UNIVERSIDADE ESTADUAL DE CAMPINAS
Faculdade de Engenharia Elétrica e de Computação

Carlos Rafael Nogueira da Silva

On the Statistics of the Product and the Ratio of Random Envelopes Taken from the α-μ, η-μ, and κ-μ Distributions

Estatísticas do Produto e da Razão de Envoltórias de Desvanecimento Tomadas das Distribuições α-μ, η-μ, e κ-μ

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Orientador: Prof. Dr. Michel Daoud Yacoub

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Abstract

Exact expression for the first order statistics, such as probability density function and cumulative distribution function, of the product and ratio of envelopes taken from the α-μ, η-μ, and κ-μ distributions are obtained in terms of the multivariable Fox H-function. Fairly simple, fast convergent series expansion are also presented as an alternative to numerically evaluate such statistics.

Several applications in wireless communications utilize the product of envelopes, e.g., multihop systems, multiple-input-multiple-output systems, cascaded channel, radar communications, to name but a few. On the other hand, the ratio of envelopes is used in multihop system modeling, spectrum sharing, co-channel interference, physical layer security among many others. Moreover, composite multipath-shadowing can be modeled as a particular case of the product of envelopes.

The α-μ, η-μ, and κ-μ distributions are general fading models encompassing several traditional fading models (Rayleigh, Nakagami-m, Hoyt, Rice and Weibull) as particular case. Thus the results presented in this thesis can be used in a wide range of fading scenarios.

Performance metrics of a cascaded channel, detection probability in UHF RFID system and secrecy capacity are a few application examples shown in this work to illustrate the usefulness and efficiency of the expressions obtained. In particular, the secrecy capacity of a Gaussian wire-tap channel used for device-to-device and vehicle-to-vehicle communications is characterized using data obtained from field measurements conducted at 5.8GHz.

In addition, miscellaneous results related to the new α-η-κ-μ fading model such as probability density function, cumulative distribution function, higher order moments, moment generating function, among others are presented in new and more efficient formulation. The α-η-κ-μ fading model is, virtually, the most complete fading distribution present in the literature which takes into consideration, non-linearities in the physical medium, clusters of multipath, power imbalance between in-phase and quadrature waves, cluster imbalances and dominant components.

Keywords: α-μ distribution; κ-μ distribution; η-μ distribution; multihop systems; MIMO systems; secrecy capacity; cascaded channel; UHF RFID.

Resumo

Expressões exatas para as estatísticas de primeira ordem, tais como função densidade de probabilidade e função distribuição cumulativa, do produto e da razão de envelopes tomadas das distribuições α-μ, η-μ, e κ-μ são obtidas em termos da função Fox-H multivariável. Expansões em séries relativamente simples e com rápida convergência também são apresentadas como alternativa para avaliar numericamente tais estatísticas.

Diversas aplicações em comunicações sem fio fazem uso do produto de envelopes, e.g., sistemas com múltiplos saltos, sistemas com múltiplas entradas e múltiplas saídas, canais em cascata, comunicações de radar, entre outras. Já a razão de envelopes é utilizada na modelagem de canais com múltiplos saltos, compartilhamento espectral, estimação de interferência co-canal, segurança em camada física entre muitas outras. Ainda, o desvanecimento composto multipercurso-sombreamento pode ser obtido como um caso particular do produto de envelopes.

As distribuições α-μ, η-μ, e κ-μ são modelos de desvanecimento genéricos que englobam diversos modelos tradicionais (Rayleigh, Nakagami-m, Hoyt, Rice e Weibull), como casos particulares. Dessa forma, os resultados apresentados nesta tese podem ser utilizados em uma vasta gama de cenários de desvanecimento.

Métricas de desempenho de um canal em cascata, probabilidade de detecção de sistema UHF RFID e capacidade de sigilo são alguns exemplos de aplicação apresentados neste trabalho para ilustrar a utilidade e eficiência das expressões obtidas. Em particular, a capacidade de sigilo de um canal de escuta gaussiano usado em comunicação dispositivo-a-dispositivo e veículo-a-veículo é caracterizado usando dados obtidos de medidas de campo realizados em 5.8 GHz.

Além disso, resultados diversos relativos ao novo modelo de desvanecimento α-η-κ-μ, tais como função densidade de probabilidade, função distribuição acumulada, momentos de maior ordem, entre outras são apresentados com nova e mais eficiente formulação. O modelo de desvanecimento α-η-κ-μ é virtualmente a distribuição de desvanecimento mais completa presente na literatura que leva em consideração não linearidade do meio físico, clusters de multipercurso, desbalanceamento de potência entre fase e quadratura, desbalanceamento de clusters e componentes dominantes.

Palavras-chaves: distribuição α-μ; distribuição κ-μ; distribuição η-μ; sistemas de múltiplos saltos; sistemas MIMO; capacidade de sigilo, canais em cascata; UHF RFID.

“The truth is out there”

Fox Mulder

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List of abbreviations and acronyms

rms	Root Mean Square
CCI	Co-Channel Interference
CDF	Cumulative Distribution Function
CR	Cognitive Radio
D2D	*Device-to-device*
LoS	*Line-of-Sight*
MGF	Moments Generating Function
MIMO	Multiple-Input Multiple-Output
PDF	Probability Density Function
RFID	Radio Frequency Identification
RV	Random Variable
SIR	Signal-to-Interference Ratio
SNR	Signal-to-noise Ratio
UHF	Ultra-High Frequency
V2V	*Vehicle-to-vehicle*

List of Symbols

${}_0F_1(;\cdot;\cdot)$	Particular case of the generalized hypergeometric function
${}_0\tilde{F}_1(;\cdot;\cdot)$	Regularized form of a particular case of the generalized hypergeometric function
${}_1F_1(\cdot;\cdot;\cdot)$	Kummer's confluent hypergeomtric function
${}_1\tilde{F}_1(\cdot;\cdot;\cdot)$	Regularized Kummer's confluent hypergeomtric function
${}_2F_1(\cdot,\cdot;\cdot;\cdot)$	Gauss' hypergeomtric function
$B(\cdot,\cdot)$	The Beta function
$E_\nu(x)$	The exponential integral function
$\mathbb{E}[\cdot]$	Expectation operator
$f^*(s)$	Mellin transform
$F_1(a;b,b';c;x,y)$	One of the Appell hypergeometric series
$\phi_3(a;b;x,y)$	Confluent form of the Appell series
$f_Z(z)$	PDF for the random variate Z
$\Gamma(\cdot)$	The gamma function
$\Gamma(\cdot,\cdot)$	The upper incomplete gamma function
$\gamma(\cdot,\cdot)$	The lower incomplete gamma function
$H_{p,q}^{m,b}[x\|-]$	The Single Variable Fox H-function
$\mathbb{H}[\cdot,\cdot,\cdot,\cdot]$	The multivariable Fox H-function
j	Imaginary particle, $j=\sqrt{-1}$

$\mathscr{L}$	Suitable contour on the complex space
$L_k^{\lambda}(x)$	Generalized Laguerre Polynomial
$\mathbb{V}[\cdot]$	Variance operator
$\mathbf{x}$, β, δ	Vectors of complex number
$\mathbb{Z}$	The set of integer numbers

Contents

Chapter 1

Introduction

Since the dawn of wireless communications with the first radio transmission performed by Marconi (arguably, the Brazilian priest Landell de Moura is claimed to precede Marconi), the wireless medium has been tirelessly studied. Path loss, interference, multipath fading and shadowing are a few examples of phenomena that affect wireless communications. In particular, shadowing is caused by the presence of large obstacles blocking the direct radio path resulting in large signal fluctuations. Its statistics are well characterized by the lognormal distribution. Due to analytical intricacies of the lognormal model, recently, the gamma and the α-μ distributions have been used to describe the shadowing phenomenon [1, 2]. On the other hand, several fading models have been used to describe the multipath fading phenomenon. Nevertheless, each fading distribution is adequate to describe a certain physical model. For instance, *line-of-sight* (LoS) is well described by Rice distribution; Hoyt is typically used to characterize imbalances between phase and quadrature waves. Recently, new fading models have been defined to generalize and include other physical phenomena, such as non-linearities and clustering of multipath. For instance, the α-μ [3] distribution considers both non-linearities and clustering of multipath. In turn the κ-μ [4] distribution includes the effect of dominant components and multipath clustering. The η-μ [4] fading model considers imbalances (or correlation) between the phase and quadrature waves along with multipath clustering.

Very recently, the α-η-κ-μ fading model [5] was proposed. It captures virtually all fading phenomena described in the literature, namely, nonlinearity of the propagation medium, scattered waves, dominant components, and multipath clustering. This way, the α-η-κ-μ model comprises an enormous amount of fading scenarios, including all of those previously cited ones, and others not yet described in the literature. Its building block was taken from [6] as the general quadrature process of a κ-μ model. The joint envelope-phase probability density function (PDF) was obtained in a closed-form expression. From the said joint PDF, envelope PDF and phase PDF can be obtained by a simple integration, and, unfortunately, no closed-form expressions for them are available. An envelope-based approach was then pursued as the sum of two independent and arbitrarily distributed squared κ-μ random variables (RV). Thence, envelope PDF was obtained both in another integral form and in a series expansion form. Envelope cumulative distribution function (CDF) in integral form and series expansion were also provided. A pure phase-based approach would not alleviate the intricacy of the formulations, and the phase PDF is only given in its original integral form.

Typically, shadowing and multipath are analyzed separately, although such approach only holds for stationary scenarios and a joint analysis is required. It is shown in [7] that the composite shadowing-multipath distribution fading can be treated as a special case of the product of RVs. In addition to composite fading, the product of RVs appears in a plethora of wireless communication process. For instance, the equivalent channel between source and destination in a multihop system is modeled as the product of the individual gains in each hop; the cascade channel [8,9] is the result of the product of several RVs; high resolution synthetic aperture radar clutter [10] is modeled as the product of two RVs; the keyhole effect [11–14] in multiple-input multiple-output (MIMO) system utilize the product of two random variates to model the distribution of the elements of the transfer matrix to name but a few examples of applications using the product of RVs.

In the next few year, a 1000-fold growth in mobile traffic is expected [15]. To address the huge capacity increment, a more efficient spectral usage is necessary. Cognitive radio (CR) [16] has been gaining great interest in research due to its ability to adapt its transmission parameters such, as bandwidth, operation frequency, modulation scheme, in accordance to channel characteristic. In this sense, CR has created the opportunity to improve spectral efficiency by dynamically allocating the transmission resources. For instance, CR is able to perform the spectrum sensing and opportunistically occupy a certain primary channel. Of course, a side effect of this is the co-channel interference (CCI) caused to the licensed network. An important metric to measure the CCI is the signal-to-interference ratio (SIR) . In a fading channel scenarios, the SIR is obtained by the ratio of random envelopes. These ratios are also used in a number of applications in wireless communications. For instance, in physical layer security, the probability of positive secrecy capacity is determined by the ratio of RVs. Other applications using the ratio of envelopes can be found in technologies such as multihop communications. Therefore, the better knowledge of the statistics of the product and ratio distribution is definitely important.

1.1 Related Work

1.1.1 Product of Random Variates

In the specialized literature, a plethora of work has been done analyzing the distribution of random envelopes. The PDF and CDF for the product of n Rayleigh RVs are obtained in [17] in terms of the Meijer G-function and it also provides series representation for the cases $n = 3, 4, 5$. This work is expanded in [8] for the product of n Nakagami-m RVs and performance metrics for the cascaded channel is derived. In [9], the PDF, CDF and moment generating function (MGF) of the product of powers of n generalized Nakagami-m are obtained in terms of the Fox H-function and used to evaluate the performance metrics of a cascaded channel. More recently, statistics for the product of two α-μ random variate were obtained in [7] in terms of finite sum of hypergeometric functions, and it established that composite shadowing-multipath fading is a particular case of the product of RVs. The work in [7] is extended for the product of three and n α-μ RVs, respectively, in [18] and [19] given in terms of both the Meijer G-function and finite sum of hypergeometric functions.

1.1.2 Ratio of Random Variates

Recently, the impact of the CCI on the outage probability metric has been of great interest. In what concerns generalized fading channels, the outage probability for the CCI over η-μ/η-μ, η-μ/κ-μ and κ-μ/η-μ is provided in [20] and [21] restricted to integer values for the parameter μ or in limited interference scenarios. The outage probability for the κ-μ/κ-μ scenario is obtained in [22] again, with restrictions for the parameter μ and limited interference. Statistics for the ratio of α-μ RVs are obtained in [23] in terms of both infinite and finite sums of hypergeometric function, in which the results are applied to the analysis of the capacity of spectrum sharing systems. Finally, the work in [23] is expanded in [24] in which the PDF and CDF for the ratio of products of α-μ RVs is obtained in terms of the Meijer G-function.

1.2 Summary of Contributions

The major contributions of this work are threefold: 1) First, the PDFs and CDFs for the ratio of two random envelopes taken from the α-μ, κ-μ and η-μ distribution is derived in exact closed-form in terms of the multivariable Fox H-function; 2) The PDFs and CDFs of the product of two random envelopes taken from the aforementioned distributions are obtained in exact closed-form in terms of the multivariable Fox H-function; and 3) the integral involving the product of a PDF and a CDF of the variates is obtained in terms of the multivariable Fox H-function. Besides, fairly simple, fast convergent series expansions are obtained as an alternative implementation of the multivariable Fox H-function. Except for the product and ratio of α-μ variates, all the results are novel and unprecedented.

Application examples regarding the ratio and product of RVs are also presented. These include performance metrics for the cascaded channel (amount of fading, outage probability, outage capacity), probability of positive secrecy capacity and probability of detection in ultra-high frequency (UHF) radio frequency identification (RFID) systems. In particular, the secrecy capacity of a Gaussian wire-tap channel used for device-to-device (D2D) and vehicle-to-vehicle (V2V) communications is characterized using data obtained from field measurements conducted at 5.8GHz at the Wireless Communications Laboratory of The Queen's University of Belfast.

Other interesting contribution concerns the first order statistics of the α-η-κ-μ fading model – namely PDF, CDF, higher order moments, moment generating function and the bit error rate. These statistics are provided in new functional form which compute faster than existing formulations. Moreover, a first approach for the parameter estimation problem is addressed.

1.3 The Fox H-Function

In 1961, in an attempt to find a most symmetrical Fourier kernel, Charles Fox [25] defined a new function involving Mellin-Barnes integral which is a generalization of the Meijer G-function. It has a vast potential of applications in several fields of science and engineering. This function generalizes a plethora of important functions, such as, exponential, Bessel-type, hypergeometric, Mittag-Leffler, Wright, hyperbolic and trigonometric functions to name but a few. Since then, the Fox H-function has been studied and new generalizations and expansions were obtained, e.g. the multivariable Fox H-function [26] and the extended $\bar{\text{H}}$-function [27]. Recently, the Fox H-function has been extensively used in the wireless communication field to obtain closed-form expressions in a plethora of applications. For instance, expressions for the capacity and bit error probabilities were obtained in terms of the Fox H-function in [28–31]. Spherically invariant channel were characterized in [32] using the Fox H-function. Closed-form expression for the symbol error probability in single and multi-branch diversity over α-μ channel were obtained in [33] and then extended for the more general H-channel in [34].

Unfortunately, the Fox H-function is yet to be implemented in the most popular mathematical packages such as MatLab or Mathematica. Nevertheless, it is possible to find implementations for the Fox H-function. For instance, the authors in [9] provide an efficient implementation for the single variable Fox H-function using an equivalence between the Fox H-function and the Meijer G-function. An alternative implementation for [9] is also propose her. A Python implementation for the multivariable Fox H-function can be found in [34] whose authors claim to efficiently and accurately evaluate the multivariable Fox H-function up to four branches in a few seconds. Alternatively, it is also possible to derive series expansions through the sum of residues [35] with the inconvenience of the number of folded summations being equal to the number of variable which, in general, renders such approach infeasible for more than three or four variables. Nevertheless, it is possible to obtain simpler series expansion depending on the parameters of the Fox H-function.

1.4 Thesis Outline

This thesis is organized as follows:

Chapter 2. This chapter presents the concepts used to develop the statistics for the ratio and product distributions. The product and ratio statistics are given in terms of the multivariable Fox H-function, whose definition is provided in this chapter. The Mellin transform (and its inverse) is the tool used to obtain the statistics for the product distribution and this is also presented. And the fading models α-μ, η-μ, κ-μ and α-η-κ-μ are revisited.

Chapter 3. This chapter presents the analytical formulation necessary to derive the PDFs and CDFs of the ratio distribution of two RVs. Specifically, expressions in terms of the multivariable Fox H-function and series expansion for each combination of ratios involving the α-μ, η-μ, and κ-μ fading distributions are presented. An application example in physical layer security is used to demonstrate the usefulness of the expressions obtained.

Chapter 4. This chapter presents novel, closed-form expressions for the PDFs and CDFs of the product of two RVs taken from the distribution α-μ, η-μ, and κ-μ distributions in terms of the multivariable Fox H-function. Fairly simple power series are presented for each combination of product distributions. An interesting integral involving the product of a PDF by a CDF which is closely related to the CDF of the product distribution is also derived both in terms of the multivariable Fox H-function as well as in terms of relatively simple infinite series. This result finds applications for instance in computing the probability of detection in UHF RFID systems.

Chapter 5. Results from different research topics are presented in this chapter. These include new, more efficient formulations for the first order statistics – namely PDF, CDF, higher order moments, moment generating function and bit error rate – for the α-η-κ-μ fading model are presented. In addition, a first approach to the parameter estimation problem is proposed.

Chapter 6 summarizes the main results, and indicates opportunities for future researches.

Appendix A. In this appendix, the derivation of the multivariable Fox H-function representation for the PDFs and CDFs of the ratio of two random envelopes taken from the α-μ, η-μ, and κ-μ distribution is provided in detail.

Appendix B. in this appendix, the derivation of the multivariable Fox H-function representation for the PDFs and the CDFs of the product of two random envelopes taken from the α-μ, η-μ, and κ-μ is provided in detail.

Appendix C. The series representation for the PDFs and CDFs of the ratio and product distributions are derived in detail in this appendix.

Appendix D. The expression for the integral involving the product of the PDF by the CDF is provided in detail in this appendix both in terms of the multivariable Fox H-function as in infinite series representation.

Appendix E. An alternative implementation for the Fox H-function is found here. As compared to that of [9], this implementation has the advantage of providing convergence at the lower tail of the distribution.

1.5 List of Publications

Journal Articles

- C. R. N. da Silva, E. J. Leonardo and M. D. Yacoub, "Product of Two Envelopes Taken From α-μ, κ-μ, and η-μ Distributions," in IEEE Transactions on Communications, vol. 66, no. 3, pp. 1284-1295, March 2018.
- C. R. N. da Silva, N. Bhargav, E. J. Leonardo, S. L. Cotton and M. D. Y., "Ratio of Two Envelopes Taken from α-μ, κ-μ and η-μ Variates and Some Practical Applications", submitted to IEEE Transaction on Vehicular Technology.
- C. R. N. da Silva, G. R. L. Tejerina, M. D. Yacoub, "The α-η-κ-μ Fading Model: New Fundamental Results", submitted to IEEE Transaction on Wireless Communications.

Co-Authored Journal Articles

- N. Bhargav, C. R. N. da Silva, Y. J. Chun, É. J. Leonardo, S. L. Cotton and M. D. Yacoub, "On the Product of Two κ-μ Random Variables and its Application to Double and Composite Fading Channels," in IEEE Transactions on Wireless Communications, vol. 17, no. 4, pp. 2457-2470, April 2018.

- N. Bhargav, C. R. N. da Silva, Y. J. Chun, S. L. Cotton and M. D. Yacoub, "Co-Channel Interference and Background Noise in κ-μ Fading Channels," in IEEE Communications Letters, vol. 21, no. 5, pp. 1215-1218, May 2017.
- N. Bhargav, C. R. N. da Silva, S. L. Cotton, P. C. Sofotasios, and M. D. Yacoub, "Double Shadowing the Rician Fading Model", Submitted to Wireless Communications Letters.

Co-Authored Conference Articles

- N. Bhargav, C. R. N. da Silva, Y. J. Chun, É. J. Leonardo, S. L. Cotton and M. D. Yacoub, "The product of two κ-μ variates and the κ-μ/κ-μ composite fading model," 2017 IEEE 28th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC), Montreal, QC, 2017, pp. 1-5.
- N. Bhargav, D. E. Simmons, C. R. N. da Silva, É. J. Leonardo, S. L. Cotton and M. D. Yacoub, "Outage probability analysis for α-μ/κ-μ and κ-μ/α-μ fading scenarios," 2017 IEEE 28th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC), Montreal, QC, 2017, pp. 1-5.

Chapter 2

Preliminaries

In this chapter, the distributions α-μ, η-μ, κ-μ, and α-η-κ-μ are briefly reviewed. Their PDFs and CDFs are rewritten in terms of the multivariable Fox H-function by replacing the exponential and Bessel functions in their formulations. In addition, the definition of the multivariable Fox H-function is presented along with some of its important properties. Finally, the Mellin transform and its connection with the generalized moments of positive RVs is presented.

2.1 The Fox H-Function

The Fox H-function in N variables is defined by multiple Mellin-Barnes contour integral in its most general form as [26]

$$\mathbb{H}[\mathbf{x};(\beta,\mathbf{B});(\delta,\mathbf{D});\mathscr{L}] = \left(\frac{1}{2\pi j}\right)^N \oint_{\mathscr{L}} \frac{\prod_{i=1}^{m}\Gamma\left(\beta_i+\sum_{k=1}^{N} b_{i,k}s_k\right)}{\prod_{i=1}^{n}\Gamma\left(\delta_i+\sum_{k=1}^{N} d_{i,k}s_k\right)} \prod_{i=1}^{N} x_i^{-s_i} ds_i, \tag{2.1}$$

in which $j=\sqrt{-1}$, $\mathbf{x}=[x_1,\ldots,x_N]$, $\beta=[\beta_1,\ldots,\beta_m]$ and $\delta=[\delta_1,\ldots,\delta_n]$ denote vectors of complex numbers and $\mathbf{B}$ and $\mathbf{D}$ are real valued matrices of order $m\times N$ and $n\times N$ respectively, $\mathscr{L}$ is an appropriate contour in the complex space, and $\Gamma(x)=\int_0^{\infty} t^{x-1}\exp(-t)dt$ $\Gamma(\cdot)$ is the gamma function [36, Equation (6.1.1)]. The function (2.1) reduces to the single variable Fox H-function for $N=1$. Properties and applications of (2.1) can be found in [37, 38], in particular for $N=2$. The convergence conditions for the integral (2.1) are described in detail in [38] for $N=2$ and then extended for an arbitrary number of variables in [26]. In [39], the single variable Fox H-function function is studied in detail providing several properties and applications in science and engineering. A comprehensive study on integral transforms involving the Fox H-function is found in [40].

2.2 The Mellin Transform

The Mellin transform and its inverse are defined as follows [39].

Definition 2.1. The Mellin transform of the function $f(x)$, denoted as $f^*(s)$, is defined by

$$f^*(s) = \int_0^\infty x^{s-1} f(x) dx, \tag{2.2}$$

provided the integral converges.

Definition 2.2. The inverse Mellin transform of the function $f^*(s)$ is defined by

$$f(x) = \frac{1}{2\pi j} \oint_{\mathcal{L}} f^*(s) x^{-s} ds \tag{2.3}$$

If $f^*(s)$ is analytic, then $f(x)$ is uniquely defined by (2.3).

2.2.1 The Mellin Transform and the Generalized Moments

From the standard probability theory, the k-th moment of a RV Z with positive support is given by

$$\mathbb{E}[Z^k] = \int_0^\infty z^k f_Z(z) dz, \tag{2.4}$$

in which $\mathbb{E}[\cdot]$ denotes the expectation operator. By comparing (2.4) with (2.2), it is easy to determine that the k-th moment is given in terms of the Mellin transform of $f_Z(z)$ as

$$\mathbb{E}[Z^k] = f^*(k+1), \tag{2.5}$$

or, equivalently,

$$f^*(s) = \mathbb{E}[Z^{s-1}]. \tag{2.6}$$

It follows from Definitions (2.1.) and (2.2.) that the PDF $f_Z(z)$ can be obtained from the generalized moments as

$$f_Z(z) = \frac{1}{2\pi j} \oint_{\mathcal{L}} \mathbb{E}[Z^{s-1}] z^{-s} ds, \tag{2.7}$$

or, alternatively, by performing the variable transformation $u = s - 1$,

$$f_Z(z) = \frac{1}{z} \frac{1}{2\pi j} \oint_{\mathcal{L}} \mathbb{E}[Z^u] z^{-u} du. \tag{2.8}$$

2.3 The α-μ Fading Model

The α-μ distribution arises from a general fading model suited to characterize non dominant components environments subject to some sort of non-linearity in the wireless medium and clustering of multipath. Let $R > 0$ be a fading envelope with α-root mean value $\hat{r} = \sqrt[\alpha]{\mathbb{E}[R^\alpha]}$. Then its PDF is given as

$$f_R(r) = \frac{\alpha\mu^\mu}{\Gamma(\mu)} \frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}} \exp\left(-\mu\frac{r^\alpha}{\hat{r}^\alpha}\right), \tag{2.9}$$

in which $\alpha > 0$ is connected to non-linearities in the wireless medium, $\mu = \mathbb{E}^2[R^\alpha]/\mathbb{V}[R^\alpha]$, and $\mathbb{V}[\cdot]$ is the variance operator. Its CDF is given as

$$F_R(r) = \frac{\gamma\left(\mu, \mu r^\alpha/\hat{r}^\alpha\right)}{\Gamma(\mu)}, \tag{2.10}$$

in which $\gamma(a, b) = \int_0^b t^{a-1}e^{-t}\,dt$ is the incomplete gamma function [36, Equation (6.5.2)]. The k-th moment is found as

$$\mathbb{E}[R^k] = \frac{\Gamma\left(\mu + k/\alpha\right)}{\Gamma(\mu)}\mathscr{A}^k, \tag{2.11}$$

in which the constant $\mathscr{A}$ is defined for simplicity as

$$\mathscr{A} = \frac{\hat{r}}{\mu^{1/\alpha}}. \tag{2.12}$$

Several important fading models are particular cases of the α-μ distribution such as Weibull ($\mu = 1$), Nakagami-m ($\mu = m$ and $\alpha = 2$), Rayleigh ($\mu = 1$ and $\alpha = 2$), gamma ($\mu = m$ and $\alpha = 1$), one-sided Gaussian ($\mu = 1/2$ and $\alpha = 2$), and exponential ($\mu = 1$ and $\alpha = 1$) to name but a few.

2.4 The κ-μ Fading Model

The κ-μ distribution arises from a general fading model suited to characterize fading signals subjected to multipath clustering with dominant components. For a fading signal $R > 0$ and root mean square (*rms*) value $\hat{r} = \sqrt{\mathbb{E}[R^2]}$, its PDF is given, alternatively to [4] by using [36, Equation (9.6.47)], as

$$f_R(r) = \frac{2(\mu(1+\kappa))^\mu}{\exp(\kappa\mu)} \frac{r^{2\mu-1}}{\hat{r}^{2\mu}} \exp\left(-\frac{r^2(1+\kappa)\,\mu}{\hat{r}^2}\right) {}_0\tilde{F}_1\left(;\mu; \frac{r^2\kappa(1+\kappa)\mu^2}{\hat{r}^2}\right), \tag{2.13}$$

in which $\kappa > 0$ is the ratio between the total power of the dominant components by the total power of the scattered-waves, $\mu = \mathbb{E}^2[R^2]/\mathbb{V}[R^2] \times (1+2\kappa)/(1+\kappa)^2$ is related to the number of multipath clusters, and ${}_0\tilde{F}_1(;a;z) = {}_0F_1(;a;x)/\Gamma(a)$ is a particular case of the generalized hypergeometric function [41, Equation (7.2.3.1)]. The CDF of the κ-μ distribution is given in terms of the Marcum-Q [4], alternatively, the CDF can be written in terms of the Fox H-function as [42]

$$F_R(r) = \left(\mu(1+\kappa)\frac{r^2}{\hat{r}^2}\right)^{\mu} \mathbf{H}[\mathbf{x};(\beta,\mathbf{B});(\delta,\mathbf{D});\mathscr{L}_{\mathbf{s}}], \tag{2.14}$$

such that, $\mathbf{x} = [\kappa\mu, \mu(1+\kappa)r^2/\hat{r}^2]$, $\beta = [\mu, 0, 0]$, $\delta = [\mu, \mu+1]$ and the matrices $\mathbf{B}$ and $\mathbf{D}$ are given as

$$\mathbf{B} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The k-th moment is obtained in closed form, alternatively to [4, Equation (5)], as

$$\mathbb{E}[R^k] = \mathscr{K}^k \frac{\Gamma(\mu + k/2)}{\Gamma(\mu)} {}_1F_1\left(-\frac{k}{2}; \mu; -\kappa\mu\right), \tag{2.15}$$

by using the identity [36, Equation (13.1.27)]. The constant $\mathscr{K}$ is defined as

$$\mathscr{K} = \frac{\hat{r}}{\sqrt{\mu(1+\kappa)}}. \tag{2.16}$$

Particular cases include the Rice distribution ($\mu = 1$), Nakagami-m ($\kappa = 0$ and $\mu = m$), Rayleigh and one-sided Gaussian.

2.5 The η-μ Fading Model

In a environment with no dominant components, a fading signal showing imbalances or correlation between the in-phase and quadrature waves subjected to clustering of multipath has its distribution characterized by the η-μ fading model. For a fading signal $R > 0$ following the η-μ distribution and *rms* $\hat{r} = \sqrt{\mathbb{E}[R^2]}$, its PDF is given, alternatively to [4, Equation (17)] by using [36, Equation (9.6.47)], as

$$f_R(r) = \frac{2^{1+2\mu}\left(h\mu^2\right)^{\mu}}{\Gamma(2\mu)} \frac{r^{4\mu-1}}{\hat{r}^{4\mu}} \exp\left(-\frac{2\mu h r^2}{\hat{r}^2}\right) {}_0F_1\left(; \mu + \frac{1}{2}; \frac{H^2\mu^2 r^4}{\hat{r}^4}\right), \tag{2.17}$$

in which $\mu = \mathbb{E}^2[R^2]/(2\mathbb{V}[R^2]) \times (1 + (H/h)^2)$ is related to the number of multipath clusters and h and H are functions of the parameter η defined according to the adopted Format of the η-μ distribution. In Format 1, $\eta > 0$ is the ratio between the power of the in-phase and quadrature scattered-waves and $h = (2 + \eta^{-1} + \eta)/4$ and $H = (\eta^{-1} - \eta)/4$; in Format 2, $-1 < \eta < -1$ is correlation coefficient between the in-phase and quadrature scattered-waves and $h = 1/(1 - \eta^2)$ and $H = \eta h$. Its CDF is given, in terms of the Fox H-function, as

$$F_R(r) = \frac{2\sqrt{\pi}}{\Gamma(\mu)} \left(\mu \frac{r^2}{\hat{r}^2}\right)^{2\mu} \mathbf{H}[\mathbf{x};(\beta,\mathbf{B});(\delta,\mathbf{D});\mathscr{L}_{\mathbf{s}}], \tag{2.18}$$

in which $\mathbf{x} = [-H^2/(4h^2), 2\mu r^2/\hat{r}^2]$, $\beta = [2\mu, 0, 0]$, $\delta = [\mu + 1/2, 1 + 2\mu]$, and the matrices $\mathbf{B}$ and $\mathbf{D}$ are given as

$$\mathbf{B} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The k-th moment is given, alternatively to [4, Equation (21)], as

$$\mathbb{E}[R^k] = \mathscr{E}^k \frac{\Gamma(2\mu + k/2)}{\Gamma(2\mu)} {}_2F_1\left(\frac{1}{2} - \frac{k}{4}, -\frac{k}{4}; \mu + \frac{1}{2}; \frac{H^2}{h^2}\right). \tag{2.19}$$

and the constant $\mathscr{E}$ is defined as

$$\mathscr{E} = \frac{\hat{r}}{\sqrt{2\mu}} \tag{2.20}$$

From the η-μ distribution it is possible to obtain other important fading models. Special cases include the Hoyt ($\mu = 1/2$), Nakagami-m ($\eta \to (0, \infty)$ and $m = \mu$ or $\eta = 1$ and $m = 2\mu$ for format 1, or $\eta = 0$ and $m = 2\mu$ or $\eta = (-1, 1)$ and $m = \mu$), from Nakagami-m is possible to obtain the Rayleigh distribution ($m = 1$) or the one-sided Gaussian ($m = 1/2$).

2.6 The α-η-κ-μ Fading Model

Accounting for virtually all short-term physical phenomena, namely nonlinearity of the propagation medium, scattered waves, dominant components, and multipath clustering, the α-η-κ-μ fading model has been recently proposed. It comprises most of the fading distributions presented in the literature and some not yet reported. This fading model has been presented in three different parametrizations. Let $R > 0$ be a α-η-κ-μ fading signal. It was recognized in [5] that the α-η-κ-μ fading envelope could be written in terms of the in-phase and quadrature waves of the complex model as $R^\alpha = X^2 + Y^2$, in which $\alpha > 0$ models the non-linearities in the physical medium and X^2 and Y^2 are the powers of two independent κ-μ distributed random variates. In its Global Parametrization, for an α-root mean $\hat{r} = \sqrt[\alpha]{\mathbb{E}[R^\alpha]}$, its PDF is given as [5]

$$f_R(r) = \frac{\alpha r^{\alpha\mu - 1}}{2^\mu \Gamma(\mu)} \exp\left(-\frac{r^\alpha}{2}\right) \sum_{k=0}^{\infty} \frac{k! c_k}{(\mu)_k} L_k^{\mu-1}(2r^\alpha), \tag{2.21}$$

in which $\mu > 0$ is the number of multipath clusters, $L_k^\lambda(\cdot)$ is the generalized Laguerre Polynomial [36, Equation (22.2.12)] and c_k is given as

$$c_k = \frac{1}{k} \sum_{i=0}^{k-1} c_i d_{k-i}, \qquad k \geq 1, \tag{2.22}$$

with c_0 and d_i given, respectively, as

$$c_0 = \frac{8^\mu \left(\frac{3\hat{r}^\alpha}{\xi\mu} + 2\right)^{-\frac{\mu}{p+1}} \left(\frac{3\eta\hat{r}^\alpha}{\xi\mu p} + 2\right)^{-\frac{\mu p}{p+1}}}{\exp\left(\frac{3\kappa\mu\hat{r}^\alpha(2\mu p\xi(\eta q+1) + 3\eta\hat{r}^\alpha(pq+1))}{\delta(2\mu\xi + 3\hat{r}^\alpha)(2\mu p\xi + 3\eta\hat{r}^\alpha)}\right)} \tag{2.23}$$

and

$$d_i = \frac{\mu}{p+1}\left(\frac{2\mu\xi - \hat{r}^{\alpha}}{2\mu\xi + 3\hat{r}^{\alpha}}\right)^{i} + \frac{p\mu}{1+p}\left(\frac{2\mu p\xi - \eta\hat{r}^{\alpha}}{2\mu p\xi + 3\eta\hat{r}^{\alpha}}\right)^{i} - \frac{8i\kappa\mu^2\eta p^2 q\xi}{\delta}\frac{\left(2\mu p\xi - \eta\hat{r}^{\alpha}\right)^{i-1}\hat{r}^{\alpha}}{\left(2\mu p\xi + 3\eta\hat{r}^{\alpha}\right)^{i+1}} - \frac{8i\kappa\mu^2\xi}{\delta}\frac{\left(2\mu\xi - \hat{r}^{\alpha}\right)^{i-1}\hat{r}^{\alpha}}{\left(2\xi\mu + 3\hat{r}^{\alpha}\right)^{i+1}} \tag{2.24}$$

wherein $\xi = (1+\eta)(1+\kappa)/(1+p)$ and $\delta = (1+p)(1+q\eta)/(1+\eta)$; $\kappa > 0$ is the ratio of the total power of dominant component and the total power of the scattered waves; $\eta > 0$ is defined as the ratio of the total power of the in-phase and quadrature scattered waves of the multipath clusters; $p > 0$ is the ratio of the number of multipath clusters of in-phase and quadrature signals; and $q > 0$ is the ratio of two ratios: the ratio of the power of the dominant components to the power of the scattered waves of the in-phase signal and its counterpart for the quadrature signal. The CDF of the fading envelope R is obtained as [5]

$$F_R(r) = \frac{r^{\alpha\mu}}{2^{\mu+1}\Gamma(\mu+1)}\exp\left(-\frac{r^{\alpha}}{2}\right)\sum_{k=0}^{\infty}\frac{k!m_k}{(\mu+1)_k}L_k^{\mu}\left(\frac{2(\mu+1)r^{\alpha}}{\mu}\right), \tag{2.25}$$

in which m_k is defined as

$$m_k = \frac{1}{k}\sum_{i=0}^{k-1} m_i q_{k-i}, \qquad k \geq 1, \tag{2.26}$$

wherein m_0 and q_i are given respectively as

$$m_0 = \frac{8^{\mu+1}(\mu+1)^{\mu+1}\left(2\mu + \frac{(3\mu+4)\hat{r}^{\alpha}}{\xi\mu}\right)^{-\frac{\mu}{p+1}}\left(2\mu + \frac{\eta\hat{r}^{\alpha}(3\mu+4)}{\mu p\xi}\right)^{-\frac{\mu p}{p+1}}}{(3\mu+4)\exp\left(\frac{\kappa\mu\hat{r}^{\alpha}(3\mu+4)(2\mu^2 p\xi(\eta q+1)+\eta\hat{r}^{\alpha}(3\mu+4)(pq+1))}{\delta(2\mu^2\xi+(3\mu+4)\hat{r}^{\alpha})(2p\mu^2\xi+\eta\hat{r}^{\alpha}(3\mu+4))}\right)} \tag{2.27}$$

and

$$q_i = \frac{\mu}{p+1}\left(\frac{\mu(2\mu\xi - \hat{r}^{\alpha})}{2\mu^2\xi + (3\mu+4)\hat{r}^{\alpha}}\right)^{i} + \frac{\mu p}{p+1}\left(\frac{\mu(2\mu p\xi - \eta\hat{r}^{\alpha})}{2p\mu^2\xi + \eta\hat{r}^{\alpha}(3\mu+4)}\right)^{i} - \frac{8i\kappa\mu^{i+2}\eta p^2 q\xi(\mu+1)}{\delta}\frac{\left(2\mu p\xi - \eta\hat{r}^{\alpha}\right)^{i-1}\hat{r}^{\alpha}}{\left(2p\mu^2\xi + \eta\hat{r}^{\alpha}(3\mu+4)\right)^{i+1}} - \frac{8i\kappa\mu^{i+2}\xi(\mu+1)}{\delta}\frac{\left(2\mu\xi - \hat{r}^{\alpha}\right)^{i-1}\hat{r}^{\alpha}}{\left(2\mu^2\xi + (3\mu+4)\hat{r}^{\alpha}\right)^{i+1}} + \left(-\frac{\mu}{3\mu+4}\right)^{i}. \tag{2.28}$$

Unfortunately, the expressions for the PDF and CDF are given in intricate formulations using recursivity, which can become quite troublesome to numerical evaluation. The α-η-κ-μ fading model comprises most of the fading distribution present in the literature. For instance, the α-κ-μ fading [43] is obtained by setting $p = \eta$; for $\kappa \to 0$ and $p = 1$, the α-η-μ fading model [43] is obtained. Other important short-term fading can be obtained from the particular case of the α-κ-μ and α-η-μ. Nevertheless, it is quite difficult to reduce the formulations provided in [5] to those of the particular cases, mainly due to the recursive particle present in (2.21) and (2.25). Of course, the reduction is straightforward by making use of physical model of these distributions.

Chapter 3

The Ratio Distribution of Two Fading Envelopes

In this chapter, the basic concepts to derive the first order statistics of the ratio of two random envelopes are shown. In particular, the PDF and CDF of the ratio of random envelopes with variates taken from the α-μ, η-μ, and κ-μ distributions are obtained in closed-form in terms of the Fox H-function. Fairly simple, fast convergent, series expansion are also derived as an alternative for implementing the Fox H-function. The chapter ends with a simple practical application example in physical layer security, namely probability of positive secrecy capacity.

3.1 The Ratio Statistics

Let $Z = X/Y > 0$ be a random variate originated from the ratio of two independent arbitrarily distributed random envelopes X and Y whose PDFs are $f_X(x)$ and $f_Y(y)$. From standard statistical procedures, the PDF of Z, denoted as $f_Z(r)$, is obtained by the integral

$$f_Z(z) = \int_0^\infty y f_X(y\ z) f_Y(y) dy, \tag{3.1}$$

and, of course, its CDF can be obtained from its definition as

$$F_Z(z) = \int_0^z f_Z(\tau) d\tau. \tag{3.2}$$

Interestingly, when the RVs X and Y are taken from the α-μ, η-μ or κ-μ distributions, both the PDF and CDF can be written in terms of a single multivariable Fox H-function given in the general form as

$$g(\mathbf{x}) = C\mathbb{H}[\mathbf{x}; (\beta, \mathbf{B}); (\delta, \mathbf{D}); \mathscr{L}]. \tag{3.3}$$

Table 3.1 – Parameters for the Fox H-Function of the Ratio PDF

Ratio	C	$\mathbf{x}$	β	$\mathbf{B}$	δ	$\mathbf{D}$
α-μ/α-μ	$\frac{1}{z\Gamma(\mu_x)\Gamma(\mu_y)}$	$\left[\frac{z}{u_{\alpha\alpha}}\right]$	$[\mu_x,\mu_y]$	$\begin{pmatrix}1/\alpha_x\\-1/\alpha_y\end{pmatrix}$	-	-
α-μ/κ-μ	$\frac{\alpha_x}{z\Gamma(\mu_x)e^{\kappa_y\mu_y}}$	$\left[\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}, -\kappa_y\mu_y\right]$	$[\mu_y,\mu_x,0]$	$\begin{pmatrix}-\alpha_x/2&-1\\1&0\\0&1\end{pmatrix}$	$[\mu_y]$	$\begin{pmatrix}0&-1\end{pmatrix}$
α-μ/η-μ	$\frac{2^{1-2\mu_y}\sqrt{\pi}\alpha_x}{z\Gamma(\mu_x)\Gamma(\mu_y)h_y^{\mu_y}}$	$\left[\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}, -\frac{H_y^2}{4h_y^2}\right]$	$[2\mu_y,\mu_x,0]$	$\begin{pmatrix}-\alpha_x/2&-2\\1&0\\0&1\end{pmatrix}$	$[\mu_y+\frac{1}{2}]$	$\begin{pmatrix}0&-1\end{pmatrix}$
κ-μ/κ-μ	$\frac{2(-\kappa_x\mu_x)^{-\mu_x}(-\kappa_y\mu_y)^{-\mu_y}}{ze^{\kappa_x\mu_x+\kappa_y\mu_y}}$	$[-\kappa_x\mu_x\nu_{\kappa\kappa}, -\kappa_y\mu_y(1-\nu_{\kappa\kappa})]$	$[0,\mu_x,\mu_y]$	$\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}$	$[0,0]$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
κ-μ/η-μ	$\frac{4\sqrt{\pi}(-1)^{-\mu_x-\mu_y}h_y^{\mu_y}z^{-1}}{\Gamma(\mu_y)e^{\kappa_x\mu_x}(\kappa_x\mu_x)^{\mu_x}(H_y^2)^{\mu_y}}$	$\left[-\kappa_x\mu_x\nu_{\kappa\eta}, -\frac{H_y^2}{4h_y^2}(1-\nu_{\kappa\eta})^2\right]$	$[0,\mu_x,\mu_y]$	$\begin{pmatrix}-1&-2\\1&0\\0&1\end{pmatrix}$	$[0,\frac{1}{2}]$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
η-μ/η-μ	$\frac{8\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_x)\Gamma(\mu_y)(H_x^2)^{\mu_x}(H_y^2)^{\mu_y}}$	$\left[-\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}, -\frac{H_y^2(1-\nu_{\eta\eta})^2}{4h_y^2}\right]$	$[0,\mu_x,\mu_y]$	$\begin{pmatrix}-2&-2\\1&0\\0&1\end{pmatrix}$	$[\frac{1}{2},\frac{1}{2}]$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

Table 3.2 – Parameters for the Fox H-Function of the Ratio CDF

Ratio	C	$\mathbf{x}$	β	$\mathbf{B}$	δ	$\mathbf{D}$
α-μ/α-μ	$\frac{1}{\Gamma(\mu_x)\Gamma(\mu_y)}$	$\left[\frac{z}{u_{\alpha\alpha}}\right]$	$[\mu_x,\mu_y,0]$	$\begin{pmatrix}1/\alpha_x\\-1/\alpha_y\\-1\end{pmatrix}$	$[1]$	(-1)
α-μ/κ-μ	$\frac{1}{\Gamma(\mu_x)e^{\kappa_y\mu_y}}$	$\left[\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}, -\kappa_y\mu_y\right]$	$[\mu_y,\mu_x,0,0]$	$\begin{pmatrix}-\alpha_x/2&-1\\1&0\\-1&0\\0&1\end{pmatrix}$	$[\mu_y,1]$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$
α-μ/η-μ	$\frac{2^{1-2\mu_y}\sqrt{\pi}}{\Gamma(\mu_x)\Gamma(\mu_y)h_y^{\mu_y}}$	$\left[\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}, -\frac{H_y^2}{4h_y^2}\right]$	$[2\mu_y,\mu_x,0,0]$	$\begin{pmatrix}-\alpha_x/2&-2\\1&0\\-1&0\\0&1\end{pmatrix}$	$[1,\mu_y+\frac{1}{2}]$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
κ-μ/κ-μ	$\frac{(-\kappa_x\mu_x)^{-\mu_x}(-\kappa_y\mu_y)^{-\mu_y}}{e^{\kappa_x\mu_x+\kappa_y\mu_y}}$	$\left[-\frac{z^2\mu_x\kappa_x}{u_{\kappa\kappa}^2}, -\mu_y\kappa_y, \frac{z^2}{u_{\kappa\kappa}^2}\right]$	$[0,0,\mu_x,\mu_y,0]$	$\begin{pmatrix}-1&-1&-1\\-1&0&-1\\&\mathbf{I}_3&\end{pmatrix}$	$[1,0,0]$	$\begin{pmatrix}-1&0&-1\\-1&0&0\\0&-1&0\end{pmatrix}$
κ-μ/η-μ	$\frac{2\sqrt{\pi}(-1)^{-\mu_x-\mu_y}(H_y^2)^{-\mu_y}}{\Gamma(\mu_y)e^{\kappa_x\mu_x}(\kappa_x\mu_x)^{\mu_x}h_y^{-\mu_y}}$	$\left[-\frac{z^2\mu_x\kappa_x}{u_{\kappa\eta}^2}, -\frac{H_y^2}{4h_y^2}, \frac{z^2}{u_{\kappa\eta}^2}\right]$	$[0,0,\mu_x,\mu_y,0]$	$\begin{pmatrix}-1&0&-1\\-1&-2&-1\\&\mathbf{I}_3&\end{pmatrix}$	$[1,0,\frac{1}{2}]$	$\begin{pmatrix}-1&0&-1\\-1&0&0\\0&-1&0\end{pmatrix}$
η-μ/η-μ	$\frac{4\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)(H_x^2)^{\mu_x}(H_y^2)^{\mu_y}}$	$\left[-\frac{z^4H_x^2}{4h_x^2u_{\eta\eta}^4}, -\frac{H_y^2}{4h_y^2}, \frac{z^2}{u_{\eta\eta}^2}\right]$	$[0,0,\mu_x,\mu_y,0]$	$\begin{pmatrix}-2&0&-1\\-2&-2&-1\\&\mathbf{I}_3&\end{pmatrix}$	$[1,\frac{1}{2},\frac{1}{2}]$	$\begin{pmatrix}-2&0&-1\\-1&0&0\\0&-1&0\end{pmatrix}$

The respective parameters for all possible combinations of ratios of RVs involving the α-μ, η-μ, and κ-μ are provided in Tables 3.1 and 3.2, respectively, for the PDF and CDF. Please see Appendix A for their mathematical derivations. The constants u_{ab} and v_{ab} with $a, b \in \{\alpha, \kappa, \eta\}$ used in the Tables 3.1 and 3.2 and elsewhere in the text are defined as

$$
\begin{aligned}
&u_{\alpha\alpha} = \frac{\mathscr{A}_x}{\mathscr{A}_y},\ u_{\alpha\kappa} = \frac{\mathscr{A}_x}{\mathscr{K}_y},\ u_{\alpha\eta} = \frac{\mathscr{A}_x\sqrt{h_y}}{\mathscr{E}_y}, \\
&u_{\kappa\kappa} = \frac{\mathscr{K}_x}{\mathscr{K}_y},\ u_{\kappa\alpha} = \frac{\mathscr{K}_x}{\mathscr{A}_y},\ u_{\kappa\eta} = \frac{\mathscr{K}_x\sqrt{h_y}}{\mathscr{E}_y}, \\
&u_{\eta\eta} = \frac{\mathscr{E}_x\sqrt{h_y}}{\sqrt{h_x}\mathscr{E}_y},\ u_{\eta\alpha} = \frac{\mathscr{E}_x}{\mathscr{A}_y\sqrt{h_x}},\ u_{\eta\kappa} = \frac{\mathscr{E}_x}{\mathscr{K}_y\sqrt{h_x}} \\
&\text{and } v_{ab} = \frac{z^2}{z^2 + u_{ab}^2}
\end{aligned}
\tag{3.4}
$$

in which $\mathscr{A}_i$, $\mathscr{K}_i$ and $\mathscr{E}_i$ with $i \in \{x, y\}$ are derived, respectively, from (2.12), (2.16) and (2.20) with the appropriate subscripts and $\mathbf{I}_n$ denotes an identity matrix of order n.

3.2 Series Representation

The implementation of the general multivariable Fox H-function can become an herculean task as the number of variables increases, and, until now, there is no such implementation. Nevertheless, some particular implementations may be found in the literature. For instance, an interesting Mathematica implementation of the single variable Fox H-function is found in [9], in which the Fox H-function is written in terms of the Meijer's G-function. In [34], a Python implementation for the multivariable Fox H-function is provided, in which the authors claim to efficiently and accurately evaluate it up to four variables. Alternatively, calculus of residues may be used to produce computable series expansions. Of course, in general, the evaluation of these series may become extremely complicated as the number of variables rises above three or four with the result being obtained from a multi-fold summation. There are some cases, though, in which the multi-fold summations can be simplified to a single sum or even to a closed-form expression.

Tables 3.3 and 3.4 present simple, fast convergent series expansions for the PDFs and CDFs, respectively, in which $B(\cdot,\cdot)$ denotes the beta function [36, Equation (6.2.1)] and $H_{p,q}^{m,n}[\cdot|-]$ represents the single variable Fox H-function [39]. These expressions compute fairly quick on an ordinary desktop computer. On the one hand, the series involving the α-μ distribution can present some complications as the overall computation time depends mostly on the efficiency of the single variable Fox H-function implementation. Although a clever computational execution of the Fox-H function can be found in [9], when $\alpha > 2$ and in the vicinity of $z = 0$ convergence is not always guaranteed. A novel algorithm has been implemented here, which guarantees convergence in all cases[1] (see Appendix E).. On the other hand, around 25 terms is enough to provide a good accuracy. The mathematical derivations for the respective series expressions can be found in Appendix C.

3.3 The Reciprocal Distributions: Ratios of κ-μ/α-μ, η-μ/α-μ and η-μ/κ-μ

In the previous sections, the PDF and CDF for the ratio distributions were obtained both in terms of the multivariate Fox H-function and as series expansions for some combinations of RVs. The remaining combinations can be obtained from the reciprocal distributions defined as $\tilde{Z} = 1/Z$. By performing a simple variable transformation the PDFs and CDFs of $\tilde{Z}$ are given as

$$f_{\tilde{Z}}(z) = \frac{1}{z^2} f_Z\left(\frac{1}{z}\right) \tag{3.5}$$

[1] When $\alpha < 2$, the Meijer G implementation of [9] is more efficient.

Table 3.3 – Series Expansion for the Ratio PDF

Ratio	Series Representation
α-μ/α-μ	Refer to [24]
α-μ/κ-μ	$f_Z(z)=\frac{\alpha_x}{z\Gamma(\mu_x)e^{\kappa_y\mu_y}}\sum_{i=0}^{\infty}\frac{(\kappa_y\mu_y)^i}{i!\Gamma(i+\mu_y)}H_{1,1}^{1,1}\left[\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\Big\vert\begin{smallmatrix}(1-i-\mu_y,\alpha_x/2)\\(\mu_x,1)\end{smallmatrix}\right]$
α-μ/η-μ	$f_Z(z)=\frac{2^{1-2\mu_y}\sqrt{\pi}\alpha_x}{z\Gamma(\mu_x)\Gamma(\mu_y)h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{1}{i!\Gamma\left(i+\mu_y+\frac{1}{2}\right)}\left(\frac{H_y}{2h_y}\right)^{2i}H_{1,1}^{1,1}\left[\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\Big\vert\begin{smallmatrix}(1-2\mu_y-2i,\alpha_x/2)\\(\mu_x,1)\end{smallmatrix}\right]$
κ-μ/κ-μ	$f_Z(z)=\frac{2\nu_{\kappa\kappa}^{\mu_x}(1-\nu_{\kappa\kappa})^{\mu_y}}{ze^{\kappa_x\mu_x+\kappa_y\mu_y}}\sum_{i=0}^{\infty}\frac{(\kappa_x\mu_x\nu_{\kappa\kappa})^i}{i!B(i+\mu_x,\mu_y)}{}_1F_1\left(i+\mu_x+\mu_y;\mu_y;\kappa_y\mu_y(1-\nu_{\kappa\kappa})\right)$
κ-μ/η-μ	$f_Z(z)=\frac{2\nu_{\kappa\eta}^{\mu_x}(1-\nu_{\kappa\eta})^{2\mu_y}}{ze^{\kappa_x\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{(\kappa_x\mu_x\nu_{\kappa\eta})^i}{i!B(i+\mu_x,2\mu_y)}{}_2F_1\left(\frac{i+\mu_x+2\mu_y}{2},\frac{i+\mu_x+2\mu_y+1}{2};\mu_y+\frac{1}{2};\frac{H_y^2(1-\nu_{\kappa\eta})^2}{h_y^2}\right)$
η-μ/η-μ	$f_Z(z)=\frac{2\nu_{\eta\eta}^{2\mu_x}(1-\nu_{\eta\eta})^{2\mu_y}}{zh_x^{\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{(\mu_x)_i}{i!B(2i+2\mu_x,2\mu_y)}\left(\frac{H_x}{h_x}\nu_{\eta\eta}\right)^{2i}$ $\times{}_2F_1\left(i+\mu_x+\mu_y,i+\mu_x+\mu_y+\frac{1}{2};\mu_y+\frac{1}{2};\frac{H_y^2}{h_y^2}(1-\nu_{\eta\eta})^2\right)$

Table 3.4 – Series Expansion for the Ratio CDF

Ratio	Series Representation
α-μ/α-μ	Refer to [24]
α-μ/κ-μ	$F_Z(z)=\frac{1}{\Gamma(\mu_x)e^{\kappa_y\mu_y}}\sum_{i=0}^{\infty}\frac{(\kappa_y\mu_y)^i}{i!\Gamma(i+\mu_y)}H_{2,2}^{1,2}\left[\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\Big\vert\begin{smallmatrix}(1,1),(1-\mu_y-i,\alpha_x/2)\\(\mu_x,1),(0,1)\end{smallmatrix}\right]$
α-μ/η-μ	$F_Z(z)=\frac{1}{\Gamma(\mu_x)h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{(\mu_y)_i}{i!\Gamma(2i+2\mu_y)}\left(\frac{H_y}{h_y}\right)^{2i}H_{2,2}^{1,2}\left[\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\Big\vert\begin{smallmatrix}(1,1),(1-2\mu_y-2i,\alpha_x/2)\\(\mu_x,1),(0,1)\end{smallmatrix}\right]$
κ-μ/κ-μ	$F_Z(z)=\frac{1}{e^{\kappa_x\mu_x+\kappa_y\mu_y}}\left(\frac{z}{u_{\kappa\kappa}}\right)^{2\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\kappa_y\mu_y)^k}{i!k!(i+\mu_x)B(i+\mu_x,k+\mu_y)}\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\kappa}^2}\right)^i$ $\times{}_2F_1\left(i+\mu_x,i+k+\mu_x+\mu_y;i+\mu_x+1;-\frac{z^2}{u_{\kappa\kappa}^2}\right)$
κ-μ/η-μ	$F_Z(z)=\frac{1}{e^{\kappa_x\mu_x}h_y^{\mu_y}}\left(\frac{z}{u_{\kappa\eta}}\right)^{2\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\mu_y)_k}{i!k!(i+\mu_x)B(i+\mu_x,2(k+\mu_y))}\left(\frac{H_y}{h_y}\right)^{2k}\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^i$ $\times{}_2F_1\left(i+\mu_x,i+2k+\mu_x+2\mu_y;i+\mu_x+1;-\frac{z^2}{u_{\kappa\eta}^2}\right)$
η-μ/η-μ	$F_Z(z)=\frac{1}{2h_x^{\mu_x}h_y^{\mu_y}}\left(\frac{z}{u_{\eta\eta}}\right)^{4\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\mu_x)_i(\mu_y)_k}{i!k!(i+\mu_x)B(2k+2\mu_y,2i+2\mu_x)}\left(\frac{z^2H_x}{u_{\eta\eta}^2h_x}\right)^{2i}\left(\frac{H_y}{h_y}\right)^{2k}$ $\times{}_2F_1\left(2i+2\mu_x,2i+2k+2\mu_x+2\mu_y;2i+2\mu_x+1;-\frac{z^2}{u_{\eta\eta}^2}\right)$

and

$$F_{\tilde{Z}}(z) = 1 - F_Z(\frac{1}{z}). \tag{3.6}$$

Hence, Tables 3.3 and 3.4 can be used directly to obtain these statistics.

3.4 Some Close-form Special Cases

For a certain set of parameters the ratio distribution presents closed-form expression. For instance, the PDF of the ratio of two Hoyt variates can be obtained by setting $\mu_x = \mu_y = 1/2$ in the η-μ/η-μ expression in Table 3.3 and then using [41, Equation (6.8.1.6)] along with the linear transformation given by [41, Eq. (7.3.1.3)], and after some algebraic manipulations, the PDF is given as

$$\begin{aligned} f_Z(z) &= \frac{2\sqrt{h_x h_y}(\hat{r}_x \hat{r}_y)^2 (h_y \hat{r}_x^2 + z^2 h_x \hat{r}_y^2) z}{(h_y \hat{r}_x^4 + 2z^2 h_x h_y \hat{r}_x^2 \hat{r}_y^2 + z^4 h_x \hat{r}_y^4)^{3/2}} \\ &\times {}_2F_1\left(\frac{3}{4}, \frac{5}{4}; 1; \frac{4z^4 H_x^2 H_y^2 (\hat{r}_x \hat{r}_y)^4}{(h_y \hat{r}_x^4 + 2z^2 h_x h_y \hat{r}_x^2 \hat{r}_y^2 + z^4 h_x \hat{r}_y^4)^2}\right). \end{aligned} \tag{3.7}$$

Closed-form expressions are also obtained for the PDF of ratios involving the Nakagami-m distribution. By setting the $\alpha_x = 2$ in the α-μ distribution the PDF of the ratio Nakagami-m/κ-μ and Nakagami-m/η-μ are given respectively as

$$f_Z(z) = \frac{2\nu_{\alpha\kappa}^{\mu_x}(1-\nu_{\alpha\kappa})^{\mu_y}}{z e^{\kappa_y \mu_y} B\left(\mu_x, \mu_y\right)} {}_1F_1\left(\mu_x + \mu_y; \mu_y; \kappa_y \mu_y (1-\nu_{\alpha\kappa})\right) \tag{3.8}$$

and

$$f_Z(z) = \frac{2\nu_{\alpha\eta}^{\mu_x}\left(1-\nu_{\alpha\eta}\right)^{2\mu_y}}{zB\left(\mu_x, 2\mu_y\right) h_y^{\mu_y}} {}_2F_1\left(\frac{\mu_x + 2\mu_y}{2}, \frac{1+\mu_x+2\mu_y}{2}; \frac{1+2\mu_y}{2}; \frac{H_y^2}{h_y^2}\left(1-\nu_{\alpha\eta}\right)^2\right). \tag{3.9}$$

Other two interesting closed-form expressions arise from the ratios involving the η-μ distribution with the parameter $\mu \in \mathbb{Z}$. By writing the ${}_2F_1$ function as the Legendre Q function using [41] and then the identity [44, Eq. (07.12.03.0028.01)], after tedious and cumbersome algebraic manipulations, the PDF of the ratio of κ-μ/η-μ and η-μ/η-μ variates are obtained respectively as

$$\begin{aligned} f_Z(z) &= \frac{2^{1-\mu_y}\left(1-\nu_{\kappa\eta}\right)^{\mu_y} \nu_{\kappa\eta}^{\mu_x}}{z e^{\kappa_x \mu_x} \left|H_y\right|^{\mu_y}} \sum_{k=0}^{\mu_y - 1} \frac{\left(1-\mu_y\right)_k \left(\mu_y\right)_k}{k!} \left(\frac{(-1)^k (1-\mathcal{G}_\kappa)^{-\mu_x-\mu_y+k} (2\mathcal{G}_\kappa)^{-k}}{B\left(\mu_x, \mu_y\right)\left(1-\mu_x-\mu_y\right)_k} \right. \\ &\times {}_1F_1\left(-k+\mu_x+\mu_y; \mu_x; \frac{\kappa_x \mu_x \nu_{\kappa\eta}}{(1-\mathcal{G}_\kappa)}\right) + \frac{(-1)^{\mu_y} \mu_x (1+\mathcal{G}_\kappa)^{-\mu_x-\mu_y}}{\left(\mu_x+\mu_y\right) B\left(\mu_y, \mu_x+\mu_y\right)} \\ &\times \left. \frac{(\mathcal{G}_\kappa - 1)^k (2\mathcal{G}_\kappa)^{-k}}{\left(1+\mu_x+\mu_y\right)_k} {}_2F_2\left(1+\mu_x, \mu_x+2\mu_y; \mu_x, 1+k+\mu_x+\mu_y; \frac{\kappa_x \mu_x \nu_{\kappa\eta}}{(1+\mathcal{G}_\kappa)}\right)\right). \end{aligned} \tag{3.10}$$

$$f_Z(z) = \frac{2^{1-\mu_y}\left(1-v_{\eta\eta}\right)^{\mu_y} v_{\eta\eta}^{2\mu_x}}{z h_x^{\mu_x}\left|H_y\right|^{\mu_y}} \sum_{k=0}^{\mu_y-1} \frac{\left(1-\mu_y\right)_k \left(\mu_y\right)_k}{k!} \left(\frac{\mathcal{G}_\eta - 1}{2\mathcal{G}_\eta}\right)^k \left(\frac{\left(1-G_\eta\right)^{-\xi_1}}{B\left(2\mu_x, \mu_y\right)(1-\xi_1)_k}\right.$$
$$\times {}_2F_1\left(\frac{\xi_1-k}{2}, \frac{\xi_1-k+1}{2}; \frac{1}{2}+\mu_x; \frac{H_x^2 v_{\eta\eta}^2}{h_x^2\left(1-G_\eta\right)^2}\right) + \frac{2(-1)^{\mu_y}\mu_x\left(1+G_\eta\right)^{-\xi_1}}{\xi_1 B\left(\mu_y, \xi_1\right)(1+\xi_1)_k}$$
$$\left.\times {}_3F_2\left(1+\mu_x, \mu_x+\mu_y, \frac{1}{2}+\mu_x+\mu_y; \frac{1+k+\xi_1}{2}, \frac{2+k+\xi_1}{2}; \frac{H_x^2 v_{\eta\eta}^2}{h_x^2\left(1+G_\eta\right)^2}\right)\right). \tag{3.11}$$

in which $\mathcal{G}_a = \left|H_y\right|\left(1-v_{a\eta}\right)/h_y$ with $a \in \{\kappa, \eta\}$ and $\xi_1 = 2\mu_x + \mu_y$. Of course the inverse of these ratios can be easily obtained as hinted earlier. Special cases of the Nakagami-m distribution can also be easily obtained by setting the appropriate parameters.

3.4.1 Asymptotic Expressions

3.4.1.1 Asymptotic PDF and CDF

To obtain closed-form expressions for the behavior of the PDF for both the lower and upper tails, the following procedure was performed. In the cases involving the α-μ distribution, the Fox H-function was expanded in a power series using the residue theorem. By taking the poles over the $\Gamma(\mu_x + t)$, the variable z will have a positive exponent. Now, consider z small enough so that it is possible to ignore all terms in the sum other than the first one resulting in an expression for the lower tail of the PDF. On the other hand, by taking the residues over the other set of poles, a negative exponent on z arises. In this case, when z is great enough all terms of the sum but the first may be ignored resulting in the upper tail of the PDF.

In the other scenarios, for small values of z, we may consider $v_{ab} \cong z^2/u_{ab}^2$ and $1 - v_{ab} \cong 1$. As z is in the vicinity of 0, summation indexes different from zero can be ignored, that is, for the lower tail asymptotic behavior only the first term in the sum is used. Proceeding like this and after some algebraic manipulations, the PDF lower tails presented in Table 3.5 are obtained. For high values of z, we consider $\lim_{z\to\infty} {}_pF_q(a_p; b_q; k(1-v_{ab})) = 1$, $v_{ab} \cong 1$ and $1 - v_{ab} \cong u_{ab}^2/z^2$. Replacing these in the expressions of Table 3.3 and performing the required summation, the PDF upper tails given in Table 3.5 are obtained. Of course, by integrating these expressions from 0 to z and z to ∞, the CDF asymptotic behavior for the lower and upper tails can also be achieved respectively and this is provided in Table 3.6. Note that the CDF upper tails are indeed equal to one, although the expressions provided in Table 3.6 can be used to achieve the lower tails of the reciprocal distribution by using (3.6).

Table 3.5 – Left and Right Tail Asymptotic Expression for the PDF of the Ratio Distribution

Ratio	Left Tail	Right Tail
α-μ/κ-μ	$\frac{\alpha_x \Gamma\left(\frac{\alpha_x\mu_x}{2}+\mu_y\right) z^{\alpha_x\mu_x-1}}{\Gamma(\mu_x) u_{\alpha\kappa}^{\alpha_x\mu_x}} {}_1\tilde{F}_1\left(-\frac{\alpha_x\mu_x}{2};\mu_y;-\kappa_y\mu_y\right)$	$\frac{2\Gamma\left(\mu_x+\frac{2\mu_y}{\alpha_x}\right)}{e^{\kappa_y\mu_y}\Gamma(\mu_x)\Gamma(\mu_y)} \frac{u_{\alpha\kappa}^{2\mu_y}}{z^{2\mu_y+1}}$
α-μ/η-μ	$\frac{\alpha_x \Gamma\left(\frac{\alpha_x\mu_x}{2}+2\mu_y\right) z^{\alpha_x\mu_x-1}}{h_y^{\mu_y}\Gamma(\mu_x)\Gamma(2\mu_y) u_{\alpha\eta}^{\alpha_x\mu_x}} {}_2F_1\left(\frac{\alpha_x\mu_x}{4}+\mu_y, \frac{1}{2}+\frac{\alpha_x\mu_x}{4}+\mu_y; \frac{1}{2}+\mu_y; \frac{H_y^2}{h_y^2}\right)$	$\frac{2h_y^{-\mu_y}\Gamma\left(\mu_x+\frac{4\mu_y}{\alpha_x}\right)u_{\alpha\eta}^{4\mu_y}}{\Gamma(\mu_x)\Gamma(2\mu_y) z^{4\mu_y+1}}$
κ-μ/κ-μ	$\frac{2\,{}_1F_1\left(-\mu_x;\mu_y;-\kappa_y\mu_y\right) z^{2\mu_x-1}}{\exp(\kappa_x\mu_x) B(\mu_x,\mu_y) u_{\kappa\kappa}^{2\mu_x}}$	$\frac{2\,{}_1F_1\left(-\mu_y;\mu_x;-\kappa_x\mu_x\right) u_{\kappa\kappa}^{2\mu_y}}{\exp(\kappa_y\mu_y) B(\mu_x,\mu_y) z^{2\mu_y+1}}$
κ-μ/η-μ	$\frac{2z^{-1+2\mu_x}\,{}_2F_1\left(\frac{\mu_x+2\mu_y}{2},\frac{1+\mu_x+2\mu_y}{2};\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right)}{\exp(\kappa_x\mu_x) h_y^{\mu_y} B(\mu_x,2\mu_y) u_{\kappa\eta}^{2\mu_x}}$	$\frac{2u_{\kappa\eta}^{4\mu_y}\,{}_1F_1\left(-2\mu_y;\mu_x;-\kappa_x\mu_x\right)}{h_y^{\mu_y} B(\mu_x,2\mu_y) z^{4\mu_y+1}}$
η-μ/η-μ	$\frac{2\,{}_2F_1\left(\mu_x+\mu_y,\frac{1}{2}+\mu_x+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right) z^{4\mu_x-1}}{h_x^{\mu_x} h_y^{\mu_y} B(2\mu_x,2\mu_y) u_{\eta\eta}^{4\mu_x}}$	$\frac{2\,{}_2F_1\left(\mu_x+\mu_y,\frac{1}{2}+\mu_x+\mu_y;\frac{1}{2}+\mu_x;\frac{H_x^2}{h_x^2}\right) u_{\eta\eta}^{4\mu_y}}{h_x^{\mu_x} h_y^{\mu_y} B(2\mu_x,2\mu_y) z^{4\mu_y+1}}$

Table 3.6 – Left and Right Tail Asymptotic Expression for the CDF of the Ratio Distribution

Ratio	Left Tail	Right Tail
α-μ/κ-μ	$\frac{\Gamma\left(\frac{\alpha_x\mu_x}{2}+\mu_y\right) z^{\alpha_x\mu_x}}{\Gamma(1+\mu_x) u_{\alpha\kappa}^{\alpha_x\mu_x}} {}_1\tilde{F}_1\left(-\frac{\alpha_x\mu_x}{2}\ \mu_y;-\kappa_y\mu_y\right)$	$1-\frac{e^{-\kappa_y\mu_y}\Gamma\left(\mu_x+\frac{2\mu_y}{\alpha_x}\right)u_{\alpha\kappa}^{2\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y+1) z^{2\mu_y}}$
α-μ/η-μ	$\frac{\Gamma\left(\frac{\alpha_x\mu_x}{2}+2\mu_y\right) z^{\alpha_x\mu_x}}{\Gamma(1+\mu_x)\Gamma(2\mu_y) u_{\alpha\eta}^{\alpha_x\mu_x} h_y^{\mu_y}} {}_2F_1\left(\frac{\alpha_x\mu_x}{4}+\mu_y, \frac{1}{2}+\frac{\alpha_x\mu_x}{4}+\mu_y; \frac{1}{2}+\mu_y; \frac{H_y^2}{h_y^2}\right)$	$1-\frac{h_y^{-\mu_y}\Gamma\left(\mu_x+\frac{4\mu_y}{\alpha_x}\right)u_{\alpha\eta}^{4\mu_y}}{\Gamma(\mu_x)\Gamma(2\mu_y+1) z^{4\mu_y}}$
κ-μ/κ-μ	$\frac{z^{2\mu_x}\,{}_1F_1\left(-\mu_x;\mu_y;-\kappa_y\mu_y\right)}{\exp(\kappa_x\mu_x)\mu_x B(\mu_x,\mu_y) u_{\kappa\kappa}^{2\mu_x}}$	$1-\frac{u_{\kappa\kappa}^{2\mu_y}\,{}_1F_1\left(-\mu_y;\mu_x;-\kappa_x\mu_x\right)}{\exp(\kappa_y\mu_y)\mu_y B(\mu_y,\mu_x) z^{2\mu_y}}$
κ-μ/η-μ	$\frac{z^{2\mu_x}\,{}_2F_1\left(\frac{\mu_x+2\mu_y}{2},\frac{1+\mu_x+2\mu_y}{2};\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right)}{\mu_x B(\mu_x,2\mu_y) u_{\kappa\eta}^{2\mu_x}\exp(\kappa_x\mu_x) h_y^{\mu_y}}$	$1-\frac{{}_1F_1\left(-2\mu_y;\mu_x;-\kappa_x\mu_x\right)u_{\kappa\eta}^{4\mu_y}}{2\mu_y B(2\mu_y,\mu_x) z^{4\mu_y} h_y^{\mu_y}}$
η-μ/η-μ	$\frac{{}_2F_1\left(\mu_x+\mu_y,\frac{1}{2}+\mu_x+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right) z^{4\mu_x}}{2\mu_x B(2\mu_x,2\mu_y) h_x^{\mu_x} h_y^{\mu_y} u_{\eta\eta}^{4\mu_x}}$	$1-\frac{{}_2F_1\left(\mu_x+\mu_y,\frac{1}{2}+\mu_x+\mu_y;\frac{1}{2}+\mu_x;\frac{H_x^2}{h_x^2}\right) u_{\eta\eta}^{4\mu_y}}{2\mu_y B(2\mu_x,2\mu_y) h_x^{\mu_x} h_y^{\mu_y} z^{4\mu_y}}$

Table 3.7 – Parameters a and b for Various Modulation/Detection Combinations [45, Table 8.1]

a \ b	1/2	1
1/2	Orthogonal coherent BFSK	Orthogonal noncoherent BFSK
1	Antipodal coherent BPSK	Antipodal differentially coherent BPSK (DPSK)
$0 \leq g \leq 1$	Correlated coherent binary signaling	–

3.4.1.2 Asymptotic Bit Error Rate

In a Gaussian channel, the bit error rate (BER) of a binary signaling may be written in a compact form as [45, Eq. (8.100)]

$$P_b(\gamma) = \frac{\Gamma(b, a\gamma)}{2\Gamma(b)}, \tag{3.12}$$

in which γ is the instantaneous SNR, the parameters a and b are defined accordingly to the modulation and detection scheme as described in Table 3.7, and $\Gamma(\cdot,\cdot)$ is the complementary gamma function [36, Eq. (6.5.3)]. Consider now a binary signal over a ratio channel $Z = h_1/h_2$ in which $h_1 \triangleq X$ and $h_2 \triangleq Y$ with X and Y as defined previously. The signal-to-noise ratio (SNR) is obtained as

Table 3.8 – Bit Error Rate on High SNR for a Binary Signal over the Ratio Channel

Ratio	Asymptotic Behavior
α-μ/α-μ	$\frac{\Gamma\left(b+\frac{\alpha_x\mu_x}{2}\right)\Gamma\left(\frac{\alpha_x\mu_x}{\alpha_y}+\mu_y\right)}{2\mu_x\Gamma(b)\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\alpha\alpha}^2 a\bar{\gamma}}\right)^{\frac{\alpha_x\mu_x}{2}}$
α-μ/κ-μ	$\frac{\Gamma\left(b+\frac{\alpha_x\mu_x}{2}\right)\Gamma\left(\frac{\alpha_x\mu_x}{2}+\mu_y\right)}{2\Gamma(b)\Gamma(1+\mu_x)}\,{}_1\tilde{F}_1\left(-\frac{\alpha_x\mu_x}{2};\mu_y;-\kappa_y\mu_y\right)\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\alpha\kappa}^2 a\bar{\gamma}}\right)^{\frac{\alpha_x\mu_x}{2}}$
κ-μ/α-μ	$\frac{\Gamma\left(\mu_y+\frac{2\mu_x}{\alpha_y}\right)\Gamma(b+\mu_x)}{2\exp(\kappa_x\mu_x)\Gamma(1+\mu_x)\Gamma\left(\mu_y\right)\Gamma(b)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\kappa\alpha}^2 a\bar{\gamma}}\right)^{\mu_x}$
α-μ/η-μ	$\frac{h_y^{-\mu_y}\Gamma\left(b+\frac{\alpha_x\mu_x}{2}\right)\Gamma\left(\frac{\alpha_x\mu_x}{2}+2\mu_y\right)}{2\Gamma(b)\Gamma(1+\mu_x)\Gamma\left(2\mu_y\right)}\,{}_2F_1\left(\frac{\alpha_x\mu_x}{4}+\mu_y,\frac{1}{2}+\frac{\alpha_x\mu_x}{4}+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right)\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\alpha\eta}^2 a\bar{\gamma}}\right)^{\frac{\alpha_x\mu_x}{2}}$
η-μ/α-μ	$\frac{h_x^{-\mu_x}\Gamma\left(\mu_y+\frac{4\mu_x}{\alpha_y}\right)\Gamma(b+2\mu_x)}{2\Gamma\left(\mu_y\right)\Gamma(1+2\mu_x)\Gamma(b)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\eta\alpha}^2 a\bar{\gamma}}\right)^{2\mu_x}$
κ-μ/κ-μ	$\frac{\Gamma(b+\mu_x)\,{}_1F_1\left(-\mu_x;\mu_y;-\kappa_y\mu_y\right)}{2\mu_x\exp(\kappa_x\mu_x)B\left(\mu_x,\mu_y\right)\Gamma(b)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\kappa\kappa}^2 a\bar{\gamma}}\right)^{\mu_x}$
κ-μ/η-μ	$\frac{\Gamma(b+\mu_x)\,{}_2F_1\left(\frac{\mu_x}{2}+\mu_y,\frac{1}{2}(1+\mu_x)+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right)}{2e^{\kappa_x\mu_x}h_y^{\mu_y}\mu_x B\left(\mu_x,2\mu_y\right)\Gamma(b)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\kappa\eta}^2 a\bar{\gamma}}\right)^{\mu_x}$
η-μ/κ-μ	$\frac{\Gamma(b+2\mu_x)\,{}_1F_1\left(-2\mu_x;\mu_y;-\kappa_y\mu_y\right)}{4\mu_x\Gamma(b)h_x^{\mu_x}B\left(\mu_y,2\mu_x\right)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\eta\kappa}^2 a\bar{\gamma}}\right)^{2\mu_x}$
η-μ/η-μ	$\frac{\Gamma(b+2\mu_x)\,{}_2F_1\left(\mu_x+\mu_y,\frac{1}{2}+\mu_x+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2}{h_y^2}\right)}{4h_x^{\mu_x}h_y^{\mu_y}\mu_x B\left(2\mu_x,2\mu_y\right)\Gamma(b)}\left(\frac{\mathbb{E}\left[Z^2\right]}{u_{\eta\eta}^2 a\bar{\gamma}}\right)^{2\mu_x}$

$$\gamma = \frac{E_s}{N_0}Z^2 = \frac{E_s}{N_0}\left(\frac{h_1}{h_2}\right)^2. \tag{3.13}$$

in which E_s is the average energy per symbol and N_0 is the noise power spectral density. The average SNR is given as

$$\bar{\gamma} = \frac{E_s}{N_0}\mathbb{E}[Z^2] = \frac{E_s}{N_0}\mathbb{E}[h_1^2]\mathbb{E}[h_2^{-2}]. \tag{3.14}$$

Therefore, the instantaneous may be written in terms of the mean SNR as

$$\gamma = \bar{\gamma}\frac{Z^2}{\mathbb{E}[Z^2]} \tag{3.15}$$

After a simple variable transformation, the PDF of the SNR is given as

$$f_\Gamma(\gamma) = \frac{\sqrt{\mathbb{E}[Z^2]}}{2\sqrt{\gamma\bar{\gamma}}}f_Z\left(\sqrt{\frac{\gamma\mathbb{E}[Z^2]}{\bar{\gamma}}}\right) \tag{3.16}$$

The average BER is obtained by averaging (3.12) over the SNR as

$$P_b = \int_0^\infty P_b(\gamma)f_\Gamma(\gamma)\,d\gamma \tag{3.17}$$

The PDF of the SNR can be written as a power series using, for instance, those in Table 3.3 and expanding the special function therein to obtain a PDF in the form of $\sum_i a_i(\gamma/\bar{\gamma})^{b_i}$. To obtain the asymptotic behavior for the BER in a high SNR regime, higher exponents in the PDF can be ignored taking only $i = 0$, which means that the BER behavior at high SNR levels depends only on the lower tail of the channel PDF. That is, to obtain an asymptotic equation for the BER at high SNR over a ratio channel, replace $f_Z(z)$ by the respective channel lower tail provided in Table 3.5. Table 3.8 gives the asymptotic behavior of the BER over a ratio channel for all possible combinations of ratios.

3.5 Numerical Results

As an illustration, Figure 3.1 shows the PDF of the ratio of κ-μ by an η-μ RVs and their asymptotic behavior at the right and left tails. It is important to remark that the slope of the left tail depends solely on the distribution of the numerator, specifically on the parameter μ (and α if involved), wherein the rate of decay on the right tail is ruled by the parameter μ of the denominator. It is interesting to note that the distributions α-μ, κ-μ and η-μ have a finite non-zero value at $z = 0$, when $\alpha\mu = 1$, $2\mu = 1$ and $4\mu = 1$, respectively. The same effect is encountered for the ratio distributions if the random variable on the numerator also satisfies the condition for finite, non-zero value at $z = 0$ and such property can be observed in Figure 3.2.

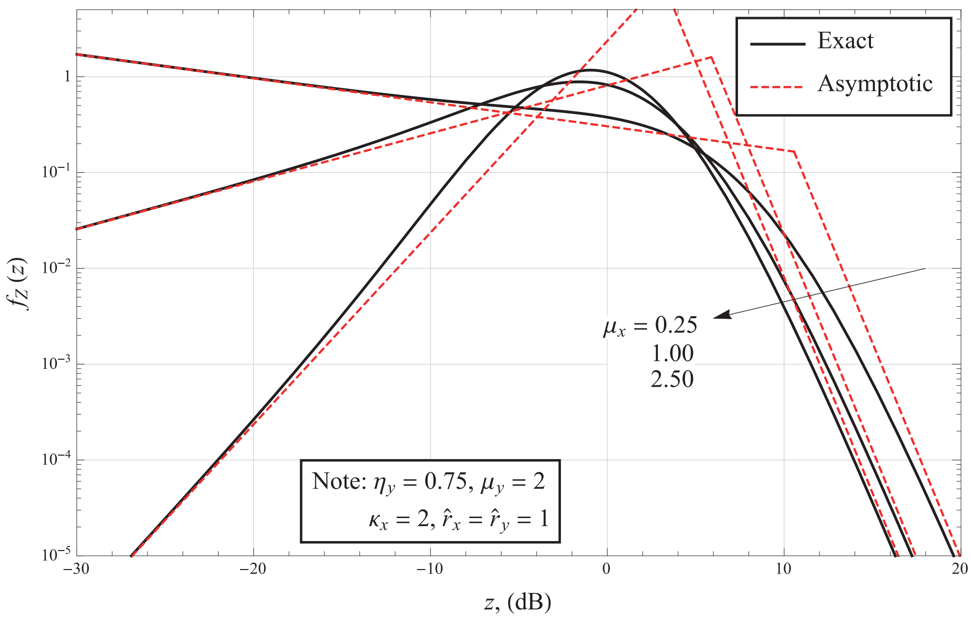

Figure 3.1 – PDF of the ratio of κ-μ and η-μ RVs with $\mu_y = 2$, $\eta_y = 0.75$, $\kappa_x = 2$, and $\mu_x = \{0.25, 1, 2.5\}$ along with their asymptotic behavior for small and large z.

In Figure 3.3, the BER is depicted for channel resulting from the ratio of a κ-μ by an η-μ distributions, in which solid lines are exact solution obtained from (3.17) and dashed lines are the asymptotic curves at high SNR. As expected, simulation results coincide with analytical curves. It can be seen from Figure 3.3 that the high SNR asymptote will start to coincide with the exact curve earlier when the numerator parameter μ (or $\alpha\mu$ if the numerator is α-μ distributed) is small.

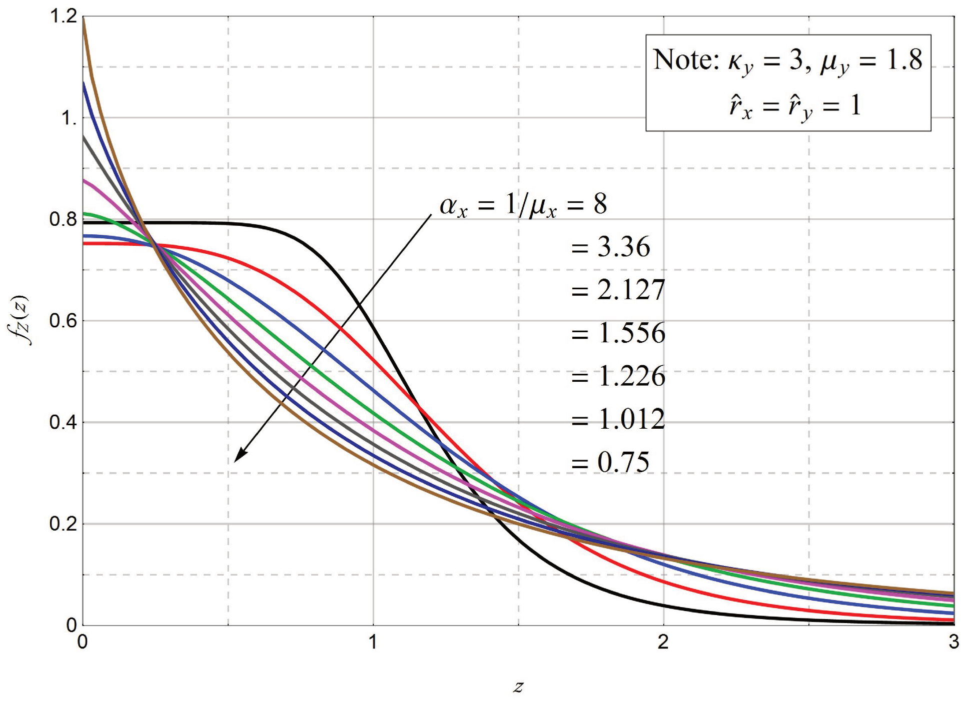

Figure 3.2 – PDF of the ratio of α-μ and κ-μ RVs with $\kappa_y = 3$, $\mu_y = 1.8$, $\hat{r}_x = \hat{r}_y = 1$ and several values for α_x and μ_x such that $\alpha_x \mu_x = 1$.

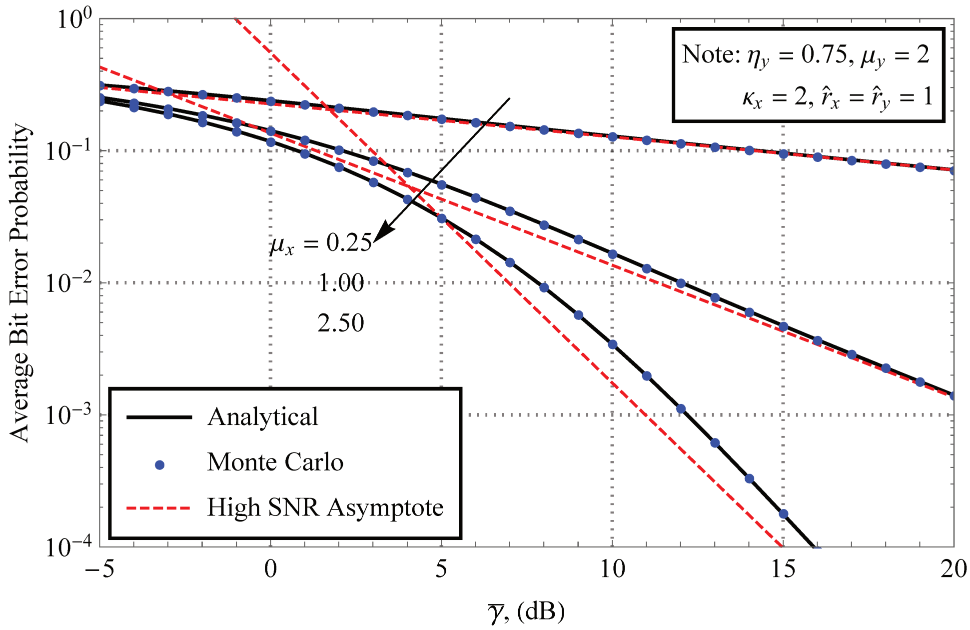

Figure 3.3 – BER of the ratio of κ-μ and η-μ RVs with $\mu_y = 2$, $\eta_y = 0.75$, $\kappa_x = 2$, and $\mu_x = \{0.25, 1, 2.5\}$ along with their asymptotic behavior for small and large z.

3.6 Practical Applications

In this section, we demonstrate the practical utility of the new formulations derived here by analysing the secrecy capacity of a Gaussian wire-tap channel [46, 47], for D2D and V2V communications, using data obtained from field measurements. The secrecy capacity can be viewed as the maximum transmission rate such that no bit of information is obtained by an eavesdropper. The experiment reported here has been conducted by Prof. Simon Cotton's team at the Wireless Communications Laboratory, Institute of Electronics, Communications and Information Technology at The Queen's University of Belfast. This is reproduced here with their due permission. The secrecy capacity for a Gaussian wire-tap channel is defined as

$$C_s = \begin{cases} \log_2\left(\dfrac{1+\gamma_m}{1+\gamma_w}\right), & \text{if } \gamma_m > \gamma_w \\ 0, & \text{if } \gamma_m \leqslant \gamma_w \end{cases} \tag{3.18}$$

in which γ_m and γ_w are the signal-to-noise ratio (SNR) for the main and the wire-tap channels, respectively, and are given by $\gamma_i = Ph_i^2/N_i$, $i = (m, w)$, in which P and N_i are the transmission and noise power, respectively, and h_i is the channel gain. The probability of positive secrecy capacity is given as

$$\begin{aligned} \Pr[C_s > 0] &= 1 - \Pr\left[\frac{h_m}{h_w} < \sqrt{\frac{N_m}{N_w}}\right] \\ &= 1 - F_Z\left(\sqrt{\frac{\bar{\gamma}_w \mathbb{E}[h_m^2]}{\bar{\gamma}_m \mathbb{E}[h_w^2]}}\right) \\ &= F_{\tilde{Z}}\left(\sqrt{\frac{\bar{\gamma}_m \mathbb{E}[h_w^2]}{\bar{\gamma}_w \mathbb{E}[h_m^2]}}\right) \end{aligned} \tag{3.19}$$

in which: (i) $F_Z(\cdot)$ is obtained from (3.3) with parameters obtained in Section 3.1 and evaluated by the exact series given in Table 3.4 in accordance with the chosen fading model for h_m and h_w; (ii) $F_{\tilde{Z}}$ is the CDF for the reciprocal distribution given in (3.6); and (iii) $\bar{\gamma}_m$ and $\bar{\gamma}_w$ are the mean values of the SNR for the main and wiretapper channels, respectively.

3.6.1 D2D Communications

The D2D measurements were conducted at 5.8 GHz within an indoor seminar room. The exact details of the measurement setup, experiments, and data analysis can be found in [48]. The measurement trial considered three persons who carried the hypothetical user equipments (UEs) A, B and E[2] and were positioned at points X, Y and Z respectively (see Figure 3.4). All three persons had the UEs positioned at their heads, and were initially stationary. The persons at positions Y and Z were then instructed to walk around randomly within a circle of radius 0.5 m from their starting positions whilst imitating a voice call. It should be noted that the channel between UEs A and B is referred to as the main channel whilst the channel between UEs A and E is referred to as the wire-tap channel.

[2] A, B and E are analogous to Alice, Bob and Eve as commonly encountered in eavesdropping analyses of wireless security applications.

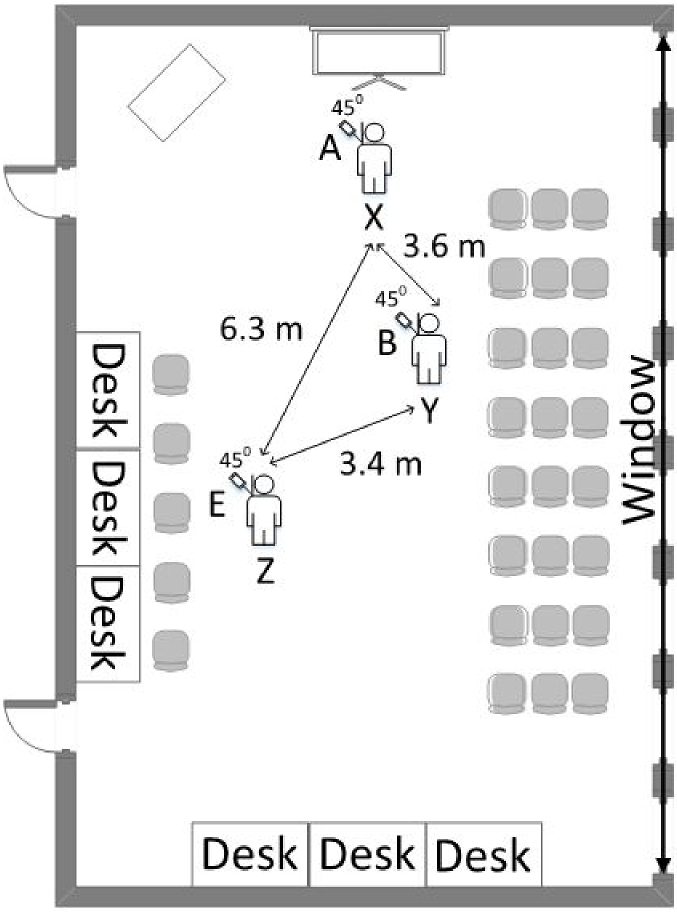

Figure 3.4 – Seminar room environment showing the position of the UEs A, B and E for the D2D scenario.

For the analysis, the data sets were normalized to their respective local means prior to parameter estimation. To determine an appropriate window size for the extraction of the local mean signal, the raw data was visually inspected and overlaid with the local mean signal for differing window sizes. In this case, a smoothing window of 500 samples was used. As an example of the data fitting process, Figs.3.5(a) and (b) show the PDF of the α-μ, κ-μ and η-μ fading models fitted to the D2D fading data for the main and the wire-tap channels, respectively. A total of 74763 samples of the received signal power were obtained and used for parameter estimation. The parameter estimates were obtained using the `lsqnonlin` function available in the optimization toolbox of MATLAB along with the α-μ, κ-μ and η-μ PDFs. To allow the reader to reproduce these plots, parameter estimates for all of measurement scenarios are given in Table 3.9. From Figure 3.5, we observe that both the α-μ and κ-μ PDFs provide a very good approximation to the measured data, whilst some disparity is noticed between the lower tail of the empirical PDF and the theoretical η-μ PDF. To select the candidate model, from the α-μ, κ-μ and η-μ distributions, most likely to have been responsible for generating the fading, the Akaike information criterion (AIC) was used. Based upon the ranking performed using the computed AIC, it was found that the

α-μ distribution was the most likely model for the main channel, whilst the κ-μ distribution was the most likely model for the wire-tap channel. The α parameter estimate for the main channel was found to be higher than that for an equivalent Nakagami-m fading channel (α = 2, $\mu = m$), while μ was found to be 0.68 suggesting a tendency towards significant fading for the main channel. This is understandable due to the constantly changing orientation and posture of both the test subjects. For the wire-tap channel, we see that the κ parameter estimate was greater than 1, suggesting that a perceptible dominant component existed. The μ parameter estimate for the wire-tap channel was found to be relatively close to 1, indicating that a single cluster of scattered multipath contributes to the signals received in the D2D (indoor) scenario.

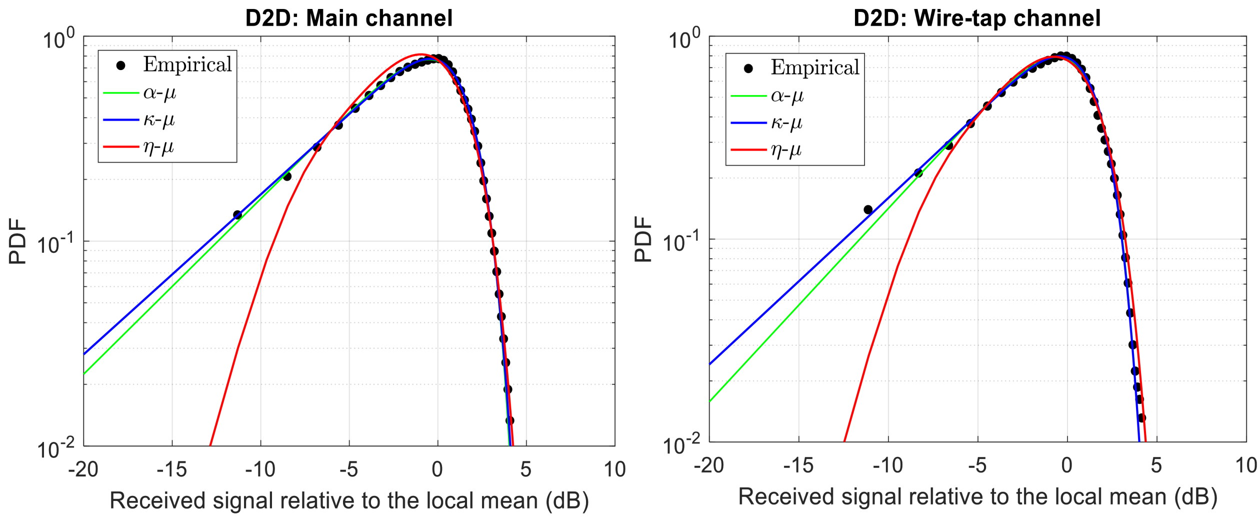

Figure 3.5 – Empirical envelope PDF of (a) the main channel and (b) the wire-tap channel compared to the α-μ, κ-μ and η-μ PDFs for the D2D channel measurements.

Utilizing the parameter estimates from the D2D field trials (see Table 3.9), Figure 3.6 depicts the estimated probability of positive secrecy capacity versus $\bar{\gamma}_w$ for a range of $\bar{\gamma}_m$, when the main and the wire-tap channels experience α-μ and κ-μ fading, respectively. We now adopt the following approach to analyse the probability of positive secrecy capacity. We perform our analysis when $\bar{\gamma}_m$ is fixed at 10 dB and for two different levels of positive secrecy capacity: 0.10 (10% level) and 0.50 (50% level), which are indicative of low and mid-range levels of positive secrecy capacity, respectively. From Figure 3.6, we observe that if the wire-tapper can improve her average SNR ($\bar{\gamma}_w$) from 5 dB to 10 dB, the probability of positive secrecy capacity will decrease from 77% to 49%. Furthermore, to ensure a positive secrecy capacity level of at least 10%, we find that the wire-tappers average SNR must not exceed 19 dB. Likewise, to ensure a mid-range positive secrecy capacity level of at least 50%, $\bar{\gamma}_w$ must not exceed 10 dB.

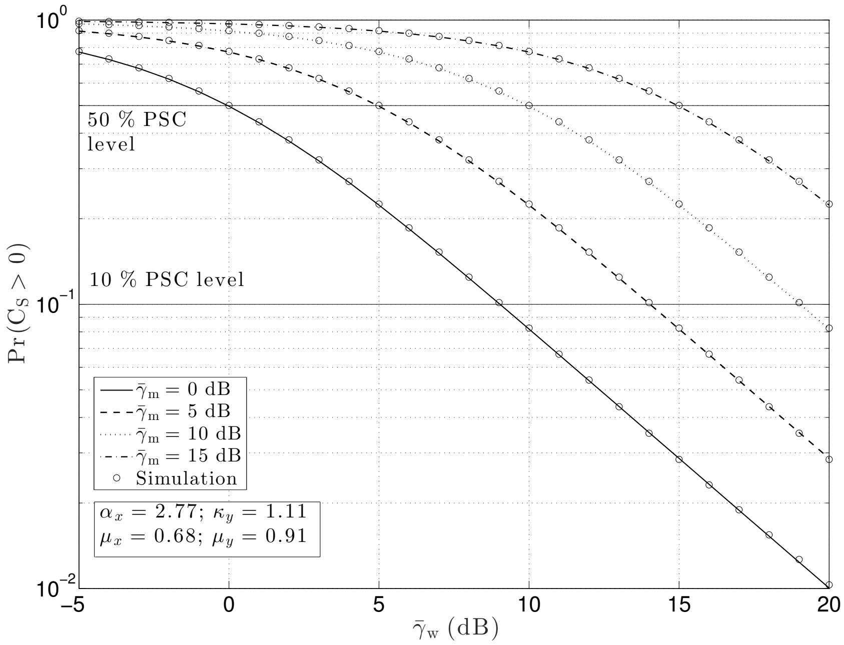

Figure 3.6 – Probability of positive secrecy capacity versus $\bar{\gamma}_w$ considering a range of $\bar{\gamma}_m$ for the D2D channel measurements when the main and wire-tap channels are assumed to undergo α-μ and κ-μ fading, respectively. Here, PSC indicates positive secrecy capacity.

Table 3.9 – Parameter estimates for the α-μ, κ-μ and η-μ fading models fitted to the D2D measured data along with the AIC Rank.

	α-μ				κ-μ				η-μ			
	α	μ	$\hat{r}$	AIC Rank	κ	μ	$\hat{r}$	AIC Rank	η	μ	$\hat{r}$	AIC Rank
D2D: Main channel	2.77	0.68	1.18	1	1.09	0.89	1.23	2	0.01	1.08	1.21	3
D2D: Wire-tap channel	2.64	0.74	1.16	2	1.11	0.91	1.20	1	0.01	1.10	1.31	3
V2V: Main channel	1.97	2.88	1.03	2	2.54	1.41	1.02	3	0.56	1.47	1.08	1
V2V: Wire-tap channel	2.78	1.55	1.04	3	41.7	0.13	0.99	2	0.80	1.39	1.02	1

3.6.2 V2V Communications

The V2V channel measurements considered in this work were conducted at 5.8 GHz. The exact details of the measurement setup and experiments can be found in [48]. Specifically, the transmitter, the legitimate receiver and the wire-tapper were positioned on the center of the dashboards within vehicles A, B and E, respectively (see [48, Figure 10]). Initially, both vehicles A and B drove towards each other at a speed of 30 mph whilst the vehicle containing node E remained parked on the side of the road. The fading data used in the analysis presented here, considered the channel acquisitions obtained when vehicles A and B drove past each other and continued their onward journey. As before, it should be noted that the channel between nodes A and B is referred to as the main channel whilst the channel between nodes A and E is referred to as the wire-tap channel.

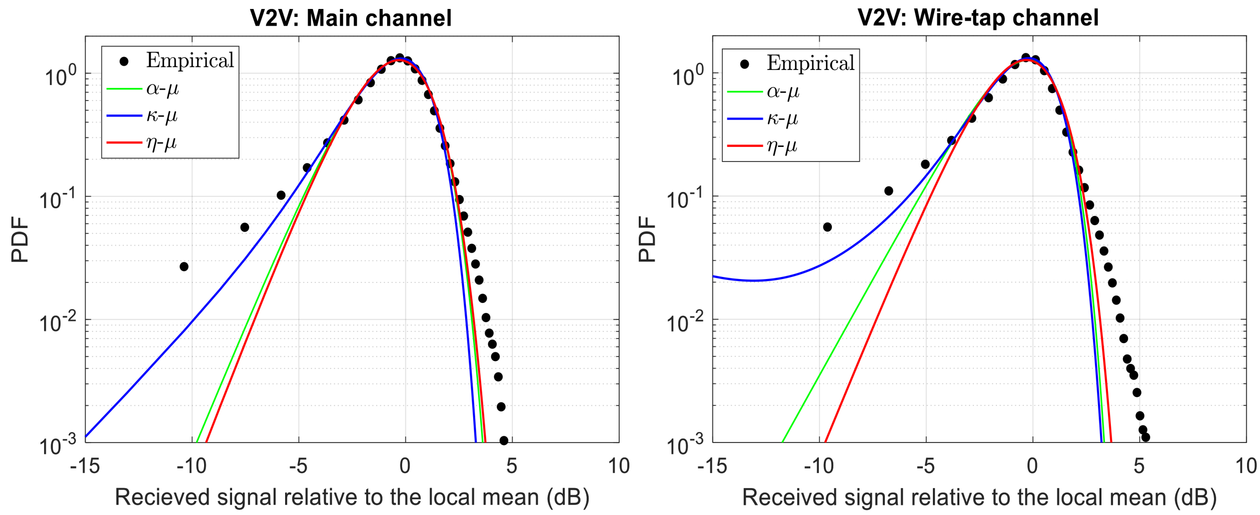

Figure 3.7 – Empirical envelope PDF of (a) the main channel and (b) the wire-tap channel compared to the α-μ, κ-μ and η-μ PDFs for the V2V channel measurements.

Similar to the D2D channel measurements, the data sets were normalized to their respective local means prior to parameter estimation. In this case, a smoothing window of 200 samples was used. Figs. 3.7(a) and (b) show the PDF of the α-μ, κ-μ and η-μ fading models fitted to the V2V data for the main and the wire-tap channels, respectively. A total of 56579 samples of the received signal power were obtained and used for parameter estimation. When compared to the α-μ and η-μ PDFs, we observe that the κ-μ distribution provides a better approximation around the lower tail of the empirical PDFs in Figs. 3.7(a) and (b). On the contrary, the η-μ PDF provides a better approximation than the α-μ and κ-μ distributions around the median (where the greatest number of fade levels occur) and also the upper tail of the empirical PDF. Accordingly, from the computed AIC rankings (see Table 3.9), it was found that the η-μ distribution was the most likely model for both the main and wire-tap channels. Inspecting the estimated η parameter for the V2V channel measurements (see Table 3.9), we observe that under the assumption of Format 1 for the η-μ fading model [4], a power imbalance existed between the in-phase and quadrature components for both the main and wire-tap channels. Under the assumption of η-μ fading, low estimates of μ were also found for both the main and wire-tap channels, indicating that minimal environmental multipath contributions offered by the surrounding environment.

Now, using the parameter estimates from V2V field trials (see Table 3.9), Figure 3.8 depicts the estimated probability of positive secrecy capacity versus $\bar{\gamma}_w$ for a range of $\bar{\gamma}_m$, when the main and the wire-tap channel experiences η-μ fading. Following a similar approach to the D2D analysis with $\bar{\gamma}_m$ fixed at 10 dB, we observe that increasing $\bar{\gamma}_w$ from 5 dB to 10 dB causes the probability of positive secrecy capacity to significantly decrease from

89% to 50%. Moreover, to ensure a positive secrecy capacity level of at least 10% or 50%, we find that $\bar{\gamma}_w$ must not exceed 15 dB and 10 dB, respectively.

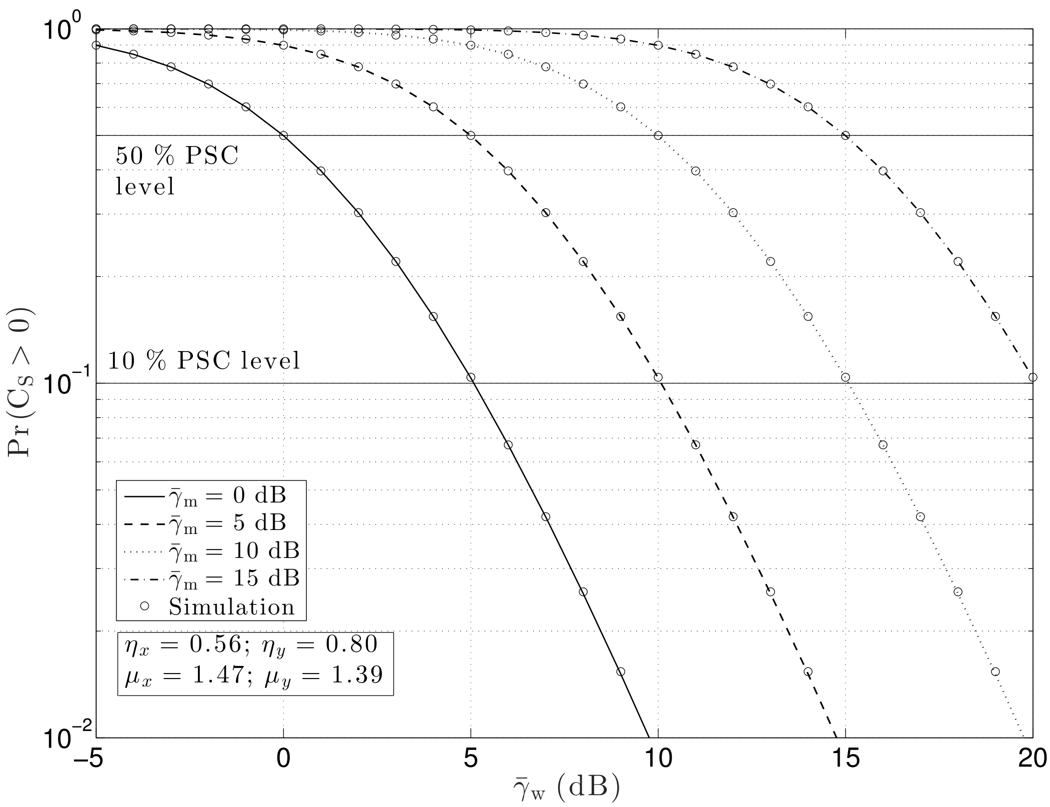

Figure 3.8 – Probability of positive secrecy capacity versus $\bar{\gamma}_w$ considering a range of $\bar{\gamma}_m$ for the V2V when it is assumed to undergo η-μ fading. Here, PSC indicates positive secrecy capacity.

3.7 Conclusions

This chapter presents novel and compact expressions for the PDF and CDF of the ratio of two fading envelopes following α-μ, κ-μ and η-μ distributions in terms of the Fox H-function. Exact, computationally efficient series representations are also provided. New, simple, exact, closed-form expressions are obtained for special cases involving the Hoyt, Nakagami-m and η-μ with integer μ parameter distributions. Applications for these results include multihop systems, spectrum sharing, the characterization of co-channel interference and physical layer security, among other areas of wireless communications. Multipath-shadowing composite fading statistics can also be obtained as special cases of the ratio distribution and the various combinations given here can be used to characterize a plethora of fading environments. Interestingly, and as well known, the sum of independent identically distributed (i.i.d.) κ-μ powers is another κ-μ. In the same way, the sum of i.i.d. η-μ is another η-μ. Therefore, the results provided here can be directly applied to maximal ratio combining systems in the presence of interference. In the same way, the sum of i.i.d. α-μ, κ-μ, and η-μ envelopes can be well approximated by the respective distributions, then rendering the results useful in the study of equal gain combining systems. Similar comments concerning the sum of independent non-identically distributed variates (power and envelope) also apply. The practical application example shown here dealt with secrecy capacity of a Gaussian wire-tap channel for D2D and V2V communications, using data from field measurements with experiments conducted at 5.8 GHz.

Chapter 4

Statistics for the Product of Fading Envelopes

In this chapter, the first order statistics such as PDFs, CDFs and moments of the product of two random envelopes taken from the α-μ, η-μ, and κ-μ distributions are presented both in terms of the multivariable Fox H-function and in fairly simple, fast convergent computable series expansions. The results, given in terms of the Fox H-function, are obtained through the inverse Mellin transform and, by the use of the sum of residues, series expansions are derived. Another interesting result provided here is the that of the definite integral involving the product of a PDF and a CDF which is closely related to the CDF of the product of two random variates. These results find application in a plethora of wireless communications systems. As an application example, performance analysis metrics namely amount of fading, outage probability, and outage capacity of a two-tap cascaded channel are derived. Additionally, the probability of detection of an UHF RFID system is also derived.

4.1 The Product of Two Random Envelopes

Consider the RV $Z = R_1 R_2 > 0$ to be the product of two positive, independent and arbitrarily distributed RVs R_1 and R_2 whose PDFs are denoted by $f_{R_1}(x)$ and $f_{R_2}(y)$. From the standard probability procedure, the PDF of the random variate Z can be obtained by the following integral

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{R_1}(\frac{z}{y}) f_{R_2}(y)\, dy, \tag{4.1}$$

and the CDF is obtained as

$$F_Z(z) = \int_0^z f_Z(\tau)\, d\tau = \int_0^\infty F_{R_1}(\frac{z}{y}) f_{R_2}(y)\, dy. \tag{4.2}$$

Table 4.1 – Parameters of the Fox H-Function for the Product PDF

Product	C	$\mathbf{x}$	β	$\mathbf{B}$	δ	$\mathbf{D}$
α-$\mu\times\alpha$-μ	$\frac{1}{z\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[\frac{z}{\mathscr{A}_1\mathscr{A}_2}\right]$	$[\mu_1,\mu_2]$	$\begin{pmatrix} 1/\alpha_1 \\ 1/\alpha_2 \end{pmatrix}$	-	-
α-$\mu\times\kappa$-μ	$\frac{2}{z\Gamma(\mu_1)}$	$\left[\kappa_2\mu_2, \frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_2,\mu_1,\mu_2]$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 2/\alpha_1 \\ 0 & 1 \end{pmatrix}$	$[\mu_2,0]$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
α-$\mu\times\eta$-μ	$\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[-\frac{H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_2,\mu_1,2\mu_2]$	$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 2/\alpha_1 \\ 0 & 1 \end{pmatrix}$	$\left[\mu_2+\frac{1}{2},0\right]$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
κ-$\mu\times\kappa$-μ	$\frac{2}{z}$	$\left[\kappa_1\mu_1,\kappa_2\mu_2, \frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,\mu_2]$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$[\mu_1,\mu_2,\mathbf{0}_2]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$
κ-$\mu\times\eta$-μ	$\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_2)}$	$\left[\kappa_1\mu_1,-\frac{H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,2\mu_2]$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$\left[\mu_1,\mu_2+\frac{1}{2},\mathbf{0}_2\right]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$
η-$\mu\times\eta$-μ	$\frac{2^{3-2\mu_1-2\mu_2}\pi}{z\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[-\frac{H_1^2}{4h_1^2},-\frac{H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,2\mu_1,2\mu_2]$	$\begin{pmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$[\mu_1+\frac{1}{2},\mu_2+\frac{1}{2},\mathbf{0}_2]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$

Table 4.2 – Parameters of the Fox H-Function for the Product CDF

Product	C	$\mathbf{x}$	β	$\mathbf{B}$	δ	$\mathbf{D}$
α-$\mu\times\alpha$-μ	$\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[\frac{z}{\mathscr{A}_1\mathscr{A}_2}\right]$	$[\mu_1,\mu_2,0]$	$\begin{pmatrix} 1/\alpha_1 \\ 1/\alpha_2 \\ -1 \end{pmatrix}$	$[1]$	(-1)
α-$\mu\times\kappa$-μ	$\frac{1}{\Gamma(\mu_1)}$	$\left[\kappa_2\mu_2, \frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_2,\mu_1,\mu_2]$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 2/\alpha_1 \\ 0 & 1 \end{pmatrix}$	$[\mu_2,1]$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
α-$\mu\times\eta$-μ	$\frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[-\frac{H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_2,\mu_1,2\mu_2]$	$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 2/\alpha_1 \\ 0 & 1 \end{pmatrix}$	$\left[\mu_2+\frac{1}{2},1\right]$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
κ-$\mu\times\kappa$-μ	1	$\left[\kappa_1\mu_1,\kappa_2\mu_2, \frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,\mu_2]$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$[\mu_1,\mu_2,1,0]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$
κ-$\mu\times\eta$-μ	$\frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}$	$\left[\kappa_1\mu_1,-\frac{H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,2\mu_2]$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$\left[\mu_1,\mu_2+\frac{1}{2},1,0\right]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$
η-$\mu\times\eta$-μ	$\frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[-\frac{H_1^2}{4h_1^2},\frac{-H_2^2}{4h_2^2}, \frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,2\mu_1,2\mu_2]$	$\begin{pmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ & \mathbf{I}_3 & \\ 0 & 0 & 1 \end{pmatrix}$	$\left[\mu_1+\frac{1}{2},\mu_2+\frac{1}{2},1,0\right]$	$\begin{pmatrix} & -\mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$

Alternatively, the inverse Mellin transform may be used to derive an expression for these statistics as discussed in Section 2.2. Since R_1 and R_2 are independent, the s-th moment of the RV Z may be obtained as

$$\mathbb{E}[Z^s] = \mathbb{E}[R_1^s]\mathbb{E}[R_2^s]. \tag{4.3}$$

Ergo, the PDF of the ratio distribution can be obtained through (2.8) whereas the CDF can be obtained with the help of (4.2). It is worth remarking that both the PDF and CDF can be represented in a compact form in terms of the multivariable Fox H-function following the general structure

$$g(\mathbf{x}) = C\mathbb{H}[\mathbf{x};(\beta,\mathbf{B});(\delta,\mathbf{D});\mathscr{L}]. \tag{4.4}$$

The parameters for the Fox H-function representation for the product PDF and CDF for any combination of two RVs are summarized in Table 4.1 and 4.2, respectively, in which $\mathbf{I}_n$ and $\mathbf{0}_n$ denotes, respectively, the identity matrix of order n and a sequence of n zeros. Please see Appendix B for their mathematical derivation.

Table 4.3 – Series Expansion for the Product PDF

Product	Series Representation
α-$\mu\times\alpha$-μ	Refer to [7]
α-$\mu\times\kappa$-μ	$f_Z(z)=\frac{2}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}\Gamma\left(\mu_1-\frac{2}{\alpha_1}(i+\mu_2)\right){}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\right.$ $\left.+\frac{\alpha_1}{2}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{\frac{\alpha_1}{2}(i+\mu_1)}\Gamma\left(\mu_2-\frac{\alpha_1}{2}(i+\mu_1)\right){}_1F_1\left(\frac{\alpha_1}{2}(i+\mu_1);\mu_2;-\kappa_2\mu_2\right)\right\}$
α-$\mu\times\eta$-μ	$f_Z(z)=\frac{2}{zh_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\left(\frac{z^2h_2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\Gamma\left(\mu_1-\frac{2}{\alpha_1}(i+2\mu_2)\right){}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.+\frac{\alpha_1}{2}\left(\frac{z^2h_2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{\frac{\alpha_1}{2}(i+\mu_1)}\Gamma\left(2\mu_2-\frac{\alpha_1}{2}(i+\mu_1)\right){}_2F_1\left(\mu_2-\frac{\alpha_1(i+\mu_1)}{4},\mu_2-\frac{\alpha_1(i+\mu_1)}{4}+\frac{1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$
κ-$\mu\times\kappa$-μ	$f_Z(z)=\frac{2}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}\Gamma(-i+\mu_1-\mu_2){}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2){}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\right.$ $\left.+\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_1}\Gamma(-i-\mu_1+\mu_2){}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1){}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right\}$
κ-$\mu\times\eta$-μ	$f_Z(z)=\frac{2\pi\csc(\pi(2\mu_2-\mu_1))}{zh_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\left(\frac{z^2h_2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{i+\mu_1}\frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(i+\mu_1-2\mu_2+1)}{}_2F_1\left(-\frac{i+\mu_1-2\mu_2}{2},-\frac{i+\mu_1-2\mu_2-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.-\left(\frac{z^2h_2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{\Gamma(i-\mu_1+2\mu_2+1)}{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$
η-$\mu\times\eta$-μ	$f_Z(z)=\frac{2\pi\csc(2\pi(\mu_2-\mu_1))}{zh_1^{\mu_1}h_2^{\mu_2}\Gamma(2\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}$ $\times\left\{\left(\frac{z^2h_1h_2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_1}\frac{{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_1+\frac{1}{2};\frac{H_1^2}{h_1^2}\right)}{\Gamma(i+2\mu_1-2\mu_2+1)}{}_2F_1\left(-\frac{i+2\mu_1-2\mu_2}{2},-\frac{i+2\mu_1-2\mu_2-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.-\left(\frac{z^2h_1h_2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\frac{{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)}{\Gamma(i-2\mu_1+2\mu_2+1)}{}_2F_1\left(-\frac{i-2\mu_1+2\mu_2}{2},-\frac{i-2\mu_1+2\mu_2-1}{2};\mu_1+\frac{1}{2};\frac{H_1^2}{h_1^2}\right)\right\}$

4.2 Series Representations

In order to facilitate the implementation of the product statistics, series expansions for the PDF and CDF of the product distribution are given in Tables 4.3 and 4.4 respectively, and their mathematical derivation is found in Appendix C.

It is important to remark that for a certain combination of the parameters α and μ, these expressions will produce a singularity. Nevertheless, these restrictions are non-prohibitive as the function exists for these parameters and a limit can be used to numerically evaluate the expressions. As expected, the parameters influence the convergence of the power series. Specifically, for a given value of z, the series will converge faster the higher the values of $\mathscr{A}$, $\mathscr{K}$ and $\mathscr{E}/\sqrt{h}$. Additionally, these series converge faster for small values of z. Table 4.5 offers the number of terms necessary to achieve a 10^{-10} accuracy for the power series in Tables 4.3 and 4.4

Table 4.4 – Series Expansion for the Product CDF

Product	Series Representation
α-$\mu\times\alpha$-μ	Refer to [7]
α-$\mu\times\kappa$-μ	$F_Z(z)=\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\frac{1}{i+\mu_2}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}\Gamma\left(\mu_1-\frac{2}{\alpha_1}(i+\mu_2)\right){}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\right.$ $\left.+\frac{1}{i+\mu_1}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{\frac{\alpha_1}{2}(i+\mu_1)}\Gamma\left(\mu_2-\frac{\alpha_1}{2}(i+\mu_1)\right){}_1F_1\left(\frac{\alpha_1}{2}(i+\mu_1);\mu_2;-\kappa_2\mu_2\right)\right\}$
α-$\mu\times\eta$-μ	$F_Z(z)=\frac{1}{h_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\frac{1}{i+2\mu_2}\left(\frac{z^2h_2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\Gamma\left(\mu_1-\frac{2}{\alpha_1}(i+2\mu_2)\right){}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.+\frac{1}{i+\mu_1}\left(\frac{z^2h_2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{\frac{\alpha_1}{2}(i+\mu_1)}\Gamma\left(2\mu_2-\frac{\alpha_1}{2}(i+\mu_1)\right){}_2F_1\left(\mu_2-\frac{\alpha_1(i+\mu_1)}{4},\mu_2-\frac{\alpha_1(i+\mu_1)}{4}+\frac{1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$
κ-$\mu\times\kappa$-μ	$F_Z(z)=\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\frac{1}{i+\mu_1}\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_1}\Gamma(-i-\mu_1+\mu_2){}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1){}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right.$ $\left.+\frac{1}{i+\mu_2}\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}\Gamma(-i+\mu_1-\mu_2){}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2){}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\right\}$
κ-$\mu\times\eta$-μ	$F_Z(z)=\frac{\pi\csc(\pi(2\mu_2-\mu_1))}{h_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\left(\frac{z^2h_2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{i+\mu_1}\frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{(i+\mu_1)\Gamma(i+\mu_1-2\mu_2+1)}{}_2F_1\left(-\frac{i+\mu_1-2\mu_2}{2},-\frac{i+\mu_1-2\mu_2-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.-\left(\frac{z^2h_2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{(i+2\mu_2)\Gamma(i-\mu_1+2\mu_2+1)}{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$
η-$\mu\times\eta$-μ	$F_Z(z)=\frac{\pi\csc(2\pi(\mu_2-\mu_1))}{h_1^{\mu_1}h_2^{\mu_2}\Gamma(2\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}$ $\times\left\{\left(\frac{z^2h_1h_2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_1}\frac{{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_1+\frac{1}{2};\frac{H_1^2}{h_1^2}\right)}{(i+2\mu_1)\Gamma(i+2\mu_1-2\mu_2+1)}{}_2F_1\left(-\frac{i+2\mu_1-2\mu_2}{2},-\frac{i+2\mu_1-2\mu_2-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.-\left(\frac{z^2h_1h_2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\frac{{}_2F_1\left(-\frac{i}{2},-\frac{i-1}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)}{(i+2\mu_2)\Gamma(i-2\mu_1+2\mu_2+1)}{}_2F_1\left(-\frac{i-2\mu_1+2\mu_2}{2},-\frac{i-2\mu_1+2\mu_2-1}{2};\mu_1+\frac{1}{2};\frac{H_1^2}{h_1^2}\right)\right\}$

Table 4.5 – Number of terms for accuracy of 10^{-10} for the PDFs of Table 4.3 and for the CDFs of Table 4.4

	α-$\mu\times\kappa$-μ		α-$\mu\times\eta$-μ		κ-$\mu\times\kappa$-μ		κ-$\mu\times\eta$-μ		η-$\mu\times\eta$-μ	
	$\mathscr{A}_1\mathscr{K}_2$	Terms	$\frac{\mathscr{A}_1\mathscr{E}_2}{h_2}$	Terms	$\mathscr{K}_1\mathscr{K}_2$	Terms	$\frac{\mathscr{K}_1\mathscr{E}_2}{h_2}$	Terms	$\frac{\mathscr{E}_1\mathscr{E}_2}{h_1h_2}$	Terms
PDF	1.1340	14	1.2320	15	0.9349	16	0.7203	19	1.3149	13
	0.2566	42	0.3953	32	0.2548	38	0.1879	57	0.2425	49
	0.0697	118	0.0757	126	0.1128	76	0.0565	170	0.0729	147
Outage Capacity	1.4062	11	1.0018	15	1.3333	12	0.6479	21	0.6130	23
	0.5315	21	0.4363	27	0.3698	27	0.3240	31	0.2890	40
	0.2526	34	0.2009	50	0.1867	46	0.1572	63	0.1159	90

4.3 Integral Involving the Product of a PDF and a CDF

Another interesting result concerns the integral of the the PDF and CDF of random envelopes, given as

$$P(\gamma_1,\gamma_2)=\int_0^{\gamma_1} f_{R_1}(r)F_{R_2}\left(\frac{\gamma_2}{r}\right)dr, \tag{4.5}$$

in which R_1 and R_2 are independent random envelopes following the α-μ, κ-μ or η-μ distribution, and $f_{R_1}(r)$ and $F_{R_2}(r)$ are, respectively, their PDF and CDF. An expression can be obtained for integral (4.5) in terms of the multivariable Fox H-function with the gen-

Table 4.6 – Parameters of the Fox H-Function for the Integral Involving the Product of PDF and CDF

Product	C	$\mathbf{x}$	β	$\mathbf{B}$	δ	$\mathbf{D}$
α-$\mu\times\alpha$-μ	$\frac{\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}, \frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right]$	$[0,\mu_1,\mu_2,0]$	$\begin{pmatrix} -\alpha_1 & \alpha_2 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$	$[1,1]$	$\begin{pmatrix} -\alpha_1 & \alpha_2 \\ 0 & -1 \end{pmatrix}$
α-$\mu\times\kappa$-μ	$\frac{\alpha_1}{\Gamma(\mu_1)}$	$\left[\kappa_2\mu_2, \frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}, \frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right]$	$[\mathbf{0}_3,\mu_2,\mu_1]$	$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 2 & -\alpha_1 \\ & \mathbf{I}_3 & \end{pmatrix}$	$[1,\mu_2,1]$	$\begin{pmatrix} 0 & 2 & -\alpha_1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
κ-$\mu\times\alpha$-μ	$\frac{2}{\Gamma(\mu_2)}$	$\left[\kappa_1\mu_1, \frac{\gamma_1^2}{\mathscr{K}_1^2}, \frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right]$	$[\mathbf{0}_3,\mu_1,\mu_2,0]$	$\begin{pmatrix} 0 & -2 & \alpha_2 \\ -1 & -1 & 0 \\ & \mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$	$[1,\mu_1,0,1]$	$\begin{pmatrix} 0 & -2 & \alpha_2 \\ & -\mathbf{I}_3 & \end{pmatrix}$
α-$\mu\times\eta$-μ	$\frac{2^{1-2\mu_2}\alpha_1\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}, -\frac{H_2^2}{4h_2^2}, \frac{\gamma_2^2}{\gamma_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_2,\mu_1,0,2\mu_2]$	$\begin{pmatrix} -\alpha_1 & 0 & 2 \\ 0 & -2 & -1 \\ & \mathbf{I}_3 & \end{pmatrix}$	$\left[1,\mu_2+\frac{1}{2},1\right]$	$\begin{pmatrix} -\alpha_1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
η-$\mu\times\alpha$-μ	$\frac{4^{1-\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[\frac{-H_1^2}{4h_1^2}, \frac{\gamma_1^2}{\mathscr{E}_1^2}, \frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right]$	$[\mathbf{0}_3,2\mu_1,\mu_2,0]$	$\begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & \alpha_2 \\ & \mathbf{I}_3 & \\ 0 & 0 & -1 \end{pmatrix}$	$\left[1,\mu_1+\frac{1}{2},0,1\right]$	$\begin{pmatrix} 0 & -2 & \alpha_2 \\ & -\mathbf{I}_3 & \end{pmatrix}$
κ-$\mu\times\kappa$-μ	1	$\left[\kappa_1\mu_1, \frac{\gamma_1^2}{\mathscr{K}_1^1}, \kappa_2\mu_2, \frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,0,\mu_2]$	$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ & \mathbf{I}_4 & & \end{pmatrix}$	$[1,\mu_1,0,\mu_2,1]$	$\begin{pmatrix} 0 & -1 & 0 & 1 \\ & -\mathbf{I}_4 & & \end{pmatrix}$
κ-$\mu\times\eta$-μ	$\frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}$	$\left[\kappa_1\mu_1, \frac{\gamma_1^2}{\mathscr{K}_1^2}, -\frac{H_2^2}{4h_2^2}, \frac{\gamma_2^2}{\gamma_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,\mu_1,0,2\mu_2]$	$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 \\ & \mathbf{I}_4 & & \end{pmatrix}$	$\left[1,\mu_1,0,\mu_2+\frac{1}{2},1\right]$	$\begin{pmatrix} 0 & -1 & 0 & 1 \\ & -\mathbf{I}_4 & & \end{pmatrix}$
η-$\mu\times\kappa$-μ	$\frac{2^{1-2\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)}$	$\left[-\frac{H_1^2}{4h_1^2}, \frac{\gamma_1^2}{\mathscr{E}_1^2}, \kappa_2\mu_2, \frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}\right]$	$[\mathbf{0}_4,2\mu_1,0,\mu_2]$	$\begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ & \mathbf{I}_4 & & \end{pmatrix}$	$\left[1,\mu_1+\frac{1}{2},0,\mu_2,1\right]$	$\begin{pmatrix} 0 & -1 & 0 & 1 \\ & -\mathbf{I}_4 & & \end{pmatrix}$
η-$\mu\times\eta$-μ	$\frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}$	$\left[-\frac{H_1^2}{4h_1^2}, \frac{\gamma_1^2}{\mathscr{E}_1^2}, -\frac{H_2^2}{4h_2^2}, \frac{\gamma_2^2}{\gamma_1^2\mathscr{E}_2^2}\right]$	$[\mathbf{0}_4,2\mu_1,0,2\mu_2]$	$\begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 \\ & \mathbf{I}_4 & & \end{pmatrix}$	$\left[1,\mu_1+\frac{1}{2},0,\mu_2+\frac{1}{2},1\right]$	$\begin{pmatrix} 0 & -1 & 0 & 1 \\ & -\mathbf{I}_4 & & \end{pmatrix}$

eral structure given in (3.3). Tables 4.6 and 4.7 provide, respectively, the parameters for the Fox H-function representation and series expansion for the integral (4.5). In Table 4.7, ${}_1\tilde{F}_1(a,b,x) = {}_1F_1(a,b,x)/\Gamma(b)$, $\gamma(a,x)$ is the lower incomplete gamma function [36, Eq. (6.5.2)], and $E_\nu(x)$ is the exponential integral [36, Equation (5.1.4)]. The corresponding mathematical derivation is provided in Appendix D. It is noteworthy that when γ_1 tends to infinity the above integral has the exact same format as the integral to obtain the CDF for the product of the random variables R_1 and R_2.

4.4 Application Examples

4.4.1 Performance Metrics for the Cascaded Channel

Consider a two-tap cascaded channel described in [9]. The instantaneous SNR is given as

$$\gamma = \frac{E_s}{N_T}(R_1R_2)^2 = \frac{E_s}{N_T}Z^2, \tag{4.6}$$

in which R_1 and R_2 are the wireless channel gain, E_s is the average energy of the transmitted symbol and N_T is the noise power spectral density. Therefore, the average SNR, defined as $\overline{\gamma} \triangleq \mathbb{E}[\gamma]$, can be computed by

$$\overline{\gamma} = \frac{Es}{N_T}\mathbb{E}[Z^2]. \tag{4.7}$$

Applying a conventional variable transformation, the PDF and CDF for the SNR can be obtained, respectively, as

$$f_\Gamma(\gamma) = \frac{1}{2}\sqrt{\frac{N_T}{\gamma E_s}} f_R\left(\sqrt{\frac{\gamma N_T}{E_s}}\right) \tag{4.8}$$

Table 4.7 – Series Expansion for the Integral Involving the Product of a PDF and a CDF in (4.5)

Product	Series Expansion
α-$\mu\times\alpha$-μ	$\frac{\gamma_1^{\alpha_1\mu_1}}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!(i+\mu_1)}\frac{\gamma_1^{i\alpha_1}}{\mathscr{A}_1^{\alpha_1(i+\mu_1)}}\left\{\gamma\left(\mu_2,\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)+\frac{\gamma_2^{\alpha_2\mu_2}}{\gamma_1^{\alpha_2\mu_2}\mathscr{A}_2^{\alpha_2\mu_2}}E_{\frac{\alpha_1}{\alpha_2}(i+\mu_1)-\mu_2+1}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)\right\}$
α-$\mu\times\kappa$-μ	$\frac{1}{\Gamma(\mu_1)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{K}_2}\right)^{2(i+\mu_2)}\frac{{}_1\tilde{F}_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)}{i+\mu_2}\gamma\left(\mu_1-\tfrac{2}{\alpha_1}(i+\mu_2),\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)\right.$ $\left.+\frac{\Gamma(\mu_2-\frac{\alpha_1}{2}(i+\mu_1))}{i+\mu_1}\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{K}_2}\right)^{\alpha_1(i+\mu_1)}{}_1\tilde{F}_1\left(\frac{\alpha_1}{2}(i+\mu_1);\mu_2;-\kappa_2\mu_2\right)\right\}$
κ-$\mu\times\alpha$-μ	$\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!(i+\mu_1)}\left(\frac{\gamma_1}{\mathscr{K}_1}\right)^{2(i+\mu_1)}{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)$ $\times\left\{\gamma\left(\mu_2,\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)+\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)^{\mu_2}E_{1-\mu_2+\frac{2}{\alpha_1}(i+\mu_1)}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)\right\}$
α-$\mu\times\eta$-μ	$\frac{1}{\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\frac{\Gamma\left(2\mu_2-\frac{\alpha_1}{2}(i+\mu_1)\right)}{i+\mu_1}\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{E}_2}\right)^{\alpha_1(i+\mu_1)}{}_2F_1\left(\tfrac{\alpha_1(i+\mu_1)}{4},\tfrac{\alpha_1(i+\mu_1)+2}{4};\mu_2+\tfrac{1}{2};\tfrac{H_2^2}{h_2^2}\right)\right.$ $\left.+\frac{\gamma\left(\mu_1-\frac{2}{\alpha_1}(i+2\mu_2),\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)}{i+2\mu_2}\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{E}_2}\right)^{2(i+2\mu_2)}{}_2F_1\left(\tfrac{i+2\mu_2}{2},\tfrac{i+1+2\mu_2}{2};\mu_2+\tfrac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$
η-$\mu\times\alpha$-μ	$\frac{h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i h_1^i}{i!(i+2\mu_1)}\left(\frac{\gamma_1}{\mathscr{E}_1}\right)^{2(i+2\mu_1)}{}_2F_1\left(\frac{1-i}{2},-\frac{i}{2};\mu_1+\frac{1}{2};\frac{H_1^2}{h_1^2}\right)$ $\times\left\{\gamma\left(\mu_2,\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)+\left(\frac{\gamma_2}{\gamma_1\mathscr{A}_2}\right)^{\alpha_2\mu_2}E_{1-\mu_2+\frac{2}{\alpha_2}(i+2\mu_1)}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)\right\}$
κ-$\mu\times\kappa$-μ	$\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\left\{\frac{\Gamma(-i-\mu_1+\mu_2)}{i+\mu_1}\left(\frac{\gamma_2}{\mathscr{K}_1\mathscr{K}_2}\right)^{2(i+\mu_1)}{}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right.$ $\left.+\left(\frac{\gamma_1}{\mathscr{K}_1}\right)^{2(i+\mu_1)}\sum_{l=0}^{\infty}\frac{(-1)^l{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(i-l+\mu_1-\mu_2)(l+\mu_2)}\left(\frac{\gamma_2}{\gamma_1\mathscr{K}_2}\right)^{2(l+\mu_2)}\right\}$
κ-$\mu\times\eta$-μ	$\frac{1}{\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\gamma_1}{\mathscr{K}_1}\right)^{2(i+\mu_1)}{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)$ $\times\left\{\frac{\Gamma(-i-\mu_1+2\mu_2)}{i+\mu_1}\left(\frac{\gamma_2}{\gamma_1\mathscr{E}_2}\right)^{2(i+\mu_1)}{}_2F_1\left(\tfrac{i+\mu_1}{2},\tfrac{1+i+\mu_1}{2};\mu_2+\tfrac{1}{2};\tfrac{H_2^2}{h_2^2}\right)\right.$ $\left.+\sum_{l=0}^{\infty}\frac{(-1)^l h_2^{l+\mu_2}{}_2F_1\left(-\frac{l}{2},\frac{1-l}{2};\mu_2+\frac{1}{2};\frac{H_2^2}{h_2^2}\right)}{l!(l+2\mu_2)(i-l+\mu_1-2\mu_2)}\left(\frac{\gamma_2}{\gamma_1\mathscr{E}_2}\right)^{2(l+2\mu_2)}\right\}$
η-$\mu\times\kappa$-μ	$\frac{h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i h_1^i}{i!}\left(\frac{\gamma_1}{\mathscr{E}_1}\right)^{2(i+2\mu_1)}{}_2F_1\left(-\tfrac{i}{2},\tfrac{1-i}{2};\tfrac{1+2\mu_1}{2};\frac{H_1^2}{h_1^2}\right)$ $\times\left\{\frac{\Gamma(-i-2\mu_1+\mu_2)}{i+2\mu_1}\left(\frac{\gamma_2}{\gamma_1\mathscr{K}_2}\right)^{2(i+2\mu_1)}{}_1F_1\left(i+2\mu_1;\mu_2-\kappa_2\mu_2\right)\right.$ $\left.+\sum_{l=0}^{\infty}\frac{(-1)^l{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(l+\mu_2)(i-l+2\mu_1-\mu_2)}\left(\frac{\gamma_2}{\gamma_1\mathscr{K}_2}\right)^{2(l+\mu_2)}\right\}$
η-$\mu\times\eta$-μ	$\frac{1}{\Gamma(2\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i h_1^{i+\mu_1}}{i!}\left(\frac{\gamma_1}{\mathscr{E}_1}\right)^{2(i+2\mu_1)}{}_2F_1\left(-\tfrac{i}{2},-\tfrac{i-1}{2};\mu_1+\tfrac{1}{2};\frac{H_1^2}{h_1^2}\right)$ $\times\left\{\frac{\Gamma(-i-2(\mu_1-\mu_2))}{i+2\mu_1}\left(\frac{\gamma_2}{\gamma_1\mathscr{E}_2}\right)^{2(i+2\mu_1)}{}_2F_1\left(\mu_1+\tfrac{i}{2},\mu_1+\tfrac{i+1}{2};\mu_2+\tfrac{1}{2};\frac{H_2^2}{h_2^2}\right)\right.$ $\left.+\sum_{l=0}^{\infty}\frac{(-1)^l h_2^{l+\mu_2}}{l!(i-l+2\mu_1-2\mu_2)(l+2\mu_2)}\left(\frac{\gamma_2}{\gamma_1\mathscr{E}_2}\right)^{2(l+2\mu_2)}{}_2F_1\left(-\tfrac{l}{2},-\tfrac{l-1}{2};\mu_2+\tfrac{1}{2};\frac{H_2^2}{h_2^2}\right)\right\}$

and

$$F_\Gamma(\gamma) = F_R\left(\sqrt{\frac{\gamma N_T}{E_s}}\right), \tag{4.9}$$

in which $f_R(r)$ may be replaced by (4.4) with parameters provided in Table 4.1 and $F_R(r)$ may be replaced by (4.4) with parameters provided in Table 4.2 according to the physical model considered, or by their respective series representation given in Tables 4.3 and 4.4.

4.4.1.1 Amount of Fading

The amount of fading (AF), which is a measure of the fading severity, is defined in [45, Eq. (1.27)] as the ratio between the variance and the square of the average of the instantaneous SNR, i.e., $AF = \mathbb{V}[\gamma]/\mathbb{E}[\gamma]^2$. Expressing it in terms of the moments of Z leads to

$$AF = \frac{\mathbb{E}[Z^4]}{\mathbb{E}[Z^2]^2} - 1, \tag{4.10}$$

in which $\mathbb{E}[Z^2]$ and $\mathbb{E}[Z^4]$ are derived from (4.3).

4.4.1.2 Outage Probability

The outage probability is the probability that the instantaneous SNR falls below a certain threshold γ_{th}, i.e.,

$$P_{\text{out}} = \Pr(0 \le \gamma \le \gamma_{\text{th}}) = \int_0^{\gamma_{\text{th}}} f_\Gamma(\gamma) d\gamma, \tag{4.11}$$

in which $f_\Gamma(\gamma)$ is given by (4.8). Therefore the outage probability is the CDF of the instantaneous SNR for γ_{th} and is given by

$$P_{\text{out}} = F_\Gamma(\gamma_{\text{th}}), \tag{4.12}$$

in which $F_\Gamma(\gamma)$ given by (4.9).

4.4.1.3 Outage Capacity

The Shannon capacity for a signal transmission over AWGN channel is defined as $C(\gamma) = W\log_2(1+\gamma)$, in which W is the signal's bandwidth and γ is the instantaneous SNR. The outage capacity is the probability that the capacity will fall below a certain threshold and may be expressed as [49]

$$C_{\text{out}} = \text{Prob}\left[C(\gamma) < \lambda\right]. \tag{4.13}$$

It can be easily shown that the outage capacity is given in terms of SNR's CDF as

$$C_{\text{out}} = F_\Gamma\left(2^{\lambda/W} - 1\right), \tag{4.14}$$

in which $F_\Gamma(\gamma)$ defined by (4.9).

4.4.2 Detection Probability in a UHF RFID System

The UHF RFID systems have been consistently applied in wireless technology to object identification. For more information on RFID system please check [50, 51] and the references therein. Figure 4.1 shows a basic configuration of an UHF-RFID system. To

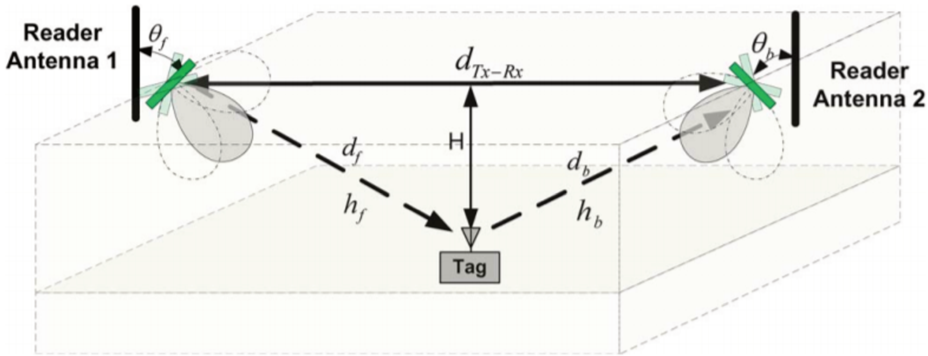

Figure 4.1 – Basic configuration of passive UHF RFID system [50].

correctly detect the RFID tag, the power received by the tag must be higher than the tag sensitivity. Upon receiving, a backscatter signal is send by the tag which must be higher than the reader sensitivity. Following the same lines as in [51, Equations (1) to (5)], the probability of detection can be written as

$$P_D = \Pr\{|h_1| > \gamma_1, |h_1||h_2| > \gamma_2\} = \int_{\gamma_1}^{\infty}\int_{\gamma_2/x}^{\infty} f_{|h_1||h_2|}(x,y)dydx, \tag{4.15}$$

in which γ_1 and γ_2 are defined in [51]; h_1 and h_2 are respectively the fading coefficient in the forward link and in the reverse link; and $f_{|h_1||h_2|}(x,y)$ is the joint PDF of h_1 and h_2. Now, let $|h_1|$ and $|h_2|$ be independent random envelopes. Hence, manipulating (4.15) for this case, the probability of detection can be rewritten as

$$P_D = 1 - F_{h_1}(\gamma_1) - F_{h_1h_2}(\gamma_2) + \int_0^{\gamma_1} f_{h_1}(x)F_{h_2}\left(\frac{\gamma_2}{x}\right)dx, \tag{4.16}$$

in which $F_{h_1}(x)$ and $F_{h_2}(x)$ are the CDF of the fading coefficients in the forward and reverse links, respectively; $f_{h_1}(x)$ is the PDF of the fading coefficient of the forward link; and $F_{h_1h_2}(x)$ is the CDF of the product of random envelopes h_1 and h_2. $F_{h_1h_2}(x)$ can be replaced by (4.4) with the adequate parameters from Table 4.2 according to the desired fading model. The remaining integral in (4.16) is the same integral solved in Section 4.3 and can be replaced by (4.4) with parameters given in Table 4.6.

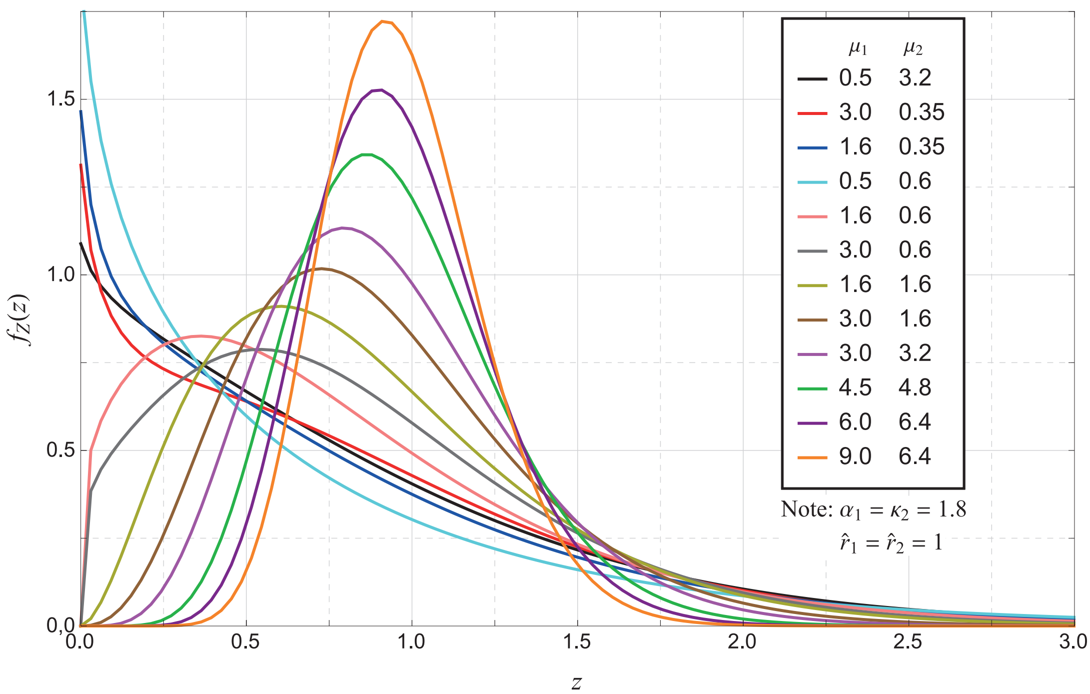

Figure 4.2 – PDF for the product of one α-μ and a κ-μ variates with $\alpha_1 = \kappa_2 = 1.8$, $\hat{r}_1 = \hat{r}_2 = 1$ and several values for μ_1 and μ_2.

4.5 Some Plots

Several plots for the product PDF with different values for the parameters α, κ, η, and μ are shown in Figs. 4.2-4.6. Without loss of generality, it is assumed that $\hat{r}_1 = \hat{r}_2 = 1$ in all plots. The parameters have been taken to show the broad range of shapes that the product PDF can exhibit. As expected, higher values of the parameters α, κ and μ tend to concentrate the curves around $\hat{r}_1\hat{r}_2$. The PDF of an α-μ random variable is finite and non-zero at $z = 0$ when $\alpha\mu = 1$. The same effect is observed when $\mu = 0.5$ or $\mu = 0.25$, respectively, for κ-μ and η-μ distributions. Interestingly, the product distribution shows the same effect if any of the random variables satisfies the conditions for non-zero PDF at $z = 0$ ($\alpha\mu = 1$, $\mu = 0.5$ and $\mu = 0.25$ for α-μ, κ-μ and η-μ, respectively).

As an application example, the outage capacity is depicted in Figs. 4.7-4.11 for the cascaded channel formed by the α-μ, κ-μ and η-μ distributions, in which the dots are Monte Carlo simulation points and the lines have been obtained from the formulations presented here. As can be seen, simulation and the exact expressions coincide with each other, showing the correctness of the formulations. These figures also show that the product distribution tends to a single distribution if one of the variables involved in the product approaches the impulse function ($\alpha \to \infty$, $\kappa \to \infty$ or $\mu \to \infty$), which gives the product distribution much more flexibility than the single distribution.

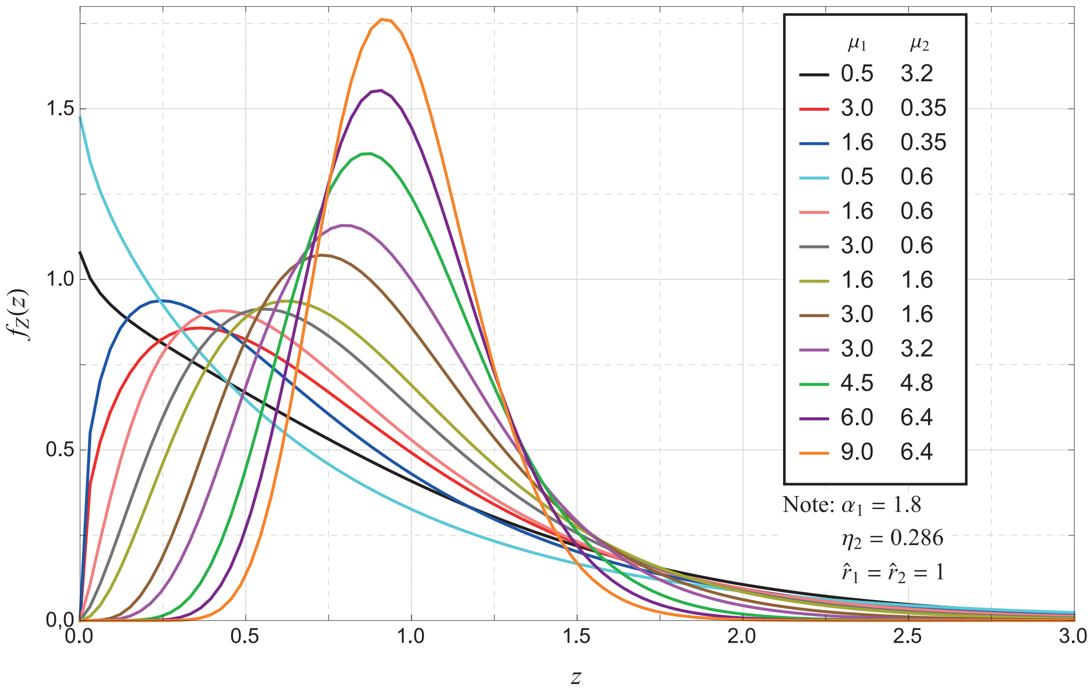

Figure 4.3 – PDF for the product of one α-μ and an η-μ variates with $\alpha_1 = 1.8$, format 2 $\eta = 0.286$, $\hat{r}_1 = \hat{r}_2 = 1$ and several values for μ_1 and μ_2,

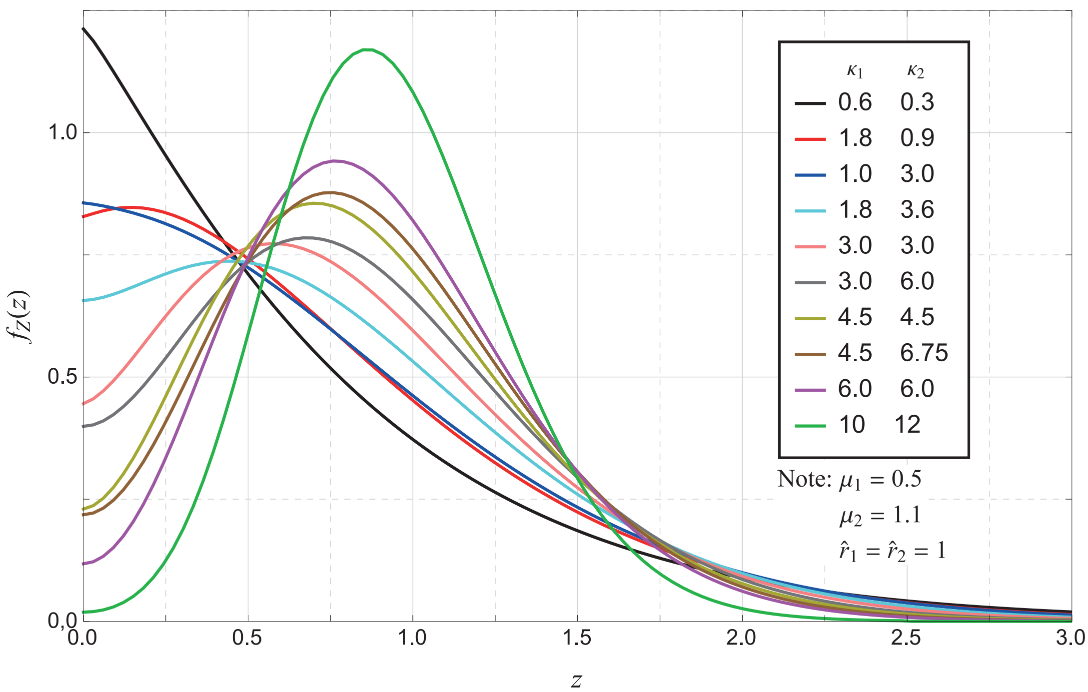

Figure 4.4 – PDF for the product of two κ-μ variates with $\mu_1 = 0.5$, $\mu_2 = 1.1$, $\hat{r}_1 = \hat{r}_2 = 1$ and several values for κ_1 and κ_2.

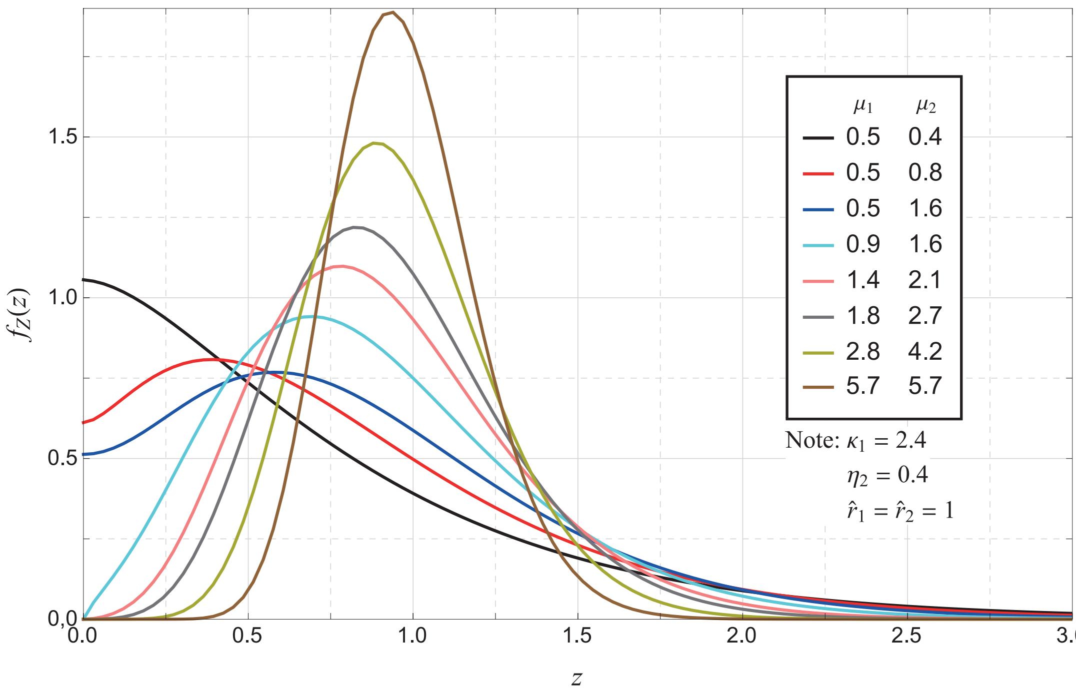

Figure 4.5 – PDF for the product of one κ-μ and one η-μ variates with $\kappa_1 = 2.4$, format 2 $\eta_2 = 0.4$, $\hat{r}_1 = \hat{r}_2 = 1$ and several values for μ_1 and μ_2.

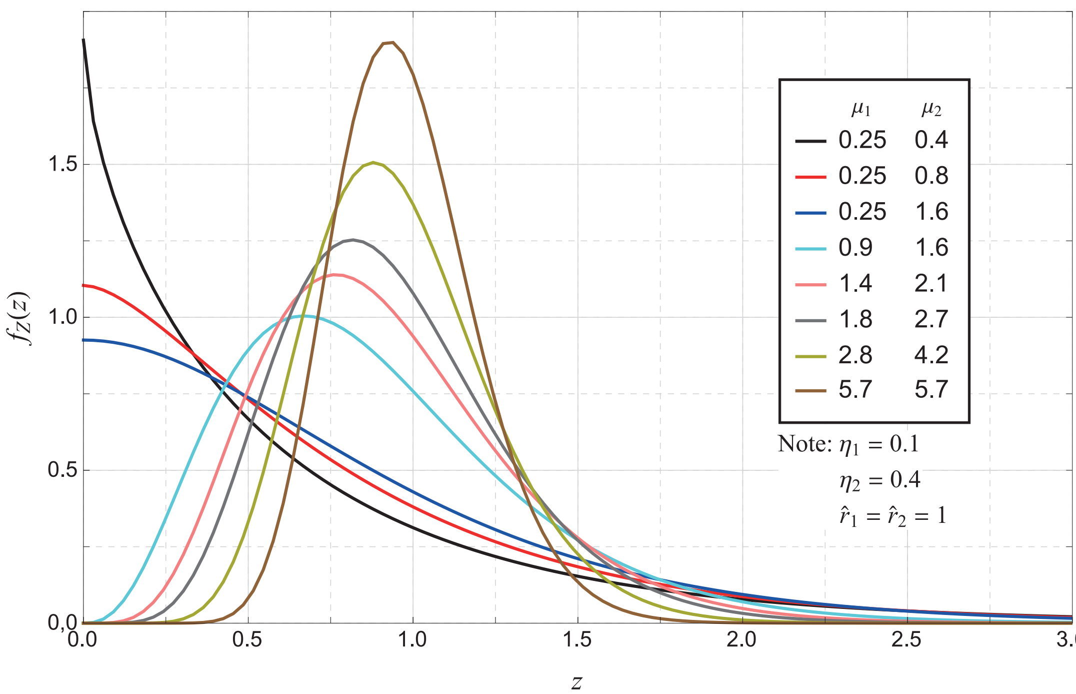

Figure 4.6 – PDF for the product of two η-μ variates with $\eta_1 = 0.1$, $\eta_2 = 0.4$, $\hat{r}_1 = \hat{r}_2 = 1$ and several values for μ_1 and μ_2.

4.6 Conclusion

This Chapter offers novel exact expressions for the probability density function and cumulative distribution function of the product of two fading envelopes following an

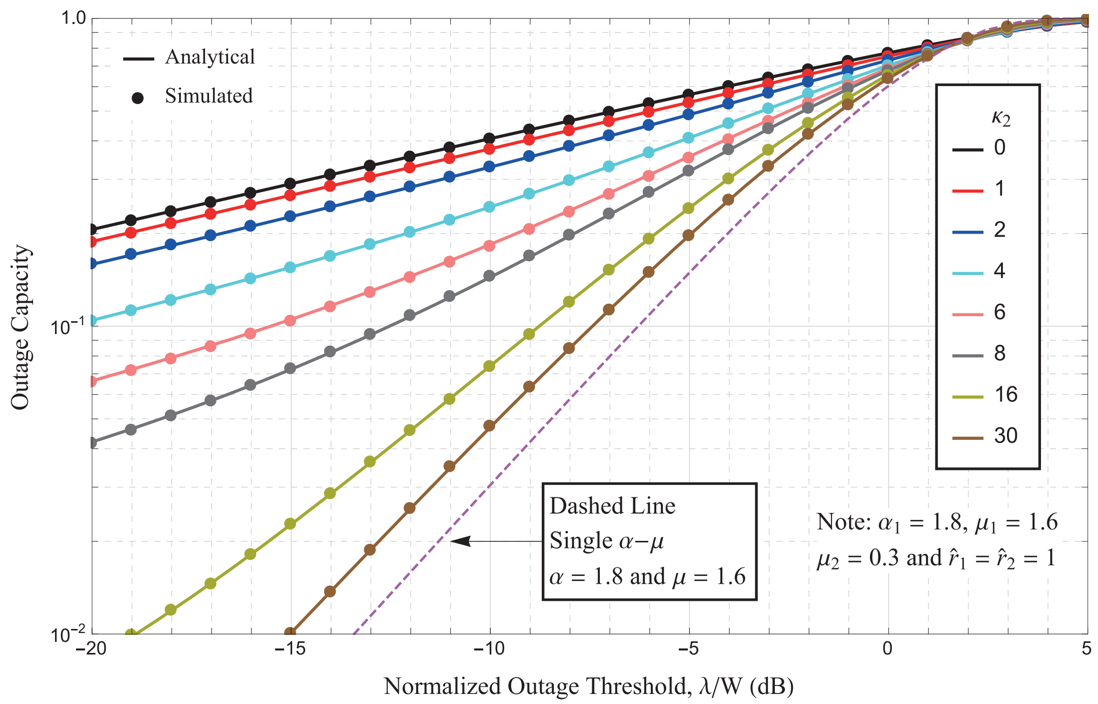

Figure 4.7 – Outage capacity for α-μ × κ-μ channel with $\alpha_1 = 1.8$, $\mu_1 = 1.6$, $\hat{r}_1 = \hat{r}_2 = 1$ and various values for κ_2.

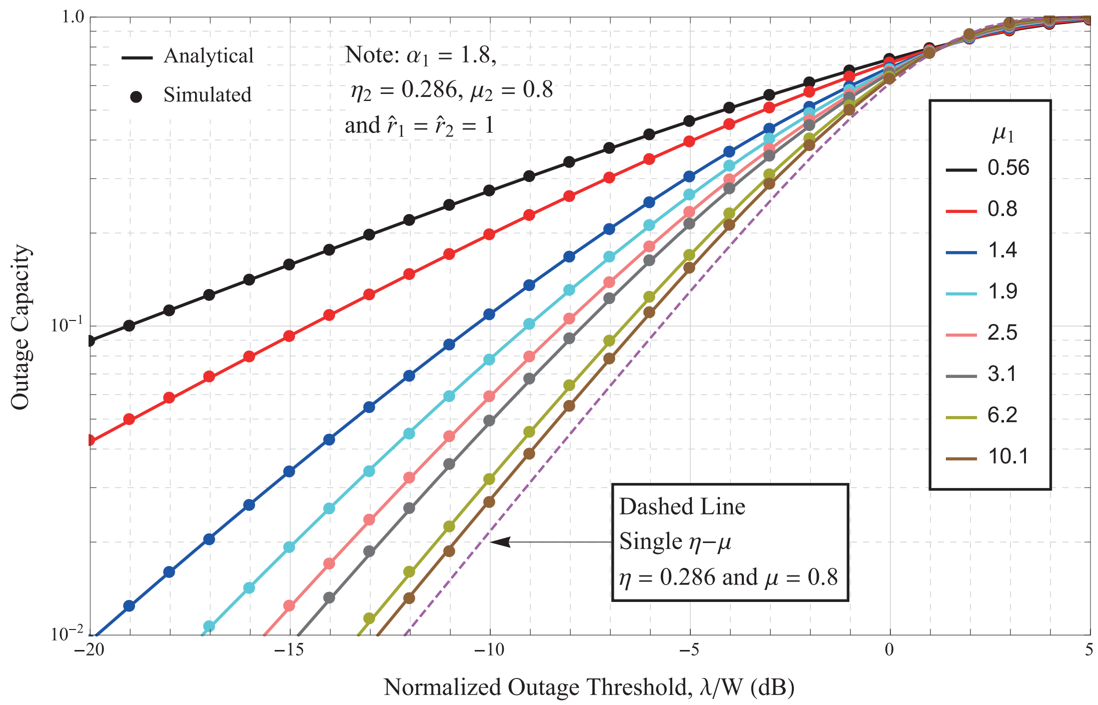

Figure 4.8 – Outage capacity for α-μ × η-μ channel with $\alpha_1 = 1.8$, $\eta_2 = 0.286$, $\mu_2 = 0.8$, $\hat{r}_1 = \hat{r}_2 = 1$ and various values for μ_1.

α-μ, κ-μ or η-μ distribution in terms of the Fox H-function. Series representations for the results are also provided. It is important to emphasize that the series in Tables 4.3 and 4.4 are not unique and other series expansions are possible. Those provided here arises naturally

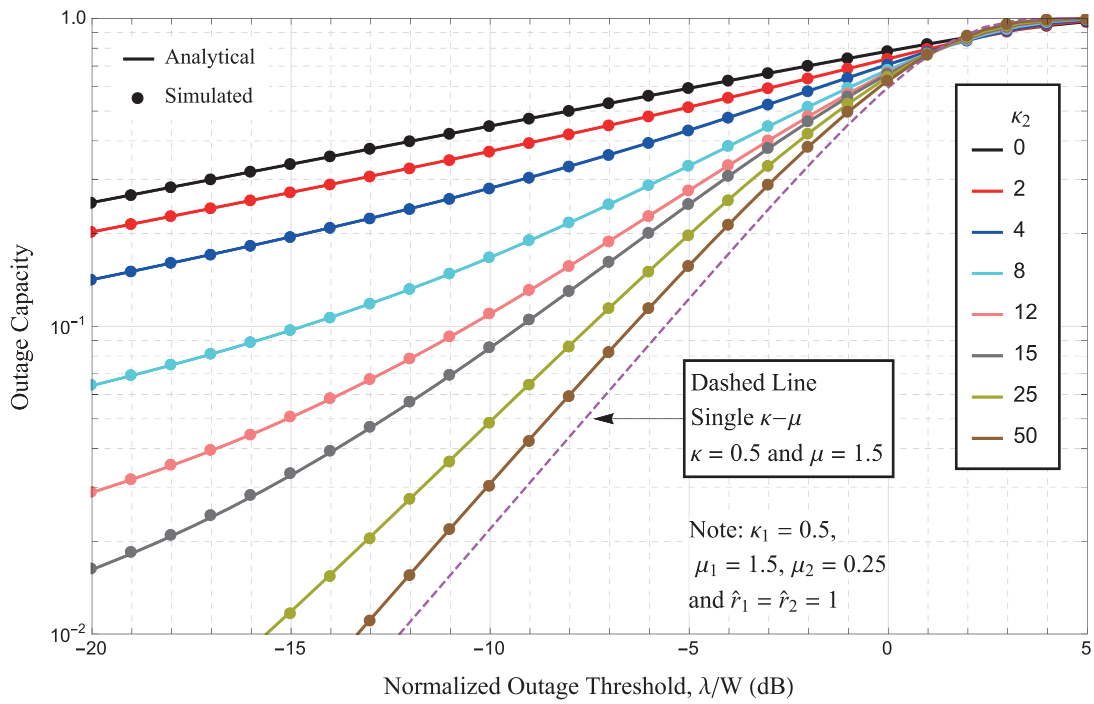

Figure 4.9 – Outage capacity for κ-μ $\times$ κ-μ channel with $\kappa_1 = 0.5$, $\mu_1 = 1.5$, $\mu_2 = 0.25$, $\hat{r}_1 = \hat{r}_2 = 1$ and various values for μ_2.

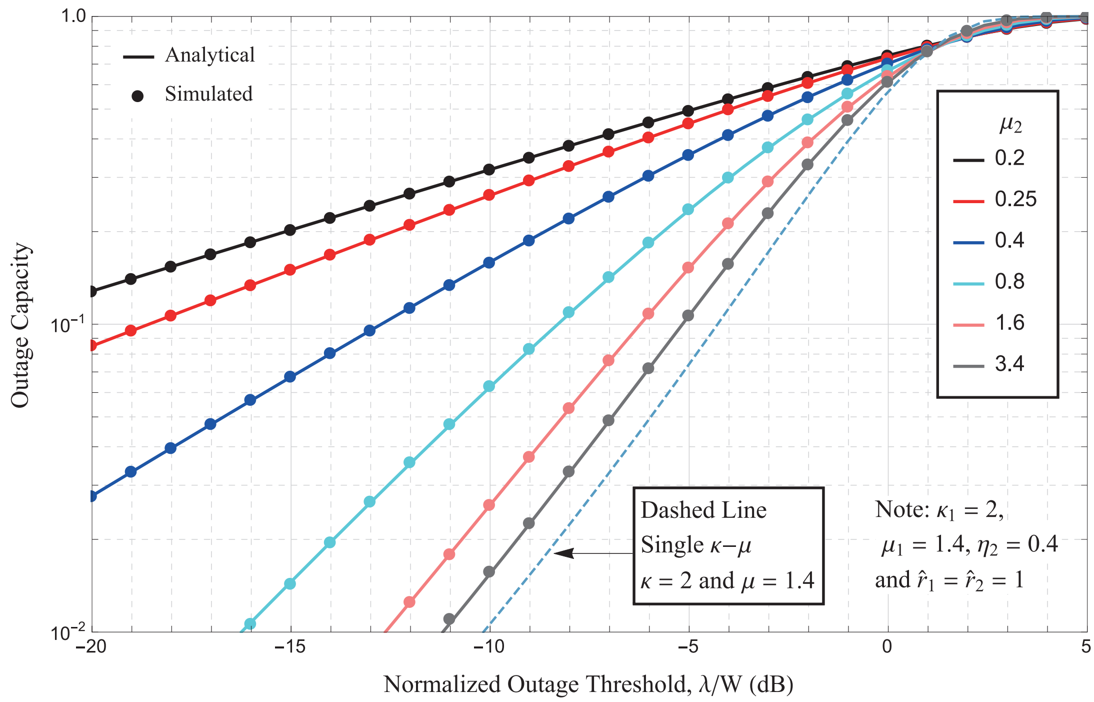

Figure 4.10 – Outage capacity for κ-μ $\times$ η-μ channel with $\kappa_1 = 2$, $\mu_1 = 1.4$, $\eta_2 = 0.4$, $\hat{r}_1 = \hat{r}_2 = 1$ and various values for μ_2

from the sum of residues without much algebraic manipulations and compute efficiently. In addition, an interesting integral involving the product of a PDF and a CDF which is related to CDF of the product of two random envelopes were derived. Said integral find application,

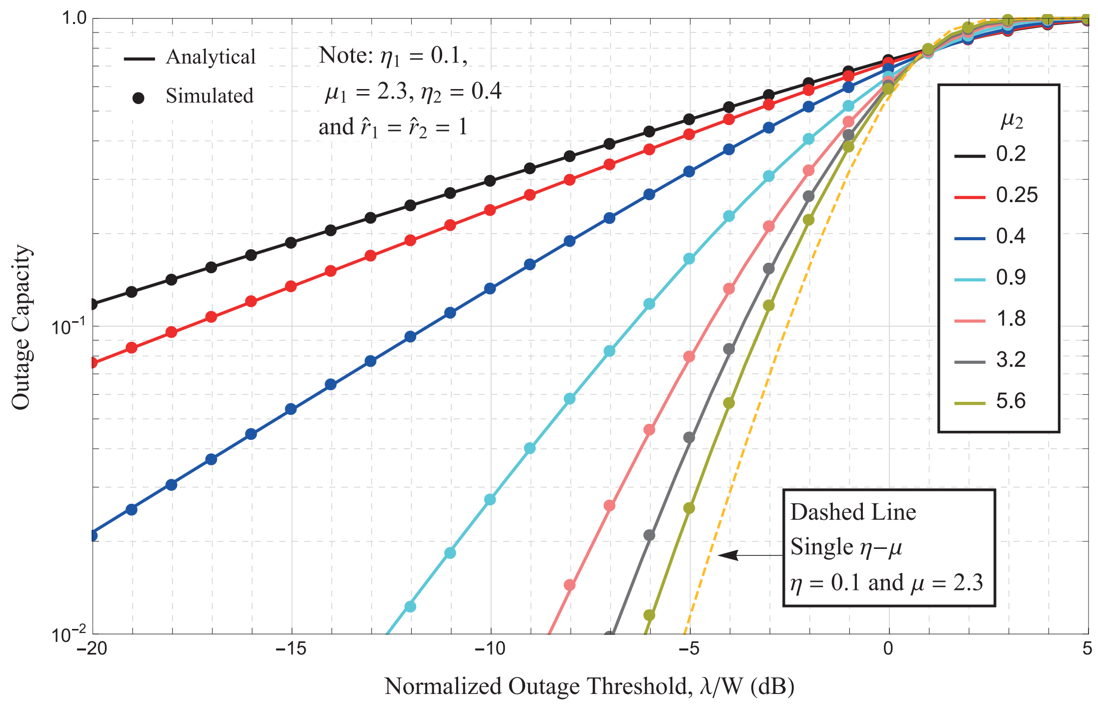

Figure 4.11 – Outage capacity for η-μ × η-μ channel with $\eta_1 = 0.1$, $\mu_1 = 2.3$, $\eta_2 = 0.4$, $\hat{r}_1 = \hat{r}_2 = 1$ and various values for μ_2

for instance in UHF-RFID systems. These results are applicable in several areas of wireless communications such as high resolution synthetic aperture radar clutter, multihop systems. It can also be used to model keyhole channels in MIMO system. Some metrics for the cascaded fading channel are provided. The various combinations of the products given here can be used as an immense multipath-shadowing class of composite fading environment.

Chapter 5

Miscellaneous Results on the α-η-κ-μ Fading Model

The efficiency of the series presented in 2.6 for the PDF and CDF of the α-η-κ-μ can be greatly compromised due to the recursivity present in the series. In this chapter, a number of new results aiming at facilitating the use of the α-η-κ-μ fading model are presented. These results include: (i) fast convergent, with no recursions, series representations for the envelope PDF and CDF; (ii) higher order moments; (iii) moment generating function; (iv) asymptotic behaviour of the PDF and also of the CDF; (v) a procedure for parameter estimation; (vi) new closed-form expressions for particular cases. As application examples, the following are shown: (i) outage probability; (ii) outage capacity; (iii) amount of fading; (iv) bit error rate, for which the asymptotic behaviour is found.

5.1 The Envelope PDF[1]

As was mentioned in Section 2.6, the α-η-κ-μ model, the relation between its envelope R and in-phase and quadrature components X and Y is given as $R^\alpha = X^2 + Y^2$, in which X^2 and Y^2 are the powers of two independent of the κ-μ variates. The respective PDFs for the modulus of X and Y follow that of (2.13) with respective parameters κ_x, $\mu_x/2$, $\hat{r}_x$ and κ_y, $\mu_y/2$ and $\hat{r}_y$. Their corresponding parameters are given in terms of those of Parametrization-2, i.e. α, κ, η, μ, p, q, and $\hat{r}$, as

$$\begin{aligned}\mu_x &= \frac{2p\mu}{1+p}, \quad \mu_y = \frac{2\mu}{1+p}, \quad \kappa_x = \frac{(1+\eta)q\kappa}{1+q\eta}, \quad \kappa_y = \frac{(1+\eta)\kappa}{1+q\eta}\\ \hat{r}_x^2 &= \frac{\eta(1+q(\eta+\kappa+\eta\kappa))\hat{r}^\alpha}{(1+\eta)(1+\kappa)(1+q\eta)}, \quad \hat{r}_y^2 = \frac{(1+\kappa+\eta(q+\kappa))\hat{r}^\alpha}{(1+\eta)(1+\kappa)(1+q\eta)}.\end{aligned} \tag{5.1}$$

[1] This has also been obtained independently in [52]

As defined in [5], $U = X^2$ and $V = Y^2$, and the PDF of the α-η-κ-μ envelope can be evaluated as

$$f_R(r) = \alpha r^{\alpha-1} \int_0^{r^\alpha} f_U(r^\alpha - v) f_V(v)\, \mathrm{d}v, \tag{5.2}$$

in which $f_U(u)$ and $f_V(v)$ are easily obtained from standard statistical procedures. Of course

$$f_R(r) = \alpha r^{\alpha-1} \int_0^{r^\alpha} f_U(u) f_V(r^\alpha - u)\, \mathrm{d}u. \tag{5.3}$$

The CDF has the integral form given as

$$F_R(r) = \int_0^{r^\alpha} F_U(r^\alpha - v) f_V(v)\, \mathrm{d}v \tag{5.4}$$

or, equivalently

$$F_R(r) = \int_0^{r^\alpha} f_U(u) F_V(r^\alpha - u)\, \mathrm{d}u. \tag{5.5}$$

In [5], these integrals were solved using series expansion formulations found in recursive forms. These recursive forms are not handy, and their convergence, although achievable, is highly dependent on the parameters. It is then convenient to find new formulations that render the use of such a flexible α-η-κ-μ model simpler.

Replacing $f_U(u)$ and $f_V(v)$ in (5.2) and with some algebraic manipulation yields

$$\begin{aligned} f_R(r) = {} & \frac{\alpha(\mu\xi)^\mu}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \frac{r^{\alpha-1}}{\hat{r}^{\alpha\mu}} \exp\left(-\frac{p\xi\mu r^\alpha}{\eta \hat{r}^\alpha}\right) \int_0^{r^\alpha} (r^\alpha - v)^{\frac{p\mu}{1+p}-1} v^{\frac{\mu}{1+p}-1} \\ & \times \exp\left(-\frac{(\eta-p)\xi\mu v}{\eta \hat{r}^\alpha}\right) {}_0\tilde{F}_1\left(;\frac{p\mu}{1+p};\frac{p^2 q\xi\kappa\mu^2(r^\alpha - v)}{\delta\eta\hat{r}^\alpha}\right) {}_0\tilde{F}_1\left(;\frac{\mu}{1+p};\frac{\kappa\xi\mu^2 v}{\delta\hat{r}^\alpha}\right) dv. \end{aligned} \tag{5.6}$$

The reader is referred to Section 2.6 for the respective definition of the parameters ξ and δ. Using the series representation for the hypergeometric function in (5.6) as given in [41, Equation (7.2.3.1)] and then, by changing the order of integration and summation, the PDF is obtained, after some algebraic manipulations, as

$$\begin{aligned} f_R(r) = {} & \frac{\alpha(\mu\xi)^\mu}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \frac{r^{\alpha-1}}{\hat{r}^{\alpha\mu}} e^{-\frac{p\xi\mu r^\alpha}{\eta\hat{r}^\alpha}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!\Gamma\left(j+\frac{\mu}{1+p}\right)\Gamma\left(i+\frac{p\mu}{1+p}\right)} \\ & \times \left(\frac{p^2 q\xi\kappa\mu^2}{\delta\eta\hat{r}^\alpha}\right)^i \left(\frac{\kappa\xi\mu^2}{\delta\hat{r}^\alpha}\right)^j \int_0^{r^\alpha} (r^\alpha - v)^{\frac{p\mu}{1+p}+i-1} v^{\frac{\mu}{1+p}+j-1} \exp\left(-\frac{(\eta-p)\xi\mu v}{\eta\hat{r}^\alpha}\right) dv. \end{aligned} \tag{5.7}$$

The inner integral can be solved with the help of [53, Equation (2.3.6.1)] resulting in

$$\begin{aligned} f_R(r) = {} & \frac{\alpha(\mu\xi)^\mu}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}} \exp\left(-\frac{p\xi\mu r^\alpha}{\eta\hat{r}^\alpha}\right) \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!k!} \\ & \times \left(\frac{p^2 q\xi\kappa\mu^2 r^\alpha}{\delta\eta\hat{r}^\alpha}\right)^i \left(\frac{\kappa\xi\mu^2 r^\alpha}{\delta\hat{r}^\alpha}\right)^k {}_1\tilde{F}_1\left(k+\frac{\mu}{1+p};i+k+\mu;\frac{r^\alpha(p-\eta)\xi\mu}{\eta\hat{r}^\alpha}\right). \end{aligned} \tag{5.8}$$

The above expression can be further simplified by writing the hypergeometric function as an infinite sum, which yields,

$$
\begin{aligned}
f_R(r) = &\frac{\alpha(\mu\xi)^{\mu}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}}\frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}}\exp\left(-\frac{p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}\right)\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{i!k!n!} \\
&\times\frac{\Gamma\left(k+n+\frac{\mu}{1+p}\right)}{\Gamma(i+k+n+\mu)\Gamma\left(k+\frac{\mu}{1+p}\right)}\left(\frac{p^2q\xi\kappa\mu^2r^{\alpha}}{\delta\eta\hat{r}^{\alpha}}\right)^{i}\left(\frac{\kappa\xi\mu^2r^{\alpha}}{\delta\hat{r}^{\alpha}}\right)^{k}\left(\frac{r^{\alpha}(p-\eta)\xi\mu}{\eta\hat{r}^{\alpha}}\right)^{n}.
\end{aligned}
\tag{5.9}
$$

By performing the summation over the index i, it results in

$$
\begin{aligned}
f_R(r) = &\frac{\alpha(\mu\xi)^{\mu}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}}\frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}}\exp\left(-\frac{p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}\right)\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma\left(k+n+\frac{\mu}{1+p}\right)}{k!n!\Gamma\left(k+\frac{\mu}{1+p}\right)} \\
&\times\left(\frac{\kappa\xi\mu^2r^{\alpha}}{\delta\hat{r}^{\alpha}}\right)^{k}\left(\frac{r^{\alpha}(p-\eta)\xi\mu}{\eta\hat{r}^{\alpha}}\right)^{n}{}_0\tilde{F}_1\left(;k+n+\mu;\frac{p^2q\xi\kappa\mu^2r^{\alpha}}{\delta\eta\hat{r}^{\alpha}}\right)
\end{aligned}
\tag{5.10}
$$

Now summing over the infinite triangle $n = n' - k$ and after some algebraic manipulations, the PDF for the α-η-κ-μ fading model is obtained as

$$
\begin{aligned}
f_R(r) = &\frac{\alpha(\mu\xi)^{\mu}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}}\frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}}\exp\left(-\frac{p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}\right)\sum_{n'=0}^{\infty}\left(\frac{r^{\alpha}(p-\eta)\xi\mu}{\eta\hat{r}^{\alpha}}\right)^{n'} \\
&\times L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right){}_0\tilde{F}_1\left(;n'+\mu;\frac{p^2qr^{\alpha}\kappa\xi\mu^2}{\delta\eta\hat{r}^{\alpha}}\right)
\end{aligned}
\tag{5.11}
$$

Interestingly, in this new form, the PDF of the α-η-κ-μ fading model can be seen as the PDF of an α-κ-μ distribution with the modified Bessel function of the first kind replaced by a linear combination of Bessel functions. Therefore, it can be conjectured that the statistics of the α-η-κ-μ distribution will be a linear combination of those of the α-κ-μ fading model. Note, however, that the formulation in (5.11) presents an indeterminacy for $p = \eta$. It is noteworthy, on the other hand, that, in the limit as $p \to \eta$, the series reduces to the exact closed-form PDF of the α-κ-μ distribution, as predicted in [5]. In addition, it is important to note that this new equation evaluates substantially faster than anyone of the formulations presented in [5] and reproduced in Section 2.6, and, in general, needs no more than 20 terms for high numerical precision.

5.2 Envelope CDF - New Series Representation

A new series representation for the α-η-κ-μ CDF can be obtained by using (5.11) in the CDF definition. The integral form for the CDF is given by

$$
\begin{aligned}
F_R(r) = &\frac{\alpha(\mu\xi)^{\mu}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}}\frac{1}{\hat{r}^{\alpha\mu}}\sum_{n'=0}^{\infty}\left(\frac{(p-\eta)\xi\mu}{\eta\hat{r}^{\alpha}}\right)^{n'}L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) \\
&\times\int_0^r\tau^{\alpha(n'+\mu)-1}\exp\left(-\frac{p\xi\mu\tau^{\alpha}}{\eta\hat{r}^{\alpha}}\right){}_0\tilde{F}_1\left(;n'+\mu;\frac{p^2q\tau^{\alpha}\kappa\xi\mu^2}{\delta\eta\hat{r}^{\alpha}}\right)d\tau,
\end{aligned}
\tag{5.12}
$$

This integral can be solved by putting the hypergeometric function in terms of the modified Bessel function of the first kind using [36, Equation (9.6.47)] and then [4, Equation (4)] which results in

$$\begin{aligned} F_R(r) = \exp\left(-\frac{\kappa\mu}{\delta}\right)\left(\frac{\eta}{p}\right)^{\frac{\mu}{1+p}} \sum_{n'=0}^{\infty}\left(1-\frac{\eta}{p}\right)^{n'} L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) \\ \times\left(1-Q_{n'+\mu}\left(\sqrt{\frac{2pq\kappa\mu}{\delta}},\sqrt{\frac{2p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}}\right)\right), \end{aligned} \tag{5.13}$$

in which $Q_\nu(a,b)$ is the Marcum-Q function [54] defined as

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}}\int_b^{\infty} x^{\nu}\exp\left(-\frac{x^2+a^2}{2}\right) I_{\nu-1}(ax)\,dx \tag{5.14}$$

The series in (5.13) converges for $0 < \eta < 2p$. By no means, this is an issue, because the α-η-κ-μ distribution presents a certain symmetry such that the envelope distribution with parameters $\{\alpha,\eta,\kappa,\mu,p,q\}$ is identical to the one with parameters $\{\alpha,1/\eta,\kappa,\mu,1/p,1/q\}$. Again, there is question concerning the parameters η and p approaching each other. As said before, this is the case in which the α-η-κ-μ reduces to the α-κ-μ, and the envelope CDF is given in a closed-form expression.

5.3 Higher Order Moments

The higher-order moments of the α-η-κ-μ fading model can be obtained as

$$\mathbb{E}[R^k] = \int_0^{\infty} r^k f_R(r)\,\mathrm{d}r \tag{5.15}$$

After replacing $f_R(r)$ with (5.11) and changing the order of integration and summation, yields

$$\begin{aligned} \mathbb{E}[R^k] = \frac{\alpha(\mu\xi)^{\mu}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \frac{1}{\hat{r}^{\alpha\mu}} \sum_{n'=0}^{\infty}\left(\frac{(p-\eta)\xi\mu}{\eta\hat{r}^{\alpha}}\right)^{n'} L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) \\ \times\int_0^{\infty} r^{-1+k+\alpha(n'+\mu)}\exp\left(-\frac{p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}\right) {}_0\tilde{F}_1\left(;n'+\mu;\frac{p^2qr^{\alpha}\kappa\xi\mu^2}{\delta\eta\hat{r}^{\alpha}}\right)dr. \end{aligned} \tag{5.16}$$

The inner integral can be solved with the help of [41, Equation (2.22.3.1)] resulting in

$$\begin{aligned} \mathbb{E}[R^k] = \frac{\hat{r}^k(\mu\xi)^{-\frac{k}{\alpha}}}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)}\left(\frac{\eta}{p}\right)^{\frac{k}{\alpha}+\frac{\mu}{1+p}} \sum_{n'=0}^{\infty}\Gamma\left(n'+\frac{k}{\alpha}+\mu\right)\left(1-\frac{\eta}{p}\right)^{n'} \\ \times L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) {}_1\tilde{F}_1\left(n'+\frac{k}{\alpha}+\mu;n'+\mu;\frac{pq\kappa\mu}{\delta}\right). \end{aligned} \tag{5.17}$$

The series in (5.17), likewise the envelope CDF, converges for $0 < \eta < 2p$. Again, this is no issue, as explained for the CDF case. And again, here the comment concerning the parameters η and p approaching each other applies, i.e. the moments in this case converge to those of the α-κ-μ distribution, which are given in closed form.

5.4 Moment Generating Function

The MGF is defined as the average of $\exp(-sR)$, i.e.,

$$\mathscr{M}(s) = \mathbb{E}[\exp(-sR)] = \int_0^\infty \exp(-sr) f_R(r)\, \mathrm{d}r. \tag{5.18}$$

The integral in (5.18) can be evaluated by replacing $f_R(r)$ with (5.11) and by rewriting the hypergeometric function as an infinite sum. After changing the order of integration and summation, the MGF can be obtained as

$$\begin{aligned}\mathscr{M}(s) = & \frac{\alpha(\mu\xi)^\mu}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \frac{1}{\hat{r}^{\alpha\mu}} \sum_{n'=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n'+\mu+k)} \left(\frac{(p-\eta)\xi\mu}{\eta\hat{r}^\alpha}\right)^{n'} \\ & \times \left(\frac{p^2 q r^\alpha \kappa\xi\mu^2}{\delta\eta\hat{r}^\alpha}\right)^k L_{n'}^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) \int_0^\infty r^{\alpha(k+n'+\mu)-1} \exp(-sr) \exp\left(-\frac{p\xi\mu r^\alpha}{\eta\hat{r}^\alpha}\right) dr.\end{aligned} \tag{5.19}$$

The inner integral can be solved in terms of the Fox H-function using [39, Equation (2.29)]. After performing a series of algebraic manipulations, the moment generating function of the α-η-κ-μ fading model is given as

$$\mathscr{M}(s) = \frac{1}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{\eta}{p}\right)^{\frac{\mu}{1+p}} \sum_{k=0}^{\infty} c_k \left(\frac{pq\kappa\mu}{\delta}\right)^k H_{1,1}^{1,1}\left[\left(\frac{\eta}{p\xi\mu}\right)^{\frac{1}{\alpha}} \hat{r}s \,\middle|\, \begin{matrix}(1-k-\mu, 1/\alpha) \\ (0,1)\end{matrix}\right], \tag{5.20}$$

in which c_k is defined as

$$c_k = \frac{1}{\Gamma(\mu+k)} \sum_{n=0}^{k} \frac{1}{(k-n)!} \left(\frac{\delta(p-\eta)}{p^2 q\kappa\mu}\right)^n \times L_n^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right). \tag{5.21}$$

5.5 Asymptotic Behavior

Expressions for the behavior of the lower portion of the PDF are found in closed-form by taking the limit of the exponential and hypergeometric function as $r \to 0$. Also, it is possible to ignore all terms in the sum other than the first one. After applying some algebraic manipulations, the lower tail envelope PDF can be obtained as

$$f_R(r) = \frac{\alpha\mu^\mu\xi^\mu}{\Gamma(\mu)\exp\left(\frac{\kappa\mu(pq+1)}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{\mu p}{p+1}} \frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}}. \tag{5.22}$$

By integrating (5.22), the corresponding CDF at the lower tail is found as

$$F_R(r) = \frac{\mu^{\mu-1}\xi^\mu}{\Gamma(\mu)\exp\left(\frac{\kappa\mu(pq+1)}{\delta}\right)} \left(\frac{p}{\eta}\right)^{\frac{\mu p}{p+1}} \left(\frac{r}{\hat{r}}\right)^{\alpha\mu}. \tag{5.23}$$

5.6 Parameter Estimation

The α-η-κ-μ fading model is described in terms of several parameters related to different physical phenomena. Parameter estimation in such a case can become a rather complicated matter, mostly due to the fact that the k-th moment of the α-η-κ-μ envelope is given in an infinite series form. Here, we propose a first approach to address the parameter estimation problem based on the moment matching technique. The idea behind this process is quite simple. It consists in first finding an estimate for the parameter α and then obtaining the remaining α-η-κ-μ parameters by using the α-moments of the α-η-κ-μ distribution, which is given in closed-form expression, as will be shown below. The tricky part now is to estimate α. We propose to estimate α by assuming that the true distribution, i.e. α-η-κ-μ, can be approximated by an α-κ-μ or an α-η-μ distribution, for which the k-th moments are given in closed-form formulas. That is, the parameter α, and only the said parameter, will be estimated using the moments of either the α-κ-μ or α-η-μ distributions. Suppose we have a set of α-η-κ-μ random envelope samples. Then, we use the relation[2]

$$\frac{\mathbb{E}[R^k]}{\mathbb{E}[R^2]^{\frac{k}{2}}} = \frac{M[k]}{M[2]^{\frac{k}{2}}}, \tag{5.24}$$

in which $M[k]$ is the k-th moment of the data, and $\mathbb{E}[R^k]$ is the k-th analytical moment of the α-η-μ [43, Eq. (2)] or α-κ-μ [43, Eq. (7)] envelope. By choosing three distinct values of k, a system of three equations and three unknowns arises. In Mathematica, a solution for each parameter can be attained using the `FindRoot` function. Another faster approach is obtained by using the `NMinimize` function, which allows for the inclusion of any objective function, e.g. the mean square error of a set of (5.24).

The α-moments of the α-η-κ-μ envelope can be found as

$$\mathbb{E}[R^{\alpha k}] = \mathbb{E}[(X^2+Y^2)^k]. \tag{5.25}$$

Using a multinomial expansion, the α-moments are then given as

$$\mathbb{E}[R^{\alpha k}] = \mathbb{E}[\sum_{i=0}^{k}\binom{k}{i}X^{2i}Y^{2(k-i)}] \tag{5.26}$$

Now, replacing the moments of the κ-μ power variates and performing the necessary calculations, the α-moments are obtained as

$$\begin{aligned}\mathbb{E}[R^{\alpha k}] = \hat{r}^{\alpha k}\sum_{i=0}^{k}&\frac{k!\Gamma\left(k-i+\frac{\mu}{1+p}\right)\Gamma\left(i+\frac{p\mu}{1+p}\right)}{i!(k-i)!p^i\eta^{-i}\xi^k\mu^k} \\ &\times {}_1\tilde{F}_1\left(-i;\frac{p\mu}{1+p};-\frac{pq\kappa\mu}{\delta}\right){}_1\tilde{F}_1\left(i-k;\frac{\mu}{1+p};-\frac{\kappa\mu}{\delta}\right)\end{aligned} \tag{5.27}$$

[2] There is an infinite number of equations that could be used, the criterion was to choose a relation such that $\hat{r}$ would vanish.

in which $k \in \mathbb{N}$. The value for $\hat{r}$ is readily obtained by setting $k = 1$. The remaining parameters can be estimated using higher order moments. Again, there are an infinite number of alternatives available. As observed, there remain five more parameters to be estimated, namely η, κ, μ, p, q. It is possible to use (5.27) and set five equations with five unknowns and solve them by means of the `FindRoot` function in Mathematica. On the other hand, it is observed that those equations become rather long as k increases, rendering the solution very difficult and unstable. By working with these moments, we have been able to find a pattern, which was then used to simplify the equations. Such a pattern led us to define what we call Moment Relation, t_k, which will be used to find the remaining parameters. Such a Moment Relation is given as

$$t_k = \left(\frac{p^k + p\eta^k}{1+p} + \frac{k\left(p^k + pq\eta^k\right)\kappa}{\delta} \right) \frac{\mu}{(\xi\mu p)^k} \tag{5.28}$$

It can be seen that $t_1 = 1$. From (5.27), we find that t_i with $i \in \{2, 6\}$ are related to the α-moments as

$$\begin{aligned}
t_2 =& \frac{\mathbb{E}[R^{2\alpha}]}{\hat{r}^{2\alpha}} - 1, \; t_3 = \frac{\mathbb{E}\left[R^{3\alpha}\right]}{2\hat{r}^{3\alpha}} - \frac{3\mathbb{E}\left[R^{2\alpha}\right]}{2\hat{r}^{2\alpha}} + 1, \\
t_4 =& \frac{\mathbb{E}\left[R^{4\alpha}\right]}{6\hat{r}^{4\alpha}} - \frac{2\mathbb{E}\left[R^{3\alpha}\right]}{3\hat{r}^{3\alpha}} + \frac{\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}}\left(2 - \frac{\mathbb{E}\left[R^{2\alpha}\right]}{2\hat{r}^{2\alpha}}\right) - 1, \\
t_5 =& \frac{\mathbb{E}\left[R^{5\alpha}\right]}{24\hat{r}^{5\alpha}} - \frac{5\mathbb{E}\left[R^{4\alpha}\right]}{24\hat{r}^{4\alpha}} - \frac{5}{12}\left(2 - \frac{\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}}\right)\left(\frac{3\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}} - \frac{\mathbb{E}\left[R^{3\alpha}\right]}{\hat{r}^{3\alpha}}\right) + 1, \\
t_6 =& \frac{\mathbb{E}\left[R^{6\alpha}\right]}{120\hat{r}^{6\alpha}} - \frac{\mathbb{E}\left[R^{5\alpha}\right]}{20\hat{r}^{5\alpha}} + \frac{\mathbb{E}\left[R^{4\alpha}\right]}{8\hat{r}^{4\alpha}}\left(2 - \frac{\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}}\right) \\
&- \frac{1}{12}\left(\frac{\mathbb{E}\left[R^{3\alpha}\right]}{\hat{r}^{3\alpha}} + 6\right)^2 + \frac{1}{4}\left(\frac{\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}} - 3\right)^3 + \frac{\mathbb{E}\left[R^{2\alpha}\right]}{\hat{r}^{2\alpha}}\left(\frac{\mathbb{E}\left[R^{3\alpha}\right]}{\hat{r}^{3\alpha}} - \frac{15}{4}\right) + \frac{35}{4}
\end{aligned} \tag{5.29}$$

Unfortunately, it is extremely difficult, if not impossible, to obtain closed-form expressions for each parameter, hence, a numerical solution is required. For such a case, Mathematica's `NMinimize` function can be used to obtain the remaining parameters as follows

$$\begin{aligned}
\{\kappa, \eta, \mu, p, q\} = \{\kappa, \eta, \mu, p, q\}/.\ \text{NMinimize}[\Big\{ \sum_{i=2}^{l} (t_i - K_i)^2, \\
\kappa > 0 \wedge \eta > 0 \wedge \mu > 0 \wedge p > 0 \wedge q > 0 \Big\}, \{\kappa, \eta, \mu, p, q\}][[2]]
\end{aligned} \tag{5.30}$$

in which t_i are given by (5.28) and K_i is the right-hand side of (5.29), obtained from the corresponding moments of the data, and $l \leq 6$. Interestingly, it may not be necessary to use all equations in (5.29) to obtain estimations for the parameters, a feature of the NMinimize function.

5.7 New Closed-Form Particular Cases

The well-known fading distributions comprised by the α-η-κ-μ fading model are described in [5]. As also mentioned in [5], a number of other fading scenarios are comprised by α-η-κ-μ that are not yet known in the literature. Here, we obtain two new closed-form expressions. This is attained as follows. By setting the parameter $q=0$, the hypergeometric function vanishes resulting in the infinite sum of Laguerre polynomials, which can also be written in terms of a confluent double Gaussian series as

$$\begin{aligned} f_R(r) =& \frac{\alpha(\xi\mu)^{\mu}\exp\left(-\frac{\xi\mu r^{\alpha}}{\hat{r}^{\alpha}}\right)}{\exp\left(\frac{\kappa\mu}{\delta'}\right)\Gamma(\mu)}\left(\frac{p}{\eta}\right)^{\frac{\mu p}{1+p}}\frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}} \\ &\times \phi_3\left(\frac{\mu p}{1+p};\mu;\frac{(\eta-p)\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}},\frac{\kappa\xi\mu^2 r^{\alpha}}{\delta'\hat{r}^{\alpha}}\right) \end{aligned} \tag{5.31}$$

in which ϕ_3 is a confluent form of the Appell series [41, Eq. (7.2.4.7)] and $\delta'=(1+p)/(1+\eta)$. This particular double hypergeometric series can be evaluated using[3]

$$\phi_3(\beta;\gamma;x,y) = \lim_{\sigma\to 0}\lim_{\epsilon\to 0} F_1\left(\frac{1}{\epsilon};\beta,\frac{1}{\sigma};\gamma;\epsilon x,\epsilon\sigma y\right) \tag{5.32}$$

in which F_1 is one of the Appell hypergeometric series [41, Eq. (7.2.4.1)] and, more importantly, is readily available in some mathematical packages such as Mathematica. A similar approach can be done for $q\to\infty$, leading to

$$\begin{aligned} f_R(r) =& \frac{\alpha(\xi\mu)^{\mu}\exp\left(-\frac{p\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}}\right)}{\exp\left(\frac{p\kappa\mu}{\delta'\eta}\right)\Gamma(\mu)}\left(\frac{p}{\eta}\right)^{\frac{\mu p}{1+p}}\frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}} \\ &\times \phi_3\left(\frac{\mu}{1+p};\mu;\frac{(p-\eta)\xi\mu r^{\alpha}}{\eta\hat{r}^{\alpha}},\frac{p^2\kappa\xi\mu^2 r^{\alpha}}{\delta'\eta^2\hat{r}^{\alpha}}\right) \end{aligned} \tag{5.33}$$

5.8 Applications

This section provides a number of interesting applications using the results derived previously in this chapter. Let γ be the instantaneous signal-to-noise ration of a signal with symbol energy E_s transmitted over a fading channel, so that

$$\gamma = \frac{E_s r^2}{N_0}, \tag{5.34}$$

in which r is the channel coefficient and N_0 is the noise power spectral density. Its mean value is then obtained as

$$\bar{\gamma} = \frac{E_s}{N_0}\mathbb{E}[R^2], \tag{5.35}$$

[3] The readers are advised that when using the Limit function in Mathematica, as it might render an incorrect result. As a suggestion, the built-in F_1 can be directly used in Mathematica, in which case ϵ and σ are set as close to zero as possible.

such that $\mathbb{E}[R^2]$ is the second moment of the channel gain and can be evaluated from (5.17). Following a standard probability procedure, the PDF for the instantaneous SNR can be evaluated from the PDF of the channel gain as

$$f_\Gamma(\gamma) = \frac{1}{2}\sqrt{\frac{\mathbb{E}[R^2]}{\gamma\bar{\gamma}}} f_R\left(\sqrt{\frac{\gamma}{\bar{\gamma}}\mathbb{E}[R^2]}\right), \tag{5.36}$$

in which $f_R(r)$ is the envelope PDF and can be evaluated using (5.2) or (5.11). The CDF of the instantaneous SNR is given as

$$F_\Gamma(\gamma) = F_R\left(\sqrt{\frac{\gamma}{\bar{\gamma}}\mathbb{E}[R^2]}\right), \tag{5.37}$$

and $F_R(r)$ is evaluated by (5.4) or (5.13).

5.8.1 Outage Probability

The probability that the SNR falls below a certain threshold defines the outage probability, i.e.

$$P_{\text{OUT}} = \Pr[\gamma < \gamma_{th}] = F_\Gamma(\gamma_{th}), \tag{5.38}$$

in which $F_\Gamma(\gamma)$ is given in (5.37).

5.8.2 Outage Capacity

Shannon capacity is defined as $C(\gamma) = W\log_2(1+\gamma)$, in which W is the channel bandwidth. Furthermore, the outage capacity is the probability that capacity falls below a certain threshold rate C_0. The outage capacity can be evaluated in terms of the CDF of the SNR as

$$C_{\text{out}} = F_\Gamma\left(2^{\frac{C_0}{W}} - 1\right). \tag{5.39}$$

5.8.3 Amount of Fading

The amount of fading (AF) [45] is a metric that indicated the severity of fading and is defined in terms of the first and second moments of the instantaneous SNR or the second and fourth moments of the envelope as

$$AF = \frac{\mathbb{E}[R^4]}{\mathbb{E}[R^2]^2} - 1 \tag{5.40}$$

which may be evaluated using (5.17).

5.8.4 Average Bit Error Rate

A general formulation for the bit error rate (BER) for binary signaling is given as [45, Eq. (8.100)]

$$P_b(\gamma) = \frac{\Gamma(b, a\gamma)}{2\Gamma(b)} \tag{5.41}$$

in which a and b are parameters chosen accordingly to the modulation and detection schemes. The authors in [45, Table 8.1] provided all possible values for a and b. The average BER can be then computed as

$$P_b = \int_0^\infty Pb(\gamma) f_\Gamma(\gamma) d\gamma = \int_0^\infty \frac{\Gamma(b, a\gamma)}{2\Gamma(b)} f_\Gamma(\gamma)\, \mathrm{d}\gamma. \tag{5.42}$$

After long algebraic manipulations, the average BER is then obtained as

$$\begin{aligned} P_b = & \frac{1}{2\Gamma(b)\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)} \left(\frac{\eta}{p}\right)^{\frac{\mu}{1+p}} \sum_{i=0}^{\infty} c_i \left(\frac{pq\kappa\mu}{\delta}\right)^i \\ & \times H_{2,2}^{1,2}\left[\frac{p\xi\mu}{\eta}\left(\frac{\mathbb{E}\left[R^2\right]}{\bar{\gamma} a \hat{r}^2}\right)^{\alpha/2} \middle| \begin{matrix} (1,1),(1-b,\alpha/2) \\ (i+\mu,1),(0,1) \end{matrix}\right] \end{aligned} \tag{5.43}$$

in which c_i was previously defined in (5.21). A closed-form asymptotic behavior at high SNR for the bit error rate can be obtained by expanding the Fox H-function through the sum of residues around the poles generated by the parameter $(i+\mu, 1)$ and then ignoring the higher exponents in the sum, the asymptotic behavior at high SNR is given as

$$P_b = \frac{2^{-1}\mu^{\mu-1}\xi^{\mu}\Gamma\left(b+\frac{\alpha\mu}{2}\right)}{\exp\left(\frac{(1+pq)\kappa\mu}{\delta}\right)\Gamma(b)\Gamma(\mu)} \left(\frac{p}{\eta}\right)^{\frac{p\mu}{1+p}} \left(\frac{\mathbb{E}\left[R^2\right]}{a\bar{\gamma}\hat{r}^2}\right)^{\frac{\alpha\mu}{2}}. \tag{5.44}$$

Now, expanding the Fox H-function using the residues around the poles created by the parameters $(1,1)$ and $(1-b, \alpha/2)$, after the necessary calculation a fairly simple infinite summation for the behavior at very low SNR arises as

$$\begin{aligned} P_b = & \frac{1}{2} - \frac{1}{2b\Gamma(b)\exp\left(\frac{\kappa\mu}{\delta}\right)} \left(\frac{\eta}{p}\right)^{\frac{\mu}{1+p}} \left(\frac{\eta}{p\mu\xi}\right)^{\frac{2b}{\alpha}} \left(\frac{a\bar{\gamma}\hat{r}^2}{\mathbb{E}[R^2]}\right)^b \sum_{n=0}^{\infty} \Gamma\left(n+\frac{2b}{\alpha}+\mu\right) \\ & \times \left(1-\frac{\eta}{p}\right)^n L_n^{\frac{\mu}{1+p}-1}\left(\frac{\eta\kappa\mu}{\delta(\eta-p)}\right) {}_1\tilde{F}_1\left(-\frac{2b}{\alpha}; n+\mu; -\frac{pq\kappa\mu}{\delta}\right). \end{aligned} \tag{5.45}$$

It is possible to note from (5.44) that the BER decreases with a power of $\alpha\mu$ as the mean SNR grows, which clearly indicates that high values of α and μ results in better channel condition, as expected.

5.9 Some Plots

In this section, plots will be presented to illustrate the use of the formulations developed here. It is opportune to mention that, because of the number of parameters of the α-η-κ-μ model, only a very limited sample of possible plots are shown.

In Figure 5.1, a few examples of shapes for the PDF are plotted with varying κ along with their respective asymptotic behavior. It is noteworthy that the asymptotic line slope is dependent only on the parameters α and μ, ergo keeping these constants will produce shapes with parallel behavior near $r = 0$ as can be seen in Figure 5.1.

The parameter estimator process described in Section 5.6 has been exercised as follows. Samples of the α-η-κ-μ envelope with some given parameters were generated. Then the parameter estimation algorithm proposed here was applied. The parameter α was obtained using the `FindRoot` function matching the data samples with the α-κ-μ distribution using a set of (5.24) with $k = \{1/3, 2/3, 1\}$. The remaining parameters were estimated through (5.30) with $l = 3$. As can be seen from Figure 5.2, the PDF using the estimated parameters fits adequately with the original PDF even though the set of estimated parameters may differ slightly from the correct one. The original and estimated parameters used to generate the curves in Figure 5.2 are given in Table 5.1.

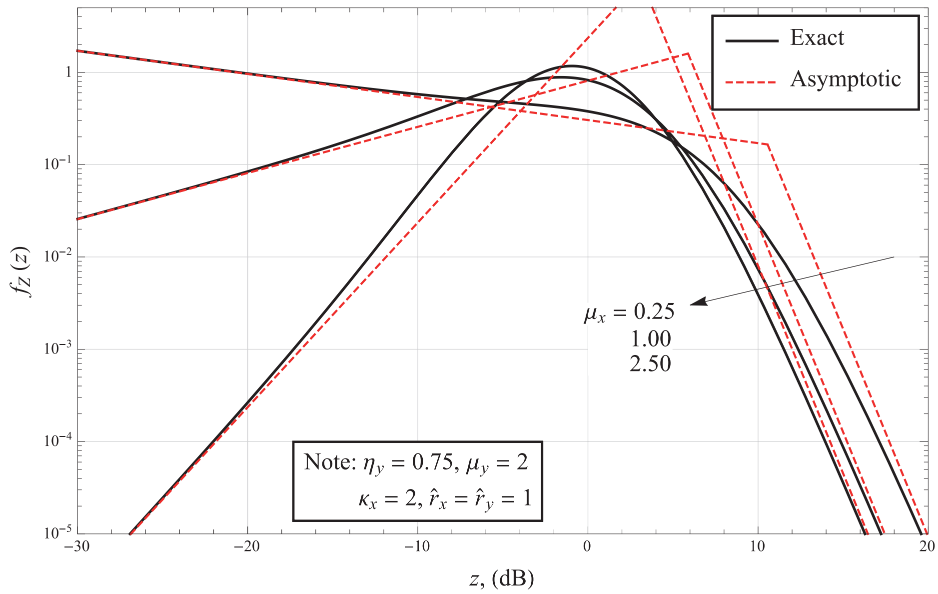

Figure 5.1 – Exact PDF (solid line) and asymptotic behavior (dashed line) for different values of κ, and $\alpha = 2.5$, $\eta = 1.2$, $\mu = 2.4$, $p = 2$ and $q = 2.5$.

Table 5.1 – Original and Estimated Parameters of Figure 5.2

	α	η	κ	μ	p	q	$\hat{r}$
Original	1.6	2.5	0.5	1.3	0.9	1.5	1.
Estimated	1.24261	1.2423	0.967155	1.44452	1.2423	2.1679	0.949505

Figure 5.3 shows the outage probability for a set of threshold values. As expected, the analytical results agree with the Monte Carlo simulation. From the asymptotic analysis, it is possible to verify that the rate at which the outage diminishes in high SNR depends only on the product $\alpha\mu$, whereas the separation between curves is a function of all the

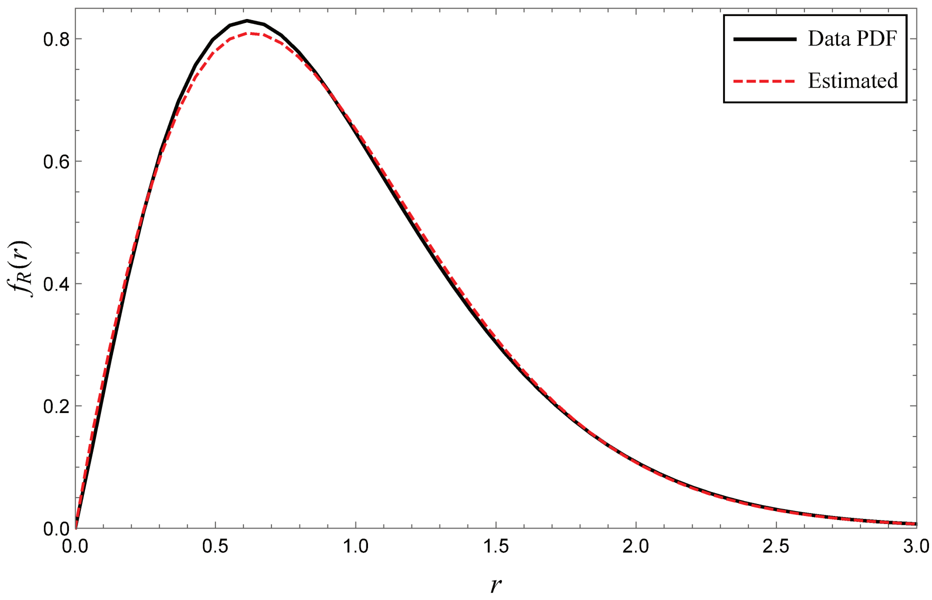

Figure 5.2 – Fitted PDF (dashed line) with estimated parameters along with the data PDF (solid line) from original set of parameters.

remaining parameters. In particular, for the scenario shown here, it is found that for a given outage probability an increase in the mean SNR leads to to an increase in the threshold by approximately the same amount.

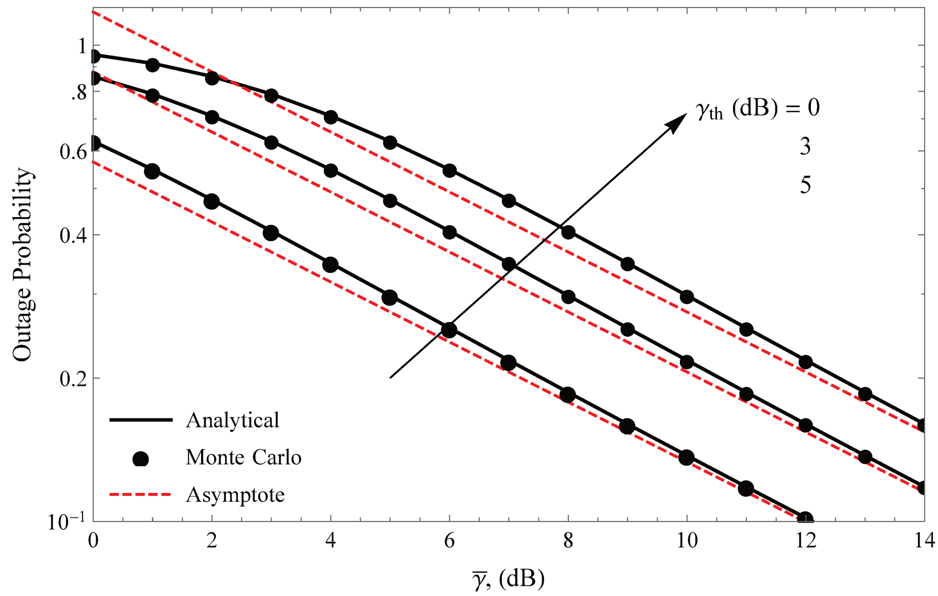

Figure 5.3 – Outage Probability for a signal subject to α-η-κ-μ fading with $\alpha = 2.1$, $\eta = 1.1$, $\kappa = 1.3$, $\mu = 0.6$, $p = 2.7$ and $q = 2.4$.

Figure 5.4 depicts the amount of fading for the α-η-κ-μ as a function of the parameter q for varying η. Interestingly, for this particular case, the behaviour of the curves depends on the ratio η/p. From this figure, the amount of fading grows on q when $\eta > p$ which indicates a more severe scenario. On the other hand, the AF diminishes for $\eta < p$ as q

increases. As said before, a number of other situations can be exercised that are not shown here.

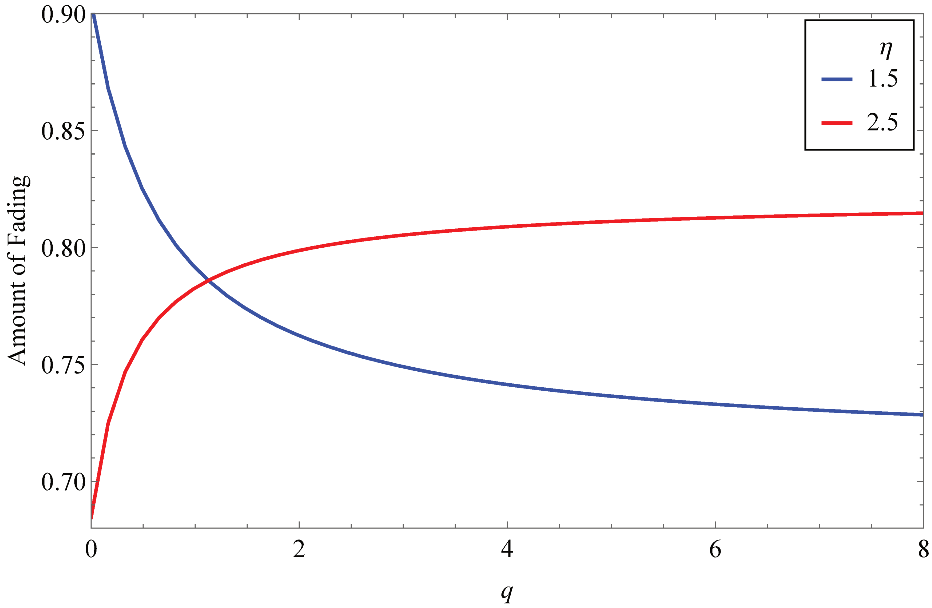

Figure 5.4 – AF for the α-η-κ-μ fading with $\alpha = 1.7$, $\kappa = 2.3$, $\mu = 0.9$, $p = 2$, $\hat{r} = 1$ and $\eta = \{1.5, 2.5\}$.

Finally, Figure 5.5 depicts the BER for a BPSK signal over the influence of the α-η-κ-μ channel and varying the parameter q. Again, for the specific scenario chosen here, the increase in q leads to a better average BER.

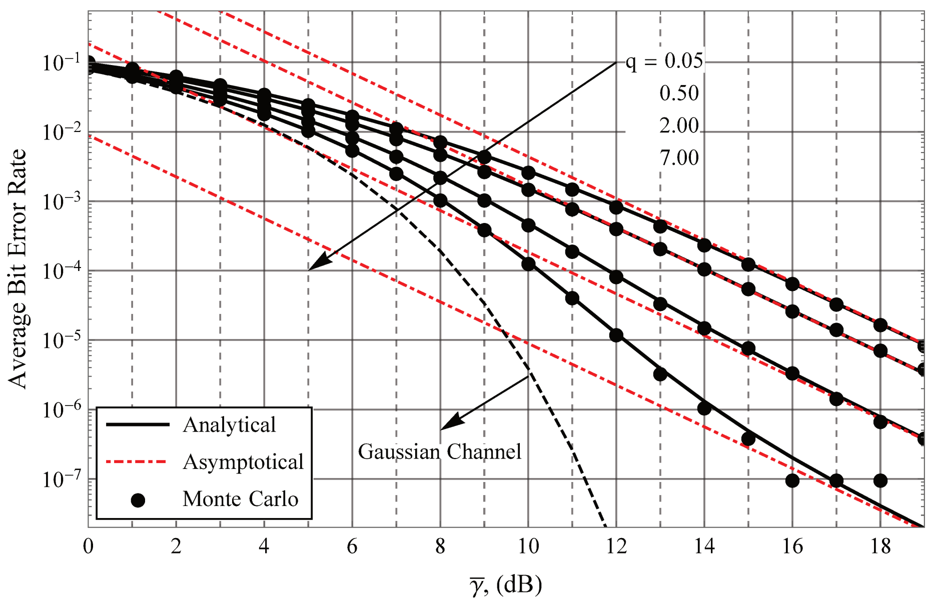

Figure 5.5 – BER for a BPSK signal over the α-η-κ-μ fading with $\alpha = 4$, $\eta = 0.25$, $\kappa = 2$, $\mu = 1.5$, $p = 3$, $\hat{r} = 1$ and varying q.

5.10 Conclusion

Several new fundamental results concerning the α-η-κ-μ model were developed in this chapter. In particular, series representations for the envelope PDF and CDF were presented in terms of well-known and easy to compute functions. Such series require no recursion and are fast convergent. Higher order moments were also given in series form. These can be used in a number of applications, including parameter estimation, bit error analyses, and others. Moment generating function was derived and, although given in terms of the Fox-H function, computes easily with the help of algorithms available in the literature. Asymptotic behavior of the PDF, CDF, and BER were given in simple closed-form formulations. An efficient procedure for parameter estimation was also given. New closed-form expressions for particular cases not shown previously in the literature were given. The use of some of the formulations was shown in applications such as outage probability, outage capacity, amount of fading, and bit error rate performance, for which the asymptotic behaviour is found.

Conclusions

In this work, the products and ratios of two fading envelopes taken from the α-μ, η-μ, and κ-μ distributions have been thoroughly explored.

In Chapter 3, closed-form expressions for the PDF and CDF of the ratio of two random envelopes taken from the α-μ, η-μ, and κ-μ distributions were obtained in terms of the multivariable Fox H-function. As the multivariable Fox H-function is yet to be implemented in the most common mathematical packages, simple, fast convergent series expressions were derived using the residues theorem to evaluate the multiple contour integral of the Fox H-function. In order to obtain further insight in the effect of the parameters, asymptotic expressions at lower and upper tails were obtained. All the expressions in Chapter 3 may be used in a plethora of wireless communication systems, such as multihop, spectrum sharing, characterization of co-channel interference, physical layer security among many others. Also, multipath-shadowing composite fading statistics can be found as particular case of the ratio of two envelopes. To illustrate their applicability, a practical example dealing with secrecy capacity of a Gaussian wire tap channel for D2D and V2V communications were conduct using data from field measurements.

In Chapter 4, first order statistics – namely PDF, CDF and moments – of the product of two independent random envelopes taken from the α-μ, η-μ, and κ-μ distributions were obtained in terms of the multivariable Fox H-function, and infinite series representation. Among the many possible applications, cascaded channel, multihop systems, MIMO's keyhole channel, high resolution synthetic aperture radar clutter stand out. In addition to the PDF and CDF of the product distribution, an integral involving the product of a PDF and a CDF related to CDF of the product distribution was found. The said integral was then used to evaluate the probability of detection in UHF RFID system. Moreover, statistics for multipath-shadowing can be obtained from the product distribution, therefore a plethora of possible scenarios of composite fading is achieved from the results here obtained.

In Chapter 5, new and more efficient series representations for the PDF and CDF of the α-η-κ-μ fading model were obtained. From the PDF new representation, novel infinite series for the higher order moments and MGF were derived. New closed-form expressions were obtained for some interesting particular cases. In addition, closed-form asymptotic expressions were obtained for the lower tail of the PDF and CDF. A first approach for the parameter estimation problem of the α-η-κ-μ model was proposed. Some application examples were also shown to demonstrate the utility of the formulations proposed here, in particular outage probability, amount of fading, outage capacity and bit error probability for which asymptotic behaviours were derived.

Future Work

Opportunities for future investigation are summarized below:

1. The first order statistics – PDF and CDF – of the product of two random envelopes were obtained here. These results can be generalized to obtain the statistics of the product of several random variates.

2. Following the same lines as before, expressions for the statistics for the ratio of products of random envelopes can be generalized to unify the results here presented.

3. Second order statistics such as level crossing rate and average fade duration are still an open issue for the product and ratio distributions.

4. Another interesting result is the derivation of complex-based model accounting for the phase and the envelope for the product and ratio distributions. It is conjectured that the phase PDF would deal with sum and difference of the random phases, respectively, for the product and ratio of complex signals.

5. The sum of fading envelopes, which finds application in several wireless communication applications, is still an open issue. The challenge regarding the sum lies in avoiding the use of multiple integrals or summations in the numerical evaluation. In the same line, the summation of random vectors is also an interesting and difficult topic of research.

6. The series presented here compute efficiently, although they are not unique and many other series representation may arise either from changing the order of summation or by altering the approach altogether. It is conjectured that the overall quality of a series may vary according to the parameters. Therefore, there may exist series expression with better performance under certain conditions or parameters and the pursuit for new, improved formulations is an interesting research opportunity.

7. Even though the series presented here compute rapidly, it is still interesting to find approximations in the closed-form fashion. In the present form, it is easy to lose sight of the impact the parameters have on the shape of the PDF. In this sense, closed-form expressions can be used to obtain an insight on the PDF shape more easily.

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Appendices

Appendix A

Derivation of the Statistics of the Ratio Distribution

In Chapter 3, the first order statistics (PDF and CDF) were presented in terms of the multivariable Fox H-function and also as simple, fast convergent infinite series. Here, the mathematical derivations for said expressions are presented.

To achieve the desired result, it will be required to write some functions in terms of their Mellin-Barnes contour integral representation, namely the exponential and the hypergeometric functions, which are given as [41, Equations (8.4.3.1) and (8.2.1.1)]

$$\exp(-x) = \frac{1}{2\pi j} \oint \Gamma(s) x^{-s}\, ds, \tag{A.1}$$

and [41, Equation (7.2.3.12)]

$$ {}_0F_1(;a;x) = \Gamma(a) \frac{1}{2\pi j} \oint_{\mathscr{L}} \frac{\Gamma(s)}{\Gamma(a-s)} (-x)^{-s}\, ds, \tag{A.2}$$

and

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \frac{\Gamma(t)\Gamma(a-t)\Gamma(b-t)}{\Gamma(c-t)} (-z)^{-t}\, dt. \tag{A.3}$$

A.1 The α-μ/α-μ Distribution

A.1.1 PDF

By replacing (2.9) in (3.1) with appropriate subscripts, the PDF for the ratio of two α-μ variates is given by the following integral

$$f_Z(z) = \frac{\alpha_x \alpha_y \mu_x^{\mu_x} \mu_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)} \frac{z^{\alpha_x \mu_x - 1}}{\hat{r}_x^{\alpha_x \mu_x} \hat{r}_y^{\alpha_y \mu_y}} \int_0^\infty y^{\alpha_x \mu_x + \alpha_y \mu_y - 1} \exp\left(-\frac{\mu_x (zy)^{\alpha_x}}{\hat{r}_x^{\alpha_x}}\right) \exp\left(-\frac{\mu_y y^{\alpha_y}}{\hat{r}_y^{\alpha_y}}\right) dy. \tag{A.4}$$

The above integral cannot be solved by elementary functions. Although, it can be interpreted as Fox H-function by putting an exponential function in terms of its Mellin-Barnes type contour integral representation using (A.1), which results in

$$f_Z(z) = \frac{\alpha_x \alpha_y \mu_x^{\mu_x} \mu_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)} \frac{z^{\alpha_x\mu_x - 1}}{\hat{r}_x^{\alpha_x\mu_x} \hat{r}_y^{\alpha_y\mu_y}} \int_0^\infty y^{\alpha_x\mu_x + \alpha_y\mu_y - 1} \times \frac{1}{2\pi j} \oint \Gamma(t) \left(\frac{\mu_x (zy)^{\alpha_x}}{\hat{r}_x^{\alpha_x}}\right)^{-t} dt \exp\left(-\frac{\mu_y y^{\alpha_y}}{\hat{r}_y^{\alpha_y}}\right) dy. \tag{A.5}$$

It is possible to change the order of integration. This shift in the integration order causes a distortion in the contour region which, in itself, is not prohibitive in this case. The inner integral can now be solved using [36, Equation (6.1.1)] reproduced here for completeness

$$\Gamma(s) = \int_0^\infty t^{s-1} \exp(t)\, dt. \tag{A.6}$$

Then, solving the inner integral, and after some algebraic manipulations and utilizing the constants defined in (3.4), results in

$$f_Z(z) = \frac{\alpha_x}{z\Gamma(\mu_x)\Gamma(\mu_y)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma(t)\Gamma\left(\frac{\alpha_x\mu_x}{\alpha_y} + \mu_y - \frac{\alpha_x t}{\alpha_y}\right) \left(\frac{z^{\alpha_x} \mathscr{A}_y^{\alpha_x}}{\mathscr{A}_x^{\alpha_x}}\right)^{-t+\mu_x} dt. \tag{A.7}$$

Finally, by performing the variable transformation $s = \alpha_x(t - \mu_x)$, the PDF for the ratio of two α-μ variates is written as

$$f_Z(z) = \frac{1}{z\Gamma(\mu_x)\Gamma(\mu_y)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_x + \frac{s}{\alpha_x}\right) \Gamma\left(\mu_y - \frac{s}{\alpha_y}\right) \left(\frac{z}{u_{\alpha\alpha}}\right)^{-s} ds. \tag{A.8}$$

The above integral can be interpreted as Fox H-function using (2.1) whose parameters are given in Table 3.1.

A.1.2 CDF

The CDF of the ratio of two α-μ variates is obtained by replacing (A.8) at the CDF's definition resulting in

$$F_Z(z) = \int_0^z \frac{1}{\tau\Gamma(\mu_x)\Gamma(\mu_y)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_x + \frac{s}{\alpha_x}\right) \Gamma\left(\mu_y - \frac{s}{\alpha_y}\right) \left(\frac{\tau}{u_{\alpha\alpha}}\right)^{-s} ds\, d\tau. \tag{A.9}$$

After changing the order of integration the inner integral can be solved using the identity $\int_0^z \tau^{-s-1} = \Gamma(-s)/\Gamma(1-s) z^{-s}$ which results in

$$F_Z(z) = \frac{1}{\Gamma(\mu_x)\Gamma(\mu_y)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \frac{\Gamma\left(\mu_x + \frac{s}{\alpha_x}\right) \Gamma\left(\mu_y - \frac{s}{\alpha_y}\right) \Gamma(-s)}{\Gamma(1-s)} \left(\frac{z}{u_{\alpha\alpha}}\right)^{-s} ds. \tag{A.10}$$

The Fox H-function parameters for the ratio of two α-μ variates are readily deduced by comparing (A.10) with (2.1).

A.2 The α-μ/κ-μ Distribution

A.2.1 PDF

Let $Z = X/Y$ be the ratio of the α-μ by the κ-μ variates with parameters $\{\alpha_x, \mu_x, \hat{r}_x\}$ and $\{\kappa_y, \mu_y, \hat{r}_y\}$, respectively. Now, replacing (2.9) and (2.13) in (3.1) results in

$$f_Z(z) = \frac{2\alpha_x \mu_x^{\mu_x} \left(\left(1+\kappa_y\right)\mu_y\right)^{\mu_y}}{\Gamma(\mu_x)\exp\left(\kappa_y \mu_y\right)} \frac{z^{\alpha_x\mu_x - 1}}{\hat{r}_x^{\alpha_x\mu_x}\hat{r}_y^{2\mu_y}} \int_0^\infty y^{\alpha_x\mu_x + 2\mu_y - 1} \exp\left(-\frac{\mu_x (zy)^{\alpha_x}}{\hat{r}_x^{\alpha_x}}\right) \times \exp\left(-\frac{\left(\mu_y\left(1+\kappa_y\right)\right)y^2}{\hat{r}_y^2}\right) {}_0\tilde{F}_1\left(;\mu_y;\frac{y^2\kappa_y\left(1+\kappa_y\right)\mu_y^2}{\hat{r}_y^2}\right) dy. \tag{A.11}$$

Except for some specific values of α_x, the above integral does not have a closed-form expression in terms of elementary functions. Again, a solution in terms of the Fox H-function is possible by replacing one exponential and the hypergeometric functions by their Mellin-Barnes contour integral representations by using (A.1) and (A.2) respectively, which gives

$$f_Z(z) = \frac{2\alpha_x \mu_x^{\mu_x} \left(\left(1+\kappa_y\right)\mu_y\right)^{\mu_y}}{\Gamma(\mu_x)\exp\left(\kappa_y \mu_y\right)} \frac{z^{\alpha_x\mu_x - 1}}{\hat{r}_x^{\alpha_x\mu_x}\hat{r}_y^{2\mu_y}} \left(\frac{1}{2\pi j}\right)^2 \int_0^\infty y^{\alpha_x\mu_x + 2\mu_y - 1} \exp\left(-\frac{\mu_y\left(1+\kappa_y\right)y^2}{\hat{r}_y^2}\right) \times \oiint_{\mathscr{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)} \left(\frac{\mu_x (zy)^{\alpha_x}}{\hat{r}_x^{\alpha_x}}\right)^{-t_1} \left(-\frac{y^2\kappa_y\left(1+\kappa_y\right)\mu_y^2}{\hat{r}_y^2}\right)^{-t_2} dt_1 dt_2 dy. \tag{A.12}$$

Again, it is possible to change the order of integration. The inner integral can solved with the help of (A.6), and after some algebraic manipulations and using the constants (3.4), it results in

$$f_Z(z) = \frac{\alpha_x}{z\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)} \left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \Gamma\left(-t_2 + \frac{1}{2}\alpha_x(-t_1+\mu_x) + \mu_y\right) \times \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)} \left(\frac{z^{\alpha_x}\mathscr{K}_y^{\alpha_x}}{\mathscr{A}_x^{\alpha_x}}\right)^{-t_1+\mu_x} \left(-\kappa_y\mu_y\right)^{-t_2} dt_1 dt_2, \tag{A.13}$$

in which $\mathscr{A}_x$ and $\mathscr{K}_y$ are derived from (2.12) and (2.16). To further simplify the above expression, the variable transformation $s = t_1 - \mu_x$ is performed, which results in

$$f_Z(z) = \frac{\alpha_x}{z\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)} \left(\frac{1}{2\pi i}\right)^2 \oint_{\mathscr{L}} \Gamma\left(\mu_y - \frac{\alpha_x}{2}s - t_2\right) \times \frac{\Gamma(\mu_x + s)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)} \left(\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\right)^{-s} \left(-\kappa_y\mu_y\right)^{-t_2} ds dt_2. \tag{A.14}$$

And, finally, the above integral can be interpreted as Fox H-function function using (2.1) with parameters provided in Table 3.1.

A.2.2 CDF

The CDF of the ratio of the α-μ by the κ-μ variates is obtained by replacing (A.14) at the CDF's definition resulting in

$$F_Z(z) = \int_0^z \frac{\alpha_x}{\tau\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\left(\frac{1}{2\pi i}\right)^2 \oint_{\mathscr{L}} \Gamma\left(\mu_y - \frac{\alpha_x}{2}s - t_2\right) \times \frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)}\left(\left(\frac{\tau}{u_{\alpha\kappa}}\right)^{\alpha_x}\right)^{-s}\left(-\kappa_y\mu_y\right)^{-t_2} ds dt_2 d\tau. \tag{A.15}$$

After changing the order of integration the inner integral can be solved using the identity $\int_0^z \tau^{-\alpha_x s-1} = \Gamma(-s)/(\alpha_x\Gamma(1-s))z^{-\alpha_x s}$ which results in

$$F_Z(z) = \frac{1}{\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\left(\frac{1}{2\pi i}\right)^2 \oint_{\mathscr{L}} \Gamma\left(\mu_y - \frac{\alpha_x}{2}s - t_2\right)\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)} \times \frac{\Gamma(-s)}{\Gamma(1-s)}\left(\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\right)^{-s}\left(-\kappa_y\mu_y\right)^{-t_2} ds dt_2. \tag{A.16}$$

The Fox H-function parameters for the ratio of the α-μ by the κ-μ variates are readily deduced by comparing (A.16) with (2.1).

A.3 The α-μ/η-μ Distribution

A.3.1 PDF

The PDF for the ratio $Z = X/Y$, in which X is α-μ distributed with parameters $\{\alpha_x, \mu_x, \hat{r}_x\}$ and Y follows the η-μ model with parameters $\{\eta_y, \mu_y, \hat{r}_y\}$, is obtained by replacing (2.9) and (2.17) in (3.1) which, after some minor algebraic manipulations, gives

$$f_Z(z) = \frac{2h_y^{\mu_y}\alpha_x}{\Gamma(\mu_x)\Gamma\left(2\mu_y\right)}\frac{z^{\alpha_x\mu_x-1}}{\mathscr{A}_x^{\alpha_x\mu_x}\mathscr{E}_y^{4\mu_y}}\int_0^\infty y^{\alpha_x\mu_x+4\mu_y-1}\exp\left(-\frac{h_y y^2}{\mathscr{E}_y^2}\right) \times \exp\left(-\frac{(zy)^{\alpha_x}}{\mathscr{A}_x^{\alpha_x}}\right){}_0F_1\left(;\mu_y+\frac{1}{2};\frac{y^4H_y^2}{4\mathscr{E}_y^4}\right)dy. \tag{A.17}$$

After replacing an exponential and the hypergeometric functions with (A.1) and (A.2) respectively, the above integral becomes

$$f_Z(z) = \frac{2h_y^{\mu_y}\alpha_x\Gamma\left(\mu_y+\frac{1}{2}\right)}{\Gamma(\mu_x)\Gamma\left(2\mu_y\right)}\frac{z^{\alpha_x\mu_x-1}}{\mathscr{A}_x^{\alpha_x\mu_x}\mathscr{E}_y^{4\mu_y}}\left(\frac{1}{2\pi j}\right)^2\int_0^\infty y^{\alpha_x\mu_x+4\mu_y-1}\exp\left(-\frac{h_y y^2}{\mathscr{E}_y^2}\right) \times \oiint \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(\frac{(zy)^{\alpha_x}}{\mathscr{A}_x^{\alpha_x}}\right)^{-t_1}\left(-\frac{y^4H_y^2}{4\mathscr{E}_y^4}\right)^{-t_2} dt_1 dt_2 dy. \tag{A.18}$$

By changing the order of integration and using (A.6) to solve the inner integral, it results, after some algebraic manipulations, in

$$\begin{aligned} f_Z(z) = & \frac{h_y^{-\mu_y}\alpha_x\Gamma\left(\mu_y+\frac{1}{2}\right)}{z\Gamma(\mu_x)\Gamma\left(2\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \Gamma\left(-2t_2+\frac{1}{2}\alpha_x\left(-t_1+\mu_x\right)+2\mu_y\right) \\ & \times\frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(\left(\frac{z\mathscr{E}_y}{\mathscr{A}_x\sqrt{h_y}}\right)^{\alpha_x}\right)^{-t_1+\mu_x}\left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} dt_1dt_2. \end{aligned} \tag{A.19}$$

Now, by performing the variable transformation $s=t_1-\mu_x$ and using the identity $\Gamma\left(\mu_y+\frac{1}{2}\right)/\Gamma\left(2\mu_y\right)=2^{1-2\mu_y}\sqrt{\pi}/\Gamma\left(\mu_y\right)$ [36, Equation (6.1.18)], the PDF is given as

$$\begin{aligned} f_Z(z) = & \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}\alpha_x}{z\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint_{\mathscr{L}} \Gamma\left(2\mu_y-\frac{\alpha_x s}{2}-2t_2\right) \\ & \times\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y+\frac{1}{2}-t_2\right)}\left(\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\right)^{-s}\left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} ds dt_2, \end{aligned} \tag{A.20}$$

which can finally be interpreted as a Fox H-function using (2.1) with parameters given in Table 3.1.

A.3.2 CDF

The CDF of the ratio of the α-μ by the η-μ variates is obtained by replacing (A.20) at the CDF's definition resulting in

$$\begin{aligned} F_Z(z) = & \int_0^z \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}\alpha_x}{\tau\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint_{\mathscr{L}} \Gamma\left(2\mu_y-\frac{\alpha_x s}{2}-2t_2\right) \\ & \times\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y+\frac{1}{2}-t_2\right)}\left(\left(\frac{\tau}{u_{\alpha\eta}}\right)^{\alpha_x}\right)^{-s}\left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} ds dt_2 d\tau. \end{aligned} \tag{A.21}$$

After changing the order of integration the inner integral can be solved using the identity $\int_0^z \tau^{-\alpha_x s-1}=\Gamma(-s)/(\alpha_x\Gamma(1-s))z^{-\alpha_x s}$ which results in

$$\begin{aligned} F_Z(z) = & \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}}{\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint_{\mathscr{L}} \Gamma\left(2\mu_y-\frac{\alpha_x s}{2}-2t_2\right)\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y+\frac{1}{2}-t_2\right)} \\ & \times\frac{\Gamma(-s)}{\Gamma(1-s)}\left(\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\right)^{-s}\left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} ds dt_2. \end{aligned} \tag{A.22}$$

The Fox H-function parameters for the ratio of two α-μ variates are readily deduced by comparing (A.22) with (2.1).

A.4 The κ-μ/κ-μ Distribution

A.4.1 PDF

Let $Z = X/Y$ be the ratio of two κ-μ variates with parameters $\{\kappa_x, \mu_x, \hat{r}_x\}$ and $\{\kappa_y, \mu_y, \hat{r}_y\}$ respectively for the variates X and Y. By replacing (2.13) in (3.1) with the appropriate parameters, it results in

$$f_Z(z) = \frac{4}{\exp\left(\kappa_x\mu_x + \kappa_y\mu_y\right)} \frac{z^{2\mu_x-1}}{\mathcal{K}_x^{2\mu_x}\mathcal{K}_y^{2\mu_y}} \int_0^\infty y^{2\left(\mu_x+\mu_y\right)-1} \exp\left(-y^2\left(\frac{z^2}{\mathcal{K}_x^2} + \frac{1}{\mathcal{K}_y^2}\right)\right) \\ \times {}_0\tilde{F}_1\left(;\mu_x;\frac{\mu_x\kappa_x(zy)^2}{\mathcal{K}_x^2}\right) {}_0\tilde{F}_1\left(;\mu_y;\frac{\mu_y\kappa_y y^2}{\mathcal{K}_y^2}\right) dy \tag{A.23}$$

Now, using the identity (A.2) for both the ${}_0F_1$ functions, it gives

$$f_Z(z) = \frac{4}{\exp\left(\kappa_x\mu_x + \kappa_y\mu_y\right)} \frac{z^{2\mu_x-1}}{\mathcal{K}_x^{2\mu_x}\mathcal{K}_y^{2\mu_y}} \left(\frac{1}{2\pi j}\right)^2 \int_0^\infty y^{2\left(\mu_x+\mu_y\right)-1} \exp\left(-y^2\left(\frac{z^2}{\mathcal{K}_x^2} + \frac{1}{\mathcal{K}_y^2}\right)\right) \\ \times \oint\!\!\oint_{\mathcal{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(-t_1+\mu_x)\Gamma\left(-t_2+\mu_y\right)} \left(-\frac{\mu_x\kappa_x(zy)^2}{\mathcal{K}_x^2}\right)^{-t_1} \left(-\frac{\mu_y\kappa_y y^2}{\mathcal{K}_y^2}\right)^{-t_2} dt_1 dt_2 dy. \tag{A.24}$$

At this point, the order of integration is shifted and the inner integral is solved with the help of (A.6) resulting, after some algebraic manipulations and using the constants (3.4), in

$$f_Z(z) = \frac{2(-\kappa_x\mu_x)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{z\exp\left(\kappa_x\mu_x + \kappa_y\mu_y\right)} \left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathcal{L}} \frac{\Gamma\left(-t_1-t_2+\mu_x+\mu_y\right)\Gamma(t_1)\Gamma(t_2)}{\Gamma(-t_1+\mu_x)\Gamma\left(-t_2+\mu_y\right)} \\ \times (-\kappa_x\mu_x\nu_{\kappa\kappa})^{-t_1+\mu_x}\left(-\kappa_y\mu_y(1-\nu_{\kappa\kappa})\right)^{-t_2+\mu_y} dt_1 dt_2. \tag{A.25}$$

Performing the variable transformations $s_1 = t_1 - \mu_x$ and $s_2 = t_2 - \mu_y$ after some algebraic manipulations results in

$$f_Z(z) = \frac{2(-\kappa_x\mu_x)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{z\exp\left(\kappa_x\mu_x + \kappa_y\mu_y\right)} \left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathcal{L}} \frac{\Gamma(-s_1-s_2)\Gamma(\mu_x+s_1)\Gamma\left(\mu_y+s_2\right)}{\Gamma(-s_1)\Gamma(-s_2)} \\ \times (-\kappa_x\mu_x\nu_{\kappa\kappa})^{-s_1}\left(-\kappa_y\mu_y(1-\nu_{\kappa\kappa})\right)^{-s_2} ds_1 ds_2. \tag{A.26}$$

The above integral can now be interpreted as a Fox H-function using (2.1), with parameters provided in Table 3.1.

A.4.2 CDF

The CDF of the ratio of two κ-μ variates is obtained by replacing (A.26) at the CDF's definition resulting in

$$F_Z(z) = \int_0^z \frac{2\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{\tau\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2)\Gamma(\mu_x+s_1)\Gamma\left(\mu_y+s_2\right)}{\Gamma(-s_1)\Gamma(-s_2)} \times\left(-\frac{\kappa_x\mu_x\tau^2}{\tau^2+u_{\kappa\kappa}^2}\right)^{-s_1}\left(-\frac{\kappa_y\mu_y u_{\kappa\kappa}^2}{\tau^2+u_{\kappa\kappa}^2}\right)^{-s_2} ds_1 ds_2 d\tau. \tag{A.27}$$

By changing the order of integration it results in

$$F_Z(z) = \frac{2\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2)\Gamma(\mu_x+s_1)\Gamma\left(\mu_y+s_2\right)}{\Gamma(-s_1)\Gamma(-s_2)} \times\left(-\kappa_x\mu_x\right)^{-s_1}\left(-\kappa_y\mu_y u_{\kappa\kappa}^2\right)^{-s_2}\int_0^z \tau^{-2s_1-1}\left(\tau^2+u_{\kappa\kappa}^2\right)^{s_1+s_2} d\tau ds_1 ds_2. \tag{A.28}$$

and the inner integral can be solved with the help of [53, Equation (1.2.4.3)] which results, after some algebraic manipulations in

$$F_Z(z) = \frac{\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2)\Gamma(\mu_x+s_1)\Gamma\left(\mu_y+s_2\right)}{\Gamma(1-s_1)\Gamma(-s_2)} \times\left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\kappa}^2}\right)^{-s_1}\left(-\kappa_y\mu_y\right)^{-s_2} {}_2F_1\left(-s_1,-s_1-s_2;1-s_1;-\frac{z^2}{u_{\kappa\kappa}^2}\right) ds_1 ds_2. \tag{A.29}$$

Finally, the hypergeometric function is written in terms of its Mellin-Barnes contour integral representation and, after some algebraic manipulations, the CDF results in

$$F_Z(z) = \frac{\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2-s_3)\Gamma(-s_1-s_3)}{\Gamma(1-s_1-s_3)\Gamma(-s_1)\Gamma(-s_2)} \times\Gamma(\mu_x+s_1)\Gamma\left(\mu_y+s_2\right)\Gamma(s_3)\left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\kappa}^2}\right)^{-s_1}\left(-\kappa_y\mu_y\right)^{-s_2}\left(\frac{z^2}{u_{\kappa\kappa}^2}\right)^{-s_3} ds_1 ds_2 ds_3. \tag{A.30}$$

The Fox H-function parameters for the ratio of two κ-μ variates are readily deduced by comparing (A.30) with (2.1).

A.5 The κ-μ/η-μ Distribution

A.5.1 PDF

Let $Z = X/Y$ be the ratio of the κ-μ by the η-μ variates with parameters $\{\kappa_x, \mu_x, \hat{r}_x\}$ and $\{\eta_y, \mu_y, \hat{r}_y\}$ respectively. Its PDF can be obtained by replacing (2.13) and (2.17) in

(3.1) which results, after some simplifications and using the constants defined in (2.16) and (2.20), in

$$f_Z(z) = \frac{4h_y^{\mu_y}}{\exp(\kappa_x\mu_x)\Gamma(2\mu_y)}\frac{z^{2\mu_x-1}}{\mathcal{K}_x^{2\mu_x}\mathcal{E}_y^{4\mu_y}}\int_0^\infty y^{2(\mu_x+2\mu_y)-1}\exp\left(-y^2\left(\frac{h_y}{\mathcal{E}_y^2}+\frac{z^2}{\mathcal{K}_x^2}\right)\right) \times {}_0\tilde{F}_1\left(;\mu_x;\frac{(zy)^2\kappa_x\mu_x}{\mathcal{K}_x^2}\right){}_0F_1\left(;\mu_y+\frac{1}{2};\frac{y^4H_y^2}{4\mathcal{E}_y^4}\right)dy. \tag{A.31}$$

This integral may be solved by replacing the hypergeometric functions with their Mellin-Barnes contour integral representations as in (A.2) resulting, after some minor algebraic manipulations, in

$$f_Z(z) = \frac{2^{3-2\mu_y}\sqrt{\pi}h_y^{\mu_y}}{\Gamma(\mu_y)\exp(\kappa_x\mu_x)}\frac{z^{2\mu_x-1}}{\mathcal{K}_x^{2\mu_x}\mathcal{E}_y^{4\mu_y}}\left(\frac{1}{2\pi j}\right)^2\int_0^\infty y^{2(\mu_x+2\mu_y)-1}\exp\left(-y^2\left(\frac{h_y}{\mathcal{E}_y^2}+\frac{z^2}{\mathcal{K}_x^2}\right)\right) \times \oint\!\!\oint_{\mathcal{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(-t_1+\mu_x)\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(-\frac{(zy)^2\kappa_x\mu_x}{\mathcal{K}_x^2}\right)^{-t_1}\left(-\frac{y^4H_y^2}{4\mathcal{E}_y^4}\right)^{-t_2}dt_1dt_2dy. \tag{A.32}$$

At this point, the order of integration is shifted and the inner integral can be solved using (A.6). After some minor algebraic manipulations, the PDF is given by

$$f_Z(z) = \frac{4\sqrt{\pi}(-1)^{-\mu_y-\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_y)\exp(\kappa_x\mu_x)\left(H_y^2\right)^{\mu_y}(\kappa_x\mu_x)^{\mu_x}}\left(\frac{1}{2\pi j}\right)^2\oint\!\!\oint_{\mathcal{L}}\Gamma\left(-t_1-2t_2+\mu_x+2\mu_y\right) \times \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(-t_1+\mu_x)\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(-\kappa_x\mu_x\nu_{\kappa\eta}\right)^{-t_1+\mu_x}\left(-\frac{H_y^2\left(1-\nu_{\kappa\eta}\right)^2}{4h_y^2}\right)^{-t_2+\mu_y}dt_1dt_2. \tag{A.33}$$

Finally, the variable transformations $s_1 = t_1 - \mu_x$ and $s_2 = t_2 - \mu_y$ are performed which gives

$$f_Z(z) = \frac{4\sqrt{\pi}(-1)^{-\mu_y-\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_y)\exp(\kappa_x\mu_x)\left(H_y^2\right)^{\mu_y}(\kappa_x\mu_x)^{\mu_x}}\left(\frac{1}{2\pi j}\right)^2\oint\!\!\oint_{\mathcal{L}}\Gamma(-s_1-2s_2) \times \frac{\Gamma(s_1+\mu_x)\Gamma(s_2+\mu_y)}{\Gamma(-s_1)\Gamma\left(\frac{1}{2}-s_2\right)}\left(-\nu_{\kappa\eta}\kappa_x\mu_x\right)^{-s_1}\left(-\frac{H_y^2}{4h_y^2}\left(1-\nu_{\kappa\eta}\right)^2\right)^{-s_2}ds_1ds_2 \tag{A.34}$$

The above integral may be interpreted as Fox H-function as in (2.1) with parameters provided in Table 3.1.

A.5.2 CDF

The CDF of the ratio of the κ-μ variate by the η-μ is obtained by replacing (A.34) at the CDF's definition resulting in

$$F_Z(z) = \int_0^z \frac{4\sqrt{\pi}(-1)^{-\mu_y-\mu_x} h_y^{\mu_y}}{\tau\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(H_y^2\right)^{\mu_y}\left(\kappa_x\mu_x\right)^{\mu_x}} \left(\frac{1}{2\pi j}\right)^2 \oint\oint_{\mathscr{L}} \Gamma\left(-s_1-2s_2\right)$$
$$\times \frac{\Gamma\left(s_1+\mu_x\right)\Gamma\left(s_2+\mu_y\right)}{\Gamma\left(-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{\kappa_x\mu_x\tau^2}{\tau^2+u_{\kappa\eta}^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\left(\frac{u_{\kappa\eta}^2}{\tau^2+u_{\kappa\eta}^2}\right)^2\right)^{-s_2} ds_1 ds_2 d\tau. \tag{A.35}$$

By changing the order of integration, it results in

$$F_Z(z) = \frac{4\sqrt{\pi}(-1)^{-\mu_y-\mu_x} h_y^{\mu_y}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(H_y^2\right)^{\mu_y}\left(\kappa_x\mu_x\right)^{\mu_x}} \left(\frac{1}{2\pi j}\right)^2 \oint\oint_{\mathscr{L}} \Gamma\left(-s_1-2s_2\right)\Gamma\left(s_1+\mu_x\right)$$
$$\times \frac{\Gamma\left(s_2+\mu_y\right)}{\Gamma\left(-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\kappa_x\mu_x\right)^{-s_1} \left(-\frac{H_y^2 u_{\kappa\eta}^4}{4h_y^2}\right)^{-s_2} \int_0^z \tau^{-2s_1-1}\left(\tau^2+u_{\kappa\eta}^2\right)^{s_1+2s_2} d\tau ds_1 ds_2, \tag{A.36}$$

and the inner integral can be solved with the help of [53, Equation (1.2.4.3)] which results, after some algebraic manipulations, in

$$F_Z(z) = \frac{2\sqrt{\pi}(-1)^{-\mu_y-\mu_x} h_y^{\mu_y}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(H_y^2\right)^{\mu_y}\left(\kappa_x\mu_x\right)^{\mu_x}} \left(\frac{1}{2\pi j}\right)^2 \oint\oint_{\mathscr{L}} \frac{\Gamma\left(-s_1-2s_2\right)\Gamma\left(s_1+\mu_x\right)}{\Gamma\left(1-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)}$$
$$\times \Gamma\left(s_2+\mu_y\right)\left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\eta}^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2} {}_2F_1\left(-s_1,-s_1-2s_2;1-s_1;-\frac{z^2}{u_{\kappa\eta}^2}\right) ds_1 ds_2. \tag{A.37}$$

Finally, the hypergeometric function is written in terms of its Mellin-Barnes contour integral representation using (A.3) and, after some algebraic manipulations, the CDF results in

$$F_Z(z) = \frac{2\sqrt{\pi}(-1)^{-\mu_y-\mu_x} h_y^{\mu_y}\left(H_y^2\right)^{-\mu_y}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(\kappa_x\mu_x\right)^{\mu_x}} \left(\frac{1}{2\pi j}\right)^3 \oint\oint\oint_{\mathscr{L}} \frac{\Gamma\left(-s_1-s_3\right)\Gamma\left(-s_1-2s_2-s_3\right)}{\Gamma\left(1-s_1-s_3\right)}$$
$$\times \frac{\Gamma\left(\mu_x+s_1\right)\Gamma\left(\mu_y+s_2\right)\Gamma\left(s_3\right)}{\Gamma\left(-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\eta}^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2} \left(\frac{z^2}{u_{\kappa\eta}^2}\right)^{-s_3} ds_1 ds_2 ds_3. \tag{A.38}$$

The Fox H-function parameters for the ratio of two κ-μ variates are readily deduced by comparing (A.38) with (2.1).

A.6 The η-μ/η-μ Distribution

A.6.1 PDF

The PDF for the ratio of two η-μ variates can be obtained by replacing (2.17) in (3.1) with the appropriate subscripts, which, after some simplifications, gives

$$f_Z(z) = \frac{4h_x^{\mu_x}h_y^{\mu_y}}{\Gamma(2\mu_x)\Gamma(2\mu_y)}\frac{z^{4\mu_x-1}}{\mathcal{E}_x^{4\mu_x}\mathcal{E}_y^{4\mu_y}}\int_0^\infty y^{4(\mu_x+\mu_y)-1}\exp\left(-y^2\left(\frac{h_x z^2}{\mathcal{E}_x^2}+\frac{h_y}{\mathcal{E}_y^2}\right)\right) \times {}_0F_1\left(;\mu_x+\frac{1}{2};\frac{(zy)^4H_x^2}{4\mathcal{E}_x^4}\right){}_0F_1\left(;\mu_y+\frac{1}{2};\frac{y^4H_y^2}{4\mathcal{E}_y^4}\right)dy. \tag{A.39}$$

A solution in terms of the Fox H-function is possible by replacing the hypergeometric functions in terms of their Mellin-Barnes contour integral using (A.2). Then, the PDF is given by

$$f_Z(z) = \frac{4^{2-\mu_x-\mu_y}\pi h_x^{\mu_x}h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)}\frac{z^{4\mu_x-1}}{\mathcal{E}_x^{4\mu_x}\mathcal{E}_y^{4\mu_y}}\left(\frac{1}{2\pi j}\right)^2\int_0^\infty y^{4(\mu_x+\mu_y)-1}\exp\left(-y^2\left(\frac{h_x z^2}{\mathcal{E}_x^2}+\frac{h_y}{\mathcal{E}_y^2}\right)\right) \times \oint\!\!\oint_{\mathcal{L}}\frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_x\right)\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(-\frac{(zy)^4H_x^2}{4\mathcal{E}_x^4}\right)^{-t_1}\left(-\frac{y^4H_y^2}{4\mathcal{E}_y^4}\right)^{-t_2}dt_1dt_2dy. \tag{A.40}$$

After changing the order of integration, the inner integral can be solved with the help of (A.6). After some algebraic manipulations and using the constants defined in (3.4), the PDF is obtained as

$$f_Z(z) = \frac{8\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}}\left(\frac{1}{2\pi j}\right)^2\oint\!\!\oint_{\mathcal{L}}\Gamma\left(-2\left(t_1+t_2-\mu_x-\mu_y\right)\right) \times \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_x\right)\Gamma\left(\frac{1}{2}-t_2+\mu_y\right)}\left(-\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{-t_1+\mu_x}\left(-\frac{H_y^2\left(1-\nu_{\eta\eta}\right)^2}{4h_y^2}\right)^{-t_2+\mu_y}dt_1dt_2. \tag{A.41}$$

Finally, the variable transformations $s_1 = t_1 - \mu_x$ and $s_2 = t_2 - \mu_y$ are performed. Then, the PDF is given as

$$f_Z(z) = \frac{8\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}}\left(\frac{1}{2\pi j}\right)^2\oint\!\!\oint_{\mathcal{L}}\Gamma(-2(s_1+s_2)) \times \frac{\Gamma(s_1+\mu_x)\Gamma(s_2+\mu_y)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)}\left(-\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{-s_1}\left(-\frac{H_y^2}{4h_y^2}\left(1-\nu_{\eta\eta}\right)^2\right)^{-s_2}ds_1ds_2, \tag{A.42}$$

which can be interpreted as Fox H-function as in (2.1) with parameters provided in Table 3.1.

A.6.2 CDF

The CDF of the ratio of two η-μ variates is obtained by replacing (A.42) at the CDF's definition which results in

$$F_Z(z) = \int_0^z \frac{8\pi(-1)^{-\mu_x-\mu_y} h_x^{\mu_x} h_y^{\mu_y}}{\tau\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \Gamma(-2(s_1+s_2))$$
$$\times \frac{\Gamma(s_1+\mu_x)\Gamma(s_2+\mu_y)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{H_x^2}{4h_x^2}\left(\frac{\tau^2}{\tau^2+u_{\eta\eta}^2}\right)^2\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\left(\frac{u_{\eta\eta}^2}{\tau^2+u_{\eta\eta}^2}\right)^2\right)^{-s_2} ds_1 ds_2 d\tau. \tag{A.43}$$

By changing the order of integration, the CDF is written as

$$F_Z(z) = \frac{8\pi(-1)^{-\mu_x-\mu_y} h_x^{\mu_x} h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \Gamma(-2(s_1+s_2))$$
$$\times \frac{\Gamma(s_1+\mu_x)\Gamma(s_2+\mu_y)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{H_x^2}{4h_x^2}\right)^{-s_1} \left(-\frac{H_y^2 u_{\eta\eta}^4}{4h_y^2}\right)^{-s_2} \int_0^z \frac{\tau^{-1-4s_1}}{\left(\tau^2+u_{\eta\eta}^2\right)^{-2s_1-2s_2}} d\tau ds_1 ds_2. \tag{A.44}$$

and the inner integral can be solved with the help of [53, Equation (1.2.4.3)].After some algebraic manipulations, the CDF results in

$$F_Z(z) = \frac{4\pi(-1)^{-\mu_x-\mu_y} h_x^{\mu_x} h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \Gamma(-2(s_1+s_2))$$
$$\times \frac{\Gamma(s_1+\mu_x)\Gamma(s_2+\mu_y)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{H_x^2 z^4}{4u_{\eta\eta}^4 h_x^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2} \frac{\Gamma(-2s_1)}{\Gamma(1-2s_1)} \tag{A.45}$$
$$\times\ {}_2F_1\left(-2s_1, -2s_1-2s_2; 1-2s_1; -\frac{z^2}{u_{\eta\eta}^2}\right) ds_1 ds_2.$$

Finally, the hypergeometric function is written in terms of its Mellin-Barnes contour integral representation using (A.3) and, after some algebraic manipulations, the CDF is obtained as

$$F_Z(z) = \frac{4\pi(-1)^{-\mu_x-\mu_y} h_x^{\mu_x} h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \left(\frac{1}{2\pi i}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-2s_1-s_3)\Gamma(-2(s_1+s_2)-s_3)}{\Gamma(1-2s_1-s_3)}$$
$$\times \frac{\Gamma(\mu_x+s_1)\Gamma(\mu_y+s_2)\Gamma(s_3)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{H_x^2 z^4}{4u_{\eta\eta}^4 h_x^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2} \left(\frac{z^2}{u_{\eta\eta}^2}\right)^{-s_3} ds_1 ds_2 ds_3. \tag{A.46}$$

The Fox H-function parameters for CDF of the ratio of two κ-μ variates are readily deduced by comparing (A.38) with (2.1).

Appendix B

Derivation of the Statistics of the Product Distribution

In Chapter 4, the first order statistics for the product of random envelopes taken from the α-μ, κ-μ and η-μ distribution were presented in terms of the multivariable Fox H-function. The mathematical derivations are provided here. Likewise the ratio statistics, the Kummer's confluent hypergeometric function ${}_1F_1$ is required to be written in terms of its Mellin-Barnes contour integral representation given in [41, Equation (7.2.3.12)]

$$ {}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \frac{\Gamma(t)\Gamma(a-t)}{\Gamma(b-t)} (-z)^{-t}\, dt \tag{B.1} $$

B.1 The α-μ × α-μ Distribution

B.1.1 PDF

Let $Z = R_1 R_2 > 0$ be the product of two α-μ variates with parameters $\{\alpha_1, \mu_1, \hat{r}_1\}$ and $\{\alpha_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.11) in (4.3) which results in

$$ \mathbb{E}[Z^s] = \frac{\Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{\alpha_2}\right)}{\Gamma(\mu_1)\Gamma(\mu_2)} \left(\frac{1}{\mathscr{A}_1 \mathscr{A}_2}\right)^{-s}, \tag{B.2} $$

in which $\mathscr{A}_1$ and $\mathscr{A}_2$ are derived from (2.12) with the appropriate subscripts. Now, the PDF of the product of two α-μ variates is obtained by replacing (B.2) in (2.8) which gives

$$ f_Z(z) = \frac{1}{z\Gamma(\mu_1)\Gamma(\mu_2)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{\alpha_2}\right)\left(\frac{z}{\mathscr{A}_1 \mathscr{A}_2}\right)^{-s} ds \tag{B.3} $$

By comparing (B.3) with (2.1), the parameters for the Fox H-function representation of the PDF of the product of two α-μ variates are readily obtained as they are provided in Table 4.1.

B.1.2 CDF

The Fox H-function representation for the CDF of the product of two α-μ variates is obtained by replacing (B.3) in (4.2) which gives

$$F_Z(z) = \int_0^z \frac{1}{\tau\Gamma(\mu_1)\Gamma(\mu_2)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{\alpha_2}\right)\left(\frac{\tau}{\mathscr{A}_1\mathscr{A}_2}\right)^{-s} ds d\tau \qquad \text{(B.4)}$$

Then changing the order of integration, the CDF results in

$$F_Z(z) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{\alpha_2}\right)\left(\frac{1}{\mathscr{A}_1\mathscr{A}_2}\right)^{-s} \int_0^z \tau^{-s-1} d\tau ds. \qquad \text{(B.5)}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-s-1} = \frac{\Gamma(-s)}{\Gamma(1-s)} z^{-s} \qquad \text{(B.6)}$$

Therefore, the CDF is obtained as

$$F_Z(z) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{\alpha_2}\right)\frac{\Gamma(-s)}{\Gamma(1-s)}\left(\frac{z}{\mathscr{A}_1\mathscr{A}_2}\right)^{-s} ds. \qquad \text{(B.7)}$$

Comparing (B.7) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

B.2 The α-μ $\times$ κ-μ Distribution

B.2.1 PDF

Let $Z = R_1R_2 > 0$ be the product of the α-μ by the κ-μ variates with parameters $\{\alpha_1, \mu_1, \hat{r}_1\}$ and $\{\kappa_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.11) and (2.15) in (4.3), resulting in

$$\mathbb{E}[Z^s] = \frac{\Gamma\left(\frac{s}{\alpha_1} + \mu_1\right)\Gamma\left(\frac{s}{2} + \mu_2\right)}{\Gamma(\mu_1)\Gamma(\mu_2)} {}_1F_1\left(-\frac{s}{2}; \mu_2; -\kappa_2\mu_2\right)\left(\frac{1}{\mathscr{A}_1\mathscr{K}_2}\right)^{-s}, \qquad \text{(B.8)}$$

in which $\mathscr{A}_1$ and $\mathscr{K}_2$ are derived from (2.12) and (2.16) respectively. The PDF of the product of the α-μ by the κ-μ variates is obtained by replacing (B.8) in (2.8), which results in

$$f_Z(z) = \frac{1}{z}\frac{1}{2\pi j} \oint_{\mathscr{L}} \frac{\Gamma\left(\frac{s}{\alpha_1} + \mu_1\right)\Gamma\left(\frac{s}{2} + \mu_2\right)}{\Gamma(\mu_1)\Gamma(\mu_2)} {}_1F_1\left(-\frac{s}{2}; \mu_2; -\kappa_2\mu_2\right)\left(\frac{z}{\mathscr{A}_1\mathscr{K}_2}\right)^{-s} ds \qquad \text{(B.9)}$$

Now, the hypergeometric function is written in terms of its Mellin-Barnes contour integral representation (B.1). After some simplifications, the PDF results in

$$f_Z(z) = \frac{1}{z\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(-t-s/2)\Gamma(t)\Gamma\left(\mu_1 + \frac{s}{\alpha_1}\right)\Gamma\left(\mu_2 + \frac{s}{2}\right)}{\Gamma(\mu_2 - t)\Gamma(-s/2)} (\kappa_2\mu_2)^{-t}\left(\frac{z}{\mathscr{A}_1\mathscr{K}_2}\right)^{-s} dtds. \qquad \text{(B.10)}$$

Of course the above integral can already be written in terms of the Fox H-function, although, to obtain the parameters provided in Table 4.1 the variable substitution $x = s/2$ is performed, which results in

$$f_Z(z) = \frac{2}{z\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t-x)\Gamma(t)\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(\mu_2+x)}{\Gamma(\mu_2-t)\Gamma(-x)}(\kappa_2\mu_2)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{-x} dt dx \tag{B.11}$$

Now, comparing (B.11) with (2.1), the parameters in Table 4.1 are obtained in an exact manner.

B.2.2 CDF

The Fox H-function representation for the CDF of the product of the α-μ by the κ-μ variates is obtained by replacing (B.11) in (4.2) which gives

$$\begin{aligned} F_Z(z) = \int_0^z \frac{2}{\tau\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t-x)\Gamma(t)}{\Gamma(\mu_2-t)\Gamma(-x)} \\ \times\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(\mu_2+x)(\kappa_2\mu_2)^{-t}\left(\frac{\tau^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{-x} dt dx d\tau. \end{aligned} \tag{B.12}$$

Then changing the order of integration, the CDF results in

$$\begin{aligned} F_Z(z) = \frac{2}{\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t-x)\Gamma(t)}{\Gamma(\mu_2-t)\Gamma(-x)} \\ \times\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(\mu_2+x)(\kappa_2\mu_2)^{-t}\left(\frac{1}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{-x}\int_0^z \tau^{-2x-1} d\tau dt dx. \end{aligned} \tag{B.13}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-2x-1} = \frac{\Gamma(-x)}{2\Gamma(1-x)} z^{-2x}. \tag{B.14}$$

Therefor, the CDF is obtained as

$$\begin{aligned} F_Z(z) = \frac{1}{\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t-x)\Gamma(t)}{\Gamma(\mu_2-t)\Gamma(1-x)} \\ \times\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(\mu_2+x)(\kappa_2\mu_2)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{-x} dt dx \end{aligned} \tag{B.15}$$

Comparing (B.15) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

B.3 The α-μ $\times$ η-μ Distribution

B.3.1 PDF

Let $Z = R_1 R_2 > 0$ be the product of the α-μ by the η-μ variates with parameters $\{\alpha_1, \mu_1, \hat{r}_1\}$ and $\{\eta_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.11) and (2.19) in (4.3) which results in

$$\mathbb{E}[Z^s] = \frac{\Gamma\left(\frac{s}{\alpha_1}+\mu_1\right)\Gamma\left(\frac{s}{2}+2\mu_2\right)}{\Gamma(\mu_1)\Gamma(2\mu_2)}\, {}_2F_1\left(\frac{1}{2}-\frac{s}{4}, -\frac{s}{4}; \mu_2+\frac{1}{2}; \frac{H_2^2}{h_2^2}\right)\left(\frac{1}{\mathscr{A}_1\mathscr{E}_2}\right)^{-s}, \tag{B.16}$$

in which $\mathscr{A}_1$ and $\mathscr{E}_2$ are derived from (2.12) and (2.20) respectively. The PDF is obtained from (2.8) by replacing the s-th moment with (B.16) which gives

$$f_Z(z) = \frac{1}{z}\frac{1}{2\pi j}\oint_{\mathscr{L}} \frac{\Gamma\left(\frac{s}{\alpha_1}+\mu_1\right)\Gamma\left(\frac{s}{2}+2\mu_2\right)}{\Gamma(\mu_1)\Gamma(2\mu_2)}\, {}_2F_1\left(\frac{1}{2}-\frac{s}{4}, -\frac{s}{4}; \mu_2+\frac{1}{2}; \frac{H_2^2}{h_2^2}\right)\left(\frac{s}{\mathscr{A}_1\mathscr{E}_2}\right)^{-s} ds, \tag{B.17}$$

At this point, the hypergeometric function is replaced by its contour integral representation (A.3). The PDF results in

$$\begin{aligned} f_Z(z) = \frac{\Gamma\left(\mu_2+\frac{1}{2}\right)}{z\Gamma(\mu_1)\Gamma(2\mu_2)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(t)\Gamma\left(\frac{1}{2}-\frac{s}{4}-t\right)\Gamma\left(-\frac{s}{4}-t\right)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)} \\ \times \frac{\Gamma\left(\mu_1+\frac{s}{\alpha_1}\right)\Gamma\left(2\mu_2+\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{4}\right)\Gamma\left(-\frac{s}{4}\right)}\left(-\frac{H_2^2}{h_2^2}\right)^{-t}\left(\frac{z}{\mathscr{A}_1\mathscr{E}_2}\right)^{-s} dt ds. \end{aligned} \tag{B.18}$$

This expression can be further simplified, by using the duplication formula of the gamma function $\Gamma\left(a+\frac{1}{2}\right)/\Gamma(2a) = 2^{1-2a}\sqrt{\pi}/\Gamma(a)$ [36, Equation (6.1.18)] and the variable transformation $x = s/2$, to

$$\begin{aligned} f_Z(z) = \frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(-2t-x)\Gamma(t)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)} \\ \times \frac{\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(2\mu_2+x)}{\Gamma(-x)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{-x} dt dx. \end{aligned} \tag{B.19}$$

The parameters for the Fox H-function representation of the PDF of the product of α-μ $\times$ η-μ is obtained by comparing (B.19) with (2.1) as they are provided in Table 4.1.

B.3.2 CDF

The Fox H-function representation for the CDF of the product of the α-μ by the η-μ variates is obtained by replacing (B.19) in (4.2), which gives

$$F_Z(z) = \int_0^z \frac{4^{1-\mu_2}\sqrt{\pi}}{\tau\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(-2t-x)\Gamma(t)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)} \times \frac{\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(2\mu_2+x)}{\Gamma(-x)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t}\left(\frac{\tau^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{-x} dt\,dx\,d\tau. \tag{B.20}$$

Then, by changing the order of integration, the CDF results in

$$F_Z(z) = \frac{4^{1-\mu_2}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(-2t-x)\Gamma(t)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)} \times \frac{\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(2\mu_2+x)}{\Gamma(-x)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t}\left(\frac{1}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{-x}\int_0^z \tau^{-2x-1}\,d\tau\,dt\,dx. \tag{B.21}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-2x-1} = \frac{\Gamma(-x)}{2\Gamma(1-x)}z^{-2x} \tag{B.22}$$

Therefore, the CDF is obtained as

$$F_Z(z) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2 \oiint_{\mathscr{L}} \frac{\Gamma(-2t-x)\Gamma(t)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)} \times \frac{\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(2\mu_2+x)}{\Gamma(1-x)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{-x} dt\,dx. \tag{B.23}$$

Comparing (B.23) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

B.4 The κ-μ × κ-μ Distribution

B.4.1 PDF

Let $Z = R_1R_2 > 0$ be the product of two κ-μ variates with parameters $\{\kappa_1, \mu_1, \hat{r}_1\}$ and $\{\kappa_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.15) in (4.3), resulting in

$$\mathbb{E}[Z^s] = \frac{\Gamma\left(\frac{s}{2}+\mu_1\right)\Gamma\left(\frac{s}{2}+\mu_2\right)}{\Gamma(\mu_1)\Gamma(\mu_2)}\,{}_1F_1\left(-\frac{s}{2};\mu_1;-\kappa_1\mu_1\right){}_1F_1\left(-\frac{s}{2};\mu_2;-\kappa_2\mu_2\right)\left(\frac{1}{\mathscr{K}_1\mathscr{K}_2}\right)^{-s}. \tag{B.24}$$

The PDF of the product of two κ-μ variates is obtained by replacing (B.24) in (2.8). After replacing the hypergeometric functions by their contour integral representations, performing the variable transformation $x = s/2$ and some algebraic manipulations, the PDF is given by

$$f_Z(z) = \frac{2}{z}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma(\mu_2-t_2)} \times \frac{\Gamma(\mu_1+x)\Gamma(\mu_2+x)}{\Gamma(-x)^2}(\kappa_1\mu_1)^{-t_1}(\kappa_2\mu_2)^{-t_2}\left(\frac{z^2}{\mathcal{K}_1^2\mathcal{K}_2^2}\right)^{-x} dt_1dt_2dx \tag{B.25}$$

Comparing (B.25) with (2.1) the parameters of the Fox H-function are obtained completing the derivation.

B.4.2 CDF

The Fox H-function representation for the CDF of the product two κ-μ variates is obtained by replacing (B.25) in (4.2), which gives

$$F_Z(z) = \int_0^z \frac{2}{\tau}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma(\mu_2-t_2)} \times \frac{\Gamma(\mu_1+x)\Gamma(\mu_2+x)}{\Gamma(-x)^2}(\kappa_1\mu_1)^{-t_1}(\kappa_2\mu_2)^{-t_2}\left(\frac{\tau^2}{\mathcal{K}_1^2\mathcal{K}_2^2}\right)^{-x} dt_1dt_2dxd\tau. \tag{B.26}$$

Then changing the order of integration, the CDF results in

$$F_Z(z) = 2\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma(\mu_2-t_2)}\frac{\Gamma(\mu_1+x)\Gamma(\mu_2+x)}{\Gamma(-x)^2} \times (\kappa_1\mu_1)^{-t_1}(\kappa_2\mu_2)^{-t_2}\left(\frac{1}{\mathcal{K}_1^2\mathcal{K}_2^2}\right)^{-x}\int_0^z \tau^{-2x-1}d\tau dt_1dt_2dx. \tag{B.27}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-2x-1} = \frac{\Gamma(-x)}{2\Gamma(1-x)}z^{-2x} \tag{B.28}$$

Therefore, the CDF is obtained as

$$F_Z(z) = \left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma(\mu_2-t_2)} \times \frac{\Gamma(\mu_1+x)\Gamma(\mu_2+x)}{\Gamma(1-x)\Gamma(-x)}(\kappa_1\mu_1)^{-t_1}(\kappa_2\mu_2)^{-t_2}\left(\frac{z^2}{\mathcal{K}_1^2\mathcal{K}_2^2}\right)^{-x} dt_1dt_2dx. \tag{B.29}$$

Comparing (B.29) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

B.5 The κ-μ $\times$ η-μ Distribution

B.5.1 PDF

Let $Z = R_1 R_2 > 0$ be the product of the κ-μ by the η-μ variate with parameters $\{\kappa_1, \mu_1, \hat{r}_1\}$ and $\{\eta_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.15) and (2.19) in (4.3) resulting in

$$\mathbb{E}[Z^s] = \frac{\Gamma\left(\mu_1 + \frac{s}{2}\right)\Gamma\left(2\mu_2 + \frac{s}{2}\right)}{\Gamma(\mu_1)\Gamma(2\mu_2)} \, {}_1F_1\left(-\frac{s}{2}; \mu_1; -\kappa_1\mu_1\right) {}_2F_1\left(\frac{1}{2} - \frac{s}{4}, -\frac{s}{4}; \mu_2 + \frac{1}{2}; \frac{H_2^2}{h_2^2}\right)\left(\frac{1}{\mathcal{K}_1\mathcal{E}_2}\right)^{-s} \tag{B.30}$$

By substituting (B.30) in (2.8), putting the hypergeometric function in terms of its Mellin-Barnes contour integral representation with the help of (B.1) and (A.3), and performing the variable transformation $x = s/2$, the PDF is given as

$$\begin{aligned} f_Z(z) = & \frac{2\Gamma\left(\frac{1}{2} + \mu_2\right)}{z\Gamma(2\mu_2)}\left(\frac{1}{2\pi j}\right) \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1 - x)\Gamma\left(-t_2 - \frac{x}{2}\right)\Gamma\left(\frac{1}{2} - t_2 - \frac{x}{2}\right)}{\Gamma(\mu_1 - t_1)\Gamma\left(\mu_2 + \frac{1}{2} - t_2\right)} \\ & \times \frac{\Gamma(t_1)\Gamma(t_2)\Gamma(\mu_1 + x)\Gamma(2\mu_2 + x)}{\Gamma(-x)\Gamma\left(\frac{1}{2} - \frac{x}{2}\right)\Gamma\left(-\frac{x}{2}\right)} (\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathcal{K}_1^2\mathcal{E}_2^2}\right)^{-x} dt_1 dt_2 dx \end{aligned} \tag{B.31}$$

Further simplifications are achieved by using the duplication formula of the gamma function, which results in

$$\begin{aligned} f_Z(z) = & \frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1 - x)\Gamma(-2t_2 - x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1 - t_1)\Gamma\left(\mu_2 + \frac{1}{2} - t_2\right)} \\ & \times \frac{\Gamma(\mu_1 + x)\Gamma(2\mu_2 + x)}{\Gamma(-x)^2} (\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathcal{K}_1^2\mathcal{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \end{aligned} \tag{B.32}$$

When comparing (B.32) with (2.1) the parameters for the Fox H-function representation for the PDF of the product of a κ-μ by an η-μ variates are obtained.

B.5.2 CDF

The Fox H-function representation for the CDF of the product of the κ-μ by the η-μ variates is obtained by replacing (B.32) in (4.2) which gives

$$\begin{aligned} F_Z(z) = & \int_0^z \frac{4^{1-\mu_2}\sqrt{\pi}}{\tau\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-t_1 - x)\Gamma(-x - 2t_2)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1 - t_1)\Gamma\left(\mu_2 + \frac{1}{2} - t_2\right)} \\ & \times \frac{\Gamma(\mu_1 + x)\Gamma(2\mu_2 + x)}{\Gamma(-x)^2} (\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{\tau^2}{\mathcal{K}_1^2\mathcal{E}_2^2}\right)^{-x} dt_1 dt_2 dx d\tau. \end{aligned} \tag{B.33}$$

By changing the order of integration, the CDF results in

$$F_Z(z) = \frac{4^{1-\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathcal{L}} \frac{\Gamma(-t_1-x)\Gamma(-x-2t_2)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma\left(\mu_2+\frac{1}{2}-t_2\right)}$$
$$\times \frac{\Gamma(\mu_1+x)\Gamma(2\mu_2+x)}{\Gamma(-x)^2}(\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{1}{\mathcal{K}_1^2\mathcal{E}_2^2}\right)^{-x}\int_0^z \tau^{-2x-1} d\tau dt_1 dt_2 dx. \tag{B.34}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-2x-1} = \frac{\Gamma(-x)}{2\Gamma(1-x)} z^{-2x}. \tag{B.35}$$

Therefore, the CDF is obtained as

$$F_Z(z) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathcal{L}} \frac{\Gamma(-t_1-x)\Gamma(-x-2t_2)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma\left(\mu_2+\frac{1}{2}-t_2\right)}$$
$$\times \frac{\Gamma(\mu_1+x)\Gamma(2\mu_2+x)}{\Gamma(1-x)\Gamma(-x)}(\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathcal{K}_1^2\mathcal{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{B.36}$$

Comparing (B.36) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

B.6 The η-μ × η-μ Distribution

B.6.1 PDF

Let $Z = R_1 R_2 > 0$ be the product of two η-μ variates with parameters $\{\eta_1, \mu_1, \hat{r}_1\}$ and $\{\eta_2, \mu_2, \hat{r}_2\}$ respectively. The s-th moment of Z is obtained by replacing (2.19), with the appropriate subscripts, in (4.3), which results in

$$\mathbb{E}[Z^s] = \frac{\Gamma\left(2\mu_1+\frac{s}{2}\right)\Gamma\left(2\mu_2+\frac{s}{2}\right)}{\Gamma(2\mu_1)\Gamma(2\mu_2)} {}_2F_1\left(\frac{1}{2}-\frac{s}{4}, -\frac{s}{4}; \mu_1+\frac{1}{2}; \frac{H_1^2}{h_1^2}\right)$$
$$\times {}_2F_1\left(\frac{1}{2}-\frac{s}{4}, -\frac{s}{4}; \mu_2+\frac{1}{2}; \frac{H_2^2}{h_2^2}\right)\left(\frac{1}{\mathcal{E}_1\mathcal{E}_2}\right)^{-s}. \tag{B.37}$$

The PDF is obtained by substituting (B.37) in (2.8), and using (A.3) to replace the hypergeometric functions with their Mellin-Barnes contour integral representations. The PDF is given by

$$f_Z(z) = \frac{\Gamma\left(\mu_1+\frac{1}{2}\right)\Gamma\left(\mu_2+\frac{1}{2}\right)}{z\Gamma(2\mu_1)\Gamma(2\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathcal{L}} \frac{\Gamma\left(-\frac{s}{4}-t_1\right)\Gamma\left(\frac{1}{2}-\frac{s}{4}-t_1\right)\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-\frac{s}{4}\right)^2\Gamma\left(-\frac{s}{4}\right)^2}$$
$$\times \frac{\Gamma\left(\frac{1}{2}-\frac{s}{4}-t_2\right)\Gamma\left(-\frac{s}{4}-t_2\right)\Gamma\left(\frac{s}{2}+2\mu_1\right)\Gamma\left(\frac{s}{2}+2\mu_2\right)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)}\left(-\frac{H_1^2}{h_1^2}\right)^{-t_1}\left(-\frac{H_2^2}{h_2^2}\right)^{-t_2}\left(\frac{z}{\mathcal{E}_1\mathcal{E}_2}\right)^{-s} dt_1 dt_2 ds. \tag{B.38}$$

Further simplifications are achieved by using the duplication formula of the gamma function, and after performing the change of variable $x = s/2$, the PDF for the product of two η-μ variates is obtained as

$$f_Z(z) = \frac{2^{3-2\mu_1-2\mu_2}\pi}{z\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-2t_1-x)\Gamma(-2t_2-x)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)} \times \frac{\Gamma(t_1)\Gamma(t_2)\Gamma(x+2\mu_1)\Gamma(x+2\mu_2)}{\Gamma(-x)^2}\left(-\frac{H_1^2}{4h_1^2}\right)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{B.39}$$

Now, by comparing (B.39) with (2.1), the parameters for the Fox H-function representation for the PDF of product of two η-μ variates are readily deduced.

B.6.2 CDF

The Fox H-function representation for the CDF of the product of two η-μ variates is obtained by replacing (B.39) in (4.2), which gives

$$F_Z(z) = \int_0^z \frac{2^{3-2\mu_1-2\mu_2}\pi}{\tau\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-2t_1-x)\Gamma(-2t_2-x)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)} \times \frac{\Gamma(t_1)\Gamma(t_2)\Gamma(x+2\mu_1)\Gamma(x+2\mu_2)}{\Gamma(-x)^2}\left(\frac{H_1^2}{4h_1^2}\right)^{-t_1}\left(\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{\tau^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{-x} dt_1 dt_2 dx d\tau. \tag{B.40}$$

By changing the order of integration, the CDF results in

$$F_Z(z) = \frac{2^{3-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-2t_1-x)\Gamma(-2t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)} \times \frac{\Gamma(x+2\mu_1)\Gamma(x+2\mu_2)}{\Gamma(-x)^2}\left(\frac{H_1^2}{4h_1^2}\right)^{-t_1}\left(\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{1}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{-x}\int_0^z \tau^{-2x-1}\, d\tau dt_1 dt_2 dx. \tag{B.41}$$

The inner integral can be easily solved using

$$\int_0^z \tau^{-2x-1} = \frac{\Gamma(-x)}{2\Gamma(1-x)} z^{-2x} \tag{B.42}$$

Therefore, the CDF is obtained as

$$F_Z(z) = \frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-2t_1-x)\Gamma(-2t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)} \times \frac{\Gamma(x+2\mu_1)\Gamma(x+2\mu_2)}{\Gamma(1-x)\Gamma(-x)}\left(\frac{H_1^2}{4h_1^2}\right)^{-t_1}\left(\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{B.43}$$

Comparing (B.43) with (2.1), the parameter for the Fox H-function are deduced as they are provided in Table 4.2.

Appendix C

Series Representation for the Ratio and Product of Random Envelopes

In the previous appendices, the derivation of the Fox H-function representations for the ratios and products of two random variates taken from the α-μ, κ-μ and η-μ distributions were presented. Here, the mathematical derivations of the series representations of each combination of ratios and products presented in Tables 3.3, 3.4, 4.3 and 4.4 are presented. These series are derived directly from their respective contour integral representations provided in the Appendices A and B through the sum of residues [35]. To obtain the series representations, it is required to know that the residues of the gamma function around its poles are given as

$$\mathrm{res}_{-i}\Gamma(x)f(x) = \frac{(-1)^i}{i!}f(-i), \qquad i \in \mathbb{N}. \tag{C.1}$$

C.1 The Ratio Distribution - PDF

The series presented in Table 3.3 were obtained as follows

C.1.1 The α-μ/κ-μ Distribution

The contour integral representation for the PDF of the ratio of the α-μ by the κ-μ variates is given as

$$\begin{aligned} f_Z(z) = {} & \frac{\alpha_x}{z\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\left(\frac{1}{2\pi i}\right)^2 \oint_{\mathscr{L}} \Gamma\left(\mu_y - \frac{\alpha_x}{2}s - t_2\right) \\ & \times \frac{\Gamma(\mu_x + s)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)}\left(\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\right)^{-s}\left(-\kappa_y\mu_y\right)^{-t_2} \, ds dt_2. \end{aligned} \tag{C.2}$$

Taking the residues around the poles of $\Gamma(t_2)$ will result in

$$f_Z(z) = \frac{\alpha_x}{z\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)} \sum_{i=0}^{\infty} \frac{\left(\kappa_y\mu_y\right)^i}{i!\Gamma\left(i+\mu_y\right)} \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma(s+\mu_x)\Gamma\left(i+\mu_y-\frac{s\alpha_x}{2}\right)\left(\frac{z^{\alpha_x}}{u_{\alpha\kappa}^{\alpha_x}}\right)^{-s} ds. \tag{C.3}$$

The inner contour integral can be interpreted as a Fox H-function and, using the notation in [39, Equation (1.2)]. Therefore the PDF results in

$$f_Z(z) = \frac{\alpha_x}{z\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)} \sum_{i=0}^{\infty} \frac{\left(\kappa_y\mu_y\right)^i}{i!\Gamma\left(i+\mu_y\right)} H_{1,1}^{1,1}\left[\frac{z^{\alpha_x}}{u_{\alpha\kappa}^{\alpha_x}} \middle| \begin{matrix} (1-i-\mu_y, \alpha_x/2) \\ (\mu_x, 1) \end{matrix}\right] \tag{C.4}$$

which is the exact same series provided in Table 3.3.

C.1.2 The α-μ/η-μ Distribution

The contour integral representation for the PDF of the ratio of the α-μ by the η-μ variates is given as

$$\begin{aligned} f_Z(z) = {} & \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}\alpha_x}{z\Gamma(\mu_x)\Gamma\left(\mu_y\right)} \left(\frac{1}{2\pi j}\right)^2 \oint_{\mathscr{L}} \Gamma\left(2\mu_y - \frac{\alpha_x s}{2} - 2t_2\right) \\ & \times \frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y+\frac{1}{2}-t_2\right)} \left(\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\right)^{-s} \left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} ds dt_2. \end{aligned} \tag{C.5}$$

Taking the residues around the poles of $\Gamma(t_2)$ will result in

$$\begin{aligned} f_Z(z) = {} & \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}\alpha_x}{z\Gamma(\mu_x)\Gamma\left(\mu_y\right)} \sum_{i=0}^{\infty} \frac{1}{i!\Gamma\left(\frac{1}{2}+i+\mu_y\right)} \left(\frac{H_y}{2h_y}\right)^{2i} \\ & \times \frac{1}{2\pi j} \oint_{\mathscr{L}} \Gamma(s+\mu_x)\Gamma\left(2i - \frac{s\alpha_x}{2} + 2\mu_y\right)\left(\frac{z^{\alpha_x}}{u_{\alpha\eta}^{\alpha_x}}\right)^{-s} ds. \end{aligned} \tag{C.6}$$

The inner contour integral can be interpreted as a Fox H-function and, using the notation in [39]. Therefore, the PDF results in

$$f_Z(z) = \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}\alpha_x}{z\Gamma(\mu_x)\Gamma\left(\mu_y\right)} \sum_{i=0}^{\infty} \frac{1}{i!\Gamma\left(\frac{1}{2}+i+\mu_y\right)} \left(\frac{H_y}{2h_y}\right)^{2i} H_{1,1}^{1,1}\left[\frac{z^{\alpha_x}}{u_{\alpha\eta}^{\alpha_x}} \middle| \begin{matrix} (1-2i-2\mu_y, \alpha_x/2) \\ (\mu_x, 1) \end{matrix}\right] \tag{C.7}$$

which is the exact same series provided in Table 3.3.

C.1.3 The κ-μ/κ-μ Distribution

The contour integral representation for the PDF of the ratio of two κ-μ variates is given as

$$f_Z(z) = \frac{2\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{z\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma\left(-s_1-s_2\right)\Gamma\left(\mu_x+s_1\right)\Gamma\left(\mu_y+s_2\right)}{\Gamma\left(-s_1\right)\Gamma\left(-s_2\right)} \times\left(-\kappa_x\mu_x\nu_{\kappa\kappa}\right)^{-s_1}\left(-\kappa_y\mu_y\left(1-\nu_{\kappa\kappa}\right)\right)^{-s_2} ds_1 ds_2. \tag{C.8}$$

Taking the residues around the poles of $\Gamma(\mu_x+s_1)$ on the positive index i and $\Gamma(\mu_y+s_2)$ on the positive index k will result in

$$f_Z(z) = \frac{2\left(-\kappa_x\mu_x\right)^{-\mu_x}\left(-\kappa_y\mu_y\right)^{-\mu_y}}{z\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma\left(i+k+\mu_x+\mu_y\right)}{i!k!\Gamma\left(i+\mu_x\right)\Gamma\left(k+\mu_y\right)} \times\left(-\nu_{\kappa\kappa}\kappa_x\mu_x\right)^{i+\mu_x}\left(-\left(1-\nu_{\kappa\kappa}\right)\kappa_y\mu_y\right)^{k+\mu_y} \tag{C.9}$$

After some algebraic manipulations, the PDF simplifies to

$$f_Z(z)=\frac{2\nu_{\kappa\kappa}^{\mu_x}\left(1-\nu_{\kappa\kappa}\right)^{\mu_y}}{z\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(i+k+\mu_x+\mu_y\right)\left(\nu_{\kappa\kappa}\kappa_x\mu_x\right)^i\left(\left(1-\nu_{\kappa\kappa}\right)\kappa_y\mu_y\right)^k}{i!k!\Gamma\left(i+\mu_x\right)\Gamma\left(k+\mu_y\right)}. \tag{C.10}$$

It is possible to further simplify the above double summation by performing the summation over either the index i or k. By choosing the index k, the PDF results in

$$f_Z(z) = \frac{2\nu_{\kappa\kappa}^{\mu_x}\left(1-\nu_{\kappa\kappa}\right)^{\mu_y}}{z\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\frac{\Gamma\left(i+\mu_x+\mu_y\right)\left(\nu_{\kappa\kappa}\kappa_x\mu_x\right)^i}{i!\Gamma\left(i+\mu_x\right)\Gamma\left(\mu_y\right)} \times {}_1F_1\left(i+\mu_x+\mu_y;\mu_y;\kappa_y\mu_y\left(1-\nu_{\kappa\kappa}\right)\right) \tag{C.11}$$

Using the definition of the beta function, the above expression is written as

$$f_Z(z) = \frac{2\nu_{\kappa\kappa}^{\mu_x}\left(1-\nu_{\kappa\kappa}\right)^{\mu_y}}{z\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\frac{\left(\nu_{\kappa\kappa}\kappa_x\mu_x\right)^i}{i!B\left(i+\mu_x,\mu_y\right)}\,{}_1F_1\left(i+\mu_x+\mu_y;\mu_y;\kappa_y\mu_y\left(1-\nu_{\kappa\kappa}\right)\right), \tag{C.12}$$

which is the exact same series provided in Table 3.3.

C.1.4 The κ-μ/η-μ Distribution

The contour integral representation for the PDF of the ratio of the κ-μ by the η-μ variates is given as

$$f_Z(z) = \frac{4\sqrt{\pi}(-1)^{-\mu_y-\mu_x}h_y^{\mu_y}}{z\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(H_y^2\right)^{\mu_y}\left(\kappa_x\mu_x\right)^{\mu_x}}\left(\frac{1}{2\pi j}\right)^2\oint\!\!\oint_{\mathscr{L}}\Gamma\left(-s_1-2s_2\right) \times\frac{\Gamma\left(s_1+\mu_x\right)\Gamma\left(s_2+\mu_y\right)}{\Gamma\left(-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)}\left(-\nu_{\kappa\eta}\kappa_x\mu_x\right)^{-s_1}\left(-\frac{H_y^2}{4h_y^2}\left(1-\nu_{\kappa\eta}\right)^2\right)^{-s_2} ds_1 ds_2. \tag{C.13}$$

Taking the residues around the poles of $\Gamma(\mu_x+s_1)$ on the positive index i and $\Gamma(\mu_y+s_2)$ on the positive index k will result in

$$f_Z(z)=\frac{4\sqrt{\pi}(-1)^{-\mu_x-\mu_y}h_y^{\mu_y}}{z\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(H_y^2\right)^{\mu_y}\left(\kappa_x\mu_x\right)^{\mu_x}}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma\left(i+2k+\mu_x+2\mu_y\right)}{i!k!\Gamma\left(i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times\left(-\frac{H_y^2\left(1-\nu_{\kappa\eta}\right)^2}{4h_y^2}\right)^{k+\mu_y}\left(-\nu_{\kappa\eta}\kappa_x\mu_x\right)^{i+\mu_x} \tag{C.14}$$

After some algebraic manipulations, the PDF simplifies to

$$f_Z(z)=\frac{2^{2-2\mu_y}\sqrt{\pi}\left(1-\nu_{\kappa\eta}\right)^{2\mu_y}\nu_{\kappa\eta}^{\mu_x}}{z\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(i+2k+\mu_x+2\mu_y\right)}{i!k!\Gamma\left(i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times\left(\nu_{\kappa\eta}\kappa_x\mu_x\right)^i\left(\frac{H_y^2}{4h_y^2}\left(1-\nu_{\kappa\eta}\right)^2\right)^k. \tag{C.15}$$

It is possible to further simplify the above double summation by performing the sum over either the index i or k. By choosing the index k, the PDF results in

$$f_Z(z)=\frac{2^{2-2\mu_y}\sqrt{\pi}\left(1-\nu_{\kappa\eta}\right)^{2\mu_y}\nu_{\kappa\eta}^{\mu_x}}{z\Gamma\left(\mu_y\right)\Gamma\left(\frac{1}{2}+\mu_y\right)\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{\Gamma\left(i+\mu_x+2\mu_y\right)\left(\nu_{\kappa\eta}\kappa_x\mu_x\right)^i}{i!\Gamma\left(i+\mu_x\right)} \times{}_2F_1\left(\frac{i}{2}+\frac{\mu_x}{2}+\mu_y,\frac{1}{2}+\frac{i}{2}+\frac{\mu_x}{2}+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2\left(1-\nu_{\kappa\eta}\right)^2}{h_y^2}\right). \tag{C.16}$$

Using the gamma's duplication formula and the definition of the beta function, the above expression reduces to

$$f_Z(z)=\frac{2\left(1-\nu_{\kappa\eta}\right)^{2\mu_y}\nu_{\kappa\eta}^{\mu_x}}{z\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{\left(\nu_{\kappa\eta}\kappa_x\mu_x\right)^i}{i!B\left(i+\mu_x,2\mu_y\right)} \times{}_2F_1\left(\frac{i}{2}+\frac{\mu_x}{2}+\mu_y,\frac{1}{2}+\frac{i}{2}+\frac{\mu_x}{2}+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2\left(1-\nu_{\kappa\eta}\right)^2}{h_y^2}\right), \tag{C.17}$$

which completes the derivation.

C.1.5 The η-μ/η-μ Distribution

The contour integral representation for the PDF of the ratio of two η-μ variates is given by

$$f_Z(z)=\frac{8\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{z\Gamma\left(\mu_x\right)\Gamma\left(\mu_y\right)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}}\left(\frac{1}{2\pi j}\right)^2\oint\oint_{\mathscr{L}}\Gamma\left(-2\left(s_1+s_2\right)\right) \times\frac{\Gamma\left(s_1+\mu_x\right)\Gamma\left(s_2+\mu_y\right)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)}\left(-\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{-s_1}\left(-\frac{H_y^2}{4h_y^2}\left(1-\nu_{\eta\eta}\right)^2\right)^{-s_2}ds_1ds_2. \tag{C.18}$$

Taking the residues around the poles $\Gamma(s_1+\mu_x)$ over the index i and $\Gamma(s_2\mu_y)$ over the index k results in

$$f_Z(z)=\frac{8\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{z\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\Gamma\left(2\left(i+k+\mu_x+\mu_y\right)\right)}{i!k!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times\left(-\frac{H_y^2\left(1-\nu_{\eta\eta}\right)2}{4h_y^2}\right)^{k+\mu_y}\left(-\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{i+\mu_x}. \tag{C.19}$$

After some algebraic manipulations, the above expression simplifies to

$$f_Z(z)=\frac{2^{3-2\mu_y-2\mu_x}\pi\nu_{\eta\eta}^{2\mu_x}\left(1-\nu_{\eta\eta}\right)^{2\mu_y}}{z\Gamma(\mu_x)\Gamma(\mu_y)h_x^{\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(2\left(i+k+\mu_x+\mu_y\right)\right)}{i!k!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times\left(\frac{H_y^2\left(1-\nu_{\eta\eta}\right)^2}{4h_y^2}\right)^{k}\left(\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{i}. \tag{C.20}$$

Performing the summation over the index k results in

$$f_Z(z)=\frac{2^{3-2\mu_y-2\mu_x}\pi\nu_{\eta\eta}^{2\mu_x}\left(1-\nu_{\eta\eta}\right)^{2\mu_y}}{z\Gamma(\mu_y)\Gamma\left(\frac{1}{2}+\mu_y\right)h_x^{\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{\Gamma\left(2\left(i+\mu_x+\mu_y\right)\right)}{i!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma(\mu_x)}\left(\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{i} \times{}_2F_1\left(i+\mu_x+\mu_y,i+\mu_x+\mu_y+\frac{1}{2};\mu_y+\frac{1}{2};\frac{H_y^2\left(1-\nu_{\eta\eta}\right)^2}{h_y^2}\right). \tag{C.21}$$

From the duplication formula of the gamma function $\Gamma\left(\mu_y\right)\Gamma\left(\frac{1}{2}+\mu_y\right)=\sqrt{\pi}2^{1-2\mu_y}\Gamma\left(2\mu_y\right)$, the above expression reduces to

$$f_Z(z)=\frac{2^{2-2\mu_x}\sqrt{\pi}\nu_{\eta\eta}^{2\mu_x}\left(1-\nu_{\eta\eta}\right)^{2\mu_y}}{z\Gamma\left(2\mu_y\right)h_x^{\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{\Gamma\left(2\left(i+\mu_x+\mu_y\right)\right)}{i!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma(\mu_x)}\left(\frac{H_x^2\nu_{\eta\eta}^2}{4h_x^2}\right)^{i} \times{}_2F_1\left(i+\mu_x+\mu_y,i+\mu_x+\mu_y+\frac{1}{2};\mu_y+\frac{1}{2};\frac{H_y^2\left(1-\nu_{\eta\eta}\right)^2}{h_y^2}\right). \tag{C.22}$$

Using once more the gamma's duplication formula, it is possible to write $\Gamma\left(\frac{1}{2}+i+\mu_x\right)=2^{1-2i-2\mu_x}\sqrt{\pi}\Gamma\left(2\left(\mu_x+i\right)\right)/\Gamma\left(i+\mu_x\right)$. After replacing this identity and using the beta function and Pochhammer symbol, the series representation for the PDF of the ratio of two η-μ variates is obtained as

$$f_Z(z)=\frac{2\nu_{\eta\eta}^{2\mu_x}\left(1-\nu_{\eta\eta}\right)^{2\mu_y}}{zh_x^{\mu_x}h_y^{\mu_y}}\sum_{i=0}^{\infty}\frac{(\mu_x)_i}{i!B\left(2\left(\mu_x+i\right),2\mu_y\right)}\left(\frac{H_x^2\nu_{\eta\eta}^2}{h_x^2}\right)^{i} \times{}_2F_1\left(i+\mu_x+\mu_y,\frac{1}{2}+i+\mu_x+\mu_y;\frac{1}{2}+\mu_y;\frac{H_y^2\left(1-\nu_{\eta\eta}\right)^2}{h_y^2}\right), \tag{C.23}$$

which completes the derivation.

C.2 The Ratio Distribution - CDF

The series provided in Table 3.4 were obtained as follows

C.2.1 The α-μ/κ-μ Distribution

The contour integral representation for the CDF of the ratio of the α-μ by the κ-μ variates is

$$F_Z(z) = \frac{1}{\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\left(\frac{1}{2\pi i}\right)^2 \oint_{\mathscr{L}} \Gamma\left(\mu_y - \frac{\alpha_x}{2}s - t_2\right)\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y - t_2\right)} \times \frac{\Gamma(-s)}{\Gamma(1-s)}\left(\left(\frac{z}{u_{\alpha\kappa}}\right)^{\alpha_x}\right)^{-s}\left(-\kappa_y\mu_y\right)^{-t_2} ds dt_2. \tag{C.24}$$

Taking the residues around the poles of $\Gamma(t_2)$, the CDF reduces to

$$F_Z(z) = \frac{1}{\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\frac{\left(\kappa_y\mu_y\right)^i}{i!\Gamma\left(i+\mu_y\right)}\frac{1}{2\pi j}\oint_{\mathscr{L}}\frac{\Gamma(-s)\Gamma(s+\mu_x)}{\Gamma(1-s)} \times \Gamma\left(i - \frac{s\alpha_x}{2} + \mu_y\right)\left(\frac{z^{\alpha_x}}{u_{\alpha\kappa}^{\alpha_x}}\right)^{-s} ds. \tag{C.25}$$

The inner contour integral can be interpreted as a Fox H-function function and, using the notation in [39], the CDF is given by

$$F_Z(z) = \frac{1}{\Gamma(\mu_x)\exp\left(\kappa_y\mu_y\right)}\sum_{i=0}^{\infty}\frac{\left(\kappa_y\mu_y\right)^i}{i!\Gamma\left(i+\mu_y\right)} H_{2,2}^{1,2}\left[\frac{z^{\alpha_x}}{u_{\alpha\kappa}^{\alpha_x}}\left|\begin{matrix}(1,1),\left(1-i-\mu_y,\frac{\alpha_x}{2}\right)\\(\mu_x,1),(0,1)\end{matrix}\right.\right], \tag{C.26}$$

which completes the derivation.

C.2.2 The α-μ/η-μ Distribution

The contour integral representation for the CDF of the ratio of the α-μ by the η-μ variates is

$$F_Z(z) = \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}}{\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\left(\frac{1}{2\pi j}\right)^2\oint_{\mathscr{L}}\Gamma\left(2\mu_y - \frac{\alpha_x s}{2} - 2t_2\right)\frac{\Gamma(\mu_x+s)\Gamma(t_2)}{\Gamma\left(\mu_y+\frac{1}{2}-t_2\right)} \times \frac{\Gamma(-s)}{\Gamma(1-s)}\left(\left(\frac{z}{u_{\alpha\eta}}\right)^{\alpha_x}\right)^{-s}\left(-\frac{H_y^2}{4h_y^2}\right)^{-t_2} ds dt_2. \tag{C.27}$$

Taking the residues around the poles of $\Gamma(t_2)$, the CDF can be written as

$$F_Z(z) = \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}}{\Gamma(\mu_x)\Gamma\left(\mu_y\right)}\sum_{i=0}^{\infty}\frac{1}{i!\Gamma\left(\frac{1}{2}+i+\mu_y\right)}\left(\frac{H_y^2}{4h_y^2}\right)^i\frac{1}{2\pi j}\oint_{\mathscr{L}}\frac{\Gamma(-s)\Gamma(s+\mu_x)}{\Gamma(1-s)} \times \Gamma\left(2i - \frac{s\alpha_x}{2} + 2\mu_y\right)\left(\frac{z^{\alpha_x}}{u_{\alpha\eta}^{\alpha_x}}\right)^{-s} ds. \tag{C.28}$$

The inner contour integral can be interpreted as a Fox H-function function and, using the notation in [39], the CDF is given as

$$F_Z(z) = \frac{2^{1-2\mu_y}\sqrt{\pi}h_y^{-\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)} \sum_{i=0}^{\infty} \frac{1}{i!\Gamma\left(\frac{1}{2}+i+\mu_y\right)} \left(\frac{H_y^2}{4h_y^2}\right)^i \times H_{2,2}^{1,2}\left[\frac{z^{\alpha_x}}{u_{\alpha\eta}^{\alpha_x}} \middle| \begin{matrix} (1,1),\left(1-2i-2\mu_y,\frac{\alpha_x}{2}\right) \\ (\mu_x,1),(0,1) \end{matrix}\right]. \tag{C.29}$$

Finally, using the duplication formula of the gamma function and the Pochhammer symbol, the CDF is simplified to

$$F_Z(z) = \frac{h_y^{-\mu_y}}{\Gamma(\mu_x)} \sum_{i=0}^{\infty} \frac{(\mu_y)_i}{i!\Gamma(2i+2\mu_y)} \left(\frac{H_y}{h_y}\right)^{2i} H_{2,2}^{1,2}\left[\frac{z^{\alpha_x}}{u_{\alpha\eta}^{\alpha_x}} \middle| \begin{matrix} (1,1),\left(1-2i-2\mu_y,\frac{\alpha_x}{2}\right) \\ (\mu_x,1),(0,1) \end{matrix}\right], \tag{C.30}$$

which completes the derivation

C.2.3 The κ-μ/κ-μ Distribution

The contour integral representation for the CDF of the ratio of two κ-μ variates is

$$F_Z(z) = \frac{(-\kappa_x\mu_x)^{-\mu_x}(-\kappa_y\mu_y)^{-\mu_y}}{\exp(\kappa_x\mu_x+\kappa_y\mu_y)} \left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2-s_3)\Gamma(-s_1-s_3)}{\Gamma(1-s_1-s_3)\Gamma(-s_1)\Gamma(-s_2)} \times\Gamma(\mu_x+s_1)\Gamma(\mu_y+s_2)\Gamma(s_3)\left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\kappa}^2}\right)^{-s_1}(-\kappa_y\mu_y)^{-s_2}\left(\frac{z^2}{u_{\kappa\kappa}^2}\right)^{-s_3} ds_1 ds_2 ds_3. \tag{C.31}$$

Here, the residues of the multivariable Fox H-function are taken around the poles of $\Gamma(\mu_x + s_1)$, $\Gamma(\mu_y + s_2)$ and $\Gamma(s_3)$, which results in the triple series

$$F_Z(z) = \frac{(-\kappa_x\mu_x)^{-\mu_x}(-\kappa_y\mu_y)^{-\mu_y}}{\exp(\kappa_x\mu_x+\kappa_y\mu_y)} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{i+k+n}\Gamma(i+k+\mu_x)}{i!k!n!\Gamma(i+\mu_x)\Gamma(1+i+n+\mu_x)} \times \frac{\Gamma(i+k+n+\mu_x+\mu_y)}{\Gamma(k+\mu_y)}\left(\frac{z^2}{u_{\kappa\kappa}^2}\right)^n\left(-\frac{z^2\kappa_x\mu_x}{u_{\kappa\kappa}^2}\right)^{i+\mu_x}(-\kappa_y\mu_y)^{k+\mu_y}. \tag{C.32}$$

After some algebraic manipulations, the CDF reduces to

$$F_Z(z) = \frac{1}{\exp(\kappa_x\mu_x+\kappa_y\mu_y)}\left(\frac{z}{u_{\kappa\kappa}}\right)^{2\mu_x} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(i+n+\mu_x)}{i!k!n!\Gamma(i+\mu_x)\Gamma(1+i+n+\mu_x)} \times \frac{\Gamma(i+k+n+\mu_x+\mu_y)}{\Gamma(k+\mu_y)}\left(\frac{z^2}{u_{\kappa\kappa}^2}\right)^n\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\kappa}^2}\right)^i(\kappa_y\mu_y)^k. \tag{C.33}$$

The summation over the index n will result in the following double summation.

$$\begin{aligned} F_Z(z) =& \frac{1}{\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{z}{u_{\kappa\kappa}}\right)^{2\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(i+k+\mu_x+\mu_y\right)}{i!k!\Gamma\left(1+i+\mu_x\right)\Gamma\left(k+\mu_y\right)} \\ &\times\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\kappa}^2}\right)^i\left(\kappa_y\mu_y\right)^k {}_2F_1\left(i+\mu_x, i+k+\mu_x+\mu_y; 1+i+\mu_x; -\frac{z^2}{u_{\kappa\kappa}^2}\right). \end{aligned} \tag{C.34}$$

Using the identity $\Gamma(a+1) = a\Gamma(a)$ and the beta function notation, the above double series can be written as

$$\begin{aligned} F_Z(z) =& \frac{1}{\exp\left(\kappa_x\mu_x+\kappa_y\mu_y\right)}\left(\frac{z}{u_{\kappa\kappa}}\right)^{2\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\kappa_y\mu_y\right)^k}{i!k!\left(i+\mu_x\right)B\left(i+\mu_x, k+\mu_y\right)} \\ &\times\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\kappa}^2}\right)^i {}_2F_1\left(i+\mu_x, i+k+\mu_x+\mu_y; 1+i+\mu_x; -\frac{z^2}{u_{\kappa\kappa}^2}\right), \end{aligned} \tag{C.35}$$

which completes the derivation.

C.2.4 The κ-μ/η-μ Distribution

The contour integral representation for the CDF of the ratio of the κ-μ by the η-μ variates is

$$\begin{aligned} F_Z(z) =& \frac{2\sqrt{\pi}(-1)^{-\mu_y-\mu_x}h_y^{\mu_y}\left(H_y^2\right)^{-\mu_y}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(\kappa_x\mu_x\right)^{\mu_x}}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma\left(-s_1-s_3\right)\Gamma\left(-s_1-2s_2-s_3\right)}{\Gamma\left(1-s_1-s_3\right)} \\ \times& \frac{\Gamma\left(\mu_x+s_1\right)\Gamma\left(\mu_y+s_2\right)\Gamma\left(s_3\right)}{\Gamma\left(-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)}\left(-\frac{\kappa_x\mu_x z^2}{u_{\kappa\eta}^2}\right)^{-s_1}\left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2}\left(\frac{z^2}{u_{\kappa\eta}^2}\right)^{-s_3} ds_1 ds_2 ds_3. \end{aligned} \tag{C.36}$$

Taking the residues around the poles of $\Gamma(\mu_x+s_1)$, $\Gamma(\mu_y+s_2)$ and $\Gamma(s_3)$, the CDF is written as the following triple summation

$$\begin{aligned} F_Z(z) =& \frac{2\sqrt{\pi}(-1)^{-\mu_x-\mu_y}h_y^{\mu_y}\left(H_y^2\right)^{-\mu_y}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)\left(\kappa_x\mu_x\right)^{\mu_x}}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{i+k+n}\Gamma\left(i+k+\mu_x\right)}{i!k!n!\Gamma\left(i+\mu_x\right)\Gamma\left(1+i+n+\mu_x\right)} \\ &\times\frac{\Gamma\left(i+2k+n+\mu_x+2\mu_y\right)}{\Gamma\left(\frac{1}{2}+k+\mu_y\right)}\left(-\frac{H_y^2}{4h_y^2}\right)^{k+\mu_y}\left(\frac{z^2}{u_{\kappa\eta}^2}\right)^n\left(-\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^{i+\mu_x}. \end{aligned} \tag{C.37}$$

After performing some algebraic manipulations, the CDF is given by

$$\begin{aligned} F_Z(z) =& \frac{2^{1-2\mu_y}\sqrt{\pi}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}}\left(\frac{z^2}{u_{\kappa\eta}^2}\right)^{\mu_x}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(i+n+\mu_x\right)}{i!k!n!\Gamma\left(i+\mu_x\right)\Gamma\left(1+i+n+\mu_x\right)} \\ &\times\frac{\Gamma\left(i+2k+n+\mu_x+2\mu_y\right)}{\Gamma\left(\frac{1}{2}+k+\mu_y\right)}\left(\frac{H_y^2}{4h_y^2}\right)^k\left(\frac{z^2}{u_{\kappa\eta}^2}\right)^n\left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^i. \end{aligned} \tag{C.38}$$

Now, performing the summation over index n will result in

$$F_Z(z) = \frac{2^{1-2\mu_y}\sqrt{\pi}}{\Gamma\left(\mu_y\right)\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(i+2k+\mu_x+2\mu_y\right)}{i!k!\Gamma\left(1+i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \left(\frac{H_y^2}{4h_y^2}\right)^k \times \left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^i {}_2F_1\left(i+\mu_x, i+2k+\mu_x+2\mu_y; 1+i+\mu_x; -\frac{z^2}{u_{\kappa\eta}^2}\right). \tag{C.39}$$

By using the gamma's duplication formula, the CDF simplifies to

$$F_Z(z) = \frac{1}{\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}} \left(\frac{z^2}{u_{\kappa\eta}^2}\right)^{\mu_x} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(i+2k+\mu_x+2\mu_y\right)\Gamma\left(k+\mu_y\right)}{i!k!\Gamma\left(1+i+\mu_x\right)\Gamma\left(2k+2\mu_y\right)\Gamma\left(\mu_y\right)} \left(\frac{H_y^2}{h_y^2}\right)^k \times \left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^i {}_2F_1\left(i+\mu_x, i+2k+\mu_x+2\mu_y; 1+i+\mu_x; -\frac{z^2}{u_{\kappa\eta}^2}\right). \tag{C.40}$$

Finally, the above expression can be further simplified by writing the gamma functions in terms of the beta function and Pochhammer symbols, and is given as

$$F_Z(z) = \frac{1}{\exp\left(\kappa_x\mu_x\right)h_y^{\mu_y}} \left(\frac{z^2}{u_{\kappa\eta}^2}\right)^{\mu_x} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\mu_y\right)_k}{i!k!\left(i+\mu_x\right)B\left(i+\mu_x, 2k+2\mu_y\right)} \left(\frac{H_y^2}{h_y^2}\right)^k \times \left(\frac{z^2\kappa_x\mu_x}{u_{\kappa\eta}^2}\right)^i {}_2F_1\left(i+\mu_x, i+2k+\mu_x+2\mu_y; 1+i+\mu_x; -\frac{z^2}{u_{\kappa\eta}^2}\right), \tag{C.41}$$

which completes the derivation.

C.2.5 The η-μ/η-μ Distribution

The contour integral representation for the CDF of the ratio of two η-μ variates is

$$F_Z(z) = \frac{4\pi(-1)^{-\mu_x-\mu_y}h_x^{\mu_x}h_y^{\mu_y}}{\Gamma\left(\mu_x\right)\Gamma\left(\mu_y\right)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma\left(-2s_1-s_3\right)\Gamma\left(-2\left(s_1+s_2\right)-s_3\right)}{\Gamma\left(1-2s_1-s_3\right)} \times \frac{\Gamma\left(\mu_x+s_1\right)\Gamma\left(\mu_y+s_2\right)\Gamma\left(s_3\right)}{\Gamma\left(\frac{1}{2}-s_1\right)\Gamma\left(\frac{1}{2}-s_2\right)} \left(-\frac{H_x^2z^4}{4u_{\eta\eta}^4h_x^2}\right)^{-s_1} \left(-\frac{H_y^2}{4h_y^2}\right)^{-s_2} \left(\frac{z^2}{u_{\eta\eta}^2}\right)^{-s_3} ds_1ds_2ds_3. \tag{C.42}$$

Taking the residues around the poles $\Gamma(\mu_x + s_1)$, $\Gamma(\mu_y + s_2)$ and $\Gamma(s_3)$, the CDF is written as a triple summation given by

$$F_Z(z) = \frac{4\pi(-1)^{-\mu_x-\mu_y} h_x^{\mu_x} h_y^{\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)\left(H_x^2\right)^{\mu_x}\left(H_y^2\right)^{\mu_y}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{i+k+n}\Gamma(2i+n+2\mu_x)}{i!k!n!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times \frac{\Gamma\left(2i+2k+n+2\mu_x+2\mu_y\right)}{\Gamma(1+2i+n+2\mu_x)} \left(-\frac{H_y^2}{4h_y^2}\right)^{k+\mu_y} \left(-\frac{z^4 H_x^2}{4h_x^2 u_{\eta\eta}^4}\right)^{i+\mu_x} \left(\frac{z^2}{u_{\eta\eta}^2}\right)^n. \tag{C.43}$$

After some algebraic manipulations, the CDF is simplified to

$$F_Z(z) = \frac{4^{1-\mu_y-\mu_x}\pi}{\Gamma(\mu_x)\Gamma(\mu_y) h_x^{\mu_x} h_y^{\mu_y}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(2i+n+2\mu_x)}{i!k!n!\Gamma\left(\frac{1}{2}+i+\mu_x\right)\Gamma\left(\frac{1}{2}+k+\mu_y\right)} \times \frac{\Gamma\left(2i+2k+n+2\mu_x+2\mu_y\right)}{\Gamma(1+2i+n+2\mu_x)} \left(\frac{H_y}{2h_y}\right)^{2k} \left(\frac{z^2 H_x}{2h_x u_{\eta\eta}^2}\right)^{2i} \left(\frac{z^2}{u_{\eta\eta}^2}\right)^n. \tag{C.44}$$

At this point, the summation over the index n is performed, which results in

$$F_Z(z) = \frac{4^{1-\mu_y-\mu_x}\pi}{\Gamma(\mu_x)\Gamma(\mu_y) h_x^{\mu_x} h_y^{\mu_y}} \left(\frac{z}{u_{\eta\eta}}\right)^{4\mu_x} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(2\left(i+k+\mu_x+\mu_y\right)\right)}{i!k!\Gamma\left(\frac{1}{2}(1+2i+2\mu_x)\right)\Gamma(1+2i+2\mu_x)} \times \frac{\Gamma(2(i+\mu_x))}{\Gamma\left(\frac{1}{2}\left(1+2k+2\mu_y\right)\right)} \left(\frac{H_y}{2h_y}\right)^{2k} \left(\frac{z^2 H_x}{2h_x u_{\eta\eta}^2}\right)^{2i} \times {}_2F_1\left(2(i+\mu_x), 2\left(i+k+\mu_x+\mu_y\right); 1+2i+2\mu_x; -\frac{z^2}{u_{\eta\eta}^2}\right). \tag{C.45}$$

Using the gamma's duplication formula and notating the gamma functions in terms of the beta function and the Pochhammer symbol, the CDF of the ratio of two η-μ variates is obtained as

$$F_Z(z) = \frac{1}{2h_x^{\mu_x} h_y^{\mu_y}} \left(\frac{z}{u_{\eta\eta}}\right)^{4\mu_x} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\mu_x)_i\left(\mu_y\right)_k}{i!k!(i+\mu_x)B\left(2i+2\mu_x, 2k+2\mu_y\right)} \left(\frac{H_y^2}{h_y^2}\right)^k \left(\frac{z^4 H_x^2}{h_x^2 u_{\eta\eta}^4}\right)^i \times {}_2F_1\left(2(i+\mu_x), 2\left(i+k+\mu_x+\mu_y\right); 1+2i+2\mu_x; -\frac{z^2}{u_{\eta\eta}^2}\right), \tag{C.46}$$

which completes the derivation.

C.3 The Product Distribution - PDF

In this Section, the expressions for the PDF of the product of two random variates taken from the α-μ, κ-μ and η-μ provided in Table 4.3 are derived.

C.3.1 The α-$\mu\times\kappa$-μ Distribution

The contour integral representation for the PDF of the product of the α-μ and the κ-μ variates is

$$f_Z(z)=\frac{2}{z\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^2\oiint_{\mathscr{L}}\frac{\Gamma(-t-x)\Gamma(t)\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(\mu_2+x)}{\Gamma(\mu_2-t)\Gamma(-x)}(\kappa_2\mu_2)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{-x}dtdx. \tag{C.47}$$

On the variable t, the residues are taken around the poles of $\Gamma(t)$ whilst on the variable x, $\Gamma(\mu_1+2x/\alpha_1)$ and $\Gamma(\mu_2+x)$ are used to obtain the residues. Ergo, the PDF can be written as the following double series

$$f_Z(z)=\frac{2}{z\Gamma(\mu_1)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!k!}\left(\frac{\alpha_1\Gamma\left(k+\frac{\alpha_1(i+\mu_1)}{2}\right)\Gamma\left(-\frac{\alpha_1(i+\mu_1)}{2}+\mu_2\right)}{2\Gamma\left(\frac{\alpha_1(i+\mu_1)}{2}\right)\Gamma(k+\mu_2)(\kappa_2\mu_2)^{-k}}\left(\frac{z^2}{\mathscr{A}^2\mathscr{K}^2}\right)^{\frac{i\alpha_1}{2}+\frac{\alpha_1\mu_1}{2}}\right.$$
$$\left.+\frac{\Gamma(i+k+\mu_2)\Gamma\left(\mu_1-\frac{2(i+\mu_2)}{\alpha_1}\right)}{\Gamma(k+\mu_2)\Gamma(i+\mu_2)}(\kappa_2\mu_2)^k\left(\frac{z^2}{\mathscr{A}^2\mathscr{K}^2}\right)^{i+\mu_2}\right). \tag{C.48}$$

Performing the summation over the index k results in

$$f_Z(z)=\frac{2}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2}{\mathscr{A}^2\mathscr{K}^2}\right)^{i+\mu_2}\Gamma\left(\mu_1-\frac{2(i+\mu_2)}{\alpha_1}\right){}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\right.$$
$$\left.+\frac{\alpha_1}{2}\left(\frac{z^2}{\mathscr{A}^2\mathscr{K}^2}\right)^{\frac{1}{2}\alpha_1(i+\mu_1)}\Gamma\left(\mu_2-\frac{1}{2}\alpha_1(i+\mu_1)\right){}_1F_1\left(\frac{\alpha_1(i+\mu_1)}{2};\mu_2;-\kappa_2\mu_2\right)\right), \tag{C.49}$$

which completes the derivation.

C.3.2 The α-$\mu\times\eta$-μ Distribution

The contour integral representation for the PDF of the product of the α-μ and the η-μ variates is

$$f_Z(z)=\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2\oiint_{\mathscr{L}}\frac{\Gamma(-2t-x)\Gamma(t)}{\Gamma\left(\mu_2+\frac{1}{2}-t\right)}$$
$$\times\frac{\Gamma\left(\mu_1+\frac{2x}{\alpha_1}\right)\Gamma(2\mu_2+x)}{\Gamma(-x)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t}\left(\frac{z^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{-x}dtdx. \tag{C.50}$$

Here, the residues are taken around the poles of $\Gamma(t)$ for the variable t and $\Gamma(\mu_1+2x/\alpha_1)$ and $\Gamma(2\mu_2+x)$ for the variable x. Ergo, the PDF is written as

$$f_Z(z)=\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i}{i!k!}\left(\frac{\alpha_1\Gamma\left(2k+\frac{\alpha_1(i+\mu_1)}{2}\right)\Gamma\left(2\mu_2-\frac{\alpha_1(i+\mu_1)}{2}\right)}{2\Gamma\left(\frac{\alpha_1(i+\mu_1)}{2}\right)\Gamma\left(\frac{1}{2}+k+\mu_2\right)}\right.$$
$$\left.\times\left(\frac{H_2^2}{4h_2^2}\right)^k\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{\frac{\alpha_1(i+\mu_1)}{2}}+\frac{\Gamma(2k+i+2\mu_2)\Gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1}\right)}{\Gamma\left(\frac{1}{2}+k+\mu_2\right)\Gamma(i+2\mu_2)}\left(\frac{H_2^2}{4h_2^2}\right)^k\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{i+2\mu_2}\right) \tag{C.51}$$

The PDF for the product of the α-μ by the η-μ variates is further simplified by performing the summation over the index k, which results in

$$f_Z(z)=\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)\Gamma\left(\frac{1}{2}+\mu_2\right)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{i+2\mu_2}\Gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1}\right)\right.$$
$$\times{}_2F_1\left(\frac{i}{2}+\mu_2,\frac{1}{2}+\frac{i}{2}+\mu_2;\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\frac{\alpha_1}{2}\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{\frac{\alpha_1(i+\mu_1)}{2}}\Gamma\left(2\mu_2-\frac{\alpha_1(i+\mu_1)}{2}\right)$$
$$\left.\times\,{}_2F_1\left(\frac{\alpha_1(i+\mu_1)}{4},\frac{1}{2}+\frac{\alpha_1(i+\mu_1)}{4};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right). \tag{C.52}$$

Further simplifications are obtained using the duplication formula of the gamma function and the linear transformation [36, Equation (15.3.3)] on the both hypergeometric functions ${}_2F_1$. Therefore, the PDF is obtained as

$$f_Z(z)=\frac{2}{z\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{i+2\mu_2}\Gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1}\right)\left(1-\frac{H_2^2}{h_2^2}\right)^{-i-\mu_2}\right.$$
$$\times{}_2F_1\left(-\frac{i}{2},\frac{1-i}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\frac{\alpha_1}{2}\left(\frac{z^2}{\mathscr{A}^2\mathscr{E}^2}\right)^{\frac{\alpha_1(i+\mu_1)}{2}}\Gamma\left(2\mu_2-\frac{\alpha_1(i+\mu_1)}{2}\right)$$
$$\left.\times\left(1-\frac{H_2^2}{h_2^2}\right)^{-\frac{\alpha_1(i+\mu_1)}{2}+\mu_2}{}_2F_1\left(\mu_2-\frac{\alpha_1(i+\mu_1)}{4},\mu_2-\frac{\alpha_1(i+\mu_1)}{4}+\frac{1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right). \tag{C.53}$$

Finally, it is easy to show that the parameters h and H of the η-μ distribution are connected by $1-H^2/h^2=1/h$. After replacing this identity at the PDF, it results in

$$f_Z(z)=\frac{2}{zh_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2h_2}{\mathscr{A}^2\mathscr{E}^2}\right)^{i+2\mu_2}\Gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1}\right)\right.$$
$$\times{}_2F_1\left(-\frac{i}{2},\frac{1-i}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\frac{\alpha_1}{2}\left(\frac{z^2h_2}{\mathscr{A}^2\mathscr{E}^2}\right)^{\frac{\alpha_1(i+\mu_1)}{2}}\Gamma\left(2\mu_2-\frac{\alpha_1(i+\mu_1)}{2}\right) \tag{C.54}$$
$$\left.\times{}_2F_1\left(\mu_2-\frac{\alpha_1(i+\mu_1)}{4},\mu_2-\frac{\alpha_1(i+\mu_1)}{4}+\frac{1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right),$$

which completes the derivation.

C.3.3 The κ-μ×κ-μ Distribution

The contour integral representation for the PDF of the product of two κ-μ variates is

$$f_Z(z) = \frac{2}{z}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-t_2-x)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma(\mu_2-t_2)} \times \frac{\Gamma(\mu_1+x)\Gamma(\mu_2+x)}{\Gamma(-x)^2}(\kappa_1\mu_1)^{-t_1}(\kappa_2\mu_2)^{-t_2}\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{C.55}$$

The residues are taken around the poles of $\Gamma(t_1)$, $\Gamma(t_2)$, $\Gamma(\mu_1+x)$ and $\Gamma(\mu_2+x)$, so that the PDF can be written as the triple summation

$$f_Z(z) = \frac{2}{z}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{n+i+k}}{n!i!k!}\left(\frac{\Gamma(n+i+\mu_1)\Gamma(i+k+\mu_1)\Gamma(-i-\mu_1+\mu_2)}{\Gamma(n+\mu_1)\Gamma(i+\mu_1)^2\Gamma(k+\mu_2)(\kappa_1\mu_1)^{-n}(\kappa_2\mu_2)^{-k}} \times\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_1} + \frac{\Gamma(-i+\mu_1-\mu_2)\Gamma(n+i+\mu_2)\Gamma(i+k+\mu_2)}{\Gamma(n+\mu_1)\Gamma(i+\mu_2)^2\Gamma(k+\mu_2)(\kappa_1\mu_1)^{-n}(\kappa_2\mu_2)^{-k}}\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}\right) \tag{C.56}$$

Performing the summation over the index n and k results in

$$f_Z(z) = \frac{2}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^{-i}}{i!}\left(\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_2+i}\Gamma(-i+\mu_1-\mu_2) \times {}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\,{}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2) + \left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_1+i}\Gamma(-i-\mu_1+\mu_2)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\,{}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right), \tag{C.57}$$

which completes the derivation.

C.3.4 The κ-μ×η-μ Distribution

The contour integral representation for the PDF of the product of the κ-μ by the η-μ variates is

$$f_Z(z) = \frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(-t_1-x)\Gamma(-x-2t_2)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_1)\Gamma\left(\mu_2+\frac{1}{2}-t_2\right)} \times \frac{\Gamma(\mu_1+x)\Gamma(2\mu_2+x)}{\Gamma(-x)^2}(\kappa_1\mu_1)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{C.58}$$

Taking the residues around the poles of $\Gamma(t_1)$, $\Gamma(t_2)$ and $\Gamma(\mu_1+x)$ and $\Gamma(\mu_2+x)$ will lead to a triple summation representation for the PDF which is given by

$$f_Z(z)=\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_2)}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+i}(\kappa_1\mu_1)^n}{n!i!k!\Gamma(n+\mu_1)\Gamma\left(\frac{1}{2}+k+\mu_2\right)}\left(\frac{\Gamma(n+i+\mu_1)}{\Gamma(i+\mu_1)^2}\right.$$
$$\times\Gamma(i+2k+\mu_1)\Gamma(-i-\mu_1+2\mu_2)\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}+\frac{\Gamma(-i+\mu_1-2\mu_2)}{\Gamma(i+2\mu_2)^2} \tag{C.59}$$
$$\left.\times\,\Gamma(n+i+2\mu_2)\Gamma(i+2k+2\mu_2)\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+2\mu_2}\right).$$

Performing the summation over the indexes n and k will result in

$$f_Z(z)=\frac{4^{1-\mu_2}\sqrt{\pi}}{z\Gamma(\mu_1)\Gamma(\mu_2)\Gamma\left(\frac{1}{2}+\mu_2\right)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\right.$$
$$\times\,\Gamma(-i-\mu_1+2\mu_2)\,{}_2F_1\left(\frac{i+\mu_1}{2},\frac{1+i+\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\Gamma(-i+\mu_1-2\mu_2) \tag{C.60}$$
$$\left.\times\,{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)\,{}_2F_1\left(\frac{i}{2}+\mu_2,\frac{1}{2}+\frac{i}{2}+\mu_2;\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+2\mu_2}\right).$$

Further simplifications are obtained by using the gamma's duplication formula, and the linear transformation [36, Equation (15.3.3)] on the ${}_2F_1$ functions. After applying these transformations, the PDF is given as

$$f_Z(z)=\frac{2}{z\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}\left(1-\frac{H_2^2}{h_2^2}\right)^{-i-\mu_1+\mu_2}\Gamma(-i-\mu_1+2\mu_2)\right.$$
$$\times\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\,{}_2F_1\left(\mu_2-\frac{i+\mu_1}{2},\mu_2+\frac{1-i-\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)$$
$$+\Gamma(-i+\mu_1-2\mu_2)\,{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)$$
$$\left.\times\,{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)\,{}_2F_1\left(-\frac{i}{2},\frac{1-i}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\left(\frac{z^2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+2\mu_2}\left(1-\frac{H_2^2}{h_2^2}\right)^{-i-\mu_2}\right). \tag{C.61}$$

Finally, it is possible to use [36, Equation (6.1.17)] and the identity $1-H^2/h^2=1/h$ to further simplify the PDF of the product of the κ-μ by the η-μ variates to

$$f_Z(z)=\frac{2\pi\csc(\pi(2\mu_2-\mu_1))}{zh_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\left(\frac{z^2h_2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}\frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i+\mu_1-2\mu_2)}\right.$$
$$\times\,{}_2F_1\left(\mu_2-\frac{i+\mu_1}{2},\mu_2+\frac{1-i-\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)-\frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i-\mu_1+2\mu_2)} \tag{C.62}$$
$$\left.\times\,{}_2F_1\left(-\frac{i}{2},\frac{1-i}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\left(\frac{z^2h_2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+2\mu_2}\right),$$

which completes the derivation.

C.3.5 The η-μ×η-μ Distribution

The contour integral representation for the PDF of the product two η-μ variates is

$$f_Z(z) = \frac{2^{3-2\mu_1-2\mu_2}\pi}{z\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-2t_1-x)\Gamma(-2t_2-x)}{\Gamma\left(\frac{1}{2}-t_1+\mu_1\right)\Gamma\left(\frac{1}{2}-t_2+\mu_2\right)} \times \frac{\Gamma(t_1)\Gamma(t_2)\Gamma(x+2\mu_1)\Gamma(x+2\mu_2)}{\Gamma(-x)^2}\left(-\frac{H_1^2}{4h_1^2}\right)^{-t_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_2}\left(\frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{-x} dt_1 dt_2 dx. \tag{C.63}$$

Taking the residues around the poles of $\Gamma(t_1)$, $\Gamma(t_2)$, $\Gamma(x+2\mu_1)$ and $\Gamma(x+2\mu_2)$ will result in the following triple series

$$f_Z(z) = \frac{2^{3-2\mu_1-2\mu_2}\pi}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i}{n!i!k!\Gamma\left(\frac{1}{2}+n+\mu_1\right)\Gamma\left(\frac{1}{2}+k+\mu_2\right)}\left(\frac{H_1^2}{4h_1^2}\right)^n\left(\frac{H_2^2}{4h_2^2}\right)^k \times\left(\frac{\Gamma(2n+i+2\mu_1)\Gamma(i+2k+2\mu_1)\Gamma(-i-2\mu_1+2\mu_2)}{\Gamma(i+2\mu_1)^2}\left(\frac{z^2}{\mathscr{E}_1\mathscr{E}_2}\right)^{i+2\mu_1} + \frac{\Gamma(-i+2\mu_1-2\mu_2)\Gamma(2n+i+2\mu_2)\Gamma(i+2k+2\mu_2)}{\Gamma(i+2\mu_2)^2}\left(\frac{z^2}{\mathscr{E}_1\mathscr{E}_2}\right)^{i+2\mu_2}\right). \tag{C.64}$$

The PDF of the product of two η-μ variates can be simplified to a single infinite series by performing the summation over the indexes n and k, which results in

$$f_Z(z) = \frac{2^{3-2\mu_1-2\mu_2}\pi}{z\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}+\mu_2\right)}\left(\left(\frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_1}\Gamma(-i-2\mu_1+2\mu_2) \times {}_2F_1\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right){}_2F_1\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right) + \Gamma(-i+2\mu_1-2\mu_2)\,{}_2F_1\left(\frac{i}{2}+\mu_2,\frac{1}{2}+\frac{i}{2}+\mu_2;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right) \times {}_2F_1\left(\frac{i}{2}+\mu_2,\frac{1}{2}+\frac{i}{2}+\mu_2;\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\left(\frac{z^2}{\mathscr{E}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}\right). \tag{C.65}$$

Further simplifications can be performed by using the gamma's duplication formula, the linear transformation [36, Equation (15.3.3)] on the hypergeometric functions, the reflection formula of the gamma function [36, Equation (6.1.17)] and the identity $(1-H^2/h^2) = 1/h$. After performing the aforementioned transformations and identities, the PDF of the product of two η-μ variates is given by

$$f_Z(z) = \frac{2\pi \csc(\pi(2\mu_2 - 2\mu_1))}{z h_1^{\mu_1} h_2^{\mu_2} \Gamma(2\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left(\frac{z^2 h_1 h_2}{\mathscr{E}_1^2 \mathscr{E}_2^2} \right)^{i+2\mu_1} \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2} + \mu_1; \frac{H_1^2}{h_1^2}\right)}{\Gamma(1+i+2\mu_1-2\mu_2)} \right.$$
$$\times {}_2F_1\left(\mu_2 - \mu_1 - \frac{i}{2}, \mu_2 - \mu_1 + \frac{1-i}{2}; \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right) - \left(\frac{z^2 h_1 h_2}{\mathscr{E}_1^2 \mathscr{E}_2^2}\right)^{i+2\mu_2} \tag{C.66}$$
$$\left. \times {}_2F_1\left(\mu_1 - \mu_2 - \frac{i}{2}, \mu_1 - \mu_2 + \frac{1-i}{2}; \frac{1}{2} + \mu_1; \frac{H_1^2}{h_1^2}\right) \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2} + \mu_2; \frac{H_2^2}{h_2^2}\right)}{\Gamma(1+i-2\mu_1+2\mu_2)} \right),$$

which completes the derivation.

C.4 The Product Distribution - CDF

From the results presented here, there are two alternatives to obtain the series representations for the CDF of the product distribution. 1) Compute the residues of the contour integral representations given in Table 4.2 in a similar way as for the PDF; or 2) by using the definition of the CDF, by integrating the PDF's series representations given in Table 4.3. The latter alternative is considerably simpler than the former. To obtain the series expressions for the CDF, the following integral is required

$$\int_0^z \tau^{x-1}\, dx = \frac{z^x}{x}. \tag{C.67}$$

C.4.1 The α-$\mu\times\kappa$-μ Distribution

The CDF for the product of the α-μ by the κ-μ variates is obtained by replacing (C.49) in the CDF's definition which results in

$$F_Z(z) = \int_0^z \left(\frac{2}{\tau\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{\tau^2}{\mathscr{A}^2 \mathscr{K}^2} \right)^{i+\mu_2} \Gamma\left(\mu_1 - \frac{2(i+\mu_2)}{\alpha_1}\right) {}_1F_1(i+\mu_2; \mu_2; -\kappa_2\mu_2) \right.\right.$$
$$\left.\left. + \frac{\alpha_1}{2} \left(\frac{\tau^2}{\mathscr{A}^2 \mathscr{K}^2}\right)^{\frac{1}{2}\alpha_1(i+\mu_1)} \Gamma\left(\mu_2 - \frac{1}{2}\alpha_1(i+\mu_1)\right) {}_1F_1\left(\frac{\alpha_1(i+\mu_1)}{2}; \mu_2; -\kappa_2\mu_2\right)\right) d\tau. \right). \tag{C.68}$$

By changing the order of integration and summation, the CDF results in

$$F_Z(z) = \frac{2}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{1}{\mathscr{A}^2 \mathscr{K}^2} \right)^{i+\mu_2} \Gamma\left(\mu_1 - \frac{2(i+\mu_2)}{\alpha_1}\right) {}_1F_1(i+\mu_2; \mu_2; -\kappa_2\mu_2) \right.$$
$$\times \int_0^z \tau^{2i+2\mu_2-1}\, d\tau + \frac{\alpha_1}{2}\left(\frac{1}{\mathscr{A}^2 \mathscr{K}^2}\right)^{\frac{1}{2}\alpha_1(i+\mu_1)} \Gamma\left(\mu_2 - \frac{1}{2}\alpha_1(i+\mu_1)\right) \tag{C.69}$$
$$\left. \times {}_1F_1\left(\frac{\alpha_1(i+\mu_1)}{2}; \mu_2; -\kappa_2\mu_2\right) \int_0^z \tau^{\alpha_1(i+\mu_1)-1}\, d\tau \right).$$

Using (C.67), the CDF is given as

$$F_Z(z) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{z^2}{\mathscr{A}^2 \mathscr{K}^2} \right)^{i+\mu_2} \frac{\Gamma\left(\mu_1 - \frac{2(i+\mu_2)}{\alpha_1}\right)}{i+\mu_2} {}_1F_1(i+\mu_2; \mu_2; -\kappa_2\mu_2) \right.$$
$$\left. + \left(\frac{z^2}{\mathscr{A}^2 \mathscr{K}^2} \right)^{\frac{\alpha_1(i+\mu_1)}{2}} \frac{\Gamma\left(\mu_2 - \frac{\alpha_1(i+\mu_1)}{2}\right)}{i+\mu_1} {}_1F_1\left(\frac{\alpha_1(i+\mu_1)}{2}; \mu_2; -\kappa_2\mu_2 \right) \right). \tag{C.70}$$

Which completes the derivation.

C.4.2 The α-μ×η-μ Distribution

The CDF for the product of the α-μ by the η-μ variates is obtained by replacing (C.54) in the CDF's definition which results in

$$F_Z(z) = \int_0^z \left(\frac{2}{\tau h_2^{\mu_2} \Gamma(\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{\tau^2 h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{i+2\mu_2} \Gamma\left(\mu_1 - \frac{2(i+2\mu_2)}{\alpha_1} \right) \right.\right.$$
$$\times {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) + \frac{\alpha_1}{2} \left(\frac{\tau^2 h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{\frac{\alpha_1(i+\mu_1)}{2}} \Gamma\left(2\mu_2 - \frac{\alpha_1(i+\mu_1)}{2} \right) \tag{C.71}$$
$$\left.\left. \times {}_2F_1\left(\mu_2 - \frac{\alpha_1(i+\mu_1)}{4}, \mu_2 - \frac{\alpha_1(i+\mu_1)}{4} + \frac{1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) \right) \right) d\tau.$$

By changing the order of integration and summation, the CDF results in

$$F_Z(z) = \frac{2}{h_2^{\mu_2} \Gamma(\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{i+2\mu_2} \Gamma\left(\mu_1 - \frac{2(i+2\mu_2)}{\alpha_1} \right) \right.$$
$$\times {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) \int_0^z \tau^{2(i+2\mu_2-1)} d\tau + \frac{\alpha_1}{2} \left(\frac{h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{\frac{\alpha_1(i+\mu_1)}{2}} \Gamma\left(2\mu_2 - \frac{\alpha_1(i+\mu_1)}{2} \right)$$
$$\left. \times {}_2F_1\left(\mu_2 - \frac{\alpha_1(i+\mu_1)}{4}, \mu_2 - \frac{\alpha_1(i+\mu_1)}{4} + \frac{1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) \right) \int_0^z \tau^{\alpha_1(i+\mu_1)-1} d\tau. \tag{C.72}$$

Using (C.67), the CDF is given as

$$F_Z(z) = \frac{1}{h_2^{\mu_2} \Gamma(\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\left(\frac{z^2 h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{i+2\mu_2} \frac{\Gamma\left(\mu_1 - \frac{2(i+2\mu_2)}{\alpha_1} \right)}{i+2\mu_2} \right.$$
$$\times {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) + \left(\frac{z^2 h_2}{\mathscr{A}^2 \mathscr{E}^2} \right)^{\frac{\alpha_1(i+\mu_1)}{2}} \frac{\Gamma\left(2\mu_2 - \frac{\alpha_1(i+\mu_1)}{2} \right)}{i+\mu_1} \tag{C.73}$$
$$\left. \times {}_2F_1\left(\mu_2 - \frac{\alpha_1(i+\mu_1)}{4}, \mu_2 - \frac{\alpha_1(i+\mu_1)}{4} + \frac{1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2} \right) \right).$$

Which completes the derivation.

C.4.3 The κ-μ×κ-μ Distribution

The CDF for the product of two κ-μ variates is obtained by replacing (C.57) in the CDF's definition which results in

$$\begin{aligned}F_Z(z) = &\int_0^z \left(\frac{2}{\tau\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^{-i}}{i!}\left(\left(\frac{\tau^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_2+i}\Gamma(-i+\mu_1-\mu_2)\right.\right.\\ &\times\ {}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\ {}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\\ &\left.\left.+\left(\frac{\tau^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_1+i}\Gamma(-i-\mu_1+\mu_2)\ {}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\ {}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right)\right).\end{aligned} \tag{C.74}$$

By changing the order of integration and summation, the CDF results in

$$\begin{aligned}F_Z(z) = &\frac{2}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^{-i}}{i!}\left(\left(\frac{1}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_2+i}\Gamma(-i+\mu_1-\mu_2)\right.\\ &\times\ {}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\ {}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\int_0^z \tau^{2(\mu_2+i)-1}\,d\tau\\ &+\left(\frac{1}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_1+i}\Gamma(-i-\mu_1+\mu_2)\ {}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\\ &\left.\times\ {}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\int_0^z \tau^{2(\mu_1+i)-1}\,d\tau\right).\end{aligned} \tag{C.75}$$

Using (C.67), the CDF is given as

$$\begin{aligned}F_Z(z) = &\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^{-i}}{i!}\left(\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_2+i}\frac{\Gamma(-i+\mu_1-\mu_2)}{i+\mu_2}\right.\\ &\times\ {}_1F_1(i+\mu_2;\mu_1;-\kappa_1\mu_1)\ {}_1F_1(i+\mu_2;\mu_2;-\kappa_2\mu_2)\\ &\left.+\left(\frac{z^2}{\mathscr{K}_1^2\mathscr{K}_2^2}\right)^{\mu_1+i}\frac{\Gamma(-i-\mu_1+\mu_2)}{i+\mu_1}\ {}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\ {}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2)\right).\end{aligned} \tag{C.76}$$

Which completes the derivation.

C.4.4 The κ-μ×η-μ Distribution

The CDF for the product of the κ-μ by the η-μ variates is obtained by replacing (C.62) in the CDF's definition which results in

$$\begin{aligned}F_Z(z) = &\int_0^z \frac{2\pi\csc(\pi(2\mu_2-\mu_1))}{\tau h_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\left(\frac{\tau^2 h_2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}\frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i+\mu_1-2\mu_2)}\right.\\ &\times\ {}_2F_1\left(\mu_2-\frac{i+\mu_1}{2},\mu_2+\frac{1-i-\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)-\frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i-\mu_1+2\mu_2)}\\ &\left.\times\ {}_2F_1\left(-\frac{i}{2},\frac{1-i}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\left(\frac{\tau^2 h_2}{\mathscr{E}_2^2\mathscr{K}_1^2}\right)^{i+2\mu_2}\right)d\tau.\end{aligned} \tag{C.77}$$

By changing the order of integration and summation, the CDF results in

$$F_Z(z) = \frac{2\pi \csc(\pi(2\mu_2-\mu_1))}{h_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \Bigg(\left(\frac{h_2}{\mathcal{E}_2^2 \mathcal{K}_1^2} \right)^{i+\mu_1} \frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i+\mu_1-2\mu_2)}$$
$$\times\ {}_2F_1\left(\mu_2 - \frac{i+\mu_1}{2}, \mu_2 + \frac{1-i-\mu_1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \int_0^z \tau^{2(i+\mu_1)-1}\, d\tau - \frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{\Gamma(1+i-\mu_1+2\mu_2)}$$
$$\times\ {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \left(\frac{h_2}{\mathcal{E}_2^2 \mathcal{K}_1^2} \right)^{i+2\mu_2} \int_0^z \tau^{2(i+2\mu_2)-1}\, d\tau \Bigg). \tag{C.78}$$

Using (C.67), the CDF is given as

$$F_Z(z) = \frac{\pi \csc(\pi(2\mu_2-\mu_1))}{h_2^{\mu_2}\Gamma(\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \Bigg(\left(\frac{z^2 h_2}{\mathcal{E}_2^2 \mathcal{K}_1^2} \right)^{i+\mu_1} \frac{{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{(i+\mu_1)\Gamma(1+i+\mu_1-2\mu_2)}$$
$$\times\ {}_2F_1\left(\mu_2 - \frac{i+\mu_1}{2}, \mu_2 + \frac{1-i-\mu_1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) - \frac{{}_1F_1(i+2\mu_2;\mu_1;-\kappa_1\mu_1)}{(i+2\mu_2)\Gamma(1+i-\mu_1+2\mu_2)} \tag{C.79}$$
$$\times\ {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \left(\frac{z^2 h_2}{\mathcal{E}_2^2 \mathcal{K}_1^2} \right)^{i+2\mu_2} \Bigg).$$

Which completes the derivation.

C.4.5 The η-μ×η-μ Distribution

The CDF for the product of two η-μ variates is obtained by replacing (C.66) in the CDF's definition which results in

$$F_Z(z) = \int_0^z \frac{2\pi \csc(\pi(2\mu_2-2\mu_1))}{\tau h_1^{\mu_1} h_2^{\mu_2}\Gamma(2\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \Bigg(\left(\frac{\tau^2 h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_1} \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)}{\Gamma(1+i+2\mu_1-2\mu_2)}$$
$$\times\ {}_2F_1\left(\mu_2-\mu_1-\frac{i}{2}, \mu_2-\mu_1+\frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) - \left(\frac{\tau^2 h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_2}$$
$$\times\ {}_2F_1\left(\mu_1-\mu_2-\frac{i}{2}, \mu_1-\mu_2+\frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)}{\Gamma(1+i-2\mu_1+2\mu_2)} \Bigg)\, d\tau. \tag{C.80}$$

by changing the order of integration and summation, the CDF results in

$$F_Z(z) = \frac{2\pi \csc(\pi(2\mu_2 - 2\mu_1))}{h_1^{\mu_1} h_2^{\mu_2} \Gamma(2\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left(\frac{h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_1} \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)}{\Gamma(1+i+2\mu_1-2\mu_2)} \right.$$
$$\times {}_2F_1\left(\mu_2-\mu_1-\frac{i}{2}, \mu_2-\mu_1+\frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \int_0^z \tau^{2(i+2\mu_1)-1}\, d\tau - \left(\frac{h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_2}$$
$$\times {}_2F_1\left(\mu_1-\mu_2-\frac{i}{2}, \mu_1-\mu_2+\frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)$$
$$\left. \times \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)}{\Gamma(1+i-2\mu_1+2\mu_2)} \int_0^z \tau^{2(i+2\mu_2)-1}\, d\tau \right). \tag{C.81}$$

Using (C.67), the CDF is given as

$$F_Z(z) = \frac{\pi \csc(\pi(2\mu_2 - 2\mu_1))}{\tau h_1^{\mu_1} h_2^{\mu_2} \Gamma(2\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left(\frac{z^2 h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_1} \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)}{(i+2\mu_1)\Gamma(1+i+2\mu_1-2\mu_2)} \right.$$
$$\times {}_2F_1\left(\mu_2-\mu_1-\frac{i}{2}, \mu_2-\mu_1+\frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) - \left(\frac{z^2 h_1 h_2}{\mathcal{E}_1^2 \mathcal{E}_2^2} \right)^{i+2\mu_2}$$
$$\left. \times {}_2F_1\left(\mu_1-\mu_2-\frac{i}{2}, \mu_1-\mu_2+\frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \frac{{}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)}{(i+2\mu_2)\Gamma(1+i-2\mu_1+2\mu_2)} \right). \tag{C.82}$$

Which completes the derivation.

Appendix D

Derivation of the Integral of the Product of a PDF by a CDF

In addition to the statistics of the product of two random envelopes, the integral of the product of the PDF by the CDF, which is closely related with the CDF of the product distributions, was presented here. The derivation of the multivariable Fox H-function and their series representation are provided here.

The desired integral is

$$P(\gamma_1,\gamma_2) = \int_0^{\gamma_1} f_{R_1}(r) F_{R_2}\left(\frac{\gamma_2}{r}\right) dr, \tag{D.1}$$

in which R_1 and R_2 are independent random envelopes.

D.1 The Integral Involving α-μ PDF $\times \alpha$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the α-μ distribution with parameters $\{\alpha_1, \mu_1, \hat{r}_1\}$ and $\{\alpha_2, \mu_2, \hat{r}_2\}$ respectively. Then (D.1) is solved by replacing (2.9) and (2.10) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2) = \int_0^{\gamma_1} \frac{\alpha_1}{\Gamma(\mu_1)} \frac{r^{\alpha_1\mu_1 - 1}}{\mathscr{A}_1^{\alpha_1\mu_1}} \exp\left(-\frac{r^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right) \frac{\Gamma\left(\mu_2, (r\mathscr{A}_2)^{-\alpha_2}\gamma_2^{\alpha_2}\right)}{\Gamma(\mu_2)} dr \tag{D.2}$$

To obtain a multivariable Fox H-function representation for the above integral, it is required to put the exponential and incomplete gamma function in terms of their Mellin-Barnes contour integral representation using (A.1) and [41, Equation (8.4.16.1)] respectively. After replacing them in (D.2), $P(\gamma_1,\gamma_2)$ reduces, after some algebraic manipulations, to

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)\mathscr{A}_1^{\alpha_1\mu_1}}\left(\frac{1}{2\pi j}\right)^2\int_0^{\gamma_1} r^{\alpha_1\mu_1-1}\oiint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(-t_2)\Gamma(t_2+\mu_2)}{\Gamma(1-t_2)} \times\left(\frac{r^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_1}\left(\frac{\gamma_2^{\alpha_2}}{(r\mathscr{A}_2)^{\alpha_2}}\right)^{-t_2}dt_1dt_2dr. \tag{D.3}$$

Changing the order of integration will result in

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)\mathscr{A}_1^{\alpha_1\mu_1}}\left(\frac{1}{2\pi j}\right)^2\oiint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(-t_2)\Gamma(t_2+\mu_2)}{\Gamma(1-t_2)} \times\left(\frac{1}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_1}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}}\right)^{-t_2}\int_0^{\gamma_1} r^{\alpha_1\mu_1-\alpha_1t_1+\alpha_2t_2-1}drdt_1dt_2. \tag{D.4}$$

Remember that the inner integral can be solved with the known result

$$\int_0^z \tau^{a\nu-1}d\tau=\frac{\Gamma(\nu)}{a\Gamma(1+\nu)}z^{a\nu} \tag{D.5}$$

Ergo, the integral $P(\gamma_1,\gamma_2)$ can be written in terms of a double Mellin-Barnes contour integral as

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1\gamma_1^{\alpha_1\mu_1}}{\Gamma(\mu_1)\Gamma(\mu_2)\mathscr{A}_1^{\alpha_1\mu_1}}\left(\frac{1}{2\pi j}\right)^2\oiint_{\mathscr{L}}\frac{\Gamma(\alpha_1(-t_1+\mu_1)+\alpha_2t_2)}{\Gamma(1+\alpha_1(-t_1+\mu_1)+t_2\alpha_2)} \times\frac{\Gamma(t_1)\Gamma(-t_2)\Gamma(t_2+\mu_2)}{\Gamma(1-t_2)}\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_1}\left(\frac{\gamma_2^{\alpha_2}}{(\gamma_1\mathscr{A}_2)^{\alpha_2}}\right)^{-t_2}dt_1dt_2. \tag{D.6}$$

Finally, the variable transformation $s_1=t_1-\mu_1$ is performed and, after some algebraic manipulations, $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^2\oiint_{\mathscr{L}}\frac{\Gamma(-\alpha_1s_1+\alpha_2t_2)}{\Gamma(1-\alpha_1s_1+\alpha_2t_2)} \times\frac{\Gamma(s_1+\mu_1)\Gamma(t_2+\mu_2)\Gamma(-t_2)}{\Gamma(1-t_2)}\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-s_1}\left(\frac{\gamma_2^{\alpha_2}}{(\gamma_1\mathscr{A}_2)^{\alpha_2}}\right)^{-t_2}ds_1dt_2. \tag{D.7}$$

Comparing (D.7) with (2.1), the parameters provided in Table 4.6 are easily deduced.

A series representation for $P(\gamma_1,\gamma_2)$ can be obtained through the sum of residues. From (D.7), the residues around the poles of $\Gamma(s_1+\mu_1)$ for the variable s_1 and for the variable t_2 the poles of $\Gamma(t_2+\mu_2)$ and $\Gamma(-\alpha_1s_1+\alpha_2t_2)$ are taken. These operation will result in the double summation

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{A}_2}\right)^{\alpha_1(i+\mu_1)}I_1(i)+\left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1(i+\mu_1)}I_2(i)\right) \tag{D.8}$$

In which $I_1(i)$ and $I_2(i)$ are given, respectively, as

$$I_1(i)=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(-\frac{k+\alpha_1(i+\mu_1)}{\alpha_2}+\mu_2\right)}{k!\Gamma(1-k)(k+\alpha_1(i+\mu_1))}\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^k, \tag{D.9}$$

and

$$I_2(i) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+\mu_2)\,(\alpha_1\,(i+\mu_1)-\alpha_2\,(k+\mu_2))} \left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2(k+\mu_2)} \tag{D.10}$$

The former can be easily solve by noting that the summand vanishes for $k > 0$ resulting in

$$I_1(i) = \frac{\Gamma\left(-\frac{\alpha_1(i+\mu_1)}{\alpha_2}+\mu_2\right)}{\alpha_1\,(i+\mu_1)} \tag{D.11}$$

On its turn, $I_2(i)$ can be solved first in terms of the hypergeometric function as

$$I_2(i) = \frac{{}_2F_2\left(\mu_2,\mu_2-\frac{\alpha_1(i+\mu_1)}{\alpha_2};\mu_2+1,1+\mu_2-\frac{\alpha_1(i+\mu_1)}{\alpha_2};-\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)}{\alpha_2\mu_2\left(\frac{\alpha_1(i+\mu_1)}{\alpha_2}-\mu_2\right)}\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2\mu_2}, \tag{D.12}$$

by using the identity $1/(a+k) = (1/a)\times(a)_k/(1+a)_k$ in which $(a)_k$ denotes the Pochhammer symbol. Further simplifications are obtained by using the identity [41, Equation (7.12.1.3)] to put I_2 in terms of the Kummer's hypergeometric function, and [41, Equation (7.11.1.12)] after applying the Kummer's transformation [36, Equation (13.1.27)]. Therefore, I_2 is given in terms of the incomplete gamma function as

$$I_2(i) = \frac{1}{\alpha_1\,(i+\mu_1)}\left(\Gamma\left(\mu_2,0,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)-\Gamma\left(\mu_2-\frac{\alpha_1\,(i+\mu_1)}{\alpha_2},0,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_1(i+\mu_1)}\right). \tag{D.13}$$

Replacing $I_1(i)$ and $I_2(i)$ in (D.8) and after some algebraic manipulations $P(\gamma_1,\gamma_2)$ is obtained as

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(i+\mu_1)} \left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1(i+\mu_1)} \Bigg(\gamma\left(\mu_2,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right) \\ +\Gamma\left(\mu_2-\frac{\alpha_1\,(i+\mu_1)}{\alpha_2},\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\left(\frac{\gamma_2}{\gamma_1\mathscr{A}_2}\right)^{\alpha_1(i+\mu_1)}\Bigg), \end{aligned} \tag{D.14}$$

in which $\Gamma(a,b)$ is the upper incomplete gamma function [36, Equation (6.5.3)]. Finally, using the identity [36, Equation (5.1.45)], the function $P(\gamma_1,\gamma_2)$ can be written as

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(i+\mu_1)} \left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1(i+\mu_1)} \Bigg(\gamma\left(\mu_2,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right) \\ + E_{1-\mu_2+\frac{\alpha_1(i+\mu_1)}{\alpha_2}}\left(\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2\mu_2}\Bigg), \end{aligned} \tag{D.15}$$

in which $E_n(x)$ is the exponential integral function [36, Equation (5.1.4)]. Equation (D.15) is the exact same provided in Table 4.7 which completes the derivation.

D.2 The Integral Involving α-μ PDF $\times\kappa$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the α-μ and κ-μ distributions respectively with parameters $\{\alpha_1,\mu_1,\hat{r}_1\}$ and $\{\kappa_2,\mu_2,\hat{r}_2\}$. Then (D.1) is solved by replacing (2.9) and (2.14) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2)=\int_0^{\gamma_1}\frac{\alpha_1}{\Gamma(\mu_1)}\frac{r^{\alpha_1\mu_1-1}}{\mathscr{A}_1^{\alpha_1\mu_1}}\exp\left(-\frac{r^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)\left(\frac{\gamma_2^2}{r^2\mathscr{K}_2^2}\right)^{\mu_2}\left(\frac{1}{2\pi j}\right)^2 \times\oiint_{\mathscr{L}}\frac{\Gamma(\mu_2-t_1-t_2)\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_2-t_1)\Gamma(\mu_2+1-t_2)}(\kappa_2\mu_2)^{-t_1}\left(\frac{\gamma_2^2}{r^2\mathscr{K}_2^2}\right)^{-t_2}dt_1dt_2dr. \tag{D.16}$$

Using (A.1), and changing the order of integration the function $P(\gamma_1,\gamma_2)$ is written as

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)}\left(\frac{1}{2\pi i}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(\mu_2-t_1-t_2)\Gamma(t_1)\Gamma(t_2)\Gamma(t_3)}{\Gamma(\mu_2-t_1)\Gamma(\mu_2+1-t_2)} \times(\kappa_2\mu_2)^{-t_1}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2}\right)^{-t_2+\mu_2}\left(\frac{1}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_3+\mu_1}\int_0^{\gamma_1}r^{-1+2t_2+\alpha_1(-t_3+\mu_1)-2\mu_2}\,drdt_1dt_2dt_3. \tag{D.17}$$

The inner integral can be solved with the help of (D.5) resulting, after some algebraic manipulations and the variable transformations $s_2=t_2-\mu_2$ and $s_3=t_3-\mu_1$, in

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-t_1-s_2)\Gamma(2s_2-s_3\alpha_1)}{\Gamma(1+2s_2-s_3\alpha_1)} \times\frac{\Gamma(t_1)\Gamma(\mu_2+s_2)\Gamma(\mu_1+s_3)}{\Gamma(\mu_2-t_1)\Gamma(1-s_2)}(\kappa_2\mu_2)^{-t_1}\left(\frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}\right)^{-s_2}\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-s_3}dt_1ds_2ds_3. \tag{D.18}$$

By comparing the above triple integral with (2.1), the parameters for the Fox H-function are readily obtained as they are provided in Table 4.6.

Series representation can be obtained by taking the residues of (D.18) around the poles of $\Gamma(t_1)$ for the variable t_1, $\Gamma(s_3+\mu_1)$ for s_3, and $\Gamma(s_2+\mu_2)$ and $\Gamma(2s_2-s_3\alpha_1)$ for the variable s_2. This will generate a triple summation representation for the function $P(\gamma_1,\gamma_2)$, which is given by

$$P(\gamma_1,\gamma_2)=\frac{\alpha_1}{\Gamma(\mu_1)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{i+\mu_2}I_1(i)+\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{K}_2}\right)^{\alpha_1(i+\mu_1)}I_2(i)\right), \tag{D.19}$$

in which $I_1(i)$ and $I_2(i)$ are given, respectively, as

$$I_1(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\Gamma(i+n+\mu_2)(\kappa_2\mu_2)^n}{(\alpha_1(k+\mu_1)-2(i+\mu_2))n!k!\Gamma(1+i+\mu_2)\Gamma(n+\mu_2)}\left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1(k+\mu_1)}, \tag{D.20}$$

and

$$I_2(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\Gamma\left(n+\frac{1}{2}(\alpha_1(i+\mu_1)+k)\right)\Gamma\left(\mu_2-\frac{1}{2}(\alpha_1(i+\mu_1)+k)\right)}{2n!k!\Gamma(1-k)\Gamma\left(1+\frac{1}{2}(\alpha_1(i+\mu_1)+k)\right)\Gamma(n+\mu_2)(\kappa_2\mu_2)^{-n}}\left(\frac{\gamma_2}{\mathscr{K}_2\gamma_1}\right)^k. \tag{D.21}$$

A closed form expression for $I_1(i)$ is obtained by performing the summation over the index n which results in the Kummer's confluent hypergeometric function and over the index k with the help of [36, Equaiton (6.5.29)], ergo the function $I_1(i)$ is obtained as

$$I_1(i) = \frac{\gamma\left(-\frac{2i}{\alpha_1}+\mu_1-\frac{2\mu_2}{\alpha_1},\left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1}\right)\,{}_1\tilde{F}_1\left(i+\mu_2;\mu_2;-\kappa_2\mu_2\right)}{\left(i+\mu_2\right)\alpha_1}\left(\frac{\gamma_1^2}{\mathscr{A}_1^2}\right)^{i+\mu_2} \tag{D.22}$$

In its turn, $I_2(i)$ vanishes for any $k>0$, and the resulting summation can be written in terms of the Kummer's hypergeometric function as

$$I_2(i) = \frac{\Gamma\left(\mu_2-\frac{\alpha_1(i+\mu_1)}{2}\right)}{\alpha_1\left(i+\mu_1\right)}\,{}_1\tilde{F}_1\left(\frac{\alpha_1\left(i+\mu_1\right)}{2};\mu_2;-\kappa_2\mu_2\right) \tag{D.23}$$

Then, $P(\gamma_1,\gamma_2)$ is obtained by replacing $I_1(i)$ and $I_2(i)$ in (D.19), which results, after some algebraic manipulations, in

$$\begin{aligned}P(\gamma_1,\gamma_2) &= \frac{1}{\Gamma\left(\mu_1\right)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{{}_1\tilde{F}_1\left(i+\mu_2;\mu_2;-\kappa_2\mu_2\right)}{i+\mu_2}\gamma\left(\mu_1-\frac{2\left(i+\mu_2\right)}{\alpha_1},\left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{\alpha_1}\right)\right.\\ &\times\left.\left(\frac{\gamma_2^2}{\mathscr{A}_1^2\mathscr{K}_2^2}\right)^{i+\mu_2}+\frac{\Gamma\left(\mu_2-\frac{1}{2}\alpha_1\left(i+\mu_1\right)\right)}{i+\mu_1}\,{}_1\tilde{F}_1\left(\frac{1}{2}\alpha_1\left(i+\mu_1\right);\mu_2;-\kappa_2\mu_2\right)\left(\frac{\gamma_2}{\mathscr{A}_1\mathscr{K}_2}\right)^{\alpha_1(i+\mu_1)}\right).\end{aligned} \tag{D.24}$$

Which completes the derivation of the series representation of the integral of the PDF of the α-μ and the CDF of the κ-μ distributions.

D.3 The Integral Involving κ-μ PDF $\times\alpha$-μ CDF

Let $R_1>0$ and $R_2>0$ be two random envelopes following the κ-μ and the α-μ distributions respectively with parameters $\{\kappa_1,\mu_1,\hat{r}_1\}$ and $\{\alpha_2,\mu_2,\hat{r}_2\}$. Then, (D.1) is solved by replacing (2.13) and (2.10) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2) = \int_0^{\gamma_1}\frac{2}{\exp\left(\kappa_1\mu_1\right)}\frac{r^{2\mu_1-1}}{\mathscr{K}_1^2}\exp\left(-\frac{r^2}{\mathscr{K}_1^2}\right){}_0\tilde{F}_1\left(;\mu_1;\frac{r^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right)\frac{\gamma\left(\mu_2,\left(\frac{\gamma_2}{r\mathscr{A}_2}\right)^{\alpha_2}\right)}{\Gamma\left(\mu_2\right)}dr. \tag{D.25}$$

This integral can be solved in terms of the Fox H-function by putting the exponential, hypergeometric and incomplete gamma functions in terms of their Mellin-Barnes contour integral representations resulting, after some algebraic manipulations, in

$$\begin{aligned}P(\gamma_1,\gamma_2) &= \frac{2}{\Gamma\left(\mu_2\right)\exp\left(\kappa_1\mu_1\right)}\left(\frac{1}{2\pi j}\right)^3\int_0^{\gamma_1}\frac{1}{r}\oiiint_{\mathscr{L}}\frac{\Gamma\left(t_1\right)\Gamma\left(t_2\right)\Gamma\left(-t_3\right)\Gamma\left(t_3+\mu_2\right)}{\Gamma\left(1-t_3\right)\Gamma\left(-t_2+\mu_1\right)}\\ &\times\left(\frac{r^2}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1}\left(-\frac{r^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2}\left(\frac{\gamma_2}{r\mathscr{A}_2}\right)^{-\alpha_2 t_3}dt_1dt_2dt_3dr.\end{aligned} \tag{D.26}$$

Changing the order of integration results in

$$P(\gamma_1,\gamma_2)=\frac{2}{\Gamma(\mu_2)\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(t_2)\Gamma(-t_3)\Gamma(t_3+\mu_2)}{\Gamma(1-t_3)\Gamma(-t_2+\mu_1)}$$
$$\times\left(\frac{1}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1}\left(-\frac{\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2}\left(\frac{\gamma_2}{\mathscr{A}_2}\right)^{-\alpha_2 t_3}\int_0^{\gamma_1} r^{2(\mu_1-t_1-t_2)+\alpha_2 t_3-1}dr\,dt_1dt_2dt_3. \tag{D.27}$$

With the help of (D.5), the inner integral can be solved and the function $P(\gamma_1,\gamma_2)$ is obtained as

$$P(\gamma_1,\gamma_2)=\frac{2}{\Gamma(\mu_2)\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-2(t_1+t_2)+t_3\alpha_2+2\mu_1)}{\Gamma(1-2(t_1+t_2-\mu_1)+t_3\alpha_2)}$$
$$\times\frac{\Gamma(t_1)\Gamma(t_2)\Gamma(\mu_2+t_3)\Gamma(-t_3)}{\Gamma(1-t_3)\Gamma(\mu_1-t_2)}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1}\left(-\frac{\gamma_1^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2}\left(\frac{\gamma_2}{\gamma_1\mathscr{A}_2}\right)^{-\alpha_2 t_3}dt_1dt_2dt_3. \tag{D.28}$$

Now, by performing the variable transformation $s_1=t_2$ and $s_2=t_1+t_2-\mu_1$, after some algebraic manipulations, the integral $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2)=\frac{2}{\Gamma(\mu_2)\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-2s_2+t_3\alpha_2)\Gamma(-s_1+s_2+\mu_1)}{\Gamma(1-t_3)\Gamma(1-2s_2+t_3\alpha_2)}$$
$$\times\frac{\Gamma(s_1)\Gamma(t_3+\mu_2)\Gamma(-t_3)}{\Gamma(-s_1+\mu_1)}(-\kappa_1\mu_1)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2}\left(\frac{\gamma_2}{\gamma_1\mathscr{A}_2}\right)^{-\alpha_2 t_3}ds_1ds_2dt_3. \tag{D.29}$$

Note that isolating the variable s_1, the contour integral resultant is the representation for the Kummer's confluent hypergeometric function. Using the Kummer's transformation the following identity holds

$$\oint_{\mathscr{L}}\frac{\Gamma(s_1)\Gamma(-s_1+s_2+\mu_1)}{\Gamma(\mu_1-s_1)}(-\kappa_1\mu_1)^{-s_1}ds_1=\exp(\kappa_1\mu_1)$$
$$\times\oint_{\mathscr{L}}\frac{\Gamma(s_1)\Gamma(-s_1-s_2)\Gamma(s_2+\mu_1)}{\Gamma(-s_2)\Gamma(-s_1+\mu_1)}(\kappa_1\mu_1)^{-s_1}ds_1 \tag{D.30}$$

Replacing this identity in (D.29), and after some algebraic manipulations, the function $P(\gamma_1,\gamma_2)$ is obtained as

$$P(\gamma_1,\gamma_2)=\frac{2}{\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-2s_2+\alpha_2 t_3)\Gamma(-s_1-s_2)\Gamma(s_1)}{\Gamma(1-2s_2+\alpha_2 t_3)\Gamma(\mu_1-s_1)}$$
$$\times\frac{\Gamma(\mu_1+s_2)\Gamma(\mu_2+t_3)\Gamma(-t_3)}{\Gamma(-s_2)\Gamma(1-t_3)}(\kappa_1\mu_1)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2}\left(\frac{\gamma_2}{\gamma_1\mathscr{A}_2}\right)^{-\alpha_2 t_3}ds_1ds_2dt_3. \tag{D.31}$$

The parameters provided in Table 4.6 for the integral of the PDF of the κ-μ by the CDF of the α-μ are readily obtained by comparing (D.31) with (2.1), completing the derivation.

A possible series representation is obtained through the sum of residues by taking the residues around the poles of $\Gamma(s_1)$ for the integration variable s_1, $\Gamma(\mu_1+s_2)$ for the variable s_2 and $\Gamma(-2s_2+\alpha_2 t_3)$ and $\Gamma(\mu_2+t_3)$ for t_3. After some algebraic manipulations, the function $P(\gamma_1,\gamma_2)$ is written as the following triple summation

$$P(\gamma_1,\gamma_2)=\frac{2}{\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma(i+\mu_1)}\left(\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{i+\mu_1}I_1(i)+\left(\frac{\gamma_2^2}{\mathscr{A}_2^2\mathscr{K}_1^2}\right)^{i+\mu_1}I_2(i)\right), \tag{D.32}$$

in which $I_1(i)$ and $I_2(i)$ are given, respectively, as

$$I_1(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\Gamma(i+n+\mu_1)(\kappa_1\mu_1)^n}{n!k!\Gamma(n+\mu_1)(k+\mu_2)(2(i+\mu_1)-\alpha_2(k+\mu_2))}\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2(k+\mu_2)}, \tag{D.33}$$

and

$$I_2(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\Gamma(i+n+\mu_1)\Gamma\left(\frac{2i+k+2\mu_1}{\alpha_2}\right)\Gamma\left(\frac{-2i-k-2\mu_1+\alpha_2\mu_2}{\alpha_2}\right)}{n!k!\Gamma(1-k)\Gamma(n+\mu_1)\Gamma\left(\frac{2i+k+\alpha_2+2\mu_1}{\alpha_2}\right)\alpha_2}\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^k(\kappa_1\mu_1)^n. \tag{D.34}$$

The function $I_1(i)$ can be solved in closed-form by performing the summation over the index n using [41, Equation (7.2.3.1)]. Over the index k, $I_1(i)$ can be solved first in terms of the ${}_2F_2$ function and then using [41, Equation (7.12.1.3)] and [41, Equation (7.11.1.12)] to write $I_1(i)$ in terms of the incomplete gamma function as

$$\begin{aligned}I_1(i)=&\frac{\Gamma(i+\mu_1)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{2\Gamma(\mu_1)(i+\mu_1)}\left(\gamma\left(\mu_2,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\right.\\&\left.-\gamma\left(\mu_2-\frac{2(i+\mu_1)}{\alpha_2},\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\left(\frac{\gamma_2^2}{\mathscr{A}_2^2\gamma_1^2}\right)^{i+\mu_1}\right)\end{aligned} \tag{D.35}$$

In its turn, $I_2(i)$ vanishes for any $k>0$ and for the index j, the summation can be solve in terms of the Kummer's confluent hypergeometric function. Ergo, $I_2(i)$ is given as

$$I_2(i)=\frac{\Gamma(i+\mu_1)\Gamma\left(-\frac{2(i+\mu_1)}{\alpha_2}+\mu_2\right)}{2\Gamma(\mu_1)(i+\mu_1)}\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1). \tag{D.36}$$

Now, by replacing $I_1(i)$ and $I_2(i)$ in (D.32) and after some algebraic manipulation, $P(\gamma_1,\gamma_2)$ is obtained as

$$\begin{aligned}P(\gamma_1,\gamma_2)=&\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{i!(i+\mu_1)}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{i+\mu_1}\\&\times\left(\gamma\left(\mu_2,\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)+\Gamma\left(\mu_2-\frac{2(i+\mu_1)}{\alpha_2},\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right)\left(\frac{\gamma_2^2}{\mathscr{A}_2^2\gamma_1^2}\right)^{i+\mu_1}\right).\end{aligned} \tag{D.37}$$

To finalize the derivation, the identity [36, Equation (5.1.45)] is used to put the upper incomplete gamma function in terms of the exponential integral function so that the function

$P(\gamma_1, \gamma_2)$ is given as

$$P(\gamma_1,\gamma_2) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i \, {}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{i!\,(i+\mu_1)} \left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{i+\mu_1} \times \left(\gamma\left(\mu_2, \left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2}\right) + E_{1+\frac{2(i+\mu_1)}{\alpha_2}-\mu_2}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2\mu_2}\right). \tag{D.38}$$

Which completes the derivation of the series representation of the integral involving the product of the PDF of the κ-μ distribution by the CDF of the α-μ distribution.

D.4 The Integral Involving α-μ PDF $\times\eta$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the α-μ and η-μ distributions respectively, with parameters $\{\alpha_1, \mu_1, \hat{r}_1\}$ and $\{\eta_2, \mu_2, \hat{r}_2\}$. Then, (D.1) is solved by replacing (2.9) and (2.18) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2) = \int_0^{\gamma_1} \frac{2^{1-2\mu_2}\sqrt{\pi}\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{r^{\alpha_1\mu_1-1}}{\mathscr{A}_1^{\alpha_1\mu_1}} \exp\left(-\frac{r^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)\left(\frac{1}{2\pi j}\right)^2 \oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(2\mu_2-2t_1-t_2)}{\Gamma\left(\mu_2+\frac{1}{2}-t_1\right)} \times \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(1+2\mu_2-t_2)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_1}\left(\frac{\gamma_2^2}{r^2\mathscr{E}_2^2}\right)^{-t_2+2\mu_2} dt_1 dt_2 dr. \tag{D.39}$$

A multivariable Fox H-function representation can be obtained by replacing the exponential function in term of its Mellin-Barnes contour integral using (A.1) and then change the order of integration. This operation results, after some algebraic manipulations, in

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(t_1)\Gamma(t_2)\Gamma(t_3)\Gamma(-2t_1-t_2+2\mu_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_2\right)\Gamma(1-t_2+2\mu_2)} \times \left(-\frac{H_2^2}{4h_2^2}\right)^{-t_1}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2}\right)^{-t_2+2\mu_2}\left(\frac{1}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_3+\mu_1}\int_0^{\gamma_1} r^{-1+2t_2+\alpha_1(-t_3+\mu_1)-4\mu_2}\,dr\,dt_1 dt_2 dt_3. \tag{D.40}$$

The inner integral can be solved with the help of (D.5) which gives

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oint\!\!\oint\!\!\oint_{\mathscr{L}} \frac{\Gamma(2(t_2-2\mu_2)+\alpha_1(-t_3+\mu_1))\Gamma(t_1)}{\Gamma(1+2(t_2-2\mu_2)+\alpha_1(-t_3+\mu_1))} \times \frac{\Gamma(t_2)\Gamma(t_3)\Gamma(-2t_1-t_2+2\mu_2)}{\Gamma\left(\frac{1}{2}-t_1+\mu_2\right)\Gamma(1-t_2+2\mu_2)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_1}\left(\frac{\gamma_2^2}{\gamma_1^2\mathscr{E}_2^2}\right)^{-t_2+2\mu_2}\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-t_3+\mu_1} dt_1 dt_2 dt_3. \tag{D.41}$$

Further simplification is obtained by performing the variable transformations $t_3 = s_1 + \mu_1$, $t_1 = s_2$ and $t_2 = s_3 + 2\mu_2$, which results in

$$P(\gamma_1, \gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3 \oiiint_{\mathscr{L}} \frac{\Gamma(-\alpha_1 s_1 + 2s_3)\Gamma(-2s_2 - s_3)}{\Gamma(1-\alpha_1 s_1 + 2s_3)\Gamma\left(\mu_2 + \frac{1}{2} - s_2\right)}$$
$$\times \frac{\Gamma(\mu_1 + s_1)\Gamma(s_2)\Gamma(2\mu_2 + s_3)}{\Gamma(1-s_3)}\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{-s_1}\left(-\frac{H_2^2}{4h_2^2}\right)^{-s_2}\left(\frac{\gamma_2^2}{\gamma_1^2 \mathscr{E}_2^2}\right)^{-s_3} ds_1 ds_2 ds_3. \tag{D.42}$$

The parameters for the Fox H-function representation are readily obtained by comparing (D.42) with (2.1), completing the derivation.

A series representation can be obtained through by summing the residues around the poles of $\Gamma(\mu_1+s_1)$ for the variable s_1, $\Gamma(s_2)$ for the integration variable s_2 and $\Gamma(2\mu_2+s_3)$ and $\Gamma(-\alpha_1 s_2 + 2s_3)$ for the s_3 variable. Therefore, a triple infinite summation arises and it is given as

$$P(\gamma_1, \gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}\alpha_1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{1}{\Gamma(1+i+2\mu_2)}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{i+2\mu_2} I_1(i) + \left(\frac{\gamma_2^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}\mathscr{E}_2^{\alpha_1}}\right)^{i+\mu_1} I_2(i)\right) \tag{D.43}$$

in which $I_1(i)$ and $I_2(i)$ are given respectively as

$$I_1(i) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(i+2n+2\mu_2)}{k!n!\Gamma\left(\frac{1}{2}+n+\mu_2\right)(\alpha_1(k+\mu_1) - 2(i+2\mu_2))}\left(\frac{H_2^2}{4h_2^2}\right)^n\left(\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)^{k+\mu_1}, \tag{D.44}$$

and

$$I_2(i) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(2n + \frac{k+\alpha_1(i+\mu_1)}{2}\right)\Gamma\left(2\mu_2 - \frac{k+\alpha_1(i+\mu_1)}{2}\right)}{2n!k!\Gamma(1-k)\Gamma\left(1+\frac{k+\alpha_1(i+\mu_1)}{2}\right)\Gamma\left(\frac{1}{2}+n+\mu_2\right)}\left(\frac{H_2^2}{4h_2^2}\right)^n\left(\frac{\gamma_2}{\mathscr{E}_2\gamma_1}\right)^k. \tag{D.45}$$

The function $I_1(i)$ can be expressed in closed-form by, first summing over the index n resulting in the Gauss' hypergeometric function ${}_2F_1$. For the index k the result can be written in terms of the incomplete gamma function using [36, Equation (6.5.29)], which results in

$$I_1(i) = \frac{\Gamma(i+2\mu_2)\gamma\left(\mu_1 - \frac{2(i+2\mu_2)}{\alpha_1}, \frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)}{\Gamma\left(\frac{1}{2}+\mu_2\right)\alpha_1}\, {}_2F_1\left(\frac{i+2\mu_2}{2}, \frac{i+2\mu_2+1}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)\left(\frac{\gamma_1}{\mathscr{A}_1}\right)^{2(i+2\mu_2)}. \tag{D.46}$$

On its turn, $I_2(i)$ can be solved by first noticing that it vanishes for any $k > 0$ and then performing the summation over the index n, which results in

$$I_2(i) = \frac{\Gamma\left(2\mu_2 - \frac{1}{2}\alpha_1(i+\mu_1)\right)}{\alpha_1(i+\mu_1)\Gamma\left(\frac{1}{2}+\mu_2\right)}\, {}_2F_1\left(\frac{1}{4}\alpha_1(i+\mu_1), \frac{1}{2}+\frac{1}{4}\alpha_1(i+\mu_1); \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \tag{D.47}$$

By replacing $I_1(i)$ and $I_2(i)$ in (D.43) and by performing some algebraic manipulations, a series representation for the integral involving the PDF of the α-μ distribution and the CDF of η-μ distribution is obtained as

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma\left(\frac{1}{2}+\mu_2\right)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1},\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)}{i+2\mu_2}\right.$$
$$\times\left(\frac{\gamma_2^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}{}_2F_1\left(\frac{i}{2}+\mu_2,\frac{i}{2}+\mu_2+\frac{1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\left(\frac{\gamma_2^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}\mathscr{E}_2^{\alpha_1}}\right)^{i+\mu_1}$$
$$\left.\times\frac{\Gamma\left(2\mu_2-\frac{1}{2}\alpha_1(i+\mu_1)\right)}{i+\mu_1}{}_2F_1\left(\frac{1}{4}\alpha_1(i+\mu_1),\frac{1}{2}+\frac{1}{4}\alpha_1(i+\mu_1);\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right).\tag{D.48}$$

By using the duplication formula of the gamma function, the function $P(\gamma_1,\gamma_2)$ is simplified to

$$P(\gamma_1,\gamma_2)=\frac{1}{\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\gamma\left(\mu_1-\frac{2(i+2\mu_2)}{\alpha_1},\frac{\gamma_1^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}}\right)}{i+2\mu_2}\right.$$
$$\times\left(\frac{\gamma_2^2}{\mathscr{A}_1^2\mathscr{E}_2^2}\right)^{i+2\mu_2}{}_2F_1\left(\frac{i}{2}+\mu_2,\frac{i}{2}+\mu_2+\frac{1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)+\left(\frac{\gamma_2^{\alpha_1}}{\mathscr{A}_1^{\alpha_1}\mathscr{E}_2^{\alpha_1}}\right)^{i+\mu_1}$$
$$\left.\times\frac{\Gamma\left(2\mu_2-\frac{1}{2}\alpha_1(i+\mu_1)\right)}{i+\mu_1}{}_2F_1\left(\frac{1}{4}\alpha_1(i+\mu_1),\frac{1}{2}+\frac{1}{4}\alpha_1(i+\mu_1);\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right),\tag{D.49}$$

which is the exact expression provided in Table 4.7 for the integral of the product of the PDF of the α-μ and CDF of the η-μ distributions.

D.5 The Integral Involving η-μ PDF $\times\alpha$-μ CDF

Let $R_1>0$ and $R_2>0$ be two random envelopes following the η-μ and the α-μ distributions respectively, with parameters $\{\eta_1,\mu_1,\hat{r}_1\}$ and $\{\alpha_2,\mu_2,\hat{r}_2\}$. Then (D.1) is solved by replacing (2.17) and (2.10) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2)=\int_0^{\gamma_1}\frac{2h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)}\frac{r^{4\mu_1-1}}{\mathscr{E}_1^{4\mu_1}}\exp\left(-\frac{h_1r^2}{\mathscr{E}_1^2}\right)$$
$$\times{}_0F_1\left(;\mu_1+\frac{1}{2};\frac{H_1^2r^4}{4\mathscr{E}_1^4}\right)\gamma\left(\mu_2,\frac{\gamma_2^{\alpha_2}}{r^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)dr\tag{D.50}$$

The Fox H-function representation for the above integral can be obtained by replacing the exponential, hypergeometric and the incomplete gamma functions with their respective Mellin-Barnes contour integral representation given at (A.1), (A.2) and [41, Equation (8.4.16.1)] respectively. The integral $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{-\mu_1}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\int_0^{\gamma_1}\frac{1}{r}\oiiint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(t_2)\Gamma(-t_3)\Gamma(t_3+\mu_2)}{\Gamma(1-t_3)\Gamma\left(\frac{1}{2}-t_2+\mu_1\right)}$$
$$\times\left(\frac{h_1r^2}{\mathscr{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2r^4}{4\mathscr{E}_1^4}\right)^{-t_2}\left(\frac{\gamma_2^{\alpha_2}}{r^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}dt_1dt_2dt_3dr.\tag{D.51}$$

Changing the order of integration will result in

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{-\mu_1}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(t_2)\Gamma(-t_3)\Gamma(t_3+\mu_2)}{\Gamma(1-t_3)\Gamma\left(\frac{1}{2}-t_2+\mu_1\right)}$$
$$\times\left(\frac{h_1}{\mathscr{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2}{4\mathscr{E}_1^4}\right)^{-t_2}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}\int_0^{\gamma_1}r^{2(-t_1+2\mu_1)-4t_2+\alpha_2t_3-1}dr\,dt_1dt_2dt_3. \tag{D.52}$$

The inner integral can be solved with the help of (D.5) resulting, after some algebraic manipulations, in

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{-\mu_1}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-2(t_1+2t_2-2\mu_1)+t_3\alpha_2)}{\Gamma(1-2(t_1+2t_2-2\mu_1)+t_3\alpha_2)}$$
$$\times\frac{\Gamma(t_1)\Gamma(t_2)\Gamma(-t_3)\Gamma(t_3+\mu_2)}{\Gamma(1-t_3)\Gamma\left(\frac{1}{2}-t_2+\mu_1\right)}\left(\frac{h_1\gamma_1^2}{\mathscr{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2\gamma_1^4}{4\mathscr{E}_1^4}\right)^{-t_2}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}dt_1dt_2dt_3. \tag{D.53}$$

Performing the variable transformations $t_1=-2s_1+s_2+2\mu_1$ and $t_2=s_1$ will result in

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{-\mu_1}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\oiiint_{\mathscr{L}}\frac{\Gamma(-2s_2+t_3\alpha_2)\Gamma(-2s_1+s_2+2\mu_1)}{\Gamma(1-2s_2+t_3\alpha_2)}$$
$$\times\frac{\Gamma(s_1)\Gamma(-t_3)\Gamma(\mu_2+t_3)}{\Gamma(1-t_3)\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{h_1\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_1}\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}ds_1ds_2dt_3. \tag{D.54}$$

Further simplifications are obtained by isolating the integral in the variable s_1 which is given as

$$\frac{1}{2\pi j}\oint_{\mathscr{L}}\frac{\Gamma(s_1)\Gamma(-2s_1+s_2+2\mu_1)}{\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}ds_1=$$
$$\frac{2^{-1+s_2+2\mu_1}}{\sqrt{\pi}}\frac{1}{2\pi j}\oint_{\mathscr{L}}\frac{\Gamma(s_1)\Gamma\left(-s_1+\frac{s_2}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}-s_1+\frac{s_2}{2}+\mu_1\right)}{\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)}\left(-\frac{H_1^2}{h_1^2}\right)^{-s_1}ds_1 \tag{D.55}$$

The above identity holds through the gamma's duplication formula. The right-hand side of (D.55) can be written in terms of the Gauss' hypergeometric function as

$$\frac{2^{-1+s_2+2\mu_1}}{\sqrt{\pi}}\frac{1}{2\pi j}\oint_{\mathscr{L}}\frac{\Gamma(s_1)\Gamma\left(-s_1+\frac{s_2}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}-s_1+\frac{s_2}{2}+\mu_1\right)}{\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)}\left(-\frac{H_1^2}{h_1^2}\right)^{-s_1}ds_1=$$
$$\frac{\Gamma(s_2+2\mu_1)}{\Gamma\left(\frac{1}{2}+\mu_1\right)}{}_2F_1\left(\frac{s_2}{2}+\mu_1,\frac{1}{2}(1+s_2)+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right). \tag{D.56}$$

By applying the linear transformation [36, Equation (15.3.3)], the hypergeometric function will reduce to

$$\frac{\Gamma(s_2+2\mu_1)}{\Gamma\left(\frac{1}{2}+\mu_1\right)}{}_2F_1\left(\frac{s_2}{2}+\mu_1,\frac{1}{2}(1+s_2)+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right)=$$
$$\frac{\Gamma(s_2+2\mu_1)}{\Gamma\left(\frac{1}{2}+\mu_1\right)}{}_2F_1\left(-\frac{s_2}{2},\frac{1-s_2}{2};\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right)\left(1-\frac{H_1^2}{h_1^2}\right)^{-s_2-\mu_1} \tag{D.57}$$

Note that due to relation amongst η with h and H of the η-μ distribution the identity $1-H^2/h^2 = 1/h$ holds. At this point the right-hand side is rewritten in terms of the Mellin-Barnes contour integral and it is replaced in (D.54). After some algebraic manipulations, the integral $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2)=\frac{2\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\iiint\limits_{\mathscr{L}}\frac{\Gamma(-2s_2+t_3\alpha_2)}{\Gamma(1-2s_2+t_3\alpha_2)}\frac{\Gamma(s_1)\Gamma\left(-s_1-\frac{s_2}{2}\right)\Gamma(-t_3)}{\Gamma\left(\frac{1}{2}(1-s_2)\right)\Gamma\left(-\frac{s_2}{2}\right)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}-s_1-\frac{s_2}{2}\right)\Gamma(s_2+2\mu_1)\Gamma(t_3+\mu_2)}{\Gamma(1-t_3)\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)}\left(-\frac{H_1^2}{h_1^2}\right)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}ds_1ds_2dt_3. \tag{D.58}$$

The final step is to use the gamma's duplication formula so that the integral $P(\gamma_1,\gamma_2)$ is simplified to

$$P(\gamma_1,\gamma_2)=\frac{4^{1-\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^3\iiint\limits_{\mathscr{L}}\frac{\Gamma(-2s_1-s_2)\Gamma(-2s_2+t_3\alpha_2)\Gamma(s_1)}{\Gamma(1-2s_2+t_3\alpha_2)\Gamma\left(\mu_1+\frac{1}{2}-s_1\right)}$$
$$\times\frac{\Gamma(2\mu_1+s_2)\Gamma(\mu_2+t_3)\Gamma(-t_3)}{\Gamma(-s_2)\Gamma(1-t_3)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}\left(\frac{\gamma_2^{\alpha_2}}{\gamma_1^{\alpha_2}\mathscr{A}_2^{\alpha_2}}\right)^{-t_3}ds_1ds_2dt_3. \tag{D.59}$$

The parameters provided in Table 4.6 can be readily obtained by comparing (D.59) with (2.1), which completes the derivation of the multivariable Fox H-function representation for the integral involving the product of the PDF of η-μ and the CDF of α-μ distributions.

A series expressions for $P(\gamma_1,\gamma_2)$ can be obtained through the sum of residues. Using the residues around the poles of $\Gamma(s_1)$ for the integration variable s_1, $\Gamma(2\mu_1+s_2)$ for s_2 and $\Gamma(\mu_2+t_3)$ and $\Gamma(-2s_2+\alpha_2t_3)$ for t_3 results in the following triple summation

$$P(\gamma_1,\gamma_2)=\frac{4^{1-\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma(i+2\mu_1)}\left(\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1}I_1(i)+\left(\frac{\gamma_2^2}{\mathscr{E}_1^2\mathscr{A}_2^2}\right)^{i+2\mu_1}I_2(i)\right), \tag{D.60}$$

in which $I_1(i)$ and $I_2(i)$ are given, respectively, as

$$I_1(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(i+2n+2\mu_1)H_1^{2n}(2h_1)^{-2n}}{n!k!\Gamma\left(\frac{1}{2}+n+\mu_1\right)(k+\mu_2)(2(i+2\mu_1)-\alpha_2(k+\mu_2))}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)^{k+\mu_2}, \tag{D.61}$$

and

$$I_2(i)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(i+2n+2\mu_1)\Gamma\left(\frac{2i+k+4\mu_1}{\alpha_2}\right)\Gamma\left(\frac{-2i-k-4\mu_1+\alpha_2\mu_2}{\alpha_2}\right)}{n!k!\Gamma(1-k)\Gamma\left(\frac{1}{2}+n+\mu_1\right)\Gamma\left(\frac{2i+k+\alpha_2+4\mu_1}{\alpha_2}\right)\alpha_2}\left(\frac{H_1^2}{4h_1^2}\right)^n\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^k. \tag{D.62}$$

For $I_1(i)$, the summation over index n can be written in terms of the Gauss' hypergeometric function by using the gamma's duplication formula. On index k, the summation is solved first in terms of the hypergeometric ${}_2F_2$ function by remembering that $1/(a+k)=(1/a)\times$

$((a)_k/(1+a)_k)$. The ${}_2F_2$ function can be simplified using [41, Equation (7.12.1.3)] and [41, Equation (7.11.1.12)]. After some algebraic manipulations $I_1(i)$ is given as

$$\begin{aligned} I_1(i) = &\frac{\Gamma(i+2\mu_1)}{2\Gamma\left(\frac{1}{2}+\mu_1\right)(i+2\mu_1)} {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\mu_1; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \\ &\times\left(\gamma\left(\mu_2, \frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right) - \gamma\left(\mu_2 - \frac{2(i+2\mu_1)}{\alpha_2}, \frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{2i+4\mu_1}\right). \end{aligned} \tag{D.63}$$

In its turn, $I_2(i)$ vanishes for any $k > 0$ and the summation over index n can be written in terms of the Gauss' hypergeometric function. Therefore, $I_2(i)$ is given as

$$I_2(i) = \frac{\Gamma(i+2\mu_1)\Gamma\left(-\frac{2(i+2\mu_1)}{\alpha_2}+\mu_2\right)}{2\Gamma\left(\frac{1}{2}+\mu_1\right)(i+2\mu_1)} {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\mu_1; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right). \tag{D.64}$$

Replacing $I_1(i)$ and $I_2(i)$ in (D.60), after some algebraic manipulations, will result in

$$\begin{aligned} P(\gamma_1,\gamma_2) = &\frac{1}{\Gamma(2\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\,(i+2\mu_1)} {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\mu_1; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \\ &\times\left(\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1}\gamma\left(\mu_2, \frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right) + \Gamma\left(\mu_2 - \frac{2(i+2\mu_1)}{\alpha_2}, \frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)\left(\frac{\gamma_2^2}{\mathscr{E}_1^2\mathscr{A}_2^2}\right)^{i+2\mu_1}\right). \end{aligned} \tag{D.65}$$

The last step is to use the linear transformation [36, Equation (15.3.3)] and put the upper incomplete gamma function in terms of the exponential integral function using [36, Equation (5.1.45)]. The integral $P(\gamma_1,\gamma_2)$ is, then, given as

$$\begin{aligned} P(\gamma_1,\gamma_2) = &\frac{h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i h_1^i}{i!\,(i+2\mu_1)} {}_2F_1\left(\frac{1-i}{2}, -\frac{i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1} \\ &\times\left(\gamma\left(\mu_2, \frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right) + E_{1-\mu_2+\frac{2(i+2\mu_1)}{\alpha_2}}\left(\frac{\gamma_2^{\alpha_2}}{\mathscr{A}_2^{\alpha_2}\gamma_1^{\alpha_2}}\right)\left(\frac{\gamma_2}{\mathscr{A}_2\gamma_1}\right)^{\alpha_2\mu_2}\right). \end{aligned} \tag{D.66}$$

Which completes the derivation of the series provided in Table 4.7 for the integral involving the product of the PDF of the η-μ and the CDF of the α-μ distributions.

D.6 The Integral Involving κ-μ PDF $\times\kappa$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the κ-μ distribution with parameters $\{\kappa_1, \mu_1, \hat{r}_1\}$ and $\{\kappa_2, \mu_2, \hat{r}_2\}$. Then (D.1) is solved by replacing (2.13) and (2.14) in it with the appropriate parameters resulting in

$$\begin{aligned} P(\gamma_1,\gamma_2) = &\frac{2}{\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^2\int_0^{\gamma_1}\frac{r^{2\mu_1-1}}{\mathscr{K}_1^{2\mu_1}}\exp\left(-\frac{r^2}{\mathscr{K}_1^2}\right){}_0\tilde{F}_1\left(;\mu_1;\frac{r^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right) \\ &\times\oint\!\!\oint_{\mathscr{L}}\frac{\Gamma(\mu_2-t_3-t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2-t_3)\Gamma(\mu_2+1-t_4)}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{r^2\mathscr{K}_2^2}\right)^{-t_4+\mu_2}dt_3dt_4dr. \end{aligned} \tag{D.67}$$

The multivariable Fox H-function representation is found by putting the exponential and the hypergeometric functions in terms of their Mellin-Barnes contour integral representations using (A.1) and (A.2). Therefore, the integral $P(\gamma_1, \gamma_2)$ is given as

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{2}{\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^4 \int_0^{\gamma_1} \frac{1}{r} \oint_{\mathscr{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1 - t_2)} \frac{\Gamma(\mu_2 - t_3 - t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2 - t_3)\Gamma(\mu_2 + 1 - t_4)} \\ \times \left(\frac{r^2}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1} \left(-\frac{r^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2} (\kappa_2\mu_2)^{-t_3} \left(\frac{\gamma_2^2}{r^2\mathscr{K}_2^2}\right)^{-t_4+\mu_2} dt_1 dt_2 dt_3 dt_4 dr \end{aligned} \quad (D.68)$$

By changing the order of integration, the integral $P(\gamma_1, \gamma_2)$ becomes

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{2}{\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1 - t_2)} \frac{\Gamma(\mu_2 - t_3 - t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2 - t_3)\Gamma(\mu_2 + 1 - t_4)} \left(\frac{1}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1} \\ \times \left(-\frac{\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2} (\kappa_2\mu_2)^{-t_3} \left(\frac{\gamma_2^2}{\mathscr{K}_2^2}\right)^{-t_4+\mu_2} \int_0^{\gamma_1} r^{-2(t_1+t_2-\mu_1)+2(t_4-\mu_2)-1} dr t_1 dt_2 dt_3 dt_4. \end{aligned}$$

(D.69)

The inner integral is solved with the help (D.5), which results in

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{1}{\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(-(t_1+t_2-t_4-\mu_1+\mu_2))}{\Gamma(1-(t_1+t_2-t_4-\mu_1+\mu_2))} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_2)} \\ \times \frac{\Gamma(\mu_2-t_3-t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2-t_3)\Gamma(\mu_2+1-t_4)} \left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-t_1+\mu_1} \left(-\frac{\gamma_1^2\kappa_1\mu_1}{\mathscr{K}_1^2}\right)^{-t_2} (\kappa_2\mu_2)^{-t_3} \left(\frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}\right)^{-t_4+\mu_2} dt_1 dt_2 dt_3 dt_4. \end{aligned} \quad (D.70)$$

Further simplifications are obtained by performing the variable transformations $t_1 = -s_1 + s_2 + \mu_1$, $t_2 = s_1$ and $t_4 = s_4 + \mu_2$, which results in

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{1}{\exp(\kappa_1\mu_1)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(-s_2+s_4)}{\Gamma(1-s_2+s_4)} \frac{\Gamma(-s_1+s_2+\mu_1)\Gamma(s_1)}{\Gamma(\mu_1-s_1)} \\ \frac{\Gamma(-s_4-t_3)\Gamma(t_3)\Gamma(s_4+\mu_2)}{\Gamma(1-s_4)\Gamma(-t_3+\mu_2)} (-\kappa_1\mu_1)^{-s_1} \left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2} (\kappa_2\mu_2)^{-t_3} \left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4. \end{aligned}$$

(D.71)

Now, the contour integral on the variable s_1 can be written in terms of the Kummer's hypergeometric function using (B.1). By using the Kummer's transformation [36, Equation (13.1.27)], the following identity is obtained

$$\begin{aligned} \oint_{\mathscr{L}} \frac{\Gamma(-s_1+s_2+\mu_1)\Gamma(s_1)}{\Gamma(\mu_1-s_1)} (-\kappa_1\mu_1)^{-s_1} ds_1 = \\ \exp(\kappa_1\mu_1) \frac{\Gamma(s_2+\mu_1)}{\Gamma(-s_2)} \oint_{\mathscr{L}} \frac{\Gamma(s_1)\Gamma(-s_1-s_2)}{\Gamma(\mu_1-s_1)} (\kappa_1\mu_1)^{-s_1} ds_1. \end{aligned} \quad (D.72)$$

Therefore, the integral $P(\gamma_1, \gamma_2)$ simplifies to

$$\begin{aligned} P(\gamma_1,\gamma_2) = \left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2)\Gamma(-t_3-s_4)\Gamma(-s_2+s_4)}{\Gamma(1-s_2+s_4)\Gamma(\mu_1-s_1)\Gamma(-s_2)} \frac{\Gamma(s_1)\Gamma(\mu_1+s_2)}{\Gamma(\mu_2-t_3)} \\ \times \frac{\Gamma(t_3)\Gamma(s_4+\mu_2)}{\Gamma(1-s_4)} (\kappa_1\mu_1)^{-s_1} \left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2} (\kappa_2\mu_2)^{-t_3} \left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4. \end{aligned} \quad (D.73)$$

The parameters of the Fox H-function representation are obtained by comparing (D.73) with (2.1) and they are provided in Table 4.6 completing its derivation.

A series representation for the integral $P(\gamma_1,\gamma_2)$ can be obtained through the sum of residues. A multi-fold summation is achieved by taking the residues around the poles of $\Gamma(s_1)$, $\Gamma(\mu_1+s_2)$ and $\Gamma(t_3)$ for the variables s_1, s_2 and t_3 respectively. For the variable s_4, the poles of $\Gamma(\mu_2+s_4)$ and $\Gamma(-s_2+s_4)$ are used. Therefore, the integral $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2)=\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+l}}{i!l!\Gamma(i+\mu_1)}\left(\frac{\gamma_1^2}{\mathcal{K}_1^2}\right)^{i+\mu_1}\left(\frac{1}{\Gamma(1+l+\mu_2)(i-l+\mu_1-\mu_2)}\right.$$
$$\left.\times\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{l+\mu_2} I_1(i,l)+\frac{\Gamma(-i-l-\mu_1+\mu_2)}{\Gamma(1-l)\Gamma(1+i+l+\mu_1)}\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{i+l+\mu_1} I_2(i,l)\right), \tag{D.74}$$

in which $I_1(i,l)$ and $I_2(i,l)$ are given, respectively, as

$$I_1(i,l)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n}\Gamma(i+k+\mu_1)\Gamma(l+n+\mu_2)(\kappa_1\mu_1)^k(\kappa_2\mu_2)^n}{k!n!\Gamma(n+\mu_2)\Gamma(k+\mu_1)}, \tag{D.75}$$

and

$$I_2(i,l)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+n}\Gamma(i+k+\mu_1)\Gamma(i+l+n+\mu_1)(\kappa_1\mu_1)^k(\kappa_2\mu_2)^n}{k!n!\Gamma(n+\mu_2)\Gamma(k+\mu_1)}. \tag{D.76}$$

Both $I_1(i,l)$ and and $I_2(i,l)$ can be written in closed form in terms of the Kummer's hypergeometric function given as

$$I_1(i,l)=\frac{\Gamma(i+\mu_1)\Gamma(l+\mu_2)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\,{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}, \tag{D.77}$$

and

$$I_2(i,l)=\frac{\Gamma(i+\mu_1)\Gamma(i+l+\mu_1)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\,{}_1F_1(i+l+\mu_1;\mu_2;-\kappa_2\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}. \tag{D.78}$$

After replacing $I_1(i,l)$ and $I_2(i,l)$ in (D.74) and some algebraic manipulations, $P(\gamma_1,\gamma_2)$ is obtained as

$$P(\gamma_1,\gamma_2)=\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\gamma_1^2}{\mathcal{K}_1^2}\right)^{i+\mu_1}{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)$$
$$\times\left(\sum_{l=0}^{\infty}\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{l+\mu_2}\frac{(-1)^l\,{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(l+\mu_2)(i-l+\mu_1-\mu_2)}+\sum_{l=0}^{\infty}\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{i+l+\mu_1}\right. \tag{D.79}$$
$$\left.\times\frac{(-1)^l\Gamma(-i-l-\mu_1+\mu_2)}{l!\Gamma(1-l)(i+l+\mu_1)}\,{}_1F_1(i+l+\mu_1;\mu_2;-\kappa_2\mu_2)\right).$$

A final simplification is found by noticing that the second summation vanishes for $l > 0$, which results in

$$P(\gamma_1,\gamma_2) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \, {}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1) \left(\frac{\Gamma(-i-\mu_1+\mu_2)}{i+\mu_1} \left(\frac{\gamma_2^2}{\mathcal{K}_1^2\mathcal{K}_2^2} \right)^{i+\mu_1} \right.$$
$$\left. \times \, {}_1F_1(i+\mu_1;\mu_2;-\kappa_2\mu_2) + \left(\frac{\gamma_1^2}{\mathcal{K}_1^2} \right)^{i+\mu_1} \sum_{l=0}^{\infty} \frac{(-1)^l \, {}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(l+\mu_2)(i-l+\mu_1-\mu_2)} \left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2} \right)^{l+\mu_2} \right), \tag{D.80}$$

completing the derivation of the series expression for the integral involving the product of the PDF and CDF of the κ-μ distribution.

D.7 The Integral Involving κ-μ PDF $\times\eta$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the κ-μ and η-μ distributions with parameters $\{\kappa_1,\mu_1,\hat{r}_1\}$ and $\{\eta_2,\mu_2,\hat{r}_2\}$. Then (D.1) is solved by replacing (2.13) and (2.18) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_2}\sqrt{\pi}}{\exp(\kappa_1\mu_1)\Gamma(\mu_2)} \int_0^{\gamma_1} \frac{r^{2\mu_1-1}}{\mathcal{K}_1^{2\mu_1}} \exp\left(-\frac{r^2}{\mathcal{K}_1^2}\right) {}_0\tilde{F}_1\left(;\mu_1;\frac{r^2\kappa_1\mu_1}{\mathcal{K}_1^2}\right) \left(\frac{1}{2\pi j}\right)^2$$
$$\times \oint\!\!\oint_{\mathcal{L}} \frac{\Gamma(2\mu_2-2t_3-t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2+\frac{1}{2}-t_3)\Gamma(1+2\mu_2-t_4)} \left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3} \left(\frac{\gamma_2^2}{r^2\mathcal{E}_2^2}\right)^{-t_4+2\mu_2} dt_3dt_4dr. \tag{D.81}$$

The Mellin-Barnes contour integral representation for $P(\gamma_1,\gamma_2)$ is obtained by replacing the exponential and hypergeometric functions with their contour integral representations using (A.1) and (A.2) respectively, and then changing the order of integration. Then, $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_2}\sqrt{\pi}}{\exp(\kappa_1\mu_1)\Gamma(\mu_2)} \left(\frac{1}{2\pi j}\right)^4 \oint_{\mathcal{L}} \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(\mu_1-t_2)} \frac{\Gamma(2\mu_2-2t_3-t_4)\Gamma(t_3)\Gamma(t_4)}{\Gamma(\mu_2+\frac{1}{2}-t_3)\Gamma(1+2\mu_2-t_4)} \tag{D.82}$$
$$\times \left(\frac{1}{\mathcal{K}_1^2}\right)^{-t_1+\mu_1} \left(-\frac{\kappa_1\mu_1}{\mathcal{K}_1^2}\right)^{-t_2} \left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3} \left(\frac{\gamma_2^2}{\mathcal{E}_2^2}\right)^{-t_4+2\mu_2} \int_0^{\gamma_1} r^{2(-t_1+\mu_1-t_2+t_4-2\mu_2)-1} dr dt_1dt_2dt_3dt_4.$$

The inner integral is solved with the help of (D.5) and $P(\gamma_1,\gamma_2)$ is obtained, after some algebraic manipulation, as

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{e^{\kappa_1\mu_1}\Gamma(\mu_2)} \left(\frac{1}{2\pi j}\right)^4 \oint_{\mathcal{L}} \frac{\Gamma(-t_1-t_2+t_4+\mu_1-2\mu_2)\Gamma(t_1)\Gamma(t_2)\Gamma(t_3)\Gamma(t_4)}{\Gamma(1-t_1-t_2+t_4+\mu_1-2\mu_2)\Gamma(\mu_1-t_2)\Gamma(\mu_2+\frac{1}{2}-t_3)}$$
$$\times \frac{\Gamma(2\mu_2-2t_3-t_4)}{\Gamma(1+2\mu_2-t_4)} \left(\frac{\gamma_1^2}{\mathcal{K}_1^2}\right)^{-t_1+\mu_1} \left(\frac{-\gamma_1^2\kappa_1\mu_1}{\mathcal{K}_1^2}\right)^{-t_2} \left(\frac{-H_2^2}{4h_2^2}\right)^{-t_3} \left(\frac{\gamma_2^2}{\gamma_1^2\mathcal{E}_2^2}\right)^{-t_4+2\mu_2} dt_1dt_2dt_3dt_4. \tag{D.83}$$

Further simplifications are achieved by performing the variable transformations $t_1 = -s_1 + s_2 + \mu_1$, $t_2 = s_1$ and $t_4 = s_4 + 2\mu_2$. After some algebraic manipulations, $P(\gamma_1, \gamma_2)$ is given as

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{e^{\kappa_1\mu_1}\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma\left(-s_2+s_4\right)}{\Gamma\left(1-s_2+s_4\right)} \frac{\Gamma\left(-s_4-2t_3\right)\Gamma(t_3)\Gamma\left(s_4+2\mu_2\right)}{\Gamma\left(1-s_4\right)\Gamma\left(\frac{1}{2}-t_3+\mu_2\right)} \times \frac{\Gamma(s_1)\Gamma(-s_1+s_2+\mu_1)}{\Gamma(-s_1+\mu_1)}(-\kappa_1\mu_1)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\gamma_1^2\mathscr{E}_2^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4. \tag{D.84}$$

Now, the integral on s_1 can be modified accordingly to (D.72) which, after some algebraic manipulations, results in

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(-s_1-s_2)\Gamma\left(-2t_3-s_4\right)\Gamma\left(-s_2+s_4\right)\Gamma(s_1)}{\Gamma\left(1-s_2+s_4\right)\Gamma(\mu_1-s_1)\Gamma(-s_2)} \times \frac{\Gamma(\mu_1+s_2)\Gamma(t_3)\Gamma\left(2\mu_2+s_4\right)}{\Gamma\left(\frac{1}{2}+\mu_2-t_3\right)\Gamma\left(1-s_4\right)}(\kappa_1\mu_1)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{-s_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4. \tag{D.85}$$

By comparing (D.85) with (2.1), the parameters for the Fox H-function representation are readily obtained and they are provided in Table 4.6 which completes the derivation.

A series representation can be obtained through the sum of residues. A multi-fold summation arises by the taking the residues around the poles of $\Gamma(s_1)$, $\Gamma(\mu_1 + s_2)$ and $\Gamma(t_3)$ for the integration variables s_1, s_2 and t_3 respectively. Those used For the variable s_4 are $\Gamma(2\mu_2 + s_4)$ and $\Gamma(-s_2 + s_4)$. Therefore, the integral $P(\gamma_1, \gamma_2)$ is given as

$$P(\gamma_1,\gamma_2) = \frac{2^{1-2\mu_2}\sqrt{\pi}}{\Gamma(\mu_2)}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+l}}{i!l!\Gamma(i+\mu_1)}\left(\frac{\left(\gamma_1^2/\mathscr{K}_1^2\right)^{i+\mu_1}}{\Gamma(1+l+2\mu_2)(i-l+\mu_1-2\mu_2)} \times \left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l+2\mu_2} I_1(i,l) + \frac{\Gamma(-i-l-\mu_1+2\mu_2)}{\Gamma(1+i+l+\mu_1)\Gamma(1-l)}\left(\frac{\gamma_2^2}{\mathscr{K}_1^2\mathscr{E}_2^2}\right)^{i+\mu_1}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l} I_2(i,l)\right), \tag{D.86}$$

in which $I_1(i,l)$ and $I_2(i,l)$ are given, respectively, as

$$I_1(i,l) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k\Gamma(i+k+\mu_1)\Gamma(l+2n+2\mu_2)}{k!n!\Gamma(k+\mu_1)\Gamma\left(\frac{1}{2}+n+\mu_2\right)(\kappa_1\mu_1)^{-k}}\left(\frac{H_2^2}{4h_2^2}\right)^n, \tag{D.87}$$

and

$$I_2(i,l) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k\Gamma(i+k+\mu_1)\Gamma(i+l+2n+\mu_1)}{k!n!\Gamma(k+\mu_1)\Gamma\left(\frac{1}{2}+n+\mu_2\right)(\kappa_1\mu_1)^{-k}}\left(\frac{H_2^2}{4h_2^2}\right)^n. \tag{D.88}$$

The functions $I_1(i,l)$ and $I_2(i,l)$ can be written in terms of the Kummer's and the Gauss' hypergeometric functions as

$$I_1(i,l) = \frac{\Gamma(i+\mu_1)\Gamma(l+2\mu_2)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(\mu_1)\Gamma\left(\frac{1}{2}+\mu_2\right)} {}_2F_1\left(\frac{l}{2}+\mu_2, \frac{1}{2}+\frac{l}{2}+\mu_2; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right), \tag{D.89}$$

and

$$I_2(i,l) = \frac{\Gamma(i+\mu_1)\Gamma(i+l+\mu_1)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(\mu_1)\Gamma\left(\frac{1}{2}+\mu_2\right)} {}_2F_1\left(\frac{i+l+\mu_1}{2},\frac{1+i+l+\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right). \tag{D.90}$$

The function $I_1(i,l)$ can be slightly simplified using the linear transformation [36, Equation (15.3.3)] and the η-μ identity $1-H^2/h^2 = 1/h$, which results in

$$I_1(i,l) = \frac{\Gamma(i+\mu_1)\Gamma(l+2\mu_2)\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)}{\Gamma(\mu_1)\Gamma\left(\frac{1}{2}+\mu_2\right)h_2^{-l-\mu_2}}\,{}_2F_1\left(-\frac{l}{2},\frac{1-l}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right). \tag{D.91}$$

After replacing $I_1(i,l)$ and $I_2(i,l)$ in (D.86) and performing some algebraic manipulations, the integral $P(\gamma_1,\gamma_2)$ is obtained as

$$\begin{aligned} P(\gamma_1,\gamma_2) = {} & \frac{1}{\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{i+\mu_1}\left(\sum_{l=0}^{\infty}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{i+\mu_1}\right. \\ & \times\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l}\frac{(-1)^l\Gamma(-i-l-\mu_1+2\mu_2)}{l!\,(i+l+\mu_1)\Gamma(1-l)}\,{}_2F_1\left(\frac{i+l+\mu_1}{2},\frac{1+i+l+\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right) \\ & \left.+\sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(l+2\mu_2)h_2^{l+\mu_2}}{l!\Gamma(1+l+2\mu_2)(i-l+\mu_1-2\mu_2)}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l+2\mu_2}{}_2F_1\left(-\frac{l}{2},\frac{1-l}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right). \end{aligned} \tag{D.92}$$

To finalize the derivation, the first summation on the index l is simplified by noting that it vanishes for any $l > 0$. Ergo $P(\gamma_1,\gamma_2)$ is written as

$$\begin{aligned} P(\gamma_1,\gamma_2) = {} & \frac{1}{\Gamma(\mu_1)\Gamma(2\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,{}_1F_1(i+\mu_1;\mu_1;-\kappa_1\mu_1)\left(\frac{\gamma_1^2}{\mathscr{K}_1^2}\right)^{i+\mu_1}\left(\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{i+\mu_1}\right. \\ & \times\frac{\Gamma(-i-\mu_1+2\mu_2)}{i+\mu_1}\,{}_2F_1\left(\frac{i+\mu_1}{2},\frac{1+i+\mu_1}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right) \\ & \left.+\sum_{l=0}^{\infty}\frac{(-1)^l h_2^{l+\mu_2}}{l!\,(l+2\mu_2)(i-l+\mu_1-2\mu_2)}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l+2\mu_2}{}_2F_1\left(-\frac{l}{2},\frac{1-l}{2};\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right)\right), \end{aligned} \tag{D.93}$$

which completes the derivation of the power series for the integral involving the product of the PDF of the κ-μ by the CDF of the η-μ distributions, which is provided in Table 4.7.

D.8 The Integral Involving η-μ PDF $\times\kappa$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the η-μ and κ-μ distributions with parameters $\{\eta_1,\mu_1,\hat{r}_1\}$ and $\{\kappa_2,\mu_2,\hat{r}_2\}$. Then (D.1) is solved by replacing

(2.17) and (2.14) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{\mu_1}}{\Gamma(2\mu_1)}\left(\frac{1}{2\pi j}\right)^2\int_0^{\gamma_1}\frac{r^{4\mu_1-1}}{\mathscr{E}_1^{4\mu_1}}\exp\left(-\frac{h_1r^2}{\mathscr{E}_1^2}\right){}_0F_1\left(;\mu_1+\frac{1}{2};\frac{H_1^2r^4}{4\mathscr{E}_1^4}\right)$$
$$\times\oiint_{\mathscr{L}}\frac{\Gamma\left(\mu_2-t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma(\mu_2-t_3)\Gamma\left(1+\mu_2-t_4\right)}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{r^2\mathscr{K}_2^2}\right)^{-t_4+\mu_2}dt_3dt_4dr. \tag{D.94}$$

A contour integral representation is found by replacing the exponential and hypergeometric functions with their equivalent contour integral using (A.1) and (A.2) respectively, and then changing the order of integration. These operations will result in

$$P(\gamma_1,\gamma_2)=\frac{2h_1^{-\mu_1}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)}\left(\frac{1}{2\pi j}\right)^4\oint_{\mathscr{L}}\frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\mu_1+\frac{1}{2}-t_2\right)}\frac{\Gamma\left(\mu_2-t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma(\mu_2-t_3)\Gamma\left(1+\mu_2-t_4\right)} \tag{D.95}$$
$$\times\left(\frac{h_1}{\mathscr{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2}{4\mathscr{E}_1^4}\right)^{-t_2}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2}\right)^{-t_4+\mu_2}\int_0^{\gamma_1}r^{2\left(-t_1+2\mu_1-2t_2+t_4-\mu_2\right)-1}drdt_1dt_2dt_3dt_4.$$

The inner integral is solved with the help of (D.5). After applying the gamma's duplication formula, the integral $P(\gamma_1,\gamma_2)$ reduces to

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_1}\sqrt{\pi}}{h_1^{\mu_1}\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^4\oint_{\mathscr{L}}\frac{\Gamma\left(-t_1-2t_2+t_4+2\mu_1-\mu_2\right)}{\Gamma\left(1-t_1-2t_2+t_4+2\mu_1-\mu_2\right)}\frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma\left(\mu_1+\frac{1}{2}-t_2\right)} \tag{D.96}$$
$$\times\frac{\Gamma\left(\mu_2-t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma(\mu_2-t_3)\Gamma\left(1+\mu_2-t_4\right)}\left(\frac{\gamma_1^2h_1}{\mathscr{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2\gamma_1^4}{4\mathscr{E}_1^4}\right)^{-t_2}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{\gamma_1^2\mathscr{K}_2^2}\right)^{-t_4+\mu_2}dt_1dt_2dt_3dt_4.$$

By performing the variable transformations $t_1=-2s_1+s_2+2\mu_1$, $t_2=s_1$ and $t_4=s_4+\mu_2$, the integral $P(\gamma_1,\gamma_2)$ is simplified to

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_1}\sqrt{\pi}}{h_1^{\mu_1}\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^4\oint_{\mathscr{L}}\frac{\Gamma\left(-s_2+s_4\right)}{\Gamma\left(1-s_2+s_4\right)}\frac{\Gamma(s_1)\Gamma(-2s_1+s_2+2\mu_1)}{\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)\Gamma\left(1-s_4\right)} \tag{D.97}$$
$$\times\frac{\Gamma\left(-s_4-t_3\right)\Gamma(t_3)\Gamma\left(s_4+\mu_2\right)}{\Gamma(-t_3+\mu_2)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{h_1\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{-s_4}ds_1ds_2dt_3ds_4.$$

The integral on the variable s_1 can be interpreted as a Gauss' hypergeometric function. Following the same steps as in (D.55) to (D.57), the integral $P(\gamma_1,\gamma_2)$ can be simplified to

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)}\left(\frac{1}{2\pi j}\right)^4\oint_{\mathscr{L}}\frac{\Gamma(-2s_1-s_2)\Gamma\left(-t_3-s_4\right)\Gamma\left(-s_2+s_4\right)\Gamma(s_1)}{\Gamma\left(1-s_2+s_4\right)\Gamma\left(\mu_1+\frac{1}{2}-s_1\right)\Gamma(-s_2)\Gamma(\mu_2-t_3)} \tag{D.98}$$
$$\times\frac{\Gamma(2\mu_1+s_2)\Gamma(t_3)\Gamma\left(\mu_2+s_4\right)}{\Gamma\left(1-s_4\right)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}(\kappa_2\mu_2)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{-s_4}ds_1ds_2dt_3ds_4.$$

The parameters provided in Table 4.6 are readily obtained by comparing (D.8) with (2.1) completing the derivation.

A series representation is found by summing the residues around the poles $\Gamma(s_1)$, $\Gamma(2\mu_1+s_2)$ and $\Gamma(t_3)$ for the variables s_1, s_2 and t_3, and $\Gamma(-s_2+s_4)$ and $\Gamma(\mu_2+s_4)$ for the variable s_4. This will result in the following multi-fold summation

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+l}}{i!l!\Gamma(i+2\mu_1)}\left(\frac{1}{\Gamma(1+l+\mu_2)(i-l+2\mu_1-\mu_2)}\right.$$
$$\left.\times\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{l+\mu_2}I_1(i,l)+\frac{\Gamma(-i-l-2\mu_1+\mu_2)}{\Gamma(1-l)\Gamma(1+i+l+2\mu_1)}\left(\frac{\gamma_2^2}{\mathscr{E}_1^2\mathscr{K}_2^2}\right)^{i+2\mu_1}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{l}I_2(i,l)\right), \tag{D.99}$$

in which $I_1(i,l)$ and $I_2(i,l)$ are defined as

$$I_1(i,l)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(i+2k+2\mu_1)\Gamma(l+n+\mu_2)(\kappa_2\mu_2)^n}{k!n!\Gamma\left(\frac{1}{2}+k+\mu_1\right)\Gamma(n+\mu_2)}\left(\frac{H_1^2}{4h_1^2}\right)^k, \tag{D.100}$$

and

$$I_2(i,l)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(i+2k+2\mu_1)\Gamma(i+l+n+2\mu_1)(\kappa_2\mu_2)^n}{k!n!\Gamma\left(\frac{1}{2}+k+\mu_1\right)\Gamma(n+\mu_2)}\left(\frac{H_1^2}{4h_1^2}\right)^k. \tag{D.101}$$

Both $I_1(i,l)$ and $I_2(i,l)$ can be written in the closed form in terms of the Kummer's hypergeometric function and the Gauss' hypergeometric function as

$$I_1(i,l)=\frac{\Gamma(i+2\mu_1)\Gamma(l+\mu_2)\,{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma(\mu_2)}\;{}_2F_1\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right), \tag{D.102}$$

and

$$I_2(i,l)=\frac{\Gamma(i+2\mu_1)\Gamma(i+l+2\mu_1)\,{}_1F_1(i+l+2\mu_1;\mu_2;-\kappa_2\mu_2)}{\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma(\mu_2)}\;{}_2F_1\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right). \tag{D.103}$$

After replacing $I_1(i,l)$ and $I_2(i,l)$ in (D.99) and, performing some algebraic manipulations, the integral $P(\gamma_1,\gamma_2)$ is obtained as

$$P(\gamma_1,\gamma_2)=\frac{2^{1-2\mu_1}\sqrt{\pi}}{\Gamma(\mu_1)\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,{}_2F_1\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right)$$
$$\times\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1}\left(\sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(-i-l-2\mu_1+\mu_2)\,{}_1F_1(i+l+2\mu_1;\mu_2;-\kappa_2\mu_2)}{l!\Gamma(1-l)(i+l+2\mu_1)}\right.$$
$$\left.\times\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{i+l+2\mu_1}+\sum_{l=0}^{\infty}\frac{(-1)^l\,{}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(l+\mu_2)(i-l+2\mu_1-\mu_2)}\left(\frac{\gamma_2^2}{\mathscr{K}_2^2\gamma_1^2}\right)^{l+\mu_2}\right) \tag{D.104}$$

The final step is to use the gamma's duplication formula, the linear transformation [36, Equation (15.3.3)] on the ${}_2F_1$ function and notice that the first summation on the index l

vanishes for $l > 0$. Therefore, the integral $P(\gamma_1,\gamma_2)$ results in

$$P(\gamma_1,\gamma_2) = \frac{h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i h_1^i}{i!}\, {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right)\left(\frac{\gamma_1^2}{\mathcal{E}_1^2}\right)^{i+2\mu_1} \times \left(\frac{\Gamma(-i-2\mu_1+\mu_2)\, {}_1F_1(i+2\mu_1;\mu_2;-\kappa_2\mu_2)}{i+2\mu_1}\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{i+2\mu_1} + \sum_{l=0}^{\infty} \frac{(-1)^l\, {}_1F_1(l+\mu_2;\mu_2;-\kappa_2\mu_2)}{l!(l+\mu_2)(i-l+2\mu_1-\mu_2)}\left(\frac{\gamma_2^2}{\mathcal{K}_2^2\gamma_1^2}\right)^{l+\mu_2}\right). \tag{D.105}$$

Which completes the derivation of the integral involving the PDF of the η-μ and the CDF of the κ-μ distributions provided in Table 4.7.

D.9 The Integral Involving η-μ PDF $\times\eta$-μ CDF

Let $R_1 > 0$ and $R_2 > 0$ be two random envelopes following the η-μ distribution with parameters $\{\eta_1, \mu_1, \hat{r}_1\}$ and $\{\eta_2, \mu_2, \hat{r}_2\}$. Then (D.1) is solved by replacing (2.17) and (2.18) in it with the appropriate parameters resulting in

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_2}\sqrt{\pi}h_1^{\mu_1}}{\Gamma(2\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi i}\right)^2 \int_0^{\gamma_1} \frac{r^{4\mu_1-1}}{\mathcal{E}_1^{4\mu_1}} \exp\left(-\frac{h_1 r^2}{\mathcal{E}_1^2}\right) {}_0F_1\left(;\mu_1+\frac{1}{2}; \frac{H_1^2 r^4}{4\mathcal{E}_1^4}\right) \times \oiint_{\mathcal{L}} \frac{\Gamma\left(2\mu_2-2t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma\left(\mu_2+\frac{1}{2}-t_3\right)\Gamma\left(1+2\mu_2-t_4\right)}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{r^2\mathcal{E}_2^2}\right)^{-t_4+2\mu_2} dt_3 dt_4 dr. \tag{D.106}$$

A contour integral representation is obtained by replacing the exponential and hypergeometric functions by their respective Mellin-Barnes integral representations using (A.1) and (A.2) respectively, and then changing the order of integration, which results in

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_2}\sqrt{\pi}\Gamma\left(\mu_1+\frac{1}{2}\right)}{\Gamma(2\mu_1)\Gamma(\mu_2)h_1^{\mu_1}}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathcal{L}} \frac{\Gamma(t_1)\Gamma(t_2)\Gamma\left(2\mu_2-2t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma\left(\mu_1+\frac{1}{2}-t_2\right)\Gamma\left(\mu_2+\frac{1}{2}-t_3\right)\Gamma\left(1+2\mu_2-t_4\right)} \left(\frac{h_1}{\mathcal{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2}{4\mathcal{E}_1^4}\right)^{-t_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\mathcal{E}_2^2}\right)^{-t_4+2\mu_2}\int_0^{\gamma_1} r^{2\left(-t_1+2\mu_1-2t_2+t_4-2\mu_2\right)-1} dr dt_1 dt_2 dt_3 dt_4. \tag{D.107}$$

The inner integral can be solved with the (D.5). After some algebraic manipulations and using the duplication formula of the gamma function, the integral $P(\gamma_1,\gamma_2)$ is given as

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_1-2\mu_2}\pi h_1^{-\mu_1}}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathcal{L}} \frac{\Gamma\left(-t_1-2t_2+t_4+2\mu_1-2\mu_2\right)\Gamma(t_1)}{\Gamma\left(1-t_1-2t_2+t_4+2\mu_1-2\mu_2\right)} \times \frac{\Gamma(t_2)\Gamma\left(2\mu_2-2t_3-t_4\right)\Gamma(t_3)\Gamma\left(t_4\right)}{\Gamma\left(\mu_1+\frac{1}{2}-t_2\right)\Gamma\left(\mu_2+\frac{1}{2}-t_3\right)\Gamma\left(1+2\mu_2-t_4\right)} \times \left(\frac{\gamma_1^2 h_1}{\mathcal{E}_1^2}\right)^{-t_1+2\mu_1}\left(-\frac{H_1^2\gamma_1^4}{4\mathcal{E}_1^4}\right)^{-t_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\gamma_1^2\mathcal{E}_2^2}\right)^{-t_4+2\mu_2} dt_1 dt_2 dt_3 dt_4. \tag{D.108}$$

Now, performing the variable transformations $t_1 = -2s_1+s_2+2\mu_1$, $t_2 = s_1$ and $t_4 = s_4+2\mu_2$, after some algebraic manipulations, will result in

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_1-2\mu_2}\pi h_1^{-\mu_1}}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma\left(-s_2+s_4\right)\Gamma(-2s_1+s_2+2\mu_1)\Gamma(s_1)}{\Gamma\left(1-s_2+s_4\right)\Gamma\left(\frac{1}{2}-s_1+\mu_1\right)\Gamma\left(1-s_4\right)} \tag{D.109}$$
$$\times \frac{\Gamma\left(-s_4-2t_3\right)\Gamma(t_3)\Gamma\left(s_4+2\mu_2\right)}{\Gamma\left(\frac{1}{2}-t_3+\mu_2\right)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{h_1\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4.$$

The above expression can be slightly simplified by following the same steps as in (D.55) to (D.57), which results in

$$P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}\left(\frac{1}{2\pi j}\right)^4 \oint_{\mathscr{L}} \frac{\Gamma(-2s_1-s_2)\Gamma\left(-2t_3-s_4\right)\Gamma\left(-s_2+s_4\right)\Gamma(s_1)}{\Gamma\left(1-s_2+s_4\right)\Gamma\left(\mu_1+\frac{1}{2}-s_1\right)\Gamma(-s_2)} \tag{D.110}$$
$$\times \frac{\Gamma(2\mu_1+s_2)\Gamma(t_3)\Gamma\left(2\mu_2+s_4\right)}{\Gamma\left(\mu_2+\frac{1}{2}-t_3\right)\Gamma\left(1-s_4\right)}\left(-\frac{H_1^2}{4h_1^2}\right)^{-s_1}\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{-s_2}\left(-\frac{H_2^2}{4h_2^2}\right)^{-t_3}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{-s_4} ds_1 ds_2 dt_3 ds_4.$$

The parameters provided in Table 4.6 are readily obtained by comparing (D.110) with (2.1).

Through the summation of residues, it is possible to obtain a series representation for the integral $P(\gamma_1,\gamma_2)$. For the integration variables s_1, s_2 and t_3, the residues around the poles of $\Gamma(s_1)$, $\Gamma(2\mu_1+s_2)$ and $\Gamma(t_3)$ are taken respectively for each variable, whilst $\Gamma(2\mu_2+s_4)$ and $\Gamma(-s_2+s_4)$ are those used for the variable s_4. The resulting multi-fold summation for the integral $P(\gamma_1,\gamma_2)$ is given as

$$\begin{aligned} P(\gamma_1,\gamma_2) = \frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+l}}{i!l!\Gamma(i+2\mu_1)}\left(\frac{\gamma_1^2}{\mathscr{E}_1^2}\right)^{i+2\mu_1}&\left(\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{l+2\mu_2}\right.\\ &\times\frac{1}{\Gamma(1+l+2\mu_2)(i-l+2\mu_1-2\mu_2)}I_1(i,l)\\ &\left.+\frac{\Gamma(-i-l-2\mu_1+2\mu_2)}{\Gamma(1-l)\Gamma(1+i+l+2\mu_1)}\left(\frac{\gamma_2^2}{\mathscr{E}_2^2\gamma_1^2}\right)^{i+l+2\mu_1} I_2(i,l)\right), \end{aligned} \tag{D.111}$$

in which $I_1(i,l)$ and $I_2(i,l)$ are given, respectively, as

$$I_1(i,l) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(i+2k+2\mu_1)\Gamma(l+2n+2\mu_2)}{k!n!\Gamma\left(\frac{1}{2}+k+\mu_1\right)\Gamma\left(\frac{1}{2}+n+\mu_2\right)}\left(\frac{H_1^2}{4h_1^2}\right)^k\left(\frac{H_2^2}{4h_2^2}\right)^n, \tag{D.112}$$

and

$$I_2(i,l) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(i+2k+2\mu_1)\Gamma(i+l+2n+2\mu_1)}{k!n!\Gamma\left(\frac{1}{2}+k+\mu_1\right)\Gamma\left(\frac{1}{2}+n+\mu_2\right)}\left(\frac{H_1^2}{4h_1^2}\right)^k\left(\frac{H_2^2}{4h_2^2}\right)^n. \tag{D.113}$$

Both $I_1(i,l)$ and $I_2(i,l)$ can be written in the closed form in terms of the Gauss' hypergeometric function as

$$\begin{aligned} I_1(i,l) = \frac{\Gamma(i+2\mu_1)\Gamma(l+2\mu_2)}{\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}+\mu_2\right)}\,{}_2F_1&\left(\frac{i}{2}+\mu_1,\frac{1}{2}+\frac{i}{2}+\mu_1;\frac{1}{2}+\mu_1;\frac{H_1^2}{h_1^2}\right)\\ &{}_2F_1\left(\frac{l}{2}+\mu_2,\frac{1}{2}+\frac{l}{2}+\mu_2;\frac{1}{2}+\mu_2;\frac{H_2^2}{h_2^2}\right), \end{aligned} \tag{D.114}$$

and

$$I_2(i,l) = \frac{\Gamma(i+2\mu_1)\Gamma(i+l+2\mu_1)}{\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}+\mu_2\right)} {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\mu_1; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) {}_2F_1\left(\frac{i}{2}+\frac{l}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\frac{l}{2}+\mu_1; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right). \tag{D.115}$$

By replacing $I_1(i,l)$ and $I_2(i,l)$ in (D.111) and performing some algebraic manipulations, the integral $P(\gamma_1, \gamma_2)$ reduces to

$$\begin{aligned} P(\gamma_1,\gamma_2) &= \frac{2^{2-2\mu_1-2\mu_2}\pi}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma\left(\frac{1}{2}+\mu_1\right)\Gamma\left(\frac{1}{2}+\mu_2\right)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\gamma_1^2}{\mathcal{E}_1^2}\right)^{i+2\mu_1} \\ &\times {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1}{2}+\frac{i}{2}+\mu_1; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \left(\sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(-i-l-2\mu_1+2\mu_2)}{l!\Gamma(1-l)(i+l+2\mu_1)}\right. \\ &\times \left(\frac{\gamma_2^2}{\mathcal{E}_2^2\gamma_1^2}\right)^{i+l+2\mu_1} {}_2F_1\left(\frac{i+l}{2}+\mu_1, \frac{1}{2}(1+i+l)+\mu_1; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) + \sum_{l=0}^{\infty} \left(\frac{\gamma_2^2}{\mathcal{E}_2^2\gamma_1^2}\right)^{l+2\mu_2} \\ &\times \left.\frac{(-1)^l}{l!(l+2\mu_2)(i-l+2\mu_1-2\mu_2)} {}_2F_1\left(\frac{l}{2}+\mu_2, \frac{1}{2}+\frac{l}{2}+\mu_2; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)\right). \end{aligned} \tag{D.116}$$

Further simplifications are obtained using the gamma's duplication formula, the linear transformation [36, Equation (15.3.3)] on the first and last Gauss' hypergeometric functions and noticing that the first summation on l vanishes for any $l > 0$. After applying these modifications, the integral $P(\gamma_1, \gamma_2)$ simplifies to

$$\begin{aligned} P(\gamma_1,\gamma_2) &= \frac{1}{\Gamma(2\mu_1)\Gamma(2\mu_2)} \sum_{i=0}^{\infty} \frac{(-1)^i h_1^{i+\mu_1}}{i!} \left(\frac{\gamma_1^2}{\mathcal{E}_1^2}\right)^{i+2\mu_1} {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}; \frac{1}{2}+\mu_1; \frac{H_1^2}{h_1^2}\right) \\ &\times \left(\frac{\Gamma(-i-2\mu_1+2\mu_2)}{i+2\mu_1} {}_2F_1\left(\frac{i}{2}+\mu_1, \frac{1+i}{2}+\mu_1; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right) \left(\frac{\gamma_2^2}{\mathcal{E}_2^2\gamma_1^2}\right)^{i+2\mu_1}\right. \\ &+ \left.\sum_{l=0}^{\infty} \frac{(-1)^l h_2^{l+\mu_2}}{l!(l+2\mu_2)(i-l+2\mu_1-2\mu_2)} \left(\frac{\gamma_2^2}{\mathcal{E}_2^2\gamma_1^2}\right)^{l+2\mu_2} {}_2F_1\left(-\frac{l}{2}, \frac{1-l}{2}; \frac{1}{2}+\mu_2; \frac{H_2^2}{h_2^2}\right)\right), \end{aligned} \tag{D.117}$$

which completes the derivation of the series representation for integral involving the product of the PDF and the CDF of the η-μ distribution provided in Table 4.7.

Appendix E

Mathematica Implementation for the Single Fox H-function

The following Mathematica program is a simple implementation for the Fox H-function used to evaluate those present in Tables 3.3 and 3.4.

```
(*    A simple Mathematica Implementation for the Fox H function    *)
foxH[a_,b_,z_,limit_] :=  Block[{F, s, R, t, Na, Nb, Da, Db, V, m, n, p, q},
    m = Length[a[[1]]]; n = Length[b[[1]]];
    p = Length[a[[2]]]; q = Length[b[[2]]];

    Na  = Product[Gamma[1-a[[1,i,1]]-a[[1,i,2]]s], {i,1,m}];
    Nb  = Product[Gamma[b[[1,i,1]]+b[[1,i,2]]s], {i,1,n}];
    Da  = Product[Gamma[a[[2,i,1]]+a[[2,i,2]]s], {i,1,p}];
    Db  = Product[Gamma[1-b[[2,i,1]]-b[[2,i,2]]s], {i,1,q}];
    F = Na Nb / Da / Db Power[z,-s];

    R = Reduce[And@@Flatten[{
          Table[1-a[[1,i,1]]-a[[1,i,2]]s>0,{i,m}],
          Table[b[[1,i,1]]+b[[1,i,2]]s>0,{i,n}]}],s];
    t = If[Length[R] == 2, Last@R +10, Mean[{First@R, Last@R}]];

    V = 1/(2 Pi I) NIntegrate[F, {s,t - I limit, t + I limit}];
    Return[V];
]
```