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Faculdade de Engenharia Elétrica e de Computação

Jonathan Matías Palma Olate

Contributions to Filter and Control design for LPV systems using LMIs

Contribuições ao projeto de filtros e controladores para sistemas LPV usando LMIs

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Tese de Doutorado apresentada à Faculdade de Engenharia Elétrica e de Computação da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Engenharia Elétrica, na Área de Energia Elétrica.

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Co-supervisor: Dr.ª Cecília de Freitas Morais

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"If It Bleeds, We Can Kill It."
Major Alan "Dutch" Schaefer
Arnold Schwarzenegger – Predator

Resumo

Esta tese de doutorado propõe três contribuições principais no contexto de controle e filtragem com ganhos robustos e escalonados de sistemas lineares discretos com parâmetros variantes no tempo (do inglês, *Linear Parameter Varying* – LPV), usando técnicas de projeto baseadas em desigualdades matriciais lineares (do inglês, *Linear Matrix Inequalities* – LMIs). A primeira consiste em um tratamento especial dos parâmetros variantes no tempo com dinâmica conhecida (tais como, funções exponenciais, trigonométricas ou periódicas). Basicamente, a taxa de variação dos parâmetros é totalmente explorada no projeto, levando a condições de síntese melhores que as baseadas em taxas de variação ilimitadas ou com limitantes finitos. Como aplicação, a abordagem é empregada em um problema de sistema controlado por rede baseado em taxas de amostragem variantes no tempo. A segunda contribuição é uma nova estratégia para aprimorar o desempenho associado ao projeto de controladores e filtros escalonados explorando, sempre que disponíveis, informações estatísticas dos parâmetros variantes no tempo. A novidade da técnica é projetar controladores e filtros tratando, de maneira independente, a estabilidade robusta (assegurada em todo o domínio) e a otimização de desempenho (que prioriza regiões de maior probabilidade de ocorrência). A principal motivação desse tópico é o problema de projeto escalonado sujeito a medições inexatas, particularmente quando os valores dos parâmetros escalonados são obtidos por identificação. Como contribuições complementares no tema, uma nova condição de projeto de filtro $\mathcal{H}_2$ e um procedimento sistemático e generalista para modelar incertezas aditivas afetando os parâmetros escalonados são propostos. Finalmente, o problema de projeto de controladores por realimentação estática de saída por meio do enriquecimento da dinâmica para sistemas LPV e sistemas lineares sujeitos a saltos markovianos com cadeias de Markov não-homogêneas é investigado. Um método de solução expresso em termos de um procedimento iterativo localmente convergente baseado em LMIs é proposto, tendo como principal novidade técnica o fato de que o controlador é tratado como uma variável de otimização do problema, permitindo a imposição de restrições de estrutura (como descentralização) e de magnitude no ganho de controle sem introduzir nenhum conservadorismo adicional. Os benefícios e vantagens das técnicas propostas são ilustrados por meio de vários experimentos numéricos, incluindo comparações com outros métodos da literatura.

Palavras-chaves: Sistemas Lineares com Parâmetros Variantes no Tempo, Sistemas Discretos no Tempo; Desigualdades Matriciais Lineares; Medições Inexatas; Controladores com Memória; Ganho Escalonado; Ganho Robusto; Controle por Realimentação de Estados; Controle por Realimentação Estática de Saída; Filtragem.

Abstract

This PhD thesis proposes three main contributions in the context of control and filtering with robust and gain-scheduled gains of linear parameter varying (LPV) discrete-time systems using linear matrix inequalities (LMIs) based techniques. The first one relies on a special treatment of time-varying parameters with known dynamics (such as, exponential, trigonometric or periodic functions). Basically, the rate of variation of the parameters is fully explored, leading to improved synthesis conditions when compared to the ones based on unlimited or bounded rates of variation. As application, the approach is employed in a networked control system problem based on time-varying sampling rate. The second contribution is a new strategy to improve performance in gain-scheduled control and filtering exploiting, whenever available, statistical information of the time-varying parameters. The novelty of the technique is to design controllers or filters treating robust stability (assured in all the domain) independently of performance optimization (which prioritizes regions with higher probability of ocurrence). The main motivation of this topic is the problem of gain-scheduled design subject to inexact measurements, particularly when the values of the scheduling parameters are obtained by identification. As complementary contributions in this topic, a new $\mathcal{H}_2$ filtering design condition and a systematic and general procedure to model additive uncertainty affecting the scheduling parameters are proposed. Finally, the problem of designing static output-feedback controllers by means of the enrichment of the system dynamics for LPV systems and non-homogeneous Markov jump linear systems (MJLS) is investigated. For this purpose, the past values of the measured outputs are included in the control law. A solution method expressed in terms of a locally convergent iterative procedure based on LMIs is proposed, having as main technical novelty the fact that the controller is treated as an optimization variable of the problem, allowing the imposition of magnitude and structural (as decentralization) constraints on the control gain without introducing additional conservativeness. The benefits and advantages of the proposed techniques are illustrated by means of several numerical experiments, including comparisons with methods from the literature.

Keywords: Linear Parameter-varying Systems; Discrete-time Systems; Linear Matrix Inequalities; Inexact measurements; Memory Controllers; Gain-scheduled; Robust Gain; State-feedback Control; Static-Output Feddback Control; Filtering.

List of Symbols and Abbreviations

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}(\mathbb{R}^{n\times m})$	Set of real vectors (matrices) with dimension $n \times 1$ $(n \times m)$
$M'(m')$	Transposition of matrix M(vector m)
$\mathrm{He}(m)$	Indicates the operation $M + M'$
$\mathrm{Tr}(M)$	Trace (sum of diagonal elements) of matrix M
$\star$	Represents symmetric blocks in square matrices
$\otimes$	Stands for Kronecker product of matrices
$M = M > 0$	M is a symmetric positive definite matrix
I	Identity matrix of appropriate dimensions
0	Matrix of appropriate dimensions with only null elemments
$\mathcal{E}\{\cdot\}$	Indicates the mathematical expectation
$(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, \Gamma)$	Fundamental probability space
Λ_N	Unit simplex of dimension N
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LTI	Linear Time Invariant
MJLS	Markov Jump Linear Systems
MSE	Mean Square Error
NCS	Networked Control Systems
PDF	Probability Distribution Function
NHMJLS	Non-Homogeneous MJLS
SDOH	Sub-Domain Optimization Heuristic

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SUMMARY

1 Introduction

There exists a class of models versatile enough to appropriately represent nonlinear systems in terms of a family of linear models, linear dynamics affected by time-varying parameters and time-varying systems obtained by mathematical models or identification strategies (DE CAIGNY *et al.*, 2009), known in the literature as *linear parameter-varying* (LPV) systems (HOFFMANN; WERNER, 2015; KVIESKA *et al.*, 2009; MOHAMMADPOUR; SCHERER, 2012; DE OLIVEIRA *et al.*, 2002). The LPV framework has been investigated for more than two decades in control theory literature, and the main reason for this fact is certainly the wide range of applicability in industrial, biologic, economic and many others engineering problems (HOFFMANN; WERNER, 2015; MOHAMMADPOUR; SCHERER, 2012). LPV models are mainly characterized by the affine dependence of the state-space matrices on bounded time-varying parameters, which can have limited or unlimited rates of variation. If the time-varying parameters only assume non-negative values and have unit sum, one has a special class of LPV systems, known as *polytopic LPV systems*. Concerning the synthesis of filters and controllers for LPV systems, generally associated to the optimization of performance criteria, such as the Mean Square Error (MSE), energy-to-peak gain, or the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, two main strategies arise. The first aims to design controllers and filters with *robust* (parameter-independent) gains, that is, the control law or the filter do not require a real-time update. This can be considered the simplest design in terms of computational resources but, on the other hand, tends to provide more conservative results. In contrast, the second strategy is based on synthesis conditions where the controller and filter matrices are scheduled by the time-varying parameters. Known as *gain-scheduled* (SHAMMA; ATHANS, 1991; APKARIAN *et al.*, 2000; LEITH; LEITHEAD, 2000; RUGH; SHAMMA, 2000; KVIESKA *et al.*, 2009; DE ARAÚJO *et al.*, 2015; BANDEIRA *et al.*, 2018), this technique clearly demands a more involved implementation since the time-varying parameters need to be available online to update the gains but, as benefit, improved performance is possible, in general being at least no more conservative when compared to the robust paradigm (BARBOSA *et al.*, 2002; DE SOUZA *et al.*, 2006; DE SOUZA; TROFINO, 2006; DE SOUZA *et al.*, 2007; DE CAIGNY *et al.*, 2010; DE CAIGNY *et al.*, 2012; LACERDA *et al.*, 2016; ROSA *et al.*, 2018).

Among the numerical tools developed for synthesis of filters and controllers for LPV systems, the methods based on Linear Matrix Inequality (LMI) optimization have received the greatest attention (APKARIAN *et al.*, 1995; APKARIAN; GAHINET, 1995; APKARIAN; ADAMS, 1998; HOFFMANN; WERNER, 2015). The first researches addressing analysis or control of LPV systems by means of LMI techniques employed quadratic stability-based approaches, where the Lyapunov matrix is constant (parameter-independent), leading to sufficient conditions for stability analysis and control that, in general, provide conservative results. The main reason for this fact is that, even if the time-varying parameters have bounded rates of variation, the conditions cannot take into account the bounds and the worst case scenario, that is, arbitrarily fast variations, is considered (KAMINER *et al.*, 1993; PERES *et al.*, 1994; MONTAGNER *et al.*, 2005b). Next, still regarding LPV systems with arbitrarily fast time-varying parameters, control design was investigated employing polytopic or piecewise-constant structures for the Lyapunov matrix (DAAFOUZ; BERNUSSOU, 2001; LEITE; PERES, 2004; MONTAGNER *et al.*, 2005a). Although those results are still only sufficient, they contain the quadratic stability as a particular case, providing less conservative evaluations. More recently, it has been proved that the asymptotic stability of discrete-time LPV systems with arbitrary rate of variation can be characterized by an increasing set of LMI conditions described in terms of path-dependent Lyapunov functions (LEE; DULLERUD, 2006; LEE, 2006), generalizing constant (quadratic stability) and affine parameter-dependent Lyapunov matrices (DAAFOUZ; BERNUSSOU, 2001). On the other hand, concerning the rate of variation of the time-varying parameters, the first improvement was to consider a bound on the rate of variation whenever it is available. Since this situation commonly appears in real world applications (AMATO; MATTEI, 2001), this more accurate modeling was employed in several publications (BARBOSA *et al.*, 2002; AMATO *et al.*, 2005; DE SOUZA *et al.*, 2006; DE SOUZA *et al.*, 2007; OLIVEIRA; PERES, 2009; DE CAIGNY *et al.*, 2010; DE CAIGNY *et al.*, 2012; BORGES *et al.*, 2010a; BORGES *et al.*, 2010b; LACERDA *et al.*, 2016; MOZELLI; ADRIANO, 2019) to solve control or filtering problems using parameter-dependent Lyapunov functions. However, polytopic and affine LPV models have a common drawback even when taking into account that the parameters have bounded rates of variation: always contain the time-invariant situation (frozen parameters) as a particular case. As a consequence, the models require the closed-loop dynamic matrix (for both control or filtering) to be stable for all the fixed values of the parameters inside the uncertain domain (GEROMEL; COLANERI, 2006; OLIVEIRA *et al.*, 2009). This feature can be quite conservative, for instance, when the parameters vary in time according with a known function, as exponential, trigonometric and periodic functions.

Concerning gain-scheduled design, a possible better performance when compared with the robust scenario comes with a price: an accurate real-time reading of the time-varying parameters. Such feature constitutes a strong requirement that may be unfeasible due to physical reasons or due to high cost for implementing sensors. Regarding the case where the on-line measurement is not an option and estimation is the only feasible strategy to obtain the values of the time-varying parameters, two important challenges can be recognized. The first one corresponds to the restrictions imposed by the computational cost associated with the identification process, since the parameter-dependent filter or controller demands a real-time update. In this sense, high performance algorithms as neural networks and genetic algorithms may be unfeasible depending on the sampling data frequency required by the process. The second one is associated to the inherent measurement or identification errors not taken into account by the classical design techniques of parameter-dependent filters and controllers for LPV systems (BORGES *et al.*, 2008; GAO *et al.*, 2005; GAO; LI, 2014). Although such simplification provides an easier modeling and, as a consequence, less complex design procedures, discrepancies between the real and the measured or estimated parameters can carry the LPV system out of its operating range, implying loss of performance or, in the worst case scenario, instability. However, advances addressing this later issue have been achieved with the approach called *control or filtering subject to inexact measurements*, where the scheduling parameters are assumed to be contaminated by additive and multiplicative noise (DAAFOUZ *et al.*, 2008; SATO, 2010; SATO; PEAUCELLE, 2013; AGULHARI *et al.*, 2013). Incorporating these uncertainties in the design procedure provides theoretical guarantees of stability and performance. On the other hand, the more complex modeling rises as the main drawback, implying basically in two negative consequences. As a rule of thumb in stability analysis and control design for uncertain systems, more parameters imply in more conservative results when applying relaxations to solve the problems (in general stated in terms of infinite-dimensional optimization problems). Moreover, the computational burden rapidly becomes prohibitive, limiting the approach to treat one or two scheduling parameters. In this sense, two relevant issues have not been sufficiently explored in the literature of gain-scheduled design subject to inexact parameters so far. The first one is if the theoretical guarantees provided by the more complex modeling can result in a level of conservativeness such that the robust filtering or control (where the time-varying parameters are not required) still is a better option. The second one occurs when the error (of measuring or estimation) contaminating the scheduling parameters is modeled as an additive bounded noise where statistical information is available. Clearly, if this information is taken into account in the synthesis conditions, improved results can be expected.

Finally, besides the influence of time-varying parameters in the system dynamics, in terms of practical applications, another important feature to be considered is an adequate modeling of abrupt changes in dynamics or operation points and information packet loss, which are recurring issues in networked control systems (NCS) (HESPANHA *et al.*, 2007) in the context of LPV systems. A class of systems that allows to simultaneously model stochastic and time-varying dynamics corresponds to Markov Jump Linear Systems (MJLS) (COSTA *et al.*, 2005) with non-homogeneous Markov chains (ABERKANE, 2011b; ABERKANE, 2011a; ABERKANE, 2013). This class can be employed to represent, for instance, MJLS with time-varying probabilities (PALMA *et al.*, 2018d) or loss of the scheduling time-varying parameters (PALMA *et al.*, 2018b).

1.1 Investigated Topics

Motivated by the potential of LPV systems in practical applications, this thesis investigates three methodologies to obtain less conservative solutions in control and filter design for linear systems affected by time-varying parameters, listed as follows: *i)* LPV systems affected by time-varying parameters whose variation in time is a known function that can be obtained as solution of a linear difference equation with constant coefficients; *ii)* LPV systems where statistical information about the time-varying parameters is available; *iii)* Memory control of LPV and Non-homogeneous MJLS systems.

Each one of these topics is motivated by practical problems, such as: NCS with time-varying sampling rates; gain-scheduled design using estimated time-varying parameters obtained by real-time identification algorithms; low probability rate of successful transmission of information packages associated with lower energy consumption and congestion in semi-reliable communication networks.

The following subsections provide a more detailed description of the investigated topics.

1.1.1 Approach to handle time-varying parameters based on the solution of difference equations

Certainly the main advantage of the methods from the literature that handle time-varying parameters in terms of polytopic or affine LPV models (parameters lying in hyperretangles) is the possibility of employing LMIs (convex optimization) to test robust stability and, in many situations, design controllers and filters. However, as mentioned before, polytopic and affine LPV models can be conservative when the parameters vary in time according with a known function, as exponential, trigonometric and periodic functions. In these situations the LPV models only take into account the minimum and maximum values of the parameters and variation rates. As a consequence, a family of functions (possibly infinity) is considered by the LPV model and the particular features of the original function are not fully taken into account. An alternative to avoid the conservative LPV modeling in continuous-time domain was proposed in Geromel e Colaneri (2006) (see also Oliveira *et al.* (2009) for a broader discussion), where time-varying parameters (generically expressed in terms of complex exponentials) are written as a solution of a linear differential equation, providing an explicit formula for their variation rate.

The first chapter of this PhD thesis aims to improve the approach from Geromel e Colaneri (2006) and to develop a new technique, similar to the continuous-time case, to deal with discrete-time parameters that can be obtained from the solution of a linear difference equation. The improvement relies on the design of the state-space system to model the time-varying parameters, more general and simpler than Geromel e Colaneri

(2006) and Oliveira *et al.* (2009). The method is then generalized to cope with multiple parameters and periodic signals by means of Fourier series. As main motivation, the proposed modeling can be used in many problems, such as control and filtering of discrete-time LPV systems and also in the NCS context, particularly in the case of time-varying sampling time (BORGES *et al.*, 2010b).

1.1.2 Gain-Scheduling Filter and control design under uncertain parameter

The second chapter of this thesis handles the problem of control and filter design of LPV systems using estimated parameters, whose synthesis technique fits in the area known as *gain-scheduled subject to inexact parameters*, where traditionally the noise affecting the scheduling parameters is incorporated in the LPV model as additional arbitrarily fast time-varying parameters.

As investigated in the conference paper Palma *et al.* (2018c), when the scheduling parameters are obtained by an identification procedure, the estimation error can be modeled as an additive bounded noise. In this case the additive noise can be considered as a random time-varying parameter with a known probability distribution function (PDF). Clearly, if the PDF information is taken into account in the synthesis conditions, improved results can be obtained. Moreover, other types of time-varying parameters, where a PDF in general is available, can be found in important problems from control theory: in design techniques dealing with time-delay (BRIAT *et al.*, 2008; ALABDULMOHSIN *et al.*, 2014; SERPEN; GAO, 2014; BRUNO; MANUEL, 2017; KHEIRANDISH *et al.*, 2017; SEURET; GOUAISBAUT, 2018), that occurs, for instance, when transmitting packets through digital networks; in NCS (HESPANHA *et al.*, 2007) and sampled-data systems (NAGHSHTABRIZI *et al.*, 2008), where the sampling period depends on the possibility of access of the communication channel, for instance, in wireless networks with high concurrency where the access channel is by TDMA based MAC protocol (ZAREEI *et al.*, 2018; KABARA; CALLE, 2012; TEIMOURI; AHMADIYAN, 2018; HOLLINGER *et al.*, 2011); in non-homogeneous MJLS (ABERKANE, 2011b; ABERKANE, 2011a), where the time-varying transition probabilities are associated to packet loss due to channel noise or distance between source and sink (PALMA *et al.*, 2018d; ABERKANE, 2013). Note that, in all those examples, the time-varying parameters usually depend on a probability function.

The main contribution of Chapter 3 is the proposition of a design procedure (state-feedback control and full-order filtering with an $\mathcal{H}_2$ performance criterion) capable to deal with three kinds of time-varying parameters: arbitrarily fast; limited rate of variation; and random but with a known PDF. To handle the latter, the PDF is taken into account using a heuristic, which basically optimizes the performance only in a certain range of values of the parameters (called sub-domain) where the probability of occurrence is higher. As a consequence, not only the guaranteed costs, but the actual worst case $\mathcal{H}_2$ norm of the closed-loop system is improved. Two additional technical contributions are presented in Chapter 3: *i)* Unlike all the approaches found in the literature dealing with additive uncertainty affecting the scheduling parameters, a generic and systematic procedure to deal with an arbitrary number of scheduling parameters for both polytopic and affine uncertainty is proposed in this thesis, considering arbitrarily fast or bounded rates of variations. Explicit formulas and an efficient algorithm are employed to consider the saturation effect of the noise parameters over the scheduling parameters without conservatism; *ii)* As a contribution in the context of $\mathcal{H}_2$ full-order LPV filtering, a new design condition based on LMIs and scalar search is proposed. The condition is only sufficient but it can be shown that a known condition from the literature can be obtained as a particular case.

1.1.3 LMI-based solution for memory static control of LPV system

Seeking to improve performance in robust control and filtering design problems, one among many strategies in the literature is the artificial enrichment of the system dynamics by introducing past values of the states or outputs in the control law or in the dynamics of the filter. This technique has been used in filtering and control of time-invariant uncertain systems (LEE *et al.*, 2009; LEE *et al.*, 2014; LEE *et al.*, 2015; FREZZATTO *et al.*, 2015; LEE *et al.*, 2015; LEE *et al.*, 2016; ROMÃO *et al.*, 2017; FREZZATTO *et al.*, 2018; FREZZATTO *et al.*, 2019). Basically applying the Lyapunov stability theory to an augmented system where the past information (states or outputs) is considered. In general, as the number of past information increases, the possibilities of providing better stabilization and performance (e.g., in terms of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms) augment.

In this sense, the enrichment of the system dynamics allied with the robust design arises as an appealing alternative to provide control laws that produce better performance, with applications in fields like engineering, mechanics, mathematics and economy (MOHAMMADPOUR; SCHERER, 2012), basically, the only price to be paid is a larger buffer for storage of past information.

Note that control design with enrichment of system dynamics can admit feasible solutions when the traditional techniques fail. Particularly in the NCS context, such feature allows to employ less reliable communication networks (lower successful transmission probabilities) implying energy and resources saving and minimizing the network traffic (DURAN-FAUNDEZ *et al.*, 2018; PALMA *et al.*, 2017; PALMA *et al.*, 2018a).

Although there are several works in the context of design of memory filters and memory control laws, Chapter 4 of this thesis aims to meet the demand of research in the field of synthesis of memory control laws applied to LPV, MJLS and non-homogeneous MJLS (NHMJLS) cases, which, to the best of the author's knowledge, have not been investigated in the literature so far. Additionally, there are few approaches in the literature that handle static output-feedback problem for systems where all matrices are affected by uncertainties. In this context, the main challenge is that the strategies used to formulate synthesis conditions in terms of LMIs in the case of time-invariant parameters, as congruence transformations and the concept of eigenvalues, are not useful to deal with systems affected by time-varying parameters, as LPV or NHMJLS systems.

2 A less conservative approach to handle discrete time-varying parameters in linear systems with applications in NCS

2.1 Introduction

This chapter presents a new modeling for time-varying parameters that can be obtained as solutions of difference equations. Among the signals that can be treated using this approach, one can cite: *i)* discrete-time exponential functions that can be used to describe, for instance, discretized systems found in NCS framework and heat transfer functions; *ii)* trigonometric functions such as cosine and sine, whose combination can describe the elliptical movement of a satellite around a celestial orb or the route of unmanned vehicles in a close circuit in industries, mineral exploration, among others; *iii)* any periodic function, which, in general, can be exactly described by Fourier series in discrete-time systems, being expressed as a sum of complex exponential terms. The main advantages of the modeling based on difference equations are: *i)* to provide solutions when methods that require stability for all the fixed values of the parameters inside the uncertain domain (pointwise stability) fail; *ii)* to yield less conservative results (in terms of better performance indexes and larger range of feasibility) when compared with other conditions available in the literature; *iii)* lower computation cost, comparable with the simpler but more conservative approaches (based on arbitrary variation) and more numerically efficient than the less conservative approaches (based on bounded rates of variations). Numerical comparisons with the standard polytopic modeling (bounded and unbounded rates of variation) in the context of $\mathcal{H}_\infty$ state-feedback control and $\mathcal{H}_\infty$ full order filtering of LPV systems are presented to illustrate the mentioned advantages.

Finally, a practical application of the proposed modeling in a current active topic of research in the NCS context is discussed in details. The proposed method can handle NCS with time-varying sampling time (BORGES *et al.*, 2010b), where the sampling period evolves according with a function that model the following behavior: to avoid an open loop control operation caused, for instance, by a network collapse (due to network congestion, buffer overflow, etc) and, at the same time, to minimize the load (packet flow per unit of time), the initial sampling rate starts from a maximum value and, as the network corrects its problems (reconfiguring itself to reduce the load), the sampling rate gradually converges to a minimum value (approaching a continuous-time representation). The same technique can be used to represent the case of NCS where the network starts from an initial setting that allows a great data flow (due to the low occupation of communication channel, the sampling frequency can be maximum) and as other devices access the network (increasing the flow of signals), the network status gradually converges to a saving work mode (where the sampling period is maximum). This behavior is illustrated by means of an experiment where a DC servo motor is controlled through a communication network.

2.2 Motivation

Consider the following discrete-time linear system affected by a time-varying parameter $\theta(k)$

$$x(k+1)=\underbrace{\begin{bmatrix}0 & 1\\ 0.5 & \rho\theta(k)\end{bmatrix}}_{A(\theta(k))}x(k),\ k\geq 0,\ \theta(k)=\lambda^k,\ \lambda<1,\ \rho\in\mathbb{R}^+. \tag{2.1}$$

It is well known that Schur stability of the dynamic matrix $A(\theta(k))$ for fixed values of k is neither necessary nor sufficient to guarantee the robust stability of the LPV system (2.1) (trajectories $x(k)$ converging to the origin as k tends to infinity). Using the Lyapunov stability theory, it is possible to formulate robust stability tests using a polytopic modeling for the dynamic matrix $A(\theta(k))$. Actually, considering that $0\leq\theta(k)\leq 1$ for all $k\geq 0$, one can apply the change of variables

$$\alpha_1(k)=\theta(k),\ \alpha_2(k)=1-\alpha_1(k),\ \alpha(k)=[\alpha_1(k)\ \alpha_2(k)]'\in\Lambda_2 \tag{2.2}$$

where

$$\Lambda_N\triangleq\left\{\xi\in\mathbb{R}^N:\sum_{i=1}^{N}\xi_i=1,\ \xi_i\geq 0,\ i=1,\ldots,N\right\}$$

is the unit simplex of dimension N. Performing the change of variables in $A(\theta(k))$ and applying a homogenization procedure, the dynamic matrix, also known as a *polytopic time-varying matrix*, can be rewritten in the form

$$A(\alpha(k))=\alpha_1(k)\underbrace{\begin{bmatrix}0 & 1\\ 0.5 & 0\end{bmatrix}}_{A_1}+\alpha_2(k)\underbrace{\begin{bmatrix}0 & 1\\ 0.5 & \rho\end{bmatrix}}_{A_2},\ \alpha(k)\in\Lambda_2,$$

where A_1 and A_2 are the vertex matrices. A simple approach to test the robust stability of matrix $A(\alpha(k))$ is to apply the well known *quadratic stability* test, that is, to search for a positive definite symmetric matrix P such that

$$A(\alpha(k))'PA(\alpha(k))-P<0,\forall\alpha(k)\in\Lambda_2. \tag{2.3}$$

Dropping the time-dependence of $\alpha(k)$ (since $\alpha(k)\in\Lambda_2,\ \forall k\geq 0$) and exploiting the fact $A(\alpha)'PA(\alpha)\leq\sum_{i=1}^{2}\alpha_iA_i'PA_i$ (convexity property), the parameter-dependent inequality

(2.3) can be numerically tested in terms of a finite set of LMIs by solving

$$A_i' P A_i - P < 0,\ \ i = 1, 2,\ \ P > 0 \tag{2.4}$$

Note that these LMIs can only be fulfilled if the vertex matrices A_i are Schur (eigenvalues strictly inside the unit circle). In other words, the polytopic modeling requires the Schur stability as a *necessary condition*, being an important source of conservativeness of the approach when $\theta(k)$ has a known dynamics. Lyapunov matrices depending on $\alpha(k)$ can be used to reduce the conservativeness. In this case, the robust stability can be checked by searching for a positive definite symmetric matrix $P(\alpha(k))$ such that

$$\begin{bmatrix} P(\alpha(k)) & A(\alpha(k))' P(\alpha(k+1)) \\ \star & P(\alpha(k+1)) \end{bmatrix} > 0. \tag{2.5}$$

Adopting the polytopic time-varying Lyapunov matrix: $P(\alpha(k)) = \alpha_1(k)P_1 + \alpha_2(k)P_2$ and considering the rate of variation of $\alpha(k)$ as arbitrary, i.e. $\alpha(k+1) = \beta(k) \in \Lambda_2$, then a finite set of LMIs that guarantees the feasibility of (2.5) (as well that $P(\alpha(k))$ is positive definite) is (DAAFOUZ; BERNUSSOU, 2001)

$$\begin{bmatrix} P_i & A_i' P_j \\ \star & P_j \end{bmatrix} > 0,\ \ i, j = 1, 2. \tag{2.6}$$

Although the LMI conditions given in (2.6) are less conservative than quadratic stability, the stability of the vertices (when $i = j$) remains necessary (possibly a source of conservatism). Besides, to consider that the time-varying parameters are arbitrarily fast is not a realistic assumption. In fact, the parameter $\theta(k) = \lambda^k$ in (2.1) cannot vary instantaneously from one to zero in one instant of time. One option to improve the modeling is to explore the definition of $\theta(k)$ and consider that $\Delta\alpha(k) = \alpha(k+1) - \alpha(k)$ is *bounded*. In this case,

$$\Delta(\alpha_1(k)) = \lambda^{k+1} - \lambda^k = \lambda^k(\lambda - 1) \Rightarrow (\lambda - 1) \leq \Delta(\alpha_1(k)) \leq 0$$

and accordingly

$$\Delta(\alpha_2(k)) = 1 - \lambda^{k+1} - 1 + \lambda^k = \lambda^k(1 - \lambda) \Rightarrow 0 \leq \Delta(\alpha_2(k)) \leq (1 - \lambda).$$

The feasible region for the pairs $(\alpha_1(k), \Delta\alpha_1(k))$ and $(\alpha_2(k), \Delta\alpha_2(k))$ is depicted in Fig. 1 and the feasible region for the vector $[\alpha_1(k)\ \alpha_2(k)\ \Delta\alpha_1(k)\ \Delta\alpha_2(k)]'$ is given by the polytope

$$\begin{bmatrix} \alpha_1(k) \\ \alpha_2(k) \\ \Delta\alpha_1(k) \\ \Delta\alpha_2(k) \end{bmatrix} \in \text{co} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \lambda - 1 \\ 1 - \lambda \end{bmatrix}, \begin{bmatrix} 1 - \lambda \\ \lambda \\ \lambda - 1 \\ 1 - \lambda \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

where co{} stands for convex hull. The convex combination of the four vertices provides

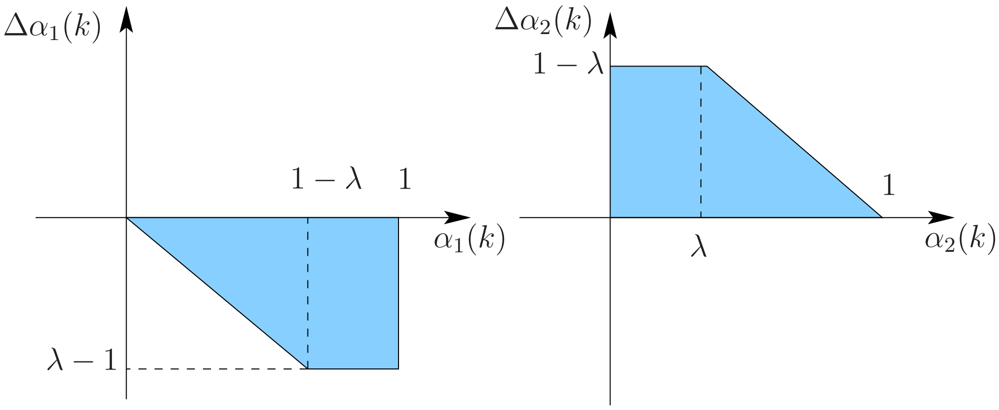


Figure 1 – Feasible region for any $(\alpha_1(k), \Delta\alpha_1(k))$ and $(\alpha_2(k), \Delta\alpha_2(k))$ pairs of time-varying parameters, obeying $\alpha_1(k), \alpha_2(k) \in [0,\ 1]$, $\Delta\alpha_1(k) \in [(\lambda-1),\ 0]$, $\Delta\alpha_2(k) \in [0,\ (1-\lambda)]$, $\alpha_1(k)+\Delta\alpha_1(k) \in [0,\ 1]$, $\alpha_2(k)+\Delta\alpha_2(k) \in [0,\ 1]$.

an explicit formula to convert the parameters $\alpha_1(k)$, $\alpha_2(k)$, $\Delta\alpha_1(k)$ and $\Delta\alpha_2(k)$ in terms of a new parameter (OLIVEIRA; PERES, 2009; DE CAIGNY *et al.*, 2010), say $\gamma(k)$, given by

$$\begin{bmatrix} \alpha_1(k) \\ \alpha_2(k) \\ \Delta\alpha_1(k) \\ \Delta\alpha_2(k) \end{bmatrix} = \gamma_1(k) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma_2(k) \begin{bmatrix} 1 \\ 0 \\ \lambda-1 \\ 1-\lambda \end{bmatrix} + \gamma_3(k) \begin{bmatrix} 1-\lambda \\ \lambda \\ \lambda-1 \\ 1-\lambda \end{bmatrix} + \gamma_4(k) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

After converting $A(\alpha(k))$ to $A(\gamma(k))$, $P(\alpha(k))$ to $P(\gamma(k))$ and $P(\alpha(k+1))$ to $P^+(\gamma(k))$, the following robust LMI can be solved to check the robust stability

$$\begin{bmatrix} P(\gamma(k)) & A(\gamma(k))'P^+(\gamma(k)) \\ \star & P^+(\gamma(k)) \end{bmatrix} > 0, \quad \forall \gamma(k) \in \Lambda_4 \tag{2.7}$$

Unfortunately, matrix $A(\gamma(k))$ is given by (the dependence on k is omitted below)

$$A(\gamma) = \gamma_1 \begin{bmatrix} 0 & 1 \\ 0.5 & \rho \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 0 & 1 \\ 0.5 & \rho\lambda \end{bmatrix} + \gamma_4 \begin{bmatrix} 0 & 1 \\ 0.5 & \rho \end{bmatrix}$$

and the first and fourth vertices of $A(\gamma)$ are equal to the second vertex of $A(\alpha)$. Consequently, both matrices $A(\alpha)$ and $A(\gamma)$ can only be Schur stable if $\rho < 0.5$ (implying that the maximum eigenvalue of A_2 is lower than one). Thus, neither of the LMI conditions presented previously can assure the robust stability of the system if $\rho \geq 0.5$. The reason for this fact is that the polytopic modeling considering both behaviors (arbitrary and bounded rates of variation), always contain the time-invariant situation as a particular case, that is, $\alpha(k+1) = \alpha(k) = \alpha$. A solution to surpass such limitation is presented in next section.

2.3 Main Contribution: More accurate modeling of time-varying parameters

The purpose of this section is to propose a less conservative approach to deal with time-varying parameters that can be expressed as complex exponentials, that is, solutions of the difference equation $\alpha(k+1) = H\alpha(k)$, being the continuous-time counterpart of the method investigated in Geromel e Colaneri (2006) and Oliveira *et al.* (2009). There are three motivations to perform this investigation: *i)* to guarantee the stability of the time-varying system without requiring Schur stability for each point (frozen values of k) inside the uncertain domain; *ii)* to propose a new approach, different from the one presented in Geromel e Colaneri (2006) and Oliveira *et al.* (2009), to obtain matrix H, that is easier to construct and also to generalize to cope with multiple parameters and periodic signals (such as square, sawtooth or periodic triangular waves) or even aperiodic signals (such as trigonometric waves with irrational frequencies) using the Fourier series; *iii)* to use the new approach to address NCS problems, particularly in the case of time-varying sampling interval.

2.3.1 Exponential time-varying parameters ($\theta(k) = \lambda^k$)

Consider the following linear difference equation

$$\theta(k+1) - \lambda\theta(k) = 0, \quad \theta(0) = 1 \tag{2.8}$$

whose solution $\theta(k) = \lambda^k$ is precisely the time-varying parameter appearing in (2.1). Besides, the difference equation provides an explicit formula for the time variation of $\theta(k)$, that is, $\theta(k+1) = \lambda\theta(k)$. Let matrix $A(\theta(k))$ in (2.1) be written in the form

$$A(\theta(k)) = A_0 + \theta(k)A_1 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} + \theta(k)\begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}$$

and consider the affine Lyapunov matrix $P(\theta(k)) = P_0 + \theta(k)P_1$ and the robust stability condition

$$\begin{bmatrix} P(\theta(k)) & A(\theta(k))'P(\theta(k+1)) \\ \star & P(\theta(k+1)) \end{bmatrix} > 0.$$

Replacing the definitions of $A(\theta(k))$ and $P(\theta(k))$, one has

$$\begin{bmatrix} P_0 + \theta(k)P_1 & (A_0 + \theta(k)A_1)'(P_0 + \theta(k+1)P_1) \\ \star & P_0 + \theta(k+1)P_1 \end{bmatrix} > 0$$

Using the expression for $\theta(k+1)$ given in (2.8), one has

$$\begin{bmatrix} P_0 + \theta(k)P_1 & (A_0 + \theta(k)A_1)'(P_0 + \lambda\theta(k)P_1) \\ \star & P_0 + \lambda\theta(k)P_1 \end{bmatrix} > 0$$

After dropping the time-dependence of $\theta(k)$, two main possibilities arise to check the positivity of the resulting quadratic polynomial matrix inequality in terms of convex optimization based on semidefinite programming (SDP). Dealing with θ as an interval parameter, Sum-of-Squares or Gram matrix relaxations could be applied (PARRILO, 2003; CHESI, 2005), resulting in a set of SDP constraints. On the other hand, employing the change of variables (2.2) to lift the uncertainty to Λ_2, it is possible to apply the extensively used "coefficient check" relaxation, also known as Pólya's relaxation (SCHERER, 2005; OLIVEIRA; PERES, 2007), providing a finite set of LMIs. The latter strategy, that presents a good trade-off between accuracy and numerical complexity (OLIVEIRA; PERES, 2007) can be automatically performed by the Robust LMI Parser (ROLMIP) (AGULHARI *et al.*, 2019), that was specially created to deal with this type of relaxation. Note that the proposed stability test is based on a Lyapunov matrix with affine dependence on θ. Higher degrees can improve the accuracy of the results (at the price of a larger computational effort) and the generalization for a Lyapunov matrix with polynomial dependence of arbitrary degree g on $\theta(k)$ is given by

$$P_g(\theta(k)) = \sum_{i=0}^{g} \theta(k)^i P_i \tag{2.9}$$

with its unit shift given by

$$P_g(\theta(k+1)) = \sum_{i=0}^{g} \lambda^i \theta(k)^i P_i.$$

2.3.1.1 Numerical experiment

The aim of this example is to find the maximum value of ρ (ρ_{max}) for a fixed $\lambda \in [0.05 \quad 1]$ such that the robust stability of system (2.1) can be guaranteed by a polynomial Lyapunov matrix given in (2.9) with $g = 1, \ldots, 3$. The results are depicted in Fig. 2. Note that as $\lambda \to 1$, the maximum value of ρ converges to 0.5 for all degrees. This is expected since in this case the parameter λ^k converges to a time-invariant parameter $\lambda = 1$. In this case, the Schur stability of matrix $A(\theta(k))$ is, actually, necessary. On the other hand, smaller values of λ provide larger stability margins in terms of ρ that cannot be certificated by the classical polytopic modeling (as discussed in the previous section, neither of the LMI conditions (2.5) and (2.7) can assure the robust stability for $\rho \geq 0.5$). The parser ROLMIP, that works jointly with Yalmip (LÖFBERG, 2004), has been employed to extract the LMIs from the polynomial positivity tests (degree $g+1$), and the SDP solver SeDuMi (STURM, 1999) was used to solve the LMIs.

2.3.2 Generalizations and guidelines

In this section it is discussed how to generalize the new modeling in terms of treating more than one parameter and also the procedure to obtain the finite dimensional

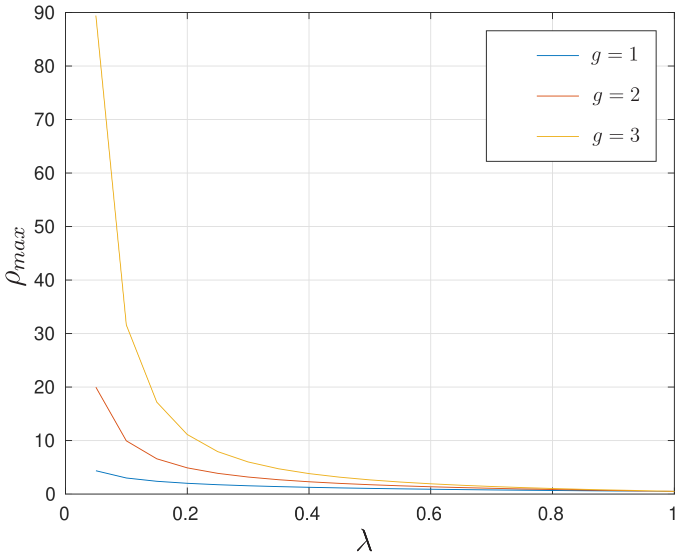


Figure 2 – Maximum value of ρ for a given $\lambda \in [0.05 \;\; 1]$ such that the robust stability of system (2.1) can be certified by a polynomial Lyapunov matrix (2.9) of degree $g = 1, \ldots, 3$.

tests systematically.

Consider system (2.1) again but with $\theta(k) = \cos(\omega k)$. This time-varying parameter[1] can be obtained as the solution of the following difference equation

$$\theta(k+2) - 2\cos(\omega)\theta(k+1) + \theta(k) = 0, \quad \theta(0) = 1, \;\; \theta(1) = \cos(\omega). \tag{2.10}$$

In the general case, that is, considering other sets of initial conditions (there are many), $\cos(\omega k)$ is given as a linear combination of $\theta(k)$ and $\theta(k+1)$ (the two linear independent signals necessary to create a base to the space of solutions). Thus, a Lyapunov matrix with affine dependence on the time-varying parameters has the structure

$$P(\theta(k), \theta(k+1)) = P_0 + \theta(k)P_1 + \theta(k+1)P_2 \tag{2.11}$$

At this point $\theta(k)$ and $\theta(k+1)$ are treated as distinct parameters lying in the intervals

$$\theta(k) \triangleq \theta_1(k) \in [-1 \;\; 1], \;\; \theta(k+1) \triangleq \theta_2(k) = \cos(\omega(k+1)) \in [-1 \; 1]$$

and using the difference equation given in (2.10), the unit time-shift of these new parameters is given in the form

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2\cos(\omega) \end{bmatrix}}_{H} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix} \tag{2.12}$$

[1] Periodic if $\omega/2\pi$ is rational.

In this new notation the Lyapunov matrix and its unit shift are given by $P(\theta_1(k), \theta_2(k)) = P_0 + \theta_1(k)P_1 + \theta_2(k)P_2$ and $P(\theta_1(k+1), \theta_2(k+1)) \triangleq P^+(\theta_1(k), \theta_2(k))$ where

$$\begin{aligned} P^+(\theta_1(k), \theta_2(k)) =& P_0 + \theta_1(k+1)P_1 + \theta_2(k+1)P_2 \\ =& P_0 + \theta_2(k)P_1 + (2\cos(\omega)\theta_2(k) - \theta_1(k))P_2 \\ =& P_0 + \theta_1(k)(-P_2) + \theta_2(k)(P_1 + 2\cos(\omega)P_2) \end{aligned}$$

and the robust LMI to certificate the robust stability is

$$\begin{bmatrix} P(\theta_1(k), \theta_2(k)) & A(\theta_1(k), \theta_2(k))' P^+(\theta_1(k), \theta_2(k)) \\ \star & P^+(\theta_1(k), \theta_2(k)) \end{bmatrix} > 0,$$

$$\forall (\theta_1(k), \theta_2(k)) \in [-1 \ \ 1] \times [-1 \ \ 1] \triangleq [-1 \ \ 1]^2 \quad (2.13)$$

Consider a slightly modified system (2.1) given by

$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ \phi(k) & \rho\theta(k) \end{bmatrix}}_{A(\theta(k),\phi(k))} x(k), \quad \forall k \geq 0, \quad (2.14)$$

with $\theta(k) = \cos(\omega k)$, $\phi(k) = \sin(\omega k)$, $\rho \in \mathbb{R}^+$. The simplest way to treat the new time-varying parameter ($\phi(k)$) is to consider the same difference equation given in (2.10) but with a different initial condition, that is,

$$\phi(k+2) - 2\cos(\omega k)\phi(k+1) + \phi(k) = 0, \qquad \phi(0) = 0, \ \phi(1) = \sin(\omega). \quad (2.15)$$

Including affine terms associated to $\phi(k)$ and $\phi(k+1)$ in the Lyapunov matrix and applying the same procedure used to treat $\theta(k)$, the resulting robust LMI to be solved (omitting the dependence on k and replacing the dependence on $k+1$ by a superscript $+$) is

$$\begin{bmatrix} P(\theta_1, \theta_2, \phi_1, \phi_2) & A(\theta_1, \theta_2, \phi_1, \phi_2)' P^+(\theta_1, \theta_2, \phi_1, \phi_2) \\ \star & P^+(\theta_1, \theta_2, \phi_1, \phi_2) \end{bmatrix} > 0, \quad \forall (\theta_1, \theta_2, \phi_1, \phi_2) \in [-1 \ \ 1]^4.$$

A different (and clever) possibility is to express $\sin(\omega k)$ in terms of a linear combination of $\theta(k)$ and $\theta(k+1)$ in (2.10). After some trigonometric manipulations[2]

$$\sin(\omega k) = \cot(\omega)\theta(k) - \csc(\omega)\theta(k+1).$$

With this option there are only two parameters, $\theta(k)$ and $\theta(k+1)$, requiring a less complex condition similar to (2.13) to be solved. The rule of thumb is to express the maximum number of parameters as the solution of the same difference equation.

Another important aspect to be discussed is more involved combinations of time-varying parameters as products and powers. Note that, using the proposed modeling, this issue is, ultimately, just a matter of dealing with higher degree polynomials.

[2] $csc(\omega)$ and $cot(\omega)$ stand for cosecant and cotangent trigonometric functions, respectively.

For instance, consider the time-varying parameter $\psi(k) = \sin(\omega_1 k)\cos(\omega_2 k)^2$. Using the material presented so far, this parameter can be expressed as $\psi(k) = \phi(k)\theta(k)^2$ with the formulas to obtain $\theta(k+2)$ and $\phi(k+2)$ in terms of $\theta(k+1)$, $\theta(k)$ and $\phi(k+1)$ and $\phi(k)$ given in (2.10) and (2.15), respectively. If $\omega_2 = \omega_1$, then

$$\psi(k) = \left(\cot(\omega_1)\theta(k) - \csc(\omega_1)\theta(k+1)\right)\theta(k)^2 = \cot(\omega_1)\theta(k)^3 - \csc(\omega_1)\theta(k)^2\theta(k+1)$$

Once all the time-varying parameters were properly modeled following the presented guidelines, the next step is to define the structure of the Lyapunov matrix, that in this chapter is considered as polynomial of arbitrary degree g on θ, more precisely

$$P(\theta(k)) = \sum_{h_1+\cdots+h_m \leq g} \theta_1^{h_1}(k)\cdots\theta_m^{h_m}(k)P_h \tag{2.16}$$

where $\theta(k) = [\theta_1(k), \ldots, \theta_m(k)]'$ is a vector of m time-varying parameters, $h = [h_1, \ldots, h_m]'$ is a vector with nonnegative integers and the sum produces all monomials up to degree g. For instance, considering $m = 3$ and $g = 2$ one has $\theta(k) = [\theta_1(k)\ \theta_2(k)\ \theta_3(k)]'$,

$$\begin{aligned} P(\theta(k)) = P_{[0,0,0]} &+ \theta_1(k)P_{[1,0,0]} + \theta_2(k)P_{[0,1,0]} + \theta_3(k)P_{[0,0,1]} \\ &+\theta_1(k)\theta_2(k)P_{[1,1,0]} + \theta_1(k)\theta_3(k)P_{[1,0,1]} + \theta_2(k)\theta_3(k)P_{[0,1,1]} \\ &+\theta_1^2(k)P_{[2,0,0]} + \theta_2^2(k)P_{[0,2,0]} + \theta_3^2(k)P_{[0,0,2]}. \end{aligned}$$

To deal with matrix polynomial positivity tests, associated to the robust stability problem or the synthesis problems presented in next sections, in the case of multiple parameters, the change of variables (2.2) can be applied in each parameter $\theta_i,\ i = 1, \ldots, m$, giving rise to parameters lying in the Cartesian product of m simplexes (named multi-simplex (OLIVEIRA *et al.*, 2008)). The advantage of this change of domain is that Pólya's relaxations can be used and all the trick polynomial manipulations are performed automatically by parser ROLMIP.

Regarding the continuous-time counterpart of the results presented in this chapter, firstly investigated in Geromel e Colaneri (2006) and Oliveira *et al.* (2009), note that the proposed approach is much easier to be applied since, differently from Geromel e Colaneri (2006) and Oliveira *et al.* (2009), the difference equation (the differential equation in Geromel e Colaneri (2006) and Oliveira *et al.* (2009)) does not need to produce parameters lying into the unit simplex. This requirement makes the problem of determining matrix H in (2.12) much more challenging, specially when dealing with multiple and discrete-time parameters. Although not discussed here, an extension of the proposed modeling to cope with continuous-time parameters can be viewed as an improvement with respect to Geromel e Colaneri (2006) and Oliveira *et al.* (2009). Finally, note that, ultimately, the proposed approach also deals with parameters in simplexes, but in this case only to check the positivity of the polynomial matrix inequalities.

2.3.2.1 Numerical experiment

Consider system (2.1) with $\theta(k) = \cos(\omega k)$. The aim is to analyze the stability margins in terms of the maximum value of ρ (ρ_{max}) for $\omega \in [0 \quad 2\pi]$, using polynomial Lyapunov matrices as in (2.16) of degrees $g = 1, \ldots, 3$. The results are shown in Fig. 3. As in the experiment given in Section 2.3.1.1, higher degrees for the Lyapunov matrix provide better results at the price of a larger computational effort. Regarding the maximum values of ρ, the best results were achieved around $\omega \approx \pi$. If a classic polytopic modeling is applied, no feasible solution can be obtained for $\rho \geq 0.5$.

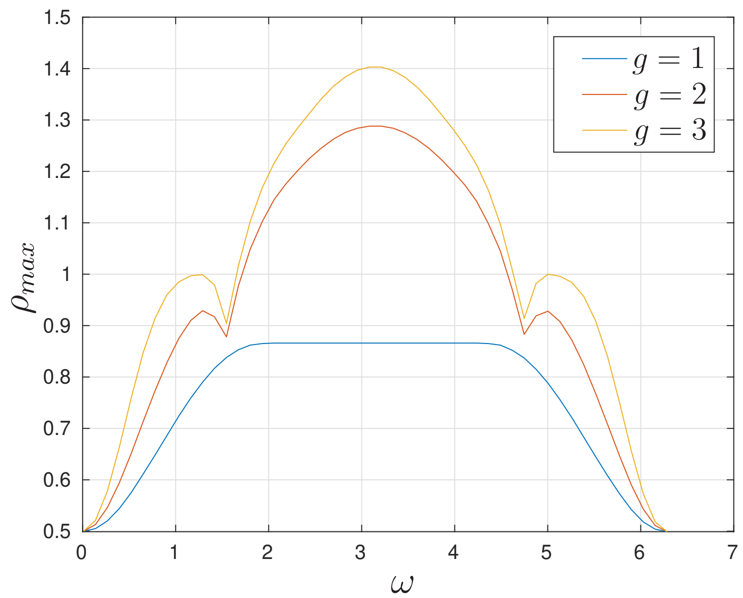


Figure 3 – Maximum value of ρ for a given $\lambda \in [0.0 \quad 2\pi]$ such that the robust stability of system (2.1) with $\theta(k) = \cos(\omega k)$ can be certified by a polynomial Lyapunov matrix of degree $g = 1, \ldots, 3$.

2.3.3 Periodic time-varying parameters

So far the proposed modeling is only capable to deal with time-varying parameters obtained as solutions of linear difference equations. However, recall that any discrete-time periodic signal can be precisely decomposed in terms of a discrete-time Fourier series, expressed as a sum of complex exponentials. Clearly, the proposed methodology can be applied to each complex exponential of the series and the price to be paid is a numerical complexity proportional to the number of terms. As an example, consider a classic time-varying signal, the *square wave* ($sw(k)$), depicted in Fig. 4 with period $N = 6$.

This signal can be represented as

$$sw(k) = \frac{1}{2} + \frac{1}{3}\cos(\frac{\pi k}{3}) + \frac{780}{1351}\sin(\frac{\pi k}{3}) + \frac{1}{6}\cos(\pi k)$$

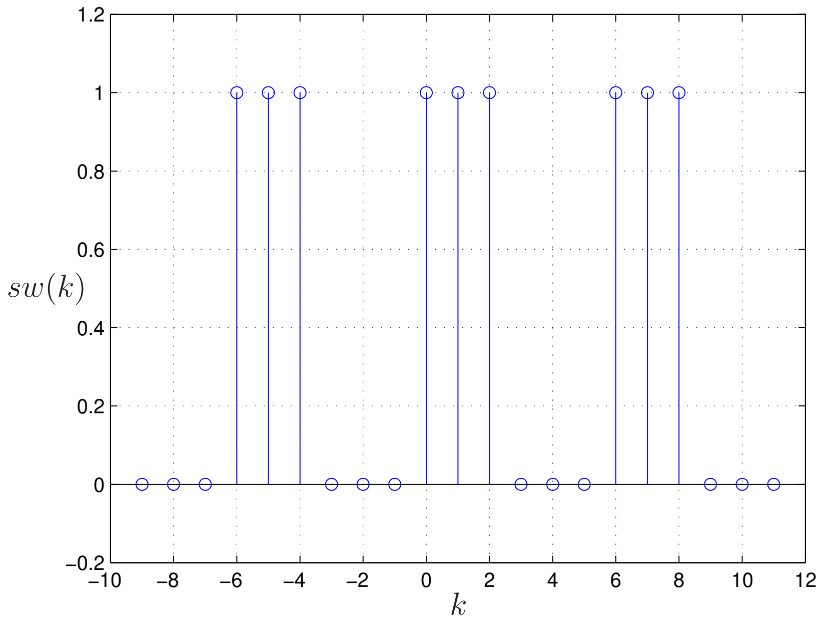


Figure 4 – Discrete-time square wave of period $N = 6$.

and, using the proposed modeling, one possible solution is

$$sw(k) = \frac{1}{2} + \frac{1}{3}\theta_1(k) + \frac{780}{1351}(\cot(\frac{\pi}{3})\theta_1(k) + \csc(\frac{\pi}{3})\theta_2(k)) + \frac{1}{6}\theta_3(k)$$

with

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \\ \theta_3(k+1) \\ \theta_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2\cos(\frac{\pi}{3}) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2\cos(\pi) \end{bmatrix} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \\ \theta_3(k) \\ \theta_4(k) \end{bmatrix}. \tag{2.17}$$

2.3.3.1 Numerical Results

The numerical experiment of previous section is revisited, but this time with $\theta(k)$ being the square wave depicted in Fig. 4. The maximum value of ρ (ρ_{max}) such that the system is robustly stable as well the numerical complexity associated are shown in Table 1 assuming polynomial Lyapunov matrices of degrees $g = 1, \ldots, 3$. Considering the less conservative result ($g = 3$), note that during three instants of time, the dynamic matrix assumes the value

$$A_{sw(k)=1} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.773 \end{bmatrix}$$

whose eigenvalues have modulus $|\lambda_1| = 0.4193$, $|\lambda_2| = 1.1923$. That is, the matrix is not Schur stable, but still the trajectories of the state converge to the origin as k tends to infinity. Observe that this analysis could not be done with the standard techniques from the literature based on polytopic LPV modeling, because they cannot assure the robust

Table 1 – Maximum values of ρ and numerical complexity (V is the number of scalar variables and L is the number of LMI rows) in the robust stability analysis of system (2.1) with $\theta(k)$ being the square wave depicted in Figure 4, using polynomial Lyapunov matrices of degrees $g = 1, \ldots, 3$..

g	ρ_{max}	V	L	Time (s)
1	0.499	15	324	0.30
2	0.638	45	1024	1.38
3	0.773	105	2500	4.64

stability if any time-invariant matrix belonging to the uncertain domain is not Schur stable.

It is also important to mention that when dealing with periodic time-varying parameters, the dynamic matrix can be viewed as a periodic time-varying matrix and, in this case, necessary and sufficient LMI conditions can be applied to test robust stability (BOLZERN; COLANERI, 1988; FARGES *et al.*, 2007). In this context the Fourier series approach can be viewed only as an alternative, probably not competitive since it is only sufficient. As an illustration of this fact, the periodic Lyapunov lemma (see, for instance, de Souza e Trofino (2000, Eq. (4))) is applied to $A(\theta(k))$ given in (2.1) considering $\theta(k) = \cos(\pi k)$ and $\theta(k) = sw(k)$ and the value of ρ is maximized such that the resulting periodic system is assymptotically stable. In this first case $\rho_{max} = 1.5$ has been obtained, being 6.84% better when compared to the value provided by the proposed method using $g = 3$. In the square wave case $\rho_{max} = 1.23$ has been achieved (59.12% better than the proposed technique with $g = 3$). Nevertheless, the proposed technique can still be useful when dealing, for instance, with periodic and aperiodic signals simultaneously. In the next two sections the proposed modeling is evaluated in synthesis problems, like $\mathcal{H}_\infty$ filtering and $\mathcal{H}_\infty$ state-feedback control.

2.4 $\mathcal{H}_\infty$ filter design

Consider the discrete-time LPV system $\mathcal{G}_f$, described by the following state-space representation

$$\mathcal{G}_f = \begin{cases} x(k+1) = A(\theta(k))x(k) + E(\theta(k))w(k), \\ y(k) = C_y(\theta(k))x(k) + E_y(\theta(k))w(k), \\ z(k) = C_z(\theta(k))x(k) + E_z(\theta(k))w(k), \end{cases} \tag{2.18}$$

where $x(k) \in \mathbb{R}^{n_x}$, $w(k) \in \mathbb{R}^{n_w}$, $z(k) \in \mathbb{R}^{n_z}$, $y(k) \in \mathbb{R}^{n_y}$, respectively denote the state, noise input, estimated output and measured output vectors. The state-space matrices depend on a vector of time-varying parameters $\theta(k) = [\theta_1(k), \ldots, \theta_N(k)]'$ that belongs to a compact convex set (as the unit simplex or the hypercube) for all $k \geq 0$. The aim is to design a full-order parameter-dependent filter given by

$$\mathcal{F} = \begin{cases} x_f(k+1) = A_f(\theta(k))x_f(k) + B_f(\theta(k))y(k), \\ z_f(k) = C_f(\theta(k))x_f(k) + D_f(\theta(k))y(k), \end{cases} \tag{2.19}$$

where $x_f(k) \in \mathbb{R}^{n_x}$, $z_f(k) \in \mathbb{R}^{n_z}$ respectively correspond to the filter state and estimated output vectors. The estimation error is defined by $e(k) = z(k) - z_f(k)$, and its dynamics is represented by the set of equations below, by connecting the filter (2.19) to the system (2.18), obtaining the augmented system

$$\mathcal{G}_{aug} = \begin{cases} \hat{x}(k+1) = \hat{A}(\theta(k))\hat{x}(k) + \hat{B}(\theta(k))w(k), \\ e(k) = \hat{C}(\theta(k))\hat{x}(k) + \hat{D}(\theta(k))w(k), \end{cases} \tag{2.20}$$

where $\hat{x}(k) = [x(k)' \ x_f(k)']' \in \mathbb{R}^{2n_x}$, with matrices given by

$$\begin{gathered} \hat{A}(\theta(k)) = \begin{bmatrix} A(\theta(k)) & 0 \\ B_f(\theta(k))C_y(\theta(k)) & A_f(\theta(k)) \end{bmatrix}, \ \hat{B}(\theta(k)) = \begin{bmatrix} E(\theta(k)) \\ B_f(\theta(k))E_y(\theta(k)) \end{bmatrix}, \\ \hat{C}(\theta(k)) = \begin{bmatrix} C_z(\theta(k)) - D_f(\theta(k))C_y(\theta(k)) & -C_f(\theta(k)) \end{bmatrix}, \\ \hat{D}(\theta(k)) = \begin{bmatrix} E_z(\theta(k)) - D_f(\theta(k))E_y(\theta(k)) \end{bmatrix}. \end{gathered} \tag{2.21}$$

As performance criterion, the filter is synthesized such that the error dynamics $e(k)$ is asymptotically stable and the $\mathcal{H}_\infty$ performance from the disturbance input $w(k)$ to the estimation error is minimized.

The concepts of $\mathcal{H}_\infty$ performance analysis for discrete-time LPV systems can be found in De Caigny *et al.* (2010) and De Caigny *et al.* (2012). The matrices of filter (2.19) can be synthesized by solving a set of parameter-dependent LMI conditions as the ones presented in Theorem 2.1.

Theorem 2.1. *If there exist symmetric positive definite matrices* $W_{11}(\theta(k))$ *and* $W_{22}(\theta(k))$, *matrices* $W_{12}(\theta(k))$, $K_{11}(\theta(k))$, $K_{21}(\theta(k))$, $\bar{K}(\theta(k))$, $H(\theta(k))$, $Z(\theta(k))$, $C_f(\theta(k))$ *and* $D_f(\theta(k))$, *with compatible dimensions, that satisfy the parameter-dependent LMI condition*

$$\begin{bmatrix} -W_{11}(\theta(k)) & \star & \star & \star & \star & \star \\ -W_{12}(\theta(k))' & -W_{22}(\theta(k)) & \star & \star & \star & \star \\ \begin{pmatrix} K_{11}(\theta(k))A(\theta(k)) \\ +Z(\theta(k))C_y(\theta(k)) \end{pmatrix} & H(\theta(k)) & \begin{pmatrix} W_{11}(\theta(k+1)) \\ -K_{11}(\theta(k)) \\ -K_{11}(\theta(k))' \end{pmatrix} & \star & \star & \star \\ \begin{pmatrix} K_{21}(\theta(k))A(\theta(k)) \\ +Z(\theta(k))C_y(\theta(k)) \end{pmatrix} & H(\theta(k)) & \begin{pmatrix} W_{12}(\theta(k+1))' \\ -\bar{K}(\theta(k)) \\ -K_{21}(\theta(k)) \end{pmatrix} & \begin{pmatrix} W_{22}(\theta(k+1)) \\ -\bar{K}(\theta(k)) - \bar{K}(\theta(k))' \end{pmatrix} & \star & \star \\ 0 & 0 & \begin{pmatrix} E(\theta(k))'K_{11}(\theta(k))' \\ +E_y(\theta(k))'Z(\theta(k)))' \end{pmatrix} & \begin{pmatrix} E(\theta(k))'K_{21}(\theta(k))' \\ +E_y(\theta(k))'Z(\theta(k))' \end{pmatrix} & -\gamma^2 I & \star \\ \begin{pmatrix} C_z(\theta(k)) \\ -D_f(\theta(k))C_y(\theta(k)) \end{pmatrix} & -C_f(\theta(k)) & 0 & 0 & \begin{pmatrix} E_z(\theta(k)) \\ -D_f(\theta(k))E_y(\theta(k))) \end{pmatrix} & -I \end{bmatrix} < 0 \tag{2.22}$$

for all $(\theta(k), \theta(k+1))$ lying in the uncertain domain, then $A_f(\theta(k)) = \bar{K}(\theta(k))^{-1}H(\theta(k))$, $B_f(\theta(k)) = \bar{K}(\theta(k))^{-1}Z(\theta(k))$, $C_f(\theta(k))$, and $D_f(\theta(k))$ are the filter matrices in (2.19) that assure a guaranteed cost γ for the $\mathcal{H}_\infty$ performance of system (2.20).

Proof. Considering the matrices in (2.21), pre- and post-multiplying (2.22), respectively, by

$$\mathcal{B} = \begin{bmatrix} I_{2n_x} & \hat{A}(\theta(k))' & 0 & 0 \\ 0 & \hat{B}(\theta(k))' & I_{n_w} & 0 \\ 0 & 0 & 0 & I_{n_z} \end{bmatrix}$$

and its transpose, one obtains a condition equivalent to the bounded real lemma given in De Caigny *et al.* (2010), De Caigny *et al.* (2012) and de Souza *et al.* (2006) for the augmented discrete-time LPV system (2.20). □

2.4.1 Numerical results

In order to evaluate the performance of the $\mathcal{H}_\infty$ filters designed using the new modeling proposed in the Section 2.3 and to compare with classical polytopic approaches based on LMIs, a numerical example is proposed. The system matrices are given by $C_z = I$, $E_z = 0$, $E = [0 \quad 2]'$,

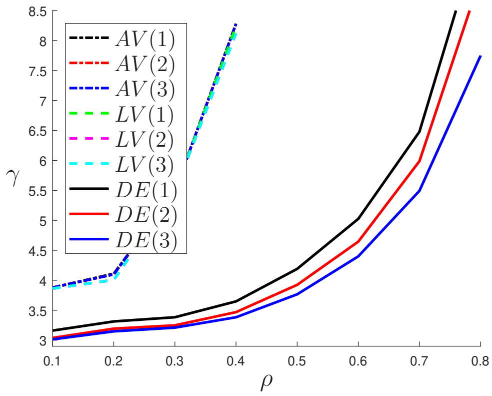
$$A(\theta(k)) = \begin{bmatrix} 0 & 1 \\ 0.5\theta_1(k) & \rho\theta_2(k) \end{bmatrix}, \quad \begin{array}{l} C_y(\theta(k)) = \begin{bmatrix} 0 & (1-\rho\theta_2(k)) \end{bmatrix}, \\ E_y(\theta(k)) = 1 - 2\theta_2(k). \end{array} \tag{2.23}$$

where $0 \leq \theta_1(k) \leq 1$ and $-1 \leq \theta_2(k) \leq 1$ are the time-varying parameters with $\rho \geq 0$. To design a full-order filter (2.19) for system (2.18), a necessary condition is that $A(\theta(k))$ is robustly stable, otherwise the dynamics of the estimation error is unstable. Note that, as discussed in Section 2.2, for $\rho > 0.5$ the spectral radius of $A(\theta(k))$ is greater than one in some instants of time and, by using the classical polytopic approaches, it is not possible to assure the robust stability of (2.18) and, consequently, the synthesis of a full-order filter is impracticable. On the other hand, the proposed modeling arises as an alternative to obtain feasible solutions.

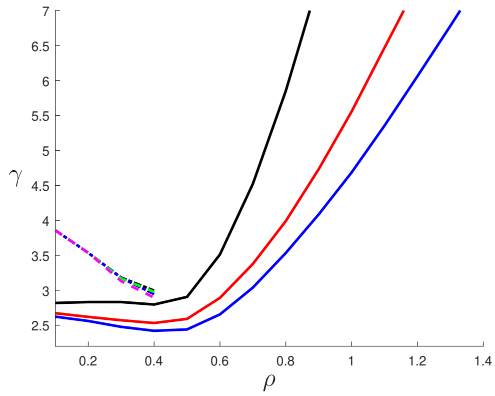
To explore the design of both gain-scheduled and robust $\mathcal{H}_\infty$ full-order filters, two cases of time-varying parameters are investigated:

- **C**ase 1: $\theta_1(k)$ is considered as a discrete-time exponential function ($\theta_1(k) = \lambda^k$), while $\theta_2(k)$ is a cosine wave ($\theta_2(k) = \cos(\omega k)$);
- **C**ase 2: $\theta_1(k)$ is considered constant ($\theta_1(k) = 1^k = 1$), while $\theta_2(k)$ is a cosine wave ($\theta_2(k) = \cos(\omega k)$).

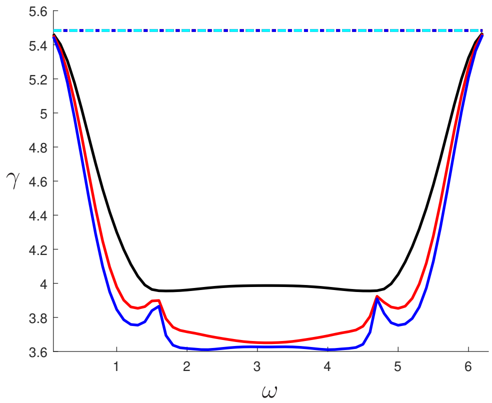
In the filter design, the Lyapunov matrix $W(\theta(k))$ and the extra-decision variables $K_{11}(\theta(k))$, $K_{21}(\theta(k))$ of condition (2.22) are described in terms of the polynomial structure (2.16) with degree $g \geq 1$. The robust filter is obtained using degree zero ($g_F = 0$) and the gain-scheduled filter is obtained with $g_F = g$ in the filter recovery variables ($\bar{K}(\theta(k))$, $H(\theta(k))$, $Z(\theta(k))$, $C_f(\theta(k))$, and $D_f(\theta(k))$) following the structure in (2.16).



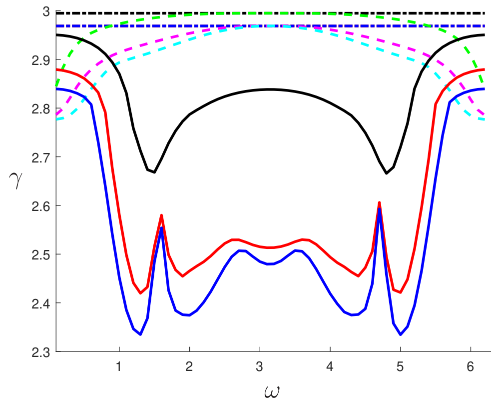
(a) Case 1: Robust.



(b) Case 1: Gain-scheduled.



(c) Case 2: Robust.



(d) Case 2: Gain-scheduled.

Figure 5 – $\mathcal{H}_\infty$ guaranteed costs (γ) computed with Theorem 2.1 for the design of (a) Robust and (b) Gain-scheduled filters for system (2.23) Case 1 ($\theta_1(k) = 0.1^k$ and $\theta_2(k) = \cos(k)$) versus $\rho > 0$; and for the design of (c) Robust and (d) Gain-scheduled filters for system (2.23) Case 2 ($\theta_1(k) = 1^k = 1$ and $\theta_2(k) = \cos(\omega k)$) versus $\omega \in [0, 2\pi]$ using AV (dash-dotted lines), LV (dashed lines), and DE (solid lines) approaches with degree g (between parentheses).

For Case 1, the $\mathcal{H}_\infty$ guaranteed cost is calculated as a function of ρ using Theorem 2.1 considering $\lambda = 0.1$ and $\omega = 1$. The proposed approach based on difference equations (DE) is compared to the classical polytopic design with arbitrary variation (AV) or limited variation (LV) for both time-varying parameters $(\theta_1(k), \theta_2(k))$. Figs. 5a and 5b show the $\mathcal{H}_\infty$ guaranteed costs (γ) computed for the synthesis of, respectively, robust ($g_F = 0$) and gain-scheduled ($g_F = g$) filters with Theorem 2.1 versus $\rho > 0$. Note that, among the three tested techniques, DE clearly provides the lowest guaranteed costs (γ), and the wider range of feasibility (greatest values for ρ) that can be further improved with the increase of g. Also observe that, differently from the proposed method, the performance indexes associated with LV and AV approaches are not noticeably improved

Table 2 – Computational cost in terms of the scalar variables (V) and the LMI rows (L) for the synthesis of robust ($g_F = 0, g = 1$) and gain-scheduled ($g_F = g = 1$) filters for system (2.23) considering Cases 1 and 2 and using AV, LV and DE approaches.

		Robust			Gain-scheduled		
		AV	LV	DE	AV	LV	DE
Case 1	V	55	55	65	103	103	137
	L	176	264	88	396	2310	297
Case 2	V	45	45	55	69	69	103
	L	44	66	44	66	231	99

with the augmentation of g. Concerning the computational effort, Table 2 shows that, to obtain results close to the ones provided by the AV modeling, the LV approach employs a greater number of LMI rows (L), and both require the same number of scalar decision variables (V). On the other hand, considering the decision variables with the same degrees ($g = 1$), the proposed technique is more efficient than AV and LV when using a solver that penalizes more the number of LMI rows and provides less conservative results (Fig. 5) than both methods at the cost of a slightly larger number of scalar variables.

For Case 2, the $\mathcal{H}_\infty$ guaranteed cost is calculated as a function of ω using Theorem 2.1 considering $\lambda = 1$ and fixing parameter $\rho = 0.3$. Figs. 5c and 5d show the values of γ computed for the synthesis of, respectively, robust ($g_F = 0$) and gain-scheduled ($g_F = g$) filters with Theorem 2.1 employing Lyapunov matrices and extra-decision variables of degree g versus the frequency $\omega \in [0, 2\pi]$. As in Case 1, the performance indexes associated with LV and AV approaches are not noticeably improved with the augmentation of g. On the other hand, in the gain-scheduled case, the increase of g provides an improvement in the three tested techniques. Furthermore, the slower is the time-varying parameter ($\omega \approx 0$ or 2π) better LV approach performs, while for fast variations ($\omega \approx \pi$) LV tends to provide the same results obtained by AV modeling. Nevertheless, in general, the DE approach presents the best results in terms of $\mathcal{H}_\infty$ guaranteed costs. It is important emphasize that, even employing a greater computational effort, in the robust case the LV approach obtains results similar to the ones provided by the AV modeling (see Table 2). Also observe that the proposed technique is capable to provide less conservative results using less LMI rows than LV approach.

2.5 $\mathcal{H}_\infty$ state-feedback design

Consider the discrete-time LPV system $\mathcal{G}$, described by the following state-space difference equations:

$$\mathcal{G} = \begin{cases} x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k) + E(\theta(k))w(k), \\ z(k) = C_z(\theta(k))x(k) + D_z(\theta(k))u(k) + E_z(\theta(k))w(k), \end{cases} \tag{2.24}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$, $w(k) \in \mathbb{R}^{n_w}$, $z(k) \in \mathbb{R}^{n_z}$, respectively denote the state, control input, noise input, and controlled output vectors. The state-space matrices depend on a vector of time-varying parameters $\theta(k) = [\theta_1(k), \ldots, \theta_N(k)]'$ that belongs to a compact convex set (as the unit simplex or the hypercube) for all $k \geq 0$. The aim is to design a state-feedback control law $u(k) = K(\theta(k))x(k)$ such that closed-loop system

$$\mathcal{G}_{cl} = \begin{cases} x(k+1) = A_{cl}(\theta(k))x(k) + B_{cl}(\theta(k))w(k), \\ z(k) = C_{cl}(\theta(k))x(k) + D_{cl}(\theta(k))w(k) \end{cases} \tag{2.25}$$

with

$$\begin{aligned} A_{cl}(\theta(k)) &= A(\theta(k)) + B(\theta(k))K(\theta(k)), & B_{cl}(\theta(k)) &= E(\theta(k)), \\ C_{cl}(\theta(k)) &= C_z(\theta(k)) + D_z(\theta(k))K(\theta(k)), & D_{cl}(\theta(k)) &= E_z(\theta(k)), \end{aligned}$$

is asymptotically stable and the $\mathcal{H}_\infty$ performance from the disturbance input $w(k)$ to the controlled output $z(k)$ is minimized. The state-feedback controller $K(\theta(k))$ can be synthesized by solving a set of parameter-dependent LMI conditions, as the ones presented in Theorem 2.2.

Theorem 2.2. *If there exist a symmetric positive definite matrix $P(\theta(k))$, matrices $G(\theta(k))$ and $Y(\theta(k))$ of compatible dimensions that satisfy the parameter-dependent LMI condition*

$$\begin{bmatrix} P(\theta(k)) - G(\theta(k)) - G(\theta(k))' & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star \\ A(\theta(k))G(\theta(k)) + B(\theta(k))Y(\theta(k)) & E(\theta(k)) & -P(\theta(k+1)) & \star \\ C_z(\theta(k))G(\theta(k)) + D_z(\theta(k))Y(\theta(k)) & E_z(\theta(k)) & 0 & -I \end{bmatrix} < 0 \tag{2.26}$$

for all $(\theta(k), \theta(k+1))$ lying in the uncertain domain, then $K(\theta(k)) = Y(\theta(k))G(\theta(k))^{-1}$ is a gain-scheduled state-feedback controller that stabilizes system (2.24) and γ is an $\mathcal{H}_\infty$ guaranteed cost for the closed-loop system (2.25).

Proof. Similar to the proof of Theorem 5 from De Caigny *et al.* (2010). □

2.5.1 Numerical results

The purpose of this section is to illustrate how the new time-varying parameter modeling affects the problem of $\mathcal{H}_\infty$ state feedback control. Consider system (2.24) with system matrices given by

$$A(\theta(k)) = \begin{bmatrix} 0 & 1 \\ 0.5\theta_1(k) & \rho\theta_2(k) \end{bmatrix}, \quad B(\theta(k)) = \begin{bmatrix} 0 & (1-\rho\theta_2(k)) \end{bmatrix}',$$

$$D_z(\theta(k)) = 0.5 + 2\theta_2(k), \quad C_z = \begin{bmatrix} 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0.5 \end{bmatrix}', \quad E_z = 0.1. \tag{2.27}$$

where $0 \leq \theta_1(k) \leq 1$ and $-1 \leq \theta_2(k) \leq 1$ are the time-varying parameters and $\rho \geq 0$. Recall that the polytopic approach only assures robust closed-loop stability when all the points inside the uncertain domain are individually stable. Knowing that, if $\rho > 1$, matrix $B(\theta(k))$ in (2.27) contains the open-loop ($B(\theta(k)) = [0\ \ 0]'$) as a particular case and matrix $A(\theta(k))$ in (2.27) contains unstable eigenvalues for fixed values of $\theta(k)$, the proposed modeling arises as an alternative to obtain feasible solutions since it is not possible to design a stabilizing controller for system (2.24) by using conventional approaches.

Following the same lines of Section 2.4.1, to explore the design of both gain-scheduled and robust $\mathcal{H}_\infty$ state-feedback controllers, two cases of time-varying parameters are investigated

- **C**ase 1: $\theta_1(k)$ is considered the *triangle wave* ($tw(k)$) with $N = 6$ depicted in Fig. 6, while $\theta_2(k) = \cos(\omega k)$;
- **C**ase 2: $\theta_1(k) = 1$ and $\theta_2(k) = \cos(\omega k)$.

The Lyapunov matrix $P(\theta(k))$ in condition (2.26) is described in terms of the polynomial structure (2.16) with degree $g \geq 1$. A robust controller is obtained using degree zero ($g_\kappa = 0$) and gain-scheduled controllers are obtained with $g_\kappa = g$ in variables $G(\theta(k))$ and $Y(\theta(k))$, following the structure given in (2.16).

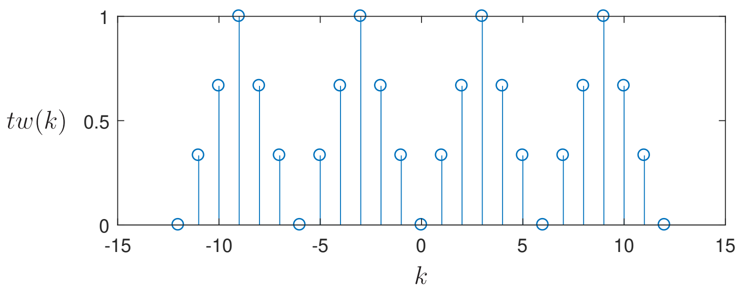


Figure 6 – Discrete-time triangle wave $tw(k)$ of period $N = 6$.

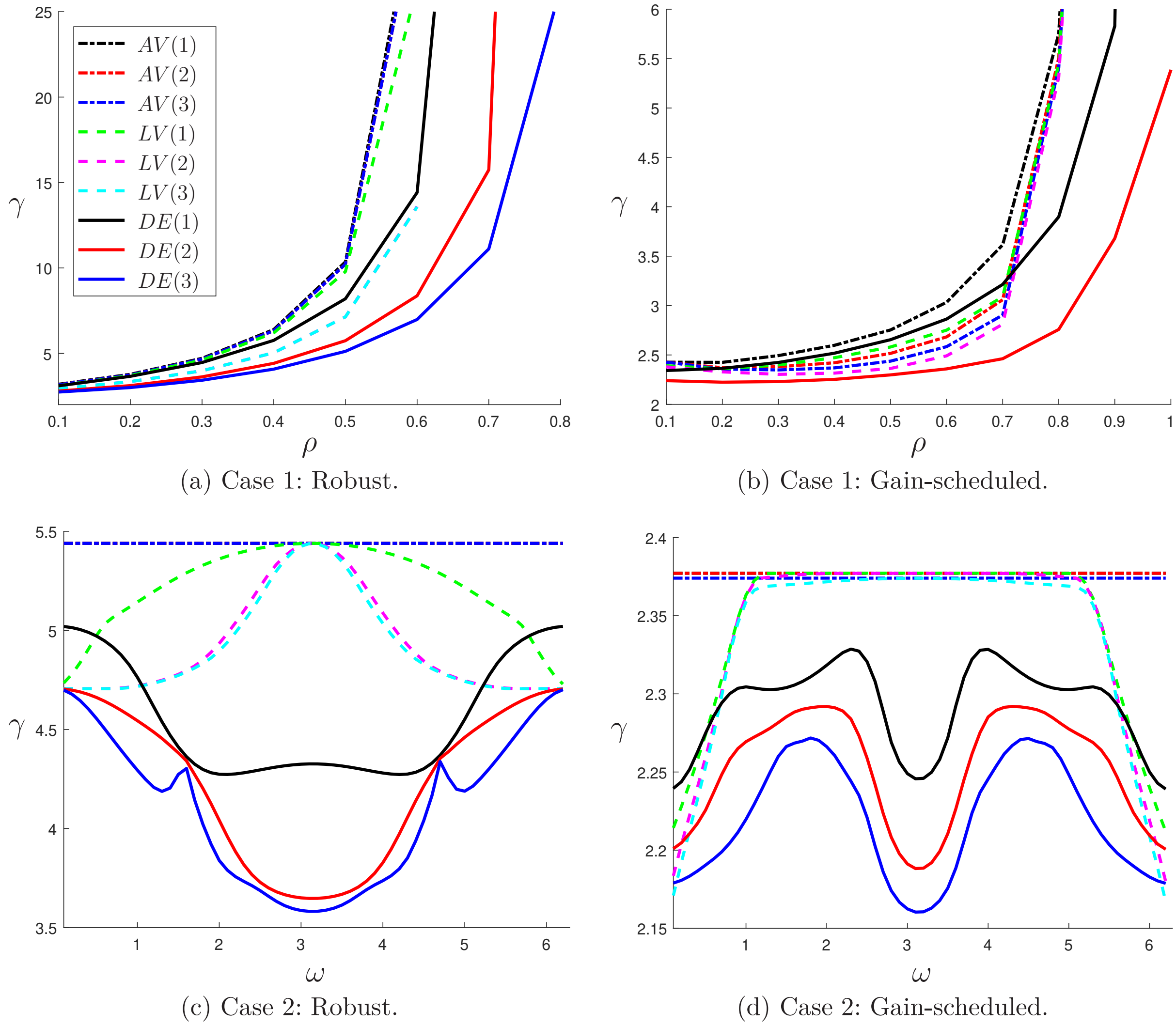
The triangle wave can be represented by

$$tw(k) = \frac{1}{2} - \frac{4}{9}\cos(\frac{\pi k}{3}) - \frac{1}{18}\cos(\pi k)$$

and, employing the proposed DE modeling, one possible solution is

$$tw(k) = \frac{1}{2} - \frac{4}{9}\phi_1(k) - \frac{1}{18}\phi_3$$

with $\phi_1(k) = \cos(\frac{\pi k}{3})$, $\phi_2(k) = \phi_1(k+1)$, $\phi_3(k) = \cos(\pi k)$, and $\phi_4(k) = \phi_3(k+1)$ following the same structure of (2.17). For Case 1 the $\mathcal{H}_\infty$ guaranteed cost is calculated by Theorem 2.2 as a function of ρ and fixing $\omega = 1$. The DE approach is compared with



(a) Case 1: Robust.
(b) Case 1: Gain-scheduled.
(c) Case 2: Robust.
(d) Case 2: Gain-scheduled.

Figure 7 – $\mathcal{H}_\infty$ guaranteed costs (γ) computed with Theorem 2.2 for the design of (a) Robust and (b) Gain-scheduled controllers for system (2.27) Case 1 ($\theta_1(k) = tw(k)$ and $\theta_2(k) = \cos(k)$) versus $\rho > 0$; and for the design of (c) Robust and (d) Gain-scheduled controllers for system (2.27) Case 2 ($\theta_1(k) = 1$ and $\theta_2(k) = \cos(\omega k)$) versus $\omega \in [0, 2\pi]$ using AV (dash-dotted lines), LV (dashed lines), and DE (solid lines) approaches with degree g (between parentheses).

AV and LV methods applied to both time-varying parameters ($\theta_1(k), \theta_2(k)$). Figs. 7a and 7b show the $\mathcal{H}_\infty$ guaranteed costs (γ) computed for the synthesis of, respectively, robust ($g_\kappa = 0$) and gain-scheduled ($g_\kappa = g$) controllers with Theorem 2.2 versus parameter $\rho > 0$. Note that, among the three tested techniques, DE clearly provides the lowest guaranteed costs (γ) that can be further improved with the increase of g. It is important to emphasize that DE approach also allows to find feasible solutions for a wider range of ρ, meaning that it can synthesize robust or gain-scheduled state-feedback controllers when the other methods fail.

For Case 2, the $\mathcal{H}_\infty$ guaranteed costs are calculated as a function of ω using

Table 3 – Computational cost in terms of the number of the scalar variables (V) and LMI rows (L) for the synthesis of robust ($g_\kappa = 0, g = 1$) and gain-scheduled ($g_\kappa = g = 1$) state-feedback controllers for system (2.27) considering Cases 1 and 2 and using AV, LV and DE approaches.

		Robust			Gain-scheduled		
		AV	LV	DE	AV	LV	DE
Case 1	V	16	16	28	28	28	64
	L	96	216	384	216	2646	4374
Case 2	V	13	13	16	19	19	28
	L	24	36	24	36	126	54

Theorem 2.2 for a fixed parameter $\rho = 0.5$. Figs. 5c and 5d show the values of γ computed for the synthesis of, respectively, robust ($g_\kappa = 0$) and gain-scheduled ($g_F = g$) gains with Theorem 2.2 employing Lyapunov matrices of degree g versus the frequency $\omega \in [0, 2\pi]$. Note that, for slow variations ($\omega \approx 0$ or 2π) the LV approach presents good performance levels, while for fast variations ($\omega \approx \pi$) LV tends to provide the same results obtained by AV modeling, as expected. Nevertheless, in general, the DE approach presents the best results in terms of $\mathcal{H}_\infty$ guaranteed costs. Observe that, both DE and LV approaches are benefited by the increase of degree g, with clearer evidence for LV in the case of robust gains.

Concerning the computational effort, Table 3 shows that, to obtain less conservative results than the ones provided by the AV modeling, the LV approach employs a greater number of LMI rows (L), both techniques use the same number of scalar decision variables (V), and DE requires more variables and LMI rows to provide the best results (with less LMI rows than LV in Case 2).

2.6 Application in networked control systems

Consider a continuous-time plant represented by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.28}$$

that must be digitally controlled through a network. Sampling with a period $T(k) \geq 0$, it is possible to obtain a discrete-time model (ÅSTRÖM; WITTENMARK, 1984), for $k \geq 0 \in \mathbb{N}$, $x(0) = 0$, and $u(v) = 0$, $v \in \{-T(0), 0\}$

$$\begin{aligned} x(k+1) &= A_d(T(k))x(k) + B_d(T(k))u(k) + E_d w(k), \\ y(k) &= C_d(T(k))x(k) + D_d(T(k))u(k) + F_d w(k). \end{aligned} \tag{2.29}$$

where $w(k) \in \mathbb{R}^{n_w}$ belongs to $\ell_2[0, \infty)$ and it is used to model noises in the process. Assuming that matrices E_d and F_d are known in discrete-time domain, matrices $A_d(T(k))$,

$B_d(T(k))$, C_d, and D_d can be computed by

$$A_d(T(k)) = \exp\left(AT(k)\right), \quad C_d = C, \quad D_d = D, B_d(T(k)) = \int_0^{T(k)} \exp\left(As\right) dsB, \quad (2.30)$$

where A, B, C, and D are the matrices of system (2.28).

As discussed in Montestruque e Antsaklis (2004), Velasco *et al.* (2005), and Borges *et al.* (2010b), the values of the sampling period may change during operation by different reasons, for example, dynamic bandwidth allocation and scheduling decisions. By considering the sampling period as a time-varying parameter, it is possible to reduce the flow of information by unit of time between sensor and actuator. In a hypothetical particular NCS scenario, consider that, due to network congestion, buffer overflow, etc., to avoid a network collapse and, consequently, open loop control and, at the same time, aiming to minimize the load (packet flow per unit of time), the initial sampling frequency starts from a maximum value (i.e., sampling period minimum) and, as the network corrects its problems (reconfiguring itself to reduce the load), the sampling frequency gradually converges to a minimum value (i.e., the time between samples is maximum). In this context, the sampling rate is time-varying and can be described by the following function

$$T(k) = T_{end} - (T_{end} - T_{ini})\sigma^k, \quad 0 < \sigma < 1, \quad k \in \mathbb{N}^+ \quad (2.31)$$

where T_{ini} and T_{end} represent, respectively, the initial and final sampling times. The main difficulty in computing the matrices in (2.29) is to solve the exponential terms of (2.30), which need to be computed for all $T(k)$ satisfying (2.31). An alternative that has already been investigated in Borges *et al.* (2010b) is to use the Cayley-Hamilton Theorem (ANTSAKLIS; MICHEL, 2006), implying that $A_d(T(k))$ and $B_d(T(k))$ can be rewritten as

$$A_d(T(k)) = \exp(AT(k)) = \sum_{i=0}^{n-1} \rho_i(k) A^i, \quad (2.32)$$

$$B_d(T(k)) = \int_0^{T(k)} \exp(As) dsB = \int_0^{T(k)} \sum_{i=0}^{n-1} \rho_i(s) A^i ds\ B, \quad (2.33)$$

$$= \sum_{i=0}^{n-1} \left(\int_0^{T(k)} \rho_i(s) ds \right) A^i B \quad (2.34)$$

$$= \sum_{i=0}^{n-1} \eta_i(k) A^i B \quad (2.35)$$

where $\eta_i(k) = \int_0^{T(k)} \rho_i(s) ds$. The coefficients $\rho_i(k)$ and $\eta_i(k)$ can be determined by solving a set of linear equations defined in terms of the eigenvalues λ_j, $j = 1, \ldots, n$, of matrix A. For instance, for a matrix with n distinct and non-null eigenvalues, one has

$$\exp(\lambda_j T(k)) = \sum_{i=0}^{n-1} \rho_i(k) \lambda_j^i, \quad j = 1, \ldots, n,$$

and

$$\int_0^{T(k)} \exp(\lambda_j s) ds = \frac{1}{\lambda_j}\left(\exp(\lambda_j T(k)) - 1\right) = \sum_{i=0}^{n-1} \eta_i(k)\lambda_j^i, \quad j = 1, \ldots, n,$$

Then, following the lines of Borges *et al.* (2010b), the discrete-time matrices are given by

$$A_d(T(k)) = \sum_{i=0}^{n-1} \rho_i(k) A^i = \sum_{i=1}^{n} \theta_i(k)\Omega_i, \quad B_d(T(k)) = \sum_{i=0}^{n-1} \eta_i(k) A^i B = \Psi_0 + \sum_{i=1}^{n} \theta_i(k)\Psi_i \tag{2.36}$$

where coefficients $\underline{\theta}_i \leq \theta_i(k) \leq \overline{\theta}_i$ represent the normal modes $(\exp(\lambda_i T(k)))$ of A with known bounds given by $\underline{\theta}_i = \min\{\exp(\lambda T(k))\}$ and $\overline{\theta}_i = \max\{\exp(\lambda T(k))\}$. Additionally, matrices $\Omega_i \in \mathbb{R}^{n\times n}$ and $\Psi_i \in \mathbb{R}^{n\times p}$ are determined collecting the terms of equation (2.36).

The main limitation of the method proposed in Borges *et al.* (2010b) is that when using a distinct parameter to represent each exponential, that is, $\theta_j(k) = \exp(\lambda_j T(k))$, one loses the intrinsic relationship between parameters $\theta_i(k)$ and $\theta_j(k)$ for $\lambda_i \neq \lambda_j$. This relation is due to their common dependence on parameter $T(k)$. As consequence, neglecting this common dependence certainly produces an uncertain domain greater than the necessary one (overbound), increasing the conservativeness of the modeling.

To take into account the common relation of the parameters with the sampling rate, it is necessary to write an expression for the discrete-time matrices such that the dependence on $T(k)$ or k is explicit. One strategy is to rewrite the normal modes of the system in terms of a Taylor series, that is,

$$\exp(\lambda_i T(k)) = \sum_{j=0}^{\infty} \frac{(\lambda_i T(k))^j}{j!} \tag{2.37}$$

$$= 1 + \lambda_i T(k) + \frac{\lambda_i^2}{2!} T(k)^2 + \ldots + \frac{\lambda_i^\ell}{\ell!} T(k)^\ell + \delta_i^\ell(k) \tag{2.38}$$

$$= \sum_{j=0}^{\ell} \frac{\lambda_i^j}{j!} T(k)^j + \delta_i^\ell(k) \tag{2.39}$$

where $\delta_i^\ell(k)$ represents the bounds of the Taylor series expansion residue of degree ℓ of the normal mode $\exp(\lambda_i T(k))$, such that it can be computed by

$$\delta_i^\ell(k) = \exp(\lambda_i T(k)) - \sum_{j=0}^{\ell} \frac{\lambda_i^j T(k)^j}{j!}. \tag{2.40}$$

Observe that, if the maximum and minimum bounds of the sampling rate are known $(T(k) \in [T_{\min}, T_{\max}])$, the bounds of $\delta_i^\ell(k)$ can also be exactly computed, such that $\delta_i^\ell(k) \in [\delta_{i_{\min}}^\ell, \delta_{i_{\max}}^\ell]$. In this sense, knowing that $a_{ij} = \lambda_i^j/j!$, the matrices of the discrete-time system can be rewritten in terms of parameters $\varphi(k) = T(k)$ and $\delta_i^\ell(k)$ as

$$\begin{aligned} A_d(\varphi(k)) &= \sum_{i=1}^{n} \Omega_i \left(\sum_{j=0}^{\ell} \left(a_{ij}\varphi(k)^j \right) + \delta_i^\ell(k) \right), \\ B_d(\varphi(k)) &= \Psi_0 + \sum_{i=1}^{n} \Psi_i \left(\sum_{j=0}^{\ell} \left(a_{ij}\varphi(k)^j \right) + \delta_i^\ell(k) \right). \end{aligned} \tag{2.41}$$

It may seem counterproductive to rewrite (2.36) as (2.41) because the number of time-varying parameters augments from n ($\theta_i(k)$, $i = 1, \dots, n$) to $n+1$ ($\delta_i^\ell(k)$, $i = 1, \dots, n$ and $\varphi(k) = T(k)$). However, a first alternative, more conservative but less computational expensive is to use a single time-varying parameter $\bar{\delta}^\ell(k)$ to represent the Taylor series expansion residues ($\delta_i^\ell(k)$, $i = 1, \dots, n$,) such that

$$\bar{\delta}^\ell(k) \in [\min_i\{\delta^\ell_{i_{\min}}(k)\},\ \max_i\{\delta^\ell_{i_{\max}}(k)\}]', \tag{2.42}$$

reducing the number of time-varying parameters from $n+1$ in (2.41) to two, regardless the order of the system. Furthermore, observe that $\delta_i^\ell(k) \to 0$ with the increase of the degree ℓ of Taylor series expansion and, for degrees ℓ large enough, the expansion residue can be neglected (see Fig. 8). In this case, a second alternative for ℓ sufficiently large, is to consider a discrete-time representation with a single time-varying parameter ($\varphi(k) = T(k)$) given by

$$A_d(\varphi(k)) = \sum_{i=1}^{n} \Omega_i \sum_{j=0}^{\ell} a_{ij}\varphi(k)^j, \qquad B_d(\varphi(k)) \;= \Psi_0 + \sum_{i=1}^{n} \Psi_i \sum_{j=0}^{\ell} a_{ij}\varphi(k)^j. \tag{2.43}$$

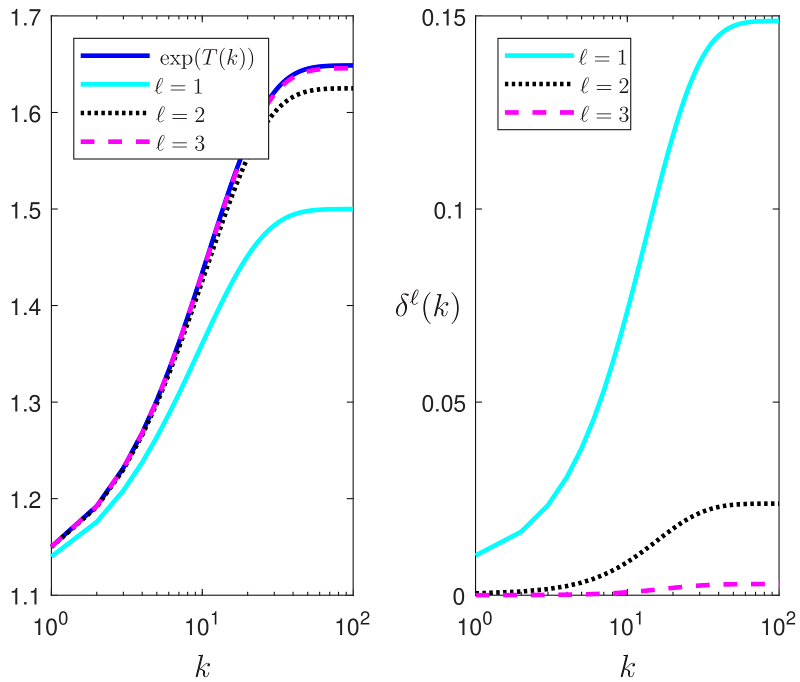


Figure 8 – Assuming $\lambda = 1$ and a time-varying sampling time described by (2.31) with $T_{ini} = 0.1$, $T_{end} = 0.5$ and $\sigma = 0.9$: (a) Comparison of $\exp(\lambda T(k))$ with the Taylor series expansion of degree ℓ; (b) residue ($\delta^\ell(k)$) of the Taylor series expansion of degree ℓ.

Additionally, following the methodology proposed in this chapter, if the function that governs the time variation of the parameter (for instance, $T(k)$ given in (2.31)) is known and can be obtained as solution of a difference equation (in this case as (2.8)), this information can be used to obtain a less conservative modeling in terms of the time instants k. Rewriting (2.39) with $T(k)$ given by (2.31), one obtains

$$\begin{aligned}\exp(\lambda_i T(k)) &= \sum_{j=0}^{\ell} \frac{\lambda_i^j}{j!}\left(T_{end} - (T_{end} - T_{ini}\sigma^k)\right)^j + \delta_i^\ell(k)\\ &= \left(1 + \lambda_i T_{end} + \frac{\lambda_i^2}{2}T_{end}^2 + \cdots + \frac{\lambda_i^\ell}{\ell!}T_{end}^\ell\right) + \cdots + (-1)^j \frac{(T_{end} - T_{ini})^j}{j!}\sum_{m=j}^{\ell}\lambda_i^m \frac{T_{end}^{(m-j)}}{(m-j)!}\sigma^{jk}\\ &\quad + \cdots + (-1)^\ell \frac{(T_{end} - T_{ini})^\ell}{\ell!}\lambda_i^\ell \sigma^{\ell k} + \delta_i^\ell(k)\\ &= \sum_{j=0}^{\ell} a_{ij}\varphi(k)^j + \delta_i^\ell(k) \end{aligned} \tag{2.44}$$

with $\delta_i^\ell(k)$ given by (2.40), $\varphi(k) = \sigma^k$ such that $0 \leq \varphi(k) \leq 1$, $\varphi(k+1) = \sigma\varphi(k)$ and

$$a_{ij} = (-1)^j \frac{(T_{end} - T_{ini})^j}{j!}\sum_{m=j}^{\ell}\lambda_i^m \frac{T_{end}^{(m-j)}}{(m-j)!}. \tag{2.45}$$

Note that with this new modeling, the discrete-time matrices can also be written as in (2.41) (considering the expansion residues δ_i^ℓ associated with each eigenvalue or the maximum expansion residue $\bar{\delta}_i^\ell$) or (2.43) (neglecting the expansion residue).

2.6.1 Application example

Consider a simplified model of an armature voltage-controlled DC servo motor consisting of a stationary field and a rotating armature and load as described in Borges *et al.* (2010b) and illustrated in Fig. 9.

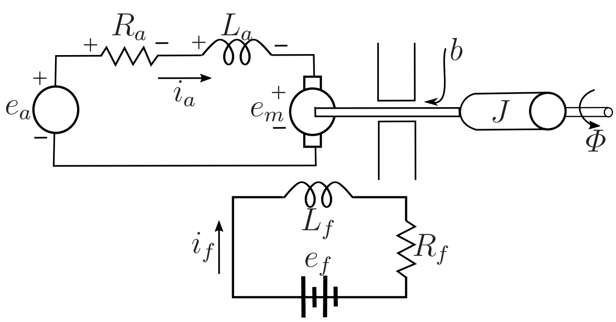


Figure 9 – DC servo motor presented in Borges *et al.* (2010b).

The aim of this example is to design an $\mathcal{H}_\infty$ state-feedback controller (robust or scheduled) for the speed of the shaft assuming that all information is sent through a communication network. The dynamics of the DC servo motor can be described by the following continuous-time state-space representation

$$\begin{bmatrix}\ddot{\Phi}\\ \ddot{\varsigma}_a\end{bmatrix} = \begin{bmatrix}-\frac{b}{J} & \frac{K_T}{J}\\ -\frac{K_\Phi}{L_a} & -\frac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}\dot{\Phi}\\ \dot{\varsigma}_a\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L_a}\end{bmatrix} e_a(t) \tag{2.46}$$

Table 4 – Definition of the parameters of the DC servo motor.

Φ	shaft position	
ς_a	armature current	
e_a	externally applied armature voltage	
R_a	resistance of the armature winding	$1\ \Omega$
L_a	armature winding inductance	$0.5\ H$
b	viscous damping due to bearing friction	$0.1\ Nms$
J	moment of inertia of the armature and load	$0.01 kgm^2/s^2$
e_m	voltage induced by the rotating armature winding	$K_\Phi \dot{\Phi}$
T	motor torque	$K_T i_a$

where $K_\Phi = K_T = 0.01 Nm/Amp$ and the other parameters are given in the Table 4.

Replacing the values of the parameters one has

$$A = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Although system (2.46) is already stable (eigenvalues $\lambda_1 = -9.9975$ and $\lambda_2 = -2.0025$), Theorem 2.2 is applied aiming to provide a state-feedback controller that guarantees robustness against not modeled $l_2[0,1)$ perturbations by minimizing the $\mathcal{H}_\infty$ performance of the closed-loop system with a time-varying sampling rate $T(k)$.

The first step is to determine the matrix coefficients using the Cayley-Hamilton theorem to obtain a discrete-time representation of the system. To compute Ω_i and Ψ_i of (2.36), first it is necessary to calculate $\rho_0(k)$, $\rho_1(k)$ by solving the linear system of equations (all the numbers were truncated with 4 decimal digits)

$$\begin{bmatrix} \lambda_1^0 & \lambda_1^1 \\ \lambda_2^0 & \lambda_2^1 \end{bmatrix} \begin{bmatrix} \rho_0(k) \\ \rho_1(k) \end{bmatrix} = \begin{bmatrix} \exp(\lambda_1 T(k)) \\ \exp(\lambda_2 T(k)) \end{bmatrix} \tag{2.47}$$

$$\begin{bmatrix} \rho_0(k) \\ \rho_1(k) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} \exp(\lambda_1 T(k)) \\ \exp(\lambda_2 T(k)) \end{bmatrix} = \begin{bmatrix} -0.2505 \exp(\lambda_1 T(k)) + 1.2505 \exp(\lambda_2 T(k)) \\ -0.1251 \exp(\lambda_1 T(k)) + 0.1251 \exp(\lambda_2 T(k)) \end{bmatrix} \tag{2.48}$$

then

$$\exp(AT(k)) = \rho_0(k) A^0 + \rho_1(k) A^1 = \rho_0(k) I + \rho_1(k) A = \Omega_1 \exp(\lambda_1 T(k)) + \Omega_2 \exp(\lambda_2 T(k))$$

with

$$\Omega_1 = \begin{bmatrix} 1.0003 & -0.1251 \\ 0.0025 & -0.0003 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -0.0003 & 0.1251 \\ -0.0025 & 1.0003 \end{bmatrix}.$$

On the other hand, $\eta_0(k)$, $\eta_1(k)$ can be obtained by solving the linear system of equations

$$\begin{bmatrix} \lambda_1^0 & \lambda_1^1 \\ \lambda_2^0 & \lambda_2^1 \end{bmatrix} \begin{bmatrix} \eta_0(k) \\ \eta_1(k) \end{bmatrix} = \begin{bmatrix} \int_0^{T(k)} \exp(\lambda_1 T(k)) ds \\ \int_0^{T(k)} \exp(\lambda_2 T(k)) ds \end{bmatrix}$$

$$\begin{bmatrix} \eta_0(k) \\ \eta_1(k) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\lambda_1} (\exp(\lambda_1 T(k)) - 1) \\ \frac{1}{\lambda_2} (\exp(\lambda_2 T(k)) - 1) \end{bmatrix} = \begin{bmatrix} 0.0251 \exp(\lambda_1 T(k)) - 0.6245 \exp(\lambda_2 T(k)) + 0.5994 \\ 0.0125 \exp(\lambda_1 T(k)) - 0.0625 \exp(\lambda_2 T(k)) + 0.0500 \end{bmatrix}$$

then

$$\int_0^{T(k)} \exp(As)dsB = \eta_0(k)A^0B + \eta_1(k)AB = \Psi_0 + \Psi_1 \exp(\lambda_1 T(k)) + \Psi_2 \exp(\lambda_2 T(k))$$

with

$$\Psi_0 = \begin{bmatrix} 0.0999 \\ 0.9990 \end{bmatrix}, \ \Psi_1 = \begin{bmatrix} 0.0250 \\ 6 \times 10^{-6} \end{bmatrix}, \ \Psi_2 = \begin{bmatrix} -0.1249 \\ -0.9991 \end{bmatrix}.$$

Assuming that the time-varying sampling rate is described by (2.31) with $T_{ini} = 0.001\ s$ and $T_{end} = 0.099\ s$ one has that, by the modeling (2.36) with $\theta_i = \exp(\lambda_i T(k))$, the time-varying coefficients belong to the following intervals

$$\theta_1(k) \in [0.3717,\ 0.9901], \quad \theta_2(k) \in [0.8202,\ 0.9980], \tag{2.49}$$

yielding a polytopic representation of four vertices by combining the extremes of $\theta_1(k)$ and $\theta_2(k)$.

In order to use the modeling (2.41), it is also necessary to calculate the bounds for parameters $\delta_i(k)$, as done in Table 5. Observe that starting from $\ell = 4$, the bounds can be neglected, that is, model (2.43) can be used instead of (2.41) without harming the controller design.

Table 5 – Maximum and minimum bounds (respectively, $\delta_{i_{\max}}$ and $\delta_{i_{\min}}$) for parameters $\delta_i(k)$ that represent the residue of the Taylor series expansion degree ℓ.

ℓ		1	2	3	4
$\delta_{i_{\max}}$	1	0.3614	-1.66×10^{-7}	0.0332	-8.31×10^{-13}
	2	0.0184	-1.34×10^{-9}	6.19×10^{-5}	-2.22×10^{-16}
$\delta_{i_{\min}}$	1	4.98×10^{-5}	-0.1284	4.15×10^{-10}	-0.0068
	2	2.00×10^{-6}	-0.0012	6.70×10^{-13}	-2.47×10^{-6}

Taking into account all the previous assumptions, the purpose is to design robust $\mathcal{H}_\infty$ state-feedback controllers considering equation (2.29) with

$$E_d = \begin{bmatrix} 0.1 & 0 \end{bmatrix}', \ C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D_d = 0, \ F_d = 1, \tag{2.50}$$

and admitting the following distinct modeling approaches:

- **C**ase 1: as suggested in Borges *et al.* (2010b), parameter $\theta_i(k) = \exp(\lambda_i T(k))$, $i = 1, 2$, is considered to vary arbitrarily fast between the interval defined in (2.49), A_d and B_d given by (2.36);
- **C**ase 2: parameter $\varphi(k) = T(k)$ is considered to vary arbitrarily fast between 0.01 s and 0.099 s and matrices A_d and B_d given by
  - (a) model (2.41) and $\delta_i^\ell(k)$ replaced by $\bar{\delta}^\ell(k)$ in (2.42);
  - (b) model (2.43) (neglecting the expansion residue);

- **C**ase 3: it is assumed that $T(k)$ is described by (2.31), the time-varying parameter $\varphi(k) = \sigma^k$ is a solution of a difference equation, and matrices A_d and B_d are given by

  (a) model (2.41) with $\delta_i^\ell(k)$ given in (2.40);

  (b) model (2.41) with $\bar{\delta}^\ell(k)$ given in (2.42);

  (c) model (2.43).

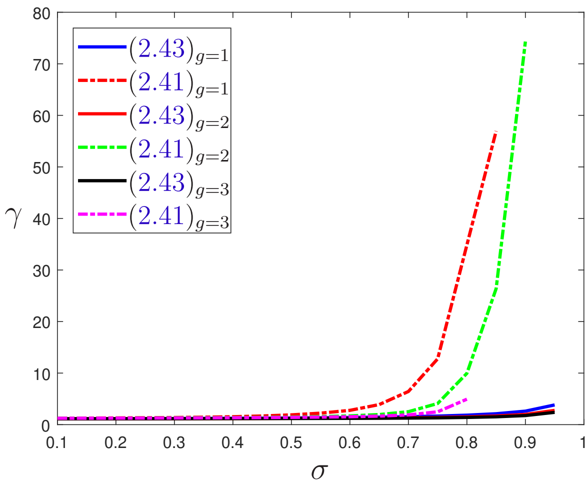
Concerning the design of a robust state-feedback controller for **C**ase 1, the guaranteed cost is independent from the degree of Taylor series expansion and from the function that governs the evolution of the sampling time, since $T(k) \in [0.001,\ 0.099]$s. Therefore, considering Theorem 2.2 with a Lyapunov matrix with $g = 1$, one obtains an $\mathcal{H}_\infty$ guaranteed cost of $\gamma = 10.9571$ (optimization problem involving 19 scalar variables and 104 LMI rows).

By using the approach of **C**ase 2 with discrete-time matrices A_d and B_d given by model (2.41) with $\bar{\delta}^\ell(k)$ or (2.43), it is expected that the $\mathcal{H}_\infty$ guaranteed cost (γ) be lower than the one obtained for **C**ase 1 (because solutions are being sought in a reduced parametric space) and that γ is improved with the increase of the Taylor series expansion degrees (because, as shown in Table 5 and Figure 8, the expansion is more accurate with higher degrees). Furthermore, since model (2.43) neglects the expansion residue, it is expected that the $\mathcal{H}_\infty$ guaranteed costs for *Case* 2(b) be lower and less computationally complex than *Case* 2(a). However, although *Case* 2(b) is not sufficient to assure the closed-loop stability, it must converge to the same result obtained by *Case* 2(a) as the residue decreases. Table 6 summarizes the results obtained by Theorem 2.2 with Lyapunov matrices of degree one for approaches of **C**ase 2.

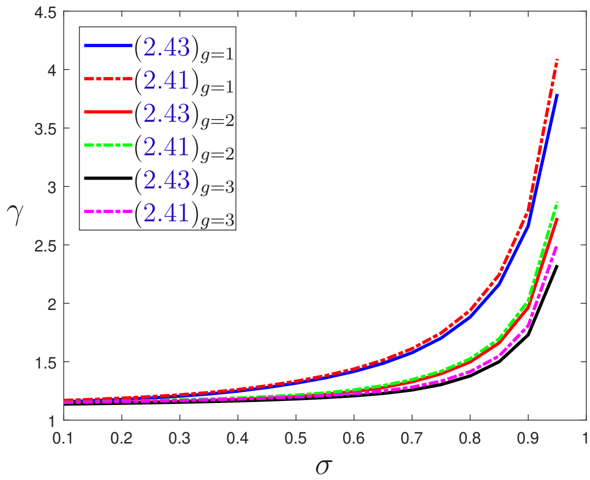
Table 6 – $\mathcal{H}_\infty$ guaranteed costs (γ), number of scalar variables (V) and LMI rows (L) computed by Theorem 2.2 with $g = 1$, $g_K = 0$ for **C**ase 2 (a) and (b) with Taylor series expansion of degree ℓ.

ℓ		1	2	3	4	5
(a)	γ	–	8.3658	6.9433	7.0843	6.7617
	V	19	19	19	19	19
	L	104	152	200	248	296
(b)	γ	7.9979	7.1873	6.7997	6.7755	6.7614
	V	13	13	13	13	13
	L	28	40	52	64	76

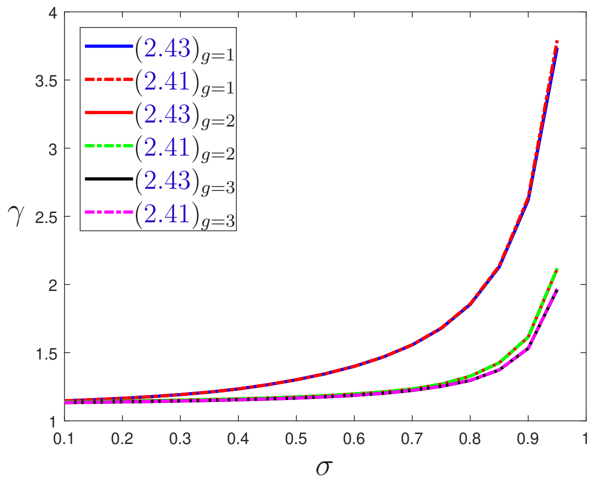
Finally, using the approach of **C**ase 3, both the degree ℓ of Taylor series expansion and the function that governs the variation of the sampling period between its minimum and maximum values are relevant, therefore the $\mathcal{H}_\infty$ guaranteed costs are presented in Fig. 10 in terms of the value of $\sigma \in [0, 1]$ in (2.31). Observe that, although



(a) $\ell = 1$.



(b) $\ell = 2$.



(c) $\ell = 5$.

Figure 10 – $\mathcal{H}_\infty$ guaranteed costs computed by Theorem 2.2 with Lyapunov matrix of degree g for approach of **C**ase 3 with Taylor series expansion of degree ℓ and matrices A_d and B_d given by (2.41) and (2.43).

for degrees ℓ sufficiently large, the uncertainties $\delta_i(k)$ can be neglected, only when they are considered it is possible to provide a theoretical certificate of stability. The graphics depicted in Figure 10 show that the difference between models from **C**ase 3 (a) and **C**ase 3 (c) reduces with the increase of ℓ. Besides, it is possible to note that, the greater the degree g of Lyapunov matrix, the less relevant is the expansion error and the better is the result (in terms of γ).

Regarding the computational cost presented in Table 7, by using model (2.43) (neglecting the Taylor series expansion residue, **C**ase 3 (c)), less scalar variables and LMI rows are required when compared with any other tested method. Nevertheless, if the model (2.41) is employed (**C**ase 3 (a)), at first, the number of time-varying parameters is the highest among all the proposed approaches, implying a large computational cost. To provide stability assurance but reducing the computational cost, an intermediate model

Table 7 – Number of scalar variables (V) and LMI rows (L) to compute the $\mathcal{H}_\infty$ state-feedback controllers using Theorem 2.2 with degree g for the Lyapunov matrix for **C**ase 3 (a), (b) and (c) with Taylor series expansion of degree ℓ.

ℓ		1			2			3			4			5		
g		1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
(a)	V	25	34	43	25	34	43	25	34	43	25	34	43	25	34	43
	L	224	1566	6400	320	1566	6400	416	2052	6400	512	2538	7936	608	3024	9472
(b)	V	19	25	31	19	25	31	19	25	31	19	25	31	19	25	31
	L	64	198	448	88	198	448	112	252	448	136	306	544	160	360	640
(c)	V	13	16	19	13	16	19	13	16	19	13	16	19	13	16	19
	L	20	30	40	26	30	40	32	36	40	38	42	46	44	48	52

(**C**ase 3 (b)) is also presented in Table 7. Note that the computational effort is similar to **C**ase 2 (a) (see Tables 6 and 7) but the $\mathcal{H}_\infty$ guaranteed costs are much lower (Fig. 10). Fig. 10 also shows that, with the increase of ℓ or g, the $\mathcal{H}_\infty$ guaranteed costs are reduced (at the price of increasing the number of variables and LMI rows, as indicated by Table 7).

To conclude, some numerical comparisons were made and it was observed that the best result of **C**ase 2 (a) (for $\ell = 5$) is 38.29% better than the result obtained using **C**ase 1. Additionally, using the same degree for the Lyapunov matrix ($g = 1$) and for the Taylor series expansion ($\ell = 5$), independently from the time-varying parameter modeling used (A_d and B_d given by (2.43) or (2.41)), even the worst result obtained for **C**ase 3 ($\sigma = 0.95$) is about 43.92% better than the guaranteed cost obtained for **C**ase 2. Additionally, in practice, to assure the stability of the closed-loop networked control system for a sampling period up to 0.099 s instead of fixed in 0.001 s allows to save network resources since it requires only 1% of the communication channel in terms of amount of information packets transmitted by unit of time. Finally, considering the case of $T(k)$ arbitrary, where the DE modeling cannot be applied, note that the proposed discretized model (2.41) can be seen as a more flexible and accurate alternative when compared to other methods based on the computation of matrix exponential (BORGES *et al.*, 2010b; BRAGA *et al.*, 2014; GEROMEL; SOUZA, 2015).

2.7 Partial Conclusion

This chapter presented a new method to model time-varying parameters that can provide less conservative stability analysis and synthesis conditions based on LMIs, allowing to provide stability certificates, stabilizing controllers or filters for LPV systems when the polytopic approaches based on LMIs (considering arbitrary or bounded rate of variation) fail. The reason for this fact is that the dynamic matrix (or the augmented dynamic matrix in filtering or closed-loop dynamic matrix in control) does not need to be Schur for all time instants. The numerical examples also show that, in control and filter design, the $\mathcal{H}_\infty$ guaranteed costs obtained by the proposed technique are much better than the ones provided by the traditional approaches, and the computational cost is commonly similar to the methods based on arbitrary variation (which present the lowest numerical complexity and higher conservatism) and much lower than the procedures based on limited rate of variation.

Finally, it was investigated a practical example based on NCS, where some phenomena common to communication networks (such as, failures, congestion and buffer overflow) were modeled with time-varying sampling rate according to an exponential function. Based on the proposed modeling, and using Cayley-Hamilton theorem and Taylor series expansion, it was possible to design a state-feedback controller associated with lower guaranteed costs and less variables and LMI rows than a specialized method from the literature. The proposed approach can be further extended in future works to handle, for instance, time-varying probabilities in non-homogeneous Markov jump linear systems.

3 $\mathcal{H}_2$ control and filtering of discrete-time LPV systems exploring statistical information of the time-varying parameters

3.1 Introduction

This chapter introduces a new strategy to improve performance in gain-scheduled control and filtering for LPV systems exploiting statistical information about the time-varying parameters whenever available. The motivation of the techniques proposed in this chapter comes precisely from the area known as *gain-scheduled subject to inexact parameters* (DAAFOUZ *et al.*, 2008), where the noise affecting the scheduling parameters is usually included in the LPV model as an additive uncertainty that varies arbitrarily fast in time. This modeling can be adopted, for instance, when the scheduling parameters are obtained by an identification procedure and the estimation error is considered as a random time-varying parameter with a known PDF. To take into account the PDF information about the additive noise (or any other type of time-varying parameter) in order to improve the synthesis results, the technique proposed in this chapter, named sub-domain optimization heuristic (SDOH), presents as main novelty the design of controllers or filters treating robust stability independently of performance. The performance (measured in terms of the $\mathcal{H}_2$ norm) is, therefore, optimized only in a sub-domain of the time-varying parameters, where a higher frequency of occurrence is expected, while the robust stability is certificated for the whole domain.

As additional contribution of this chapter in the context of inexact measurements, a more complete modeling for the additive uncertainty is given, generalizing previous results from the literature for two types of uncertainties: polytopic and affine. It is also important to mention that a new design condition for $\mathcal{H}_2$ full-order LPV filtering is proposed. Although the condition is only sufficient, it can be shown that it includes a known condition from the literature as particular case. All the synthesis conditions (handling $\mathcal{H}_2$ state-feedback control or $\mathcal{H}_2$ full-order filtering) presented in this chapter are given in terms of parameter-dependent LMIs, avoiding the introduction of trick notations to deal with polynomial inequalities. Alternatively, Chapter 3 also discusses in details how to obtain the programmable set of LMIs, basically fixing the optimization variables as polynomials of arbitrary degree and applying polynomial relaxations using a software. Several numerical examples are provided to illustrate the advantages of the proposed methodology when compared with previous techniques.

This chapter is organized as follows: Section 3.2 presents the necessary notations and definitions; Section 3.3 introduces the LPV model and the concept of sub-domain, a key ingredient for the main achievements of the chapter; In Section 3.4 a new condition to compute a bound to the $\mathcal{H}_2$ norm using the concept of sub-domain is presented; Section 3.5 presents a generic procedure to model the additive uncertainty. Section 3.6 investigates the problem of $\mathcal{H}_2$ state-feedback control, presenting two numerical examples. The proposed heuristic is extended to deal with $\mathcal{H}_2$ full-order filtering in Section 3.7, with two numerical examples. Finally, some conclusions are drawn in Section 3.8.

3.2 Notations and Definitions for SDOH

Let $\mathsf{B} \in \mathbb{R}^{m\times 2}$ be a matrix with the following property: $\mathsf{B}_{i1} < \mathsf{B}_{i2}$, $i = 1,\dots,m$. Consider the following definition for a hyperrectangle

$$\Delta_m(\mathsf{B}) \triangleq \Big\{(\xi_1,\dots,\xi_m) \in \mathbb{R}^m : \mathsf{B}_{i1} \le \xi_i \le \mathsf{B}_{i2},\ i = 1,\dots,m\Big\}. \tag{3.1}$$

Note that the elements of B define the bounds for the intervals of ξ_i, $i = 1,\dots,m$. The unit simplex of dimension N is given by

$$\Lambda_N \triangleq \Big\{\xi \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \ge 0,\ i = 1,\dots,N\Big\}. \tag{3.2}$$

3.3 System description

Consider the following LPV discrete-time system

$$\mathcal{G} = \begin{cases} x(k+1) &= A(\theta(k))x(k) + B(\theta(k))u(k) + E(\theta(k))w(k) \\ z(k) &= C_z(\theta(k))x(k) + D_z(\theta(k))u(k) + E_z(\theta(k))w(k) \\ y(k) &= C_y(\theta(k))x(k) + E_y(\theta(k))w(k),\ k \ge 0 \end{cases} \tag{3.3}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$, $w(k) \in \mathbb{R}^{n_w}$, $z(k) \in \mathbb{R}^{n_z}$ and $y(k) \in \mathbb{R}^{n_y}$, respectively denote the state, control input, external disturbance, controlled or estimated output and the measured output vectors. In this chapter, two different structures for the state-space matrices are considered:

$$X(\theta(k)) = X_0 + \sum_{i=1}^{m} \theta_i(k) X_i,\ \theta(k) \in \Delta_m(\mathsf{B}) \tag{3.4}$$

known as *affine* uncertainty ($\mathsf{B} \in \mathbb{R}^{m\times 2}$ is given) and

$$X(\theta(k)) = \sum_{i=1}^{N} \theta_i(k) X_i,\ \theta(k) \in \Lambda_N \tag{3.5}$$

known as *polytopic* uncertainty, where $X(\theta(k))$ represents any matrix of system (3.3). These two types of uncertainties are the most used in the LPV literature dealing with the same problems investigated in this chapter. $\theta(k)$ is a vector of bounded time-varying parameters lying in a hyperrectangle ($\Delta_m(\mathsf{B})$) or in the unit simplex (Λ_N) for all $k \geq 0$. Moreover, to increase the generality of the approach, the rate of variation of parameters $\theta_i(k)$, whenever available, is modeled in the form

$$\mathsf{B}_{i1} - \mathsf{B}_{i12} \leq \underline{b}_i \leq \theta_i(k+1) - \theta_i(k) \leq \overline{b}_i \leq \mathsf{B}_{i2} - \mathsf{B}_{i1}, \quad \text{affine uncertainty} \quad (3.6)$$

$$0 \leq \underline{b}_i \leq \theta_i(k+1) - \theta_i(k) \leq \overline{b}_i \leq 1, \quad \text{polytopic uncertainty} \quad (3.7)$$

If $\underline{b}_i$ and $\overline{b}_i$ assume their extreme values, then the parameter is considered to vary *arbitrarily fast*.

Next section presents an important concept for the purposes of the main achievements of this chapter.

3.3.1 Sub-domain

Considering that additional information about the behavior of $\theta(k)$ is available, for instance, a PDF, it is important to define a systematic procedure to create a sub-domain inside the original domain (the hyperrectangle or the unit simplex). For this purpose, consider that the sub-domain is defined in terms of a bounded polyhedron (polytope)

$$\mathcal{S}_r(\mathsf{H}) \triangleq \left\{ \zeta \in \mathbb{R}^q : \zeta = \sum_{i=1}^{r} \xi_i h^i, \ \xi \in \Lambda_r \right\}, \ \mathsf{H} = \left[h^1 \cdots h^r \right] \in \mathbb{R}^{q \times r}$$

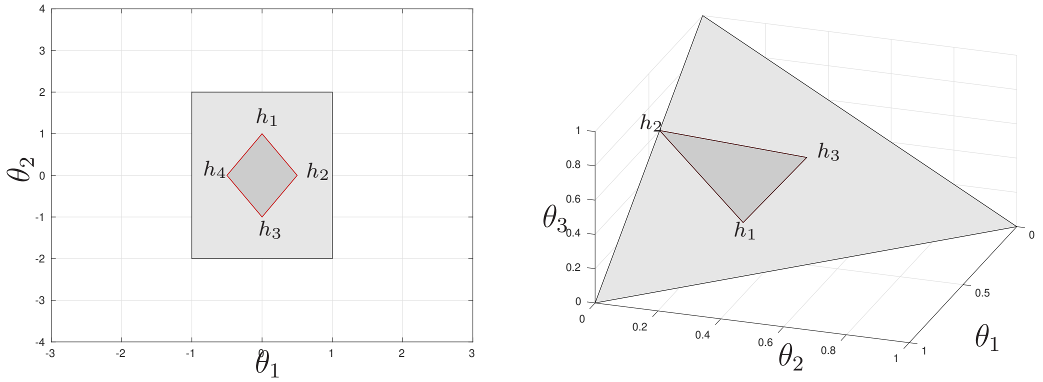
where h^i are the vertices of the polytope. As illustrative examples, consider the following domains $\Delta_2([1\ 2]' \otimes [-1\ 1])$ and Λ_3, their associated sub-domains

$$\mathcal{S}_4(\mathsf{H}_a), \mathsf{H}_a = \begin{bmatrix} 0 & 0.5 & 0 & -0.5 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathcal{S}_3(\mathsf{H}_b), \mathsf{H}_b = \frac{1}{10} \begin{bmatrix} 5 & 4 & 2 \\ 3 & 0 & 4 \\ 2 & 6 & 4 \end{bmatrix}$$

respectively, whose geometric representations are depicted in Fig. 11. Since both the domains ($\Delta_m(\mathsf{B})$ and Λ_N) and sub-domains are convex, the relation $\mathcal{S}_r(\mathsf{H}) \subset \Delta_m(\mathsf{B})$ holds if and only if $h^i \in \Delta_m(\mathsf{B})$, $i = 1, \ldots, r$. Similarly $\mathcal{S}_r(\mathsf{H}) \subset \Lambda_N \Leftrightarrow h^i \in \Lambda_N$, $i = 1, \ldots, r$.

An important feature of the definition of the sub-domain $\mathcal{S}_r(\mathsf{H})$ is an implicit linear transformation that can be used to consider the original matrices only in points lying inside the sub-domain. For instance, consider (the dependence on k is omitted) $A(\theta) = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3$, $\theta \in \Lambda_3$ and the sub-domain $\mathcal{S}_3(\mathsf{H}_b)$ previously defined. In this case, any point $(\theta_1, \theta_2, \theta_3)$ lying in the sub-domain $\mathcal{S}_3(\mathsf{H}_b)$, can be written as

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \xi_1 \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} + \xi_2 \begin{bmatrix} 0.4 \\ 0 \\ 0.6 \end{bmatrix} + \xi_3 \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.3 & 0 & 0.4 \\ 0.2 & 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$



(a) Hyperrectangle (b) Simplex

Figure 11 – Convex sub-domains inside a hyperrectangle of dimension two and a unit simplex of dimension three.

Thus, it is immediate to obtain a representation of $A(\theta)$ with $\theta \in \mathcal{S}_3(\mathsf{H}_b)$, that is,

$$\begin{aligned} A(\theta) = A(\xi) &= (0.5\xi_1 + 0.4\xi_2 + 0.2\xi_3)A_1 + (0.3\xi_1 + 0\xi_2 + 0.4\xi_3)A_2 + (0.2\xi_1 + 0.6\xi_2 + 0.4\xi_3)A_3 \\ &= (0.5A_1 + 0.3A_2 + 0.2A_3)\xi_1 + (0.4A_1 + 0.6A_3)\xi_2 + (0.2A_1 + 0.6A_2 + 0.4A_3)\xi_3 \\ &= \hat{A}_1\xi_1 + \hat{A}_2\xi_2 + \hat{A}_3\xi_3, \quad (\xi_1, \xi_2, \xi_3) \in \Lambda_3 \end{aligned}$$

3.4 Robust Performance in Sub-domains

In this section a new strategy to access robust performance of the LPV system (3.3) exploiting the concept of sub-domain is proposed. Next theorem presents a set of parameter-dependent LMI conditions to assure robust stability in the whole domain and to compute an $\mathcal{H}_2$ guaranteed cost only inside the union of a set of pre-specified sub-domains.

Theorem 3.1. *Consider system (3.3) with $\theta(k) \in \Delta_m(\mathsf{B})$ and a set of sub-domains $\mathcal{S}_{r_i}(\mathsf{H}_i)$, $i = 1, \ldots, s$ where B and H_i are given matrices. For a given positive scalar ρ, if there exist parameter-dependent symmetric positive definite matrices $P(\theta(k))$, $P_{s_i}(\theta(k))$ and $W_i(\theta(k))$, and matrices $G(\theta(k))$ and $G_{s_i}(\theta(k))$, $i = 1, \ldots, s$, such that*

$$\begin{bmatrix} P(\theta(k+1)) & \star \\ A(\theta(k))G(\theta(k)) & P(\theta(k)) - G(\theta(k)) - G(\theta(k))' \end{bmatrix} > 0 \tag{3.8}$$

holds for all $(\theta(k), \theta(k+1)) \in \Delta_m(\mathsf{B}) \times \Delta_m(\mathsf{B})$, and for $i = 1, \ldots, s$

$$\begin{bmatrix} P_s(\xi(k+1)) & \star & \star \\ A(\xi(k))G_s(\xi(k)) & P_s(\xi(k)) - G_s(\xi(k)) - G_s(\xi(k))' & \star \\ B(\xi(k)) & 0 & I \end{bmatrix} > 0, \tag{3.9}$$

$$\begin{bmatrix} W(\xi(k)) - E_z(\xi(k))E_z(\xi(k))' & \star \\ C_z(\xi(k))G_s(\xi(k)) & G_s(\xi(k)) + G_s(\xi(k))' - P_s(\xi(k)) \end{bmatrix} > 0, \tag{3.10}$$

$$Tr(W(\xi(k))) < \rho^2 \tag{3.11}$$

hold for all $(\xi(k), \xi(k+1)) \in \Lambda_{r_i} \times \Lambda_{r_i}$, where $\theta(k) = \mathsf{H}_i\xi(k)$, then the system is robustly stable for all $\theta(k) \in \Delta_m(\mathsf{B})$ and ρ is an $\mathcal{H}_2$ guaranteed cost valid only inside the union of sub-domains $\mathcal{S}_{r_i}$, $i = 1, \ldots, s$.

Proof. Similar to the proof of Theorem 6 from De Caigny *et al.* (2010), with adaptations to cope with sub-domains in inequalities (3.9)-(3.11), responsible for the $\mathcal{H}_2$ performance □

The conditions of Theorem 3.1 are the extension of Theorem 6 from De Caigny *et al.* (2010) to deal with $\mathcal{H}_2$ performance only in a prespecified region of the original domain. All matrices depending on $\xi(k)$ or $\xi(k+1)$ in Theorem 3.1 are readily obtained from the corresponding matrices depending on $\theta(k)$ or $\theta(k+1)$ through the linear transformation $\theta(k) = \mathsf{H}_i\xi(k)$, as explained in the previous section. Note that inequality (3.8) guarantees the robust stability for the whole domain, and the other inequalities assure the guaranteed cost only in the considered sub-domains. The extension of Theorem 3.1 to deal with the unit simplex as main domain is immediate and is not presented (just replace $\Delta_m(\mathsf{B})$ by Λ_N). Regarding the numerical implementation, note that all inequalities in Theorem 3.1 are parameter-dependent LMI conditions, which are infinite dimensional problems. The procedure to obtain a programmable set of LMIs is explained in Section 3.6.1.

The purpose of Theorem 3.1 is to treat time-varying parameters for which not only the bounds are known (possibly the rates of variation), but also some statistical information in terms of a PDF. As a consequence, it is possible to exploit the fact that the parameters lie in a sub-domain (or multiples sub-domains) most of the time. As a practical example, consider the problem of gain-scheduled control or filtering, where the designer has access to statistical information assuring that in most of the time the estimation or measurement corresponds to the actual value of the scheduling parameter, indicating that the probability distribution of the error is concentrated around zero. This problem is formally introduced in next section.

3.5 LPV filtering and control with contaminated scheduling parameters

In the context of designing gain-scheduled filters and controllers subject to inexact measurements or inaccurate estimates, the scheduling parameters are assumed to be contaminated by additive and/or multiplicative noises (DAAFOUZ *et al.*, 2008; SATO, 2010; SATO, 2015a; SATO, 2015b; SATO; PEAUCELLE, 2013; AGULHARI *et al.*, 2013; LACERDA *et al.*, 2016; SADEGHZADEH, 2018; PALMA *et al.*, 2018c). As a matter of fact, if the effect of the noise is not taken into account during the design phase, the performance associated to the controllers and filters scheduled in terms of those inaccurate parameters can lead the closed-loop system (or augmented system in filtering) out of its operating range, implying loss of performance or, in the worst case scenario, instability.

The design procedures for gain-scheduled filters and controllers investigated in this chapter assume that the time-varying parameter $\theta(k)$ is known by means of real-time measurements or estimations obtained through an identification technique. It is important to emphasize that both strategies are associated to inherent measurement or identification errors, which are not taken into account by the classical gain-scheduled design methods for LPV systems. The measured or estimated value of the time-varying parameter, denoted by $\tilde{\theta}(k)$, is modelled in the form

$$\tilde{\theta}_i(k) = \theta_i(k) + \delta_i(k), \tag{3.12}$$

where

$$\underline{\delta}_i \leq \delta_i(k) \leq \overline{\delta}_i,\ \ \overline{\delta}_i > \underline{\delta}_i,\ \ 0 \in [\underline{\delta}_i, \overline{\delta}_i] \tag{3.13}$$

is the additive uncertainty representing the error arising in the identification, instrumentation faults or other issues. In order to avoid introducing extra conservativeness in the analysis, the physical bounds of $\theta(k)$ must be respected, i.e.,

$$\mathsf{B}_{i1} \leq \theta_i(k) + \delta_i(k) \leq \mathsf{B}_{i2}, \qquad \text{affine uncertainty} \tag{3.14}$$

$$0 \leq \theta_i(k) + \delta_i(k) \leq 1, \quad \text{polytopic uncertainty} \tag{3.15}$$

As a consequence, the feasible region of pairs $(\theta_i(k), \delta_i(k))$ is not necessarily $[\mathsf{B}_{i1}, \mathsf{B}_{i2}] \times [\underline{\delta}_i, \overline{\delta}_i]$ since some points inside this region (rectangle) do not fulfill (3.14) (the same applies to the polytopic uncertainty). In other words, the saturation effect produced by $\delta_i(k)$ over $\theta_i(k)$ is not taken into account. Some previous techniques from the literature provided the shape of the true feasible region in some particular cases (SATO, 2011; LACERDA *et al.*, 2013; LACERDA *et al.*, 2016). One of the contributions of this chapter is to provide a systematic procedure to deal with the more general case for both affine and polytopic uncertainties and also including the bounded rates of variation given in (3.6) and (3.7). For instance, consider affine uncertainty and the constraints $\mathsf{B}_{i1} \leq \theta_i(k) \leq \mathsf{B}_{i2}$, (3.13) and (3.14). Since all constraints are linear, it is possible to rewrite them in the form $\mathbf{A}x \leq \mathbf{b}$ where $x = \begin{bmatrix}\theta_1(k) & \cdots & \theta_m(k) & \delta_1(k) & \cdots & \delta_m(k)\end{bmatrix}'$ and $\mathbf{A}$ and $\mathbf{b}$ are known matrix and vector. The set $\{x : \mathbf{A}x \leq \mathbf{b}\}$ defines a bounded polyhedron, whose vertices can be systematically obtained by means of a vertex enumeration algorithm (AVIS; FUKUDA, 1992), which is an efficient technique (polynomial complexity) based on linear programming methods. The definitions of $\mathbf{A}$ and $\mathbf{b}$ for affine and polytopic uncertainties, considering arbitrarily fast or bounded rates of variation are given in the Section 3.9. This is an important contribution of the chapter, mainly for software development. Since it is not useful to consider bounded rates of variation for the additive uncertainty in the problems investigated in this chapter (parameter with random behavior), only the arbitrarily fast case is considered for $\delta(k)$. However, the approach could handle the bounded case immediately, merely including additional linear constraints in the problem. Regarding the multiplicative uncertainty considered in some previous papers, note that the vertex enumeration algorithm, unfortunately, cannot be applied, since nonlinear constraints are needed to define the feasible region.

Although the proposed modeling for the additive uncertainty provides the exact region where the parameters $\theta(k)$ and $\delta(k)$ can assume values, the number of vertices of the feasible region grows very rapidly with the increase of N or m (more accentuated in the polytopic case). One possibility to alleviate this issue is, for instance, to disregard the constraints (3.14) or (3.15), that is, to consider that the pairs $(\theta_i(k), \delta_i(k))$ lie in the rectangle (overbounded region). On the other hand, conservativeness is introduced in the analysis, as discussed in details in the paper Palma *et al.* (2018c) and in the second numerical example of the next section. As a result, the user must trade-off the need for accuracy and the computational complexity. Next section presents state-feedback synthesis conditions exploiting the concept of sub-domain and Section 3.6.2 presents a numerical example exploiting the proposed strategy to deal with additive uncertainty.

3.6 $\mathcal{H}_2$ State-Feedback Design

Motivated by the problem discussed in the previous section, the problem of $\mathcal{H}_2$ gain-scheduled state-feedback control subject to inexact measurements is investigated. Consider system (3.3) with $C_y(\theta(k)) = I$, $E_y(\theta(k)) = 0$, and a gain-scheduled control law $u(k) = K(\tilde{\theta}(k))x(k)$ where $\tilde{\theta}(k)$ represents time-varying parameters $\theta(k)$ contaminated by additive noise parameters $\delta(k)$ as in (3.12). The aim is to asymptotically stabilize the system for the whole domain of $(\theta(k), \delta(k))$ and to minimize an $\mathcal{H}_2$ guaranteed cost from the disturbance input $w(k)$ to the controlled output $z(k)$ only in a sub-domain (or set of sub-domains) of $(\theta(k), \delta(k))$, exploiting some PDF associated to $\delta(k)$.

Theorem 3.2. *Let the augmented vector $\Theta(k) = (\theta(k)', \delta(k)')'$ and the linear transformation $\tilde{\theta}(k) = ([1, 1] \otimes I_m)\Theta(k)$ be given. Consider system (3.3) with $\Theta(k) \in \Delta_{2m}(\mathsf{B})$ and a set of sub-domains $\mathcal{S}_{r_i}(\mathsf{H_i})$, $i = 1, \ldots, s$, where B and H_i are given matrices. For a given positive scalar ρ, if there exist symmetric positive definite parameter-dependent matrices $P(\Theta(k))$, $P_{s_i}(\Theta(k))$ and $W_i(\Theta(k))$, and matrices $G(\tilde{\theta}(k))$ and $Y(\tilde{\theta}(k))$, $i = 1, \ldots, s$, such* *that*

$$\begin{bmatrix} P(\Theta(k+1)) & \star \\ A(\theta(k))G(\Theta(k)) + B(\theta(k))Y(\tilde{\theta}(k)) & P(\Theta(k)) - G(\tilde{\theta}(k)) - G(\tilde{\theta}(k))' \end{bmatrix} > 0 \quad (3.16)$$

holds for all $(\Theta(k), \Theta(k+1)) \in \Delta_{2m}(\mathsf{B}) \times \Delta_{2m}(\mathsf{B})$, *and for* $i = 1, \ldots, s$

$$\begin{bmatrix} P_{s_i}(\xi(k+1)) & \star & \star \\ A(\xi(k))G(\xi(k)) + B(\xi(k))Y(\xi(k)) & P_{s_i}(\xi(k)) - G(\xi(k)) - G(\xi(k))' & \star \\ E(\xi(k)) & 0 & I \end{bmatrix} > 0, \quad (3.17)$$

$$\begin{bmatrix} W_i(\xi(k)) - E_z(\xi(k))E_z(\xi(k))' & \star \\ C_z(\xi(k))G(\xi(k)) + D_u(\xi(k))Y(\xi(k)) & -P_{s_i}(\xi(k)) + G(\xi(k)) + G(\xi(k))' \end{bmatrix} > 0, \quad (3.18)$$

$$Tr(W_i(\xi(k))) < \rho^2 \quad (3.19)$$

hold for all $(\xi(k), \xi(k+1)) \in \Lambda_{r_i} \times \Lambda_{r_i}$, *where* $\Theta(k) = \mathsf{H}_i\xi(k)$, $i = 1, \ldots, s$, *then* $K(\tilde{\theta}(k)) = Y(\tilde{\theta}(k))G(\tilde{\theta}(k))^{-1}$ *is a gain-scheduled state-feedback gain that asymptotically stabilizes system* (3.3) *for all* $\Theta(k) \in \Delta_{2m}(\mathsf{B})$ *and* ρ *is an* $\mathcal{H}_2$ *guaranteed cost valid only inside the union of sub-domains* $\mathcal{S}_{r_i}(\mathsf{H_i})$, $i = 1, \ldots, s$.

Proof. Follows immediately from the conditions of Theorem 3.1 considering $A(\theta(k)) = A(\theta(k)) + B(\theta(k))K(\tilde{\theta}(k))$, $C_z(\theta(k)) = C_z(\theta(k)) + D_u(\theta(k))K(\tilde{\theta}(k))$ and the change of variable $Y(\tilde{\theta}(k)) = K(\tilde{\theta}(k))G(\tilde{\theta}(k))$. □

The conditions of Theorem 3.2 were specially produced to deal with the additive uncertainty. To handle other problems where a PDF is available for $\theta(k)$, just consider $\Theta(k) = \theta(k) = \tilde{\theta}(k)$. If the time-varying parameters $\theta(k)$ cannot be estimated or measured, a robust control gain is still viable, just fixing matrices $Y(\theta(k))$ and $G(\theta(k))$ as parameter-independent. Note that even in this case the heuristic based on sub-domains can be applied whenever a PDF is available. From this point, the proposed heuristic is called *Sub-Domain Optimization Heuristic* (SDOH).

3.6.1 Numerical implementation

All conditions proposed in this chapter have been given in terms of parameter-dependent LMIs to keep the notation as simple as possible, making a clean presentation. On the other hand, parameter-dependent LMIs cannot be numerically tested (infinite dimensional optimization problem). However, as already routine in the literature, relaxations based on a finite set of LMIs, also known as *LMI relaxations*, can be obtained by imposing polynomial structures (arbitrary degree) for the optimization variables (BLIMAN, 2004; BLIMAN *et al.*, 2006) and applying well established positivity tests for the resulting polynomial matrix inequalities (OLIVEIRA; PERES, 2007). Besides, such procedure is already performed by software, for instance, by the Robust LMI Parser (ROLMIP) (AGULHARI *et al.*, 2019), that works jointly with Yalmip (LÖFBERG, 2004).

Regarding the definition of variables depending on $\tilde{\theta}(k)$ in Theorem 3.2, note that, ultimately, these variables depends on $\theta(k)$ and $\delta(k)$ using the informed linear transformation. The polynomial degrees of the optimization variables associated to each $\theta_i(k)$ and $\delta_i(k)$ could be different, but in the numerical examples presented in this chapter, all degrees have been considered the same. Particularly with respect to variables $G(\tilde{\theta})$ and $Y(\tilde{\theta})$, the degrees are all zero when a robust gain (parameter-independent) is desired.

3.6.2 $\mathcal{H}_2$ State-Feedback Design Examples

In this section, two case studies are investigated for $\mathcal{H}_2$ state-feedback control design. The first one is concerned with the evaluation of the actual worst case $\mathcal{H}_2$ closed-loop norm (obtained via Monte Carlo (MC) simulation) associated with two distinct robust $\mathcal{H}_2$ state-feedback controllers: the classical one and the one synthesized using the SDOH technique, that is, optimizing the time-response using statistical information about the time-varying parameter. The second case study aims to evaluate the $\mathcal{H}_2$ guaranteed cost of the closed-loop system with a gain-scheduled controller subject to additive noise and to compare the methods of $\mathcal{H}_2$ control synthesis associated to two distinct parametric models: *i)* the saturation effect of $\delta(k)$ on $\theta(k)$ is disregarded (classical modeling); *ii)* the saturation effect is taken into account using the proposed modeling (by means of (3.14)).

3.6.2.1 Example of $\mathcal{H}_2$ state-feedback control using the SDOH

The purpose of this example is to illustrate the use of the SDOH methodology in robust (parameter-independent) control design when the PDF distribution of the time-varying parameter is known. For this purpose, consider system (3.3) with polytopic uncertainty where $C_y(\theta(k)) = I$, $E_y(\theta(k)) = 0$, and the other state-space matrices with vertices given by

$$\left[\begin{array}{c|c|c|c|c|c} A_1 & A_2 & B_1 & B_2 & E_1 & E_2 \\ \hline C_{z1} & C_{z2} & D_{z1} & D_{z2} & E_{z1} & E_{z2} \end{array}\right] = \left[\begin{array}{ccc|ccc|cc|cc|cc|cc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & -2 \\ -1.7 & 0.15 & 6.2 & 2.7 & -0.3 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 5 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0,3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{3.20}$$

An important observation is that the dynamic matrix $\theta_1(k)A_1 + \theta_2(k)A_2$ is not Schur stable for all $\theta(k) \in \Lambda_2$. Suppose that the time-varying parameters $\theta_1(k)$ and $\theta_2(k)$ are random variables with normal distribution with mean γ and standard deviation σ, denoted by $\mu(\gamma, \sigma)$ (discrete-time uncorrelated random variables). Two scenarios are evaluated. In the first one, $\theta_1(k) = \mu(0.5, \sigma) \in [0,\ 1]$. The histogram of $\theta_1(k)$ is presented in Fig. 13a for $\sigma = 0.15$. In the second scenario, parameter $\theta_1(k) = \{1 - \mu(1, \sigma)\}$ such that $\mu(1, \sigma) \in [0,\ 1]$

is an asymmetric random variable centered on $\theta_1(k) = 1$ and standard deviation given by σ and the histogram is presented in Fig. 13e for $\sigma = 0.25$. In both cases $\theta_2(k) = 1 - \theta_1(k)$, since the state-space matrices are polytopic, meaning that $\theta(k) = (\theta_1(k), \theta_2(k))$ belongs to the unit simplex Λ_2.

The classical robust controller is obtained using the conditions of Theorem 3.2 considering only one sub-domain, equal to the main domain, i.e., $\mathcal{S}_2 = \Lambda_2$ (in this case only (3.17), (3.18) and (3.19) are solved). With respect to the polynomial degrees of the decision variables, matrices $G(\cdot)$ and $Y(\cdot)$ are set to degree zero and the Lyapunov matrix $P_{s_1}(\Theta)$ and matrix $M_1(\Theta)$ are set to degree three. The synthesized gain is given by (truncated with 4 decimal digits)

$$K_r = \begin{bmatrix} -0.6520 & 0.8173 & -2.9060 \\ 1.6225 & -0.1693 & -2.2685 \end{bmatrix}$$

Observe that, in this situation, the synthesis procedure does not consider the information about the temporal behavior of the time-varying parameters (only their bounds). To obtain a robust controller with improved performance, the propose heuristic can be employed to explore the statistical information about the time-varying parameters that is available, basically optimizing the $\mathcal{H}_2$ cost in a sub-domain constructed according to the probability distribution of the parameters. In this example, the task of selecting the sub-domain systematically employs first and second order statistics (respectively, mean and standard deviation), such that the sub-domain $\mathcal{S}_r(\mathsf{H})$ always represents a region defined in terms of the mean and the standard deviation, as illustrated in Fig. 12 for generic values of γ and σ. Note that when dealing with two parameters lying in the unit simplex, only one pair (γ, σ) is enough to specify the sub-domain, since $\theta_2 = 1 - \theta_1$.

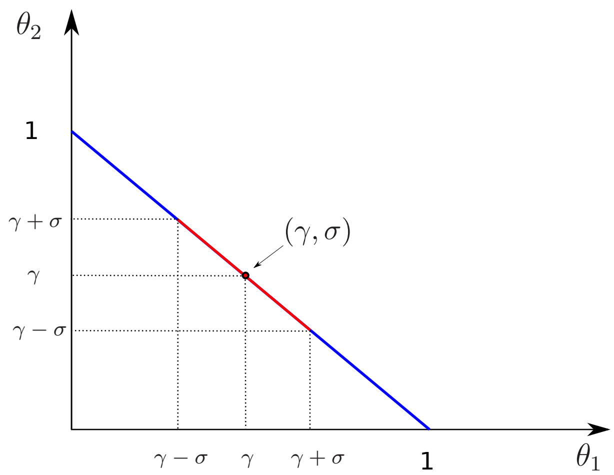


Figure 12 – Domain Λ_2 with sub-domain given by $\mathcal{S}(\mathsf{H})$ with $H = [h^1 \quad h^2]$, $h^1 = [\gamma - \sigma, \gamma + \sigma]'$ and $h^2 = [\gamma + \sigma, \gamma - \sigma]'$.

Also observe that in Case 1 ($\theta_1(k) = \mu(0.5, \sigma)$), independently from the value of the standard deviation σ, the mean of $\theta_1(k)$ is always the same (0.5) because $\theta(k)$ is a random variable with symmetric distribution. On the other hand, in Case 2 ($\theta_1(k) = \{1-\mu(1,\sigma)\}$), the variable $\theta_1(k)$ has asymmetric distribution, therefore, the mean depends on the standard deviation σ. Employing Theorem 3.2 with the same polynomial degrees for the decision variables and using appropriate sub-domains to handle Cases 1 and 2, the following robust gains are obtained respectively

$$K_s = \begin{bmatrix} -0.4730 & 0.7814 & -3.1657 \\ 1.4591 & -0.1500 & -2.0566 \end{bmatrix}, \quad K_a = \begin{bmatrix} -0.3652 & 0.7878 & -3.3314 \\ 1.5193 & -0.1428 & -2.1498 \end{bmatrix}$$

Although numerically different, the three gains are capable to stabilize the closed-loop system in the whole domain Λ_2. However, the most important feature to be analyzed is the close-loop performance provided by the controllers. In terms of guaranteed costs, it is expected that K_s and K_a yield lower values but, ultimately, the interest is the actual $\mathcal{H}_2$ norm of the closed-loop system. Knowing that a reasonably accurate estimate for this value can be obtained via MC simulation, consider the results of Fig. 13b, 13c, 13f and 13g, which present the mean and the associated standard deviation of[1] $z(k)'z(k)$, where $z(k)$ is the controlled output of the closed-loop system, for each choice of controller (K_r, K_s, or K_a) versus time (k). Those results where obtained using 10^5 MC realizations and a time-window of $k = 150$ samples.

Considering Case 1, where the time-varying parameter has symmetric distribution and standard deviation $\sigma = 0.15$ (Fig. 13a), the value of the actual $\mathcal{H}_2$ norm obtained through MC simulation of the closed-loop system using the classical robust controller K_r (Fig. 13b) is $\mathcal{H}_2$= 25.2901. On the other hand, using the gain K_s synthesized using the proposed heuristic method (Fig. 13c), one obtains $\mathcal{H}_2$= 21.6094, which implies in a reduction of 14.55% when compared with the classical design (K_r). Regarding Case 2, where the time-varying parameter has asymmetric distribution and standard deviation $\sigma = 0.25$ (Fig. 13e), the value of the actual worst case $\mathcal{H}_2$ norm computed using the classical robust controller K_r (Fig. 13f) is $\mathcal{H}_2$= 22.8117, while employing gain K_a designed using the proposed heuristic method (Fig. 13f), that takes into account the region of greater probability distribution of the parameter, the $\mathcal{H}_2$ norm is 17.6364, which represents an improvement of 22.68% when compared with the classical design. Furthermore, note that the standard deviation (dashed line in Fig. 13b, 13c, 13f, 13g), whose value is inversely proportional to the confidence interval of the project, is noticeably higher when using the classical gain (K_r) than with the employment of SDOH robust controllers (K_s or K_a), representing another advantage of the proposed approach.

For the purpose of exposing the general benefits of using the proposed technique, Fig. 13d and 13h show the percentage improvement of the $\mathcal{H}_2$ norm obtained by MC simulation as a function of $\sigma \in [0.05,\ 0.5]$:

$$\mathcal{H}_2(\sigma)\% = \frac{\mathcal{H}_2(K_r) - \mathcal{H}_2(K_\bullet)}{\mathcal{H}_2(K_r)} \times 100, \tag{3.21}$$

[1] The square of $\mathcal{H}_2$ norm is the area under the curve described by $z(k)'z(k)$.

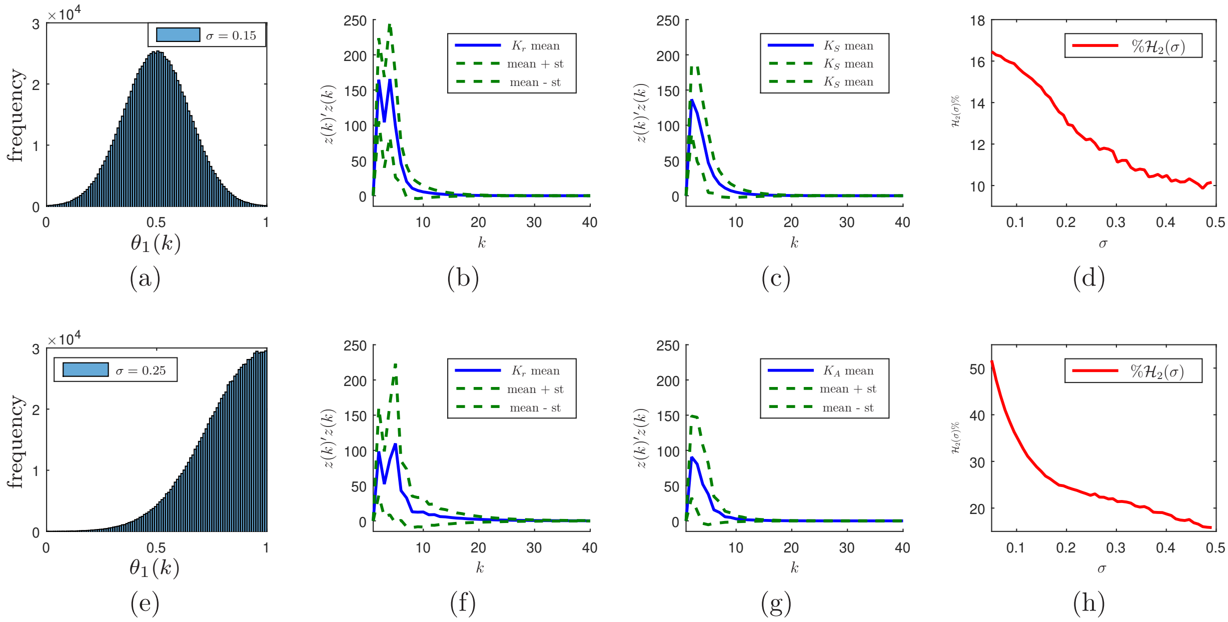


Figure 13 – Graphs of Example 3.6.2.1: Histogram of the random time-varying parameter (a) $\theta_1(k) = \mu(0.5, 0.15)$ and (e) $\theta_1(k) = 1 - \mu(1, 0.25)$; Mean and standard deviation of $z(k)'z(k) \times k$ obtained through MC simulation of the closed-loop system with (b) $\theta_1(k) = \mu(0.5, 0.15)$ and gain K_r, (c) $\theta_1(k) = \mu(0.5, 0.15)$ and gain K_s, (f) $\theta_1(k) = 1 - \mu(1, 0.25)$ and gain K_r, (g) $\theta_1(k) = 1 - \mu(1, 0.25)$ and gain K_a; Percentage improvement of the $\mathcal{H}_2$ norm ($\mathcal{H}_2(\sigma)\%$) of the closed-loop system with (d) K_s and (h) K_a when compared with K_r versus σ.

where $K_\bullet$ can be replaced by K_s or K_a. The range of the standard deviation σ of the distribution of $\theta_1(k)$ was chosen $[0.05, 0.5]$ because for values lower than 0.05 the behavior of the parameter is approximately time-invariant, and for values greater than 0.5 the distribution of $\theta_1(k)$ tends to a uniform distribution, that is, the frequency of occurrence of $\theta_1(k)$ is approximately the same for the whole domain, meaning that the heuristic method recovers the classical approach. In Case 1 (symmetric distribution) the $\mathcal{H}_2$ norm computed by MC simulation is between 9.86% to 16.46% lower using the heuristic than using the classical design method. The improvement is more pronounced in Case 2 (asymmetric distribution) where the $\mathcal{H}_2$ norm is about 15.82% to 51.64% better by using the heuristic technique.

3.6.2.2 Comparisons with other sub-domains for the heuristics

The aim of this example is to compare the performance of the proposed SDOH formulation with other design methods that, in principle, could also employ statistical information of the parameter in different ways. For this purpose, consider Case 2 of the previous example (Subsection 3.6.2.1), where the time-varying parameter ($\theta_1(k) = \{1 - \mu(1, \sigma)\}$) has an asymmetric distribution centered in $\gamma = 1$ with $\sigma = 0.3$. The value of the actual worst case $\mathcal{H}_2$ closed-loop norm associated with each controller is computed through MC simulation with 10^4 realizations of $k = 250$ samples and are depicted in Fig. 14. The following design methods are evaluated: (a) A robust SDOH controller considering the sub-domain defined in Fig. 12; (b) Classical robust controller; (c) Optimal controller considering (3.20) as a precisely known LTI system with a fixed operation point corresponding to the value of the parameter $\theta_1(k)$ of maximum frequency ($\theta_1(k) = 1$); (d) A robust controller designed considering only the sub-domain (solving only LMIs (3.17), (3.18) and (3.19) of Theorem 3.2), disregarding the robust stability requirement for the whole domain (as required by the proposed SDOH formulation); (e) A robust controller assuring stability for the entire domain but optimizing a bound to the $\mathcal{H}_2$ norm in a single point corresponding to the value of the parameter $\theta_1(k)$ of maximum frequency ($\theta_1(k) = 1$); (f) The previous case considering the mean value of the parameter, that is, $\theta_1(k) = 0.7611$.

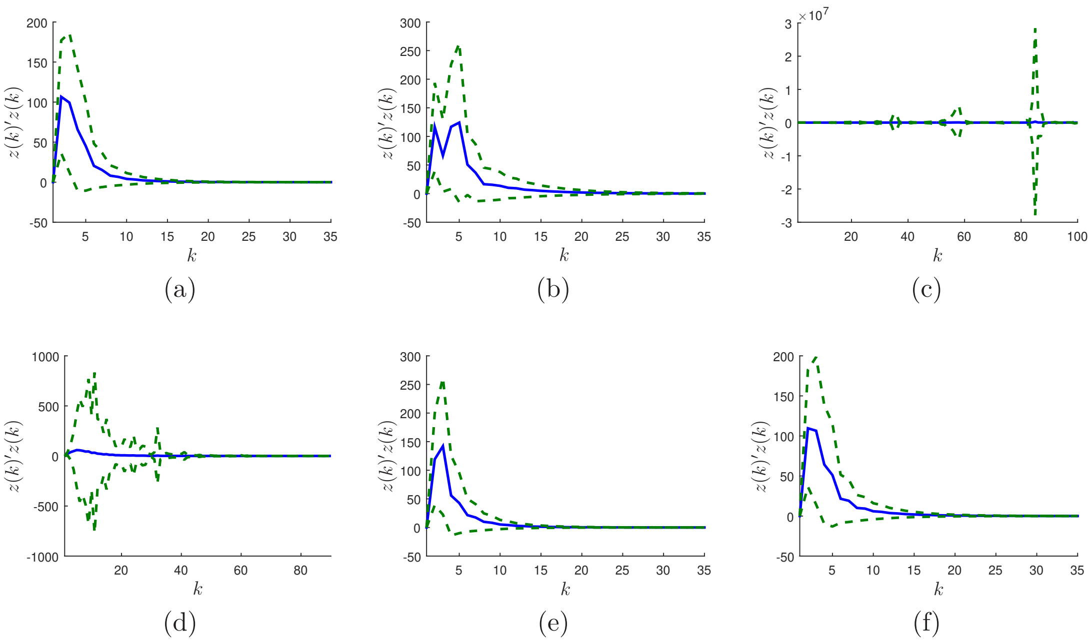


Figure 14 – Mean (blue) and standard deviation (dashed green) of $z(k)'z(k) \times k$ obtained through MC simulation of the closed-loop system of Example 3.6.2.2 with (a) SDOH formulation; (b) Classical robust controller; (c) Optimal controller for $\theta_1(k)$ of maximum frequency; (d) Robust controller designed for the sub-domain disregarding the stability guarantee for all domain; Robust controller assuring the stability for all domain but optimizing the $\mathcal{H}_2$ cost in a single operation point (e) maximum frequency and (f) expected value.

Observe that the methods (b), (c), (e) and (f) (whose results are respectively presented in Figs. 14b, 14c, 14e, and 14f) are simpler than the proposed one (method (a) associated with Fig. 14a) because they do not require the concept of sub-domain (Subsection 3.3.1). Additionally, although the approach (d) (whose results are depicted in Fig. 14d) employs the sub-domain notion, it presents the same disadvantage as technique (c): it does not guarantee stability for all the parametric domain. This feature is more evident in Fig. 14c, revealing that the system in closed-loop with the optimal controller provided by method (c) is unstable. Furthermore, despite Fig. 14d does not represent an unstable case, it illustrates an equally bad case where the standard deviation is almost 5 times greater and the $\mathcal{H}_2$ norm is 25.00% higher than those obtained by the SDOH technique (a). Note that, even optimizing the guaranteed cost in the sub-domain, the method (d) provides an actual worst case $\mathcal{H}_2$ norm of 24.5460, as bad as the classical robust case (b), whose value is $\mathcal{H}_2$= 24.8760, being 26.88% bigger than the SDOH technique ($\mathcal{H}_2$= 19.6037). Finally, even though approaches (e) and (f) assure stability for all the parametric domain and use the statistical information of the probability distribution of the parameter in some way, a loss of performance is obtained: $\mathcal{H}_2$= 20.9387 for case (e) and $\mathcal{H}_2$= 20.5434 for technique (f), respectively 6.81% and 4.69% worst than the SDOH method. Note that, to optimize the $\mathcal{H}_2$ guaranteed cost in a single operation point (maximum frequency or expected value, as done by (e) and (f) and shown in Figs. 14e and 14f) can provide results associated with higher standard deviations than the one yielded when optimizing the performance criterion in a sub-domain (as done by the proposed SDOH method (a)).

3.6.2.3 $\mathcal{H}_2$ state-feedback control design using the accurate modeling of the additive noise

This example aims to illustrate that an accurate modeling of the additive uncertainty can improve the design of the stabilizing controller, allowing to obtain lower estimates for the closed-loop $\mathcal{H}_2$ guaranteed cost and also to provide feasible solutions when other methods fail. First the model disregards (3.14), that is, the effect of $\delta(k)$ over $\theta(k)$, providing a more conservative representation but with a less complex model. In the second case (3.14) is taken into account, representing precisely the feasible region of the pair $(\theta(k), \delta(k))$ but requiring a more complex model and, consequently, a higher computational cost. The concept of sub-domain is not used in this example and the conditions of Theorem 3.2 are solved considering only (3.17), (3.18) and (3.19).

Consider system (3.3) with polytopic uncertainty and the problem of state-feedback control ($C_y(\theta(k)) = I$, $E_y(\theta(k)) = 0$) with the coefficients of the state-space matrices given by

$$\left[\begin{array}{c|c|c|c|c|c} A_1 & A_2 & B_1 & B_2 & E_1 & E_2 \\ \hline C_{z1} & C_{z2} & D_{z1} & D_{z2} & D_{z1} & D_{z2} \end{array}\right] = \left[\begin{array}{cc|cc|c|c|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 1 & 1 & -1 & 1 & 1 \\ \hline 0 & 4 & 0 & 0 & 0.5 & 2 & 0.5 & 2 \end{array}\right] \tag{3.22}$$

with the time-varying parameters $(\theta_1(k), \theta_2(k)) \in \Lambda_2$. At least for optimization variables $(P_{s_1}(\Theta(k))$ and $W_1(\Theta(k))$ up to degree 3, Theorem 3.2 is not capable to provide a robust ($G(\cdot)$ and $Y(\cdot)$ of degree zero) state-feedback control law for this example. Possibly this system can only be stabilized by a gain-scheduled control law and, if a real-time identification procedure is applied, the existence of an additive noise error inherent to the estimation process must be taken into account. In this example, the additive uncertainty is investigated considering $-\bar{\delta} \leq \delta_i(k) \leq \bar{\delta}$, $\bar{\delta} \in [0, 0.12]$, $i = 1, 2$.

The consideration of the saturation effect of $\delta_1(k)$ over $\theta_1(k)$ produces a six-vertex polytope, depicted by the light-grey area in Fig. 15 when $\bar{\delta} = 0.12$. Disregarding the saturation, the feasible region comprises the light-grey area plus the dark-grey area, yielding a four-vertex polytope (rectangle). Note that the rectangle model tends to be more and more conservative as the value $\bar{\delta}$ increases and the contrary also holds (less and less conservative as $\bar{\delta} \to 0$). The experiment relies on computing gain-scheduled gains

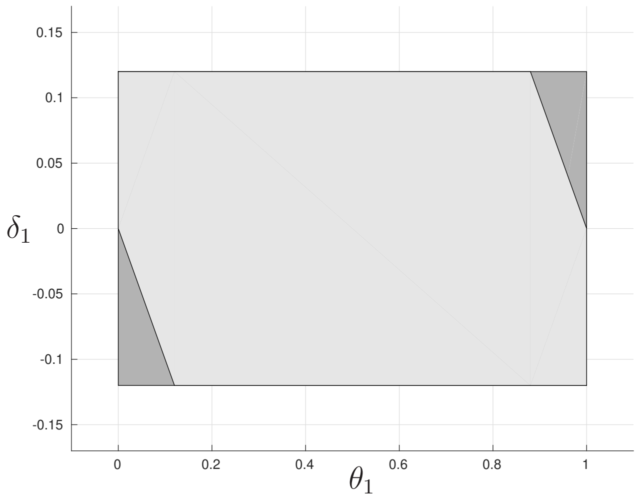


Figure 15 – Feasible region of the pair (θ_1, δ_1) where $(\theta_1, \theta_2) \in \Lambda_2$ and $-0.12 \leq \delta_i \leq 0.12$, $i = 1, 2$, considering (light-grey) and not considering (light-grey plus dark-grey) (3.15). A similar region is obtained for the pair (θ_2, δ_2).

considering both models, analyzing the trade-off between conservativeness in terms of $\mathcal{H}_2$ guaranteed costs and the computational burden. The conditions of Theorem 3.2 are employed considering degree one for all optimization variables (no improvements were observed for higher degrees). The $\mathcal{H}_2$ guaranteed costs associated to the designed controllers are shown in Fig. 16 as a function of $\bar{\delta} \in [0,\ 0.12]$.

Observe that, when no additive uncertainty is considered ($\bar{\delta} = 0$), both models are associated with the same $\mathcal{H}_2$ guaranteed cost $\rho = 72.1971$. Regarding only the stabilization problem, note that the four-vertex model can be stabilized using Theorem 3.2 only if $\bar{\delta} \in [0,\ 0.0780]$. On the other hand, the six-vertex model can be stabilized through Theorem 3.2 for $\bar{\delta(k)} \in [0,\ 0.1140]$, which represents an increase around 69.42% in the feasibility region. It is important to emphasize that a control design associated to a wider range of additive noise (6% of the fluctuation of the time-varying parameter against 4.1% when employing the conservative model) allows the use of less accurate estimation algorithms or measurement sensors, which usually imply faster parameter estimation or hardware of lower costs in practical implementations.

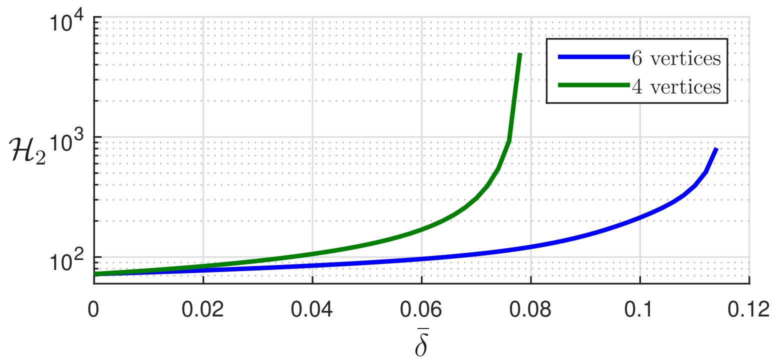


Figure 16 – $\mathcal{H}_2$ guaranteed costs (ρ) obtained by Theorem 3.2 as a function of $\bar{\delta} \in [0,\ 0.12]$ considering the conservative model (4 vertices – green) and the accurate model of the additive uncertainty (6 vertices – blue).

In terms of guaranteed costs, the six-vertex model always provides less conservative results, with a more pronounced improvement for larger values of $\bar{\delta}$ (note that the $\mathcal{H}_2$ guaranteed cost axis of the graph depicted in Fig. 16 is in logarithmic scale). For instance, assuming $\bar{\delta} = 0.07$, the $\mathcal{H}_2$ guaranteed cost is about 3 times lower by using the six-vertex model ($\mu = 308.9$ for the four-vertex model and $\mu = 103.8$ for the six-vertex model).

To conclude the analysis, it is necessary to evaluate the computational complexity demanded by the models. The conditions of Theorem 2 using degree $g = 3$ in all optimization variables required $V = 317$ decision variables and $L = 2336$ LMI rows for the four-vertex model and $V = 3317$ and $L = 28336$ for the six-vertex model. As expected, less conservative results come with a price, that is, a more complex optimization problem.

3.7 $\mathcal{H}_2$ Full-Order Filter Design

Consider the gain-scheduled full-order filter

$$\mathcal{F} = \begin{cases} x_f(k+1) = A_f(\tilde{\theta}(k))x(k) + B_f(\tilde{\theta}(k))y(k), \\ z_f(k) = C_f(\tilde{\theta}(k))x(k) + D_f(\tilde{\theta}(k))y(k), \end{cases} \tag{3.23}$$

where $x_f(k) \in \mathbb{R}^{n_x}$, $z_f(k) \in \mathbb{R}^{n_z}$ respectively denote the filter state and the estimated output vectors. As in the state-feedback case, the matrices of the filter are scheduled by time-varying parameters contaminated by noise in the form of (3.12). By connecting the filter with the LPV system (3.3) with $B(\theta(k)) = 0$, $D_z(\theta(k)) = 0$, one obtains the augmented system

$$\mathcal{G}_{aug} = \begin{cases} \tilde{x}(k+1) = \tilde{A}(\theta(k), \tilde{\theta}(k))\tilde{x}(k) + \tilde{B}(\theta(k), \tilde{\theta}(k))w(k), \\ e(k) = \tilde{C}(\theta(k), \tilde{\theta}(k))\tilde{x}(k) + \tilde{D}(\theta(k), \tilde{\theta}(k))w(k), \end{cases} \tag{3.24}$$

where $\tilde{x}(k) = [x(k)'\ x_f(k)']'$ is the augmented state vector, $e(k) = z(k) - z_f(k)$ represents the error dynamics, and the augmented state-space matrices are

$$\tilde{A}(\theta(k), \tilde{\theta}(k)) = \begin{bmatrix} A(\theta(k)) & 0 \\ B_f(\tilde{\theta}(k))C_y(\theta(k)) & A_f(\tilde{\theta}(k)) \end{bmatrix},\ \tilde{B}(\theta(k), \tilde{\theta}(k)) = \begin{bmatrix} E(\theta(k)) \\ B_f(\tilde{\theta}(k))E_y(\theta(k)) \end{bmatrix}, \tag{3.25}$$

$$\tilde{C}(\theta(k), \tilde{\theta}(k)) = \begin{bmatrix} C_z(\theta(k)) - D_f(\tilde{\theta}(k))C_y(\theta(k)) & -C_f(\tilde{\theta}(k)) \end{bmatrix}$$

and $\tilde{D}(\theta(k), \tilde{\theta}(k)) = E_z(\theta(k)) - D_f(\tilde{\theta}(k))E_y(\theta(k))$. The aim is to synthesize matrices $A_f(\tilde{\theta}(k))$, $B_f(\tilde{\theta}(k))$, $C_f(\tilde{\theta}(k))$ and $D_f(\tilde{\theta}(k))$ such that the augmented system (3.24) is asymptotically stable for the whole domain of uncertainty and to minimize an $\mathcal{H}_2$ guaranteed cost from the disturbance input $w(k)$ to the estimation error $e(k)$ only in a subdomain (or multiple sub-domains).

Consider the following partitioning scheme

$$W(\bullet) = \begin{bmatrix} W_{11}(\bullet) & \star \\ W_{21}(\bullet) & W_{22}(\bullet) \end{bmatrix}, \quad X(\bullet) = \begin{bmatrix} X_1(\bullet) & X_3(\bullet) \\ X_2(\bullet) & X_4(\bullet) \end{bmatrix} \tag{3.26}$$

Theorem 3.3. *Let the augmented vector $\Theta(k) = (\theta(k)', \delta(k)')'$ and the linear transformation $\tilde{\theta}(k) = ([1,1] \otimes I_m)\Theta(k)$ be given. Consider system (3.3) ($B(\theta(k)) = 0$, $D_z(\theta(k)) = 0$) with $\Theta(k) \in \Delta_{2m}(\mathsf{B})$ and a set of subdomains $\mathcal{S}_{r_i}(\mathsf{H_i})$, $i = 1, \ldots, s$, where B and H_i are given matrices. For given scalars $\rho > 0$, γ_A, γ_B and $\gamma_q \in (-1, 1)$, if there exist symmetric positive definite parameter-dependent matrices $W(\Theta(k))$, $W_{s_i}(\Theta(k))$ and $M(\Theta(k))$, matrices $X(\Theta(k))$, and $X_{s_i}(\Theta(k))$, $i = 1, \ldots, s$, structured as in (3.26) and matrices $S_A(\tilde{\theta}(k))$, $S_B(\tilde{\theta}(k))$, $L_A(\tilde{\theta}(k))$, $L_B(\tilde{\theta}(k))$, $C_f(\tilde{\theta}(k))$ and $D_f(\tilde{\theta}(k))$ such that*

$$\begin{bmatrix} Q_{11}(\Theta(k), W, X, L_A(\tilde{\theta}(k)), L_B(\tilde{\theta}(k))) & \star \\ Q_{31}(\Theta(k), L_A(\tilde{\theta}(k)), L_B(\tilde{\theta}(k))) & 0 \end{bmatrix} + He\left(\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_3 \end{bmatrix} \begin{bmatrix} \mathcal{B}_1(\theta(k), X, S_A(\tilde{\theta}(k)), S_B(\tilde{\theta}(k))) \\ \mathcal{B}_3(S_A(\tilde{\theta}(k)), S_B(\tilde{\theta}(k))) \end{bmatrix}'\right) < 0, \tag{3.27}$$

holds for all $(\Theta(k), \Theta(k+1)) \in \Delta_{2m}(\mathsf{B}) \times \Delta_{2m}(\mathsf{B})$, and for $i = 1, \ldots, s$

$$\begin{bmatrix} M(\xi(k)) & \star & \star & \star \\ (C_z(\xi(k)) - D_f(\xi(k))C_y(\xi(k)))^T & W_{si_{11}}(\xi(k)) & \star & \star \\ -C_f(\xi(k))^T & W_{si_{21}}(\xi(k)) & W_{si_{22}}(\xi(k)) & \star \\ (E_z(\xi(k)) - D_f(\xi(k))E_y(\xi(k)))^T & 0 & 0 & I \end{bmatrix} > 0, \tag{3.28}$$

$$\begin{bmatrix} Q_{11}(\xi(k), W_{s_i}, X_{s_i}, L_A(\xi(k)), L_B(\xi(k))) & \star & \star \\ Q_{21}(\xi(k), W_{s_i}, X_{s_i}, L_A(\xi(k)), L_B(\xi(k))) & Q_{22} & \star \\ Q_{31}(\xi(k), L_A(\xi(k)), L_B(\xi(k))) & Q_{32}(\xi(k), L_B(\xi(k)) & 0 \end{bmatrix} \tag{3.29}$$

$$+He\left(\begin{bmatrix}\mathcal{X}_1\\ \mathcal{X}_2\\ \mathcal{X}_3\end{bmatrix}\begin{bmatrix}\mathcal{B}_1(\xi(k), X_{s_i}, S_A(\xi(k)), S_B(\xi(k)))\\ \mathcal{B}_2\\ \mathcal{B}_3(S_A(\xi(k)), S_B(\xi(k)))\end{bmatrix}'\right) < 0, \tag{3.30}$$

$$Tr(M(\xi(k))) < \rho^2, \tag{3.31}$$

hold for all $(\xi(k), \xi(k+1)) \in \Lambda_{r_i} \times \Lambda_{r_i}$, *where*[2]

$$\left[\begin{array}{c|c|c} Q_{11}(\bullet, W, X, L_A(\blacktriangle), L_B(\blacktriangle)) & \star & \star \\ \hline Q_{21}(\bullet, W, X, L_A(\blacktriangle), L_B(\blacktriangle)) & Q_{22} & \star \\ \hline Q_{31}(\bullet, L_A(\blacktriangle), L_B(\blacktriangle)) & Q_{32}(\bullet, L_B(\blacktriangle)) & 0 \end{array}\right] = \left[\begin{array}{cccc|c|cc} \Psi_{11} & \star & \star & \star & \star & \star & \star \\ \Psi_{21} & \Psi_{22} & \star & \star & \star & \star & \star \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \star & \star & \star & \star \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} & \star & \star & \star \\ \hline \Psi_{51} & \Psi_{52} & \Psi_{53} & \Psi_{54} & -I & \star & \star \\ \hline \Psi_{61} & 0 & 0 & 0 & \Psi_{65} & 0 & \star \\ 0 & L_A(\blacktriangle) & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{3.32}$$

$$\Psi_{11} = \begin{array}{c}\gamma_q He(X_1(\bullet)A(\bullet) \\ +L_B(\blacktriangle)C_y(\bullet) - W_{11}(\bullet))\end{array} \qquad \Psi_{21} = \begin{array}{c}\gamma_q(X_2(\bullet)A(\bullet) + L_B(\blacktriangle) \\ C_y(\bullet) + L_A(\blacktriangle)') - W_{21}(\bullet)\end{array}$$

$$\Psi_{31} = -\gamma_q X_1(\bullet)' + X_1(\bullet)A(\bullet) + L_B(\blacktriangle)C_y(\bullet) \qquad \Psi_{41} = -\gamma_q X_3(\bullet)' + X_2(\bullet)A(\bullet) + L_B(\blacktriangle)C_y(\bullet)$$

$$\Psi_{51} = \gamma_q(E(\bullet)X_1(\bullet)' + E_y(\bullet)'L_B(\blacktriangle)') \qquad \Psi_{61} = L_B(\blacktriangle)C_y(\bullet)$$

$$\Psi_{22} = \gamma_q He(L_A(\blacktriangle)) - W_{22}(\bullet) \qquad \Psi_{32} = -\gamma_q X_2(\bullet)' + L_A(\blacktriangle)$$

$$\Psi_{42} = -\gamma_q X_4(\bullet)' + L_A(\blacktriangle) \qquad \Psi_{52} = \gamma_q(E(\bullet)'X_2(\bullet)' + E_y(\bullet)'L_B(\blacktriangle)')$$

$$\Psi_{33} = W_{11}^+(\bullet) - He(X_1(\bullet)) \qquad \Psi_{43} = W_{21}^+(\bullet) - X_2(\bullet) - X_3(\bullet)'$$

$$\Psi_{53} = E(\bullet)'X_1(\bullet)' + E_y(\bullet)'L_B(\blacktriangle)' \qquad \Psi_{44} = W_{22}^+(\bullet) - He(X_4(\bullet))$$

$$\Psi_{54} = E(\bullet)'X_2(\bullet)' + E_y(\bullet)'L_B(\blacktriangle)' \qquad \Psi_{65} = L_B(\blacktriangle)E_y(\bullet),$$

[2] The symbols $\bullet$ and $\blacktriangle$ are used to represent the blocks of matrices Q and $\mathcal{B}$ generically in terms of $\Theta(k)$, $\tilde{\theta}(k)$ and $\xi(k)$. Moreover, the superscript $^+$ is used to denote the block matrices W_{ij} one time-instant ahead.

$$\begin{bmatrix} \mathcal{B}_1(\bullet, X, S_A(\blacktriangle), S_B(\blacktriangle))' \\ \hline \mathcal{B}_2' \\ \hline \mathcal{B}_3(S_A(\blacktriangle), S_B(\blacktriangle))' \end{bmatrix} = \begin{bmatrix} \gamma_q\left(X_3(\bullet) - S_B(\blacktriangle)\right) & \gamma_q\left(X_3(\bullet) - S_A(\blacktriangle)\right) \\ \gamma_q\left(X_4(\bullet) - S_B(\blacktriangle)\right) & \gamma_q\left(X_4(\bullet) - S_A(\blacktriangle)\right) \\ X_3(\bullet) - S_B(\blacktriangle) & X_3(\bullet) - S_A(\blacktriangle) \\ X_4(\bullet) - S_B(\blacktriangle) & X_4(\bullet) - S_A(\blacktriangle) \\ \hline 0 & 0 \\ \hline -S_B(\blacktriangle) & 0 \\ 0 & -S_A(\blacktriangle) \end{bmatrix}, \quad \begin{bmatrix} \mathcal{X}_1 \\ \hline \mathcal{X}_2 \\ \hline \mathcal{X}_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ I & 0 \\ 0 & I \\ \hline 0 & 0 \\ \hline \gamma_B I & 0 \\ 0 & \gamma_A I \end{bmatrix}, \tag{3.33}$$

and $\Theta(k) = \mathsf{H}_i\xi(k)$, $i = 1, \ldots, s$, *then* $A_f(\tilde{\theta}(k)) = S_A^{-1}(\tilde{\theta}(k))L_A(\tilde{\theta}(k))$, $B_f(\theta(k)) = S_B^{-1}(\tilde{\theta}(k))L_B(\tilde{\theta}(k))$, $C_f(\tilde{\theta}(k))$ *and* $D_f(\tilde{\theta}(k))$ *are the filter matrices that guarantee that the filtered system* (3.24) *is asymptotically stable for all* $\Theta(k) \in \Delta_{2m}(\mathsf{B})$ *and* ρ *is an* $\mathcal{H}_2$ *guaranteed cost valid only inside the union of sub-domains* $\mathcal{S}_{r_i}(\mathsf{H_i})$, $i = 1, \ldots, s$.

Proof. Firstly, multiply (3.27) on the left by $\mathcal{B}^\perp(X(\theta(k)), S_A(\tilde{\theta}(k)), S_B(\tilde{\theta}(k)))'$ and on the right by the transpose with

$$\mathcal{B}^\perp(X(\bullet), S_A(\blacktriangle), S_B(\blacktriangle)) = \begin{bmatrix} \mathcal{B}_{11}^\perp \\ \hline \mathcal{B}_{31}^\perp \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline \gamma_q \Phi_{B3} & \gamma_q \Phi_{B4} & \Phi_{B3} & \Phi_{B4} \\ \gamma_q \Phi_{A3} & \gamma_q \Phi_{A4} & \Phi_{A3} & \Phi_{A4} \end{bmatrix}, \quad \begin{array}{l} \text{with} \\ \Phi_{A3} = \left(S_A(\blacktriangle)^{-1}X_3(\bullet) - I\right)', \\ \Phi_{A4} = \left(S_A(\blacktriangle)^{-1}X_4(\bullet) - I\right)', \\ \Phi_{B3} = \left(S_B(\blacktriangle)^{-1}X_3(\bullet) - I\right)', \\ \Phi_{B4} = \left(S_B(\blacktriangle)^{-1}X_4(\bullet) - I\right)', \end{array} \tag{3.34}$$

to obtain

$$\begin{bmatrix} -W(\Theta(k)) & \star \\ 0 & W(\Theta(k+1)) \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \gamma_q X(\Theta(k)) \\ X(\Theta(k)) \end{bmatrix} \begin{bmatrix} \tilde{A}(\theta(k), \tilde{\theta}(k)) & -I \end{bmatrix} \right) < 0 \tag{3.35}$$

where $\tilde{A}(\theta(k), \tilde{\theta}(k))$ is given by (3.25) replacing $A_f(\tilde{\theta}(k))$ by $S_A^{-1}(\tilde{\theta})(k)L_A(\tilde{\theta}(k))$ and $B_f(\tilde{\theta}(k))$ by $S_B^{-1}(\tilde{\theta}(k))L_B(\tilde{\theta}(k))$. Pre- and post-multiplying (3.35) respectively by

$$N_A' = \begin{bmatrix} I & \tilde{A}(\theta(k), \tilde{\theta}(k))' \end{bmatrix}$$

and N_A, one has

$$\tilde{A}(\theta(k), \tilde{\theta}(k))'W(\Theta(k+1))\tilde{A}(\theta(k), \tilde{\theta}(k)) - W(\Theta(k)) < 0$$

that assures the asymptotic stability of the augmented system (3.24). The existence of the inverse of matrices $S_A(\tilde{\theta}(k))$ and $S_B(\tilde{\theta}(k))$ can be proved by observing that $\gamma_A(S_A(\tilde{\theta}(k)) + S_A(\tilde{\theta}(k))')$ and $\gamma_B(S_B(\tilde{\theta}(k)) + S_B(\tilde{\theta}(k))')$ appear in diagonal blocks of (3.27). On the other hand, by multiplying (3.29) on the left by $\mathcal{B}^{\perp}(X_{s_i}(\xi(k)), S_A(\xi), S_B(\xi))'$ and on the right by the transpose, with

$$\mathcal{B}^{\perp}(X(\bullet), S_A(\blacktriangle), S_B(\blacktriangle)) = \left[\begin{array}{c|c} \mathcal{B}_{11}^{\perp} & \mathcal{B}_{12}^{\perp} \\ \hline \mathcal{B}_{21}^{\perp} & \mathcal{B}_{22}^{\perp} \\ \hline \mathcal{B}_{31}^{\perp} & \mathcal{B}_{32}^{\perp} \end{array}\right], \text{ with } \begin{array}{ll} \mathcal{B}_{12}^{\perp} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}', & \mathcal{B}_{11}^{\perp} \text{ from (3.34)}, \\ \mathcal{B}_{21}^{\perp} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, & \mathcal{B}_{31}^{\perp} \text{ from (3.34)}, \\ \mathcal{B}_{32}^{\perp} = \begin{bmatrix} 0 & 0 \end{bmatrix}', & \mathcal{B}_{22}^{\perp} = I, \end{array} \tag{3.36}$$

one gets

$$\begin{bmatrix} -W_{s_i}(\xi(k)) & \star & \star \\ 0 & W_{s_i}(\xi(k+1)) & \star \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left(\underbrace{\begin{bmatrix} \gamma_q X_{s_i}(\xi(k)) \\ X_{s_i}(\xi(k)) \\ 0 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \tilde{A}(\xi(k)) & -I & \tilde{B}(\xi(k)) \end{bmatrix}}_{U} \right) < 0 \tag{3.37}$$

where $\tilde{A}(\xi(k))$ and $\tilde{B}(\xi(k))$ are given by (3.25) replacing $A_f(\tilde{\theta}(k))$ by $S_A^{-1}(\tilde{\theta}(k))L_A(\tilde{\theta}(k))$ and $B_f(\tilde{\theta}(k))$ by $S_B^{-1}(\tilde{\theta}(k))L_B(\tilde{\theta}(k))$ and using the linear transformation $\Theta(k) = \mathsf{H}_i\xi(k)$.

Choosing the following bases for the null spaces of U and V

$$N_U' = \begin{bmatrix} I & \tilde{A}(\xi(k))' & 0 \\ 0 & \tilde{B}(\xi(k))' & I \end{bmatrix}, \quad N_V' = \begin{bmatrix} -I & \gamma_q I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{3.38}$$

pre- and post-multiplying (3.37) respectively by N_U' and N_U, and applying the Schur's Complement, one obtains the controllability gramian of the LPV system (3.24) (DE CAIGNY *et al.*, 2010, Theorem 2), valid only inside the set of sub-domains. Additionally, pre- and post-multiplying (3.37), respectively, by N_V^T and N_V, one gets the bounds for the scalar parameter $\gamma_q \in (-1, 1)$. Finally, note that (3.28) corresponds to the cost condition for the computation of the $\mathcal{H}_2$ guaranteed cost of (3.24) (DE CAIGNY *et al.*, 2010, Theorem 2) valid only inside the set of sub-domains. □

One of the contributions of this chapter is the new filter synthesis conditions proposed in Theorem 3.3 where, differently from the routine adopted in the literature, no structural constraints are imposed to the slack variables $X(\Theta(k))$ and $X_{si}(\Theta(k))$. In this conditions, the filter matrices $A_f(\tilde{\theta}(k))$ and $B_f(\tilde{\theta}(k))$ are recovered by a change of variables whose variables are totally independent of each other (usually, $S_A(\tilde{\theta}(k))$ and $S_B(\tilde{\theta}(k))$ are the same). Furthermore, next lemma demonstrates that conditions (3.28)-(3.31) of Theorem 3.3 (considered for only one sub-domain and equal to the whole domain, i.e., all variables depending on $\Theta(k)$) encompass a method from the literature for the synthesis of $\mathcal{H}_2$ full-order filter by means of particular choices of the scalars and variables.

Lemma 3.1. *If Theorem 1 from Palma* et al. *(2018c) has a solution, then Theorem 3.3 also does by means of the following choices $\gamma_q = \xi$, $\gamma_A = \gamma_B \to \infty$, $X_3(\bullet) = X_4(\bullet) =$*

$S_A(\bullet) = S_B(\bullet) = \bar{X}(\Theta(k))$, $L_A(\bullet) = H(\Theta(k))$, $L_B(\bullet) = Z(\Theta(k))$, $X_1(\bullet) = X_{11}(\Theta(k))$, $X_2(\bullet) = X_{21}(\Theta(k))$.

Proof. Firstly observe that conditions (3.31) and (3.28) of Theorem 3.3 are respectively the same of (8) and (9) from Palma *et al.* (2018c). Adopting the choices given in Lemma 3.1 in (3.29), one has

$$\left[\begin{array}{c|c} \mathit{\Pi}_A & \star \\ \hline \mathit{\Pi}_B & \mathit{\Pi}_C \end{array}\right] = \left[\begin{array}{ccccc|cc} \Pi_{11} & \star & \star & \star & \star & \star & \star \\ \Pi_{21} & \Pi_{22} & \star & \star & \star & \star & \star \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \star & \star & \star & \star \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & \star & \star & \star \\ \xi\Pi_{53} & \xi\Pi_{54} & \Pi_{53} & \Pi_{54} & \Pi_{55} & \star & \star \\ \hline \Pi_{61} & 0 & \Pi_{63} & 0 & \Pi_{65} & \Pi_{66} & \star \\ 0 & \Pi_{72} & 0 & \Pi_{63} & 0 & 0 & \Pi_{77} \end{array}\right] < 0, \tag{3.39}$$

with

$\Pi_{11} = \xi \mathrm{He}\left(X_{11}(\Theta(k))A(\Theta(k)) + Z(\Theta(k))C_y(\Theta(k))\right) - W_{11}(\Theta(k))$,
$\Pi_{72} = H(\Theta(k)) - \bar{X}(\Theta(k))'$,
$\Pi_{21} = \xi\left(X_{21}(\Theta(k))A(\Theta(k)) + Z(\Theta(k))C_y(\Theta(k)) + H(\Theta(k))'\right) - W_{21}(\Theta(k))$,
$\Pi_{33} = \tilde{W}_{11}(\Theta(k)) - \mathrm{He}\left(X_{11}(\Theta(k))\right)$,
$\Pi_{31} = X_{11}(\Theta(k))A(\Theta(k)) + Z(\Theta(k))C_y(\Theta(k)) - \xi X_{11}(\Theta(k))'$,
$\Pi_{43} = \tilde{W}_{21}(\Theta(k)) - X_{21}(\Theta(k)) - \bar{X}(\Theta(k))'$,
$\Pi_{41} = X_{21}(\Theta(k))A(\Theta(k)) + Z(\Theta(k))C_y(\Theta(k)) - \xi\bar{X}(\Theta(k))'$,
$\Pi_{63} = -\bar{X}(\Theta(k))'$,
$\Pi_{53} = (X_{11}(\Theta(k))E(\Theta(k)) + Z(\Theta(k))E_y(\Theta(k)))'$,
$\Pi_{55} = -I$,
$\Pi_{61} = Z(\Theta(k))C_y(\Theta(k)) - \bar{X}(\Theta(k))'$,
$\Pi_{65} = Z(\Theta(k))E_y(\Theta(k))$,
$\Pi_{22} = \xi\mathrm{He}(H(\Theta(k))) - W_{22}(\Theta(k))$,
$\Pi_{66} = -\gamma_B\mathrm{He}\left(\bar{X}(\Theta(k))\right)$,
$\Pi_{32} = H(\Theta(k)) - \xi X_{21}(\Theta(k))'$,
$\Pi_{42} = H(\Theta(k)) - \xi\bar{X}(\Theta(k))'$,
$\Pi_{54} = (X_{21}(\Theta(k))E(\Theta(k)) + Z(\Theta(k))E_y(\Theta(k)))'$,
$\Pi_{77} = -\gamma_A\mathrm{He}(\bar{X}(\Theta(k)))$.

By applying a Schur's complement in (3.39) one obtains

$$\mathit{\Pi}_A - \mathit{\Pi}_B'\left(\mathit{\Pi}_C\right)^{-1}\mathit{\Pi}_B < 0 \quad\Rightarrow\quad \mathit{\Pi}_A - \mathit{\Pi}_B'\begin{bmatrix} -\gamma_B\mathrm{He}\left(\bar{X}(\Theta(k))\right) & 0 \\ 0 & -\gamma_A\mathrm{He}(\bar{X}(\Theta(k))) \end{bmatrix}^{-1}\mathit{\Pi}_B < 0. \tag{3.40}$$

As suggested in Lemma 3.1, $\gamma_A \to \infty$ and $\gamma_B \to \infty$, implying that $\mathit{\Pi}_B'\left(\mathit{\Pi}_C\right)^{-1}\mathit{\Pi}_B \to 0$ and (3.40) $\to \mathit{\Pi}_A < 0$, recovering the parameter-dependent LMI (10) from Palma *et al.* (2018c).

□

Instead of following the main stream of methods available in the literature, where the routine is to introduce more and more slack variables to reduce the conservativeness, a different approach was pursued in Theorem 3.3. The slack variable X in Palma *et al.* (2018c) was kept completely parameter-dependent (in Palma *et al.* (2018c) the second column is fixed) and the filter matrices are introduced independently. As important consequences, Theorem 3.3 is no more conservative than Palma *et al.* (2018c) and an extra degree of freedom is obtained in the construction of $A_f(\tilde{\theta}(k))$ and $B_f(\tilde{\theta}(k))$ with $S_A(\tilde{\theta}(k))$ and $S_B(\tilde{\theta}(k))$. Finally, the technique employed in the development of Theorem 3 is general in the sense it can be applied in other design methods with more slack variables, for instance, in the method of Lacerda *et al.* (2011).

3.7.1 $\mathcal{H}_2$ Filter Design Examples

An example borrowed from the literature is investigated in this section with the purpose of evaluating the proposed approach in the context of $\mathcal{H}_2$ gain-scheduled filtering with estimated parameters. Two investigations are carried out. The first one is the analysis of the saturation effect of $\delta(k)$ over $\theta(k)$. To enrich the analysis, both robust and gain-scheduled filters are designed and their performances are evaluated in terms of guaranteed costs and the MSE of the estimation error. The second study aims to evaluate the actual worst case $\mathcal{H}_2$ closed-loop norm (obtained via MC simulation) associated with three distinct $\mathcal{H}_2$ full-order filters: the robust one and two gain-scheduled filters synthesized using the SDOH approach (considering the whole additive uncertain interval or a sub-domain defined by the standard deviation of the identification error histogram).

3.7.1.1 $\mathcal{H}_2$ filter design using estimated parameters

Consider system (3.3) (with $u(k) = 0$) with affine uncertainty and state-space matrices (borrowed from Palma *et al.* (2018c)) given by

$$\left[\begin{array}{c|c|c|c|c|c} A_0 & A_1 & A_2 & E_0 & E_1 & E_2 \\ \hline C_{y0} & C_{y1} & C_{y2} & E_{y0} & E_{y1} & E_{y2} \\ \hline C_{z0} & C_{z1} & C_{z2} & E_{z0} & E_{z1} & E_{z2} \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc|cc|cc|cc} 0 & 1 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 1 & -0.2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0.7 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.25 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.41}$$

The time-varying parameters are $\theta_1(k) = -0.5 - 0.2\sin((2\pi/1800)k)$ (Fig. 17a) and $\theta_2(k)$ is a triangular wave centered around 0.3 with a period of $k = 4000$ samples and amplitude 0.1 (Fig. 17d). Thus,

$$\mathsf{B} = \begin{bmatrix} -0.7 & -0.3 \\ 0.2 & 0.4 \end{bmatrix}$$

By measuring the output $y(k)$ and knowing the input $w(k) = [w_1(k)' \; w_2(k)']'$, it is possible to identify the time-varying parameters in real-time through an inverse identification scheme using the Recursive-Least-Squares (RLS) Algorithm following the steps given in Section V from Palma *et al.* (2018c). For the MC simulation, the exogenous input $w_1(k)$ is composed by a discrete-time sine wave with magnitude 8.8 and a sampling period of $k = 20$ samples plus a Gaussian noise with null mean and standard deviation 3, while $w_2(k)$ is a Gaussian noise with null mean and standard deviation 0.25.

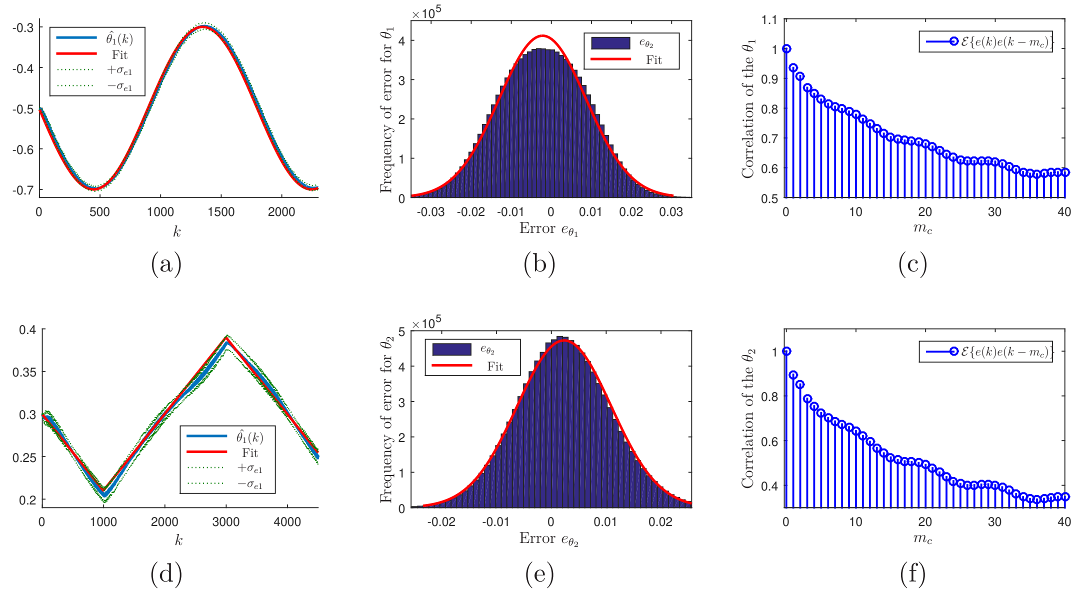


Figure 17 – Results of the identification process for the parameters: Fig. 17a and Fig. 17d show the actual parameter (red curve), mean (blue curve) and standard deviation (green curve). Fig. 17b and Fig. 17e expose the histograms of the identification error and the normal approximation for the data (red curve). Fig. 17c and Fig. 17f show the correlation of the identification error computed by $\mathcal{E}\{e_{\theta_i}(k)e_{\theta_i}(k-m_c)\}$.

The results of the identification process were obtained by employing 10^3 iterations of MC for 10^4 samples. Fig. 17a and Fig. 17d expose the actual value $(\theta_i(k))$, the mean $(\hat{\theta}_i(k))$ and the standard deviation $(\pm\sigma_{ei})$ of the identified parameters, whereas Fig. 17b and Fig. 17e show the histograms of the identification error (e_{θ_i}) and the normal fit (red curve) in the data. Other relevant information, which was not presented in Palma *et al.* (2018c), is the correlation of the identification error shown in Fig. 17c and Fig. 17f. The statistical information obtained from the MC simulations that is used to design the filter corresponds to the mean error $(\bar{e}_{\theta_i})$ and the standard deviation (σ_{ei}), such that $\bar{e}_{\theta_1} = -0.0022$, $\sigma_{e_1} = 0.0109$, $\bar{e}_{\theta_2} = 0.0023$ and $\sigma_{e_2} = 0.0087$. Note that both error histograms present shapes close to the normal distribution, with a negligible discrepancy in the values of higher frequency of θ_1. Nevertheless, it is not assumed that those identification errors have a normal distribution since the current value of the error has a strong correlation with the past values (Fig. 17c and Fig. 17f). In this sense, the filter design must consider the entire domain of the parameters, that is, the bounds for the additive uncertainty must be given by the maximum value coming from the MC simulation: $\max(|e_{\theta_1}|) = 0.0659$ and $\max(|e_{\theta_2}|) = 0.0648$.

First case study: comparison between gain-scheduled and robust filters

Traditionally, design methods for gain-scheduled filtering using inexact measurements aim to guarantee stability and performance for the whole domain of the scheduling and noise parameters. However, this case study investigates if the filter design considering the whole domain of the additive uncertainty ($\max(|e_\theta|)$ is capable of improving or not the temporal behavior of the estimation error ($e(k)$ in (3.24)) in terms of MSE. Although it is not theoretically feasible to obtain the probability distribution of the identification error, by using the information depicted in Fig. 17, it is possible to determine a range where the additive error lies. For instance, one can consider $\delta(k) \in (-\max(|e_\theta|),\ \max(|e_\theta|))$, and synthesize a robust filter valid for the entire additive uncertainty domain (denoted by the sub-index $(\bullet)_{\max}$). Another option, suggested in Palma *et al.* (2018c), is to assume that $\delta(k) \in [-3\sigma_e,\ 3\sigma_e]$ (denoted by the sub-index $(\bullet)_{3\sigma}$), since the frequency of the identification error is concentrated around zero and close to a normal curve[3], meaning that more than 99.5% of the histogram data is located in this interval. To design a filter using Theorem 3.3, the time-varying parameter vector $\Theta(k) = (\theta_1(k), \theta_2(k), \delta_1(k), \delta_2(k))$ is considered to belong to one of the following domains

$$\Delta_4(\mathsf{B}_{\max}) = \begin{bmatrix} \underline{\theta}_1 & \bar{\theta}_1 \\ \underline{\theta}_2 & \bar{\theta}_2 \\ -\max(|e_{\theta_1}|) & \max(|e_{\theta_1}|) \\ -\max(|e_{\theta_2}|) & \max(|e_{\theta_2}|) \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 \\ -0.1 & 0.1 \\ -0.0659 & 0.0659 \\ -0.0648 & 0.0648 \end{bmatrix}, \quad \Delta_4(\mathsf{B}_{3\sigma}) = \begin{bmatrix} \underline{\theta}_1 & \bar{\theta}_1 \\ \underline{\theta}_2 & \bar{\theta}_2 \\ -3\sigma_{e1} & 3\sigma_{e1} \\ -3\sigma_{e2} & 3\sigma_{e2} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 \\ -0.1 & 0.1 \\ -0.0327 & 0.0327 \\ -0.0261 & 0.0261 \end{bmatrix} \tag{3.42}$$

Five different filter settings are investigated: *i)* classical robust filter (parameter-independent denoted by *rob*) that is obtained by fixing zero degree in the variables associated to matrices of the the filter; *ii)* gain-scheduled filter with affine dependency on the time-varying parameters whose parametric domain is given by $\Delta_4(\mathsf{B}_{max})$ and the overbounded additive noise model (polytope of four vertices); *iii)* the previous one but using the less conservative additive noise model (polytope of six vertices); *iv)* Affine gain-scheduled filter with parametric domain given by $\Delta_4(\mathsf{B}_{3\sigma})$ and overbounded additive noise model; *v)* Affine gain-scheduled filter with parametric domain given by $\Delta_4(\mathsf{B}_{3\sigma})$ and less conserva-

[3] Figs. 17c and 17f do not represent exactly a normal distribution, since the correlation function is not null for $m_c > 0$.

tive additive noise model. In the case of gain-scheduled filters; all filters are designed with matrices $S_A(\bullet)$ and $S_B(\bullet)$ of degree zero and matrices $L_A(\bullet)$ and $L_B(\bullet)$ with degree one. The other variables of Theorem 3.3 are fixed as polynomials of degree one.

Table 8 presents the $\mathcal{H}_2$ guaranteed costs (ρ) provided by Theorem 3.3, the MSE of the estimation error and its standard deviation (SD_{MSE}). Additionally, Table 8 also highlights the percentage improvement in those performance criteria ($\rho\%$, $MSE\%$, $SD_{MSE}\%$) when comparing gain-scheduled and robust filters.

Table 8 – $\mathcal{H}_2$ guaranteed cost (ρ), MSE and the associated standard deviation (SD_{MSE}), and the percentage improvements ($\rho\%$, $MSE\%$, $SD_{MSE}\%$) of gain-scheduled filters when compared with the robust filter considering different models of the additive uncertainty (four and six vertices, respectively, equations (3.12) and (3.14)) and domains of additive uncertainty ($\Delta_4(\mathsf{B}_{\max})$ and $\Delta_4(\mathsf{B}_{3\sigma})$).

		Additive Uncertainty			
		four vertices		six vertices	
Structure	rob	lpv$_{max}$	lpv$_{3\sigma}$	lpv$_{max}$	lpv$_{3\sigma}$
ρ	1.2231	1.0651	0.7315	1.0056	0.6859
$\rho\%$	-	12.92	40.19	17.78	43.92
MSE	4.4159	3.4439	1.8085	2.9677	1.5931
$MSE\%$	-	22.01	59.05	32.80	63.92
SD_{MSE}	1.4541	1.1772	0.6032	1.0085	0.5290
$SD_{MSE}\%$	-	19.04	58.52	30.64	63.62

Clearly, when the online measurement of the time-varying parameters is not an option, the design of a robust filter is the simplest alternative. However, if it is possible to implement a real-time identification, one can expect less conservative results in terms of $\mathcal{H}_2$ guaranteed costs and better performance in time-domain (MSE and SD_{MSE}) by designing filters scheduled by the identified parameters. The results informed in Table 8 corroborate the fact that the performance improvement achieved with the gain-scheduled filters justifies the implementation of an identification procedure. The percentage difference is above 12% in the $\mathcal{H}_2$ guaranteed cost and above 22% in the MSE. Similarly to the conclusions obtained in Example 3.6.2.3, a more accurate modeling of the additive uncertainty can improve the results. Finally, it is important to emphasize that the amount of information lost when considering the bounds of the additive uncertainty as $\pm 3\sigma_e$ is negligible (less than 1% of the samples). However, the performance difference is enormous, being at least twice as efficient as the gain-scheduled filter (lpv$_{max}$) when compared to the robust one. This fact indicates that incorporating the probability distribution information (obtained by the histogram of the parameter identification) can be useful when looking for performance similar to that produced by the ideal gain-scheduled filter (accurate reading of the time-varying parameters). The SDOH technique proposed in this chapter comes in handy in this context, because it delivers theoretical guarantees of stability for the whole domain ($\Delta_4(\mathsf{B}_{max})$) and an optimized performance in a sub-domain associated with higher data frequency. This claim is better investigated in the second case study.

Second case study: gain-scheduled filter using the SDOH methodology

Consider system (3.41). As previously discussed, the best performance can be obtained by a gain-scheduled filter that employs a perfect reading of the parameters, that is, an ideal gain-scheduled filter (called lpv in this case study). The lpv filter designed by Theorem 3.3 considering null additive uncertainties ($\delta_i = 0$) yields $\rho = 0.3449$, $MSE = 0.4004$ and $SD_{MSE} = 0.0519$, corresponding to lower bounds of the performance that can be attained by a scheduled filter that takes into account the additive uncertainties. On the other hand, the performance of the robust filter ($\rho = 1.2231$, $MSE = 4.4159$ and $SD_{MSE} = 1.4541$) establishes an upper bound for the scheduled filter in terms of $\mathcal{H}_2$ guaranteed cost, MSE and standard deviation of the MSE.

In this case study, two SDOH filters are designed using the four-vertex additive uncertainty model (neglecting the saturation of $\delta(k)$ over $\theta(k)$). While the first design assures the stability of the filter solving (3.27) for the whole additive uncertain domain ($\Delta_4(\mathsf{B}_{\max})$), the second one provides a stability certificate only for $\Delta_4(\mathsf{B}_{3\sigma})$. Regarding the optimization of the $\mathcal{H}_2$ guaranteed cost, both filters were synthesized considering the same sub-domain (assuming that the additive uncertainty lies in $\delta_i \in [-\sigma_{e_i},\ +\sigma_{e_i}]$) with $\sigma_{e_1} = 0.0109$ and $\sigma_{e_2} = 0.0087$. Since the value of ρ provided by Theorem 3.3 is only valid for the sub-domain, it constitutes a lower bound for the $\mathcal{H}_2$ "worst case" norm, because the time-varying parameters can assume any value in the whole domain. Despite that, when the filter is scheduled in terms of identified parameters, the $\mathcal{H}_2$ worst case norm cannot be computed by MC simulation. Actually, the $\mathcal{H}_2$ norm is obtained by computing the area of $z(k)z(k)'$ after applying a delta of Dirac in the exogenous input. However, note that an impulsive input does not allow to provide a real-time parametric identification, that is an assumption of this example. In this sense, the best alternative to evaluate the filter performance is to compute the MSE (a performance index more frequently used in practical applications) of the estimation error and the standard deviation of the MSE.

Table 9 summarizes the results of this example, presenting the value of ρ computed by Theorem 3.3, the MSE and SD_{MSE} (computed through MC simulation performed as described in Palma *et al.* (2018c)), associated with the synthesis of a robust filter (rob), an ideal gain-scheduled filter (lpv), three gain-scheduled filters with inexact parameters: two designed using the SDOH technique (lpv($\mathcal{S}(\mathsf{B}_{\max})$) and lpv($\mathcal{S}(\mathsf{B}_{3\sigma})$)) and the best result of the first case study (lpv$^{6}_{3\sigma}$).

For the sub-domains constructed with $\mathcal{S}(\mathsf{B}_{\max})$ and $\mathcal{S}(\mathsf{B}_{3\sigma})$ the filter performance is the same (identical filter matrices). This means that, to use the full domain of uncertainty does not imply introducing conservatism in the design, differently from what was observed in the previous case study (Table 8). The filters synthesized by the SDOH

Table 9 – Values of ρ, MSE and SD_{MSE} associated with the filters: rob, lpv($\mathcal{S}(\mathsf{B}_{\max})$), lpv($\mathcal{S}(\mathsf{B}_{3\sigma})$) and $\text{lpv}^6_{3\sigma}$ and the incremental ratio between those performance indexes and the one obtained by the ideal design lpv (MSE_{In}, $SD_{MSE_{In}}$).

Structure	lpv	lpv($\mathcal{S}(\mathsf{B}_{\max})$)	lpv($\mathcal{S}(\mathsf{B}_{3\sigma})$)	$\text{lpv}^6_{3\sigma}$	rob
ρ	0.3449	0.4184	0.4184	0.6858	1.2231
MSE	0.4004	0.6337	0.6337	1.5931	4.4159
MSE_{In}	-	1.5827	1.5827	3.9788	11.0287
SD_{MSE}	0.0519	0.1719	0.1719	0.5290	1.4541
$SD_{MSE_{In}}$	-	3.3121	3.3121	10.1927	28.0173

methodology yield, respectively, a MSE and an SD_{MSE} about 2.51 and 3.07 times lower than the results obtained by the best standard approach with additive uncertainty (filter $\text{lpv}_{3\sigma}$ with six-vertex in Table 8), and 6.96 and 8.45 better than the robust filter.

It is also possible to verify in Table 9 that the results employing the SDOH methodology are much closer to the ideal project (lpv) than the best standard approach with additive uncertainty ($\text{lpv}^6_{3\sigma}$) or the robust filter. While the MSE provided by the SDOH approach is only 1.58 times the MSE from ideal lpv design, the standard gain-scheduled and robust approaches are 3.97 and 11.02 times greater. Likewise, concerning SD_{MSE}, the SDOH method is 3.31 higher than the lpv, while the $\text{lpv}^6_{3\sigma}$ and rob are 10.19 and 28.01 higher than the value obtained by lpv.

3.8 Partial Conclusion

The first contribution of this chapter is a systematic and generalist modeling of LPV systems where the time-varying parameters can have polytopic or affine structure and present arbitrarily fast or bounded rates of variation. The scheduling parameters are assumed to be affected by additive uncertainties, which can be modeled in a traditional way (considering that they are independent) or in a less conservative manner (where the saturation is taken into account). In this sense, a control design example showed that if the saturation is considered, a finite and lower bound to the $\mathcal{H}_2$ norm is achieved, and when the effect of the saturation is neglected, the system may be not stabilized. Similarly, a filtering example illustrated that both the $\mathcal{H}_2$ cost of the augmented system and the associated time-dependent performance indexes (MSE and SD_{MSE}) are reduced by considering the saturation.

The second main contribution of the chapter is a novel design methodology (for gain-scheduled $\mathcal{H}_2$ filtering and state-feedback control), called SDOH, that relies on optimizing the performance criterion in a sub-domain associated with the higher frequency of probability distribution of the random time-varying parameters, while the stability of the closed-loop system (in case of control) or of the estimation error system (in case of filtering) is assured for the whole domain of uncertainty. In order to allow the employment of this technique, that is, to address gain-scheduled problems where statistical information about the time-varying parameters is available, this chapter has introduced the concept of sub-domain, proposing all the technical framework to handle this specificity and showing how the procedure can be numerically implemented using the available software. The control synthesis examples have shown that even in the context of robust (parameter-independent) design, the SDOH technique can provide an improvement of 10% to 50% (or even greater if the time-varying parameters are random variables of lower dispersion) with respect to the actual worst case $\mathcal{H}_2$ closed-loop norm. In the context of filters scheduled in terms of parameters contaminated by additive noise (inexact measurements or inherent identification error), besides providing a new and more general LMI-based design condition that includes another one from the literature as a particular case, the examples have shown that the employment of SDOH can provide a temporal behavior closer to the ideal LPV technique (perfect reading or estimation) than to the robust filter or the gain-scheduled filter using the techniques available in the literature.

3.9 General additive uncertainty modeling

The vertices of the polytope defined by the set $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ can be obtained systematically by a vertex enumeration algorithm (AVIS; FUKUDA, 1992). A numerical implementation that works with Matlab can be found in the Multi-Parametric Toolbox (HERCEG *et al.*, 2013) through the script `Polyhedron}`.The input parameters are matrix $\mathbf{A}$ and vector $\mathbf{b}$. In the next sections $\mathbf{A}$ and $\mathbf{b}$ are defined for affine and polytopic uncertainty, considering limited and arbitrarily fast rates of variation.

3.9.1 Affine uncertainty

In the case of arbitrarily fast variation of $\theta(k)$, there is no need to consider the limits for the rate of variation and the computation of the vertices of the polytope that defines the feasible region of the pair (θ_i, δ_i) is performed considering

$$x = \begin{bmatrix} \theta_i \\ \delta_i \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} \mathsf{B}_{i2} \\ -\mathsf{B}_{i1} \\ \overline{\delta}_i \\ -\underline{\delta}_i \\ \mathsf{B}_{i2} \\ -\mathsf{B}_{i1} \end{bmatrix}$$

If the parameters present bounded rates of variation as in (3.6), the constraints $\mathsf{B}_{i1} \leq \theta_i(k) \leq \mathsf{B}_{i2}$, $\mathsf{B}_{i1} \leq \theta_i(k+1) \leq \mathsf{B}_{i2}$, (3.6), (3.13), (3.14) and $\mathsf{B}_{i1} \leq \theta_i(k+1) + \delta_i(k) \leq \mathsf{B}_{i2}$

have to be considered, yielding the following definitions

$$x = \begin{bmatrix} \theta_i \\ \theta_i^+ \\ \delta_i \end{bmatrix}, \quad A = \begin{bmatrix} \tilde{A} \\ \hat{A} \\ \bar{A} \end{bmatrix}, \; \tilde{A} = \begin{bmatrix} I_3 \\ -I_3 \end{bmatrix}, \; \hat{A} = \begin{bmatrix} I_2 & \mathbb{1}_2 \\ -I_2 & -\mathbb{1}_2 \end{bmatrix}, \; \bar{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} \mathsf{B}_{i2} & \mathsf{B}_{i2} & \overline{\delta}_i & -\mathsf{B}_{i1} & -\mathsf{B}_{i1} & -\underline{\delta}_i & \mathsf{B}_{i2} & \mathsf{B}_{i2} & -\mathsf{B}_{i1} & -\mathsf{B}_{i1} & \overline{b}_i & -\underline{b}_i \end{bmatrix}'$$

where $\theta_i^+ = \theta(k+1)$ and $\mathbb{1}_r$ denotes a vector of ones with dimension r.

3.9.2 Polytopic Uncertainty

For arbitrarily fast rates of variation, the following constraints need to be considered: $0 \le \theta_i \le 1$, (3.13), and (3.15). Note that the unitary sum of polytopic parameters allows that the last parameter $\theta_N(k)$ be removed from the problem, although the additive uncertainty associated to δ_N still can be considered. In this case, the following constraints are included

$$0 \le 1 - \theta_1 - \cdots - \theta_{N-1} \le 1, \; 0 \le 1 - \theta_1 - \cdots - \theta_{N-1} + \delta_N \le 1 \tag{3.43}$$

As in this last set of constraints all parameters $\theta_i, \; i = 1, \ldots, N-1$ are present, it is not possible to build each pair (θ_i, δ_i) independently, as in the affine case. As a consequence all parameters must be determined simultaneously. Taking into account all the necessary constraints, one has

$$x = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{N-1} \\ \delta_1 \\ \vdots \\ \delta_N \end{bmatrix}, \; A = \begin{bmatrix} I_{2N-1} \\ -I_{2N-1} \\ \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbb{1}'_{N-1} & 0_2 \end{bmatrix} \\ \begin{bmatrix} I_{N-1} & 0_{N-1} \\ -I_{N-1} & 0_{N-1} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbb{1}'_{N-1} & \begin{matrix} 1 \\ -1 \end{matrix} \end{bmatrix} \end{bmatrix},$$

$$b = \begin{bmatrix} \mathbb{1}'_{N-1} & \overline{\delta}_1 & \cdots & \overline{\delta}_N & 0'_{N-1} & -\underline{\delta}_1 & \cdots & -\underline{\delta}_N & 0 & 1 & 0'_{2N-1} & 0 & 1 \end{bmatrix}'$$

Once the set of vertices that defines the polyhedral region is obtained, the coordinate θ_N of each vertex can be determined using $\theta_N = 1 - \theta_1 - \cdots - \theta_{N-1}$.

Bounded rates of variation can be taken into account by considering (3.7). Using the unitary sum, θ_N can be removed and it is considered (3.7) for $i = 1, \ldots, N-1$ and

$$\underline{b}_N \le 1 - \sum_{i=1}^{N-1} \theta_i(k+1) - (1 - \sum_{i=1}^{N-1} \theta_i(k)) \le \overline{b}_N \Rightarrow \underline{b}_N \le - \sum_{i=1}^{N-1} \theta_i(k+1) + \sum_{i=1}^{N-1} \theta_i(k) \le \overline{b}_N \tag{3.44}$$

Besides, it is also necessary to limit $\theta_i(k+1)+\delta_i$ by means of

$$0 \leq \alpha_i(k+1)+\delta_i \leq 1,\ i=1,\ldots,N-1,\ 0 \leq 1-\sum_{i=1}^{N-1}\alpha_i(k+1)+\delta_N \leq 1 \tag{3.45}$$

Taking into account all the constraints in the arbitrarily fast case plus (3.44), (3.45) and $0 \leq \theta_i(k+1) \leq 1$, one has the following definitions

$$x = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{N-1} \\ \alpha_1^+ \\ \vdots \\ \alpha_{N-1}^+ \\ \delta_1 \\ \vdots \\ \delta_N \end{bmatrix},\ A = \begin{bmatrix} I_{3N-2} \\ -I_{3N-2} \\ \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbb{1}'_{N-1} & \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \otimes \mathbb{1}'_{N-1} & 0_{4\times N} \end{bmatrix} \\ \begin{bmatrix} I_{2(N-1)} & \begin{matrix} I_{N-1} \\ I_{N-1} \end{matrix} & 0_{2(N-1)\times 1} \end{bmatrix} \\ \begin{bmatrix} -I_{2(N-1)} & \begin{matrix} -I_{N-1} \\ -I_{N-1} \end{matrix} & 0_{2(N-1)\times 1} \end{bmatrix} \\ \begin{bmatrix} \begin{matrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbb{1}'_{N-1} \\ \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \otimes \mathbb{1}'_{N-1} \end{matrix} & 0_{4\times N-1} & \mathbb{1}_2 \otimes \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{matrix} -I_{N-1} & I_{N-1} \\ I_{N-1} & -I_{N-1} \end{matrix} & 0_{2(N-1)\times N} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \mathbb{1}'_{N-1} & 0_{2\times N} \end{bmatrix} \end{bmatrix}$$

$$b = \begin{bmatrix} \hat{b} \\ \tilde{b} \\ \check{b} \end{bmatrix},\ \hat{b} = \begin{bmatrix} \mathbb{1}_{2(N-1)} \\ \overline{\delta}_1 \\ \vdots \\ \overline{\delta}_N \\ 0_{2(N-1)} \\ -\underline{\delta}_1 \\ \vdots \\ -\underline{\delta}_N \end{bmatrix},\ \tilde{b} = \begin{bmatrix} (\mathbb{1}_2 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\ \mathbb{1}_{2(N-1)} \\ 0_{2(N-1)} \\ (\mathbb{1}_2 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \end{bmatrix},\ \check{b} = \begin{bmatrix} \overline{b}_1 \\ \vdots \\ \overline{b}_{N-1} \\ -\underline{b}_1 \\ \vdots \\ -\underline{b}_{N-1} \\ \overline{b}_N \\ -\underline{b}_N \end{bmatrix}$$

4 LMI-based solution for memory static output-feedback control of discrete-time linear systems affected by time-varying parameters

This chapter proposes a solution for the problem of designing static output-feedback memory control laws for uncertain LTI systems, LPV systems and Markov jump linear systems with time-varying probabilities (non-homogeneous Markov chain). Differently from the LTI case, where the concept of eigenvalue, duality and congruence transformations are useful to construct convex synthesis conditions for memory controllers, the time-varying case poses a much more challenging scenario. To circumvent this problem, the inspiration is the approach given in Felipe *et al.* (2016), Felipe (2017), which has as main issue the fact that the control gain is treated as an optimization variable of the problem, and not retrieved *a posteriori* by means of a change of variables, as the main stream of methods for LPV and MJLS available in the literature. Therefore, as a unique feature, the proposed technique allows the imposition of structures (decentralization) or magnitude constraints in the control gain without restricting any other optimization variable of the problem. New relaxations are provided to cope with the particular structure of the closed-loop matrices due to the memory, yielding synthesis conditions formulated in terms of a locally convergent iterative procedure based on LMIs. As by-products, other important problems and particular cases as state-feedback or gain-scheduled control with or without memory applied to LTI, LPV, MJLS and NHMJLS (mode-dependent, mode-independent or considering partial information about the operation modes) can also be dealt with. The applicability and advantages of the proposed technique are illustrated by means of numerical examples borrowed from the literature and statistical comparisons demonstrating the low conservativeness of the proposed method (with or without memory) when compared with techniques from the literature in LTI, LPV, MJLS and NHMJLS scenarios.

4.1 LPV systems

Consider the following linear discrete-time system affected by time-varying parameters

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B(\alpha(k))u(k) \\ y(k) &= C_y(\alpha(k))x(k) \end{aligned} \tag{4.1}$$

where $x(k) \in \mathbb{R}^{n_x}$ represents the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, and $\alpha(k) = [\alpha_1(k), \ldots, \alpha_N(k)]'$ is a vector of bounded time-varying parameters, which lies in the unit simplex given by

$$\Lambda_N = \Big\{\xi \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0,\ i = 1, \ldots, N\Big\}, \tag{4.2}$$

for all $k \geq 0$. The rate of variation of the parameters is assumed arbitrarily fast, which includes the class of switched systems, but bounded rates of variation could be addressed as well following strategies available in the literature (DE CAIGNY *et al.*, 2010). The state-space matrices of system (4.1) are given as a convex combination of N known matrices (called vertices) as $M(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)M_i,\ \alpha(k) \in \Lambda_N$. The aim is to determine a scheduled (or robust, when parameter-independent) static output-feedback control law

$$u(k) = \sum_{\ell=0}^{m-1} K_\ell(\alpha(k))y(k-\ell) \tag{4.3}$$

that stabilizes system (4.1), where $K_\ell(\alpha(k)) \in \mathbb{R}^{n_u \times n_y}$, $\ell = 0, \ldots, m-1$ are the static-output feedback gains to be determined and $m \in \mathbb{N}^+$ is a given positive integer corresponding to the number of measured outputs that compose the control law (4.3). For simplicity, each gain K_ℓ depends only on the current value of $\alpha(k)$ and there is no need to store the past values of $\alpha(k-\ell)$, only the past measured output values $y(k-\ell)$. The choice $m = 1$ corresponds to the design of a standard gain-scheduled static output-feedback control law, that is, $u(k) = K_0(\alpha(k))y(k)$.

Applying the proposed control law (4.3) (with $m > 1$) in system (4.1), one obtains the closed-loop system given by

$$s(k+1) = \underbrace{\begin{bmatrix} \varphi_1 & \varphi_2 \\ B(\alpha_k)\mathcal{K} & A(\alpha_k) + B(\alpha_k)K_0(\alpha_k)C(\alpha_k) \end{bmatrix}}_{A_{cl}(\alpha(k))} s(k) \tag{4.4}$$

where[1]

$$s(k) = \Big[x(k-m+1)' \quad \cdots \quad x(k-1)' \quad x(k)'\Big]', \tag{4.5}$$

$$\mathcal{K} = \Big[K_{m-1}(\alpha_k)C_y(\alpha_{k-m+1}) \quad \cdots \quad K_1(\alpha_k)C_y(\alpha_{k-1})\Big] \tag{4.6}$$

$$\varphi_1 = \begin{bmatrix} 0 & I_{(m-2)n_x} \\ 0_{n_x} & 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ I_{n_x} \end{bmatrix}. \tag{4.7}$$

[1] Hereafter, the dependence on $\alpha(k-\ell)$ is replaced by $\alpha_{k-\ell}$.

Note that the past values of the outputs introduced in the closed-loop system (4.4) by the control law (4.3) provide an augmentation in the system dynamics, producing an equivalent system with greater dimensions and a particular structure that will be conveniently explored.

4.1.1 SOF Memory Control Design for LPV systems

Before presenting the proposed control design conditions for the LPV system (4.1), define:

$$P(\alpha_k) = \begin{bmatrix} P_1(\alpha_k) & \star \\ P_2(\alpha_k) & P_3(\alpha_k) \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} I_{(m-1)n_x} & 0 & 0 \\ 0 & I_{n_x} & 0 \\ \varphi_1 & \varphi_2 & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix},$$
$$\Theta_{11}(\bar{\alpha}_k) = \mathcal{N}' \begin{bmatrix} -P(\alpha_k) & 0 \\ 0 & P(\alpha_{k+1}) \end{bmatrix} \mathcal{N}, \quad \bar{\alpha}_k = (\alpha_k, \alpha_{k+1}),$$
$$\Theta_{21}(\alpha_k) = \begin{bmatrix} B(\alpha_k)\mathcal{K} & A(\alpha_k) + B(\alpha_k)K_0(\alpha_k)C(\alpha_k) & -I_{n_x} \end{bmatrix}, \tag{4.8}$$

where $P_1(\alpha_k) \in \mathbb{R}^{(m-1)n_x \times (m-1)n_x}$, $P_2(\alpha_k) \in \mathbb{R}^{n_x \times (m-1)n_x}$, and $P_3(\alpha_k) \in \mathbb{R}^{n_x \times n_x}$, and $\mathcal{K}$ is given by (4.6).

The first contribution of this chapter is a new sufficient parameter-dependent LMI condition for the synthesis of the gains $K_\ell(\alpha_k)$ associated to the control law (4.3).

Theorem 4.1. *For given matrices $Y_1(\bar{\alpha}_k) \in \mathbb{R}^{n_x \times mn_x}$, $Y_2(\bar{\alpha}_k)$ and $Y_3(\bar{\alpha}_k) \in \mathbb{R}^{n_x \times n_x}$, if there exist parameter-dependent matrices $P(\alpha_k) = P(\alpha_k)' > 0$ given in (4.8), $X_1(\bar{\alpha}_k) \in \mathbb{R}^{mn_x \times n_x}$, $X_2(\bar{\alpha}_k)$ and $X_3(\bar{\alpha}_k) \in \mathbb{R}^{n_x \times n_x}$, and $K_\ell(\alpha_k) \in \mathbb{R}^{n_u \times n_y}$ such that*

$$\mathcal{Q}(\bar{\alpha}_k) + He\Big(\mathcal{X}(\bar{\alpha}_k)\mathcal{B}(\bar{\alpha}_k)\Big) < 0, \tag{4.9}$$

is verified for all $\bar{\alpha}_k = (\alpha_k, \alpha_{k+1}) \in \Lambda_N \times \Lambda_N$, where

$$\mathcal{B}(\bar{\alpha}_k) = \begin{bmatrix} Y_1(\bar{\alpha}_k) & Y_2(\bar{\alpha}_k) & Y_3(\bar{\alpha}_k) \end{bmatrix},$$
$$\mathcal{X}(\bar{\alpha}_k) = \begin{bmatrix} X_1'(\bar{\alpha}_k) & X_2'(\bar{\alpha}_k) & X_3'(\bar{\alpha}_k) \end{bmatrix}',$$
$$\mathcal{Q}(\bar{\alpha}_k) = \begin{bmatrix} \Theta_{11}(\bar{\alpha}_k) & \star \\ \Theta_{21}(\alpha_k) & 0 \end{bmatrix}, \tag{4.10}$$

with $\Theta_{11}(\bar{\alpha}_k)$ and $\Theta_{21}(\alpha_k)$ given in (4.8), then $u(k)$ given in (4.3) is a stabilizing static output-feedback gain-scheduled control law for system (4.1).

Proof. Knowing that if (4.9) is satisfied, $Y_3(\bar{\alpha}_k)$ has full-rank and, therefore, $\mathcal{B}(\bar{\alpha}_k)$ can be rewritten as $\mathcal{B}(\bar{\alpha}_k) = Y_3(\bar{\alpha}_k)\begin{bmatrix} -F_1(\bar{\alpha}_k) & -F_2(\bar{\alpha}_k) & I_{n_x} \end{bmatrix}$. Then, pre- and post-multiplying (4.9) respectively by $\mathcal{B}^{\perp\prime}(\bar{\alpha}_k)$ and $\mathcal{B}^{\perp}(\bar{\alpha}_k)$ with $\mathcal{B}^{\perp\prime}(\bar{\alpha}_k) = \begin{bmatrix} I_{(m+1)n_x} & \mathcal{F}'(\bar{\alpha}_k) \end{bmatrix}$, where $\mathcal{F}(\bar{\alpha}_k) = \begin{bmatrix} F_1(\bar{\alpha}_k) & F_2(\bar{\alpha}_k) \end{bmatrix}$ one obtains

$$\Theta_{11}(\bar{\alpha}_k) + He\left(\mathcal{F}'(\bar{\alpha}_k)\Theta_{21}(\alpha_k)\right) < 0 \tag{4.11}$$

By multiplying (4.11) on the right by $\Theta_{21}^{\perp}(\alpha_k)$ and on the left by $\Theta_{21}^{\perp}{}'(\alpha_k)$, with

$$\Theta_{21}^{\perp}(\alpha_k) = \begin{bmatrix} I_{(m-1)n_x} & 0 \\ 0 & I_{n_x} \\ B(\alpha_k)\mathcal{K} & A(\alpha_k) + B(\alpha_k)K_0(\alpha_k)C_y(\alpha_k) \end{bmatrix} \tag{4.12}$$

and $A_{cl}(\alpha_k)$ given in (4.4), one has

$$\begin{aligned} \Theta_{21}^{\perp}{}'(\alpha_k)\mathcal{N}' \begin{bmatrix} -P(\alpha_k) & 0 \\ 0 & P(\alpha_{k+1}) \end{bmatrix} \mathcal{N}\Theta_{21}^{\perp}(\alpha_k) \\ = \begin{bmatrix} I_{mn_x} & A_{cl}'(\alpha_k) \end{bmatrix} \begin{bmatrix} -P(\alpha_k) & 0 \\ 0 & P(\alpha_{k+1}) \end{bmatrix} \begin{bmatrix} I_{mn_x} \\ A_{cl}(\alpha_k) \end{bmatrix} \\ = A_{cl}'(\alpha_k)P(\alpha_{k+1})A_{cl}(\alpha_k) - P(\alpha_k) < 0 \end{aligned} \tag{4.13}$$

that assures the asymptotic stability of the closed-loop system (4.4) and, consequently, of system (4.1) under the control law (4.3). □

Although the inequality in (4.9) is linear with respect to the decision variables, the fact that matrices $Y_{1,2,3}(\bar{\alpha}_k)$ are fixed imposes a great level of conservativeness. To circumvent this problem, a relaxation is introduced. Unfortunately, the relaxation proposed in Felipe *et al.* (2016) (for continuous-time systems) is not useful because it only works for time-invariant parameters, where the concept of eigenvalue is representative. A new relaxation is proposed in the sequence, capable to cope with both LPV and NHMJLS. First, considering that $A_{cl}(\alpha_k)$ and $P(\alpha_k)$ can also be partitioned as

$$A_{cl}(\alpha_k) = \begin{bmatrix} 0 & I_{(m-1)n_x} \\ H_1 & H_2 \end{bmatrix}, \ P(\alpha_k) = \begin{bmatrix} \hat{P}_1(\alpha_k) & \star \\ \hat{P}_2(\alpha_k) & \hat{P}_3(\alpha_k) \end{bmatrix}, \tag{4.14}$$

with $\hat{P}_1(\alpha_k) \in \mathbb{R}^{n_x \times n_x}$, $\hat{P}_2(\alpha_k) \in \mathbb{R}^{(m-1)n_x \times n_x}$, $\hat{P}_3(\alpha_k) \in \mathbb{R}^{(m-1)n_x \times (m-1)n_x}$, the closed-loop stability condition (4.13) can be rewritten as

$$\begin{bmatrix} H_1'\hat{P}_3(\alpha_{k+1})H_1 & \star \\ \begin{pmatrix} \hat{P}_2'(\alpha_{k+1})H_1 \\ +H_2'\hat{P}_3(\alpha_{k+1})H_1 \end{pmatrix} & \begin{pmatrix} \hat{P}_1(\alpha_{k+1}) \\ +\text{He}(H_2'\hat{P}_2(\alpha_{k+1})) \\ +H_2'\hat{P}_3(\alpha_{k+1})H_2 \end{pmatrix} \end{bmatrix} - \begin{bmatrix} \hat{P}_1(\alpha_k) & \star \\ \hat{P}_2(\alpha_k) & \hat{P}_3(\alpha_k) \end{bmatrix} < 0 \tag{4.15}$$

or

$$M_1 < M_3 - M_2 \tag{4.16}$$

where

$$M_1 = \begin{bmatrix} H_1' & 0 \\ 0 & H_2' \end{bmatrix} \begin{bmatrix} \hat{P}_3(\alpha_{k+1}) & \hat{P}_3(\alpha_{k+1}) \\ \hat{P}_3(\alpha_{k+1}) & \hat{P}_3(\alpha_{k+1}) \end{bmatrix} \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 0 & \star \\ \hat{P}_2'(\alpha_{k+1})H_1 & \mathrm{He}(H_2'\hat{P}_2(\alpha_{k+1})) \end{bmatrix}, \tag{4.17}$$
$$M_3 = \begin{bmatrix} \hat{P}_1(\alpha_k) & \star \\ \hat{P}_2(\alpha_k) & \hat{P}_3(\alpha_k) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \hat{P}_1(\alpha_{k+1}) \end{bmatrix}.$$

The proposed relaxation consists in solving (4.9) with $A_{cl}(\alpha_k)$ given in (4.14) replaced by

$$A_{cl}(\alpha_k) = \begin{bmatrix} 0 & I_{(m-1)n_x} \\ H_1/\rho & H_2/\rho \end{bmatrix}, \ \rho > 0$$

and including the additional constraint $M_3 > 0$. In the case of a feasible solution, (4.13) leads to (instead of (4.16))

$$M_1 < \rho^2 M_3 - \rho M_2. \tag{4.18}$$

The first benefit of the relaxation is that the relaxation level ρ allows to construct initial choices for $Y_{1,2,3}(\bar{\alpha})$ that guarantee a feasible solution for Theorem 4.1, as demonstrated in Theorem 4.2 presented in the sequence. Moreover, another important property can be demonstrated: if there is a feasible solution for $\rho = \bar{\rho}$, (4.18) remains feasible for any $\rho > \bar{\rho}$. First observe that $M_1 \geq 0$ by construction. Thus, if

$$M_1 < \bar{\rho}^2 M_3 - \bar{\rho} M_2 \tag{4.19}$$

is feasible, then

$$\bar{\rho}^2 M_3 - \bar{\rho} M_2 > 0 \Rightarrow \bar{\rho} M_3 > M_2. \tag{4.20}$$

To guarantee that (4.19) remains feasible for values greater than $\bar{\rho}$, it is necessary that

$$(\bar{\rho}+\epsilon)^2 M_3 - (\bar{\rho}+\epsilon)M_2 > \bar{\rho}^2 M_3 - \bar{\rho} M_2, \ \forall \epsilon > 0$$

which is simplified to

$$\bar{\rho} M_3 + (\bar{\rho}+\epsilon) M_3 > M_2$$

Finally, as (4.20) holds, by summing up a positive term (since $\bar{\rho}$, ϵ and M_3 are positive definite), one has that last inequality is always feasible for any positive ϵ.

Also note that, for $\rho = 1$, to solve the relaxed condition (4.19) is the same as satisfying the original stability condition (4.16), meaning that the closed-loop LPV system is stable. As a consequence of the mentioned facts, if the conditions of Theorem 4.1 are tested with the proposed relaxation and a solution with $\rho \leq 1$ is found, the resulting controller is stabilizing. Second and more important, it is possible to apply the changes

$$\hat{A}(\alpha_k) = A(\alpha_k)/\rho \quad \text{and} \quad \hat{K}_\ell(\alpha_k) = K_\ell(\alpha_k)/\rho, \tag{4.21}$$

and condition (4.9) can be solved with $\hat{A}(\alpha_k)$ and $\hat{K}_\ell(\alpha_k)$ considering ρ as an objective function, since ρ appears linearly in (4.9). Regarding the additional constraint $M_3 > 0$, in principle it does not seem to be hard and its only purpose is to guarantee that if a solution with $\rho \leq 1$ is found, the controller is actually stabilizing. An alternative would be not to considering $M_3 > 0$ (probably sparing some computational effort) and apply a stability analysis condition, *a posteriori*, in the closed-loop dynamic matrix to assure the robust stability.

Before presenting an algorithm that exploits the minimization of ρ in Theorem 4.1, the following result proposes a particular choice for the initial conditions $Y_{1,2,3}(\bar{\alpha}_k)$ that assures the existence of a feasible solution for (4.9) using the proposed relaxation in terms of ρ.

Theorem 4.2. *The choice* $\mathcal{B}(\bar{\alpha}_k) = \mathcal{B}_0 = \begin{bmatrix} 0 & I_{n_x} & -I_{n_x} \end{bmatrix}$ *assures the existence of a feasible solution for Theorem 4.1 with* $A(\alpha_k)$ *and* $K_\ell(\alpha_k)$ *replaced respectively by* $\hat{A}(\alpha_k)$ *and* $\hat{K}_\ell(\alpha_k)$, $\forall \ell = 0, \ldots, m-1$ *as in* (4.21).

Proof. First, consider the following partitions for the blocks $P_1(\alpha_k)$ and $P_2(\alpha_k)$ of $P(\alpha_k)$ from (4.8)

$$P_1(\alpha_k) = \begin{bmatrix} P_{11}(\alpha_k) & P_{12}'(\alpha_k) \\ P_{12}(\alpha_k) & P_{13}(\alpha_k) \end{bmatrix}, \; P_2(\alpha_k) = \begin{bmatrix} P_{21}(\alpha_k) & P_{22}(\alpha_k) \end{bmatrix},$$

meaning that $\Theta_{11}(\bar{\alpha}_k)$ can be rewritten as

$$\Theta_{11}(\bar{\alpha}_k) = \begin{bmatrix} -P_{11}(\alpha_k) & \star & \star & \star \\ P_{12}(\alpha_k) & P_{11}(\alpha_{k+1}) - P_{13}(\alpha_k) & \star & \star \\ -P_{21}(\alpha_k) & P_{12}(\alpha_{k+1}) - P_{22}(\alpha_k) & P_{13}(\alpha_{k+1}) - P_3(\alpha_k) & \star \\ 0 & P_{21}(\alpha_{k+1}) & P_{22}(\alpha_{k+1}) & P_3(\alpha_{k+1}) \end{bmatrix}.$$

Then, for $\mathcal{B}(\bar{\alpha}_k) = \mathcal{B}_0$, the following choices for the variables of Theorem 4.1 such that (4.9) is always verified can be done: a large enough ρ such that $\hat{A}(\alpha_k) = 0$ and $\hat{K}_\ell(\alpha_k) = 0$, $\mathcal{X}(\bar{\alpha}_k) = -0.5\mathcal{B}_0'$,

$$\begin{aligned} P_{12}(\alpha_k) &= P_{12}(\alpha_{k+1}) = 0, & P_{11}(\alpha_k) &= P_{11}(\alpha_{k+1}) = \epsilon_1 I, \\ P_{21}(\alpha_k) &= P_{21}(\alpha_{k+1}) = 0, & P_{13}(\alpha_k) &= P_{13}(\alpha_{k+1}) = \epsilon_2 I, \\ P_{22}(\alpha_k) &= P_{22}(\alpha_{k+1}) = 0, & P_3(\alpha_k) &= P_3(\alpha_{k+1}) = \epsilon_3 I, \end{aligned}$$

with $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1$. By plugging the above values in (4.9) with $A(\alpha_k)$ and $K_\ell(\alpha_k)$ replaced respectively by $\hat{A}(\alpha_k) = 0$ and $\hat{K}_\ell(\alpha_k) = 0$, one has

$$\begin{bmatrix} -\epsilon_1 I & 0 & 0 & 0 & 0 \\ 0 & (\epsilon_1 - \epsilon_2)I & 0 & 0 & 0 \\ 0 & 0 & (\epsilon_2 - \epsilon_3)I & 0 & 0 \\ 0 & 0 & 0 & (\epsilon_3 - 1)I & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{4.22}$$

that is always verified for $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1$. □

With the result of Theorem 4.2, Theorem 4.1 initialized with $\mathcal{B}(\bar{\alpha}_k) = \mathcal{B}_0$ always provides a feasible solution with a finite value for ρ. If $\rho \leq 1$, the synthesized gain is robustly stabilizing. On the other hand, if $\rho > 1$, nothing can be concluded. Next theorem provides the the last ingredient necessary to establish an iterative procedure using the conditions of Theorem 4.1.

Theorem 4.3. *Suppose that the conditions of Theorem 4.1 are feasible, providing $P_f(\alpha_k)$, $\mathcal{X}_f(\bar{\alpha}_k)$ and $\bar{\rho}_f$ as solution. Then, a new feasible solution with $\rho \leq \bar{\rho}_f$ is always obtained by choosing $\mathcal{B}(\bar{\alpha}_k) = \mathcal{X}_f(\bar{\alpha}_k)'$.*

Proof. Straightforward, since $He\Big(\mathcal{X}(\bar{\alpha}_k)\mathcal{B}(\bar{\alpha}_k)\Big) = He\Big(\mathcal{B}(\bar{\alpha}_k)'\mathcal{X}(\bar{\alpha}_k)'\Big)$ in (4.9). The update $\mathcal{X}(\bar{\alpha}_k) = \mathcal{B}(\bar{\alpha}_k)'$ always provides a feasible solution if Theorem 4.1 is tested again and the new value of ρ cannot be greater than the previous one. □

Based on the above results, a locally convergent iterative algorithm is proposed (see Algorithm 1), being the main contribution of the chapter. For a maximum number of iterations it_{max} and an initial condition $\mathcal{B}(\bar{\alpha}_k) = \mathcal{B}_0$, minimize ρ subject to (4.9) using the replacements indicated in (4.21). Update $\mathcal{B}(\bar{\alpha}_k) = \mathcal{X}'(\bar{\alpha}_k)$ at each iteration, while $\rho > 1$ and the number of iterations $it < it_{max}$. If $\rho \leq 1$, the asymptotic stability of $A_{cl}(\alpha_k)$ is assured, and the iterative procedure is interrupted providing the control gain matrices $K_\ell(\alpha_k) = \rho\hat{K}_\ell(\alpha_k)$. If $\rho > 1$, then a stabilizing gain could not be obtained.

As a final important property, it is shown that the increase of the memory size cannot produce worse (higher) values of ρ if a proper initialization of $\mathcal{B}(\bar{\alpha}_k)$ is adopted. Note that condition (4.9) considering m memories can be rewritten as

$$\begin{bmatrix} Q_{11}(\bar{\alpha}_k) & \star \\ Q_{21}(\bar{\alpha}_k) & \begin{bmatrix} Q_{\tilde{m}}(\bar{\alpha}_k) \end{bmatrix} \end{bmatrix} + He\left(\begin{bmatrix} X(\bar{\alpha}_k) \\ \begin{bmatrix} \mathcal{X}_{\tilde{m}}(\bar{\alpha}_k) \end{bmatrix} \end{bmatrix} \begin{bmatrix} B(\bar{\alpha}_k) & \begin{bmatrix} \mathcal{B}_{\tilde{m}}(\bar{\alpha}_k) \end{bmatrix} \end{bmatrix} \right) < 0, \tag{4.23}$$

where $Q_{\tilde{m}}(\bar{\alpha}_k)$, $\mathcal{X}_{\tilde{m}}(\bar{\alpha}_k)$ and $\mathcal{B}_{\tilde{m}}(\bar{\alpha}_k)$ are the solutions for Theorem 4.1 with $\tilde{m} < m$ memories. Observe that, if $Q_{\tilde{m}}(\bar{\alpha}_k)$, $\mathcal{X}_{\tilde{m}}(\bar{\alpha}_k)$ and $\mathcal{B}_{\tilde{m}}(\bar{\alpha}_k)$ are solutions for Theorem 4.1, (4.23) can always be satisfied with $Q_{11}(\bar{\alpha}_k) = -\epsilon I$, $\epsilon > 0$, $Q_{21}(\bar{\alpha}_k) = 0$, $X(\bar{\alpha}_k) = 0$, $B(\bar{\alpha}_k) = 0$, meaning that another valid initial condition for Theorem 4.2 with m memories is given by $\mathcal{B}(\bar{\alpha}_k) = [0 \ \mathcal{B}_{\tilde{m}}(\bar{\alpha}_k)]$. Furthermore, this particular choice assures that if Theorem 4.2

has a solution with $\tilde{m}$ memories, the same holds for $m > \tilde{m}$.

```
funtion Input((A, B, C_y)(ᾱ_k), it_max, B_0(ᾱ_k))
B(ᾱ_k) = B_0(ᾱ_k)
A(α_k) = A(α_k)/ρ
while it < it_max do
    Minimize ρ subject to
      Theorem 4.1 and M_3 > 0 (Eq. (4.16))
    if ρ > 1 then
        B(ᾱ_k) = X'(ᾱ_k)
    else
        K_ℓ(α_k) = ρK_ℓ(α_k)
        return
    end
end
```

Algorithm 1: SOF control of LPV system.

4.1.2 Numerical Implementation

The synthesis conditions of Theorem 4.1 are given in terms of parameter-dependent LMIs because, currently, a finite set of LMIs (programmable test) assuring the feasibility of the parameter-dependent LMIs can be automatically obtained by means of polynomial approximations for the optimization variables as well as by relaxations for the polynomial positivity test. Particularly for parameter-dependent LMIs with parameters lying in the unit simplex, the parser ROLMIP (Robust LMI Parser) (AGULHARI *et al.*, 2019) that works jointly with YALMIP (LÖFBERG, 2004) and the semi-definite programming solvers Mosek (MOSEK ApS, 2015) and SeDuMi (STURM, 1999), can be used, and the only task of the user is to choose the polynomial degrees of the variables. The following choices have been made in all examples presented in the next sections: $P(\alpha_k)$ is affine (degree one) on α_k; $X_i(\bar{\alpha}_k)$ is affine on both α_k and α_{k+1} (i.e., multi-affine); $K(\alpha_k)$ is independent (degree zero) of α_k when designing robust controllers or affine on α_k when synthesizing gain-scheduled controllers. The reason for these choices is that the conditions from the literature also employ affine structures for the optimization variables.

4.1.3 Numerical examples

This section presents numerical examples to illustrate the applicability of the proposed method to cope with robust and gain-scheduled stabilization of LTI and LPV systems.

4.1.3.1 Statistical comparisons of the stabilization methods for polytopic LTI and LPV systems

The aim of this example is to statistically compare the effectiveness of some output-feedback control design techniques from the literature with the method given in this paper for both LTI and LPV polytopic systems.

To this end, numerical stabilization tests were performed using a database of time-invariant polytopic systems composed by open loop unstable systems, proposed by Morais *et al.* (2013b), which are guaranteed to be stabilized by some robust (parameter-independent) state-feedback gain but that are not quadratically stabilized. The proposed analysis, adapted to deal exclusively with output-feedback, follows the experiment presented in Rosa *et al.* (2018), considering 100 systems for each combination of $n_x \in \{2,3\}$ states and $N \in \{2,\dots,5\}$ vertices. Two cases are investigated: **Case 1:** Robust static output-feedback (SOF) stabilization considering the systems as time-invariant; **Case 2:** Gain-scheduled (affine dependence on the scheduling parameters) SOF stabilization considering the parameters as time-varying with arbitrarily fast rates of variation. It was shown in Rosa *et al.* (2018) that, concerning Case 1, Corollary 1 from Morais *et al.* (2013b) (tested with nineteen values of ξ equally spaced in the interval $[-0.9, 0.9]$) and Theorem 3 from de Oliveira *et al.* (1999) provide a lower feasibility rate than Theorem 1 from Rosa *et al.* (2018) with $\gamma = -10^5$, $\xi = \{-0.2, -0.1, 0, 0.1, 0.2\}$ (RT), which is slightly more conservative than the heuristic technique based on pole location using Algorithm 1 from Rosa *et al.* (2018) with $\rho = \{1.05,\ 1.1\}$ and degP=1 (RC). Similarly, the results reported in Rosa *et al.* (2018) have shown that the condition adapted from Equation (49) of De Caigny *et al.* (2010) (by eliminating the last column and row) for gain-scheduled output-feedback control is less efficient than (RT) and (RC).

Aiming to evaluate the benefits of control laws with memory, the technique proposed in this paper (A1) with $it_{\max} = 10$ and the one developed in Frezzatto *et al.* (2018) (FOP), considering nineteen values of λ equally spaced in the interval $[-0.9, 0.9]$, are compared only with the methods (RT) and (RC), which have provided the best results for this database of systems in the time-invariant case.

Table 10 shows the results for Case 1, regarding robust SOF stabilization of polytopic LTI systems. The first important observation is that Algorithm 1 outperformed all the other methods without resorting to a memory control law (feasibility rate of 99.3%). When adopting memory, note that both FOP and A1 provided better results when including past outputs in the control law, which can be considered an interesting alternative, since only a slightly larger buffer (when compared with the case without memory) and a few extra arithmetic operations are required in terms of implementation. Additionally, even requiring fewer tests (a maximum of 10 iterations imposed to Algorithm 1 (A1) against 19 values of scalar search used by FOP), the proposed technique is the less con-

Table 10 – Robust SOF stabilization results for polytopic LTI systems.

n_x	N	**RT**	**RC**	FOP(m)				A1(m)	
				(1)	(2)	(3)	(4)	(1)	(2)
2	2	55	68	55	63	66	67	99	100
	3	57	74	62	68	70	70	100	100
	4	69	76	68	74	75	76	99	99
	5	69	85	64	72	72	72	99	99
3	2	42	71	57	67	70	70	98	100
	3	47	66	66	71	74	74	99	100
	4	50	75	75	78	78	78	100	100
	5	57	77	72	77	77	78	100	100
Total (%)		55.8	74	64.9	71.3	72.8	73.1	99.3	99.8

Table 11 – Gain-scheduled SOF stabilization results for polytopic LPV systems.

n_x	N	**RT**	**RC**	A1(m)	
				(1)	(2)
2	2	14	18	31	35
	3	21	28	42	53
	4	17	19	38	53
	5	25	28	38	46
3	2	3	10	19	24
	3	4	9	25	35
	4	8	16	39	47
	5	10	17	30	39
Total (%)		12.8	18.1	32.8	41.5

servative one, outperforming FOP by more than 25%.

Before commenting Case 2, note that there is no guarantee of existence of feasible controllers when considering the parameters as time-varying because the database was created only to deal with time-invariant systems. However, the fact of considering gain-scheduled gains increases the chances of finding feasible controllers. Regarding the results, Table 11 shows that the proposed method stabilizes more than twice the number of systems stabilized by RC. The advantage of using memory can also be observed.

4.1.3.2 Comparison with Frezzatto *et al.* (2018) and magnitude constraint on the gain entries

An important feature of the proposed method is the capability of imposing magnitude constraints in the gain without restricting other variables of the problem, differently from most of available methods, such as the one from Frezzatto *et al.* (2018) where both matrices of gain recovery (F and Z_ℓ, $\ell = 0, \ldots, m-1$) are affected. To evaluate how this property affects the conservativeness of the methods, consider again the system given in Example 1 from Frezzatto *et al.* (2018) and the problem of establishing the minimum magnitude (M) that can be imposed to the gains $|K_\ell| < M$, $\ell = 1, \ldots, m-1$, and still obtain feasible solutions. The results are presented in Fig. 18.

Note that the magnitude constraint could be imposed in the method from Frezzatto *et al.* (2018) just because in this example the control gain is a number (SISO system). If the gain is a vector or a matrix, all the synthesis conditions available in the literature that computes the gain *a posteriori* with a change of variables are not suitable to cope with magnitude constraints, differently from the condition proposed in this chapter, where the control gain is an optimization variable.

4.1.3.3 Structural constraints

Consider a linearized dynamic model of a VTOL (Vertical Take-off and Landing) helicopter adapted from Keel *et al.* (1988). The states $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and the control inputs $u_1(t)$, $u_2(t)$ represent, respectively, the horizontal and vertical velocities (knots), the pitch rate (degree/s), pitch angle (degrees), the collective pitch control and the longitudinal cyclic pitch control, such that the continuous-time state-space matrices $(\bar{A}, \bar{B}, \bar{C})$ are given by

$$\bar{A}=\begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & p \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}, \quad \bar{B}=\begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0.0000 & 0.0000 \end{bmatrix},$$
$$\bar{C}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{4.24}$$

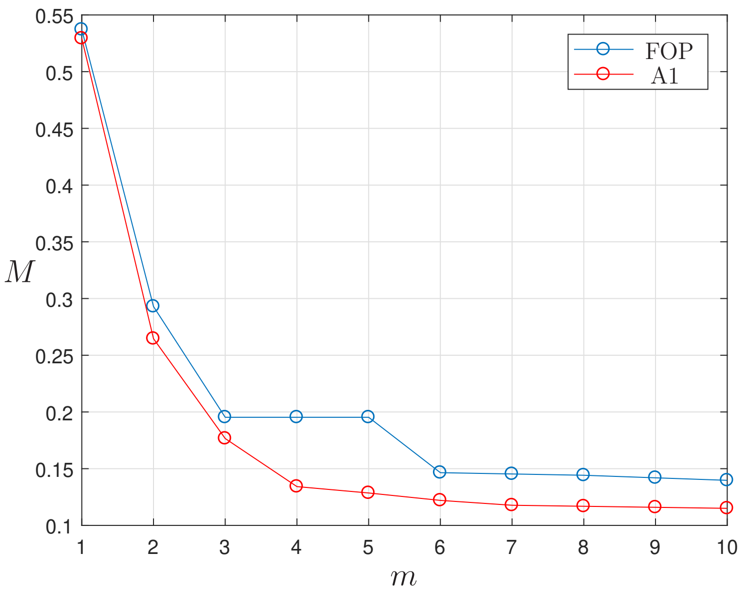


Figure 18 – Minimum magnitude (M) to obtain gains $|K_i| < M$, $i = 1, \ldots, m-1$ employing the methods from Frezzatto *et al.* (2018) (FOP) and the one proposed here (A1).

where $p = 1.4200 \pm \Delta$. The system is discretized using a first order Taylor expansion approximation associated to a zero order hold method, meaning that the discrete-time matrices (A, B, C) are given by $A = \mathrm{I} + \bar{A}T$, $B = T\bar{B}$, and $C = \bar{C}$, where $T = 1s$ is the sampling period.

The aim of this example is to verify the maximum value of Δ such that the memory control conditions from Frezzatto *et al.* (2018) (FOP) and Algorithm 1 are capable of stabilizing the discretized model by introducing past values of the states. The following situations are investigated: synthesis of state-feedback (SF) gains (full or structured) or static-output feedback (SOF) gains considering that Δ is either time-invariant (only robust case is considered) or time-varying (gain-scheduled and robust cases are evaluated) interval uncertainty. The results for full gains are presented in Table 12.

Table 12 – Maximum value of Δ such that the method from Frezzatto *et al.* (2018) (FOP) and Algorithm (A1) with $it_{\max} = 13$ can provide a stabilizing robust memory control law.

Method		m	1	2	3	4	5
LTI	SF	FOP	1.000	1.000	1.299	1.414	1.414
		A1	1.042	1.430	1.871	1.927	1.961
	SOF	FOP	–	–	–	–	–
		A1	–	0.028	0.282	0.382	0.434
LPV	SF	$A1_{rob}$	1.000	1.000	1.000	1.000	1.000
	SOF	$A1_{rob}$	–	0.015	0.200	0.262	0.271
		$A1_{sch}$	–	0.033	0.472	0.489	0.993

Observe that the increase of m allows to attain less conservative solutions in both approaches. The method FOP, capable to deal with only time-invariant systems, in this example provided stabilizing solutions only in SF control. However, even in this particular case, the range of Δ associated with feasible solutions is from 1% to 40% lower than the feasible range yielded by the proposed method. As expected, a loss of performance is observed in SOF control. However, when the values of the time-varying parameter are available for feedback purposes, the gain-scheduled approach can produce solutions for larger ranges of Δ.

In the second part of this example, the challenging problem of designing a structured gain is investigated. Consider the following decentralized gain

$$K_\ell = \begin{bmatrix} k_{11}^\ell & k_{12}^\ell & 0 & 0 \\ 0 & 0 & k_{23}^\ell & k_{24}^\ell \end{bmatrix}$$

for the discretized model of (4.24) assuming p as time-invariant. The method FOP fails in providing a feasible solution when $m = 1, \ldots, 5$, while the proposed technique is able to provide stabilizing controllers for different ranges of Δ according with the chosen number of memories. For instance, by fixing $\Delta = 1.2$, Algorithm 1 can only stabilize the discrete-time model by employing $m \geq 2$. For instance, considering $m = 2$, one obtains the following decentralized gains

$$K_0 = \begin{bmatrix} -0.0298 & 0.0709 & 0 & 0 \\ 0 & 0 & -0.3035 & -0.5141 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.0369 & -0.0360 & 0 & 0 \\ 0 & 0 & -0.0430 & -0.1315 \end{bmatrix}.$$

To conclude, consider the problem of determining the minimum value of M, $|k_{ij}^{\ell}| \leq M$, such that the system can be stabilized using the same decentralized structure. This type of constraint is very difficulty to cope with using the LMI-based strategies from the literature for multiple-input multiple-output systems. However, using the proposed approach, the magnitude constraints are easily treated by simply including 8 additional linear constraints in the optimization problem. Applying Algorithm 1 to solve the problem, $M_{min} = 0.2237$ has been obtained ($it_{\max} = 13$) with the following gains

$$K_0 = \begin{bmatrix} -0.2116 & 0.0814 & 0 & 0 \\ 0 & 0 & -0.2236 & -0.2236 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.2236 & -0.0309 & 0 & 0 \\ 0 & 0 & -0.2236 & -0.2236 \end{bmatrix}.$$

4.2 Non-Homogeneous MJLS

Consider the discrete-time MJLS given by

$$\mathcal{G} = \begin{cases} x(k+1) = A_{\theta_k}(\beta_k)x(k) + B_{\theta_k}(\beta_k)u(k) \\ y(k) = C_y(\beta_k)x(k) \end{cases} \tag{4.25}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, and $y(k) \in \mathbb{R}^{n_z}$ is the measured output. The process $\{\theta_k;\ k \geq 0\}$ is described by a discrete-time Markov chain with finite state-space $\mathbb{K} = \{1, \ldots, \sigma\}$ associated with a transition probability matrix $\Gamma(\alpha_k) = [p_{ij}(\alpha_k)]$, $\forall i, j \in \mathbb{K}$, whose elements are given by

$$p_{ij}(\alpha_k) = Pr\left(\theta_{k+1} = j \mid \theta_k = i\right), \quad \forall k \geq 0,$$

satisfying $p_{ij}(\alpha_k) \geq 0$ and $\sum_{j=1}^{\sigma} p_{ij}(\alpha_k) = 1$. When $\Gamma(\alpha_k) = \Gamma(\alpha), \forall k \geq 0$ (i.e., time-invariant), the Markov chain is *homogeneous*, otherwise, the Markov chain is called *non-homogeneous* (i.e., time-varying probabilities) (IOSIFESCU, 1980). Aiming a stabilizing method as general as possible, each element $p_{ij}(\alpha_k)$ can vary between two known bounds

($0 \leq \underline{p}_{ij} \leq p_{ij}(\alpha_k) \leq \overline{p}_{ij} \leq 1$), or be completely unknown ($p_{ij}(\alpha_k) =?$), or the time-varying transition probability matrix $\Gamma(\alpha_k)$ can be considered polytopic (been expressed as a convex combination of known vertices). Hence, to systematically describe $\Gamma(\alpha_k)$, each one of its u uncertain rows is considered composed by time-varying parameters $\alpha_r(k) = [\alpha_{r1}(k), \ldots, \alpha_{rN_r}(k)]'$ belonging to a distinct unit simplex Λ_{N_r}, $r = 1, \ldots, u$, given in (4.2) with $N = N_r$, by simply imposing that all elements of a row must sum up to one, that is, $\sum_{j\in\mathbb{K}} p_{ij}(\alpha_k) = 1$, similarly to what was done in Morais *et al.* (2013a) for the homogeneous case ($p_{ij}(\alpha_k) = p_{ij}(\alpha), \forall k \geq 0$). Then, the time-varying parameters associated to the partly unknown rows of $\Gamma(\alpha_k)$ are grouped into one single domain, given by the Cartesian product of u unit simplexes $\Lambda_N = \Lambda_{N_1} \times \cdots \times \Lambda_{N_u}$, named as multi-simplex (OLIVEIRA *et al.*, 2008).

The state space matrices of (4.25) depend upon the discrete-time non-homogeneous Markov chain $\{\theta_k;\ k \geq 0\}$ and on the time-varying parameter $\beta_k = [\beta_{k_1}, \ldots, \beta_{k_L}]'$ belonging to a new unit simplex Λ_L of dimension L. For conciseness, whenever $\theta(k) = i$, $\forall i \in \mathbb{K}$, the notation $\big(A_{\theta(k)}, B_{\theta(k)}\big)(\beta_k) = \big(A_i, B_i\big)(\beta_k)$ is used.[2]

It is important to emphasize that the concepts of stability and norm computation for MJLS with non-homogeneous Markov chain are distinct from the homogeneous case, that are derived from the second moment stability (SMS) concept, due to the arbitrary variation of the transition probabilities. As a consequence, system (4.25) is said to be exponentially stable in the mean square sense with conditioning of type I (ESMS-CI) (see Aberkane (2011a), Aberkane (2013) and Page 68 Definition 3.1(c) from Dragan *et al.* (2010) for further details about this definition).

Before presenting the main results of this paper for non-homogeneous MJLS, an extension of the stability condition for discrete-time non-homogeneous MJLS with transition probabilities affected by arbitrarily fast time-varying parameters with polytopic structure is presented in the next lemma (ABERKANE, 2011a; ABERKANE, 2013).

Lemma 4.1. *System (4.25) with $u(k) \equiv 0$ is ESMS-CI if and only if there exist symmetric positive definite matrices $P_i(\alpha_k, \beta_k)$, such that the parameter-dependent inequalities*

$$A_i(\beta_k)' P_{pi}(\alpha_{k+1}, \beta_{k+1}) A_i(\beta_k) - P_i(\alpha_k, \beta_k) < 0 \qquad (4.26)$$

hold for each $i \in \mathbb{K}$ and for all $(\beta_k, \alpha_k) \in \Lambda_L \times \Lambda_N$, $\forall k \geq 0$, being $P_{pi}(\alpha_k, \beta_k) = \sum_{j=1}^{\sigma} p_{ij}(\alpha_k) P_j(\alpha_k, \beta_k)$.

For design purposes, it is also considered that the Markov chain may or may not be completely accessible, meaning that the σ operation modes can be divided in $\sigma_c \leq \sigma$ disjoint groups (clusters) whose union generates the set $\mathbb{K}$, i.e., $\mathbb{K} = \cup_{q\in\mathbb{Q}} \mathbb{U}_q$ such that $\cap_{q\in\mathbb{Q}} \mathbb{U}_q$ with indexes $q \in \mathbb{Q} = \{1, 2, \ldots, \sigma_c\}$. Additionally, the controller to be

[2] Note that (4.25) is suitable to model problems of *local sensor - remote actuator* (LSRA) (AMORIM

designed can be considered robust (parameter-independent) or gain-scheduled in terms of the time-varying parameters β_k and probabilities α_k (assuming that they can be measured or estimated in real-time), such that the control law is given by

$$u(k) = \sum_{\ell=0}^{m-1} K_\ell^q(\omega_k)y(k-\ell), \ \omega_k = (\alpha_k, \beta_k) \in \Lambda_N \times \Lambda_L, \tag{4.27}$$

where $K_\ell^q(\omega_k) \in \mathbb{R}^{n_u \times n_y}$, $q \in \mathbb{Q}$, $\ell = 0, \ldots, m-1$, are the static-output feedback partially mode-dependent gains to be determined and $m \in \mathbb{N}^+$ is a given positive integer corresponding to the number of measured outputs used in the control law. As an important observation, note that since the output matrix $C_y(\beta_k)$ in (4.25) does not depend on θ_k, the output feedback control law (4.27) does not lose the Markovian property, depending only on the previous state of the Markov chain. Furthermore, the implementation of (4.27) does not require the storage of the past values of the time-varying parameters $(\omega_{k-\ell})$, only the past values of the measured output $(y(k-\ell))$ and, as discussed in the LPV section, the choice $m = 1$ corresponds to the design of a standard gain-scheduled partially mode-dependent static output-feedback control law, that is, $u(k) = K_0^q(\omega_k)y(k)$, $q \in \mathbb{Q} = \{1, 2, \ldots, \sigma_c\}$.

Applying the proposed control law (4.27) in system (4.25), one obtains the closed-loop system given by

$$s(k+1) = A_{cl\,i}(\omega_k)s(k) \tag{4.28}$$

where $s(k)$ has the same structure as in (4.5) and

$$A_{cl\,i}(\omega_k) = \begin{bmatrix} \varphi_1 & \varphi_2 \\ B_i(\beta_k)\mathcal{K}^q & A_i(\beta_k) + B_i(\beta_k)K_0^q(\omega_k)C(\beta_k) \end{bmatrix} \tag{4.29}$$

with φ_1 and φ_2 given in (4.7), and

$$\mathcal{K}^q = \begin{bmatrix} K_{m-1}^q(\omega_k)C_y(\beta_{k-m+1}) & \cdots & K_1^q(\omega_k)C_y(\beta_{k-1}) \end{bmatrix}. \tag{4.30}$$

4.2.1 SOF Memory Control Design for NHMJLS

Before presenting the proposed control design conditions for the NHMJLS system (4.25), define:

$$\begin{aligned} P_i(\omega_k) &= \begin{bmatrix} P_{1i}(\omega_k) & \star \\ P_{2i}(\omega_k) & P_{3i}(\omega_k) \end{bmatrix}, \\ \Omega_{11}^i(\bar{\omega}_k) &= \mathcal{N}' \begin{bmatrix} -P_i(\omega_k) & 0 \\ 0 & P_{pi}(\omega_{k+1}) \end{bmatrix} \mathcal{N}, \quad \bar{\omega}_k = (\omega_k, \omega_{k+1}), \\ \Omega_{21}^i(\omega_k) &= \begin{bmatrix} B_i(\beta_k)\mathcal{K}^q & A_i(\beta_k) + B_i(\beta_k)K_0^q(\omega_k)C(\beta_k) & -I_{n_x} \end{bmatrix}, \end{aligned} \tag{4.31}$$

et al., 2016), since the output matrix $C_y(\beta_k)$ does not depend on θ_k.

where $P_{1i}(\omega_k) \in \mathbb{R}^{(m-1)n_x \times (m-1)n_x}$, $P_{2i}(\omega_k) \in \mathbb{R}^{n_x \times (m-1)n_x}$, and $P_{3i}(\omega_k) \in \mathbb{R}^{n_x \times n_x}$, $P_{pi}(\omega_k) = \sum_{j=1}^{\sigma} p_{ij}(\alpha_k) P_j(\omega_k)$, $\mathcal{N}$ is given in (4.8), and $\mathcal{K}^q$ is given in (4.30).

The second contribution of this chapter is a new sufficient parameter-dependent LMI condition for the synthesis of the memory static output-feedback gain-scheduled control law (4.27) for the discrete-time NHMJLS (4.25), as presented next.

Theorem 4.4. *For given matrices $Y_{1i}(\bar{\omega}_k) \in \mathbb{R}^{n_x \times mn_x}$, $Y_{2i}(\bar{\omega}_k)$ and $Y_{3i}(\bar{\omega}_k) \in \mathbb{R}^{n_x \times n_x}$, if there exist parameter-dependent matrices $P_i(\omega_k) = P_i(\omega_k)' > 0$ given in (4.31), $X_{1i}(\bar{\omega}_k) \in \mathbb{R}^{mn_x \times n_x}$, $X_{2i}(\bar{\omega}_k)$ and $X_{3i}(\bar{\omega}_k) \in \mathbb{R}^{n_x \times n_x}$, $\forall i \in \mathbb{K}$, and $K_\ell^q(\omega_k) \in \mathbb{R}^{n_u \times n_y}$, $\forall q \in \mathbb{Q}$, such that*

$$\mathcal{Q}_i(\bar{\omega}_k) + He\Big(\mathcal{X}_i(\bar{\omega}_k)\mathcal{B}_i(\bar{\omega}_k)\Big) < 0, \tag{4.32}$$

are verified for all $i \in \mathbb{K}$, and for all $\bar{\omega}_k = (\omega_k, \omega_{k+1}) \in (\Lambda_N \times \Lambda_L) \times (\Lambda_N \times \Lambda_L)$, where

$$\begin{aligned} \mathcal{B}_i(\bar{\omega}_k) &= \begin{bmatrix} Y_{1i}(\bar{\omega}_k) & Y_{2i}(\bar{\omega}_k) & Y_{3i}(\bar{\omega}_k) \end{bmatrix}, \\ \mathcal{X}_i(\bar{\omega}_k) &= \begin{bmatrix} X_{1i}'(\bar{\omega}_k) & X_{2i}'(\bar{\omega}_k) & X_{3i}'(\bar{\omega}_k) \end{bmatrix}', \\ \mathcal{Q}_i(\bar{\omega}_k) &= \begin{bmatrix} \Omega_{11}^i(\bar{\omega}_k) & \star \\ \Omega_{21}^i(\omega_k) & 0 \end{bmatrix}, \end{aligned} \tag{4.33}$$

with $\Omega_{11}^i(\bar{\omega}_k)$ and $\Omega_{21}^i(\omega_k)$ given in (4.31), then $u(k)$ given in (4.27) is a partially mode-dependent static output-feedback gain-scheduled control law that stabilizes system (4.25).

Proof. If (4.32) is satisfied then $Y_{31}(\bar{\omega}_k)$ has full-rank and, therefore, $\mathcal{B}_i(\bar{\omega}_k)$ can be rewritten as $\mathcal{B}_i(\bar{\omega}_k) = Y_{3i}(\bar{\omega}_k)\begin{bmatrix} -G_{1i}(\bar{\omega}_k) & -G_{2i}(\bar{\omega}_k) & I_{n_x} \end{bmatrix}$. Pre- and post-multiplying (4.32) respectively by $\mathcal{B}_i^{\perp\prime}(\bar{\omega}_k)$ and $\mathcal{B}_i^{\perp}(\bar{\omega}_k)$ with $\mathcal{B}_i^{\perp\prime}(\bar{\omega}_k) = \begin{bmatrix} I_{(m+1)n_x} & \mathcal{G}_i'(\bar{\omega}_k) \end{bmatrix}$, where $\mathcal{G}_i(\bar{\omega}_k) = \begin{bmatrix} G_{1i}(\bar{\omega}_k) & G_{2i}(\bar{\omega}_k) \end{bmatrix}$ one obtains

$$\Omega_{11}^i(\bar{\omega}_k) + He\left(\mathcal{G}_i'(\bar{\omega}_k)\Omega_{21}^i(\omega_k)\right) < 0 \tag{4.34}$$

By multiplying (4.34) on the right by $\Omega_{21}^{i\perp}(\omega_k)$ and on the left by $\Omega_{21}^{i\perp\prime}(\omega_k)$, with

$$\Omega_{21}^{i\perp}(\omega_k) = \begin{bmatrix} I_{(m-1)n_x} & 0 \\ 0 & I_{n_x} \\ B_i(\beta_k)\mathcal{K}^q & A_i(\beta_k) + B_i(\beta_k)K_0^q(\omega_k)C_y(\beta_k) \end{bmatrix} \tag{4.35}$$

and $A_{cl_i}(\omega_k)$ given in (4.29), one has

$$\begin{aligned} &\Omega_{21}^{i\perp\prime}(\omega_k)\mathcal{N}' \begin{bmatrix} -P_i(\omega_k) & 0 \\ 0 & P_{pi}(\omega_{k+1}) \end{bmatrix} \mathcal{N}\Omega_{21}^{i\perp}(\omega_k) \\ &\qquad = \begin{bmatrix} I_{mn_x} & A_{cl_i}'(\omega_k) \end{bmatrix} \begin{bmatrix} -P_i(\omega_k) & 0 \\ 0 & P_{pi}(\omega_{k+1}) \end{bmatrix} \begin{bmatrix} I_{mn_x} \\ A_{cl_i}(\omega_k) \end{bmatrix} \\ &\qquad = A_{cl_i}'(\omega_k)P_{pi}(\omega_{k+1})A_{cl_i}(\omega_k) - P_i(\omega_k) < 0 \end{aligned} \tag{4.36}$$

that assures that the closed-loop system (4.28) is ESMS-CI. □ □

As done in the LPV case, the conditions of Theorem 4.4 need to be relaxed to compensate the fact that $Y_{1,2,3_i}(\bar{\omega}(k))$ are fixed matrices. Considering that $A_{cl\,i}(\omega_k)$ can be partitioned as

$$A_{cl\,i}(\omega_k) = \begin{bmatrix} 0 & I_{(m-1)n_x} \\ H_{1i} & H_{2i} \end{bmatrix}, \tag{4.37}$$

the ESMS-CI condition (4.36) for the closed-loop NHMJLS can be rewritten as

$$\begin{bmatrix} H_{1i}'P_{3pi}(\omega_{k+1})H_{1i} & \star \\ \begin{pmatrix} P_{2pi}'(\omega_{k+1})H_{1i} \\ +H_{2i}'P_{3pi}(\omega_{k+1})H_{1i} \end{pmatrix} & \begin{pmatrix} P_{1pi}(\omega_{k+1}) \\ +\mathrm{He}(H_{2i}'P_{2pi}(\omega_{k+1}) \\ +H_{2i}'P_{3pi}(\omega_{k+1})H_{2i} \end{pmatrix} \end{bmatrix} - \begin{bmatrix} P_{1i}(\omega_k) & \star \\ P_{2i}(\omega_k) & P_{3i}(\omega_k) \end{bmatrix} < 0 \tag{4.38}$$

or

$$M_{1i} < M_{3i} - M_{2i} \tag{4.39}$$

where

$$\begin{aligned} M_{1i} &= \begin{bmatrix} H_{1i}' & 0 \\ 0 & H_{2i}' \end{bmatrix} \begin{bmatrix} P_{3pi}(\omega_{k+1}) & P_{3pi}(\omega_{k+1}) \\ P_{3pi}(\omega_{k+1}) & P_{3pi}(\omega_{k+1}) \end{bmatrix} \begin{bmatrix} H_{1i} & 0 \\ 0 & H_{2i} \end{bmatrix}, \\ M_{2i} &= \begin{bmatrix} 0 & \star \\ P_{2pi}(\omega_{k+1})H_{1i} & \mathrm{He}(H_{2i}'P_{2pi}(\omega_{k+1}) \end{bmatrix}, \\ M_{3i} &= \begin{bmatrix} P_{1i}(\omega_k) & \star \\ P_{2i}(\omega_k) & P_{3i}(\omega_k) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_{1pi}(\omega_{k+1}) \end{bmatrix}. \end{aligned} \tag{4.40}$$

Introducing the relaxation level $\rho > 0$ in $A_{cli}(\omega_k)$ given in (4.37), that is,

$$A_{cl\,i}(\omega_k) = \begin{bmatrix} 0 & I_{(m-1)n_x} \\ H_{1i}/\rho & H_{2i}/\rho \end{bmatrix}, \tag{4.41}$$

it is possible to show that if the conditions of Theorem 4.4 solved jointly with $M_{3i} > 0$, $i \in \mathbb{K}$, are feasible for a given ρ, then inequalities remain feasible for any $\bar{\rho} > \rho$, following the same arguments presented in the LPV case. As a consequence, if the conditions of Theorem 4.4 are tested replacing $A_i(\beta_k)$ by $\hat{A}_i(\beta_k) = A_i(\beta_k)\rho$ and $K_\ell^q(\omega_k)$ by $\hat{K}_\ell^q(\omega_k)/\rho$, a feasible solution for any $\rho \leq 1$ assures that the closed-loop system (4.28) is ESMS-CI.

For completeness, it is also proved that a particular choice for the matrices $Y_{1,2,3_i}(\bar{\omega}_k)$ assures that the conditions of Theorem 4.4 tested with the relaxation level ρ always provide a feasible solution.

Theorem 4.5. *The choice $\mathcal{B}_i(\bar{\omega}_k) = \mathcal{B}_{i0} = \begin{bmatrix} 0 & I_{n_x} & -I_{n_x} \end{bmatrix}$ assures the existence of a feasible solution for Theorem 4.4 with $A_i(\beta_k)$ and $K_\ell^q(\omega_k)$ replaced respectively by $\hat{A}_i(\beta_k)$ and $\hat{K}_\ell^q(\omega_k)$, $\forall \ell = 0, \ldots, m-1$, $q \in \mathbb{Q}$, $i \in \mathbb{K}$.*

Proof. First, consider the following partitions for the blocks $P_{1i}(\omega_k)$ and $P_{2i}(\omega_k)$ of $P_i(\omega_k)$ from (4.31)

$$P_{1i}(\omega_k) = \begin{bmatrix} P_{11\,i}(\omega_k) & \star \\ P_{12\,i}(\omega_k) & P_{13\,i}(\omega_k) \end{bmatrix}, \; P_{2i}(\omega_k) = \begin{bmatrix} P_{21\,i}(\omega_k) & P_{22\,i}(\omega_k) \end{bmatrix},$$

meaning that $\Omega_{11}^i(\bar{\omega}_k)$ from (4.31) can be rewritten as

$$\Omega_{11}^i(\bar{\omega}_k) = \begin{bmatrix} -P_{11i}(\omega_k) & \star & \star & \star \\ P_{12i}(\omega_k) & \begin{pmatrix} P_{11pi}(\omega_{k+1}) \\ -P_{13i}(\omega_k) \end{pmatrix} & \star & \star \\ -P_{21i}(\omega_k) & \begin{pmatrix} P_{12pi}(\omega_{k+1}) \\ -P_{22i}(\omega_k) \end{pmatrix} & \begin{pmatrix} P_{13pi}(\omega_{k+1}) \\ -P_{3i}(\omega_k) \end{pmatrix} & \star \\ 0 & P_{21pi}(\omega_{k+1}) & P_{22pi}(\omega_{k+1}) & P_{3pi}(\omega_{k+1}) \end{bmatrix}.$$

Then, for $\mathcal{B}_i(\bar{\omega}_k) = \mathcal{B}_{0i}$, the following choices for the variables of Theorem 4.4 such that (4.32) holds can be made: a large enough ρ such that $\hat{A}_i(\beta_k) = 0$ and $\hat{K}_\ell^q(\omega_k) = 0$, $\mathcal{X}_i(\bar{\omega}_k) = -0.5\mathcal{B}_{0i}'$,

$$\begin{aligned} P_{12i}(\omega_k) = P_{12pi}(\omega_{k+1}) = 0, \quad P_{11i}(\omega_k) = \epsilon_1 I, \\ P_{21i}(\omega_k) = P_{21pi}(\omega_{k+1}) = 0, \quad P_{13i}(\omega_k) = \epsilon_2 I, \\ P_{22i}(\omega_k) = P_{22pi}(\omega_{k+1}) = 0, \quad P_{3i}(\omega_k) = \epsilon_3 I, \\ P_{11pi}(\omega_{k+1}) = \epsilon_1 p_{ii}(\alpha_k) I, \quad P_{13pi}(\omega_{k+1}) = \epsilon_2 p_{ii}(\alpha_k) I, \\ P_{3pi}(\omega_{k+1}) = \epsilon_3 p_{ii}(\alpha_k) I, \end{aligned}$$

with $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1$. By replacing the mentioned values in (4.32) with $A_i(\beta_k) = A_i(\beta_k)/\rho$ and $K_\ell^q(\beta_k) = K_\ell^q(\beta_k)/\rho$ with $\rho \to \infty$, one has

$$\begin{bmatrix} -\epsilon_1 I & 0 & 0 & 0 & 0 \\ 0 & (\epsilon_1 p_{ii}(\alpha_k) - \epsilon_2) I & 0 & 0 & 0 \\ 0 & 0 & (\epsilon_2 p_{ii}(\alpha_k) - \epsilon_3) I & 0 & 0 \\ 0 & 0 & 0 & (\epsilon_3 - 1) I & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{4.42}$$

that is always verified since $p_{ii}(\alpha_k) \leq 1$ and $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1$. □ □

As in the LPV case, the conditions of Theorem 4.4 can be solved iteratively, starting with $\mathcal{B}_{i0}$ proposed in Theorem 4.5 and using the update $\mathcal{B}_i(\bar{\omega}_k) = \mathcal{X}_i'(\bar{\omega}_k)$ in each new iteration, as formally presented by Algorithm 2. Local convergence and guarantees that the increase of the memory cannot produce worse results can be easily demonstrated following the material presented for the LPV case. Concerning the structure of the controllers, note that two polynomial degrees must be chosen by the designer: one associated to β_k and the other associated to α_k. If both degrees are zero, then one has the classical gain (mode-dependent or independent) suitable to deal with homogeneous MJLS where both α_k and β_k are uncertain but time-invariant.

funtion Input$\Big(\big(A_i, B_i, C_y\big)(\bar{\beta}_k), it_{max}, \mathcal{B}_{0i}(\bar{\omega}_k)\Big)$
$\mathcal{B}_i(\bar{\omega}_k) = \mathcal{B}_{0i}(\bar{\omega}_k)$
$A_i(\beta_k) = A_i(\beta_k)/\rho$ initialization;
while $it < it_{max}$ **do**
  Minimize ρ **subject to**
    Theorem 4.4 and $M_{3i} > 0$ (Eq. (4.39))
  if $\rho > 1$ **then**
    $\mathcal{B}_i(\bar{\omega}_k) = \mathcal{X}_i'(\bar{\omega}_k)$
  else
    $K_\ell^q(\omega_k) = \rho K_\ell^q(\omega_k)$
    return
  end
end

Algorithm 2: SOF control of NHMJL.

4.2.2 Numerical example of memory SOF mode-independent robust control of NHMJLS

In NCS, where the communication among the components (sensors, controllers and actuators) is performed through non ideal channels such as semi-reliable networks, the MJLS is a suitable class of models to represent the common phenomena arising in this scenario. In this example, the *local sensor – remote actuator* (LSRA) (AMORIM *et al.*, 2016) problem is investigated in the context of SOF control of a Vertical Take-Off and Landing (VTOL) helicopter.

The linearized state-space dynamic model of the VTOL helicopter is given in (4.24), adapted from Keel *et al.* (1988), with $p_1 = 0.3$, $p_2 = p_3 = 1.42$. The equivalent discrete-time system is obtained by a zero-order hold discretization method considering the sampling time $T = 0.5$s. For static output-feedback control design, consider that only the horizontal and vertical velocities ($x_1(t)$ and $x_2(t)$) are measured (by radar, image processing or other method). The control law is calculated remotely and sent, through wireless communication between the remote control and the VTOL helicopter, constituting an LSRA control architecture. The control law scheme is depicted in Fig. 19. Observe that the output measurement is always successfully performed, but a packet loss can occur in the transmission of the control signal, which is done, for instance, by a standard wireless communication, such as IEEE 802.15.4 usually employed in Wireless Sensor Network (WSN).

The probability of the helicopter receiving a successful transmission of the control signal can be considered as a time-varying parameter since it depends on the distance between the controller and the VTOL helicopter, and also because of the noise

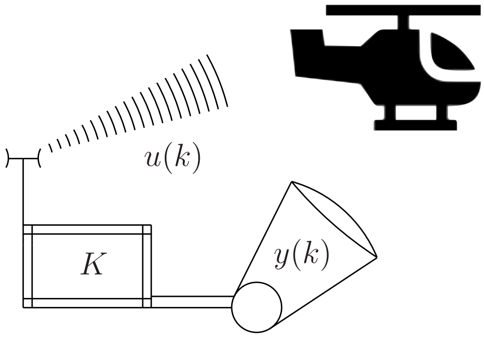
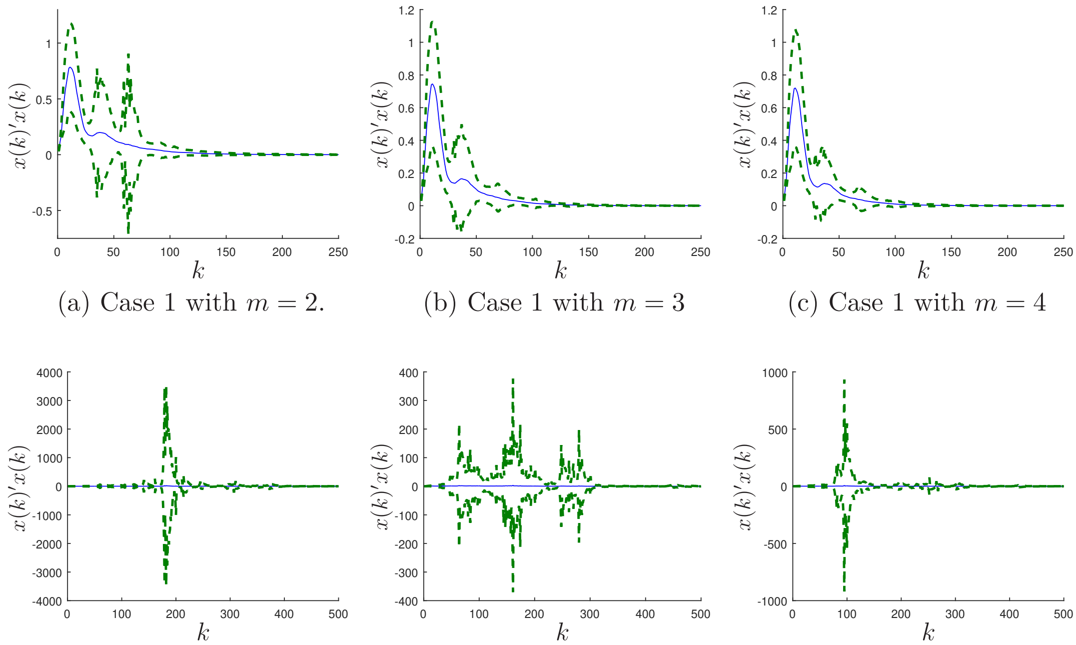


Figure 19 – *Local sensor - remote actuator* (LSRA) for networked output-feedback control of the VTOL helicopter.

introduced by the communication channel. Among the existing Markovian channel models



(a) Case 1 with $m = 2$. (b) Case 1 with $m = 3$ (c) Case 1 with $m = 4$

(d) Case 2 with $m = 2$ (e) Case 2 with $m = 3$ (f) Case 2 with $m = 4$

Figure 20 – Mean (blue) and standard deviation (dashed green) of $x(k)'x(k)$ obtained through MC simulation of the *Local sensor - remote actuator* (LSRA) closed-loop system for VTOL helicopter with: Case one with (a) $m = 2$, (b) $m = 3$, (c) $m = 4$; Case two with (d) $m = 2$, (e) $m = 3$, (f) $m = 4$.

in the literature for packet loss representation (GONÇALVES *et al.*, 2010), in this example a simplified Gilbert-Elliot model, appropriate to represent both time-varying probabilities and burst failures, is used. In this sense, the time-varying transition probability matrix is given by

$$\mathbb{P}(\alpha(k)) = \begin{bmatrix} p(\alpha(k)) & 1 - p(\alpha(k)) \\ 1 - q(\alpha(k)) & q(\alpha(k)) \end{bmatrix}, \; p(\alpha(k)), \; q(\alpha(k)) \in [0, \; 1], \tag{4.43}$$

where $p(\alpha(k))$ corresponds to the probability of a successful transmission be followed by another successful transmission, $q(\alpha(k))$ represents the probability of two consecutive transmission failures, while their respective complements, $1-p(\alpha(k))$ and $1-q(\alpha(k))$, are related to a change of states (success-failure and failure-success).

Note that the adopted Markov model considers two operation modes: the first one corresponds to a successful transmission of the control signal, while the second one represents failure. In this example, the packet loss of the control signal is modeled by the *Zero-input* (SCHENATO, 2009) approach, which is equivalent to clear the elements of the control matrix in the mode of the Markov chain that corresponds to transmission failure ($B_2 = 0$) and maintaining its actual value ($B_1 = B$ given in (4.24)) in the operation mode that represents success. Assuming that the probabilities $p(\alpha(k))$ and $q(\alpha(k))$ from (4.43) are bounded time-varying parameters whose intervals are given by $p(\alpha(k)) \in [0.55,\ 1]$ and $q(\alpha(k)) \in [0,\ 0.45]$, the time-varying probability matrix (4.43) can be described by a polytope of four vertices or, using multi-simplex representation, since each row of $\mathbb{P}(\alpha(k))$ is independent from each other, by the Cartesian product of two simplexes of two vertices each ($\Lambda_2 \times \Lambda_2$). Since it is not possible to know when a correct or a failure transmission happened, the designed controller is considered mode-independent. When no memory is considered ($m = 1$), Theorem 4.4 with $it_{\max} = 200$ is not capable of providing a feasible solution. On the other hand, when using at least one memory ($m \geq 2$), the following stabilizing mode-independent control gains are found (truncated with 4 decimal digits and respectively considering $m = 2$, $m = 3$ and $m = 4$):

$$
\begin{gathered}
\left[\begin{array}{c|c} K_0 & K_1 \end{array}\right] = \left[\begin{array}{cc|cc} .0503 & .0535 & -.1065 & .0028 \\ -.0044 & .4220 & -.4112 & .0136 \end{array}\right] \\
\left[\begin{array}{c|c|c} K_0 & K_1 & K_2 \end{array}\right] = \left[\begin{array}{cc|cc|cc} .0405 & .0517 & -.0801 & .0038 & -.0166 & .0001 \\ .0015 & .4212 & -.3967 & .0153 & -.0208 & .0001 \end{array}\right] \\
\left[\begin{array}{c|c|c|c} K_0 & K_1 & K_2 & K_3 \end{array}\right] = \\
\left[\begin{array}{cc|cc|cc|cc} .0402 & .0509 & -.0723 & .0049 & -.0167 & .0010 & -.0089 & -.0000 \\ .0339 & .4202 & -.3952 & .0195 & -.0493 & .0009 & -.0094 & -.0001 \end{array}\right]
\end{gathered}
\tag{4.44}
$$

For additional validation, the closed-loop system stability is checked by a stability analysis condition derived from Aberkane (2011b).

Aiming to illustrate the temporal behavior of the closed-loop states, Fig. 20 shows the graphics of $x(k)'x(k)$ using 10^4 iterations of a Monte Carlo simulation and the following initial condition $x(0) = \begin{bmatrix} 0.1 & -0.1 & 0 & -0.1 \end{bmatrix}$. Two case studies are investigated. The first one corresponds to assume the following functions for the time-varying probabilities

$$p(\alpha(k)) = 0.55 + 0.45\cos(0.45k)^2, \tag{4.45}$$

$$q(\alpha(k)) = 0.45\cos(0.1k)^2. \tag{4.46}$$

The second case study relies on evaluating the temporal behavior considering the most aggressive probability value for NCS design, that is the lower successful packet rate $p(\alpha(k)) = 0.55$ and the maximum probability of failure $q(\alpha(k)) = 0.45$. The results of the first and second study cases are respectively presented in the first and second rows of the graphics depicted in Fig. 20, where the green curves represent the standard deviation and the blue curves the mean. Note that, although no performance criterion is used in the design procedure (that regards only stabilization), when increasing the number of memories considered in the controller, the dispersion around the mean of $x(k)'x(k)$ is clearly improved. This difference is more evident in the second row of Fig. 20, when the probabilities are considered constant. Finally, observe that even though the inclusion of memories in the control law augments the dimensions of the closed-loop system (demanding a larger computational effort to solve the problem), since no solution is obtained when considering no memory, the proposed methodology can be considered advantageous because it can provide stabilizing controllers when the conventional methods fail.

4.3 Partial Conclusion

This chapter presented a new approach for static output-feedback control whose main novelty is the use of past information of the states or measured outputs in the control law for uncertain systems with time-varying parameters. The main contribution is an iterative procedure given in terms of parameter-dependent linear matrix inequalities that handles the gain directly as an optimization variable, allowing to generate sufficient convex conditions for control design of memory controllers for LPV systems, that do not exist in the literature. The results are extended for non-homogeneous MJLS, where the time-varying parameters can affect the dynamics (space-state matrices) and probabilities (Markov chain). The existence of feasible initial conditions, local convergence for the iterative procedures and the validity of a relaxation strategy adopted in the algorithms aiming to increase the effectiveness of the method were demonstrated. In the numerical examples, the proposed approach proved to be superior to other techniques from the literature in terms of effectiveness considering memory in the control law or not. In the context of non-homogeneous MJLS, a practical example was presented to show that the proposed approach can provide stabilizing controllers when the other methods fail.

5 Conclusion and Future Works

Different design methodologies were proposed in this PhD Thesis to provide improved control and filtering solutions for LPV systems through convex optimization based on LMIs. The first contribution, presented in Chapter 2, consisted in the proposition of a new modeling by exploring explicit formulas for the variation rate of time-varying parameters. In this strategy, the time-varying parameters are written as solution of difference equations (usually expressed in terms of complex exponentials). Some numerical examples have illustrated the effectiveness and efficiency of the proposed approach in stability analysis and synthesis problems (control and filtering) when compared with the traditional polytopic modeling (based on arbitrary or bounded rates of variations). It was also shown that the proposed modeling is capable to provide, with lower computational complexity, state-feedback controllers assuring improved performance for an example with practical appeal (NCS with time-varying sampling rate), when compared with other specialized methods from the literature.

The second research topic, presented in Chapter 3, was concerned with the problem of $\mathcal{H}_2$ state-feedback control and full-order filter design for LPV systems subject to inexact information about the time-varying parameters. A special attention has been paid to the case where the additive error (or the time-varying parameter itself) is associated with a known PDF, which was taken into account in the synthesis conditions in order to improve performance (measured in terms of the closed-loop $\mathcal{H}_2$ norm or MSE). The proposal developed in Chapter 3, called Sub-Domain Optimization Heuristic (SDOH), basically relies on optimizing performance only in a certain range of values of the parameters (called sub-domain) where the probability of occurrence is higher, while the stability of the closed-loop system (in case of control) or the estimation error system (when handling filter design) is assured for all uncertain domain. Some numerical examples showed that this strategy can provide great percentage improvements in the closed-loop $\mathcal{H}_2$ worst case norm, MSE and SD_{MSE} when compared with the methods from the literature.

Chapter 4 has proposed an iterative procedure based on parameter-dependent LMI conditions for the synthesis of memory static output-feedback control laws to stabilize LPV systems. The main novelty is the fact that the control gain is an optimization variable of the problem (no change of variables is necessary), which is useful to deal with practical requirements as magnitude constraints for the entries of the control gain. The numerical examples have also shown that, besides more general, this approach is able to obtain less conservative solutions (stabilizing a larger number of systems with or without structure or magnitude constraints), when compared with other methods from the literature that can only cope with LTI systems. Chapter 4 has also proposed a generalization of the method to deal with memory control of NHMJLS, which is an important feature since this class of systems allows to simultaneously model stochastic and time-varying dynamics, being appropriate to represent several problems in the NCS framework.

5.1 Main Papers that compose the PhD Thesis

This PhD research has produced five papers that constitute the most relevant contributions of the thesis:

1. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ control and filtering of discrete-time LPV systems exploring statistical information of the time-varying parameters. Accepted in Journal of the Franklin Institute ISSN: 0016-0032.

2. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. A less conservative approach to handle time-varying parameters in discrete-time LPV systems with applications in NCS. Accepted in International Journal of Robust and Nonlinear Control ISSN: 1099-1239.

3. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. LMI-based solution for memory static output-feedback control of discrete-time linear systems affected by time-varying parameters. Submitted to Automatica ISSN: 0005-1098.

4. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ performance comparison for gain-scheduled design using inexact information of the scheduling parameters: standard parametric model versus saturation model. In Proceedings of the 2019 IEEE CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON), Valparaiso, Chile, November 2019.

5. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ gain-scheduled filtering for discrete-time LPV systems using estimated time-varying parameters. In *Proceedings of the 2018 American Control Conference*, pages $4367-4372$, Milwaukee, WI, USA, June 2018. <10.23919/ACC.2018.8431838>.

5.2 Other related subjects investigated during doctorate

In addition to the main topics addressed in this thesis, the PhD candidate also worked on related areas, producing the following articles.

5.2.1 Less conservative design for generalized Bernoulli jump systems

Many practical problems arising in networked control systems can be suitably modeled by linear stochastic systems described in terms of discrete-time generalized

Bernoulli models (PALMA *et al.*, 2015; PALMA *et al.*, 2016; DURAN-FAUNDEZ *et al.*, 2018), that are a particular case of MJLS. Motivated by real world applications where the transition probability matrix is uncertain and the Markov chain obeys a generalized Bernoulli distribution, a general framework to deal with the problems of filter and control design was investigated, providing synthesis conditions for state-feedback control, full- or reduced-order filtering that are sufficient in the uncertain case and also necessary (optimal) for precisely known models. Those propositions were summarized in

1. C. F. Morais, **J. M. Palma**, P. L. D. Peres, R. C. L. F. Oliveira, An LMI approach for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ reduced-order filtering of uncertain discrete-time Markov and Bernoulli jump linear systems. *Automatica*, 95(9):463-471, September 2018.

2. C. F. Morais, **J. M. Palma**, P. L. D. Peres, R. C. L. F. Oliveira, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ mode-independent state-feedback control of generalized Bernoulli jump systems with uncertain probabilities. In *Proceedings of the 2018 American Control Conference*, pages 5724-5729, Milwaukee, WI, USA, June 2018.

The conditions are based on parameter-dependent linear matrix inequality conditions associated with a scalar parameter that are sufficient to provide mode-dependent, partially mode-dependent or mode-independent in controller and filter project.

5.2.2 $\mathcal{H}_\infty$ control design for NHMJLS

Non-homogeneous Markov jump linear systems (ABERKANE, 2011b) are suitable for modeling discrete-time jump systems with time-varying operation modes. In this topic two main problems regarding NHMJLS are evaluated: The first one is concerned with discrete-time MJLS whose stochastic process that rules the jumps between the operation modes is non-homogeneous (time-varying transition probabilities). The second problem deals with control or filter design of LPV systems whose dynamics is subject to jumps. The main purpose is to emphasize networked control problems in which the parameters and output or input signals may not be available due to packet loss. The main motivation to study those problems is their wide application in the NCS context. As result of the investigations in this research field, the following papers have been published.

1. **J. M. Palma**, C. F. Morais, R. C. L. F. Oliveira, $\mathcal{H}_\infty$ state-feedback gain-scheduled control for MJLS with non-homogeneous Markov chains. In *Proceedings of the 2018 American Control Conference*, pages 5718-5723, Milwaukee, WI, USA, June 2018.

2. **J. M. Palma**, C. F. Morais, R. C. L. F. Oliveira, Gain-scheduled control for LPV systems with scheduling parameters transmitted through a Markov channel. In *Proceedings of the Joint 9th IFAC Symposium on Robust Control Design and 2nd IFAC*

Workshop on Linear Parameter Varying Systems, pages 549-554, Florianópolis, SC, Brasil, September 2018.

5.2.3 Development of resource savings methods for wireless sensor networks

In order to optimize performance of systems controlled through digital communication networks, the PhD candidate has proposed new protocols to improve energy-efficiency in NCS (by evaluating the trade-off between the network consumption and performance degradation) and has also investigated a new model of Markov Channel to properly represent multi-route networks. These topics are summarized in

1. **J. M. Palma**, L. de P. Carvalho, C. F. Morais, R. C. L. F. Oliveira, E. Rubio, K. Herman, Protocol for Energy-Efficiency in Networked Control Systems Based on WSN. *Sensors (Switzerland)*, 18(8):2590, August 2018.

2. **J. M. Palma**, L. P. Carvalho, T. E. Rosa, C. F. Morais, and R. C. L. F. Oliveira. "$\mathcal{H}_2$ and $\mathcal{H}_\infty$ state-feedback control through Multi-Hop Networks: Trade-Off Analysis Between the Network Load and Performance Degradation". In *IEEE Latin America Transactions*. Vol. 16, no. 9 pp. 2377 – 2384, September 2018.

3. **J. M. Palma**, L. de P. Carvalho, T. E. Rosa, C. F. Morais, R. C. L. F. Oliveira, "$\mathcal{H}_2$ filtering through multi-hop networks: Trade-off analysis between the network consumption and performance degradation". In *Proceedings of the CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)*, Pucon, RA, Chile, October 2017.

4. **J. M. Palma**, C. F. Morais, L. de P. Carvalho, R. C. L. F. Oliveira, "Modelo de canal Markoviano para projetos de filtros $\mathcal{H}_\infty$ em redes multi-rotas". In *Simpósio Brasileiro de Automaçãoao Inteligente - SBAI 2017*, pages 554-560, Porto Alegre, RS, Brasil, Outubro 2017.

The first three papers are concerned with the proposition of new communication protocols for output-feedback and state-feedback control or filtering through multi-hop communication networks by optimizing two conflicting criteria: the network energy consumption and the stability/performance of the closed-loop system. In particular, the second and third papers evaluate the influence of the initial probability distribution on the design of filter and state-feedback controllers and the corresponding performance in time-domain. The last article investigates the problem of filtering through a multi-path network, proposing a modeling that minimizes the impact of the packet loss in the performance of the designed filter.

5.2.4 Robust control of combustion systems

The PhD candidate also investigated the robust control of combustion systems. A non-linear model obtained through an identification technique by optical instrumentation of the plant was used to propose a linear model around an operating condition, for which an output-feedback controller was designed.

1. **J. M. Palma**, H. O. Garces, A. J. Rojas, C. F. Morais, R. C. L. F. Oliveira, $\mathcal{H}_\infty$ output-feedback control design for combustion systems using optical instrumentation. In *Proceedings of the Joint 9th IFAC Symposium on Robust Control Design and 2nd IFAC Workshop on Linear Parameter Varying Systems*, pages 207-2012, Florianópolis, SC, Brasil, September 2018.

2. **J. M. Palma**, H. O. Garces, C. F. Morais, R. C. L. F. Oliveira, A new approach of $\mathcal{H}_\infty$ filtering for combustion systems using optical instrumentation. Submitted to ISA Transactions ISSN: 0019-0578.

5.2.5 LPV modeling of Biomathematical systems

Mathematical modeling of biological problems is a vast area of study in applied mathematics. In the stability analysis for biological dynamic systems, the relationship between different types of populations is commonly investigated in the literature using LTI approximations of nonlinear dynamics. During the doctorate, the PhD candidate has proposed a stability analysis methodology based on LPV (or polytopic LTI) models for biological systems, with nonlinear dynamics, linearized around multiple equilibrium points. Additionally, the differences between the nonlinear and the LPV representations, particularly for the case of *fish population*, were evaluated in time domain. The research in Biomathematical context can be found in the following publications.

1. R. Lobo, **J. M. Palma**, C. F. Morais, L. de P. Carvalho, M. E. Valle and R. C. L. F. Oliveira. A Brief Tutorial on Quadratic Stability of Linear Parameter-Varying Model for Biomathematical Systems. Accepted for publication in CHILECON 2019, Valparaiso, Chile, October 2019.

2. D. E. Sánchez, **J. M. Palma**, R. A. Lobo, C. F. Morais, J. Meyer, A. Rojas-Palma and R. C. L. F. Oliveira. Modeling and Stability Analysis of Salmon Mortality due to Microalgae Bloom using Linear Parameter-Varying Structure. Accepted for publication in CHILECON 2019, Valparaiso, Chile, October 2019.

5.3 Future Works

The different topics studied in the PhD program allow to expand the research areas and future works can be carried out. For instance, it is possible to generalize the modeling of time-varying parameters based on solutions of difference equations for all kinds of time-functions by means of a truncated Fourier series plus an approximation error (representing the neglected terms). Note that, the inclusion of an additive uncertainty (representing the error) in the model is an issue investigated in this thesis.

Another possible future contribution is to apply the modeling of stochastic time-varying parameters (and the information about their PDF) in the NHMJLS context, aiming to improve control and filtering synthesis techniques. One can also develop LMI conditions for gain-scheduled control or filtering of LPV systems with time-varying parameters transmitted through a Markov channel, since it is known that, when some information is transmitted through a digital network, usually occurs the incorporation of a discrete time-delay (d_k) in the packet of information. Thus, it is reasonable to consider that the controller or filter is scheduled by $\theta(k-d_k)$, that is, by the delayed time-varying parameter $\theta(k)$, where d_k represents the time-delay. This delay can be modeled by using the results of Palma *et al.* (2018b), connecting the theory of LPV systems subject to inexact measurements and the information about bounded rates of variation in the parameters and/or the model proposed in terms of difference equations (specially when the time-varying parameters or probabilities are a known function, such as a periodic one).

To provide an extra improvement in the performance associated with LMI-based control design methods, one can also use the artificial enrichment of the system dynamics by employing memory control laws. The main challenge of this research topic is to develop memory control and filtering synthesis conditions associated with the optimization of performance criteria, such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed costs.

5.4 Full List of Publications

In short, the PhD candidate has produced 31 works between March 2016 and November 2019. Five of those papers constitute the most relevant contributions of the PhD thesis (PALMA *et al.*, 2018c; PALMA *et al.*, 2019a; PALMA *et al.*, 2019b; PALMA *et al.*, 2019c; PALMA *et al.*, 2019d). The partial results of this thesis, which are listed below, correspond to eighteen papers developed under supervision of Prof. Dr. Ricardo C. L. F. Oliveira and Dra. Cecília F. Morais.

Three papers published in international journals:

1. **J. M. Palma**, L. de P. Carvalho, T. E. Rosa, C. F. Morais, and R. C. L. F. Oliveira. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state-feedback control through Multi-Hop Networks: Trade-Off Analysis Between the Network Load and Performance Degradation. In *IEEE Latin America*

Transactions. Vol. 16, no. 9 pp. 2377 – 2384, September 2018. <10.1109/TLA.2018.8789558>

2. **J. M. Palma**, C. Duran-Faundez, L. de P. Carvalho, C. F. Morais, R. C. L. F. Oliveira, E. Rubio and K. Herman, Protocol for Energy-Efficiency in Networked Control Systems based on WSN. In *Sensors (Basel)*, Vol. 18, no. 8 pp. 2590, August 2018. <https://doi.org/10.3390/s18082590>

3. C. F. Morais, **J. M. Palma**, P. L. D Peres and R. C.L.F. Oliveira. An LMI approach for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ reduced-order filtering of uncertain discrete-time Markov and Bernoulli Jump Linear Systems. In *Automatica*. Vol. 95, pp. 463 – 471, September 2018. <https://doi.org/10.1016/j.automatica.2018.06.014>

Five papers submitted to international journals:

1. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ control and filtering of discrete-time LPV systems exploring statistical information of the time-varying parameters. Accepted in Journal of the Franklin Institute ISSN: 0016-0032.

2. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. A less conservative approach to handle time-varying parameters in discrete-time LPV systems with applications in NCS. Accepted in International Journal of Robust and Nonlinear Control ISSN: 1099-1239.

3. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. LMI-based solution for memory static output-feedback control of discrete-time linear systems affected by time-varying parameters. Submitted Automatica ISSN: 0005-1098.

4. C. E. Galarza, **J. M. Palma**, C. F. Morais, J Utura, L. de P. Carvalho and R. C. L. F. Oliveira. A novel probabilistic model for opportunistic routing and applications in computation of energy consumption in WSN. Submitted in the IEEE/ACM Transactions on Networking ISSN: 1063-6692.

5. **J. M. Palma**, H. O. Garces, C. F. Morais and R. C. L. F. Oliveira. A new approach of $\mathcal{H}_\infty$ filtering for combustion systems using optical instrumentation. Submitted in ISA Transactions ISSN: 0019-0578.

Nine papers published in the proceedings of international conferences:

1. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ performance comparison for gain-scheduled design using inexact information of the scheduling parameters: standard parametric model versus saturation model. Proceedings of the *2019 IEEE CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)*, Valparaiso, Chile, November 2019.

2. R. Lobo, **J. M. Palma**, C. F. Morais, L. de P. Carvalho, M. E. Valle and R. C. L. F. Oliveira. A Brief Tutorial on Quadratic Stability of Linear Parameter-Varying Model for Biomathematical Systems. Proceedings of the *2019 IEEE CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)*, Valparaiso, Chile, November 2019.

3. D. E. Sánchez, **J. M. Palma**, R. A. Lobo, C. F. Morais, J. Meyer, A. Rojas-Palma and R. C. L. F. Oliveira. Modeling and Stability Analysis of Salmon Mortality due to Microalgae Bloom using Linear Parameter-Varying Structure. Proceedings of the *2019 IEEE CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)*, Valparaiso, Chile, November 2019.

4. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. Gain-scheduled control for LPV system with parametric information transmitted through a Markov channel. Proceedings of the *1nd IFAC Workshop on Linear Parameter Varying Systems LPVS 2018*. Vol. 51, no 26 pp. 167 − 172, Florianópolis, Brazil, 3-5 September 2018. <https://doi.org/10.1016/j.ifacol.2018.11.144>

5. **J. M. Palma**, H. O. Garces, A. J. Rojas, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_\infty$ output-feedback control design for combustion systems using optical instrumentation. Proc. *in 9th IFAC Symposium on Robust Control Design ROCOND 2018* Vol. 51, no 25 pp. 134 − 139, Florianópolis, Brazil, 3-5 September 2018. <https://doi.org/10.1016/j.ifacol.2018.11.094>

6. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_\infty$ state-feedback gain-scheduled control for MJLS with non-homogeneous Markov chains. Proc. *in the 2018 American Control Conference*, pp. 5717 − 5723, Milwaukee, WI, USA, June 2018. <10.23919/ACC.2018.8430913>.

7. **J. M. Palma**, C. F. Morais and R. C. L. F. Oliveira. $\mathcal{H}_2$ gain-scheduled filtering for discrete-time LPV systems using estimated time-varying parameters. Proc. *in the 2018 American Control Conference*, pp. 4367 − 4372, Milwaukee, WI, USA, June 2018. <10.23919/ACC.2018.8431838>.

8. C. F. Morais, **J. M. Palma**, P. L. D. Peres and Ricardo C. L. F. Oliveira. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ mode-independent state-feedback control of generalized Bernoulli jump systems with uncertain probabilities. Proc. *in the 2018 American Control Conference*, pp. 5724 − 5729, Milwaukee, WI, USA, June 2018. <10.23919/ACC.2018.8431634>.

9. **J. M. Palma**, L. de P. Carvalho, T. E. Rosa, C. F. Morais, and R. C. L. F. Oliveira. $\mathcal{H}_2$ filtering through multi-hop networks: trade-off analysis between the network

consumption and performance degradation. Proc. *in 2017 CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)* Pucon, Chile, 18-20 October 2017. <10.1109/CHILECON.2017.8229621>.

One paper published in a national conference

1. **J. M. Palma**, C. F. Morais, L. de P. Carvalho and R. C. L. F. Oliveira. "Proposta de modelo de canal markoviano para projetos de filtros H_∞ *in Anais do XIII Congresso Brasileiro de Automação Inteligente*, pp. $554 - 560$, Porto Alegre, RS, Brasil, Outubro 2017.

As result of partnerships with other institutions and researchers, the PhD candidate also has produced ten papers (two of them published in international journals and eight in conferences proceedings) between March-2016 to November-2019. The journal papers are listed below:

1. L. de P. Carvalho, **J. M. Palma**, C. Duran-Faundez and A. P. C. Gonçalves. Vehicle Following problem: A Control Approach for Uncertain Systems with Lossy Networks. In *IEEE Latin America Transactions*. Vol. 16, no. 9 pp. $2392 - 2399$, September 2018. <10.1109/TLA.2018.8789560>

2. **J. M. Palma**, L. de P. Carvalho, A. P. C. Gonçalves, C. E. Galarza and A. M. de Oliveira. Networked Control Systems Application: Minimization of the global number of interactions, transmissions and receptions in Multi-Hop network using Discrete-Time Markovian Jump Linear Systems. In *IEEE Latin America Transactions*. Vol. 14, no. 6 pp. $2675 - 2680$, August 2016. <10.1109/TLA.2016.7555237>

The conference papers are listed next:

1. L. de P. Carvalho, **J. M. Palma** and O. L. V. Costa. $\mathcal{H}_\infty/\mathcal{H}_-$ Residual Generator for Markovian Jump Linear Systems. Proceedings of the *2019 IEEE CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)*, Valparaiso, Chile, November 2019.

2. H. O. Garces, **J. M. Palma**, A. Rojas and V. Valdebenito . Model Analysis of Heat Transfer by Hammerstein systems and optical instrumentation. Accepted *to the 12th IFAC Symposium on Dynamics and Control of Process Systems, including Biosystems*. Florianópolis, Brazil, April 2019.

3. F. J. Uribe, C. F. Morais and **J. M. Palma**. Protocol for Energy-Efficiency using Robust Control on WSN. Expanded Abstract, Proc. *in XIV IEEE Latin American Summer School on Computational Intelligence (EVIC)*. USACH Santiago-Chile, 12-14 December 2018. Available in arXiv:1812.07011.

4. L. de P. Carvalho, **J. M. Palma**, C. Duran-Faundez and A. P. C. Gonçalves. Applying Polytopic Uncertainty in the Vehicle-Following Problem with Lossy Networks. *Proc. IEEE CHILECON2017*. Proc. *in 2017 CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)* Pucon, Chile, 18-20 October 2017.<10.1109/CHILECON.2017.8229623>.

5. V. Gutierrez, L. Gonzalez, C. Hernandez, C. Duran-Faundez, E. Rubio, A. M. Sosa, K. Herman and **J. M. Palma**. Wireless control of a coupled tanks system: A case study. Proc. *in 2017 CHILEAN Conference on Electrical, Electronics Engineering, Information and Communication Technologies (CHILECON)* Pucon, Chile, 18-20 October 2017. <10.1109/CHILECON.2017.8229628>.

6. **J. M. Palma**, L. de P. Carvalho and A. P. C. Gonçalves. Proposta de Modelo de Consumo de Energia para Redes WSN. Proc. *in* 9^{th} *Encontro dos Alunos e Docentes do Departamento de Engenharia de Computação e Automação Industrial (IX-EADCA)*. 10-11 September, 2016. Campinas, SP-Brazil.

7. L. de P. Carvalho, **J. M. Palma**, Lucas and A. P. C. Gonçalves. Influence of Burst Failures on the H_∞ Norm. Proc. *in* 9^{th} *Encontro dos Alunos e Docentes do Departamento de Engenharia de Computação e Automação Industrial (IX-EADCA)*. 10-11 September, 2016. Campinas, SP-Brazil.

8. L. de P. Carvalho, **J. M. Palma** and A. P. C. Gonçalves. Output Feedback Control applied to Car Pursuit problem with lossy network. Proc. *in 2016 IEEE Biennial Congress of Argentina (ARGENCON)*, 15-17 June 2016. Buenos Aires – Argentina. <10.1109/ARGENCON.2016.7585264>

Finally, other relevant productions of the PhD candidate correspond to three R-Package presented in the sequence:

1. C. E. Galarza and **J. M. Palma**. Package '`Opportunistic`': Routing Distribution, Broadcasts, Transmissions and Receptions in an Opportunistic Network, versions $1-1.2$. The version 1.2 is available online at <https://CRAN.R-project.org/package=Opportunistic>, published: $2017-06-27$.

2. C. E. Galarza and **J. M. Palma**. Package '`hopbyhop`': Transmissions and Receptions in a Hop by Hop Network, versions $1-2.1$. The version 2.1 is available online at <https://CRAN.R-project.org/package=hopbyhop>, published: $2016-09-13$.

3. C. E. Galarza and **J. M. Palma**. Package '`endtoend`': Transmissions and Receptions in an End to End Network, version 1. The version 1 is available online at <https://CRAN.R-project.org/package=endtoend>, published: $2016-09-11$.

1 The R Project for Statistical Computing

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A Appendix - Mathematical Framework: Stability and Performance Analysis for discrete-time LPV Systems

This Appendix presents parameter-dependent LMI conditions for stability analysis and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm computation for discrete-time LPV systems.

Consider the following discrete-time LPV system

$$\mathcal{G} = \begin{cases} x(k+1) & = A(\theta(k))x(k) + E(\theta(k))w(k) \\ z(k) & = C_z(\theta(k))x(k) + E_z(\theta(k))w(k) \end{cases} \tag{A.1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $w(k) \in \mathbb{R}^{n_w}$ and $z(k) \in \mathbb{R}^{n_z}$, respectively denote the state, external disturbance and output vectors. The matrices of the system depend polynomially on the time-varying parameter vector $\theta(k)$, which lies in a compact set Λ for all $k \geq 0$.

Next corollary presents a stability analysis condition expressed in terms of parameter-dependent LMIs (DE CAIGNY *et al.*, 2010).

Theorem A.1. *System* (A.1) *is asymptotically stable if there exists a symmetric positive definite parameter-dependent matrix* $P(\theta(k))$ *such that*

$$A(\theta(k))'P(\theta((k+1))A(\theta(k)) - P(\theta(k)) < 0, \tag{A.2}$$

hold for all $(\theta(k) \times \theta(k+1)) \in \Lambda \times \Lambda$.

Theorem (A.1) is based on the existence of a quadratic Lyapunov function depending generically on $\theta(k)$ and constitutes an infinite dimensional problem. In the context of LPV systems, the computation of performance indexes based on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are also important.

A.0.1 $\mathcal{H}_2$ guaranteed cost analysis

For an asymptotically stable system in the form (A.1), its $\mathcal{H}_2$ performance in time-domain is defined as

$$||\mathcal{G}||_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\{z(k)'z(k)\}, \tag{A.3}$$

when the input $w(k)$ is driven by a white-noise Gaussian process with null mean and identity covariance. See De Caigny *et al.* (2010) for more details. Next theorem , borrowed from De Caigny *et al.* (2010) (where the proof can be found) presents parameter-dependent LMIs that can be used to compute upper bounds (guaranteed costs) for the $\mathcal{H}_2$ norm of system (A.1).

Theorem A.2. *Consider system (A.1) as asymptotically stable,*

(a) *for a given positive scalar ρ, if there exist symmetric positive definite parameter-dependent matrices $P(\theta(k))$ and $W(\theta(k))$, such that*

$$\begin{bmatrix} P(\theta(k+1)) - A(\theta(k))P(\theta(k))A(\theta(k))' & \star \\ E(\theta(k))' & I \end{bmatrix} > 0 \tag{A.4}$$

and

$$\begin{bmatrix} W(\theta(k)) - E_z(\theta(k))E_z(\theta(k))' & \star \\ P(\theta(k))C_z(\theta(k))' & P(\theta(k)) \end{bmatrix} > 0, \tag{A.5}$$

$$Tr(W(\theta(k))) < \rho^2 \tag{A.6}$$

hold for all $(\theta(k), \theta(k+1)) \in \Lambda \times \Lambda$, then ρ is an upper bound for the $\mathcal{H}_2$ norm.

(b) *for a given positive scalar γ, if there exist symmetric positive definite parameter-dependent matrices $S(\theta(k))$ and $Q(\theta(k))$, such that*

$$\begin{bmatrix} Q(\theta(k)) - A(\theta(k))'Q(\theta(k+1))A(\theta(k)) & \star \\ C_z(\theta(k))' & I \end{bmatrix} > 0 \tag{A.7}$$

and

$$\begin{bmatrix} S(\theta(k)) - E_z(\theta(k))E_z(\theta(k))' & \star \\ Q(\theta(k+1))E(\theta(k)) & Q(\theta(k+1)) \end{bmatrix} > 0, \tag{A.8}$$

$$Tr(S(\theta(k))) < \gamma^2 \tag{A.9}$$

hold for all $(\theta(k), \theta(k+1)) \in \Lambda \times \Lambda$, then γ is an upper bound for the $\mathcal{H}_2$ norm.

Next section presents parameter-dependent LMIs that can be used to compute upper bounds (guaranteed costs) for the $\mathcal{H}_\infty$ norm of system (A.1).

A.0.2 $\mathcal{H}_\infty$ guaranteed cost analysis

Another important performance criterion used in robust control and filter design is the $\mathcal{H}_\infty$ norm. In the context of LPV systems, it is defined by the quantity

$$\|\mathcal{G}\|_\infty = \sup_{\|w(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2}, \quad w(k) \in \mathbb{L}_2 \tag{A.10}$$

where $\mathbb{L}_2$ is the class of square-summable sequences. Next theorem, borrowed from de Souza *et al.* (2006) (where the proof can be found) provides a procedure to compute guaranteed costs for the $\mathcal{H}_\infty$ norm using parameter-dependent LMIs.

Theorem A.3. *Consider system (A.1) as asymptotically stable. If there exist a symmetric positive definite parameter-dependent matrix $P(\theta(k))$ and a parameter-dependent matrix*

$G(\theta(k))$, such that

$$\begin{bmatrix} P(\theta(k+1)) & \star & \star & \star \\ G(\theta(k))'A(\theta(k))' & G(\theta(k))' + G(\theta(k)) - P(\theta(k)) & \star & \star \\ E(\theta(k)) & 0 & \eta I & \star \\ 0 & C_z(\theta(k))G(\theta(k)) & E_z(\theta(k)) & \eta I \end{bmatrix} > 0 \qquad \text{(A.11)}$$

hold for all $(\theta(k), \theta(k+1)) \in \Lambda \times \Lambda$, then η is an upper bound for the $\mathcal{H}_\infty$ norm.

Finite dimensional tests based on LMIs to solve the conditions of Theorems A.1, A.2 and A.3 can be obtained by employing polynomial structures for the optimization variables and applying polynomial positivity tests according to the structure of the set Λ (OLIVEIRA; PERES, 2007; AGULHARI *et al.*, 2019).

B Appendix - Computational Framework: solving parameter-dependent LMIs using ROLMIP

This Appendix presents the computational framework to solve parameter-dependent LMIs using the Robust LMI Parser (ROLMIP) (AGULHARI *et al.*, 2019), which was employed in all numerical experiments of this thesis. As an example, the problem of robust stability analysis of a discrete-time LPV system with polytopic structure is presented.

Consider the following discrete-time LPV system

$$x(k+1) = A(\theta(k))x(k) \tag{B.1}$$

where $x(k) \in \mathbb{R}^{n_x}$ denotes the state vector and $A(\theta(k))$ is a time-varying polytopic matrix given in the form

$$A(\theta(k)) = \sum_{i=1}^{N} \theta_i(k)A_i, \;\; \theta(k) \in \Lambda_N, \tag{B.2}$$

where Λ_N is the unit simplex of dimension N. The robust stability of system (B.1) can be checked by the conditions presented next, which are the same used in Theorem 3.1.

Corollary B.1. *If there exist a symmetric positive definite parameter-dependent matrix $P(\theta(k))$ and a parameter-dependent matrix $G(\theta(k))$ such that*

$$\begin{bmatrix} P(\theta(k+1)) & \star \\ A(\theta(k))G(\theta(k)) & P(\theta(k)) - G(\theta(k)) - G(\theta(k))' \end{bmatrix} > 0 \tag{B.3}$$

holds for all $(\theta(k), \theta(k+1)) \in \Lambda_N \times \Lambda_N$, then system (B.1) is robustly stable.

The first step to derive finite dimensional tests is to impose a particular structure for the optimization variables $P(\theta(k))$ and $G(\theta(k))$. For instance, consider $\theta(k) \in \Lambda_2$ and assume that $P(\theta(k))$ and $G(\theta(k))$ depend affinely on $\theta(k)$, that is,

$$P(\theta(k)) = \theta_1(k)P_1 + \theta_2(k)P_2, \quad G(\theta(k)) = \theta_1(k)G_1 + \theta_2(k)G_2, \quad \theta(k) \in \Lambda_2, \tag{B.4}$$

Considering that $\theta(k)$ varies arbitrarily fast over time, the value of $\theta(k+1)$ can be considered completely independent of $\theta(k+1)$, say $\beta(k) \in \Lambda_2$. In this case one has

$$P(\theta(k+1)) = P(\beta(k)) = \beta_1(k)P_1 + \beta_2(k)P_2. \tag{B.5}$$

Adopting the structures given in (B.4) and the hypothesis established in (B.5), inequality (B.3) can be programmed using the Robust LMI Parser, that applies Pólya's relaxations to check the positivity of the polynomial matrix inequality (depending on θ and β). The code presented next illustrates how the programming procedure can be done. It is assumed that the dynamic matrix $A(\theta)$ is informed in terms of its vertices, given as a cell array.

Matlab code.

```
function output = stab_d(A)

% >> Determines the stability of the system
%           dx(t)/dt = A(\theta(k))x(k)
% >> solving
% [P(\theta(k+1))                G(\theta(k))'A(\theta(k))'                  ;
%  A(\theta(k))G(\theta(k))      P(\theta(k))-G(\theta(k))-G(\theta(k))']>0
%
% Using a Lyapunov matrix P(\theta(k)) and a slack variable G(\theta(k))
% with affine parameter-dependency (degree one) on \theta(k).

vertices_A = size(A,2);
order_A = size(A{1},1);

% Converting the vertices of matrix A in a rolmipvar object
A = rolmipvar(A, 'A', vertices_A, 1);

%   Variable
G = rolmipvar(order_A,order_A,'G(k)','full',vertices_A,1);

% Lyapunov function P(\theta(k))
P = rolmipvar(order_A,order_A,'P(k)','symmetric',vertices_A,1);

% Lyapunov function in k+1, P(\theta(k+1)), arbitrarily fast variation
Pk1 = fork(P,'P(k+1)')




% LMI condition
LMIstd = [Pk1, (A*G)' ;
          A*G, P-G'-G] ;
LMIs = LMIstd >= 0;
```

```
sol = optimize(LMIs,[],sdpsettings('verbose',0,'solver','mosek'));

if min(checkset(LMIs)) > 0
print('the system is stable');
end
end
```

Note that the command `fork` is being used to shift matrix $P(\theta)$ to a different simplex, providing $P(\beta)$. If higher degrees for the optimization variables are required, pottentially leading to less conservative results, only the last input parameter of command `rolmipvar` needs to be changed (the rest of the code is the same). See (AGULHARI *et al.*, 2019) for more details about how to deal with continuous- and discrete-time LPV systems, including the case of bounded rates of variations.

See also (LOBO *et al.*, 2019) for a tutorial about using ROLMIP to deal with biomathematical systems modeled as LPV systems, available online at the GitHub Repository[1].

[1] Repository "Linear Parameter Varying system Toolbook for Biomathematical" in GitHub, Link <https://github.com/JonathanMPalma/Linear-Parameter-Varying-system-Toolbook-for-Biomathematical>