UNICAMP

SBU
SISTEMA DE BIBLIOTECAS DA UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS
SISTEMA DE BIBLIOTECAS DA UNICAMP
REPOSITÓRIO DA PRODUÇÃO CIENTIFICA E INTELECTUAL DA UNICAMP

Versão do arquivo anexado / Version of attached file:

Versão do Editor / Published Version

Mais informações no site da editora / Further information on publisher's website:
https://epubs.siam.org/doi/10.1137/130943662

DOI: 10.1137/130943662

Direitos autorais / Publisher's copyright statement:
©2014 by Society for Industrial and Applied Mathematics. All rights reserved.

DIRETORIA DE TRATAMENTO DA INFORMAÇÃO

Cidade Universitária Zeferino Vaz Barão Geraldo
CEP 13083-970 – Campinas SP
Fone: (19) 3521-6493
http://www.repositorio.unicamp.br

SIAM J. CONTROL OPTIM.
Vol. 52, No. 6, pp. 3917–3934
© 2014 Society for Industrial and Applied Mathematics

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

OUTER INVARIANCE ENTROPY FOR LINEAR SYSTEMS ON LIE GROUPS*

ADRIANO J. DA SILVA†

Abstract. Linear systems on Lie groups are a natural generalization of linear systems on Euclidian spaces. For such systems, this paper studies what happens with the outer invariance entropy introduced by Colonius and Kawan [*SIAM J. Control Optim.*, 48 (2009), pp. 1701–1721]. It is shown that, as for the linear Euclidean case, the outer invariance entropy is given by the sum of the positive real parts of the eigenvalues of a linear derivation $\mathcal{D}$ that is associated to the drift of the system.

Key words. invariance entropy, linear systems, Lie groups

AMS subject classifications. 16W25, 93C15, 37C60

DOI. 10.1137/130935379

1. Introduction.

The notion of outer invariance entropy was defined in [4] as a measure of how often open-loop control functions have to be updated in order to achieve invariance of a given compact and controlled invariant subset Q of the state space for a fixed set of initial states $K \subset Q$. Such notion is closely related with the notion of feedback entropy defined by Nair, et al. [14]. General bounds for this new concept of entropy were established in Kawan [9], [10], [11], but there are just a few examples of explicit formulas for it (cf. [12, Theorem 3.1]). In this paper we study the outer invariance entropy of linear systems on Lie groups, as introduced in [1] and [2]. The drift of such system generates a flow of automorphisms which allow us to define an associated derivation of the Lie algebra. As for the linear Euclidean case we can show that the outer invariance entropy is given in terms of the positive real parts of the eigenvalues associated with this derivation.

The contents of this paper are as follows. Section 2 introduces the definition of control systems and the outer invariance entropy and states some main results that will be used in later sections. In section 3 we define linear systems on connected Lie groups and homogeneous spaces and prove some results for such systems. We consider also a linear system on the adjoint group of a Lie group G that is semiconjugated to the original system on G. In section 4 we prove the main theorem of this paper (Theorem 4.7). We start by obtaining an upper bound in terms of the positive real parts of the eigenvalues of the associated derivation and also a first lower bound in terms of the real parts of all eigenvalues. In order to get rid of the negative terms we consider an associated system on a homogeneous space where we have an invariant measure. Section 5 gives some examples of linear systems on Lie groups when the derivation associated is inner and use it to compute the outer invariance entropy.

Notation. If (X, ϱ) is a metric space we denote by $N_\varepsilon(Q)$ the ε-neighborhood of a subset Q of X, that is,

$$N_\varepsilon(Q) := \{x \in M; \exists y \in Q \text{ with } \varrho(x, y) < \varepsilon\}.$$

*Received by the editors September 3, 2013; accepted for publication (in revised form) September 15, 2014; published electronically December 10, 2014. This research was supported by FAPESP grant 2013/19756-8. The work also was partially supported by CAPES (grant 4229/10-0) and CNPq (grant 142082/2013-9).
http://www.siam.org/journals/sicon/52-6/93537.html
†Instituto de Matemática, Universidade Estadual de Campinas, Cx. Postal 6065, 13.081-970 Campinas-SP, Brasil (ajsilvamat@hotmail.com).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

The term *smooth* will stand for $\mathcal{C}^\infty$. A smooth manifold will be a connected, finite-dimensional, second-countable, topological Hausdorff manifold endowed with a smooth differentiable structure.

2. Preliminaries. In this section we will introduce the concept of control systems and its outer invariance entropy. For the theory of control systems we refer to [5] and [16] and for more on outer invariance entropy the reader can consult [4], [9], [10], [11], and [12].

2.1. Control systems. Let M be a d dimensional smooth manifold. By a control system in M we understand a family

$$\dot{x}(t) = f(x(t), u(t)), \quad u \in \mathcal{U}, \tag{1}$$

of ordinary differential equations, with a right-hand side $f : M \times \mathbb{R}^m \to TM$ satisfying $f_u := f(\cdot, u) \in \mathcal{X}(M)$ for all $u \in \mathbb{R}^m$. For simplicity, we assume that f is smooth.

The family $\mathcal{U}$ of admissible control functions is given by

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m; \quad u \text{ measurable and } u(t) \in \Omega \text{ a.e.}\},$$

where $\Omega \subset \mathbb{R}^m$ is a compact convex set called the *control range* of the system. Smoothness of f in the first argument guarantees that for each control function $u \in \mathcal{U}$ and each initial value $x \in M$ there exists a unique solution $\phi(t, x, u)$ satisfying $\phi(0, x, u) = x$, defined on an open interval containing $t = 0$. Note that in general $\phi(t, x, u)$ is only a solution in the sense of Carathéodory, i.e., a locally absolutely continuous curve satisfying the corresponding differential equation almost everywhere. We assume that all such solutions can be extended to the whole real line. In fact, for the purpose of studying outer invariance entropy, we may assume this without loss of generality, since we consider only solutions which do not leave a small neighborhood of a compact set. Hence, we obtain a mapping

$$\phi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (t, x, u) \mapsto \phi(t, x, u),$$

satisfying the *cocycle property*

$$\phi(t + s, x, u) = \phi(t, \phi(s, x, u), \Theta_s u)$$

for all $t, s \in \mathbb{R}$, $x \in M$, $u \in \mathcal{U}$, where for $t \in \mathbb{R}$ the map Θ_t denotes the *shift flow* on $\mathcal{U}$ defined by

$$(\Theta_t u)(s) := u(t + s).$$

Instead of $\phi(t, x, u)$ we will usually write $\phi_{t,u}(x)$. Note that smoothness of the right-hand side f implies smoothness of $\phi_{t,u}$.

2.2. Outer invariance entropy. Consider the control system (1), and let ϱ be a metric on M compatible with the given topology. Let $K, Q \subset M$ be nonempty sets. We say that (K, Q) is an *admissible pair for the system* (1) if

1. the set K is compact;
2. for each $x \in K$ there exists $u \in \mathcal{U}$ such that $\phi(\mathbb{R}^+, x, u) \subset Q$ (in particular, $K \subset Q$).

For given $\tau, \varepsilon > 0$ a set $S \subset \mathcal{U}$ of control functions is called $(\tau, \varepsilon, K, Q)$-spanning if for all $x \in K$ there exists $u \in S$ with $\phi(t, x, u) \in N_\varepsilon(Q)$ for all $t \in [0, \tau]$. The minimal

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

cardinality of such a set is denoted by $r_{\text{inv}}(\tau,\varepsilon,K,Q)$, and the *outer invariance entropy of* (K,Q) is defined as

$$h_{\text{inv,out}}(K,Q;\varrho) := \lim_{\varepsilon\searrow 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\ln r_{\text{inv}}(\tau,\varepsilon,K,Q).$$

The next proposition states that the definition of the outer invariance entropy is independent of uniformly equivalent metrics and that the times τ can be chosen as integer multiplies of a fixed time. The proof can be found in [9, Propositions 2.5 and 2.6].

PROPOSITION 2.1. *Let (K,Q) be an admissible pair for the system* (1). *The following hold:*

(i) *$h_{\text{inv,out}}(K,Q;\varrho)$ is independent of uniformly equivalent metrics on Q. In particular, if Q is a compact set the outer invariance entropy is independent of the metric.*

(ii) *For all $\tau>0$*

$$h_{\text{inv,out}}(K,Q;\varrho) := \lim_{\varepsilon\searrow 0}\limsup_{n\to\infty}\frac{1}{n\tau}\ln r_{\text{inv}}(n\tau,\varepsilon,K,Q). \tag{2}$$

If the metric is clear from the context we write simply $h_{\text{inv,out}}(K,Q)$ for the outer invariance entropy. The next theorem shows that the invariance entropy cannot increase under semiconjugation.

THEOREM 2.2. *Consider two control systems $\dot{x}=f_1(x,u)$ and $\dot{y}=f_2(y,v)$ on M and N with corresponding solutions $\phi_1(t,x,u)$ and $\phi_2(t,y,v)$ and control spaces $\mathcal{U}$ and $\mathcal{V}$ corresponding to control ranges U and V. Let $\pi: M\to N$ be a continuous map and $h: U\to V$ any map with the semiconjugation property*

$$\pi(\phi_1(t,x,u)) = \phi_2(t,\pi(x),h(u)) \quad \text{for all } x\in M, u\in\mathcal{U}, t\geq 0. \tag{3}$$

Then, for an admissible pair (K,Q) of the system $\dot{x}=f_1(x,u)$ on M such that Q is a compact set, it holds that

$$h_{\text{inv,out}}(\pi(K),\pi(Q)) \leq h_{\text{inv,out}}(K,Q).$$

The proof can be found in [12, Proposition 2.13].

3. Linear systems.

In this section we consider linear systems as introduced in [1] and [2]. Associated with the drift of a linear system we have a flow of automorphisms and that allows us to associate a derivation of the Lie algebra to it. Our purpose is to show that the outer invariance entropy is given by the positive real parts of the eigenvalues of such derivation as for the linear Euclidean case (cf. [12, Theorem 3.1]).

3.1. Linear systems on Lie groups.

A linear control system on a Lie group G is a family of differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t)X_j(g(t)), \tag{4}$$

where the drift vector field $\mathcal{X}$, called the *linear vector field*, is an infinitesimal automorphism,[1] X_j are right invariant vector fields, and $u=(u_1,\dots,u_m)$ belongs to the class of admissible controls $\mathcal{U}$.

[1] We say that a vector field is an infinitesimal automorphism if its solutions are a family of automorphisms of the group.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

The usual example of a linear system is the one on $\mathbb{R}^n$ given by

$$\dot{x}(t) = Ax(t) + Bu(t); \quad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}.$$

Since the right (left) invariant vector fields on the abelian Lie algebra $\mathbb{R}^n$ are given by constant vectors we have

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)b_i, \qquad B = (b_1|b_2|\cdots|b_m),$$

that is, it is a linear system in the sense of (4).

Let $(\psi_t)_{t\in\mathbb{R}}$ denote the one parameter group of automorphisms of G generated by $\mathcal{X}$ and denote by $e \in G$ the identity element of G. For a right invariant vector field Y, we have

$$[\mathcal{X}, Y](e) = \frac{d}{dt}_{|t=0}(d\psi_{-t})_{\psi_t(e)}Y(\psi_t(e)) = \frac{d}{dt}_{|t=0}(d\psi_{-t})_e Y(e) \tag{5}$$

since $\psi_t(e) = e$ for all $t \in \mathbb{R}$. Also, since $\psi_{-t} \circ R_{\psi_t(g)} = R_g \circ \psi_{-t}$ we have at any point g that

$$\begin{aligned}[\mathcal{X}, Y](g) &= \frac{d}{dt}_{|t=0}(d\psi_{-t})_{\psi_t(g)}Y(\psi_t(g)) = \frac{d}{dt}_{|t=0}(d\psi_{-t})_{\psi_t(g)}(dR_{\psi_t(g)})_e Y(e)\\ &= \frac{d}{dt}_{|t=0}(dR_g)_e(d\psi_{-t})_e Y(e) = (dR_g)_e[\mathcal{X}, Y](e).\end{aligned}$$

Then for a given linear vector field $\mathcal{X}$, one can associate the derivation $\mathcal{D}$ of $\mathfrak{g}$ by

$$\mathcal{D}Y = -[\mathcal{X}, Y] \ \forall Y \in \mathfrak{g},$$

that is, $\mathcal{D} = -\mathrm{ad}(\mathcal{X})$.

Remark 3.1. Note that for the linear Euclidean case the formula $[Ax, b] = -Ab$ implies that the derivation $\mathcal{D}$ coincides with A.

The minus sign in $\mathcal{D} = -\mathrm{ad}(\mathcal{X})$ comes from the formula in the Euclidian case, $[Ax, b] = -Ab$, and it is also used in order to avoid a minus sign in the equality

$$\psi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y) \ \forall t \in \mathbb{R}, Y \in \mathfrak{g},$$

stated in the next proposition (Proposition 2 of [8]).

PROPOSITION 3.2. *For all* $t \in \mathbb{R}$

$$(d\psi_t)_e = \mathrm{e}^{t\mathcal{D}}$$

and consequently

$$\psi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y) \ \forall t \in \mathbb{R}, Y \in \mathfrak{g}.$$

We have also the following proposition concerning the solutions of (4).

PROPOSITION 3.3. *For a given* $u \in \mathcal{U}$, $t \in \mathbb{R}$, *let us denote by* $\phi_{t,u} := \phi_{t,u}(e)$ *the solution of* (4) *starting at the origin* $e \in G$. *Then, for each* $g \in G$ *the solutions of* (4) *satisfies*

$$\phi(t, g, u) = \phi_{t,u} \cdot \psi_t(g) = L_{\phi_{t,u}}(\psi_t(g)),$$

where L_g *is the left translation on* G.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Proof. Let us consider the curve $\alpha(t)$ given by

$$\alpha(t) = \phi_{t,u} \cdot \psi_t(g).$$

We have that $\alpha(0) = g$ and

$$\begin{aligned}
\dot{\alpha}(t) &= (dL_{\phi_{t,u}})_{\psi_t(g)} \frac{d}{dt}\psi_t(g) + (dR_{\psi_t(g)})_{\phi_{t,u}} \frac{d}{dt}\phi_{t,u} \\
&= (dL_{\phi_{t,u}})_{\psi_t(g)} \mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\phi_{t,u}} \left\{ \mathcal{X}(\phi_{t,u}) + \sum_{j=1}^m u_j(t) f_j(\phi_{t,u}) \right\} \\
&= \left\{ (dL_{\phi_{t,u}})_{\psi_t(g)} \mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\phi_{t,u}} \mathcal{X}(\phi_{t,u}) \right\} + \sum_{j=1}^m u_j(t) f_j(\alpha(t)).
\end{aligned}$$

Since ψ_t is a flow of automorphism we have that

$$\mathcal{X}(kh) = \frac{d}{ds}_{|s=0} \psi_s(kh) = (dL_k)_h \mathcal{X}(h) + (dR_h)_k \mathcal{X}(k).$$

This gives us, taking $k = \phi_{t,u}$ and $h = \psi_t(g)$,

$$(dL_{\phi_{t,u}})_{\psi_t(g)} \mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\phi_{t,u}} \mathcal{X}(\phi_{t,u}) = \mathcal{X}(\phi_{t,u} \cdot \psi_t(g)) = \mathcal{X}(\alpha(t))$$

and consequently

$$\dot{\alpha}(t) = \mathcal{X}(\alpha(t)) + \sum_{j=1}^m f_j(\alpha(t)).$$

By the unicity of the solution, we have the desired result. ∎

Remark 3.4. The formula for the solution of the linear system (4) in the above proposition corresponds to the variation-of-constants formula in the Euclidean case.

3.2. Linear systems on homogeneous spaces. Let $H \subset G$ be a closed connected Lie group with Lie algebra $\mathfrak{h}$ and consider the homogeneous space G/H. Denote by π the projection onto G/H.

For a given linear vector field $\mathcal{X}$ on G, we want to ensure the existence of a vector field on G/H that is π-related to $\mathcal{X}$. Such a vector field exists if and only if

$$\text{for all } x \in G \text{ and } y \in H, \qquad \pi(\psi_t(xy)) = \pi(\psi_t(x)).$$

But $\pi(\psi_t(xy)) = \psi_t(x)\psi_t(y)H$ and the preceding condition is equivalent to

$$\text{for all } y \in H, t \in \mathbb{R}, \qquad \psi_t(y) \in H.$$

Thus $\mathcal{X}$ is π-related to a vector field f on G/H if and only if H is invariant under the flow of $\mathcal{X}$.

Since H is connected its elements are products of exponentials. The invariance of H under ψ_t is written

$$\forall Y \in \mathfrak{h}, \ \forall t \in \mathbb{R}, \quad \psi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}Y}) \in H.$$

This is equivalent to

$$\forall Y \in \mathfrak{h}, \ \forall t \in \mathbb{R}, \quad \mathrm{e}^{t\mathcal{D}Y} \in \mathfrak{h},$$

and finally to the invariance of $\mathfrak{h}$ under $\mathcal{D}$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

For a given $\mathcal{D}$-invariant Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $H \subset G$ be the connect Lie subgroup with Lie algebra $\mathfrak{h}$ and assume that H is closed. Since $\mathcal{X}$ and $f = (d\pi)_*\mathcal{X}$ are π-related, its respective solutions ψ_t and Ψ_t satisfy

$$\Psi_t \circ \pi = \pi \circ \psi_t, \quad t \in \mathbb{R}.$$

We have then the *induced linear system on* G/H given by

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{6}$$

where $f_i = (d\pi)_* X_i$ for $i = 1, \dots m$ and $u = (u_1, \dots, u_m) \in \mathcal{U}$.

Since the systems (4) and (6) are π-related, we have that

$$\pi(\phi(t, g, u)) = \Phi(t, \pi(g), u), \quad t \in \mathbb{R}, g \in G, u \in \mathcal{U}$$

where Φ is the solution of the induced system on G/H. For any $g \in G$ we denote by $\mathcal{L}_g$ the translation on G/H. We have then for any $t \in \mathbb{R}$, $x = gH \in G/H$, and $u \in \mathcal{U}$ that

$$\Phi(t, x, u) = \pi(\phi_{t,u} \cdot \psi_t(g)) = \phi_{t,u} \cdot \pi(\psi_t(g)) = \mathcal{L}_{\phi_{t,u}}(\Psi_t(x)),$$

where $\phi_{t,u}$ is as before the solution of the linear system (4) at the neutral element e of G.

Let us assume that $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_i$ are $\mathcal{D}$-invariant Lie subalgebras, $i = 1, 2$. If we consider the quotient $\mathfrak{g}/\mathfrak{g}_2 \sim \mathfrak{g}_1$ we have a well-defined linear application $\bar{\mathcal{D}} : \mathfrak{g}/\mathfrak{g}_2 \to \mathfrak{g}/\mathfrak{g}_2$ given by $\bar{\mathcal{D}}((d\pi)_e(X)) := (d\pi)_e(\mathcal{D}(X))$. By invariance we can identify $\bar{\mathcal{D}}$ with $\mathcal{D}|_{\mathfrak{g}_1}$.

PROPOSITION 3.5. *Let $\mathfrak{g}_i$, $i = 1, 2$ be $\mathcal{D}$-invariant Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let $G_2 \subset G$ be the connected Lie group with Lie algebra $\mathfrak{g}_2$ and assume that G_2 is closed. Then, for the homogeneous space G/G_2 we have*

$$(d\Psi_t)_{G_2} = \mathrm{e}^{t\mathcal{D}|_{\mathfrak{g}_1}}.$$

Proof. Since $\Psi_t \circ \pi = \pi \circ \psi_t$ we have, using Proposition 3.2, that for any $X \in \mathfrak{g}$,

$$\begin{aligned}(d\Psi_t)_{G_2}((d\pi)_e(X)) &= d(\Psi_t \circ \pi)_e(X) = d(\pi \circ \psi_t)_e(X) = (d\pi)_e((d\psi_t)_e(X)) \\ &= (d\pi)_e(\mathrm{e}^{t\mathcal{D}}(X)) = \mathrm{e}^{t(d\pi)_e \circ \mathcal{D}}(X) = \mathrm{e}^{t\bar{\mathcal{D}}}((d\pi)_e(X)),\end{aligned}$$

and since $(d\pi)_e$ is onto $T_{G_2}(G/G_2) = \mathfrak{g}/\mathfrak{g}_2$ and $\bar{\mathcal{D}}$ can be identified with $\mathcal{D}|_{\mathfrak{g}_1}$ we get

$$(d\Psi_t)_{G_2} = \mathrm{e}^{t\mathcal{D}|_{\mathfrak{g}_1}}$$

as stated. ∎

3.3. The adjoint linear system. For a linear system on G we have in a natural way a semiconjugated system on the adjoint group of G as follows.

Denote by $\varphi : G \to \varphi(G) \subset \mathrm{Gl}(\mathfrak{g})$ the adjoint homomorphism given by $g \mapsto \varphi(g) := \mathrm{Ad}(g)$.

We can define a linear system on $\varphi(G)$ by

$$g(t) := \mathcal{X}^{\mathrm{Ad}}(g(t)) + \sum_{i=1}^{m} u_i(t) X_i^{\mathrm{Ad}}(g(t)), \tag{7}$$

where $\mathcal{X}^{\mathrm{Ad}}(\varphi(g)) := (d\varphi)_g \mathcal{X}(g)$ and $X_i^{\mathrm{Ad}}(\varphi(g)) := (d\varphi)_g(X_i(g))$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

We note that $\mathcal{X}^{\mathrm{Ad}}$ generates the flow of automorphisms ψ_t^{Ad} on $\varphi(G)$ given by $\psi_t^{\mathrm{Ad}}(\varphi(g)) = \varphi(\psi_t(g))$. Also, since X_i, $i = 1, \dots, m$, are right invariant vector fields on G we have

$$X_i^{\mathrm{Ad}}(\varphi(g)) := (d\varphi)_g(X_i(g)) = (d\varphi)_g \circ (dR_g)_e X_i(e)$$
$$= (dR_{\varphi(g)})_{\varphi(e)} \circ (d\varphi)_e X_i(e) =: (dR_{\varphi(g)})_{\varphi(e)} X_i^{\mathrm{Ad}}(\varphi(e)),$$

showing that X_i^{Ad} are right invariant vector fields on $\varphi(G)$ and that (7) is in fact a linear system.

If we denote by ϕ^{Ad} the solution of the above system, we have that

$$\phi^{\mathrm{Ad}}(t, \varphi(g), u) := \varphi(\phi(t, g, u)) = L_{\varphi(\phi_{t,u})}(\psi^{\mathrm{Ad}}(\varphi(g))),$$

where $\zeta_{t,u}$ is as before the solution of the system (4) at the neutral element of G. We have then that φ is a semiconjugation between the systems (4) and (7).

PROPOSITION 3.6. *For any $\mathcal{D}$-invariant subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we have that*

$$\det\Big((d\psi_t^{\mathrm{Ad}})_{\varphi(e)}\Big)|_{\mathrm{ad}(\mathfrak{h})} = \det\Big((d\psi_t)_e\Big)|_{\mathfrak{h}} = \mathrm{e}^{\mathrm{tr}\left(\mathcal{D}|_{\mathfrak{h}}\right)}.$$

Proof. It follows directly from the fact that $\psi_t^{\mathrm{Ad}} \circ \varphi = \varphi \circ \psi_t$. In fact, if $\{X_1, \dots, X_m\}$ is a basis of $\mathfrak{h}$, then $\{\mathrm{ad}(X_1), \dots, \mathrm{ad}(X_m)\}$ is a basis of $\mathrm{ad}(\mathfrak{h})$. By the above formula, we have that

$$(d\psi_t^{\mathrm{Ad}})_{\varphi(e)} \circ (d\varphi)_e = (d\varphi)_e \circ (d\psi_t)_e,$$

and since $(d\varphi)_e = \mathrm{ad}$ we have, for $j = 1, \dots, m$, that

$$(d\psi_t^{\mathrm{Ad}})_{\varphi(e)} \circ (\mathrm{ad}(X_j)) = (d\varphi)_e \circ (d\psi_t)_e(X_j) = \sum_{i=1}^{m} a_{ij}\, \mathrm{ad}(X_j),$$

where $(d\psi_t)_e(X_j) = \sum_{i=1}^m a_{ij}\, X_j$. The $\mathcal{D}$-invariance of $\mathfrak{h}$ together with the formula $(d\psi_t)_e = \mathrm{e}^{t\mathcal{D}}$ implies that the matrix of $(d\psi_t)_e$ restricted to $\mathfrak{h}$ is $A = (a_{ij})$. Consequently, the matrix of $(d\psi_t^{\mathrm{Ad}})_{\varphi(e)}$ in $\mathrm{ad}(\mathfrak{h})$ is also A and the result follows. ∎

For any given left invariant Haar measure d_G on G we can define a left invariant measure on $\varphi(G)$ as

$$d_{\mathrm{Ad}(G)}(B) := d_G(\varphi^{-1}(B))$$

for any Borel set B of $\varphi(G)$. Since, up to scalar, the Haar measure on a Lie group is unique we can assume that $d_{\mathrm{Ad}(G)}$ as above is the Haar measure on $\varphi(G)$. Then, for any $K \subset G$ it holds that $d_{\mathrm{Ad}(G)}(\varphi(K)) \geq d_G(K)$, since $K \subset \varphi^{-1}(\varphi(K))$.

4. Upper and lower bounds.

Our goal now is to show that the outer invariance entropy of the linear system (4) is given in terms of the positive real parts of the eigenvalues of the associated derivation $\mathcal{D}$.

4.1. Upper bound.

In order to give an upper bound for the outer invariance entropy we will recall the notion of topological entropy.

Let (X, d) be a metric space and $\psi : \mathbb{R} \times X \to X$ be a flow over X. For a given compact set $K \subset X$ and $\varepsilon, \tau > 0$ we say that a set $S_{\mathrm{top}} \subset K$ is a (τ, ε)-spanning set[2] for K if for every $y \in K$ there exist $x \in S_{\mathrm{top}}$ such that

$$\varrho(\psi_t(x), \psi_t(y)) < \varepsilon \quad \forall \quad t \in [0, \tau].$$

[2] The usual definition is that $S_{\mathrm{top}} \subset X$, but there is no loss of generality in assuming $S_{\mathrm{top}} \subset K$; hence the definition of topological entropy for both cases coincides.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

If we denote by $r_\tau(\varepsilon, K)$ the minimal cardinality of a spanning set, the topological entropy of ψ over K is defined as

$$h_{\text{top}}(\psi, K; \varrho) := \lim_{\varepsilon \searrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} r_\tau(\varepsilon, K)$$

and the *topological entropy* of ϕ as

$$h_{\text{top}}(\psi; \varrho) := \sup_{K \text{compact}} h_{\text{top}}(\psi, K; \varrho).$$

We should note that the topological entropy $h_{\text{top}}(\psi; \varrho)$ is independent of uniformly equivalent metrics.

By Lemma 2.1 in [4] we have that the topological entropy of a flow ψ coincides with the entropy of the time-one map ψ, that is, $h_{\text{top}}(\psi; \varrho) = h_{\text{top}}(\psi_1; \varrho)$. Also, Corollary 16 together with Proposition 10 of [3] ensures that for any automorphism A of a Lie group G, with Lie algebra $\mathfrak{g}$, we have that

$$h_{\text{top}}(A; \varrho_L) = \sum_{|\lambda_i|>1} \log |\lambda_i|,$$

where λ_i are the eigenvalues of $dA|_{\mathfrak{g}}$ and ϱ_L is a left invariant metric on G.

Remark 4.1. The result proved in Corollary 16 of [3] is actually for a right invariant metric on G and for an endomorphism of G. But since on a Lie group we always have right and left invariant metrics and right and left invariant Haar measures, and we are assuming that A is an automorphism, Proposition 10 of [3] implies that the topological entropy of an automorphism coincides for both metrics.

We have then the following theorem.

THEOREM 4.2. *Let (K, Q) be an admissible pair of the linear system* (4) *on G and let ϱ_L to be a left invariant metric on G. Then, the outer invariance entropy satisfies*

$$h_{\text{inv,out}}(K, Q; \varrho_L) \leq \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}},$$

where $\lambda_{\mathcal{D}}$ are the real parts of the eigenvalues of the derivation $\mathcal{D}$. In particular, when Q is a compact set, we have

$$h_{\text{inv,out}}(K, Q) \leq \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}.$$

Proof. By Proposition 3.3 and the left invariance of the metric we have that two solutions $\phi_{t,u}(g)$ and $\phi_{t,u}(g')$ satisfy

$$\varrho_L(\phi_{t,u}(g'), \phi_{t,u}(g)) = \varrho_L(\phi_{t,u} \cdot \psi_t(g'), \phi_{t,u} \cdot \psi_t(g)) = \varrho_L(\psi_t(g'), \psi_t(g)).$$

Let then $S_{\text{top}} \subset K$ be a minimal (τ, ε)-spanning set for K of the flow ψ_t, that is, for all $g' \in K$ there exists $g \in S_{\text{top}}$ such that

$$\varrho_L(\psi_t(g), \psi_t(g')) < \varepsilon \quad \forall t \in [0, \tau].$$

Since (K, Q) is admissible and $S_{\text{top}} \subset K$ there exists, for each $g \in S_{\text{top}}$, $u_g \in \mathcal{U}$ such that $\phi_{t,u_g}(g) \in Q$ for all $t \in \mathbb{R}$. Then for all $g' \in K$, there exists u_g such that

$$\varrho_L(\phi_{t,u_g}(g'), \phi_{t,u_g}(g)) = \varrho_L(\psi_{t,u_g} \cdot \psi_t(g'), \phi_{t,u_g} \cdot \phi_t(g)) = \varrho_L(\psi_t(g), \psi_t(g)) < \varepsilon$$

for all $t \in [0, \tau]$, that is, $\phi_{t,u_g}(g') \in N_\varepsilon(Q)$ showing that $\{u_g;\ g \in S_{\text{top}}\}$ is a (τ, ε)-

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

spanning set for (K, Q) and implying that

$$h_{\mathrm{inv,out}}(K, Q; \varrho_L) \leq h_{\mathrm{top}}(K, \psi; \varrho_L) \leq h_{\mathrm{top}}(\psi; \varrho_L) = h_{\mathrm{top}}(\psi_1; \varrho_L).$$

Since ψ_1 is an automorphism we have

$$h_{\mathrm{top}}(\psi_1; \varrho_L) = \sum_{\alpha;\, |\alpha|>1} \log |\alpha|,$$

where α are the eigenvalues of $(d\psi_1)_e$. By Proposition 3.2

$$(d\psi_1)_e = \mathrm{e}^{\mathcal{D}},$$

which implies that the eigenvalues of $(d\psi_1)_e$ are given by the exponential of the eigenvalues of $\mathcal{D}$ and consequently $|\alpha| = \mathrm{e}^{\lambda_{\mathcal{D}}}$, where $\lambda_{\mathcal{D}}$ is the real part of an eigenvalue of $\mathcal{D}$. Then

$$h_{\mathrm{top}}(\psi_1; \varrho_L) = \sum_{\alpha;\, |\alpha|>1} \log |\alpha| = \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}},$$

showing the theorem. ∎

4.2. Lower bound. As for the upper bound, the following theorem give us a first lower bound in terms of the real parts of the eigenvalues of $\mathcal{D}$.

THEOREM 4.3. *Let (K, Q) be an admissible pair of the linear system* (4) *and assume that $d_G(K) > 0$, where d_G is the left invariant Haar measure of G, and for some $\gamma > 0$ we have $d_G(N_\gamma(Q)) < \infty$. Then the estimate*

$$h_{\mathrm{inv,out}}(K, Q) \geq \sum \lambda_{\mathcal{D}}$$

holds, where $\lambda_{\mathcal{D}}$ are the real parts of the eigenvalues of the derivation $\mathcal{D}$.

Proof. By Proposition 2.1 we can consider just spanning sets for natural numbers. Let then $n \in \mathbb{N}$, $0 < \varepsilon < \gamma$, and consider a minimal (n, ε)-spanning set $S = \{u_1, \ldots, u_k\}$ for (K, Q). Define the sets

$$K_j = \{g \in K; \phi([0, n], g, u_j) \subset N_\varepsilon(Q)\}, \qquad j = 1, \ldots, k.$$

We have that $K = \cup_j K_j$. Using the continuity of $\phi_{t,u}$ it is easy to show that the sets K_j are Borel sets, and since the solution map $\phi_{n,u_j} : G \to G$ is a diffeomorphism, $\phi_{n,u_j}(K_j)$ is also a Borel set and satisfies

$$d_G(\phi_{n,u_j}(K_j)) \leq d_G(N_\varepsilon(Q)).$$

Using the left invariance of d_G, we have that

$$d_G(\phi_{n,u_j}(K_j)) = d_G(L_{\phi_{n,u_j}} \psi_n(K_j)) = d_G(\psi_n(K_j))$$

and also that $\det(dL_g)_h = 1$ for each $g, h \in G$.

Since $\psi_n \circ L_g = L_{\psi_n(g)} \circ \psi_n$, we have

$$|\det(d\psi_n)_g| = |\det(dL_{\psi_n(g)})_e||\det(d\psi_n)_e||\det(dL_{g^{-1}})_g| = \mathrm{e}^{n \sum \lambda_{\mathcal{D}}}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

and consequently

$$d_G(\psi_n(K_j)) = \int_{\psi_n(K_j)} d_G(g) = \int_{K_j} |\det(d\psi_n)_g| d_G(g)$$
$$= e^{n \sum \lambda_{\mathcal{D}}} \int_{K_j} d_G(g) = e^{n \sum \lambda_{\mathcal{D}}} d_G(K_j).$$

Taking $j_0 \in \{1, \dots, k\}$ such that $d_G(K_j) \leq d_G(K_{j_0})$ for $j \in \{1, \dots, k\}$ we have

$$d_G(K) \leq \sum_{j=1}^{k} d_G(K_j) \leq k \cdot d_G(K_{j_0}) = k \cdot \frac{d_G(N_\varepsilon(Q))}{e^{n \sum \lambda_{\mathcal{D}}}},$$

which implies

$$r_{\text{inv,out}}(n, \varepsilon, K, Q) \geq e^{n \sum \lambda_{\mathcal{D}}} \cdot \frac{d_G(K)}{d_G(N_\varepsilon(Q))},$$

and for $0 < \varepsilon \leq \gamma$ we have

$$0 < \frac{d_G(K)}{d_G(N_\gamma(Q))} \leq \frac{d_G(K)}{d_G(N_\varepsilon(Q))} \leq \frac{d_G(K)}{d_G(Q)} < \infty,$$

which implies

$$h_{\text{inv,out}}(K, Q) \geq \sum \lambda_{\mathcal{D}}$$

as desired. ◻

Using Theorem 4.2 and the above we have the following corollary.

COROLLARY 4.4. *Let (K, Q) be an admissible pair of the linear system (4) with the assumptions of the above theorem. Assume that the derivation $\mathcal{D}$ has just eigenvalues with positive real parts. Then*

$$h_{\text{inv,out}}(K, Q; \varrho_L) = \sum_{\lambda_{\mathcal{D}} > 0} \lambda_{\mathcal{D}},$$

where ϱ_L is a left invariant metric on G. Moreover, if Q is a compact set it holds that

$$h_{\text{inv,out}}(K, Q) = \sum_{\lambda_{\mathcal{D}} > 0} \lambda_{\mathcal{D}}.$$

Lower bound improvement. We will prove that the outer invariance entropy of an admissible pair (K, Q) with the assumptions that K has positive Haar measure and Q is a compact set satisfies

$$h_{\text{inv,out}}(K, Q) = \sum_{\lambda_{\mathcal{D}} > 0} \lambda_{\mathcal{D}},$$

where $\lambda_{\mathcal{D}}$ are the real parts of the eigenvalues of the derivation $\mathcal{D}$ associated with $\mathcal{X}$.

Consider the generalized eigenspaces associated to the derivation $\mathcal{D}$ given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; (\mathcal{D} - \alpha)^n X = 0 \text{ for some } n \geq 1\},$$

where α is an eigenvalue of $\mathcal{D}$. We can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}^{+,0} \oplus \mathfrak{n},$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

where

$$\mathfrak{g}^{+,0} = \bigoplus_{\alpha;\,\mathrm{Re}(\alpha)\geq 0} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\alpha;\,\mathrm{Re}(\alpha)<0} \mathfrak{g}_\alpha.$$

The next proposition show us that the vector spaces $\mathfrak{g}^{+,0}$ and $\mathfrak{n}$ are Lie algebras and that $\mathfrak{n}$ is actually nilpotent. The proof can be found in [15, Proposition 3.1].

PROPOSITION 4.5. *Let $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ a derivation of the Lie algebra $\mathfrak{g}$ of finite dimension over a closed field. Consider the decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the generalized eigenspace associated to the eigenvalue α. Then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{8}$$

with $\mathfrak{g}_{\alpha+\beta} = 0$ in case $\alpha + \beta$ is not an eigenvalue of $\mathcal{D}$.

Consider then the derivation $\mathcal{D}$ associated with a linear vector field $\mathcal{X}$ on G. Since we are also interested in the case where $\mathfrak{g}$ is a real Lie algebra, denote by $\mathfrak{g}_\mathbb{C}$ its complexification. Since the elements of $\mathfrak{g}_\mathbb{C}$ are of the form $X = \sum a_i X_i$ with $a_i \in \mathbb{C}$, $X_i \in \mathfrak{g}$, we can extend $\mathcal{D}$ by linearity to a derivation $\mathcal{D}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. It is not hard to show using the properties of $\mathbb{C}$ that $\mathcal{D}$ and $\mathcal{D}_\mathbb{C}$ have the same eigenvalues and that

$$(\mathfrak{g}_\alpha)_\mathbb{C} = (\mathfrak{g}_\mathbb{C})_\alpha,$$

where $\mathfrak{g}_\alpha$ and $(\mathfrak{g}_\mathbb{C})_\alpha$ are the generalized eigenspace associated respectively with $\mathcal{D}$ and $\mathcal{D}_\mathbb{C}$, and that implies that the above proposition is also valid when $\mathfrak{g}$ is a real Lie algebra.

Consider then the Lie algebra $\mathfrak{n}$ and let $N \subset G$ be the associated connected subgroup. By the above $\mathfrak{n}$ is a nilpotent Lie algebra and consequently N is a nilpotent Lie subgroup of G. Also if we assume that N is closed we have a well-defined linear system on G/N, since $\mathfrak{n}$ is $\mathcal{D}$-invariant.

A measure μ on a homogeneous space G/H is said to be a *G-invariant* Borel measure if for any $g \in G$ and any Borel set A of G/H we have that $\mu(\mathcal{L}_g(A)) = \mu(A)$. It is a well-known fact (cf. [13, Theorem 8.36]) that when such measure exists we have, for any continuous function f with compact support, that

$$\int_G f(g) d_G(g) = \int_{G/H} \int_H f(gh)\, d_H(h)\, d\mu(\bar{g}),$$

where d_G, d_H are the left invariant Haar measures on G and H, respectively. Since for any compact set $K \subset G$, $\bar{\mathbb{1}}_K(\bar{g}) = \int_H \mathbb{1}_K(gh)\, d_H(h)$ is a bounded positive function and $\bar{\mathbb{1}}_K > 0$ if and only if $\mathbb{1}_{\pi(K)} > 0$ we have that

$$\mu(\pi(K)) > 0 \quad \text{if} \quad d_G(K) > 0.$$

Let us assume that the nilpotent subgroup N is closed. Our goal is to prove that the quotient G/N admits a G-invariant Borel measure and then proceed as in the proof of Theorem 4.3 to obtain the result about the outer invariance entropy.

We define the *modular function* Δ_G of a Lie group G as

$$\Delta_G(g) = |\det(\varphi(g))|,$$

where φ is as before the adjoint Ad of G.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

We note that since the Lie algebra of $\varphi(G)=\mathrm{Ad}(G)$ is $\mathrm{ad}(\mathfrak{g})$ we have

$$\mathrm{Ad}_{\mathrm{Ad}(G)}(\varphi(g))(\mathrm{ad}(X))=\varphi(g)\mathrm{ad}(X)\varphi(g)^{-1}=\mathrm{ad}(\varphi(g)(X)),$$

which implies that $\Delta_{\mathrm{Ad}(G)}(\varphi(g))=\Delta_G(g)$.

By Theorem 8.36 of [13] a quotient space G/H admits a unique (up to scalar) G-invariant Borel measure if and only if $(\Delta_G)_{|H}=\Delta_H$. Using that, we have the following.

Proposition 4.6. *Let N defined as before with the assumption that N is closed. The homogeneous space G/N admits a G-invariant measure.*

Proof. By the above we just have to show that $(\Delta_G)_{|N}=\Delta_N$. Since N is a nilpotent Lie group, we have $\Delta_N\equiv 1$ and then we have to show that $(\Delta_G)_{|N}\equiv 1$. Since N is connected, its elements are products of exponentials and since Δ_G is a homomorphism of groups we just have to prove that $\Delta_G(\exp X)=1$ for any $X\in\mathfrak{n}$. We have

$$\Delta_G(\exp X)=|\det(\mathrm{Ad}(\exp X))|=|\det(\mathrm{e}^{\mathrm{ad}(X)})|=|\mathrm{e}^{\mathrm{tr}(\mathrm{ad}(X))}|$$

and $|\mathrm{e}^{\mathrm{tr}(\mathrm{ad}(X))}|=1$ when $\mathrm{tr}(\mathrm{ad}(X))=0$.

By (8) we have for any two eigenvalues α,β of $\mathcal{D}$ that

$$\mathrm{ad}(X_\alpha)^n X_\beta\subset\mathfrak{g}_{n\alpha+\beta},$$

which implies that if $\alpha\neq 0$ and $\dim\mathfrak{g}<\infty$, the map $\mathrm{ad}(X_\alpha)$ is nilpotent. Then, for any $X\in\mathfrak{n}$ we have that $X=\sum_\alpha a_\alpha X_\alpha$, where $\mathrm{Re}(\alpha)<0$ and hence that

$$\mathrm{tr}(\mathrm{ad}(X))=\sum_\alpha a_\alpha\mathrm{tr}(\mathrm{ad}(X_\alpha))=0$$

once that $\mathrm{ad}(X_\alpha)$ is a nilpotent map. □

With the above lemma we are in conditions to give the main result of this paper.

Theorem 4.7. *Let (K,Q) be an admissible pair of the system* (4) *such that $d_G(K)>0$ and Q is a compact set. It holds that*

$$h_{\mathrm{inv,out}}(K,Q)=\sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}},$$

where $\lambda_{\mathcal{D}}$ are the real parts of the eigenvalues of $\mathcal{D}$.

Proof. We just have to prove the inequality

$$h_{\mathrm{inv,out}}(K,Q)\geq\sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}}.$$

Let us first assume that N is closed in G. Since the linear system (4) on G is π-related with the linear system (6) on G/N and Q is compact, Theorem 2.2 implies that

$$h_{\mathrm{inv,out}}(K,Q)\geq h_{\mathrm{out,inv}}(\bar{K},\bar{Q}),$$

where $\bar{K}=\pi(K)$ and $\bar{Q}=\pi(Q)$.

For given $n,\varepsilon>0$ let $S=\{u_1,\dots,u_k\}\subset\mathcal{U}$ be a minimal (n,ε)-spanning set of the admissible pair $(\bar{K},\bar{Q})$ and consider as before the sets

$$K_j:=\{x\in\bar{K};\Phi([0,n],x,u_j)\subset N_\varepsilon(\bar{Q})\}.$$

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

For every $j=1,\ldots,k$, the sets K_j are Borel sets, $\bar{K}=\cup_j K_j$ and $\Phi_{n,u_j}(K_j)\subset N_\varepsilon(\bar{Q})$. Consider the G-invariant measure μ on G/N ensured by Proposition 4.6 above. We have

$$\mu(\Phi_{n,u_j}(K_j))=\mu(\mathcal{L}_{\phi_{n,u_j}}(\Psi_n(K_j)))=\mu(\Psi_n(K_j))=\int_{K_j}|\det(d\Psi_n)_{\bar{g}}|d\mu(\bar{g}).$$

Since for any $g\in G$ the translation $\mathcal{L}_g$ on G/N and Ψ_t satisfies $\Psi_n\circ\mathcal{L}_g=\mathcal{L}_{\psi_n(g)}\circ\Psi_n$ we have

$$(d\Psi_n)_{\bar{g}}=(d\mathcal{L}_{\psi_n(g)})_N\circ(d\Psi_n)_N\circ(d\mathcal{L}_{g^{-1}})_{\bar{g}}.$$

The G-invariance of μ implies that $|\det(d\mathcal{L}_g)_{\bar{h}}|=1$ for any $\bar{h}\in G/N$. This fact together with Proposition 3.5 implies

$$\det(d\Psi_n)_{\bar{g}}=\det(d\Psi_n)_N=\det(\mathrm{e}^{n\mathcal{D}|\mathfrak{g}^{+,0}})=\mathrm{e}^{n\,\mathrm{tr}(\mathcal{D}|\mathfrak{g}^{+,0})}=\mathrm{e}^{n\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}},$$

which give us

$$\mu(\Phi_{n,u_j}(K_j))=\mathrm{e}^{n\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}}\mu(K_j).$$

Let $j_0\in\{1,\ldots,k\}$ such that $\mu(K_{j_0})\geq\mu(K_j)$ for $j=1,\ldots,k$. Since $\Phi_{n,u_j}(K_j)\subset N_\varepsilon(\bar{Q})$ we have $\mu(\Phi_{n,u_j}(K_j))\leq\mu(N_\varepsilon(\bar{Q}))$ and consequently

$$\begin{aligned}\mu(\bar{K})\leq\sum_{j=1}^k\mu(K_j)\leq k\cdot\mu(K_{j_0})&=k\cdot\mu(\Phi_{n,u_{j_0}}(K_{j_0}))\,\mathrm{e}^{-n\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}}\\&\leq k\cdot\mu(N_\varepsilon(\bar{Q}))\mathrm{e}^{-n\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}},\end{aligned}$$

which implies

$$r_{\mathrm{inv}}(\tau,\varepsilon,\bar{K},\bar{Q})\geq\frac{\mu(\bar{K})}{\mu(N_\varepsilon(\bar{Q}))}\,\mathrm{e}^{n\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}}.$$

Since by hypothesis $d_G(K)>0$ we have that $\mu(\bar{K})=\mu(\pi(K))>0$. Also, since Q is a compact set, $\bar{Q}=\pi(Q)$ is also a compact set and consequently, for $\varepsilon>0$ small enough, $\mu(N_\varepsilon(\bar{Q}))<\infty$. Dividing by n, taking the logarithm and the limsup we have then that

$$\limsup_{n\to\infty}\frac{1}{n}\frac{\mu(\bar{K})}{\mu(N_\varepsilon(\bar{Q}))}=0,$$

which gives us

$$h_{\mathrm{inv,out}}(\bar{K},\bar{Q})\geq\sum_{\lambda_\mathcal{D}>0}\lambda_\mathcal{D}$$

and concludes the proof with the assumption that N is closed.

Assume now that N is not closed. Since N is a nilpotent Lie group we have that $\varphi(N)$ is closed in $\mathrm{Gl}(\mathfrak{n})$ (cf. [17, Theorem 2.2.1]). Since we have the embedding $\mathrm{Gl}(\mathfrak{n})\hookrightarrow\mathrm{Gl}(\mathfrak{g})$ given by

$$A\in\mathrm{Gl}(\mathfrak{n})\mapsto\begin{pmatrix}A&0\\0&I\end{pmatrix}\in\mathrm{Gl}(\mathfrak{g}),$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

where I is the identity $(\dim \mathfrak{g} - \dim \mathfrak{n})$-matrix, we have that $\varphi(N)$ is closed in $\mathrm{Gl}(\mathfrak{g})$. But then $\varphi(N)$ must be closed in $\varphi(G)$, since the topology of $\varphi(G)$ is stronger than the topology of $\mathrm{Gl}(\mathfrak{g})$, once that $\varphi(G)$ is a Lie subgroup of $\mathrm{Gl}(\mathfrak{g})$.

Consider the adjoint linear system (7) on $\varphi(G)$. By compacity of Q we have by Theorem 2.2 that

$$h_{\mathrm{inv,out}}(K, Q) \geq h_{\mathrm{inv,out}}(\varphi(K), \varphi(Q)).$$

Also, since $d_{\mathrm{Ad}(G)}(\varphi(K)) \geq d_G(K) > 0$ we have that the admissible pair $(\varphi(K), \varphi(Q))$ satisfies the assumptions of the theorem. Moreover,

$$(\Delta_{\mathrm{Ad}(G)})|_{\mathrm{Ad}(N)} = (\Delta_G)|_N = 1 = \Delta_N = \Delta_{\mathrm{Ad}(N)}$$

and we have an invariant measure on the quotient space $\mathrm{Ad}(G)/\mathrm{Ad}(N)$. Proceeding analogously as the case N closed and using Proposition 3.6 we obtain

$$h_{\mathrm{inv,out}}(\varphi(K), \varphi(Q)) \geq \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}},$$

which concludes the proof. □

We note that the condition that N is a closed set is needed in order to give to the quotient G/N a structure of differentiable manifold.

Example 4.8 (existence of admissible pairs). Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by the vectors $X_1, \ldots, X_m$ of the linear system (4) and denote by $\mathcal{D}\mathfrak{h}$ the smallest, nontrivial, $\mathcal{D}$-invariant subspace of $\mathfrak{g}$. We say that the linear system satisfies the *ad-rank condition* if $\mathcal{D}\mathfrak{h} = \mathfrak{g}$.

Assuming that $0 \in \operatorname{int}\Omega$ and that the linear system (4) satisfies the ad-rank condition, Theorem 3.5 of [2] implies that for any $\tau > 0$ the set

$$\mathcal{C}_\tau := \{g \in G; \phi_{\tau,u}(g) = e;\ \text{for some } u \in \mathcal{U}\}$$

is a neighborhood of the origin $e \in G$. Let then $Q_1 \subset \operatorname{int}\mathcal{C}_\tau$ be a compact set with nonempty interior that contains the origin $e \in G$. For any $g \in Q_1$ there exists $u_g \in \mathcal{U}$ such that $\phi_{\tau,u_g}(g) = e$. Consider the set

$$Q := \{\phi(t, g, u_g);\ \ g \in Q_1 \text{ and } t \in [0, \tau]\}.$$

Since $Q_1 \subset Q$ and $\operatorname{int} Q_1 \neq \emptyset$ we have that Q has nonempty interior. Also, for any sequence $\{z_n\} \subset Q$ there exists $g_n \in Q$ and $s_n \in [0, \tau]$ such that $\phi(s_n, g_n, u_n) = z_n$, where $u_n = u_{g_n}$. By compactness of $[0, \tau]\times Q\times\mathcal{U}$ we have a subsequence $(s_{n_k}, g_{n_k}, u_{n_k})$ that converges to a point (s^*, g^*, u^*) as $k \to \infty$. Continuity of ϕ implies that

$$\phi(t, g_{n_k}, u_{n_k}) \to \phi(t, g^*, u^*) \ \text{ for } \ t \in [0, \tau],$$

which implies $u^* = u_{g^*}$ and that $\phi(t, g^*, u^*) \in Q$ for $t \in [0, \tau]$. Also by continuity we have that

$$z_{n_k} = \phi(s_{n_k}, g_{n_k}, u_{n_k}) \to \phi(s^*, g^*, u^*) = z^* \in Q,$$

which implies that Q is a compact set. For a given u_g as above, we can define $\omega_g \in \mathcal{U}$ as

$$\omega_g(t) := \begin{cases} u_g(t), & t < \tau, \\ 0, & t \geq \tau. \end{cases}$$

Since $\phi(t, g, \omega_g)$ just depends on $\omega_g|_{[0,t)}$ we have that $\phi(\mathbb{R}^+, g, \omega_g) \subset Q$. In fact, for $t \in [0, \tau)$ we have that $\phi(t, g, \omega_g) = \phi(t, g, u_g) \in Q$ by the construction of Q. For $t \geq \tau$ we have that $\phi(t, g, \omega_g) \equiv e \in Q_1 \subset Q$.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Therefore, for any compact set $K \subset Q$ the pair (K, Q) is an admissible pair of the linear system (4), proving the existence of such pairs under the ad-rank condition.

Remark 4.9. One should note that for any $\tau > 0$ we can construct a set Q as above.

5. Examples: Inner derivations. In this section we consider linear systems in some specific types of groups and assume that the derivation $\mathcal{D}$ is an inner derivation.

5.1. The nilpotent case. Let N be a nilpotent Lie group and consider the linear system

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{i=1}^{m} u_i(t) X_i(t)$$

on N. Assume that the derivation $\mathcal{D}$ associated with $\mathcal{X}$ is inner, that is, $\mathcal{D} = \mathrm{ad}(X)$ for some $X \in \mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of N. Since $\mathfrak{n}$ is a nilpotent Lie algebra we have that $\mathrm{ad}(X)$ is a linear nilpotent map and consequently $\mathcal{D}$ has just zero eigenvalues. For any admissible pair (K, Q) of the above linear system on N we have by Theorem 4.2 that

$$h_{\mathrm{inv,out}}(K, Q; \varrho_L) = 0.$$

When Q is a compact set, the above is true for any distance ϱ compatible with the topology.

5.2. The $\mathbf{Gl}(n, \mathbb{R})$ case. Let $G = \mathrm{Gl}(n, \mathbb{R})$ be the set of the invertible real matrices. Its Lie algebra is $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$, the set of $n \times n$ matrices. For any $A \in \mathfrak{g}$ we can define the vector field

$$\mathcal{X} : G \to TG, \qquad g \mapsto Ag - gA.$$

Such vector field generates the flow $\psi_t : G \to G$ defined as

$$\psi_t(g) = \mathrm{e}^{tA} g \mathrm{e}^{-tA}.$$

In fact

$$\frac{d}{dt}\psi_t(g) = (dL_{\mathrm{e}^{tA}})_{g\mathrm{e}^{-tA}} \frac{d}{dt} g\mathrm{e}^{-tA} + (dR_{g\mathrm{e}^{-tA}})_{\mathrm{e}^{tA}} \frac{d}{dt}\mathrm{e}^{tA}.$$

Since

$$\frac{d}{dt} g\mathrm{e}^{-tA} = (dL_g)_{\mathrm{e}^{-tA}} \frac{d}{dt}\mathrm{e}^{-tA}, \qquad \frac{d}{dt}\mathrm{e}^{\pm tA} = \pm A\mathrm{e}^{\pm tA}$$

and the left and right translations are linear maps, we get

$$\frac{d}{dt}\psi_t(g) = \mathrm{e}^{tA} g(-A\mathrm{e}^{-tA}) + A\mathrm{e}^{tA} g\mathrm{e}^{-tA} = -\psi_t(g)A + A\psi_t(g) = \mathcal{X}(\psi_t(g)).$$

Is easy to see that $(\psi_t)_{t \in \mathbb{R}}$ is a family of automorphisms and consequently that $\mathcal{X}$ is a linear vector field.

On G we have then the linear system

$$\dot{g}(t) = Ag(t) - g(t)A + \sum_{i=1}^{m} u_i(t) B_i g(t) = \left(A - \sum_{i=1}^{m} u_i(t) B_i\right) g(t) - g(t)A, \tag{9}$$

where $B_i \in \mathfrak{g}$, $i = 1, \ldots, m$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

The derivation associated with $\mathcal{X}$ is inner and given by $\mathcal{D} = [A, \cdot]$, where $[\cdot, \cdot]$ is the matrix commutator. Since we are just interested in the eigenvalues of A we can assume that $A = (a_{ij})$ is an upper triangular matrix in $\mathfrak{gl}(\mathbb{C}, n)$. Considering the matrix canonical base E_{ij} we have

$$[A, E_{ij}] = \sum_{k,l=1}^{n} a_{kl}(\delta_{li}E_{kj} - \delta_{jk}E_{il}) = \sum_{k=1}^{n} a_{ki}E_{kj} - \sum_{l=1}^{n} a_{jl}E_{il},$$

where δ_{ij} is the Kronecker delta. Since we are assuming that A is upper triangular, we have $a_{ij} = 0$ when $i > j$, which give us

$$[A, E_{ij}] = \sum_{k \geq i}^{n} a_{ki}E_{ki} - \sum_{l=1}^{j} a_{jl}E_{il}.$$

An easy calculation show us that the matrix of $\mathcal{D}$ in the canonical base is block diagonal,

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & A_n \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} a_{ii} - a_{11} & -a_{21} & \cdots & -a_{i1} & \cdots & -a_{n1} \\ 0 & a_{ii} - a_{22} & \cdots & -a_{i2} & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & -a_{ni} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & a_{ii} - a_{nn} \end{pmatrix}.$$

The above implies that the eigenvalues of $\mathcal{D}$ are 0 and $a_{ii} - a_{jj}$, $i, j = 1, \ldots, n$, that is, the eigenvalues of $\mathcal{D}$ are the difference between the eigenvalues of A. If we assume that

$$\mathrm{Re}(a_{i_1 i_1}) \geq \mathrm{Re}(a_{i_2 i_2}) \geq \cdots \geq \mathrm{Re}(a_{i_n i_n})$$

and let $\lambda_k = \mathrm{Re}(a_{i_k i_k})$ we have by Theorem 4.7 that

$$h_{\mathrm{out,inv}}(K, Q) = \sum_{\substack{i<j \\ \lambda_i \neq \lambda_j}} \lambda_i - \lambda_j$$

for any admissible pair (K, Q) of the system (9) with $d_{\mathrm{Gl}(\mathbb{R},\mathrm{n})}(K) > 0$ and Q compact.

5.3. The semisimple case. Concerning the theory of semisimple Lie groups we refer to Duistermat, Kolk, and Varadarajan [6], Helgason [7], Knapp [13], and Warner [18].

Let G be a semisimple Lie group and consider a linear system

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{i=1}^{m} u_i(t)X_i(t)$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

on G. Since $\mathfrak{g}$ is a semisimple Lie algebra we have that any derivation is inner, and then there exists $X \in \mathfrak{g}$ such that $\mathcal{D} = \mathrm{ad}(X)$, where $\mathcal{D}$ is the derivation associated with $\mathcal{X}$. There exists a decomposition $X = E + H + N$, called *Jordan decomposition*, such that $\mathrm{ad}(E)$, $\mathrm{ad}(H)$, and $\mathrm{ad}(N)$ commute with each other, and for some Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ we have that $H \in \mathrm{cl}\,\mathfrak{a}^+$, $E \in \mathfrak{k}_H$, and $N \in \mathfrak{n}^+$. Moreover, $\mathrm{ad}(H)$ is diagonal with real eigenvalues and $\mathrm{ad}(K)$ has just imaginary eigenvalues (cf. [7, Chapter 9, Lemma 3.1]).

If we consider the root system associated with $\mathfrak{a}^+$ we have

$$\mathfrak{g} = \mathfrak{n}_\Theta^- \oplus \mathfrak{n}(\Theta) \oplus \mathfrak{n}_\Theta^+,$$

where $\Theta := \{\alpha \in \Sigma;\ \alpha(H) = 0\}$, Σ is the set of simple roots, and

$$\mathfrak{n}_\Theta^\pm = \bigoplus_\alpha \{\mathfrak{g}_\alpha;\ \ \alpha \in \Pi^\pm \setminus \langle\Theta\rangle\} \qquad \text{and} \qquad \mathfrak{n}(\Theta) = \bigoplus_\alpha \{\mathfrak{g}_\alpha;\ \ \alpha \in \langle\Theta\rangle\}$$

are the sum of the eigenspaces of $\mathrm{ad}(H)$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ we have for any given $X_\alpha \in \mathfrak{g}_\alpha$ that

$$\mathrm{ad}(N)(X_\alpha) \in \bigoplus_{\beta;\ \beta \neq \alpha} \mathfrak{g}_\beta,$$

which implies that the matrix of $\mathrm{ad}(N)$ is lower triangular with 0's in the main diagonal. Also, since $\mathrm{ad}(K)$ commutes with $\mathrm{ad}(H)$, we have that the eigenspaces $\mathfrak{g}_\alpha$ are invariant by $\mathrm{ad}(K)$ and then we can find a basis for each $\mathfrak{g}_\alpha$ such that the matrix of $\mathrm{ad}(K)$ is lower triangular. Consequently, the eigenvalues of $\mathcal{D}$ are given by the sum of the eigenvalues of $\mathrm{ad}(K)$ with the eigenvalues of $\mathrm{ad}(H)$. Since $\mathrm{ad}(K)$ has just imaginary eigenvalues, we have that the real parts of the eigenvalues of $\mathcal{D}$ are given by $\alpha(H)$. Then, for any admissible pair (K, Q) of a linear system on a semisimple Lie group G such that $d_G(K) > 0$ and Q is compact, we have that

$$h_{\mathrm{out,inv}}(K, Q) = \sum_{\alpha \in \Pi^+ \setminus \langle\Theta\rangle} (\dim \mathfrak{g}_\alpha) \cdot \alpha(H).$$

Acknowledgments. I would like to thank Prof. Fritz Colonius and Prof. Luiz San Martin, my Ph.D. advisors, and Dr. Christoph Kawan for many fruitful discussions about control and Lie theory.

REFERENCES

[1] V. Ayala and L.A.B. San Martin, *Controllability properties of a class of control systems on Lie groups*, in Nonlinear Control in the Year 2000, Lecture Notes in Control and Inform. Sci. 258, Springer, Berlin, 2001, pp. 83–92.
[2] V. Ayala and J. Tirao, *Linear Control Systems on Lie Groups and Controllability*, G. Ferreyra et al., eds., AMS, Providence, RI, 1999.
[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc., 153 (1971), pp. 401–414.
[4] F. Colonius and C. Kawan, *Invariance entropy for control systems*, SIAM J. Control Optim., 48 (2009), pp. 1701–1721.
[5] F. Colonius and C. Kliemann, *The Dynamics of Control*, in Systems & Control: Foundations & Applications, Birkhäuser Boston, Boston, MA, 2000.
[6] J.J. Duistermat, J.A.C. Kolk, and V.S. Varadarajan, *Functions, flows and oscillatory integral on flag manifolds*, Compositio Math., 49 (1983), pp. 309–398.
[7] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, AMS, Providence, RI, 2001.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[8] P. Jouan, *Equivalence of control systems with linear systems on Lie groups and homogeneous spaces*, ESAIM Control Optim. Calc. Var., 16 (2010), pp. 956–973.
[9] C. Kawan, *Upper and lower estimates for invariance entropy*, Discrete Contin. Dyn. Syst., 30 (2011), pp. 169–186.
[10] C. Kawan, *Invariance entropy of control sets*, SIAM J. Control Optim., 49 (2011), pp. 732–751.
[11] C. Kawan, *Lower bounds for the strict invariant entropy*, Nonlinearity, 24 (2011), 1909–1935.
[12] C. Kawan, *Invariance Entropy for Deterministic Control Systems: An Introduction*, Lecture Notes in Math. 2089, Springer, Berlin, 2013.
[13] A.W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Birkhäuser, Berlin, 2004.
[14] G. Nair, R.J. Evans, I. Mareels, and W. Moran, *Topological feedback entropy and nonlinear stabilization*, IEEE Trans. Automat. Control, 49 (2004), pp. 1585–1597.
[15] L.A.B. San Martin, *Algebras de Lie*, 2nd ed., Editora da Unicamp, 2010.
[16] E.D. Sontag, *Mathematical Control Theory. Deterministic Finite-Dimensional Systems*, Springer, Berlin, 1998.
[17] W.T. Van Est, *Dense imbeddings of Lie groups*, Proc. Kon. Ned. Ak., 54, 1951, pp. 321–328.
[18] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups*, vol. I, Springer, Berlin, 1972.

Downloaded 05/21/15 to 143.106.108.185. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.