SS-traveltime parameters from PP and PS reflections

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ABSTRACT

The SS-wave traveltimes can be derived from PP- and PS-wave data with the previously derived PP + PS = SS method. We have extended this method as follows. (1) The previous requirement that sources and receivers be located on a common acquisition surface is removed, which makes the method directly applicable to PS-waves recorded on the ocean bottom and PP-waves recorded at the ocean surface. (2) By using the concept and properties of surface-to-surface propagator matrices, the second-order traveltime derivatives of the SS-waves are obtained. In the same way as for the original PP + PS = SS method, the proposed extension is valid for arbitrary anisotropic media. The propagator matrix and geometric spreading of an SS-wave reflected at a given point on a target reflector are obtained explicitly from the propagators of the PP- and PS-waves reflected at the same point. These additional parameters provided by the extended PP + PS = SS method can be used for a partial reconstruction of the SS-wave amplitude as well as for tomographic estimation of the elastic velocity model. A full simulation of the SS-wave, which includes reflection and transmission coefficients, cannot be obtained directly from recorded PP- and PS-wave amplitudes.

INTRODUCTION

In most situations, combined use of P- and S-wave information is widely recognized as crucial for reliable imaging and inversion of seismic data of reservoir exploration and monitoring interest. Conventional processing of PP-waves alone is not enough to assess structural properties such as anisotropy, nor to account for lithology complications, such as gas clouds in reservoir overburden. Present acquisition, e.g., from ocean-bottom and land seismics, rely on P-wave source excitations giving rise essentially to PP- and PS-wavelfield data. In this way, shear-wave velocities can be derived only from converted PS-waves included in the data. This fact has been responsible for the high interest in the development of processing/imaging methodologies and tools that can effectively extend the classical ones available for nonconverted waves. An overview of the various converted-wave processing methods and their application in seismic exploration is presented in Stewart et al. (2002).

As is well known (see, e.g., Tsvankin and Behle, 1988; Tsvankin and Thomsen, 1994; Thomsen, 1999), the treatment of PS-converted waves has significant differences from the one applied to nonconverted PP-waves. Because of asymmetrical raypaths, conversion-point determinations, and sensitivity to anisotropy, PS-reflection moveouts are essentially nonhyperbolic and multiparametric, leading to more involved velocity analysis and parameter estimations. Nevertheless, especially in recent years, significant progress has been made, allowing processing and imaging of converted waves to be carried out in a similar way as for nonconverted waves. See, e.g., Dai et al. (2007), Li et al. (2007), and further references therein.

In principle, processing of SS-waves, if available in the seismic data, would parallel the one routinely carried out for PP-waves to provide corresponding S-wave information. In the near-offset situation, NMO velocities or normal-incidence-point (NIP) wave curvatures (Hubral, 1983) can be found from a conventional (time-domain) velocity analysis. The NMO velocities correspond to second-order derivatives of nonconverted wave traveltimes, assumed to be of type PP or SS. In this way, one avoids dedicated traveltime processing to obtain these velocities. Along with the traveltimes and slopes, the NIP wave curvatures can be applied to tomographic inversion in isotropic models for the corresponding P- and S-wave velocity fields. See, e.g., Iversen and Gjøystdal (1984) and Duveneck (2004) for the PP situation.

Grechka and Tsvankin (2002b) introduced a method to (kinematically) simulate SS reflections by means of a suitable combination of PP and PS reflections. More specifically, the method, referred to as
the PP + PS = SS method, selects identified traveltimes and slopes of PP and PS reflections of the same reflector to produce the corresponding SS-reflection traveltimes and slopes from that reflector.

The obtained SS reflections can be incorporated into the original seismic volume as SS-reflection data and processed in the same way as PP-waves. The PP + PS = SS methodology has been used in Foss et al. (2005) for depth-consistent tomography of PP and PS reflections. For a few key reflectors, the zero-offset PP and PS traveltimes were used to estimate the SS-reflection times. Then these were used along with the PP traveltimes for reflector cophasing.

A natural question is whether the knowledge of the second-order traveltime derivatives of PP- and PS-waves, as provided by their ray propagator matrices, leads to the ray propagator matrix of the corresponding SS-wave. In this paper, we show that the answer to this question is affirmative. By using the algebra of ray propagator matrices, as described in, e.g., Červený (2001), an explicit relationship between the involved PP-, PS-, and SS-ray propagator matrices is achieved. By examining the relationships that exist between the coefficients of the second-order Taylor expansion of traveltime and the submatrix components of the PP-, PS-, and SS-ray propagator matrices, we see that the new results provide the second-order derivatives of SS traveltime, thus extending the counterpart zero- and first-order derivatives provided by the original PP + PS = SS method of Grechka and Tsvankin (2002b). Moreover, the assumption that sources and receivers are located on a common planar acquisition surface is removed, which makes the method directly applicable to PS-waves recorded on the ocean bottom and PP-waves recorded at the ocean surface.

A main application of the proposed extension of the PP + PS = SS method is that it opens the way for tomographic reconstruction of the elastic velocity model using first-order derivatives (slopes), as well as second-order derivatives (curvatures) of SS traveltimes. Furthermore, the described method yields the relative geometric spreading for the SS-wave. We remark, in passing, that the relative geometric spreading in anisotropic media is controlled by parameters that are responsible also for time-domain processing (Ursin and Hoks-tad, 2003; Xu et al., 2005).

The SS-wave relative geometric spreading is estimated here with very limited knowledge required about the velocity model. More specifically, for anisotropic conditions along the acquisition surface, the only formal requirement is that the parameters determining the S-wave propagation, e.g., P and S velocities along an axis and related Thomsen-type parameters, are known along this surface (see Appendix A). Therefore, by providing the relative geometric spreading for the SS-wave, our method permits a partial reconstruction of the S-wave amplitude. It is, however, not possible to obtain a full simulation of the SS-wave, including the effects of reflection and transmission coefficients, based only on observed PP- and PS-wave amplitudes. We discuss this topic in Appendix B.

The method makes it possible to perform an accurate prediction of the SS-wave traveltime parameters and, with the knowledge of the S-wave velocity at the surface, the SS-geometric spreading. The SS-reflection coefficient cannot be retrieved, however. Thus, the method can be regarded as a partial modeling approach, with the great advantage that it requires very limited knowledge of the velocity model. It remains to be seen whether the obtained attributes also can be useful for seismic imaging: this is an interesting topic for further research.

In the first section, we review the basic properties of surface-to-surface traveltime approximations. Then we derive in two subsequent sections the SS-wave surface-to-surface propagator matrix corresponding to the finite-offset and zero-offset situations. The section following these is devoted to the presentation of a technique applying slope matching of PP and PS events. This is the key procedure for determining (virtual) source and receiver positions of the SS events. Finally, a numerical-example section describes a test of the method for a model similar to the one used by Grechka and Tsvankin (2002b).

**SURFACE-TO-SURFACE TRAVELTIME APPROXIMATIONS**

The formulation and main derivations of the extended PP + PS = SS method proposed here can be conveniently described using the concepts and basic properties of surface-to-surface propagator matrices. These are summarized briefly below, following Červený (2001) as a main reference. For the zero-offset situation, Bortfeld (1989) and Iversen (2006) are to be cited. Specific treatment of traveltime is given in Schleicher et al. (1993).

Figure 1 shows a fixed (central) ray that connects a (central) source point at an anterior surface \( \Sigma^1 \) to a (central) receiver point at a posterior surface \( \Sigma^2 \). The central ray traverses a medium consisting of inhomogeneous anisotropic layers bounded by curved interfaces. We assume that both anterior and posterior surfaces, as well as all interfaces and the medium within the layers, are sufficiently smooth so that wave propagation is well described by zero-order ray theory. Orthogonal curvilinear coordinates \((x_1^r, x_2^r, x_3^r)\) and \((x_1^r, x_2^r, x_3^r)\) are associated with, respectively, the anterior and posterior surfaces, in such a way that \( x_3^r \) and \( x_3^r \) are both constants (e.g., equal zero) along these surfaces. For the first two components of the curvilinear coordinates, we use the vector/matrix notations \( x^r = (x_1^r, x_2^r)^T \) and \( x^r = (x_1^r, x_2^r)^T \), where superscript T means transposition. To avoid an unnecessary complication of terminology, we have not introduced a specific, parallel notation for points in 3D space. This means that we refer to such a point as, e.g., “the point \( x^r \),” although, strictly speaking, the vector \( x^r \) specifies only two (curvilinear) coordinates of the point under consideration.

![Figure 1. Central and paraxial rays from anterior \( \Sigma^1 \) through posterior \( \Sigma^2 \) surfaces. Points \( x^r \) and \( x^r \) are central and paraxial sources specified in orthogonal curvilinear coordinates on the anterior surface. Points \( x^r \) and \( x^r \) are corresponding central and paraxial receivers specified on the posterior surface.](image-url)
Figure 1 finally shows an arbitrary paraxial ray that starts on surface \( \Sigma^1 \) and ends on surface \( \Sigma^2 \). By definition, the paraxial ray has the same wave-mode signature and a sufficiently close trajectory to the central ray. Under the present assumptions of smooth model parameters and interfaces, any such paraxial ray is determined completely by the differences in the curvilinear coordinates of the initial and end points relative to the central source and receiver points, respectively.

Attached to the central ray, the \( 4 \times 4 \) surface-to-surface ray propagator matrix has the form (see Červený, 2001, Section 4.4.7)

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where \( A, B, C, \) and \( D \) are \( 2 \times 2 \) constant submatrices. Matrix \( T \) incorporates the dynamic quantities (second-order derivatives of traveltime) of the central ray, as well as the properties of the anterior and posterior surfaces, and those of the medium in which the central ray propagates. The propagator matrix then expresses the first-order relationship

\[
\begin{pmatrix} \Delta x' \\ \Delta p' \end{pmatrix} = T \begin{pmatrix} \Delta x \\ \Delta p \end{pmatrix}
\]

between the relative differences with respect to the central ray of the projections of the position and slowness vectors onto the anterior and posterior surfaces. More explicitly, for any given paraxial ray, these relative differences can be written as

\[
\begin{pmatrix} \Delta x' \\ \Delta p' \end{pmatrix} = \begin{pmatrix} x' - x^0 \\ p' - p^0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Delta x \\ \Delta p \end{pmatrix} = \begin{pmatrix} x - x^0 \\ p - p^0 \end{pmatrix},
\]

where entities labeled with superscript zero belong to the central ray.

The propagator matrix satisfies the relationships (see Červený, 2001, Section 4.4.7)

\[
T^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},
\]

and

\[
T^{rev} = \begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix},
\]

which are natural consequences of ray theory. In equation 5, the operation signified by the superscript “rev” implies that the resulting propagator matrix on the left-hand side corresponds to the reverse ray direction, i.e., the direction from \( x' \) to \( x \) rather than from \( x \) to \( x' \). From equations 1 and 4, we obtain

\[
C = (DA^{-T} - I)B^{-T},
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and the superscript \( -T \) denotes the transpose of the inverse matrix.

The propagator matrix also satisfies the important continuation property or chain rule, which states that if \( T(x', x^0) \) is the propagator matrix of a ray that connects \( x^0 \) to \( x' \), and for a given intermediate point \( x \) along the ray, \( T(x', x) \) and \( T(x', x^0) \) are the propagator matrices for the ray segments from \( x \) through \( x' \) and from \( x^0 \) through \( x' \), respectively, then (see Červený, 2001, Section 4.3.4 and 4.4.7)

\[
T(x^0, x^a) = T(x^0, x^a)T(x^a, x').
\]

The second-order Taylor expansion of traveltime of a paraxial ray in terms of its relative source and receiver coordinates, \( \Delta x' \) and \( \Delta x^0 \), is given by (see, e.g., Schleicher et al., 1993)

\[
\begin{align*}
\tau(x', x^0) &= \tau^0 + (p^0)^T \Delta x' - (p^0)^T \Delta x^0 + (\Delta x')^T M'^r \Delta x' \\
&\quad + \frac{1}{2} (\Delta x')^T M''^{ss} \Delta x' + \frac{1}{2} (\Delta x^0)^T M''^{ss} \Delta x^0,
\end{align*}
\]

where \( \tau^0 = \tau(x^0, x^0) \) is the traveltime along the central ray and

\[
\begin{align*}
p^0 &= \left( \frac{\partial \tau}{\partial x'} \right) \\
p^0 &= \left( \frac{\partial \tau}{\partial x^0} \right)
\end{align*}
\]

are the coefficients of the linear terms or slowness. Moreover,

\[
\begin{align*}
M'^r &= \left( \frac{\partial^2 \tau}{\partial x' \partial x'} \right), \\
M''^{rs} &= \left( \frac{\partial^2 \tau}{\partial x' \partial x^0} \right), \\
M''^{sr} &= \left( \frac{\partial^2 \tau}{\partial x^0 \partial x'} \right), \\
M''^{ss} &= \left( \frac{\partial^2 \tau}{\partial x^0 \partial x^0} \right)
\end{align*}
\]

are the coefficients of the quadratic terms. All the derivatives are evaluated for \( x' = x^0 \) and \( x' = x^0 \). The symmetrical matrix \( M'^r \) is related to the wavefront curvatures at \( x^0 \) for a point source at \( x^0 \), and to the wavefront curvatures at \( x^0 \) for a point source at \( x^0 \). The matrix \( M''^{ss} \) of second-order mixed derivatives is related to the relative geometric spreading (see Appendix A).

By squaring equation 8 and retaining the terms as high as second order only, we obtain the more commonly used Taylor series for traveltime squared:

\[
\begin{align*}
\tau(x', x^0)^2 &= \left[ \tau^0 + (p^0)^T \Delta x' - (p^0)^T \Delta x^0 \right]^2 \\
&\quad + \tau^0 \cdot [2(\Delta x')^T M''^{ss} \Delta x' + (\Delta x')^T M''^{ss} \Delta x'] \\
&\quad + (\Delta x')^T M''^{ss} \Delta x'.
\end{align*}
\]

From the basics of Taylor series expansions, the approximations 8 and 11 are valid for sufficiently small arguments, \( x' \) and \( x^0 \), meaning that paraxial rays should be sufficiently close to the central ray. A general quantification of the accuracy of the approximations is impossible, as it depends on the properties of the medium, basically smoothness of medium parameters and interfaces. In spite of their limited accuracy, Taylor approximations have been used with quite good success in seismic imaging.

The traveltime approximations in equations 8 and 11 have the same structure as the ones defined for paraxial rays with arbitrary 3D relative source and receiver coordinates (see, e.g., Ursin, 1982). In this situation, the linear coefficients \( p^0 \) and \( p^0 \) would be given by three-component vectors, and the quadratic coefficients \( M'^r \), \( M''^{ss} \), and \( M''^{ss} \) would be \( 3 \times 3 \) matrices. The explicit relationship between the 3D and surface-to-surface traveltime Taylor expansions is presented in Schleicher et al. (1993). The expansions 8 and 11, in surface-to-surface or generally three dimensions, are valid for a single traveltime branch in a heterogeneous anisotropic medium.

As shown in numerical examples by Gjistvedal et al. (1984), in the 3D situation, there are cases when equation 8 is more accurate than equation 11 and other cases when the opposite is true. The range of validity of these equations might be very small, however, or even nonexistent. This was shown by Tygel et al. (2007) for quasi-SV reflections in a horizontally layered vertical transversely isotropic (VTI) medium. In this case, there can be a traveltime triplication on
the vertical axis, which strongly limits the range of validity of the power series expansion.

With the help of the propagator matrix, we have the important relationships (see Červený, 2001, Section 4.4.7; Schleicher et al., 1993)

\[ M'' = DB^{-1}, \quad M''' = B^{-1}A, \quad \text{and} \quad M'' = -B^{-1}. \]  

(12)

From these, we obtain

\[ B = -(M'')^{-1}, \quad A = -(M'')^{-1}M''' \quad \text{and} \quad D = -M''(M'')^{-1}, \]  

(13)

so that from equation 6,

\[ C = (M'')^T - M''(M'')^{-1}M''. \]  

(14)

In the following, we shall assume that the traveltime Taylor series expansions are valid, and that the traveltime parameters \( \rho^, p^, p^, M^, M'', \) \( M'' \) and \( M''' \) can be estimated from the seismic data.

**SS-TRAVELTIME PARAMETERS**

We consider survey configurations with source and receiver points distributed on two surfaces, as follows. Acquisition surface 1 (e.g., the ocean surface), denoted as \( \Sigma^1 \), shall consist of source points for PP-reflected and PS-reflected waves and receiver points for PP-reflected waves. Acquisition surface 2 (typically the ocean bottom), denoted as \( \Sigma^2 \), shall consist of receiver points for PS-reflected waves. As a special case (land seismics), these two surfaces coincide. For the common situation in ocean-bottom seismics that the PP-reflected waves actually have been recorded on the ocean bottom (surface \( \Sigma^2 \)), it will be necessary either to perform receiver-redatuming of the PP-reflected events to surface \( \Sigma^1 \), or to perform source-redatuming of all PP- and PS-reflected events to surface \( \Sigma^2 \).

With the above requirements on sources and receivers, we consider one PP-wave and two PS-waves for the same reflection point \( y \), as outlined in Figure 2. The slowness vectors and the normal vector \( n \) of the reflecting interface at the point \( y \) all lie in a common plane (Snell’s law). The source and receiver points corresponding to the PP reflection at \( y \) are denoted, respectively, as \( x^b \) and \( x^d \). Correspondingly, the receiver points of the two PS-waves reflected at \( y \) are denoted \( x^s \) and \( x^d \). As also shown in Figure 2, the PP ray intersects surface \( \Sigma^2 \) at the points \( x^b \) and \( x^d \), respectively.

In the above perspective, the points \( x^b \) and \( x^d \) can be considered as virtual source and receiver points of an SS-wave reflected at the point \( y \). Moreover, for any given PP-wave source/receiver couple \( (x^b, x^d) \), the corresponding SS-wave virtual source/receiver couple \( (x^s, x^d) \), will be unknown. To determine \( (x^s, x^d) \), we start by identifying a PP reflection for the couple \( (x^s, x^d) \) and estimate its traveltime parameters. Next the PS-waves from \( x^b \) through \( x^s \) and from \( x^d \) through \( x^s \) are identified so that the slowness vectors of the P-waves at \( x^b \) and \( x^d \) are parallel to the corresponding PP-wave slowness vectors at the same points (Grechka et al., 2002; Grechka and Tsvankin, 2002b). The traveltime parameters for the two PS-waves are estimated also. Then it is straightforward to show that the traveltime for the SS-wave satisfies the equation derived by Grechka and Tsvankin (2002b) under the assumption of a common acquisition surface for all sources and receivers, namely,

\[ f^{SS}(x^s, x^d) = f^{SP}(x^s, x^d) + f^{SP}(x^b, x^d) - f^{PP}(x^b, x^d). \]  

(15)

Using the continuation property of the propagator matrix (see equation 7), the surface-to-surface ray propagators of PP and PS reflections are given by

\[ T^{PP}(x^d, x^b) = T^P(x^d, x^b)T^P(x^b, y)T^P(y, x^d), \]  

\[ T^{SP}(x^d, x^b) = T^S(x^d, y)T^P(y, x^b)T^P(x^b, x^d), \]  

\[ T^{SP}(x^b, x^d) = T^S(x^b, y)T^P(y, x^d)T^P(x^d, x^b). \]  

(16)

Here, the superscripts denote the wave types and the indexing is from right to left (as in equation 15). For further use, we also note that

\[ [T^{SP}(x^b, x^d)]^{-1} = T^{PS}(x^d, x^b) = T^P(x^d, x^b)T^P(x^b, y)T^S(y, x^b), \]  

(17)

and hence,

\[ T^{S}(y, x^d) = [T^{P}(x^d, y)]^{-1}[T^{P}(x^d, x^b)]^{-1}[T^{SP}(x^b, x^d)]^{-1}, \]  

(18)

where the inverse and reverse matrices are given in equations 4 and 5, respectively.

Furthermore, the first two subrelations in equation 16 can be rearranged as

\[ T^{P}(x^d, y) = [T^{P}(x^d, x^b)]^{-1}T^{PP}(x^d, x^b) \]  

\[ \times [T^{P}(x^b, x^d)]^{-1}[T^{P}(y, x^b)]^{-1}, \]  

\[ T^{S}(x^b, y) = T^{SP}(x^b, x^d) \]  

\[ \times [T^{P}(x^d, x^b)]^{-1}[T^{P}(y, x^b)]^{-1}. \]  

(19)

Using the continuation property in equation 7, the SS-wave propagator matrix can be factored as

\[ T^{SS}(x^s, x^b) = T^{S}(y, x^b)T^{S}(y, x^d). \]  

(20)

Then, by inserting equations 18 and 19 into equation 20, it follows that the surface-to-surface ray propagator for the SS-wave from \( x^b \) through \( x^d \) is given by
\[ T^{SS}(x', x^h) = T^{SP}(x', x^0)[(T^{PP}(x^0, x^0))^{-1}][T^{SP}(x^h, x^0)]^{rev}. \] (21)

Equation 21 constitutes the main theoretical result of this paper. From the ray propagator matrix for the SS-wave, we can compute the second-order traveltime parameters using equation 12.

**NORMAL-INCIDENCE REFLECTIONS**

It is well known that traveltime moves out such as the Taylor expansions considered above considerably simplify when the central ray is a normal-incidence ray. In particular, the moves out depend on a reduced number of coefficients, leading to more efficient estimation and inversion. These good properties justify a separate description of our proposed methodology to the case of normal-incidence reflections.

For PP events with sources and receivers along surface \( \Sigma^1 \), we can apply the commonly used transformation to midpoint/half-offset coordinates

\[ x = \frac{1}{2}(x' + x^0) \text{ and } h = \frac{1}{2}(x' - x^0). \] (22)

The same type of relations applies to SS events with sources and receivers along surface \( \Sigma^2 \); however, in that case, we label all quantities with a prime (e.g., \( x', h' \)) to distinguish the two situations. Because the source/receiver coordinates are curvilinear, this also will be the case for the midpoint/half-offset coordinates. For a normal-incidence ray (which is reflected at an interface so that the slowness vector is parallel to the interface normal, and the ray code up is equal to the ray code down), the source and receiver points coincide. The Taylor series for traveltime then becomes (see, e.g., Schleicher et al., 1993, equation 17)

\[ t(x, h) = t^0 + p^{0T} \Delta x + \frac{1}{2}(\Delta x M^{xx} \Delta x + h M^{hh} h), \] (23)

where \( t^0 = t(x^0, h = 0) \) is the two-way traveltime along the normal ray. Moreover, the two-component vector \( p^0 \) and \( 2 \times 2 \) matrix \( M^{xx} \) constitute, respectively, the first- and second-order traveltime derivatives for a given event in the zero-offset data cube. Finally, the \( 2 \times 2 \) matrix \( M^{hh} \) is composed of the second-order traveltime derivatives of the given event with respect to half-offset in the common-midpoint (CMP) configuration around the central (coincident) source-receiver pair.

Traveltime squared is given, always in second-order approximation, by (see, e.g., Schleicher et al., 1993, equation 18, and Ursin, 1982, equation 33, for the general 3D situation)

\[ t(x, h)^2 = (t^0 + p^{0T} \Delta x)^2 + t^0(\Delta x M^{xx} \Delta x + h M^{hh} h). \] (24)

We see that \( M^{hh} \) is closely related to the square of NMO velocities (the NMO ellipse; see Grechka and Tsvankin, 2002a). In this case (see also Ursin, 1982; Hubral, 1983),

\[ M^{rr} = \frac{1}{4}(M^{xx} + M^{hh}) = M^{rr} \text{ and } M^{rz} = \frac{1}{4}(M^{xx} - M^{hh}) = (M^{rr})^T. \] (25)

For the surface-to-surface ray propagator, we have (Iversen, 2006)

\[ A = D^T, \quad B = B^T, \quad \text{and } C = C^T. \] (26)

Using equations 13, 14, and 25, and applying some elementary matrix algebra, we obtain

\[ B = 4(M^{hh} - M^{rr})^{-1}, \quad A = (M^{hh} - M^{xx})^{-1}(M^{hh} + M^{xx}) = D^T, \quad C = M^{hh}(M^{hh} - M^{xx})^{-1}M^{xx}. \] (27)

This also means that matrix \( C \) satisfies the relation

\[ C^{-1} = (M^{xx})^{-1} - (M^{hh})^{-1}. \] (28)

Figure 3 shows a PP and an SS reflection with the same normal-incidence reflection point \( y \).

To obtain the ray propagator matrix for zero-offset SS-wave \( T^{SS}(x', x^0) \), we proceed as follows. First we identify the PP normal-incidence reflection at the surface point \( x \) and estimate its traveltime parameters. Next we identify the PS reflection with the receiver point at \( x' \) that has reflection slopes as the PP reflection we have identified, and then estimate the traveltime parameters for the PS-reflected wave. Then the traveltime for the normal-incidence SS reflection is (Foss et al., 2005)

\[ t^{SS}(x', x') = 2t^{SP}(x', x) + t^{PP}(x, x). \] (29)

The slope vector at \( x' \) for the SS reflection is

\[ p^{s'} = 2p', \] (30)

where \( p' \) is the corresponding slope vector at the receiver for the PS-reflected wave. The surface-to-surface ray propagator for the normal-incidence SS-wave is given by equation 21 with \( x^0 = x^0 = x \) and \( x^h = x = x' \), namely,

\[ T^{SS}(x', x') = T^{SP}(x', x)[T^{PP}(x, x)]^{-1}[T^{SP}(x', x)]^{rev}. \] (31)

![Diagram](https://via.placeholder.com/150)
In equation 31, the surface-to-surface ray propagator for the PS-wave must be computed with the general traveltime approximations 8 or 11 and with equation 27. The surface-to-surface ray propagator for the normal-incidence PP-wave can be computed from the simplified traveltime approximations 23 or 24 and using equation 27.

**SLOPE MATCHING BASED ON SECOND-ORDER TRAVELTIME DERIVATIVES**

The extended PP + PS = SS method described above makes use of computed second derivatives of PP- and PS-wave traveltimes. This offers the possibility of using such derivatives to match reflection slopes of PP and PS reflections.

Consider the problem of finding a root $\xi$ of the nonlinear vector equation $f(\xi) = 0$ using an iteration technique of the Newton-Raphson type. Under the assumption of equal dimensionality of $\xi$ and $f$, the inherent linearization of each iteration step yields the following update of $\xi$ with respect to the current solution $\xi^0$:

$$
\xi = \xi^0 - \left[ \frac{\partial f}{\partial \xi}(\xi^0) \right]^{-1} f(\xi^0). \quad (32)
$$

The slope matching consists of two independent steps, which collectively make use of equation 32. In the first step, we consider a PS-wave for which the source point is located at $x^*$ (Figure 2). The function $f$ then has the definition

$$
f(\xi) = \frac{\partial^2 P_P}{\partial x^0} (x^0, x^*) - \frac{\partial^2 S_P}{\partial x^0} (\xi, x^*), \quad (33)
$$

with the first derivatives given by

$$
\frac{\partial f}{\partial \xi} (\xi) = \frac{\partial^2 S_P}{\partial x^0} (\xi, x^*) = - M^r(\xi, x^*). \quad (34)
$$

The matrix $M^r$ in equation 34 belongs to the PS-wave. We let the iteration process proceed until the criterion $f(\xi) = 0$ is satisfied, which yields the sought solution $\xi = x^*$.

In the second step, we consider another PS-wave, having $x^*$ as its source point (Figure 2). The function $f$ and its derivatives now can be specified by

$$
f(\xi) = \frac{\partial^2 P_P}{\partial x^0} (x^*, x^0) - \frac{\partial^2 S_P}{\partial x^0} (\xi, x^0), \quad (35)
$$

$$
\frac{\partial f}{\partial \xi} (\xi) = \frac{\partial^2 S_P}{\partial \xi} (\xi, x^0) = - M^r(\xi, x^0). \quad (36)
$$

Again, the matrix $M^r$ belongs to the PS-wave. The output from this second iteration step is the solution $\xi = x^*$. Note especially that only one iterative slope-matching step is required in the normal-incidence situation.

In this way, the slope-matching procedure finds the (virtual) SS-wave source and receiver points $x^*$ and $x^0$, which correspond to the (real) source and receiver points for the recorded PP-waves. The convergence of this Newton-Raphson formulation is fast, requiring that the applied second-order traveltime derivatives are smooth functions. An alternative is to use a nonlinear inversion technique that does not require computation of the first derivatives of the function $f$ (and thus the second derivatives of the traveltimes). Such techniques generally are slower, although more robust, than the Newton-Raphson approach.

**NUMERICAL EXAMPLE**

In this section, we present a numerical example demonstrating the extended PP + PS = SS method. Our experiment is conducted with a model similar to the one used by Grechka and Tsvankin (2002b). The model is two-dimensional and consists of three homogeneous VTI layers. The layers are separated by smoothly curved interfaces, which were generated by digitizing the interfaces plotted in Grechka and Tsvankin’s (2002b) paper. For the latter reason, the models used by Grechka and Tsvankin (2002b) and by us are not exactly the same; however, in practice they can be considered equal. We consider quasi-P- and quasi-SV-wave types. Hence, in each layer, the wave propagation is described by four parameters, specified using Thomson’s (1986) representation:

- Top layer: $V_{P0} = 2.0 \text{ km/s}$, $V_{S0} = 0.8 \text{ km/s}$, $\alpha = 0.20$, $\delta = 0.10$
- Middle layer: $V_{P0} = 2.5 \text{ km/s}$, $V_{S0} = 1.25 \text{ km/s}$, $\alpha = 0.25$, $\delta = 0.05$
- Bottom layer: $V_{P0} = 3.0 \text{ km/s}$, $V_{S0} = 1.8 \text{ km/s}$, $\alpha = 0.15$, $\delta = 0.10$

**Test using noise-free input data**

Traveltime observations corresponding to PP and PS reflections from the lowest interface of the bottom layer were generated using ray tracing. Figure 4 shows (a) PP rays, and (b) PS rays for a common receiver at the horizontal coordinate $x = 0.5 \text{ km}$. The corresponding traveltime observations for all source and receiver locations are shown, respectively, in Figure 5a and b. The simulated true traveltimes for the SS reflections are displayed in Figure 5c. Comparing to Figures 5 and 8 in Grechka and Tsvankin (2002b), one finds a convincing consistency between the PP, PS, and SS traveltime maps and our models. We remark that the maps in Figure 5 are represented by a fifth-order B-spline function, motivated by the objective of obtaining reliable second derivatives to be used in the extended PP + PS = SS method. Apart from very small numerical errors resulting from the ray-tracing procedure and smoothing effects caused by the B-spline representation, the traveltime data in Figure 5 can be considered noise free.

Figure 6a-c shows the PP-, PS-, and SS-wave relative geometric spreading computed using the traveltime data in the corresponding subfigures of Figure 5. The simulated true SS-wave relative geometric spreading in Figure 6c is used below for comparison with the estimated results obtained using the extended PP + PS = SS method. The simulated relative geometric spreading for the PP- and PS-waves (Figure 6a and b) are not used in this method, but the two plots nevertheless serve to indicate the stability of the second derivatives of the observed PP- and PS-traveltime functions.

As shown in Appendix A, the relative geometric spreading $\mathcal{L}(x', x^0)$ can be factored in terms of the surface-to-surface relative geometric spreading $\mathcal{L}(x', x^*)$, and an obliquity factor $C(x', x^*) = |\cos \phi(x')\cos \phi(x^*)|^{1/2}$, containing the cosines of the phase angles at the source and receiver position, so that

$$
\mathcal{L}(x', x^0) = C(x', x^0) \mathcal{L}(x', x^*). \quad (37)
$$

Figure 7 shows the obliquity factor corresponding to the SS-wave relative geometric spreading plotted in Figure 6c. We observe that
the contribution of this factor to the relative geometric spreading is, at most, 3% in the current experiment.

In Figure 8, we have plotted the reconstructed SS-wave source-receiver pairs that form the basis of the estimation of SS-wave travel-times and relative geometric-spreading factors. This plot is to be compared with Figure 6 in Grechka and Tsvankin (2002b), and again, we find very similar results. Figure 9a shows the SS-wave travel-times estimated using the previously published PP + PS = SS method, which is to be compared with Figure 5c. The discrepancy between the two results is displayed in Figure 9b. We observe that the traveltime error introduced by the PP + PS = SS method, using noise-free input traveltimes, has a mean value of −0.002% and a standard deviation of 0.003%. Figure 10 shows results for the extended PP + PS = SS method. Figure 10a shows the estimated SS-wave relative geometric spreading and is to be compared with Figure 6c. Figure 10b shows the errors, which have a mean value of 0.1% and a standard deviation of 0.7%.

Figure 4. (a) PP-wave rays, and (b) PS-wave rays for a common receiver at x = 0.5 km in a model similar to the one used by Grechka and Tsvankin (2002b).

Figure 5. Simulated traveltime data (s) for waves reflected at the lowermost model interface: (a) PP-wave, (b) PS-wave, and (c) SS-wave.
The errors in Figure 10b mainly come from the fifth-order B-spline representation, which gives rise to a certain smoothing of the input PP and PS traveltimes and the true SS traveltimes. These three independent smoothing operations collectively affect the shown errors. We find the rise of the error level, compared to the errors of the estimated SS-wave traveltimes, quite natural in view that the second derivatives of PP- and PS-wave traveltimes have been used to estimate the SS-wave relative geometric spreading.

Test using input data with noise added

To get an impression of the robustness of the extended PP + PS = SS method, we added Gaussian noise with a standard deviation of 2 ms to the input data, i.e., the PP and PS traveltimes in Figure 5a and

Figure 6. Simulated relative geometric-spreading data (km²/s) for waves reflected at the lowermost model interface: (a) PP-wave, (b) PS-wave, and (c) SS-wave.

Figure 7. Obliquity factor (dimensionless) contributing to the SS-wave relative geometric spreading in Figure 6c.

Figure 8. Reconstructed SS-wave source-receiver pairs obtained by slope matching on noise-free PP- and PS-traveltime input data.
b. For this noise level, the inherent smoothing provided by the B-spline representation was not sufficient to ensure stable calculation of second derivatives. Therefore, we introduced additional smoothing in the form of a repeated application of a Hamming operator, applied independently to the various coordinate directions. This Hamming-operator smoothing is constrained by a certain aperture, defined as the maximum distance within which a given data sample will contribute to the smoothing of neighboring samples.

Figure 11 shows the estimated virtual source and receiver positions for the SS-wave using (a) no Hamming-operator smoothing, (b) Hamming-operator smoothing with aperture 0.4 km, and (c) Hamming-operator smoothing with aperture 0.8 km prior to the slope matching process. These plots are to be compared with the point distribution in Figure 8 corresponding to noise-free input data.

Using Hamming-operator smoothing with aperture 0.8 km on the noisy-input traveltime data, we regenerated the results in Figures 9 and 10 and displayed them, respectively, in Figures 12 and 13. The mean errors now are 0.08% in traveltime and 0.6% in relative geometric spreading, whereas the corresponding standard deviations are 0.46% and 12.3%. An interesting aspect is related to the error result for relative geometric spreading. Because the contribution of the obliquity factor (Figure 7) is at most 3%, our result suggests that, at least for small to moderate phase angles of incidence/departure at the acquisition surface, we could omit the calculation and simply disregard this factor. Using this approximation, the estimation of relative geometric spreading by the extended PP + PS = SS method can be performed completely without knowledge of the velocity model.

Figure 9. (a) Estimated SS-wave traveltime (s) for noise-free PP- and PS-traveltime input data. (b) Error in estimated SS-wave traveltime (10^{-3}%). Mean value of the errors: −0.002%. Standard deviation of the errors: 0.003%.

Figure 10. (a) Estimated SS-wave relative geometric spreading (km^2/s) for noise-free PP- and PS-traveltime input data. (b) Error in estimated SS-wave relative geometric spreading (%). Mean value of the errors: 0.1%. Standard deviation of the errors: 0.7%.
A final comment

This numerical-example section has been formed on the basis of the model and survey used by Grechka and Tsvankin (2002b), and therefore is limited to a maximum source-receiver offset of 2 km for the PP-waves. As concluded by Grechka and Tsvankin (2002b), it is necessary to use long-offset input PP data to ensure that the SS data are reconstructed for a sufficiently wide range of offsets. Their recommendation is that the PP-wave maximum offset is at least twice the reflector depth. For further discussions concerning limitations of the PP + PS = SS method and possible pitfalls, see Grechka and Tsvankin (2002b).

Figure 11. Reconstructed SS-wave source-receiver pairs obtained by slope matching on PP- and PS-traveltime input data containing noise. Degree of Hamming-operator smoothing before slope matching: (a) no such smoothing, (b) smoothing with aperture 0.4 km, and (c) smoothing with aperture 0.8 km.

Figure 12. (a) Estimated SS-wave traveltime (s) for PP- and PS-traveltime input data containing noise. Applied Hamming-operator smoothing aperture: 0.8 km. (b) Error in estimated SS-wave traveltime (%). Mean value of the errors: 0.08%. Standard deviation of the errors: 0.46%.
CONCLUSIONS

For a given target reflector, the full surface-to-surface propagator matrix of an SS-wave can be obtained from the corresponding surface-to-surface propagator matrices of the PP- and PS-waves of the same reflector. This new result, which captures the second-order derivatives of SS-wave traveltime, extends the counterpart scheme of retrieving the SS-wave traveltime and slope. This is known in the literature as the PP + PS = SS method.

The knowledge of the second-order derivatives of the SS traveltime permits us to determine, in addition to the relative geometric spreading, the common-source and common-receiver traveltime curvatures of the SS-wave. In the same way as for PP-waves, these quantities represent useful constraints for the construction of a seismic velocity model by means of tomographic methods.

Our investigation showed, however, that the proposed approach cannot provide the full amplitude of the SS-wave. The reason is that the SS-wave reflection and transmission coefficients are impossible to retrieve from the recorded PP and PS amplitudes. As a basis for obtaining further information of the SS-wave reflection, it will be necessary to perform prestack amplitude-versus-angle migration/inversion of PP and PS data at the reflection point.

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APPENDIX A

RELATIVE GEOMETRIC SPREADING

The relative geometric spreading is given by (Červený, 2001)

\[ L(x',x^*) = |\det Q_2(x',x^*)|^{1/2} = |\cos \varphi(x')\cos \varphi(x^*)|^{1/2}|\det B(x',x^*)|^{1/2}, \]

(A-1)

where \( \varphi(x') \) and \( \varphi(x^*) \) are the phase angles of incidence and departure upon the acquisition surface at the source and receiver, respectively. These phase angles cannot be extracted readily from the traveltime observations; hence, they need to be estimated under the assumption that the elastic parameters determining the phase velocities at the points \( x' \) and \( x^* \) are known.

Equation A-1 shows that the relative geometric spreading can be factored as

\[ L(x',x^*) = C(x',x^*)L^2(x',x^*), \]

(A-2)

where

\[ C(x',x^*) = |\cos \varphi(x')\cos \varphi(x^*)|^{1/2} \]

and

\[ L^2(x',x^*) = |\det B(x',x^*)|^{1/2} \]

(A-3)

are referred to here, respectively, as the obliquity factor and the surface-to-surface relative geometric spreading.

For simplicity and without loss of generality, we assume that the acquisition surface is a plane \( z = 0 \), and consider a projection \( (p_1,p_2) \) of the slowness vector \( (p_1,p_2,p_3) \) onto this surface. At the source and the receiver, the projections \( (p_1,p_2)' \) and \( (p_1,p_2)'' \) are known from the corresponding slopes of the observed traveltime function \( t(x_1,x_2,x_3) \). The components \( p_1' \) and \( p_1'' \), however, are not known from this traveltime function, but they are needed to obtain the phase angles in equation A-1 for the relative geometric spreading. Along the acquisition surface, the component \( p_1 \) for a given wave type is a function of the four variables \( (x_1,x_2,p_1,p_2) \).

Figure 13. (a) Estimated SS-wave relative geometric spreading (km²/s) for PP- and PS-traveltime input data containing noise. Applied Hamming-operator smoothing aperture: 0.8 km. (b) Error in estimated SS-wave relative geometric spreading (%). Mean value of the errors: 0.6%. Standard deviation of the errors: 12.3%.
Furthermore, the simplicity of the computation of the component \( p_1 \) at the points \( x' \) or \( x'' \) depends on the nature of the velocity model. For example, in the case of arbitrary anisotropy, the component \( p_1 \) appears as a root of a sixth-order polynomial equation with coefficients constituted by the elastic moduli. For an isotropic medium, the component \( p_1 \) results from the relation

\[
[p_3(x_1,x_2,p_1,p_2)]^2 = V^{-2}(x_1,x_2) - (p_1)^2 - (p_2)^2, \tag{A-4}
\]

where \( V \) denotes the velocity of the P- or S-wave under consideration. Having computed \( p_1(x_1,x_2,p_1,p_2) \), the corresponding cosine factor for the phase angle of incidence/departure required in equation A-1 is given by

\[
\cos \varphi = \frac{V(x_1,x_2)[p_3(x_1,x_2,p_1,p_2)]}{[p_3(x_1,x_2,p_1,p_2)]^2}. \tag{A-5}
\]

Applying equation A-5 at the source as well as at the receiver, and using equations 13, A-2, and A-3, one can compute the relative geometric spreading from the relation

\[
\mathcal{L}(x',x') = C(x',x')[\det(M^{bb})]^{-1/2}. \tag{A-6}
\]

Given that the phase angles of incidence/departure upon the acquisition surface are small to moderate, one might justify that \( C \) equals one to make the approximation. In this situation, an approximation for the relative geometric spreading can be obtained exclusively from travelt ime data.

Finally, for a normal-incidence reflected wave at the point \( x \), we use equation 25 to obtain (Hubral, 1983)

\[
\mathcal{L}(x,x) = 4|\cos \varphi(x)||\det(M^{bb} - M^{bc})|^{-1/2}. \tag{A-7}
\]

**APPENDIX B**

**REFLECTION AND TRANSMISSION AMPLITUDES**

In this appendix, we discuss whether it is possible to estimate the effects of SS-wave reflection/transmission coefficients from observations of amplitudes for PP- and PS-waves. To illuminate this topic, it is sufficient to consider a simplified scenario involving only elementary ray-theory S-wave Green’s functions (Červený, 2001). Moreover, we exclude from the discussion the effects of the polarization vectors, relative geometric spreads, and phase shifts arising because of caustics (the KMAH index). This leaves us with amplitude coefficients which only consist of products of reflection and transmission coefficients.

For the PP-primary reflection shown in Figure 2, this amplitude coefficient is

\[
a^{PP}(x',y,x') = T^{PP}(x',y)R^{PP}(y)R^{PP}(y,x'), \tag{B-1}
\]

where \( R^{PP} \) and \( T^{PP} \) denote, respectively, the reflection coefficient and the transmission coefficient products for the PP-wave. Using similar notation, the amplitude coefficients for the two PS-primary reflections are

\[
a^{SP}(x',y,x') = T^{SS}(x',y)R^{SP}(y)R^{PP}(y,x'), \tag{B-2}
\]

\[
a^{SP}(x',y,x') = T^{SS}(x',y)R^{SP}(y)R^{PP}(y,x'). \tag{B-3}
\]

Let us now introduce the dependence of the \( R^{PS} \) and \( R^{PP} \) coefficients with respect to the slowness vector components \( (p_1,p_2) \) along the reflector. We use flux-normalized reflection coefficients (Chapman, 1994) so that

\[
R^{PS}(p_1,p_2) = R^{SP}(-p_1,-p_2). \tag{B-4}
\]

This explains the use of the reflection coefficient for the PS-wave from \( x' \) to \( x'' \) in equation B-3. It is implicitly assumed that all the coefficients in equations B-1–B-3 correspond to the same tangential slowness \( (p_1,p_2) \) with respect to the reflector at the point \( y \).

We can combine the above three amplitude coefficients into

\[
\frac{a^{PS}(x',y,x')}{a^{PP}(x',y,x')} = \frac{T^{SS}(x',y)R^{SP}(y)R^{PP}(y)}{R^{PP}(y)}. \tag{B-5}
\]

On the other hand, the amplitude of the reflected SS-wave is

\[
a^{SS}(x',y,x') = T^{SS}(x',y)R^{SS}(y)R^{SP}(y,x'). \tag{B-6}
\]

Based on the above assumptions, the left-hand side of equation B-5 is known. However, the five entities on the right-hand side are all unknowns, and equation B-6 contains only unknowns.

Although equations B-5 and B-6 have the same form, they are indeed different, even for an isotropic medium and linearized reflection coefficients (Aki and Richards, 2002; Stovas and Ursin, 2001, 2003). The reason for this is that \( R^{PP} \) depends on contrasts in the Lamé parameters \( \lambda \) and \( \mu \), and the density \( \rho \), but \( R^{SS} \) depends on \( \mu \) and \( \rho \) only. Furthermore, for normal-incidence or small angles, \( R^{PS} \) and \( R^{SP} \) go to zero, whereas \( R^{PP} \) and \( R^{SS} \) go to nonzero values. This means that the expression in equation B-5 goes to zero, but \( a^{SS} \) does not. In summary, this simplified analysis demonstrates that the SS-amplitude coefficient \( a^{SS} \), reflection coefficient \( R^{SS} \), or transmission coefficient products \( T^{SS} \) cannot be estimated on the basis of the amplitude coefficients \( a^{PP} \) and \( a^{SP} \) only.

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