to be a promising one. Indeed, the fact that exactly one of $T \vDash \phi \to \psi, T \vDash \phi \to \neg\psi$ holds does not imply that, for each model $\mathfrak{A}$, exactly one of $\mathfrak{A} \vDash \phi \to \psi, \mathfrak{A} \vDash \phi \to \neg\psi$ holds and this is the feature that provides a link between atomic theories and atomic models in the classical case.

3.3 Craig's Interpolation Theorem

This section presents Craig's Interpolation Theorem, a result that can be extended to QmbC with the same proof as that for the classical case. The validity of this important result for QmbC^- is a plausible conjecture. However, the classical proof does not work in QmbC^-, not even with well chosen adaptations.

The theorem will be enunciated below and will be followed by a discussion on the technical difficulties that rise.

Theorem 3.3.1 (Craig's Interpolation Theorem)**.** *Let $\langle \phi, \psi \rangle$ be a pair of sentences such that $\phi \vDash \psi$. Then, there is a sentence θ such that:*

1. *$\phi \vDash \theta$ and $\theta \vDash \psi$;*
2. *every constant, function symbol or predicate symbol (except for the identity '$\approx$') that occurs in θ occurs simultaneously in ϕ and ψ.*

A sketch of the proof in [18] will be presented in order to point the difficulties that arise in the context of Qmbc^-.

The proof consists in taking a pair of sentences $\langle \phi, \psi \rangle$ that admits no Craig interpolant and building a model satisfying ϕ but not ψ.

For that, the concept of inseparable sets of sentences is introduced: Two sets of sentences Θ and Γ are inseparable iff there is no sentence δ such that $\Theta \vDash \delta$ and $\Gamma \vDash \neg\delta$. In the sequel, it is proven that $\{\phi\}$ and $\{\neg\psi\}$ are inseparable and a pair of chains $\{\phi\} = \Phi_0 \subseteq \Phi_1 \ldots$ and $\{\neg\psi\} = \Psi_0 \subseteq \Psi_1 \ldots$ of sets is constructed from that pair of sets in a richer signature so that Φ_i and Ψ_i are inseparable for every i. The respective unions of these two chains are separable and maximal consistent (or nontrivial) sets and so are their intersection. At this point, a model of that intersection can be taken and the reduct of this model to the convenient signature is the desired model.

The fact that the referred sets are inseparable plays a central role in the proof of their maximal consistency and this prevents the strategy from being adapted to QmbC^-. The fact that, for two sets of sentences Θ and Γ, $\Theta \vDash \delta$ and $\Gamma \vDash \neg\delta$ does not perform the role it should, for $\Gamma \vDash \neg\delta$ does not yield $\Gamma \nvDash \delta$. That is the point! The natural adaptation would be to define that a sentence δ separates Θ and Γ iff $\Theta \vDash \delta$ and $\Gamma \nvDash \delta$. But this is still not enough; For the argumentation to work properly, it is needed to define that a sentence δ separates Θ and Γ iff $\Theta \vDash \delta$ and, for every model of Γ with valuation v, $v(\delta) = 0$. But this is not possible, by Proposition 3.1.1.

The first classical application of Craig's Interpolation Theorem is Beth's Theorem, which is valid in the QmbC context again with the same proof. In the context of PRS, things are a little bit more delicate. Craig's Interpolation Theorem is valid, again with the same proof. With regard to Beth's Theorem, there is a problem. The point is that the structure does not define valuation for every interpretation of every basic formula. For the sake of clearness, a simpler system will be considered in order to illustrate the problem and the possible solutions for it. Consider the system obtained by simplifying PRS, just dropping **vPred** from QmbC and assuming **vPredPos** and **vPredNeg** in the same fashion as defined in PRS.

vPredPos If $\vec{a} \in A_{P1}$, then $v(P(\vec{a})) = 1$;

vPredNeg If $\vec{a} \in A_{P2}$, then $v(P(\vec{a})) = 0$.

A system so defined does not completely control valuation of basic formulae from the structures. In other words, given a model $\mathfrak{A}$ and an interpreted formula $P(\vec{a})$, it is not necessarily the case that $P(\vec{a}) \in Pk(\mathfrak{A})$.

In the rest of this section, R will stand to the relation of the predicate symbol P. Moreover, the relation R for P (of arity n) will be said to be complete iff, for every $\vec{a} \in \bar{A}$ (with length n), $\vec{a} \in A_{P1}$ or $\vec{a} \in A_{P2}$.

The definition above is a classical one.

Definition 3.3.2 (Implicit Definition of a Predicate)**.** *Let $\Gamma(P)$ be a set of sentences in L_{Σ_P} and let $\Gamma(P)$ be the same set of sentences in $L_{\Sigma_{P'}}$ (shifting P' for P). Then, $\Gamma(P)$ is said to* implicitly define *P if the following condition, that will be referred to as ID, is fulfilled*

$$\Gamma(P) \cup \Gamma(P') \vDash \forall \vec{x}(P(\vec{x}) \leftrightarrow P'(\vec{x}))$$

in $L_{\Sigma_{P,P'}}$.

The validity of the proposition below depends on the completeness of the relations. In QmbC or in classical logic, it is just the classical proposition, for every relation is complete in those contexts.

Proposition 3.3.3. *A set of sentences $\Gamma(P)$ implicitly defines P iff: If R and R' are complete relations and $\langle \mathfrak{A}, R\rangle$ and $\langle \mathfrak{A}, R'\rangle$ are models for $\Gamma(P)$, then $R = R'$.*

Proof. Let R and R' be complete relations.
($\Rightarrow$) Assume ID and let $\langle \mathfrak{A}, R\rangle$ and $\langle \mathfrak{A}, R'\rangle$ be two models for $\Gamma(P)$. Thus, $\mathfrak{A}' = \langle \mathfrak{A}, R, R'\rangle$ is a $\Sigma_{P,P'}$-model for $\Gamma(P) \cup \Gamma(P')$, whence $\langle \mathfrak{A}, R, R'\rangle \vDash \forall \vec{x}(P(\vec{x}) \leftrightarrow P'(\vec{x}))$, by ID. As R and R' are complete, for every $\vec{a} \in \bar{A}$, $\vec{a} \in A^{\mathfrak{B}'}_{P1}$ or $\vec{a} \in A^{\mathfrak{B}'}_{P2}$, whether R or R' is the relation considered. Therefore, for every $\vec{a}$, $\vec{a} \in A^{\mathfrak{A}'}_{P1}$ iff* $v_{\mathfrak{A}'}(P(\vec{a})) = 1$ iff $v_{\mathfrak{A}'}(P'(\vec{a})) = 1$ iff* $\vec{a} \in A^{\mathfrak{A}'}_{P'1}$ and $\vec{a} \in A^{\mathfrak{A}'}_{P2}$ iff* $v_{\mathfrak{A}'}(P(\vec{a})) = 0$ iff $v_{\mathfrak{A}'}(P'(\vec{a})) = 0$ iff* $\vec{a} \in A^{\mathfrak{A}'}_{P'2}$, which means that $R = R'$, as desired.
($\Leftarrow$) Let $\mathfrak{A}' = \langle \mathfrak{A}, R, R'\rangle$ be a $\Sigma_{P,P'}$-model for $\Gamma(P) \cup \Gamma(P')$. Then, $\langle \mathfrak{A}, R\rangle$ and $\langle \mathfrak{A}, R'\rangle$ are models for $\Gamma(P)$, whence $R = R'$, by hypothesis. As R and R' are complete, for every $\vec{a}$, $v_{\mathfrak{A}'}(P(\vec{a})) = 1$ iff* $\vec{a} \in A^{\mathfrak{A}'}_{P1}$ iff $\vec{a} \in A^{\mathfrak{A}'}_{P'1}$ iff* $v_{\mathfrak{A}'}(P'(\vec{a})) = 1$ and $v_{\mathfrak{A}'}(P(\vec{a})) = 0$ iff* $\vec{a} \in A^{\mathfrak{A}'}_{P2}$ iff $\vec{a} \in A^{\mathfrak{A}'}_{P'2}$ iff* $v_{\mathfrak{A}'}(P'(\vec{a})) = 0$. Therefore, $\mathfrak{A}' \vDash \forall \vec{x}(P(\vec{x}) \leftrightarrow P'(\vec{x}))$, as desired. $\square$

Note that the double-way implications marked with * in the proof would be just single-way implications if R and R' were not complete relations.

The definition above is again a classical one.

Definition 3.3.4 (Explicit Definition of a Predicate)**.** *A set* $\Gamma(P)$ *is said to* explicitly define P *if there exists a formula* $\phi(\vec{x})$ *such that*

$$\Gamma(P) \vDash \forall\vec{x}(P(\vec{x}) \leftrightarrow \phi(\vec{x}))$$

Note that, like the concept of implicit definition, the concept of explicit definition does not depend on the notion of complete relation. On the other hand, like Proposition 3.3.3, Beth's Theorem does.

Theorem 3.3.5 (Beth's Theorem)**.** $\Gamma(P)$ *implicitly defines* P *iff* $\Gamma(P)$ *explicitly defines* P*.*

The 'if' half, that is, 'if $\Gamma(P)$ explicitly defines P, then $\Gamma(P)$ implicitly defines P', does not depend on relation completeness and its proof is pretty easy. The converse, that is, the 'only if' half, does. The proof is the same classical one, provided that Proposition 3.3.3, which is used in the proof, is taken as enunciated here.

All the discussion that has just been done took place in the system that was figured for it. Nevertheless, it can be done in PRS as well. Actually, it could be done even in QmbC$^-$, were Craig's Interpolation Theorem proven for this system.

This section will be closed with the enunciation of another classical theorem that is also an application of Craig's Interpolation Theorem. This result is valid in QmbC as well as in PRS with the same proof, provided, once again, that the negation be understood as the auxiliar classical negation. This result does not depend on the completeness of any relation. Its proof could not be transposed to QmbC$^-$.

Theorem 3.3.6. *Let* Σ_1 *and* Σ_2 *be distinct signatures. Let* Σ *and* Σ' *be their intersection and their union, respectively. Let* L_1*,* L_2*,* L *and* L' *be the respective languages. Suppose that* T *is a complete theory in* L *and that* $T_1 \supset T$ *and* $T_2 \supset T$ *are complete theories in* L_1 *and* L_2*. Then,* $T_1 \cup T_2$ *is consistent in* L'*.*

3.4 Elementary Extensions

In the last two sections, the attempt to work with QmbC$^-$ revealed severe technical difficulties.

The discussion above shows beyond doubt that QmbC$^-$ is very intractable a system from the point of view of model theory, but one can still argue that the problems presented do not provide enough grounds to disqualify it as a worthy system. The same discussion suggests that QmbC is a plainly tractable system that recovers all the classical features with respect to model theory through a defined classical negation. Indeed, in the previous sections, two among the most celebrated results from classical model theory were successfully transposed to the QmbC context, basically with the same proof. Unfortunately, there is a blatant trouble, namely that quasi-isomorphisms preserve very little in terms of validity from one model to another one that is quasi-isomorphic to the first. This problem turns out to be a sensible handicap, for many of the important results in model theory involve isomorphic models and perhaps the most powerful techniques hinge on the broad capacity of preservation of validity that isomorphisms have. Fortunately, there remains a hope: The existence of a consistency operator provides a way of delimiting some set of sentences whose validities are to be preserved by quasi-isomorphisms, in the fashion of Chapter 2. This possibility comes to the rescue of QmbC or at least of some variation of it, but not of QmbC$^-$. This is a serious problem for QmbC$^-$ and may rule it out as a worthy system from the point of view of model theory. Maybe the loss of QmbC$^-$ as a possibility for modeling the world does not call for so much disappointment. In fact, for a class of models to be deprived of a classical negation, it is necessary that it admit models with a single element (for otherwise $\forall x(c \approx x)$ is a bottom particle) and theories that describe whatever aspect of the world (whether a scientific of mathematical world) must refer to a great plurality of objects. Being so, hopes turn back to QmbC exclusively.

As the previous sections suggest, QmbC is indeed a worthy system, but maybe in a somewhat enriched version that shall offer an acceptable range of preservation through quasi-isomorphisms. Here, the focus will be on the Paraconsistent Reasoning System (PRS) defined in Chapter 2.

The next steps lead to an alternative version of the Elementary Extension Theorem presented in Chapter 2. The version presented in Chapter 2 focuses on the possibility of extending a state of knowledge (a model) while preserving the whole of the knowledge it possesses. Now, the focus shifts to what knowledge a given extension preserves and propagates through quasi-isomorphism.

Firstly, some work is required:

Definition 3.4.1 ($S_0^{\mathfrak{A}}$). *Let $\mathfrak{A}$ be a paraconsistent reasoning model. Then, $S_0^{\mathfrak{A}}$ is the set of interpreted formulae defined such that $\theta(\vec{a})$ belongs to $S_0^{\mathfrak{A}}$ iff one of the following cases takes place:*

1. $\theta(\vec{a}) = (\tau_1(\vec{a}) \approx \tau_2(\vec{a}))$;
2. $\theta(\vec{a}) = P(\tau_1 \ldots, \tau_n)[\vec{a}]$ *and* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$;

3. $\theta(\vec{a}) = P(\tau_1 \ldots, \tau_n)[\vec{a}]$ *and* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P3}^{\mathfrak{A}}$;

4. $\theta(\vec{a}) = \circ P(\tau_1 \ldots, \tau_n)[\vec{a}]$ *and* $(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P2}^{\mathfrak{A}}$;

5. $\theta(\vec{a}) = P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \to P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}])$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P4}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P5}^{\mathfrak{A}}$;

6. $\theta(\vec{a}) = P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \vee P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}])$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P6}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P7}^{\mathfrak{A}}$;

7. $\theta(\vec{a}) = P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \wedge P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}])$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P8}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P9}^{\mathfrak{A}}$;

8. $\theta(\vec{a}) = \circ(P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \to P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}]))$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P4}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P5}^{\mathfrak{A}}$;

9. $\theta(\vec{a}) = \circ(P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \vee P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}]))$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P6}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P7}^{\mathfrak{A}}$;

10. $\theta(\vec{a}) = \circ(P(\tau_1[\vec{a}], \ldots, \tau_n[\vec{a}]) \wedge P'(\tau_1[\vec{a}], \ldots, \tau_m[\vec{a}]))$, $(P', \tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in A_{P8}^{\mathfrak{A}}$ *and* $(P, \tau_1^{\mathfrak{A}}, \ldots, \tau_m^{\mathfrak{A}}) \in A_{P9}^{\mathfrak{A}}$;

11. $\theta(\vec{a})$ *is valid or logically false in PRS.*

Lemma 3.4.2. *Let* $\mathfrak{A}$ *be a paraconsistent reasoning model. Let* $\mathfrak{B}$ *be an extension of* $\mathfrak{A}$ *(let* h *be the partial quasi-isomorphism). Then, for every* $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$, $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ *and* $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$. *For basic formulae, the converse holds, that is, if* $\theta(\vec{a})$ *has complexity 0 and* $\theta(\vec{a}) \in Pk(\mathfrak{A})$, *then* $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$.

Proof. Let $\mathfrak{A}'$ and $\mathfrak{B}'$ be such that $\mathfrak{A}' \approxeq \mathfrak{A}$ and $\mathfrak{B}' \approxeq \mathfrak{B}$. Let μ and ν be the respective quasi-isomorphism. As $\mathfrak{B}$ is an extension of $\mathfrak{A}$, $\mathfrak{A}$ and $\mathfrak{A}'$ are quasi-isomorphic and so are $\mathfrak{B}$ and $\mathfrak{B}'$, two facts follow:
Fact 1: For every interpreted term $\tau(\vec{a})$, $I_{\mathfrak{A}'}(\tau(\mu(\vec{a}))) = \mu(I_{\mathfrak{A}}(\tau(\vec{a})))$, $I_{\mathfrak{B}}(\tau(h(\vec{a}))) = h(I_{\mathfrak{A}}(\tau(\vec{a})))$ and $I_{\mathfrak{B}'}(\tau(\nu(h(\vec{a})))) = \nu(I_{\mathfrak{B}}(\tau(h(\vec{a}))))$.
Fact 2: For every sequence $\vec{d}$ of elements in A, $\mu(\vec{d}) \in A_i^{\mathfrak{A}'}$ iff $\vec{d} \in A_i^{\mathfrak{A}}$ iff $h(\vec{d}) \in A_i^{\mathfrak{B}}$ iff $\nu(h(\vec{a})) \in A_i^{\mathfrak{B}'}$.
Let $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$. One of the cases in Definition 3.4.1 takes place and the possibilities must be analized separately:
Case 1: Suppose $\theta(\vec{a}) = (\tau_1(\vec{a}) \approx \tau_2(\vec{a}))$. By fact 1, if $\tau_1(\vec{a})^{\mathfrak{A}} = \tau_2(\vec{a})^{\mathfrak{A}}$, then $\tau_1(\mu(\vec{a}))^{\mathfrak{A}'} = \mu(\tau_1(\vec{a})^{\mathfrak{A}}) = \mu(\tau_2(\vec{a})^{\mathfrak{A}}) = \tau_2(\mu(\vec{a}))^{\mathfrak{A}'}$. Conversely, if $\tau_1(\mu(\vec{a}))^{\mathfrak{A}'} = \tau_2(\mu(\vec{a}))^{\mathfrak{A}'}$, then $\mu(\tau_1(\vec{a})^{\mathfrak{A}}) = \tau_1(\mu(\vec{a})^{\mathfrak{A}'}) = \tau_2(\mu(\vec{a})^{\mathfrak{A}'}) = \mu(\tau_2(\vec{a})^{\mathfrak{A}})$. As μ is an injective function, $\tau_1(\vec{a})^{\mathfrak{A}} = \tau_2(\vec{a})^{\mathfrak{A}}$. Thus, $\tau_1(\mu(\vec{a}))^{\mathfrak{A}'} = \tau_2(\mu(\vec{a}))^{\mathfrak{A}'}$ iff $\tau_1(\vec{a})^{\mathfrak{A}} = \tau_2(\vec{a})^{\mathfrak{A}}$. Applying this same reasoning twice again, it follows that $(\tau_1(\mu(\vec{a}))^{\mathfrak{A}'} = \tau_2(\mu(\vec{a}))^{\mathfrak{A}'})$ iff $(\tau_1(\vec{a})^{\mathfrak{A}} = \tau_2(\vec{a})^{\mathfrak{A}})$ iff $(\tau_1(h(\vec{a}))^{\mathfrak{B}} = \tau_2(h(\vec{a}))^{\mathfrak{B}})$ iff $(\tau_1(\nu(h(\vec{a})))^{\mathfrak{B}'} =$

$\tau_2(\nu(h(\vec{a})))^{\mathfrak{B}'})$. This yields that $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$.
Case 2: Suppose $\theta(\vec{a}) = P(\tau_1 \dots, \tau_n)[\vec{a}]$ and $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$.
By Fact 1, $(\tau_1^{\mathfrak{A}'}[\mu(\vec{a})], \dots, \tau_n^{\mathfrak{A}'}[\mu(\vec{a})]) = (\mu(\tau_1^{\mathfrak{A}}[\vec{a}]), \dots, \mu(\tau_n^{\mathfrak{A}}[\vec{a}]))$,
$(h(\tau_1^{\mathfrak{A}}[\vec{a}]), \dots, h(\tau_n^{\mathfrak{A}}[\vec{a}])) = (\tau_1^{\mathfrak{B}}[h(\vec{a})], \dots, \tau_n^{\mathfrak{B}}[h(\vec{a})])$ and
$(\tau_1^{\mathfrak{B}'}[\nu(h(\vec{a}))], \dots, \tau_n^{\mathfrak{B}'}[\nu(h(\vec{a}))]) = (\nu(\tau_1^{\mathfrak{B}}[h(\vec{a})]), \dots, \nu(\tau_n^{\mathfrak{B}}[h(\vec{a})]))$.
Joining this with Fact 2, $(\tau_1^{\mathfrak{A}'}[\mu(\vec{a})], \dots, \tau_n^{\mathfrak{A}'}[\mu(\vec{a})]) \in A_{P1}^{\mathfrak{A}'}$ iff $(\tau_1^{\mathfrak{A}}[\vec{a}], \dots, \tau_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$ iff $(\tau_1^{\mathfrak{B}}[h(\vec{a})], \dots, \tau_n^{\mathfrak{B}}[h(\vec{a})]) \in A_{P1}^{\mathfrak{B}}$ iff $(\tau_1^{\mathfrak{B}'}[\nu(h(\vec{a}))], \dots, \tau_n^{\mathfrak{B}'}[\nu(h(\vec{a}))]) \in A_{P1}^{\mathfrak{B}'}$. As $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\mu(h(\vec{a}))) = 1$. Therefore, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$, $\theta(\vec{a}) \in Pk(\mathfrak{A})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$, as desired.
Cases from 2 to 10: Analogous to Case 1.
Case 11: Immediate.
Thus, the first part is proven, that is, for every $\theta(\vec{a}) \in S_0$, $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$.
Now, suppose $\theta(\vec{a})$ has complexity 0 and $\theta(\vec{a}) \in Pk(\mathfrak{A})$. It will be proven that $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$.
In fact, if $\theta(\vec{a}) = (\tau_1(\vec{a}) \approx \tau_2(\vec{a}))$, then $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$. If $\theta(\vec{a}) = P(\tau_1, \dots, \tau_n)[\vec{a}]$ and $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P1}^{\mathfrak{A}}$ or $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \in A_{P3}^{\mathfrak{A}}$, then $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$. If $\theta(\vec{a}) = P(\tau_1, \dots, \tau_n)[\vec{a}]$, $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \notin A_{P1}^{\mathfrak{A}}$ and $(t_1^{\mathfrak{A}}[\vec{a}], \dots, t_n^{\mathfrak{A}}[\vec{a}]) \notin A_{P3}^{\mathfrak{A}}$, then $\theta(\vec{a}) \notin Pk(\mathfrak{A})$. These cases exaust the possibilities for $\theta(\vec{a})$ with complexity 0. In any case, $\theta(\vec{a}) \in S_0^{\mathfrak{A}}$ whenever $\theta(\vec{a}) \in Pk(\mathfrak{A})$, as desired. $\square$

The next definition extends the previous one.

Definition 3.4.3 ($S_k^{\mathfrak{A}}$)**.** *Let $\mathfrak{A}$ be a paraconsistent reasoning model. Then, $S_k^{\mathfrak{A}}$ is the set of interpreted formulae defined recursively as follows:*

- *$S_0^{\mathfrak{A}}$ is set as in Definition 3.4.1;*
- *S_{k+1} is so that $\theta(\vec{a}) \in S_{k+1}$ iff*

1. *$\theta(\vec{a}) \in S_k^{\mathfrak{A}}$;*
2. *$\theta(\vec{a}) = (\alpha \vee \beta)(\vec{a})$ and $\{\alpha(\vec{a}), \beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$;*
3. *$\theta(\vec{a}) = (\alpha \vee \beta)(\vec{a})$, $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\alpha(\vec{a})) = 1$ (or $\beta(\vec{a}) \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\beta(\vec{a})) = 1$);*
4. *$\theta(\vec{a}) = (\alpha \wedge \beta)(\vec{a})$ and $\{\alpha(\vec{a}), \beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$;*
5. *$\theta(\vec{a}) = (\alpha \wedge \beta)(\vec{a})$, $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\alpha(\vec{a})) = 0$ (or $\beta(\vec{a}) \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\beta(\vec{a})) = 0$);*

6. $\theta(\vec{a}) = (\alpha \to \beta)(\vec{a})$ *and* $\{\alpha(\vec{a}), \beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$;
7. $\theta(\vec{a}) = (\alpha \to \beta)(\vec{a})$, $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$ *and* $v_{\mathfrak{A}}(\alpha(\vec{a})) = 0$;
8. $\theta(\vec{a}) = (\alpha \to \beta)(\vec{a})$, $\beta(\vec{a}) \in S_k^{\mathfrak{A}}$ *and* $v_{\mathfrak{A}}(\beta(\vec{a})) = 1$;
9. $\theta(\vec{a}) = \exists x\alpha(x)[\vec{a}]$ *and, for every* $a \in A$, $\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$;
10. $\theta(\vec{a}) = \forall x\alpha[\vec{a}]$ *and, for every* $a \in A$, $\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$;
11. $\theta(\vec{a}) = \circ(\alpha \wedge \beta)(\vec{a})$, $\{\circ\alpha(\vec{a}), \circ\beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$ *and* $v_{\mathfrak{A}}(\circ\alpha(\vec{a})) = v_{\mathfrak{A}}(\circ\beta(\vec{a})) = 1$;
12. $\theta(\vec{a}) = \circ(\alpha \vee \beta)(\vec{a})$, $\{\circ\alpha(\vec{a}), \circ\beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$ *and* $v_{\mathfrak{A}}(\circ\alpha(\vec{a})) = v_{\mathfrak{A}}(\circ\beta(\vec{a})) = 1$;
13. $\theta(\vec{a}) = \circ(\alpha \to \beta)(\vec{a})$, $\{\circ\alpha(\vec{a}), \circ\beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$ *and* $v_{\mathfrak{A}}(\circ\alpha(\vec{a})) = v_{\mathfrak{A}}(\circ\beta(\vec{a})) = 1$;
14. $\theta(\vec{a}) = \circ(\exists x\alpha(x))[\vec{a}]$ *and, for every* $a \in A$, $\circ\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$ *and* $v(\circ\alpha[a, \vec{a}]) = 1$;
15. $\theta(\vec{a}) = \circ(\forall x\alpha)[\vec{a}]$ *and, for every* $a \in A$, $\circ\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$ *and* $v(\circ\alpha[a, \vec{a}]) = 1$;
16. $\theta(\vec{a}) = \sim \alpha(\vec{a})$ *and* $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$.

The union of this sequence of sets is designated by $S^{\mathfrak{A}}$ ($S^{\mathfrak{A}} = \cup_{i=1}^{\infty} S_i^{\mathfrak{A}}$).

This definition plays an important role in what follows. It allows a reformulation of the concept of elementary extension.

Definition 3.4.4 (Elementary Extension)**.** *Let* $\mathfrak{A}$ *be a reasoning model. Then,* $\mathfrak{B}$ *is said to be an* elementary extension *of* $\mathfrak{A}$ (*this is denoted by* $\mathfrak{A} \prec \mathfrak{B}$) *iff* $\mathfrak{B} \vDash S^{\mathfrak{A}}$ *and* $S^{\mathfrak{A}} \subset Pk(\mathfrak{B})$.

The next definition is linked to the previous one.

Definition 3.4.5 (Witnessing Extension)**.** *An extension* $\mathfrak{B}$ *of a model* $\mathfrak{A}$ *is said to be a* witnessing extension *iff, given* $\exists x\phi(x, \vec{x})[\vec{a}] \in Pk(\mathfrak{A})$, *the following condition is fulfilled:*
If $v_{\mathfrak{B}'}(\phi(\nu(b), \nu(h(\vec{a})))) = 1$ *for some* $\mathfrak{B}'$ *that is quasi-isomorphic to* $\mathfrak{B}$ ($\mathfrak{B} \approxeq_\nu \mathfrak{B}'$) *and for some* $b \in B$, *then* $v_{\mathfrak{A}}(\phi(a, \vec{a})) = 1$ *for some* $a \in A$.

The next lemma shows the power of the auxiliar negation once again.

Lemma 3.4.6. *Let* $\mathfrak{B}$ *be a witnessing extension of a model* $\mathfrak{A}$ *and let* $\phi(a, \vec{a}) \in S^{\mathfrak{A}}$ *for every* $a \in A$. *If* $v_{\mathfrak{B}'}(\phi(\nu(b), \nu(h(\vec{a})))) = 0$ *for some* $\mathfrak{B}'$ *that is quasi-isomorphic to* $\mathfrak{B}$ ($\mathfrak{B} \approxeq_\nu \mathfrak{B}'$) *and for some* $b \in B$, *then* $v_{\mathfrak{A}}(\phi(a, \vec{a})) = 0$ *for some* $a \in A$.

Proof. If $\phi(a,\vec{a}) \in S^{\mathfrak{A}}$ for every $a \in A$, then, by the definition of $S^{\mathfrak{A}}$, $\sim \phi(a,\vec{a}) \in S^{\mathfrak{A}}$ for every $a \in A$, whence $\exists x \sim \phi(x)[\vec{a}] \in S^{\mathfrak{A}}$.
If $v_{\mathfrak{B}'}(\phi(\nu(b),\nu(h(\vec{a})))) = 0$ for some $b \in B$, then $v_{\mathfrak{B}'}(\sim \phi(\nu(b),\nu(h(\vec{a})))) = 1$ for such b. By the definition of witnessing extension, $v_{\mathfrak{A}}(\sim \phi(a,\vec{a})) = 1$ for some $a \in A$, which implies that $v_{\mathfrak{A}}(\phi(a,\vec{a})) = 0$ for such a, as desired. $\square$

Finally, the main result:

Theorem 3.4.7 (Elementary Extension Theorem II). *Let $\mathfrak{A} = \langle A, I_{\mathfrak{A}}, v_{\mathfrak{A}}\rangle$ be a reasoning model. Then,*

- $S^{\mathfrak{A}} \in Pk(\mathfrak{A})$.

Let still $\mathfrak{B} = \langle B, I_{\mathfrak{B}}, v_{\mathfrak{B}}\rangle$ be an extension of $\mathfrak{A}$ (let h be the partial quasi-isomorphism). Then,

- *$\mathfrak{B}$ is an elementary extension iff it is a witnessing extension.*

Proof. Let $\mathfrak{B}$ be an extension of a given model $\mathfrak{A}$, where h is the partial quasi-isomorphism. The proof will be divided in two parts and the work will start from the easiest one:
($\Rightarrow$) Assume that $\mathfrak{B}$ is an elementary extension. Take $\exists x\theta(x,\vec{x})[\vec{a}] \in S^{\mathfrak{A}}$ and suppose that $v_{\mathfrak{B}'}(\theta(\nu(b),\nu(h(\vec{a})))) = 1$ for some $\mathfrak{B}'$ such that $\mathfrak{B} \approxeq_{\nu} \mathfrak{B}'$ and for some $b \in B$. As $\mathfrak{B}$ is an elementary extension, $v_{\mathfrak{A}}(\exists x\theta(x,\vec{x})[\vec{a}]) = v_{\mathfrak{B}}(\exists x\theta(x,\vec{x})[\vec{a}])$ and $\exists x\theta(x,\vec{x})[\vec{a}] \in Pk(\mathfrak{B})$, which implies $v_{\mathfrak{B}}(\exists x\theta(x,\vec{x})[\vec{a}]) = v_{\mathfrak{B}'}(\exists x\theta(x,\vec{x})[\vec{a}])$. As $v_{\mathfrak{B}'}(\exists x\theta(x,\vec{x})[\vec{a}]) = 1$, it holds that $v_{\mathfrak{A}}(\exists x\theta(x,\vec{x})[\vec{a}]) = 1$, whence $v_{\mathfrak{A}}(\theta(a,\vec{a})) = 1$ for some $a \in A$, as desired.

Now, the difficult part:
($\Leftarrow$) Assume that $\mathfrak{B}$ is a witnessing extension. It will be proven that $\mathfrak{B}$ is an elementary extension. In addition, it will be proven that $S^{\mathfrak{A}} \subset Pk(\mathfrak{A})$. This is the same as to prove that, for every $\theta(\vec{a}) \in S^{\mathfrak{A}}$, $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$. The first step is to prove the result by induction for every $S^{\mathfrak{A}}_k$. Lemma 3.4.2 states it for $S^{\mathfrak{A}}_0$. Suppose the result holds for $S^{\mathfrak{A}}_k$. In order to prove it for $S^{\mathfrak{A}}_{k+1}$, let $\mathfrak{A}'$ be an arbitrary model that is quasi-isomorphic to $\mathfrak{A}$ (let μ be the quasi-isomorphism), let $\mathfrak{B}'$ be an arbitrary model that is quasi-isomorphic to $\mathfrak{B}$ (let ν be the quasi-isomorphism) and let $\theta(\vec{a})$ be an arbitrary interpreted formula that belongs to $S^{\mathfrak{A}}_{k+1}$. For each possibility for $\theta(\vec{a})$, it will be proven that $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a}))))$. This yields that $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$, as desired.

The fact that $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a}))$ holds in $S^{\mathfrak{A}}_0$, by Lemma 3.4.2. In the second part of the induction, that will be presented below, the fact that $\mathfrak{B}$ is a witnessing extension is used to prove that $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta\nu(h((\vec{a})))),$

but it is not used to prove that $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a}))$. Therefore, the conclusion that $S^{\mathfrak{A}} \subset Pk(\mathfrak{A})$ does not depend on the fact that $\mathfrak{B}$ is a witnessing extension. Actually, it does not depend even on the fact that there is an extension of $\mathfrak{A}$ at all. Now, the second step of induction follows:

Case 1: It is just the inductive hypothesis;

Case 2: Suppose that $\theta(\vec{a}) = (\alpha \vee \beta)(\vec{a})$ and $\{\alpha(\vec{a}), \beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$. By the inductive hypothesis, $v_{\mathfrak{A}}(\alpha(\vec{a})) = v_{\mathfrak{B}}(\alpha(h(\vec{a})))$, $v_{\mathfrak{A}}(\beta(\vec{a})) = v_{\mathfrak{B}}(\beta(h(\vec{a})))$, $\{\alpha(\vec{a}), \beta(\vec{a})\} \subset Pk(\mathfrak{A})$ and $\{\alpha(h(\vec{a})), \beta(h(\vec{a}))\} \subset Pk(\mathfrak{B})$. Therefore, by **vOr**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a}))))$, as desired.

Case 3: Suppose that $\theta(\vec{a}) = (\alpha \vee \beta)(\vec{a})$, $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\alpha(\vec{a})) = 1$. By the inductive hypothesis, $v_{\mathfrak{A}}(\alpha(\vec{a})) = v_{\mathfrak{B}}(\alpha(h(\vec{a})))$, $\alpha(h(\vec{a})) \in Pk(\mathfrak{A})$ and $\alpha(h(\vec{a})) \in Pk(\mathfrak{B})$, which yields $v_{\mathfrak{A}'}(\alpha(\mu(\vec{a}))) = v_{\mathfrak{A}}(\alpha(\vec{a})) = v_{\mathfrak{B}}(\alpha(h(\vec{a}))) = v_{\mathfrak{B}'}(\alpha(\nu(h(\vec{a})))) = 1$, which yields, by **vOr**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 1$, as desired. If $\beta(\vec{a}) \in Pk(\mathfrak{A})$ and $v_{\mathfrak{A}}(\beta(\vec{a})) = 1$, the proof is identical.

Cases 4 and 6: Analogous to case 2.

Cases 5, 7 and 8: Analogous to case 3.

Case 9: Suppose that $\theta(\vec{a}) = \exists x \alpha(x)[\vec{a}]$ and, for every $a \in A$, $\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$.
If $v_{\mathfrak{A}}(\theta(\vec{a})) = 1$, then, for some $a \in A$, $v_{\mathfrak{A}}(\alpha[a, \vec{a}]) = 1$. By the inductive hypothesis, $\alpha[a, \vec{a}] \in Pk(\mathfrak{A})$, $\alpha[h(a), h(\vec{a})] \in Pk(\mathfrak{B})$ and $v_{\mathfrak{B}}(\alpha[h(a), h(\vec{a})]) = 1$, whence, by **vEx**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 1$, as desired.

If $v_{\mathfrak{A}}(\theta(\vec{a})) = 0$, then suppose, for the sake of contradiction, that $v_{\mathfrak{B}'}(\alpha[\nu(b), \nu(h(\vec{a}))]) = 1$ for some $b \in B$. By hypothesis ($\mathfrak{B}$ is a witnessing extension), $v_{\mathfrak{A}}(\alpha[a, \vec{a}]) = 1$ for some $a \in A$, which is a contradiction against the fact that $v_{\mathfrak{A}}(\exists x \alpha(x, \vec{x})[\vec{a}]) = 0$. Hence, $v_{\mathfrak{B}'}(\alpha[\nu(b), \nu(h(\vec{a}))]) = 0$ for every $b \in B$. In the same way, $v_{\mathfrak{B}}(\alpha[b, h(\vec{a})]) = 0$ for every $b \in B$, for $\mathfrak{B} \approxeq \mathfrak{B}$. Moreover, $v_{\mathfrak{A}'}(\alpha[\mu(a), \mu(\vec{a})]) = 0$ for every $a \in A$, for $\alpha[a, \vec{a}] \in Pk(\mathfrak{A})$ for every $a \in A$, by the inductive hypothesis. Therefore, by **vEx**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 0$, as desired.

Case 10: Suppose that $\theta(\vec{a}) = \forall x \alpha[\vec{a}]$ and, for every $a \in A$, $\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$.

If $v_{\mathfrak{A}}(\theta(\vec{a})) = 1$, then $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = 1$, for, by the inductive hypothesis, $\alpha[a, \vec{a}] \in Pk(\mathfrak{A})$ for every $a \in A$. Suppose, for the sake of contradiction, that $v_{\mathfrak{B}'}(\alpha[\nu(b), \nu(h(\vec{a})])) = 0$ for some $b \in B$. By Lemma 3.4.6 ($\mathfrak{B}$ is a witnessing extension), $v_{\mathfrak{A}}(\alpha[a, \vec{a}]) = 0$ for some $a \in A$, which is a contradiction against the fact that $v_{\mathfrak{A}}(\forall x \alpha(x, \vec{x})[\vec{a}]) = 1$. Hence, $v_{\mathfrak{B}'}(\alpha[\nu(b), \nu(h(\vec{a}))]) = 1$ for every $b \in B$. In the same way, $v_{\mathfrak{B}}(\alpha[b, h(\vec{a})]) = 1$ for every $b \in B$, for $\mathfrak{B} \approxeq \mathfrak{B}$. Therefore, by **vUni**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 1$, as desired.

If $v_{\mathfrak{A}}(\theta(\vec{a})) = 0$, then, for some $a \in A$, $v_{\mathfrak{A}}(\alpha[a, \vec{a}]) = 0$. By the inductive hypothesis, $\alpha[a, \vec{a}] \in Pk(\mathfrak{A})$, $\alpha[h(a), h(\vec{a})] \in Pk(\mathfrak{B})$ and $v_{\mathfrak{B}}(\alpha[h(a), h(\vec{a})]) = 0$, whence, by **vUni**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 0$, as desired.

Case 11: Suppose that $\theta(\vec{a}) = \circ(\alpha \wedge \beta)(\vec{a})$, $\{\circ\alpha(\vec{a}), \circ\beta(\vec{a})\} \subset S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\circ\alpha(\vec{a})) =$

$v_{\mathfrak{A}}(\circ\beta(\vec{a})) = 1$. By the inductive hypothesis, $v_{\mathfrak{A}}(\circ\alpha(\vec{a})) = v_{\mathfrak{B}}(\circ\alpha(h(\vec{a}))) = 1$, $v_{\mathfrak{A}}(\circ\beta(\vec{a})) = v_{\mathfrak{B}}(\circ\beta(h(\vec{a}))) = 1$ and $\{\circ\alpha(h(\vec{a})), \circ\beta(h(\vec{a}))\} \subset Pk(\mathfrak{B})$.
Hence, $v_{\mathfrak{A}'}(\circ\alpha(\mu(\vec{a}))) = v_{\mathfrak{A}}(\circ\alpha(\vec{a})) = v_{\mathfrak{B}}(\circ\alpha(h(\vec{a}))) = v_{\mathfrak{B}'}(\circ\alpha(\nu(h(\vec{a})))) = 1$ and $v_{\mathfrak{A}'}(\circ\beta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\circ\beta(\vec{a})) = v_{\mathfrak{B}}(\circ\beta(h(\vec{a}))) = v_{\mathfrak{B}'}(\circ\beta(\nu(h(\vec{a})))) = 1$. By **vPropOr**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 1$, as desired.
Cases 12 and 13: Analogous to case 11.
Case 14: Suppose that $\theta(\vec{a}) = \circ(\exists x\alpha(x))[\vec{a}]$ and, for every $a \in A$, $\circ\alpha[a, \vec{a}] \in S_k^{\mathfrak{A}}$ and $v_{\mathfrak{A}}(\circ\alpha[a, \vec{a}]) = 1$. By the inductive hypothesis, for every $a \in A$, $\circ\alpha[a, \vec{a}] \in Pk(\mathfrak{A})$ and $\circ\alpha[h(a), h(\vec{a})] \in Pk(\mathfrak{B})$, whence $v_{\mathfrak{A}'}(\circ\alpha[h(a), h(\vec{a})]) = 1$. Suppose, for the sake of contradiction, that $v_{\mathfrak{B}'}(\circ\alpha[\nu(b), \nu(h(\vec{a}))]) = 0$ for some $b \in B$. By Lemma 3.4.6, $v_{\mathfrak{A}}(\circ\alpha[a, \vec{a}]) = 0$, for some $a \in A$, which is a contradiction against the fact that $v_{\mathfrak{A}}(\circ\alpha[a, \vec{a}]) = 1$ for every $a \in A$. Hence, $v_{\mathfrak{B}'}(\circ\alpha[\nu(b), \nu(h(\vec{a})]) = 1$ for every $b \in B$. By **vPropEx**, $v_{\mathfrak{A}'}(\theta(\mu(\vec{a}))) = v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a}))) = v_{\mathfrak{B}'}(\theta(\nu(h(\vec{a})))) = 1$, as desired.
Case 15: Analogous to case 14.
Case 16: Suppose that $\theta(\vec{a}) =\sim \alpha(\vec{a})$ and $\alpha(\vec{a}) \in S_k^{\mathfrak{A}}$. By the inductive hypothesis, $v_{\mathfrak{A}}(\alpha(\vec{a})) = v_{\mathfrak{B}}(\alpha(h(\vec{a})))$, $\alpha(\vec{a}) \in Pk(\mathfrak{A})$ and $\alpha(h(\vec{a})) \in Pk(\mathfrak{B})$, whence $v_{\mathfrak{A}'}(\alpha(\mu(\vec{a}))) = v_{\mathfrak{A}}(\alpha(\vec{a})) = v_{\mathfrak{B}}(\alpha(h(\vec{a}))) = v_{\mathfrak{B}'}(\alpha(\nu(h(\vec{a}))))$. As $\sim$ behaves like a classical negation, this implies that $v_{\mathfrak{A}'}(\sim \alpha(\mu(\vec{a}))) = v_{\mathfrak{A}}(\sim \alpha(\vec{a})) = v_{\mathfrak{B}}(\sim \alpha(h(\vec{a}))) = v_{\mathfrak{B}'}(\sim \alpha(\nu(h(\vec{a}))))$, as desired. The induction is complete.

Now, let $\theta(\vec{a}) \in S^{\mathfrak{A}} = \cup_{i=1}^{\infty} S_i^{\mathfrak{A}}$. For some indice k, $\theta(\vec{a}) \in S_k^{\mathfrak{A}}$. The result just proven states that $\theta(\vec{a}) \in Pk(\mathfrak{A})$, $\theta(h(\vec{a})) \in Pk(\mathfrak{B})$ and $v_{\mathfrak{A}}(\theta(\vec{a})) = v_{\mathfrak{B}}(\theta(h(\vec{a})))$, as desired. □

Two models $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *equivalents* when $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \prec \mathfrak{A}$. The notation $\mathfrak{A} \equiv \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ are equivalents.

A very interesting classical result is that a family of equivalent models possesses a model that is an elementary extension of each model of the family. In view of the meaning intended for models as states of knowledge, this means that a family of states admits a state that embraces the whole family. In other words, it is possible to extend the scientific discourse in a sound way.

The discussion of extensions of models as a tool for enlarging the scientific discourse will be closed with the transposition of the classical result to the environment of PRS. Firstly, some work is required.

Definition 3.4.8 (Elementary Diagram)**.** *Let $\mathfrak{A}$ be a model in the language L_Σ with domain of interpretation A. Let $C_A = \{c_a\}_{a \in A}$. The* elementary diagram *of $\mathfrak{A}$ is the set of sentences $\Gamma_A = \{\phi | \mathfrak{A}_A \vDash \phi\}$. For each $\phi \in \mathfrak{A}_A$, there is $\psi(x_1, \ldots .x_n) \in L_\Sigma$ such that $\phi = \psi(c_{a_1}, \ldots, c_{a_n})$. Let $\phi' = \psi(x_1, \ldots, x_n)[a_1, \ldots, a_n]$ be the correspondent interpreted formula. If K is a set of interpreted formulae, the K elementary* *diagram of* $\mathfrak{A}$ *is the set of sentences* $\Gamma_A(K) = \{\phi | \phi \in \Gamma_A \text{ and } \phi' \in K\}$.

Lemma 3.4.9. *Let* $\mathfrak{F}$ *be a family of models with disjoint domains and* $\Delta = \cup \Gamma_A(S^{\mathfrak{A}})$. *Suppose that* $\mathfrak{B} \vDash \Delta$. *Then, there exists a model* $\mathfrak{B}'$ *that fulfills the following property:*
Property: *If* $\mathfrak{A} \in \mathfrak{F}$, $\phi(a_1, \ldots, a_n) \in S^{\mathfrak{A}}$ *and* $\mathfrak{D}$ *is quasi-isomorphic to* $\mathfrak{B}'$ ($\mathfrak{B}' \approxeq_\nu \mathfrak{D}$), *then* $v_{\mathfrak{A}}(\phi(a_1, \ldots, a_n)) = v_{\mathfrak{B}'}(\phi(c_{a_1}, \ldots, c_{a_n})) = v_{\mathfrak{D}}(\phi(c_{a_1}, \ldots, c_{a_n}))$.

Before the proof, two observations are in order:
(1) The property of Lemma 3.4.9 can be rephrased as $\mathfrak{B}' \vDash \Delta$ and $\Delta \subset S^{\mathfrak{B}'}$.
(2) The family $\mathfrak{F}$ may be a unary one.

Proof. Define $\mathfrak{B}' = \langle B', I_{\mathfrak{B}'}, v_{\mathfrak{B}'} \rangle$ in the following way: $B' = B$; $v_{\mathfrak{B}'} = v_{\mathfrak{B}}$ and $I_{\mathfrak{B}'}$ is so that, given a predicate symbol P, a model $\mathfrak{A} \in \mathfrak{F}$ and a sequence $(a_1, \ldots, a_n) \in \bar{A}$, if $(b_1, \ldots, b_n) \in \bar{B}'$ interprets the sequence of constants $(c_{a_1}, \ldots, c_{a_n})$ in $\mathfrak{B}$, then $(b_1, \ldots, b_n) \in A^{\mathfrak{B}'}_{Pi}$ iff $(a_1, \ldots, a_n) \in A^{\mathfrak{A}}_{Pi}$.

So defined, $I_{\mathfrak{B}'}$ does not conflict with $v_{\mathfrak{B}'}$. In fact, the sets $A^{\mathfrak{A}}_{Pi}$ determine valuations of interpreted formulae in $S^{\mathfrak{A}}$ and the sets $A^{\mathfrak{B}'}_{Pi}$ determine valuations of interpreted formulae in $S^{\mathfrak{B}'}$. Thus, the definition of $I_{\mathfrak{B}'}$ determines that, if $\phi(a_1, \ldots, a_n) \in S^{\mathfrak{A}}$, then $v_{\mathfrak{B}'}(\phi(c_{a_1}, \ldots, c_{a_n})) = v_{\mathfrak{A}}(\phi(a_1, \ldots, a_n))$, which is in accordance with the defintion of $\mathfrak{B}'$, and that $\phi(c_{a_1}, \ldots, c_{a_n}) \in Pk(\mathfrak{B}')$.
Therefore, $v_{\mathfrak{A}}(\phi(a_1, \ldots, a_n)) = v_{\mathfrak{B}}(\phi(c_{a_1}, \ldots, c_{a_n})) = v_{\mathfrak{D}}(\phi(c_{a_1}, \ldots, c_{a_n}))$ (for $\phi(a_1, \ldots, a_n) \in S^{\mathfrak{A}}$ and $\mathfrak{B} \vDash S^{\mathfrak{A}}$) and $v_{\mathfrak{B}'}(\phi(c_{a_1}, \ldots, c_{a_n})) = v_{\mathfrak{D}}(\phi(c_{a_1}, \ldots, c_{a_n}))$ (for $\phi(c_{a_1}, \ldots, c_{a_n}) \in Pk(\mathfrak{B}')$), as desired. □

Proposition 3.4.10. *Let* $\mathfrak{B}$ *be an extension of* $\mathfrak{A}$. *Then,* $\mathfrak{A} \prec \mathfrak{B}$ *iff there is an extension* $\mathfrak{B}'$ *of* $\mathfrak{B}$ *in* L_{Σ_A} *that is a model of* $\Gamma_A(S^{\mathfrak{A}})$ ($\mathfrak{B}' \vDash \Gamma_A(S^{\mathfrak{A}})$).

Proof. ($\Rightarrow$) Assume that h is an elementary immersion of $\mathfrak{A}$ in $\mathfrak{B}$. It will be proven that $\mathfrak{B}'' = \langle \mathfrak{B}; h(a) \rangle_{a \in A}$ is a model for $\Gamma_A(S^{\mathfrak{A}})$. In fact, let $\phi(c_{a_1}, \ldots, c_{a_n})$ be an arbitrary sentence in $\Gamma_A(S^{\mathfrak{A}})$. Then, $\mathfrak{A} \vDash \phi(a_1, \ldots, a_n)$, by the definition of $\Gamma_A(S^{\mathfrak{A}})$, which implies that $\mathfrak{B} \vDash \phi(h(a_1), \ldots, h(a_n))$, which, on its turn, implies that $\mathfrak{B}'' \vDash \phi(c_{a_1}, \ldots, c_{a_n})$. Therefore, $\mathfrak{B}'' \vDash \Gamma_A(S^{\mathfrak{A}})$. By Lemma 3.4.9, there is a model $\mathfrak{B}'$ such that $\mathfrak{B}' \vDash \Gamma_A(S^{\mathfrak{A}})$ and $\Gamma_A(S^{\mathfrak{A}}) \subset S^{\mathfrak{B}'} \subset Pk(\mathfrak{B}')$.
($\Leftarrow$) Let $\mathfrak{B}' = \langle \mathfrak{B}; h(a) \rangle_{a \in A}$ be an extension of $\mathfrak{B}$ that is a model for $\Gamma_A(S^{\mathfrak{A}})$. By Lemma 3.4.9, it can be assumed that $\mathfrak{B}$ is such that

- For every predicate symbol P and for every $\vec{a} \in \bar{A}$ (with the proper lenth), $\vec{a} \in A^{\mathfrak{A}}_{Pi} \Leftrightarrow h(\vec{a}) \in A^{\mathfrak{B}}_{Pi}$ $(1 \leq i \leq 9)$.

This implies that $\Gamma_A(S^{\mathfrak{A}}) \subset Pk(\mathfrak{B})$. In order to prove that $h : \mathfrak{A} \to \mathfrak{B}$ is a quasi-homomorphism, two points remain to be proven:

- Let c be an arbritrary constant in the original signature. Then, $c^{\mathfrak{A}} = a$ for some $a \in A$. Thus, $\mathfrak{A}_A \vDash (c \approx c_a)$, whence $\mathfrak{B}' \vDash (c \approx c_a)$ (for $(c \approx c_a) \in \Gamma_A(S^{\mathfrak{A}})$). This means that $c^{\mathfrak{B}} = h(a) = h(c^{\mathfrak{A}})$;

- Let f be a function symbol of arity n and let $(a_1, \ldots, a_n)$ be a tuple of elements from A. Then, $f^{\mathfrak{A}}(a_1, \ldots, a_n) = a$ for some $a \in A$. Thus, $\mathfrak{A}_A \vDash (f(c_{a_1}, \ldots, c_{a_n}) \approx c_a)$, whence $\mathfrak{B}' \vDash (f(c_{a_1}, \ldots, c_{a_n}) \approx c_a)$ (for $(f(c_{a_1}, \ldots, c_{a_n}) \approx c_a) \in \Gamma_A(S^{\mathfrak{A}})$). This means that $f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) = h(a) = h(f^{\mathfrak{A}}(a_1, \ldots, a_n))$.

It remains to be proven that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$. By Theorem 3.4.7, it is enough to prove that $\mathfrak{B}$ is an $S^{\mathfrak{A}}$-witnessing extension of $\mathfrak{A}$.

In fact, let $\phi(x, \vec{x})$, $\mathfrak{D}$, $b \in B$ and $\vec{a} \in \bar{A}$ be such that $\exists x \phi(x, \vec{x})[\vec{a}] \in S^{\mathfrak{A}}$, $\mathfrak{B} \approxeq_\nu \mathfrak{D}$ and $\mathfrak{D} \vDash \phi(\nu(b), \nu(h(\vec{a})))$. Then, $\mathfrak{D} \vDash \phi(x, c_{\nu(a_1)}, \ldots, c_{\nu(a_n)})[\nu(b)]$, for $\mathfrak{D} \vDash \phi(\nu(b), \nu(h(a_1)), \ldots, \nu(h(a_1)))$. This is the same as
$\mathfrak{D} \vDash \phi(x, c_{a_1}, \ldots, c_{a_n})[\nu(b)]$, for $c_{\nu(h(a_i))} = c_{a_i}$ $(1 \leq i \leq n)$. Hence,
$\mathfrak{D}' \vDash \exists x \phi(x, c_{a_1}, \ldots, c_{a_n})$. By the way how $\mathfrak{B}$ was taken, as allowed by Lemma 3.4.9, it is the case that $\mathfrak{A} \vDash \exists x \phi(x, c_{a_1}, \ldots, c_{a_n})$. Therefore, $\mathfrak{A} \vDash \phi(a, \vec{a})$ for some $a \in A$, as desired. $\square$

Finally, the promised result.

Proposition 3.4.11. *Let $\mathfrak{F} \neq \emptyset$ be a family of equivalent models. Then, there exists a model $\mathfrak{B}$ such that every model $\mathfrak{A} \in \mathfrak{F}$ is elementarily immersed in $\mathfrak{B}$.*

Proof. For each $\mathfrak{A} \in \mathfrak{F}$, let $\Gamma_A(S^{\mathfrak{A}})$ be its $S^{\mathfrak{A}}$ elementary diagram, as in Definition 3.4.8. Assume that, if $\mathfrak{A} \neq \mathfrak{A}'$, then $\{c_a | a \in A\} \cap \{c_a | a \in A'\} = \emptyset$. Let $L_{\mathfrak{F}}$ be the language obtained from the union of the signatures Σ_A $(\mathfrak{A} \in \mathfrak{F})$ and let $\Delta = \cup_{\mathfrak{A} \in \mathfrak{F}} \Gamma_A(S^{\mathfrak{A}})$. It will be proven that Δ is a consistent set of sentences in $L_{\mathfrak{F}}$.

Let $\{\phi_1, \ldots, \phi_n\}$ be a finite subset of Δ. It can be supposed that $\phi_i \in \Gamma_{A_i}$ for each indice i; If fact, if it is not the case, ψ_i can be defined as the conjunction of the formulae in Δ that belong to Γ_{A_i}, for each indice i.

It can be assumed that, for each $i \in \{1, \ldots, n\}$, there is $k \in \mathbb{N}$ such that $\phi_i = \phi'_i(a_{i,1}, \ldots, a_{i,k})$, where $a_{i,j} \in A_i$ and $\phi'_i(a_{i,1}, \ldots, a_{i,k}) \in S^{\mathfrak{A}_i}$.

As, for each i, $\mathfrak{A}_1 \equiv \mathfrak{A}_i$ and $\exists x_1 \ldots \exists x_k \phi'_i \in S^{\mathfrak{A}_i}$, it holds that

$$\mathfrak{A}_1 \vDash (\exists x_1 \ldots \exists x_k \phi'_1(\vec{x})) \wedge \cdots \wedge (\exists x_1 \ldots \exists x_k \phi'_n(\vec{x}))$$

Let $(b_{1,1}, \ldots, b_{1,k}, b_{2,1}, \ldots, b_{2,k}, \ldots, b_{n,1}, \ldots, b_{n,k})$ be the sequence of elements of A_1 that satisfy the sentence $(\exists x_1 \ldots \exists x_k \phi'_1(\vec{x})) \wedge \cdots \wedge (\exists x_1 \ldots \exists x_k \phi'_n(\vec{x}))$. Then,

$$\langle \mathfrak{A}_1; b_{i,j} \rangle \vDash \phi_1 \wedge \cdots \wedge \phi_n.$$

This shows that $\{\phi_1, \dots, \phi_n\}$ has a model.

By the Compacity Theorem, Δ has a model $\mathfrak{B}' = \langle B', I_{\mathfrak{B}'}, v_{\mathfrak{B}'}\rangle$ in $L_{\mathfrak{F}}$. Take the reduct $\mathfrak{B}$ of $\mathfrak{B}'$ to L_Σ. Thus, for every $\mathfrak{A} \in \mathfrak{F}$, $\mathfrak{B}'$ is an extension of $\mathfrak{B}$ that is a model for $\Gamma_A(S^{\mathfrak{A}})$. By Proposition 3.4.10, $\mathfrak{B}$ is an elementary extension of each $\mathfrak{A} \in \mathfrak{F}$, as desired. □

Of course, the definition of $S^{\mathfrak{A}}$ is just one possible way of delimiting a set of sentences whose valuation may be propagated. Anyway, it seems a good choice, for it captures the whole set of basic interpreted formulae that are preserved and also the propagation of those formulae through all the connectives. This illustrates once again the difficulties in delimiting the whole preservation kernel of a model.

3.5 Chains of Models

Having talked about extensions of models, it is quite natural to talk about chains of models. This section will perform a brief discussion of the subject.

Definition 3.5.1 (Chain of Models)**.** *A* chain of models *of length α is an increasing sequence*

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots \subseteq \mathfrak{A}_\beta \subseteq \dots \quad (\beta \in \alpha)$$

If each extensions is an elementary one, then the referred chain is an elementary one.

Taking the union of a chain of models is quite a natural idea.

Definition 3.5.2 (Union of a Chain of Models)**.** *The union $\mathfrak{A} = \cup_{\beta\in\alpha}\mathfrak{A}_\beta$ of a chain of models*

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots \subseteq \mathfrak{A}_\beta \subseteq \dots \quad (\beta \in \alpha)$$

is the model $\mathfrak{A} = \langle A, I_{\mathfrak{A}}, v_{\mathfrak{A}}\rangle$, where

- $A = \cup_{\beta\in\alpha} A_\beta$;
- *For every $f \in \bar{F}$, $f^{\mathfrak{A}} = \cup_{\beta\in\alpha} f^{\mathfrak{A}_\beta}$;*
- *For every $P \in \bar{P}$, $A^{\mathfrak{A}}_{Pi} = \cup_{\beta\in\alpha} A^{\mathfrak{A}_\beta}_{Pi}$ $(1 \le i \le 9)$;*
- *For every constant $c \in \Sigma$, $I_{\mathfrak{A}}(c) = I_{\mathfrak{A}_0}(c)$.*

The union of an elementary chain of models results in a model that is an elementary extension of each model in the chain.

Theorem 3.5.3. *Let $(\mathfrak{A}_\beta)_{\beta\in\alpha}$ be an elementary chain of models. Then, $\mathfrak{A}_\beta \prec \cup_{\eta\in\alpha}\mathfrak{A}_\eta$ for every $\beta \in \alpha$.*

Proof. By the Elementary Extension Theorem II, it must be proven that $\mathfrak{A} = \cup_{\eta\in\alpha}\mathfrak{A}_\eta$ is a witnessing extension of $\mathfrak{A}_\beta$. It is clear that it is an extension, for $A^{\mathfrak{A}_\beta}_{Pi} = A^{\mathfrak{A}}_{Pi} \cap \bar{A}_\beta$ $(1 \leq i \leq 9)$ for every $P \in \bar{P}$, $I_{\mathfrak{A}}(f(\vec{a})) = I_{\mathfrak{A}_\beta}(f(\vec{a}))$ for every $f \in \bar{F}$ and $I_{\mathfrak{A}}(c) = I_{\mathfrak{A}_\beta}(c)$ for every constant c. For the same reason, if $\vec{a} \in A_\beta$ and $\phi(\alpha)$ is a given interpreted formula, then $\phi(\vec{a}) \in S_0^{\mathfrak{A}_\beta}$ iff $\phi(\vec{a}) \in S_0^{\mathfrak{A}}$. This will be called 'fact 0'.

In order to prove that $\mathfrak{A}$ is indeed a witnessing extension, a fact will come to the rescue:
Fact: Let $\psi(\vec{a})$ be an interpreted formula. If $\psi(\vec{a}) \in S^{\mathfrak{A}}$, then, for every $\eta \in \alpha$, $\psi(\vec{a}) \in S^{\mathfrak{A}_\eta}$ and $v_{\mathfrak{A}_\eta}(\psi(\vec{a})) = v_{\mathfrak{A}}(\psi(\vec{a}))$, whenever $\vec{a} \in A_\eta$.

The result will be proven for each $S_k^{\mathfrak{A}}$ by induction on k. Let $\psi(\vec{a}) \in S_0^{\mathfrak{A}}$. Fact 0 states the desired result.

Suppose the fact holds for $S_k^{\mathfrak{A}}$. Let $\psi(\vec{a}) \in S_{k+1}^{\mathfrak{A}}$ and $\vec{a} \in A_\eta$. Like in the proof of the Elementary Extension Theorem II, each case must be treated separately. The situation now is not altogether analogous, but it is very similar, anyway. For this reason, only Case 9 will be treated. Case 10 is analogous to case 9. Those are the most delicate cases. The non-quantified cases are easier.
Case 9: Suppose that $\psi(\vec{a}) = \exists x\alpha(x,\vec{x})[\vec{a}]$ and, for every $a \in A$, $\alpha(a,\vec{a}) \in S_k^{\mathfrak{A}}$.

If $v_{\mathfrak{A}}(\psi(\vec{a})) = 1$, then $v_{\mathfrak{A}}(\alpha(b,\vec{a})) = 1$, for some $b \in A$. For some $\eta' \geq \eta$, $(b,\vec{a}) \in A_{\eta'}$. By the inductive hypothesis, $v_{\mathfrak{A}_{\eta'}}(\alpha(b,\vec{a})) = 1$, whence $v_{\mathfrak{A}_{\eta'}}(\psi(\vec{a})) = 1$. Still by the inductive hypothesis, $\alpha(a,\vec{a}) \in S_k^{\mathfrak{A}_{\eta'}}$ for every $a \in A_{\eta'}$. By Case 9 of the definition of S_{k+1}, $v_{\mathfrak{A}_{\eta'}}(\psi(\vec{a})) = 1$ and $\psi(\vec{a}) \in S_{k+1}^{\mathfrak{A}_{\eta'}}$. By the inductive hypothesis, $\alpha(a,\vec{a}) \in S_k^{\mathfrak{A}_\eta}$ for every $a \in A_\eta$, which yields that $\psi(\vec{a}) \in S_{k+1}^{\mathfrak{A}_\eta}$. As $\mathfrak{A}_\eta \prec \mathfrak{A}_{\eta'}$, it is the case that $v_{\mathfrak{A}_{\eta'}}(\psi(\vec{a})) = 1$, as desired.

If $v_{\mathfrak{A}}(\psi(\vec{a})) = 0$, then $v_{\mathfrak{A}}(\alpha(b,\vec{a})) = 0$, for every $b \in A$. In particular, $v_{\mathfrak{A}}(\alpha(a,\vec{a})) = 0$, for every $a \in A_\eta$. By the inductive hypothesis, for every $a \in A_\eta$, $\alpha(a,\vec{a}) \in S_k^{\mathfrak{A}_\eta}$ and $v_{\mathfrak{A}_\eta}(\alpha(a,\vec{a})) = 0$. By Case 9 of the definition of S_{k+1}, $v_{\mathfrak{A}_\eta}(\psi(\vec{a})) = 0$ and $\psi(\vec{a}) \in S_{k+1}^{\mathfrak{A}_\eta}$.
In any case, $\psi(\vec{a}) \in S_{k+1}^{\mathfrak{A}_\eta}$ and $v_{\mathfrak{A}_\eta}(\psi(\vec{a})) = v_{\mathfrak{A}}(\psi(\vec{a}))$, as desired.

Finally, it is time to prove that $\mathfrak{A}$ is indeed a witnessing extension of $\mathfrak{A}_\beta$. For that, suppose $\mathfrak{A} \approxeq_\nu \mathfrak{B}$, $\vec{a} \in \bar{A}_\beta$, $b \in A$, $\exists x\phi(x,\vec{x})[\vec{a}] \in S^{\mathfrak{A}_\beta}$ and $v_{\mathfrak{A}}(\phi(b,\vec{a})) = 1$.

Fact 0 states that $S_0^{\mathfrak{A}_\beta} \subset S^{\mathfrak{A}}$. It is easy to prove by induction that $S_k^{\mathfrak{A}_\beta} \subset S^{\mathfrak{A}}$ for every k and, in the sequel, to prove that $S^{\mathfrak{A}_\beta} = \cup_{k\in\mathbb{N}} S_k^{\mathfrak{A}_\beta} \subset S^{\mathfrak{A}}$. This implies that $\exists x\phi(x,\vec{x})[\vec{a}] \in S^{\mathfrak{A}} \subset Pk(\mathfrak{A})$, which implies that $v_{\mathfrak{A}}(\exists x\phi(x,\vec{x})[\vec{a}]) =$

$v_{\mathfrak{B}}(\exists x\phi(x,\vec{x})[\nu(\vec{a})]) = 1$. By the fact, $v_{\mathfrak{A}_\beta}(\exists x\phi(x,\vec{x})[\vec{a}]) = v_{\mathfrak{A}}(\exists x\phi(x,\vec{x})[\vec{a}]) = 1$, which implies that $v_{\mathfrak{A}_\beta}(\phi(a,\vec{a})) = 1$, for some $a \in A_{\mathfrak{A}_\beta}$, as desired. $\square$

A non-denumerable chain makes little sense in the context of scientific discussion, where even energy is countable. In a mathematical context, however, it does make a lot of sense. For this reason, such a concept was presented, even though in the whole of this work the efforts were concentrated in constructing an environment for scientific discourse.

In classical model theory, there is an alternative kind of chain that is similar to that of elementary chains. On the one hand, it preserves a smaller set of formulae. On the other hand, it requires no extra conditions concerning witnesses for quantified formulae. This interesting alternative concept can be transposed to the context of PRS. Firstly, the formulae involved will be presented.

Definition 3.5.4 (Σ^0_0-formula and Π^0_0-formula)**.** *A formula ϕ is a* Σ^0_0-formula *or a* Π^0_0-formula *iff it has no connectives.*

- *If ϕ is a Π^0_n-formula, then $\psi = \exists\vec{x}\phi$ is a Σ^0_{n+1}-formula;*
- *If ϕ is a Σ^0_n-formula, then $\psi = \forall\vec{x}\phi$ is a Π^0_{n+1}-formula.*

Now, the new kind of chain can be defined.

Definition 3.5.5 (Σ^0_n-extension)**.** *An extension $\mathfrak{B}$ of a model $\mathfrak{A}$ is said to be a* Σ^0_n-extension *iff, for every Σ^0_n-formula $\phi(\vec{x})$ and for every $\vec{a}$, if $\phi(\vec{a}) \in S^{\mathfrak{A}}$ and $\mathfrak{A} \vDash \phi(\vec{a})$, then $\phi(\vec{a}) \in S^{\mathfrak{B}}$ and $\mathfrak{B} \vDash \phi(\vec{a})$. A chain of models is said to be a Σ^0_n-chain iff the extensions are Σ^0_n-extensions.*

A theorem that plays the same role with respect to Σ^0_n-chains as Theorem 3.5.3 does with respect to elementary chains will be presented. Firstly, a lemma is in order.

Lemma 3.5.6. *Let $\mathfrak{A}$ be a model in a language L_Σ, $\phi(x_1,\ldots,x_n)$ a formula in L_Σ and $\vec{a} = (a_1,\ldots,a_n)$ a sequence in $\bar{A}$. Let $C = \{c_{a_1},\ldots,c_{a_n}\}$ be a set of new constants and $\mathfrak{A}_C$ as usual. Then, $\phi(a_1,\ldots,a_n) \in S^{\mathfrak{A}}$ iff $\phi(a_1,\ldots,a_n) \in S^{\mathfrak{A}_C}$ iff $\phi(c_{a_1},\ldots,c_{a_n}) \in S^{\mathfrak{A}_C}$. Moreover, if $\phi(a_1,\ldots,a_n) \in S^{\mathfrak{A}}$, then $v_{\mathfrak{A}}(\phi(a_1,\ldots,a_n)) = v_{\mathfrak{A}_C}(\phi(a_1,\ldots,a_n)) = v_{\mathfrak{A}_C}(\phi(c_{a_1},\ldots,c_{a_n}))$.*

Proof. Let $\mathfrak{A}$, ϕ, $\vec{a}$, C and $\mathfrak{A}_C$ be as in the enunciation.

Firstly, for every term $\tau(\vec{x})$, $(\tau(a_1,\ldots,a_n))^{\mathfrak{A}} = (\tau(a_1,\ldots,a_n))^{\mathfrak{A}_C} = (\tau(c_{a_1},\ldots,c_{a_n}))^{\mathfrak{A}_C}$. The proof of this fact is the same as that of the classical case.

Now, the proof can be performed for $S^{\mathfrak{A}}_k$ and $S^{\mathfrak{A}_C}_k$ by induction on k.

First step of induction: $P(x_1,\dots,x_n)[a_1,\dots,a_n] \in S_0^{\mathfrak{A}}$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}}$ or $(a_1,\dots,a_n) \in A_{P3}^{\mathfrak{A}}$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}_C}$ or $(a_1,\dots,a_n) \in A_{P3}^{\mathfrak{A}_C}$ iff $P(x_1,\dots,x_n)[a_1,\dots,a_n] \in S_0^{\mathfrak{A}_C}$. Also, $P(c_{a_1},\dots,c_{a_n}) \in S_0^{\mathfrak{A}_C}$ iff $((c_{a_1})^{\mathfrak{A}_C},\dots,(c_{a_n})^{\mathfrak{A}_C}) \in A_{P1}^{\mathfrak{A}_C}$ or $((c_{a_1})^{\mathfrak{A}_C},\dots,(c_{a_n})^{\mathfrak{A}_C}) \in A_{P3}^{\mathfrak{A}_C}$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}_C}$ or $(a_1,\dots,a_n) \in A_{P3}^{\mathfrak{A}_C}$. This proves the result in case $\phi(\vec{a}) \in S_0^{\mathfrak{A}_C}$ is an interpreted formula of the form $P(\vec{a})$. The other cases for which $\phi(\vec{a}) \in S_0^{\mathfrak{A}_C}$ are analogous.

Finally, the second step of induction is straightforward.

The proof of the second part follows the same line: If $P(c_{a_1},\dots,c_{a_n}) \in S_0^{\mathfrak{A}_C}$, then $v_{\mathfrak{A}}(\phi(a_1,\dots,a_n)) = 1$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}}$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}_C}$ iff $v_{\mathfrak{A}_C}(\phi(a_1,\dots,a_n)) = 1$. Also, $v_{\mathfrak{A}}(\phi(c_{a_1},\dots,c_{a_n}) = 1$ iff $(a_1,\dots,a_n) \in A_{P1}^{\mathfrak{A}_C}$. And so on... □

Finally, the theorem.

Theorem 3.5.7. *Let $(\mathfrak{A}_\beta)_{\beta\in\alpha}$ be a Σ_n^0-chain of models and let $\mathfrak{A} = \cup_{\beta\in\alpha}\mathfrak{A}_\beta$. Then,*

- *$\mathfrak{A}$ is a Σ_n^0-extension of each $\mathfrak{A}_\beta$;*
- *If ψ is a Π_{n+1}^0-sentence such that $\mathfrak{A}_\beta \vDash \psi$ and $\psi \in S^{\mathfrak{A}_\beta}$ for every $\beta \in \alpha$, then $\mathfrak{A} \vDash \psi$ and $\psi \in S^{\mathfrak{A}}$.*

Proof. The proof will be performed by induction on n. The fact that $\mathfrak{A}$ is a Σ_0^0-extension of each $\mathfrak{A}_\beta$ follows easily by induction. In fact, Lemma 3.4.2 guarantees that the interpreted Σ_0^0-formulae in $S_0^{\mathfrak{A}_\beta}$ that are valid in $\mathfrak{A}_\beta$ are also valid in $\mathfrak{A}$ and belong to $S_0^{\mathfrak{A}}$. The rest of the interpreted Σ_0^0-formulae in $S^{\mathfrak{A}_\beta}$ can be recursively obtained from that set without using quantifiers. This fact renders the induction straightforward. Now, let $\psi = \forall\vec{x}\phi(\vec{x})$ be a Π_1^0-sentence such that, for every $\beta \in \alpha$, $\mathfrak{A}_\beta \vDash \psi$ and $\psi \in S^{\mathfrak{A}_\beta}$. Let $\vec{a} \in \vec{A}$ be an arbitrary sequence with the same lenth of $\vec{x}$. For some $\beta \in \alpha$, $\vec{a} \in \bar{A}_\beta$. As $\mathfrak{A}_\beta \vDash \psi$ and $\psi \in S^{\mathfrak{A}_\beta}$, it holds that $\phi(\vec{a}) \in S^{\mathfrak{A}_\beta}$ and $\mathfrak{A}_\beta \vDash \phi(\vec{a})$. As $\phi(\vec{x})$ is a Σ_0^0-formula, it follows that $\mathfrak{A} \vDash \phi(\vec{a})$ and $\phi(\vec{a}) \in S^{\mathfrak{A}}$. As $\vec{a}$ is arbitrary, it is the case that $\mathfrak{A} \vDash \phi(\vec{a})$ and $\phi(\vec{a}) \in S^{\mathfrak{A}}$ for every $\vec{a}$. Therefore, $\mathfrak{A} \vDash \psi$ and $\psi \in S^{\mathfrak{A}}$, as desired. So, the proof is done for $n = 0$.

Suppose the result holds for every $k \leq n$. Let $\psi(\vec{y}) = \exists\vec{x}\phi(\vec{x},\vec{y})$ be a Σ_n^0-formula. Thus, $\phi(\vec{x},\vec{y})$ is a Π_{n-1}^0-formula. Suppose that $\mathfrak{A}_\beta \vDash \psi(\vec{b})$ and $\psi(\vec{b}) \in S^{\mathfrak{A}_\beta}$. It must be proven that $\mathfrak{A} \vDash \psi(\vec{b})$ and $\psi(\vec{b}) \in S^{\mathfrak{A}}$. It follows from $\mathfrak{A}_\beta \vDash \psi(\vec{b})$ that $\mathfrak{A}_\beta \vDash \phi(\vec{a},\vec{b})$ for some $\vec{a} \in \bar{A}_\beta$ and it follows from $\psi(\vec{b}) \in S^{\mathfrak{A}_\beta}$ that $\phi(\vec{a},\vec{b}) \in S^{\mathfrak{A}_\beta}$ for every $\vec{a} \in \bar{A}_\beta$. Let $Y = \{\vec{a},\vec{b}\} = \{a_1,\dots,a_m,b_1,\dots,b_p\}$ and consider

$$(\mathfrak{A}_\beta)_Y \subseteq (\mathfrak{A}_{\beta+1})_Y \subseteq \dots (\mathfrak{A}_\lambda)_Y \subseteq \dots \quad (*)$$

This is a Σ^0_n-chain. If fact, let $\theta(\vec{x}, c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p})$ be a Σ^0_n-formula in L_Y such that $(\mathfrak{A}_\eta)_Y \vDash \theta(\vec{x}, c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p})[\vec{u}]$ and $\theta(\vec{x}, c_{a_1}, \dots, c_{a_m}, c_{b_1} \dots, c_{b_p})[\vec{u}] \in S^{(\mathfrak{A}_\eta)_Y}$ (where $\eta \in \alpha$ and $\beta \leq \eta$). Thus, $(\mathfrak{A}_\eta)_Y \vDash \theta(\vec{x}, \vec{y}, \vec{z})[\vec{u}, \vec{a}, \vec{b}]$ and $\theta(\vec{x}, \vec{y}, \vec{z})$ is a Σ^0_n-formula. Moreover, $\theta(\vec{u}, \vec{a}, \vec{b}) \in S^{(\mathfrak{A}_\eta)_Y}$, by Lemma 3.5.6. Hence, if $\lambda \in \alpha$ and $\eta \leq \lambda$, then $(\mathfrak{A}_\lambda)_Y \vDash \theta(\vec{x}, \vec{y}, \vec{z})[\vec{u}, \vec{a}, \vec{b}]$ and $\theta(\vec{u}, \vec{a}, \vec{b}) \in S^{(\mathfrak{A}_\eta)_Y}$, for $(\mathfrak{A}_\beta)_{\beta\in\alpha}$ is a Σ^0_n-chain.
It follows that $(\mathfrak{A}_\lambda)_Y \vDash \theta(\vec{x}, c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p})[\vec{u}]$ and $\theta(\vec{x}, c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p})[\vec{u}] \in S^{(\mathfrak{A}_\lambda)_Y}$. Hence, (*) is a Σ^0_n-chain. It is immediate that $\mathfrak{A}_Y$ is the union of (*).

Summing up, $\phi(c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p})$ is a Π^0_{n-1}-sentence in L_Y that valid in every $(\mathfrak{A}_\eta)_Y$ and that belongs to every $S^{(\mathfrak{A}_\eta)_Y}$. By the inductive hypothesis applied to (*), $(\mathfrak{A})_Y \vDash \phi(c_{a_1}, \dots, c_{a_m}, c_{b_1} \dots, c_{b_p})$ and $\phi(c_{a_1}, \dots, c_{a_m}, c_{b_1}, \dots, c_{b_p}) \in S^{(\mathfrak{A})_Y}$. Therefore, $\mathfrak{A} \vDash \exists\vec{x}\phi(\vec{x}, \vec{y})[\vec{b}]$ and $\phi(\vec{a}, \vec{b}) \in S^{(\mathfrak{A})_Y}$ (again by Lemma 3.5.6). As $\vec{a}$ is arbitrary, it follows that, for every $\vec{a}$, $\phi(\vec{a}, \vec{b}) \in S^{(\mathfrak{A})_Y}$ and, therefore, $\phi(\vec{a}, \vec{b}) \in S^{(\mathfrak{A})_Y}$. This finally implies that $\mathfrak{A} \vDash \exists\vec{x}\phi(\vec{x}, \vec{y})[\vec{b}]$ and $\exists\vec{x}\phi(\vec{x}, \vec{b}) \in S^{(\mathfrak{A})}$, that is, $\mathfrak{A} \vDash \psi(\vec{b})$ and $\psi(\vec{b}) \in S^{(\mathfrak{A})_Y}$. This proves that $\mathfrak{A}$ is a Σ^0_n-extension of $\mathfrak{A}_\beta$.

For the second part, let $\forall\vec{x}\theta(\vec{x})$ be a Π^0_{n+1}-sentence that is valid in every $\mathfrak{A}_\beta$. Naturally, $\theta(\vec{x})$ is a Σ^0_n-formula. Fix $\vec{a} \in \bar{A}$. For some $\beta \in \alpha$, $\vec{a} \in \bar{A}_\beta$. For such a β, $\mathfrak{A}_\beta \vDash \theta(\vec{a})$ and $\theta(\vec{a}) \in S^{\mathfrak{A}_\beta}$. As θ is a Σ^0_n-formula and $\mathfrak{A}$ is a Σ^0_n-extension, $\mathfrak{A} \vDash \theta(\vec{a})$ and $\theta(\vec{a}) \in S^{\mathfrak{A}}$. As $\vec{a}$ is arbitrary, $\mathfrak{A} \vDash \forall\vec{x}\theta(\vec{x})$ and $\forall\vec{x}\theta(\vec{x}) \in S^{\mathfrak{A}}$, as desired. □

3.6 Axiomatization and Elementary Equivalence

The idea of working with unsafe knowledge is quite natural in science. In Mathematics, it may sound somewhat weird in principle. But it does make sense, at least for two reasons: First, science does not exist without Mathematics. Second, Mathematics itself is full of doubts: Gödel's Incompleteness Theorem states that it must be so. The idea of some knowledge that is plausible while unsafe appears naturally in the form of conjectures.

Being so, the answer to the question of what mathematical paraconsistent reasoning should be is that it should be the same as scientifical paraconsistent reasoning, that is, the act of splitting what is supposedly known into safe and unsafe knowledge and treating the second part carefully.

The most obvious concern of mathematical model theory is that of describing the class of models that satisfy a given theory and describing the set of sentences that are satisfied by a given class of models. This is what this section is about.

Again, the concept of preservation kernel plays a crucial role.

Definition 3.6.1. *Let Γ be a set of sentences in a language L_Σ.*

- *The* collection of models *of Γ is defined to be the class of models $MOD(\Gamma) = \{\mathfrak{A}$ is a model in$L_\Sigma | \mathfrak{A} \models \Gamma\}$.*
- *The* preservation domain *of Γ is defined to be the class of models $Pd(\Gamma) = \{\mathfrak{A}$ is a model in $L_\Sigma | \Gamma \in Pk(\mathfrak{A})\}$.*
- *The* collection of preservation models *of Γ is denoted by $PMOD(\Gamma)$ and is defined to be the intersection of $MOD(\Gamma)$ and $Pd(\Gamma)$. In other terms, $PMOD(\Gamma) = MOD(\Gamma) \cap Pd(\Gamma) = \{\mathfrak{A}$ is a model in$L_\Sigma | \mathfrak{A} \models \Gamma$ and $\Gamma \in Pk(\mathfrak{A})\}$.*

Let M be a collection of models in a language L_Σ.

- *The* theory *of M is defined to be the set of sentences $Th(M) = \{\phi \in L_\Sigma | M \models \phi\}$.*
- *The* preservation kernel *of M is denoted by $Pk(M)$ and is defined to be the intersection of the preservation kernels in M, that is, $Pk(M) = \cap_{\mathfrak{A}\in M} Pk(\mathfrak{A})$.*
- *The* preserved theory *of M is denoted by $PTh(M)$ and is defined to be the intersection of $Th(M)$ and $Pk(M)$. In other terms, $PTh(M) = \{\phi \in L_\Sigma | M \models \phi$ and $\phi \in Pk(M)\}$.*

The proposition below states some basic but useful facts involving the concepts presented in the definition above.

Proposition 3.6.2. .

(i) $\Gamma \subseteq \Gamma'$ *implies* $MOD(\Gamma') \subseteq MOD(\Gamma)$;

(ii) $M \subseteq M'$ *implies* $Th(M') \subseteq Th(M)$;

(iii) $\Gamma \subseteq Th(MOD(\Gamma))$ *and* $MOD(Th(MOD(\Gamma))) = MOD(\Gamma)$;

(iv) $M \subseteq MOD(Th(M))$ *and* $Th(MOD(Th(M))) = Th(M)$;

(i') $\Gamma \subseteq \Gamma'$ *implies* $Pd(\Gamma') \subseteq Pd(\Gamma)$;

(ii') $M \subseteq M'$ *implies* $Pk(M') \subseteq Pk(M)$;

(iii') $\Gamma \subseteq Pk(Pd(\Gamma))$ *and* $Pd(Pk(Pd(\Gamma))) = Pd(\Gamma)$;

(iv') $M \subseteq Pd(Pk(M))$ *and* $Pk(Pd(Pk(M))) = Pk(M)$;

(i") $\Gamma \subseteq \Gamma'$ *implies* $PMOD(\Gamma') \subseteq PMOD(\Gamma)$*;*

(ii") $M \subseteq M'$ *implies* $PTh(M') \subseteq PTh(M)$*;*

(iii") $\Gamma \subseteq PTh(PMOD(\Gamma))$ *and* $PMOD(PTh(PMOD(\Gamma))) = PMOD(\Gamma)$*;*

(iv") $M \subseteq PMOD(PTh(M))$ *and* $PTh(PMOD(PTh(M))) = PTh(M)$*.*

Proof. Let Γ and Γ' be two sets of sentences such that $\Gamma \subseteq \Gamma'$. Let $\mathfrak{A} \in MOD(\Gamma')$ and $\phi \in \Gamma$. Then, $\phi \in \Gamma'$, for $\Gamma \subseteq \Gamma'$. Hence, $\mathfrak{A} \vDash \phi$, for $\mathfrak{A} \vDash \Gamma'$. As ϕ is an arbitrary sentence in Γ, $\mathfrak{A} \vDash \Gamma$, that is, $\mathfrak{A} \in MOD(\Gamma)$. As $\mathfrak{A}$ is an arbitrary model in $MOD(\Gamma')$, $MOD(\Gamma') \subseteq MOD(\Gamma)$. Therefore, (i) is proven.

The proof of (ii) is analogous to the proof of (i). The proofs of (i') and (ii') are analogous to the proofs of (i) and (ii). Item (i'') is an immediate consequence of (i) and (i'). In fact, if $\Gamma \subseteq \Gamma'$, then $MOD(\Gamma') \subseteq MOD(\Gamma)$, by (i), and $Pd(\Gamma') \subseteq Pd(\Gamma)$, by (ii). Hence, $MOD(\Gamma') \cap Pd(\Gamma') \subseteq MOD(\Gamma) \cap Pd(\Gamma)$, that is, $PMOD(\Gamma') \subseteq PMOD(\Gamma)$. Analogously, (ii'') follows immediately from (ii) and (ii').

Now, let $\phi \in \Gamma$ and $\mathfrak{A} \in MOD(\Gamma)$. Then, $\mathfrak{A} \vDash \phi$. As $\mathfrak{A}$ is an arbitrary model in $MOD(\Gamma)$, $\phi \in Th(MOD(\Gamma))$. As ϕ is an arbitray sentence in Γ, $\Gamma \subseteq Th(MOD(\Gamma))$. Applying (i) to the inclusion just proven yields $MOD(Th(MOD(\Gamma))) \subseteq MOD(\Gamma)$ (a). Again, let $\mathfrak{A} \in MOD(\Gamma)$. If $\phi \in Th(MOD(\Gamma))$, then $\mathfrak{A} \vDash \phi$. As ϕ is an arbitray sentence in $Th(MOD(\Gamma))$, $\mathfrak{A} \vDash Th(MOD(\Gamma))$, whence $\mathfrak{A} \in MOD(Th(MOD(\Gamma)))$. As $\mathfrak{A}$ is an arbitrary model in $MOD(\Gamma)$, $MOD(\Gamma) \subseteq MOD(Th(MOD(\Gamma)))$ (b). Joining (a) and (b) yields $MOD(Th(MOD(\Gamma))) = MOD(\Gamma)$. Therefore, (iii) is proven. The proof of (iii') is analogous to the proof of (iii). The proofs of (iv) and (iv') are analogous to the proofs of (iii) and (iii').

The proof of (iii'') does not follow easily from (iii) and (iii') in the same fashion as (i'') and (ii'') follow, respectively, from (i) and (i') and (ii) and (ii'). In order to prove (iii''), let $\phi \in \Gamma$ and $\mathfrak{A} \in PMOD(\Gamma)$. Then, $\mathfrak{A} \vDash \phi$ and $\phi \in Pk(\mathfrak{A})$. As $\mathfrak{A}$ is an arbitrary model in $PMOD(\Gamma)$, $\phi \in Th(PMOD(\Gamma))$ and $\phi \in Pk(PMOD(\Gamma))$, that is, $\phi \in Th(PMOD(\Gamma)) \cap Pk(PMOD(\Gamma)) = PTh(PMOD(\Gamma))$. As ϕ is an arbitrary sentence in Γ, $\Gamma \subseteq PTh(PMOD(\Gamma))$. Applying (i'') to the inclusion just proven yields $PMOD(PTh(PMOD(\Gamma))) \subseteq PMOD(\Gamma)$ (a"). Again, let $\mathfrak{A} \in PMOD(\Gamma)$. If $\phi \in PTh(PMOD(\Gamma))$, then $\mathfrak{A} \vDash \phi$ and $\phi \in Pk(PMOD(\Gamma))$. As ϕ is an arbitrary sentence in $Th(MOD(\Gamma))$, $\mathfrak{A} \in PMOD(Pk(PMOD(\Gamma)))$. As $\mathfrak{A}$ is an arbitrary model in $MOD(\Gamma)$, $MOD(\Gamma) \subseteq MOD(Th(MOD(\Gamma)))$ (b"). Joining (a") and (b") yields $MOD(Th(MOD(\Gamma))) = MOD(\Gamma)$. Therefore, (iii'') is proven. The proor of (iv'') is analogous. $\square$

In classical model theory, a class of models M is said to be axiomatizable when there exists a set of sentences Γ such that $M = MOD(\Gamma)$. In the paraconsistent context, a concept of axiomatization in terms of preservation kernels shall be a much more promising one.

Definition 3.6.3 (Axiomatizable Class of Models)**.** *A class of models M is said to be* axiomatizable *iff there exists a set of sentences Γ such that $M = PMOD(\Gamma)$. If there is a finite set Γ such that $M = PMOD(\Gamma)$, then M is said to be* finitely axiomatizable*. Finally, a mathematical concept is said to be* (finitely) expressable *in the language L_Σ iff the class of models that is the reference of that concept is (finitely) axiomatizable in L_Σ.*

In classical model theory, a class of models M is axiomatizable iff $M = MOD(Th(M))$. This result can be easily transposed to the paraconsistent constext thanks to Proposition 3.6.2.

Proposition 3.6.4. *A class of models M is axiomatizable iff $M = PMOD(PTh(M))$.*

Proof. ($\Leftarrow$) Immediate.
($\Rightarrow$) Let M be an axiomatizable class of models. Then, $M = PMOD(\Gamma)$, for some set of sentences Γ. 'Applying' $PMOD(PTh(\cdot))$ to both members of the equality above yields

$$PMOD(PTh(M)) = PMOD(PTh(PMOD(\Gamma))) = PMOD(\Gamma) = M.$$

□

The next result is again a straight transposition of a classical result.

Proposition 3.6.5. *$PMOD(PTh(M))$ is the smallest axiomatizable class that contains M, that is: if $M \subseteq M'$ and M' is axiomatizable, then $PMOD(PTh(M)) \subseteq M'$.*

Proof. If $M \subseteq M'$, then $PTh(M') \subseteq PTh(M)$, by item (ii'') of Proposition 3.6.2. Hence, by item (i'') of the same proposition, $PMOD(PTh(M)) \subseteq PMOD(PTh(M')) = M'$, as desired. □

It is in order to point that the discussion above fits for PRS as well as for QmbC or whatever paraconsistent system where the concept of preservation kernel is well settled. It is always to be held in mind that PRS is just one possibility among infinitely many of a system that copes with truth propagation through quasi-isomorphisms in paraconsistent environments.

As things are posed, it shall be clear that the classical characterization of axiomatizable classes of models in terms of MOD and Th would not be convenient. The problem is that, when the system under consideration is a one that may control sentences 'from the beginning' and may not do this control, like PRS, if there is a model that satisfies a set of sentences Γ, then there is a model that satisfies Γ and does not control any sentence 'from the beginning', that is, a model whose preservation kernel contains only sentences that are logically determined. In other words, $Pk(MOD(\Gamma))$ is trivial in a certain sense. This is because 'belonging to the preservation kernel' is a notion of the metalanguage.

There are two main alternatives for one who wants to work out the concept of axiomatization in terms of MOD and Th. The first one is to give up propagation through quasi-isomorphisms and concentrate on consistency. The second one is to bring the notion of 'belonging to the preservation kernel' into the language.

The natural way of working out the first alternative is to substitute the concept of $PMOD$ by an analog concept in terms of consistency. For that, it is needed a system that shall be able to guarantee the consistency of logically determined formulae and the closeness of consistency under entailment. In such a context, a the new concept that suits the task is: $CMOD(\Gamma) = \{\mathfrak{A} | \mathfrak{A} \vDash \Gamma \cup \circ\Gamma\}$, where $\circ\Gamma = \{\circ\phi | \phi \in \Gamma\}$. The concept of $CMOD$ shall play the same role as $PMOD$. In this context, a class of models M would be said to be axiomatizable when there is a set of sentences Γ such that $M = CMOD(\Gamma)$.

In order to work out the second alternative, it would be necessary to count with new a symbol, say '$\boxdot$', that should impose that $v_{\mathfrak{A}}(\boxdot\phi) = 1$ iff $\phi \in Pk(\mathfrak{A})$.

A blend of the two alternatives may be a good choice. In fact, if the available apparatus in the structures guarantees that consistency is closed under entailment and propagates through quasi-isomorphisms, then the consistency symbol brings the notion of 'belonging to the preservation kernel' into the language, at least to some extent.

The very promise of this chapter is to work out paraconsistent model theory. The first practical case to be analyzed is that of the class of models in the standard signature of arithmetic and the concept of interest is that of being a field. The interpretation proposed for a paraconsistent model in science is that of a state of knowledge where some assertions are consistent and some are not. For mathematical models, the same interpretation fits well. The result that follows is a corollary of Proposition 3.6.4. Firstly, a lemma is in order.

Lemma 3.6.6. *If $M = PMOD(\Gamma)$ and M is finitely axiomatizable, then $M = PMOD(\Gamma_0)$ for some finite $\Gamma_0 \subseteq \Gamma$.*

Proof. Suppose there exists a sentence σ such that $M = PMOD(\Gamma) = PMOD(\sigma)$.

Then, for every model $\mathfrak{A}$, $\mathfrak{A} \vDash \Gamma$ and $\Gamma \in Pk(\mathfrak{A})$ iff $\mathfrak{A} \vDash \sigma$ and $\sigma \in Pk(\mathfrak{A})$. By the 'only if' part, $\Gamma \vDash \sigma$. By compacity, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \sigma$. If $\mathfrak{A} \vDash \sigma$, then $\mathfrak{A} \vDash \Gamma$; in particular, $\mathfrak{A} \vDash \Gamma_0$, whence $MOD(\sigma) \subseteq MOD(\Gamma_0)$. Conversely, if $\mathfrak{A} \vDash \Gamma_0$, then $\mathfrak{A} \vDash \sigma$, whence $MOD(\Gamma_0) \subseteq MOD(\sigma)$. Joining the two parts, $MOD(\Gamma_0) = MOD(\sigma)$. Now, consider an arbitrary $\mathfrak{A} \in Pd(\Gamma_0)$. Consider an arbitrary $\mathfrak{A}' \approxeq \mathfrak{A}$. Then, $\mathfrak{A}' \vDash \Gamma_0$, whence $\mathfrak{A}' \vDash \sigma$, for $\Gamma_0 \vDash \sigma$. Thus, $\sigma \in Pk(\mathfrak{A})$, which means that $\mathfrak{A} \in Pd(\sigma)$. Therefore, $Pd(\Gamma_0) \subseteq Pd(\sigma)$. Analogously, $Pd(\sigma) \subseteq Pd(\Gamma_0)$. Joining the two parts, $Pd(\Gamma_0) = Pd(\sigma)$. Finally, $MOD(\Gamma_0) \cap Pd(\Gamma_0) = MOD(\sigma) \cap Pd(\sigma)$, that is, $PMOD(\Gamma_0) = PMOD(\sigma) = PMOD(\Gamma)$, as desired. □

Finally, the corollary.

Corollary 3.6.7. *The property of being a field of characteristic 0 is axiomatizable, but not finitely axiomatizable.*

Proof. Consider the standard signature Σ and the usual set Γ of axioms of field theory. Let $\Gamma^* = \Gamma \cup \{\sim (p.1 \approx 0) | p \text{ is a prime}\}$. Clearly, $M = MOD(\Gamma^*)$ is the class of models of characteristic 0 and $M = PMOD(\Gamma^*)$ is the class of fields of characteristic 0 whose quasi-isomorphic models are fields of characteristic 0 as well.

Suppose, for the sake of contradiction, that M is finitely axiomatizable. By Lemma 3.6.6, $M = PMOD(\Gamma_0)$ for some finite $\Gamma_0 \subseteq \Gamma$. As Γ_0 is a finite set, there are finitely many sentences of the form $\sim (p.1 \approx 0)$, whence some prime p_0 is greater than any prime in Γ_0. Hence, any field of characteristic p_0, say $\mathbb{Z}_p$, is a model for Γ^*. But $\mathbb{Z}_p \notin M$, for it does not have characteristic 0. That is a contradiction! □

The perspective of expressing a concept by $PMOD$ is a striking one. In fact, if a given concept is expressed by a given set of sentences, then the class of models obtained is that of models that not only satisfy the properties that characterize that concept, but also preserve that concept through quasi-isomorphisms. Say, for instance, that Γ expresses the concept of being a field. Then, $PMOD(\Gamma)$ is the the class of models that are fields and whose quasi-isomorphic models are fields as well. The approaches can be meshed if it is convenient to do so. Say, for instance, that Γ expresses the concept of being a group and $\Gamma \cup \Gamma'$ expresses the concept of being an abelian group. Then, $MOD(\Gamma \cup \Gamma') \cap PMOD(\Gamma)$ is the class of abelian groups whose quasi-isomorphisms are groups, but not necessarily abelian groups.

As already pointed, expressibility can be stated in terms of $CMOD$, so that the focus is pinned on consistency. In fact, the whole work that has been carried out concerning preservation through quasi-isomorphisms can be carried out with the concern pinned on consistency. This approach tastes like paraconsistent reasoning more than an approach concerned rather on preservation through quasi-isomorphims. It also seems to grasp more properly the philosophical intension of paraconsistent reasoning. The point is that preservation through quasi-isomorphisms is a vital feature for developing model theory. Thus, it is highly desirable to reach a compromise between the two approaches. The natural way to do so is to blend $PMOD$ and $CMOD$: Define $PCMOD(\Gamma) = PMOD(\Gamma \cup \circ\Gamma)$. Then, a class of models M is axiomatizable when there is a set of sentences Γ such that $M = PCMOD(\Gamma)$.

This definition seems to be the pursued one. If a set Γ axiomatizes a class of models M in this sense, then not only does Γ hold in every model of M, but it is also consistent and is preserved through quasi-isomorphisms. That is, if a concept is axiomatizable in this sense, then it is a consistent concept and is preserved through quasi-isomorphisms. Moreover, it is surely a manageable one, for it relies on the concept of $PMOD$, which means that the developments of this section are applicable to this point of view.

It shall be clear that, for $PCMOD$ to be a sound definition, it is necessary that the logical system under consideration be able to propagate consistency through quasi-isomorphisms, which is the same as to be able to control consistency formulae 'from the beginning'. The developments of this section do not presuppose any specific logical system, although PRS has been standing as the reference. Regarding this system, a great amount of consistency formulae can be controlled from the beginning. Actually, every formula with complexity 0 can have its consistency controlled and it also applies to most of the cases where the complexity is 1. Thus, it may be a suitable system for many situations. For others, some stronger system (in the sense of controlling consistency) may be required.

The sense of paraconsistency in mathematics shall be the same one as in science: some facts are safe knowledge and some are not. The meaning of 'safe' or 'unsafe' dwells in the realm of epistemology. In this line, to say that the set of sentences Γ axiomatizes a given concept is to say that, if Γ is held in a model $\mathfrak{A}$ as safe knowledge, then $\mathfrak{A}$ consistently fulfills that concept.

The approach proposed so far refers to certainty or doubt about sentences as a whole, that is, considering the whole domain of each model. There is, however, another approach for paraconsistency, which was the focus in the first two chapters: some fact is consistent for some element of a given domain and inconsistent for some other element of the same domain. It would be the case, for instance, that some portion of a domain behaves like a field. In other words, it is a modal proposal. In order to give life to this approach, fix a domain A and consider the class (actually, the set) of models that have A as their domain. Let $B \subseteq A$. Consider the signature Σ' obtained from Σ (where Σ is the signature under consideration) by substituing $\exists$ by $\exists_B$ and $\forall$ by $\forall_B$. The new quantifiers work in the following way: $\mathfrak{A} \vDash \exists_B x\phi(x)$

iff $\mathfrak{A} \vDash \phi(b)$ for some $b \in B$ and $\mathfrak{A} \vDash \forall_B x\phi(x)$ iff $\mathfrak{A} \vDash \phi(b)$ for every $b \in B$. Now, for each $\phi \in L_\Sigma$, let ϕ_B be the formula in L_B obtained by substituting each occurrence of $\exists$ in ϕ by $\exists_B$ and each occurrence of $\forall$ by $\forall_B$. Naturally, if Γ is a set of formulae, then $\Gamma_B = \{\phi_B | \phi \in \Gamma\}$. Finally, define B-satisfaction as follows: $\mathfrak{A} \vDash_B \phi(\vec{a})$ iff $\mathfrak{A} \vDash \phi_B(\vec{a})$, where $\exists_B$ and $\forall_B$ work as described above. Obviously, if $\Gamma \cup \{\phi\} \subseteq L_\Sigma$, then $\Gamma \vDash_B \phi$ iff $\Gamma_B \vDash \phi_B$. In light of what has just been done, define $MOD_B(\Gamma) = \{\mathfrak{A} | \mathfrak{A} \vDash_B \Gamma\}$.

Having provided the necessary definitions, the proposed modal concept of axiomatization can be stated: A set of sentences Γ axiomatizes a class of models M with respect to B iff $M = PCMOD_B(\Gamma)$. This modal concept is able to capture the idea that doubt may lie on elements of the domain, rather than on sentences as a whole.

Chapter 4

Toward a Paraconsistent Reasoning Prolog

The system developed in Chapter 2 under the name Paraconsistent Reasoning System (PRS) is a proposal of a tool to be used in real life situations where inconsistency inexorably bulges. In Chapter 2, the system was developed and was shown to be philosophically plausible. In Chapter 3, it was shown that no weaker system would satisfactorily perform the task, that is, PRS is indeed what it should be. In this chapter, the idea is to look for an application of PRS. Such an application will be found in computer science; more precisely, a paraconsistent version of PROLOG will be proposed.

It is in order to remark that other works, previous to this one, have pointed for what a paraconsistent PROLOG should be and have indeed reached interesting results (see [37] and [38]). The developments in this chapter go in a quite different direction.

The whole theory that is needed to understand what follows can be found in [32]. Section 1.5 makes a brief presentation of the subject.

Before starting the journey, the definition of Herbrand interpretation will be presented. It is so important as to deserve a distinguished presentation.

Definition 4.0.1 (Herbrand Interpretation)**.** *A interpretation I is a Herbrand interpretation if the domain of interpretation D is the set of terms in the language and*

- *If c is a constant, then $I(c) = c$;*

- *If f is a function symbol of arity n and $\tau_1, \ldots, \tau_n$ is a sequence of terms, then $I(f)(\tau_1, \ldots, \tau_n) = f(\tau_1, \ldots, \tau_n)$*

In other words, a Herbrand interpretation is a one that interprets each term as itself.

4.1 An Essay on Paraconsistent Programming

In [32], a program that sorts a list of non-negative integers into a list in which the elements are in increasing order is presented. The base language comprises a single constant '0', which is intended to work as the zero in the integers, a single function symbol 'f' (of arity 1), which is intented to work as the successor function and five predicate symbols, namely, 'sorted' (of arity 1), '$\leq$', 'perm' and 'sort' (of arity 2) and 'delete' (of arity 3). Variables designate lists of terms. Those lists must be unary when applied to 'f' and '$\leq$'. The symbol '*nil*' designate the empty list and two lists with a dot between them designates the concatenation of those lists. As usual, the pre-interpretation considered is that of Herbrand and the valuation function is fully determined by the program.

SLOWSORT PROGRAM

sort$(x, y) \leftarrow$sorted(y), perm(x, y)

sorted$(nil) \leftarrow$

sorted$(x.nil) \leftarrow$

sorted$(x.y.z) \leftarrow x \leq y$, sorted$(y.z)$

perm$(nil, nil) \leftarrow$

perm$(x.y, u.v) \leftarrow$ delete$(u, x.y, z)$, perm(z, v)

delete$(x, x.y, y) \leftarrow$

delete$(x, y.z, y.w) \leftarrow$ delete(x, z, w)

$0 \leq x$

$f(x) \leq f(y) \leftarrow x \leq y$

As defined above, delete(x, y, z) holds if z is the list obtained by deleting the element x from the list y; perm(x, y) holds if the list y is a permutation of the list x, sorted(x) holds if the list x is in the increasing order and sort(x, y) holds if the list y is the permutation of the list x that is in increasing order.

Now, suppose a group of people is given the task of sorting a list of members of that group into a list in which people are listed in increasing order of height. In order to accomplish the task, an observer is asked to compare with the naked eye each pair of people from that group and say who is the tallest one. It is likely that the observer will not always be sure about his or her judgement. So, he or she will also be asked to say whether that was a confident judgement or a doubtful one. This time, the model to be considered has a constant for each member in the group of people and no function symbols. Also, the valuation for '$\leq$' is not determined by a program, but rather is given by a database.

The great novelty with respect to the first case is that there is doubtful information and some new tool is needed to cope with the problem. In view of the subject of this work, the natural tool is the notion of consistency, that is, if ϕ is a formula, then $\circ\phi$ is a formula that is true if the truth value of ϕ is known as a safe piece of information. It will be convenient to treat the consistency of an atom as an atom itself. The consistency of an atom of the form $a \leq b$ is given by the same database that gives its truth value. An atom of the form $\mathrm{perm}(a, b)$ or $\mathrm{delete}(a, b, c)$ should always be consistent, for an uncertainty about who between two people in a list is the tallest one does not carry any uncertainty into the question of whether a list is a permutation of another or of whether a list is obtained by deleting an element from another. However, it does carry uncertainty into the question of whether a list is in increasing order of height. So, the consistency of atoms of the form $\mathrm{sorted}(a)$ or $\mathrm{sort}(a, b)$ should depend on the consistency of the comparison of the elements from the lists. A suitable program to treat the situation can be obtained by eliminating the last two lines (that define '$\leq$' and 'f') and adding the following extra lines in order to rule consistency.

$\circ\mathrm{sort}(x, y) \leftarrow \circ\mathrm{sorted}(y)$

$\circ\mathrm{sorted}(nil) \leftarrow$

$\circ\mathrm{sorted}(x.nil) \leftarrow$

$\circ\mathrm{sorted}(x.y.z) \leftarrow \circ(x \leq y),\ \circ\mathrm{sorted}(y.z)$

$\circ\mathrm{perm}(x, y) \leftarrow$

$\circ\mathrm{delete}(x, y, z) \leftarrow$

The idea of treating the consistency of atoms as atoms themselves matches the idea of enriching the structure of LFI models to control consistency 'from the beginning'. The idea of controlling implications 'from the beginning' is innate in Prolog and so is the idea of controlling a conjunction by attributing it a positive value. Controlling a disjunction by attributing it a positive value is only possible when dealing with disjunctive Prolog and controlling conjunctions and disjunctions in general depends on controlling negative information, which may compromise the monotonicity of the application $T \uparrow$ (see section 1.5). Conveying negative information is a delicate problem to classical Prolog too.

In the light of the considerations above, it is fair to call the program for ordering people PARACONSISTENT SLOWSORT PROGRAM and the kind of approach that is being proposed Paraconsistent Prolog. The task now is to pursue some more features of the Paraconsistent Reasoning System (PRS) defined in Chapter 2.

The first feature to be pursued is the notion of quasi-isomorphism. In classical model theory, isomorphisms preserve truth value for every formula. In PRS, quasi-isomorphisms preserve truth value for a preservation kernel. In Classical Prolog, predicate symbols are defined by a program. In the proposal of a Paraconsistent Prolog that has been gaining shape through these lines, predicate symbols can be defined by a program or by a database, which consists of a sequence of sets $A_1, \ldots, A_n$ that rule the truth values of the predicate symbols that are not defined by the program and of the predicates that indicate their consistency. In order to give life to the idea of preservation kernel, the concept of kernel will be introduced together with that of *Paraconsistent Reasoning Prolog Base*.

Firstly, some auxiliar definitions will be required:

1. A Prolog program $Prog'$ is a subprogram of another Prolog program $Prog$ if the set of clauses of $Prog'$ is a subset of the set of clauses of $Prog$. This is denoted by $Prog' \subseteq Prog$;

2. An interpretation function I for a language L and a set A is a function whose domain is the set of constants, function symbols and predicate symbols from L and

   - The image of a constant is an element of A;
   - The image of a function symbol of arity n is a function from A^n to A;
   - The image of a predicate symbol of arity n is a set of subsets of A^n;

3. An interpretation function I' for a language L and a set A is a subinterpretation of an interpretation I for the same language and the set B if there is a function $h : A \to B$ such that,

   (a) For each constant c in L, $I_{\mathfrak{B}}(c) = h(I_{\mathfrak{A}}(c))$;

   (b) For each function symbol f of arity n and for each $(a_1, \ldots, a_n) \in A^n$, $I_{\mathfrak{B}}(f)(h(a_1), \ldots, h(a_n)) = h(I_{\mathfrak{A}}(f)(a_1, \ldots, a_n))$;

   (c) For each predicate symbol P of arity n and for each $(a_1, \ldots, a_n) \in A^n$, $(h(a_1), \ldots, h(a_n)) \in I_{\mathfrak{B}}(P)$ if $(a_1, \ldots, a_n) \in I_{\mathfrak{A}}(P)$.

This is denoted by $I' \subseteq I$ and h is called an *inclusion function*.
If $I' \subseteq I$ and $I \subseteq I'$, I and I' are said to be equal and this is denoted by $I' = I$.

4. A pair $\langle Prog', I' \rangle$ of a Prolog program is said to be contained in the pair $< Prog, I >$ of the same kind if $Prog' \subseteq Prog$ and $I' \subseteq I$.

Item c in the third definition above may be rephased as $I'(P) \subseteq I(P)$, for every predicate symbol P in L. The case where $A = B$ is worth noting. This is the case for $I_{\mathfrak{A}}$ and $I'_{\mathfrak{A}}$ in the definition below.

Definition 4.1.1 (Paraconsistent Reasoning Prolog Base)**.** *A Paraconsistent Reasoning Prolog Base for a language L is a 4-tuple*
$\mathfrak{A} = \langle A, Prog_{\mathfrak{A}}, I_{\mathfrak{A}}, Ker_{\mathfrak{A}} \rangle$*, where*

- *A is a set, which will be called domain of interpretation;*
- *$Prog$ is a Prolog program;*
- *$I_{\mathfrak{A}}$ is an interpretation relation for L and A;*
- *$Ker_{\mathfrak{A}} =< Prog'_{\mathfrak{A}}, I'_{\mathfrak{A}} >$, where $I'_{\mathfrak{A}}$ is an interpretation function for L and A such that $I'_{\mathfrak{A}} \subseteq I_{\mathfrak{A}}$ and $Prog'_{\mathfrak{A}}$ is a Prolog program such that $Prog'_{\mathfrak{A}} \subseteq Prog_{\mathfrak{A}}$.*

Obs.: $Ker_{\mathfrak{A}}$ is said to be the kernel of $\mathfrak{A}$. If the context is clear, the subscript $\mathfrak{A}$ may be omitted, so that $Prog$, $Prog'$, I, I' and Ker may be written instead of $Prog_{\mathfrak{A}}$, $Prog'_{\mathfrak{A}}$, $I_{\mathfrak{A}}$, $I'_{\mathfrak{A}}$ and $Ker_{\mathfrak{A}}$. Again, if the context is clear, a paraconsistent reasoning prolog model will be just called a model.

The notion of interpretation is exactly the same as the one for the classical case and will not be presented. The notion of valuation of atomic formulae brings something new and will be presented below.

Definition 4.1.2 (Validity in a Paraconsistent Reasoning Prolog Base)**.** *Let $\mathfrak{A} = \langle A, Prop, I, Ker \rangle$ be a paraconsistent reasoning prolog base for a language L, $P \in L$ a predicate of arity n and $\tau_1(\vec{x}), \ldots, \tau_n(\vec{x})$ and $\vec{a}$ a sequence of elements of A with the same length as $\vec{x}$.*

Then, $P(\vec{a})$ is valid in $\mathfrak{A}$ if

1. *$(\tau_1(\vec{a}), \ldots, \tau_n(\vec{a})) \in I(P)$ or*
2. *$P(\tau_1(\vec{a}), \ldots, \tau_n(\vec{a}))$ is a consequence of $Prop$ together with the set of atoms that are valid by item 1.*

As usual, a paraconsistent reasoning prolog base will be just called a base, if confusion is not to rise.

Note that a base does not fully determine a valuation. This is the reason why it would not be appropriate to call it a model. In fact, programs do not validate negative information. Moreover, the definition of base does not require that every positive atom have its validity defined, neither by the program nor by the interpretation function. Finally, the term 'model' is used with different senses in Model Theory and Prolog Theory.

Definition 4.1.3 (Valuation). *A valuation v for a language L is a function from the set IF of interpreted formulae in L to the set $\{V, F\}$ ($v : S \to \{V, F\}$). A valuation v is said to be coherent with the base $\mathfrak{A}$ if, for each predicate symbol P of arity n and each string of interpreted terms $(\tau_1(\vec{a}), \ldots, \tau_n(\vec{a}))$, $v(P(\tau_1(\vec{a}), \ldots, \tau_n(\vec{a}))) = V$ if $(I_{\mathfrak{A}}(\tau_1(\vec{a})), \ldots, I_{\mathfrak{A}}(\tau_n(\vec{a}))) \in I_{\mathfrak{A}}(P)$ and, moreover, the clauses of $Prog_{\mathfrak{A}}$ are fulfilled.*

The definition that follows is the analog to the definition of model in model theory.

Definition 4.1.4 (Valuated Base). *A base $\mathfrak{A}$ is said to be a* valuated base if it is endowed with a valuation, that is, $\mathfrak{A} = \langle A, Prog_{\mathfrak{A}}, I_{\mathfrak{A}}, Ker_{\mathfrak{A}}, v_{\mathfrak{A}} \rangle$, where $\langle A, Prog_{\mathfrak{A}}, I_{\mathfrak{A}}, Ker_{\mathfrak{A}} \rangle$ is a base and $v_{\mathfrak{A}}$ is a valuation that is coherent with it.

The definition below is analog to the definition of model in Prolog theory. Along this chapter, 'model' will signify 'Prolog model'.

Definition 4.1.5 (Prolog Model). *A set of interpreted formulae M is a* Prolog Model for a base $\mathfrak{A}$ if there exists a valuation v such that v is coherent with $\mathfrak{A}$ and $v(\phi(\vec{a})) = V$, for every $\phi(\vec{a}) \in M$.

The proposal of Chapter 2 is to regard models as states of knowledge. Valuated bases are the natural candidates to be regarded as states of knowledge here. However, it would not match the spirit of Prolog, which is the spirit of considering to be known only information that is actually available, that is, information that can be actually obtained by a finite computation through the program, and, in the case that is in focus here, by a finite verification in the sets of the interpretation function. In accordance with the spirit of Prolog, bases are the suitable entities to be regarded as states of knowledge. Each base defines the class of models that can be originated from it. That class is actually a set and can be regarded as the set of possible worlds from a state of knowledge.

If bases are to be regarded as states of knowledge, the next step is to compare them in the same fashion as models were compared in Chapter 2. There, a quasi-homomorphims between two models exists when the preservation kernel of one is contained in the preservation kernel of the other. Here, the kernel of a base was defined to play that role. In fact, it is clear that the larger are the interpretation function and the program, the larger is the set of interpreted formulae it validates. This discussion leads to the following definition:

Definition 4.1.6 (Quasi-Homomorphism and Quasi-Isomorphism)**.** *A base* $\mathfrak{A}$ *is* quasi-homomorphic to a base $\mathfrak{B}$ if $ker_{\mathfrak{A}} \subseteq ker_{\mathfrak{B}}$. Any given inclusion function is a quasi-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
A base $\mathfrak{A}$ is quasi-isomorphic to a base $\mathfrak{B}$ if $ker_{\mathfrak{A}} = ker_{\mathfrak{B}}$. Any given inclusion function is an *quasi-isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$. In this case, $\mathfrak{A}$ and $\mathfrak{B}$ are said to be quasi-isomorphic.

Some considerations are in order:

If h is a quasi-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then h is invertible and its inverse is a quasi-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. The proof of this fact is exactly the same as that for the case of classical isomorphisms.

The concepts of quasi-homomorphism and quasi-isomorphism could have been defined in a more direct way, without the aid of a kernel. However, introducing a kernel provides flexibility to the concepts and is a worthy idea. Anyway, the definition without kernel is equivalent to the particular case where $I'_{\mathfrak{A}} = I_{\mathfrak{B}}$ and $Prog' = Prog$.

Mixing a program and an interpretation function in defining predicate symbols renders the concept of quasi-homomorphism an even more proficuous one. In fact, the definition of a predicate symbol through interpretation function can be extended while some of its feature is enforced by a program. For example, the clause $P(x, z) \leftarrow P(x, y), P(y, z)$ enforces transitivity, that is, the database that composes the interpretation function can be extended as long as transistivity in P is preserved.

The kernel of a base gives rise to a kernel in the same sense as it is introduced in Paraconsistent Reasoning System, there is, an amount of information that is preserved through quasi-isomorphisms.

The introduction of an interpretation function in Prolog does not introduce any criterion to validate negative clauses. In this way, the monotonicity of Prolog is preserved in Paraconsistent Reasoning Prolog. In other words, if there is a quasi-homorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then $\mathfrak{B}$ preserves more information through quasi-isomorphism than $\mathfrak{A}$.

The notion of typed variables is the same as that in the classical case and does not require further explanations.

The developments reached so far are concerned with declarative semantics. In order to reach procedural semantics, it will be convenient to work with Herbrand interpretations, as usual.

One of the most remarkable features that distinguish bases from programs is that each base is linked to an interpretation domain and, being endowed with an interpretation function, determines validity of interpreted formulae directly. Programs, on their turn, determine validity of interpreted formulae only indirectly. At first sight, this fact seems to prevent one to apply the classical methods of SD-resolution. However, the difficulty can be overcome provided that each element of the domain of interpretation that is referred in the interpretation function interprets some constant in the language. Being so, the condition $(a_1, \ldots, a_n) \in I_{\mathfrak{A}}(P)$ can be substituted by the clause $P(c_1, \ldots, c_n) \leftarrow$, where, for each $1 \leq i \leq n$, c_i is a constant that is interpreted by the element a_i ($I_{\mathfrak{A}}(c_i) = a_i$). Even in this case, there remains a difference: The set $I_{\mathfrak{A}}(P)$ may be infinite, so that the solution proposed of reducing bases to programs may lead to an infinite program. In this way, bases can be viewed as programs that may have infinitely many clauses of the form $P(\vec{c}) \leftarrow$.

Herbrand interpretations fulfill the condition that every element in the domain interprets some constant. As this is the case that really matters as long as procedural methods are concerned, bases will be regarded as programs that may have infinitely many clauses of the form $P(\vec{c}) \leftarrow$. It is to be noted that bases, as well as programs, do not convey negative information, which explains the fact that paraconsistency has not raised any strange behavior so far.

The following classical results remain the same and their proofs are also the same, even considering possibly infinitely many sets of clauses.

Proposition 4.1.7. *Let S be a set of clauses. If S has a model, then S has a Herbrand model.*

Corollary 4.1.8. *Let $\mathfrak{A}$ be a base. Then $\mathfrak{A}$ is unsatisfiable iff $\mathfrak{A}$ has no Herbrand models.*

Minimal models will play a central role in the paraconsitent reasoning version of Prolog as it does in the classical version.

Definition 4.1.9 (Minimal Model)**.** *The minimal model $M_{\mathfrak{A}}$ of a set of clauses S is the intersection of all Herbrand models for S.*

It is straightforward to prove that the minimal model is a model.

The definition of minimal model was designed to satisfy the following result, whose proof is the same as in the classical case, thanks to the existence of an auxiliar classical negation. As always, '$\sim$' stands to the auxiliar classical negation, that is, '$\sim \alpha$' is an abbreviation for '$\alpha \to (\circ\alpha \wedge (\alpha \wedge \neg\alpha))$'.

Proposition 4.1.10. *Let $\mathfrak{A}$ be a Herbrand base and $B_{\mathfrak{A}}$ the set of all Herbrand sentences in the language of $\mathfrak{A}$. Then $M_{\mathfrak{A}} = \{\alpha \in B_{\mathfrak{A}} | \alpha$ is a logical consequence of $\mathfrak{A}\}$*

Proof. α is a logical consequence of $\mathfrak{A}$
iff α belongs to every model for $\mathfrak{A}$
iff $\sim \alpha$ belongs to no model for $\mathfrak{A}$
iff $\sim \alpha$ belongs to no Herbrand model for $\mathfrak{A}$, by Proposition 4.1.7
iff α belongs to every Herbrand model for $\mathfrak{A}$
iff $\alpha \in M_{\mathfrak{A}}$. □

Now, the focus will return to the case of ordering a group of people from the guesses of an observer.

As the database generated from the observations is about people height, an order relation should hold, that is, reflexivity, antisymmetry and transitivity should hold. This fact may be used in two different directions: If the database is incomplete, it may be used to complete it; If it is complete, it may be used to verify information. In the first case, adding the following clauses is a straightforward solution:

$x \leq x \leftarrow$

$\circ(x \leq x) \leftarrow$

$x = y \leftarrow x \leq y, y \leq x$

$\circ(x = y) \leftarrow \circ(x \leq y), \circ(y \leq x)$

$x \leq z \leftarrow x \leq y, y \leq z$

$\circ(x \leq z) \leftarrow \circ(x \leq y), \circ(y \leq z)$

Besides the straightforward clauses, new ones may be added in order to gain extra assertions that are likely to be true. For instance, if the observer guessed that $x \leq y$ and $y \leq z$ and he or she is sure about one of these assertions, it is likely that $x \leq z$ even though the other assertion is not safe. In fact, if the observer is sure that $x \leq y$, it is probably the case that there is a considerable difference between x and y and, in case the guess that $y \leq z$ is wrong, it would hardly be the case that the difference between y and z be greater than the former difference. Another reasonable supposition is that, if $x \leq y$, $y \leq z$ and $z \leq w$, then $x \leq w$. This is the same as to consider that the observer does not accumulate mistakes, in the sense that is not likely that two out of three observations are wrong or that one of them is wrong and the error is big enough to exceed the margin of the two correct observations. The clauses that have just been discussed are

$\circ(x \leq z) \leftarrow x \leq y, y \leq z, \circ(x \leq y)$

$\circ(x \leq z) \leftarrow x \leq y, y \leq z, \circ(y \leq z)$

$\circ(x \leq w) \leftarrow x \leq y, y \leq z, z \leq w$

The idea of using clauses to complement a database is in line with the idea of gaining information from inside explored in Chapter 2. In a certain sense, it can be said that a base is a kind of paraconsistent reasoning model endowed with a refinement.

Above, it was said that the fact that the database in question must constitute an order relation could be used in two directions. The firts one, just explored, is to complement an incomplete database. The second one, which will be briefly discussed, is to correct an incorrect database. It is reasonable that, if it is the case that $x \leq y$ and $y \leq z$ but it is not the case that $x \leq z$, then neither $x \leq y$ nor $y \leq z$ can be consistent. That would lead to clauses such as

$\neg \circ (x \leq y) \leftarrow x \leq y, y \leq z, \neg(x \leq z)$

$\neg \circ (y \leq z) \leftarrow x \leq y, y \leq z, \neg(x \leq z)$

However, such clauses introduce negative information. For this reason, a deeper discussion on the subject will not be made.

This section will be closed with two remarks, both of which concerned with revising assertions and the existence of a kernel:

First: The possibility of correcting the database calls for the existence of a kernel. In fact, if every assertion could be revised, the idea of safe knowledge would make little sense.

Second: The interpretation function can be defined with multiple databases for the same purpose. It may be the case, for instance, that several observers be asked to guess about the height of a group of people. Then, an assertion can be considered safely true when most observers agree on it. The existence of a kernel is an interesting aid when dealing with multiple databases too, for it is reasonable to require that the databases have a common kernel of agreement.

4.2 Paraconsistency from a Procedural Point of View

In Section 2, the idea of dealing with uncertainty derived from a database was explored. There is still another kind of uncertainty that appears naturally in Prolog. The point is that a program is intended to compute answers in a finite number of steps. However, some computations may require a large number of steps or even infinitely many steps. For this reason, if a computation has performed a large number of steps and has not succeeded yet, it remains unclear wether it will ever succeed or not.

The circumstance that an atom A is a consequence of a program P does not entail that its negation $\sim A$ be a consequence of P. That is the sense of the assertion that programs do not convey negative information. One natural way of solving this problem is to assume the so-called *Closed World Assumption* (CWA), which rules that if A is not a consequence of P, then $\sim A$ is true. This criterion does provide a classical negation. As long as a declarative semantic is concerned, such a solution is plainly satisfactory. From a procedural point of view, it is necessary to guarantee that decisions can be made within a finite and assessable number of steps.

The way of finding out wether an atom A is a consequence of a program P or not is to give it the goal $\leftarrow A$ and try to find a successful or a failed SLD-resolution. The procedural criterion that corresponds to CWA is that A is true iff there is a successful SLD-resolution and $\sim A$ is true iff there is a failed SLD-resolution to A. The problem is that there may be only infinite successful or failed resolutions to A. An alternative criterion would be to consider that A is true iff there is a finite successful resolution to A and $\sim A$ is true iff there is a failed resolution to A. This would yield a negation that is not complementar, for it could be the case that A and $\sim A$ be simultaneously false. From a practical point of view, it would be completely odd to consider to be false an atom that is valid, in the sense of being a consequence of P. It would not be problematic however to consider $\sim A$ to be false without knowing wether there is a failed resolution or not. A solution could be to propose that A is true iff A is a consequence of P and that $\sim A$ is true iff there is a finitely failed SLD-resolution for A. Unfortunately, a finitely failed SLD-resolution may be hugely long. So, it would be convenient to set a limit for the process. This leads the search for defining that A is true iff A is a consequence of P and $\sim A$ is true iff there exists a failed SLD-resolution for A of depth $\leq k$ for some given k. An SLD-resolution is said to have depth k if each branch has k nodes at most. Fixed k, this negation will be denoted by $\sim_k$.

An interesting feature of $\sim_k$ is that it satisfies a weak form of de Morgan.

Proposition 4.2.1. *Let P be a definite program and $A_1, \ldots, A_n \in B_P$. Then, $P \cup \{\leftarrow A_1, \ldots, A_n\}$ has a refutation of depth $\leq k$ iff some $A_i (1 \leq i \leq n)$ has a refutation of depth $\leq k$.*

Proof. ($\Rightarrow$) The proof will be performed by induction on k.
First Part: Suppose $P \cup \{\leftarrow A_1, \ldots, A_n\}$ has a refutation Ref of depth 1. Let A_i be the selected atom in the only step of Ref. Then, Ref is a refutation of A_i of length 1.

Second Part: Suppose that, if $P \cup \{\leftarrow A_1, \dots, A_n\}$ has a refutation of depth $\leq k$, then some A_i has a refutation of length $\leq k$. Suppose $P \cup \{\leftarrow A_1, \dots, A_n\}$ has a refutation Ref of depth $k+1$. It must be proven that some A_i has a refutation of length $\leq k+1$.
For that, take such a refutation, say Ref and let A_i be the atom selected in the first step, l the number of branches of that step and θ_j the unifier of the branch number j. If, for some $p \neq i$, A_p has a refutation of length $\leq k$, the work is done. Otherwise, let $\{\leftarrow A_1\theta_1, \dots, A_{i-1}\theta_1, B_1^1\theta_1, \dots, B_{m_1}^1\theta_1, A_{i+1}\theta_1, \dots, A_n\theta_1, \dots,$
$\leftarrow A_1\theta_l, \dots, A_{i-1}\theta_l, B_1^l\theta_l, \dots, B_{m_l}^l\theta_l, A_{i+1}\theta_l, \dots, A_n\theta_l\}$ be the set of goals for the branches of the first step of Ref. For each $1 \leq j \leq l$, $m_j > 0$.
In fact, if $m_j = 0$, the application of the inductive hypothesis to the goal $\leftarrow A_1\theta_j, \dots, A_{i-1}\theta_j, A_{i+1}\theta_j, \dots, A_n\theta_j$ yields that some atom among $A_1\theta_j, \dots, A_{i-1}\theta_j, A_{i+1}\theta_j, \dots, A_n\theta_j$ has a refutation of length $\leq k$, which implies that some atom among $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ has a refutation of length $\leq k$, which is a contradiction against the supposition that this is not the case. Hence, by the inductive hypothesis, for each $1 \leq j \leq l$, there is some $B_{q_j}^j\theta_j$ (belonging to the goal $\leftarrow A_1\theta_j, \dots, A_{i-1}\theta_j, B_1^j\theta_j, \dots, B_{m_j}^j\theta_j, A_{i+1}\theta_j, \dots,$
$A_n\theta_j$) that has a refutation Ref_j of depth $\leq k$. A refutation can be built in the following manner: Take the first step just as in Ref. For each branch j, select the atom $B_{q_j}^j\theta_j$ and follow as in Ref_j. This is clearly a refutation of A_i of depth $\leq k+1$, as desired.

($\Leftarrow$) This is the easy part. If Ref is a refutation of depth $\leq k$ for A_i, then Ref' obtained from Ref by exchanging each goal $\leftarrow B_1\theta_1 \dots \theta_m, \dots, B_l\theta_1 \dots \theta_m$ by $\leftarrow B_1\theta_1 \dots \theta_m, \dots, B_l\theta_1 \dots \theta_m, A_1\theta_1 \dots \theta_m, \dots, A_{i-1}\theta_1 \dots \theta_m, A_{i+1}\theta_1 \dots \theta_m, \dots, A_n\theta_1 \dots \theta_m$ is a refutation of depth $\leq k$ for $\leftarrow A_1, \dots, A_n$. □

Corollary 4.2.2. *If $A_1, \dots, A_n$ are atoms, then $\sim_k (A_1 \wedge \dots \wedge A_n)$ is equivalent to $\sim_k A_1 \vee \dots \vee \sim_k A_n$.*

Proof. Let $A_1, \dots, A_n$ be atoms. Then $\sim_k(A_1 \wedge \dots \wedge A_n)$ is valid iff $\leftarrow A_1, \dots, A_n$ has a refutation of length $\leq k$ iff some $A_i (1 \leq i \leq n)$ has a refutation of length $\leq k$ iff $\sim_k A_i$ is valid for some $A_i (1 \leq i \leq n)$ iff $\sim_k A_1 \vee \dots \vee \sim_k A_n$ is valid. □

It should not be surprising that $\sim_k$ behaves classically in some aspects, for the truth of $\sim_k A$ is always safe information. In fact, $\sim_k A$ is true only if it is actually verifiable that A is not a consequence of P.

So defined, $\sim_k$ is a supplementing negation but not a complementing one. In fact, A and $\sim_k A$ may be simultaneously false, although never simultaneously true.

A kind of 'mirror definition' of $\sim_k$ is that of a negation $\daleth_k$ where $\daleth_k\, A$ is true when there is no finite SLD-resolution of depth $\leq k$ for A. This one is a paraconsistent negation. Unlike $\sim_k$, $\daleth_k$ does not respect de Morgan, even in a weak form. With this regard, it is clear that $\daleth_k\, A_1 \vee \cdots \vee \daleth_k\, A_n$ implies $\daleth_k\, (A_1 \wedge \cdots \wedge A_n)$. In fact, a resolution for $A_1, \ldots, A_n$ would be a resolution for each A_i. However, the converse does not hold. In fact, consider the program

$A_1 \leftarrow$

$A_2 \leftarrow$.

Clearly, there is a resolution of depth 1 for $\leftarrow A_1$ and $\leftarrow A_2$, but there is no resolution of depth 1 for $\leftarrow A_1, A_2$, which means that $\daleth_1\, (A_1 \wedge A_2)$ is valid while $\daleth_1\, A_1 \vee \daleth_1\, A_2$ is not.

Joining the ideas behind the proposals above, a negation $\neg_k$ can be defined so that A is true when there exists no failed SLD-resolution for A of depth $\leq k$ and $\neg_k$ is true if there exists no succeeded SLD-resolution of depth $\leq k$. The result is a paraconsistent negation that accommodates the clause **vCiw** (discussed in Chapter 2) in a very natural way.

Ciw $\circ\alpha \vee (\alpha \wedge \neg\alpha)$

A model based on such a negation can well be called a state of knowledge. Better yet, it can be called a state of computation of depth k. Being so, a whole discussion akin to that one performed in Section 2 can be made within a procedural point of view. Moreover, the two kinds of uncertainty presented in these sections can be mixed in order to cope with both at the same time.

Summing up, paraconsistency is a sound concept and a useful tool in a Prolog environment.

Final Considerations

The proposal of this thesis is to develop Model Theory, which is a highly developed classical theory, in paraconsistent basis. This is a theory that was born in the field of philosophy and soon became one of the most mathematical branchs of logic. For this reason, it is mandatory to start giving a philosophical account to the subject and then treat it in mathematical terms. Treating it in mathematical terms shall mean to transpose classical results to the new context. For a new theory to be worth consideration, it is necessary that some application be revealed. Being so, it is also mandatory to search for applications of the new theory developed. As computing is the field where the tools provided by logic are usually applied, it is quite natural to search for applications in this area. Summing up, three tasks impose themselves: First, to give an account of paraconsistent model theory. Second, to show that classical results are amenable to be transposed to a paraconsistent environment and to transpose as many classical results as possible. Third, to find applications.

The answer to the philosophical task of providing an account to paraconsistent model theory was a proposal of a system for paraconsistent reasoning. Starting from the premise that paraconsistency is an ubiquitous phenomenon that belongs to the realm of epistemology, models are proposed as states of knowledge.

At start, it was clear that, although the classical concept of isomorphism would make perfect sense in the paraconsistent context, it would be completely unfruitful, for very little would be preserved in terms of truth value. So, the problem of re-defining isomorphim imposed itself. The solution was to define quasi-isomorphism in an enriched structure, keeping track of truth preserving by the new concept of 'preservation kernel'. The strategy turned out to be a winning one, opening the door to defining a methold for refining knowledge. The account provided by this strategy revealed to be sound and promising. The first task was successfully accomplished.

The solution of defining quasi-isomorphism in enriched structures turned out to raise good results in mathematical issues too. Good classical results that do not involve isomophism, namely Omitting Types Theorem and Craig's Theorem toghether with some consequences, were obtained while a discussion were made on what a system for paraconsistent model theory should be. Results envolving isomorphism were possible thanks to the definition of quasi-isomorphism, which revealed suitable for mathematical besides philosophical purposes. An idea is worthy when it responds to the needs that are imposed by working practice. The idea of quasi-isomorphism in enriched structures responds to working practice in two different fields, which shows that it is the suitable one. The second task was successfully accomplished.

The area that was chosen for the search for applications was that of Logic Programming. The system constructed in the first chapter was not designed to that area, but it fitted so finely to that study that it seems it was. A blend of program and structure was the basic concept for working out the ideas. The concepts developed in the first chapters meshed perfectly with the techniques of Logic Programming. The third task was successfully accomplished.

Further Developments

The task accomplished by this thesis was not that of solving a specific problem. It was rather that of providing a tool for solving problems. For this reason, this is a work that is destined to be continued.

As philosophical, mathematical and computational problems were raised, it is to be expected that questions of these three natures call for further investigation. In the next lines, some of these questions will be posed.

Within a philosophical line of research, questions for investigation pullulate and an acknowledgement is due to Professor Abílio Azambuja Rodrigues Filho for having raised a handful of them.

Among the many points that are worth investigation, the one that deserves the closest attention is how consistency shall propagate. The advantages and disadvantages of assuming **Cwi** is a point that merits exploration. In Chapter 2, its use was considered incovenient for technical reasons; In Chapter 4, it turned out to be fulfilled by be the negation that presented itself as the most suitable one.

Another point that deserves attention is the possibility or the convenience of having different negations coexisting in an ecumenical system.

Within a mathematical line of research, the main task shall be obviously the search for more classical concepts and results that apply to the paraconsistent system that was proposed and how handable they are in such an environment.

The second task is to give a sense to what paraconsistent mathematics would be like. The last section of Chapter 3 points in that direction.

Within a computational line of research, the task is to look fo applications. As the construction of a paraconsistent Prolog was started in Chapter 4, it sounds sensible to start from the point the developments reached.

The monotonicity of the application $T\uparrow$ plays an important role in Logic Programming. In definite programs, where negated predicates are forbidden in the body of a clause, monotonicity holds. In normal programs, where negated predicates are allowed in the body of a clause, monotonicity does not hold.

The introduction of predicates that are negated by a paraconsitent negation, on its turn, does not destroy monotonicity. In fact, programs do not determine the falsity of a predicate, which means that they do not determine the falsity of a predicate and its negation at the same time. This would be a problem for a paraconsistent negation. Monotonicity is lost with a classical negation because a normal program may determine the truth of a predicate and its negation at the same time. But this is not a problem for a paraconsistent negation at all. However ,when consistency clauses are allowed, problems may rise.

Being so, exploring the behavior of paraconsistent negation in systems endowed with a consistency operator with respect to the monotonicity of the operator $T\uparrow$ may be a good line of research. Firstly, because it is an essencial point for continuing the construction of the Paraconsistent Reasoning Prolog; Second, because it may serve to many other alternative constructions. Thus, this is a research that is worthy for its own.

Logic Programming is definitely a fertile soil and many other important questions can be treated. One of those is how paracosnsitency would behave in disjunctive programs. It is to be expected that new possibilities sprout, for this kind of program seems to be the most suitble one to accomodate the propagation clauses, as commented in Chapter 4.

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