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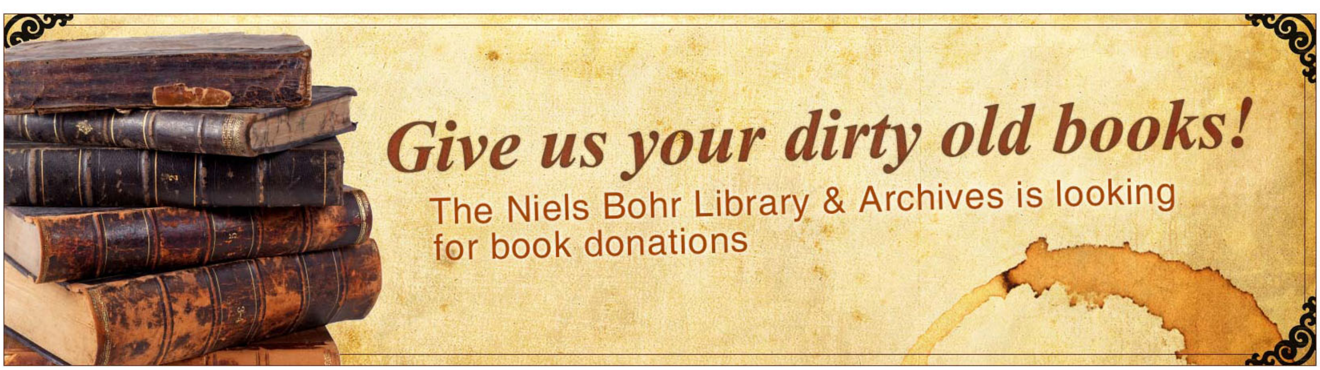

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On some fractional Green's functions

R. Figueiredo Camargo,[1,a)] R. Charnet,[2,b)] and E. Capelas de Oliveira[3,c)]
[1]Departamento de Matemática, IMECC, UNICAMP, CP 6065, Campinas, Sao Paulo 13081-970, Brazil
[2]Departamento de Estatística, IMECC, UNICAMP, CP 6065, Campinas, Sao Paulo 13081-970, Brazil
[3]Departamento de Matemática Aplicada, IMECC, UNICAMP, CP 6065, Campinas, Sao Paulo 13081-970, Brazil

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In this paper we discuss some fractional Green's functions associated with the fractional differential equations which appear in several fields of science, more precisely, the so-called wave reaction-diffusion equation and some of its particular cases. The methodology presented is the juxtaposition of integral transforms, in particular, the Laplace and the Fourier integral transforms. Some recent results involving the reaction-diffusion equation are pointed out. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3119484]

I. INTRODUCTION

Fractional calculus is one of the most accurate tools to refine the description of natural phenomena. The usual way to use this tool is to replace the integer order derivatives of the differential equation that describes one specific phenomenon by a derivative of noninteger order. Numerous important results and generalizations were obtained using this procedure in several areas such as genetic algorithm,[29] robotics,[5] diffusion-wave equations,[17] reaction-diffusion,[20,27] dynamical systems,[30] and others.[23] In order to discuss a fractional differential equation we suggest Refs. 13 and 23.

In the solution of fractional partial differential equations, the integral transform methodology plays an important role, especially to constant coefficient equations that possess time and spatial dependence, since after applying the juxtaposition of Laplace transform in the time variable and the Fourier (or Mellin or Hankel) transform in the spatial variable, we obtain an algebraic equation. After solving this algebraic equation, we recover the original solution by means of the corresponding inverse transforms.[7]

On the other hand, some years ago appeared in the literature an interesting paper by Manne *et al.*[18] where, for the first time, were discussed some nonlinear waves in reaction-diffusion systems, particularly related to the effect of transport memory associated with a physical problem involving stress distributions in granular materials. This type of equation is also important in the context of the anomalous diffusion.[11,21]

In a recent paper, Mathai *et al.*[19] discussed the solution of unified fractional reaction-diffusion system and treated, as a particular case, the result obtained by Manne *et al.* in Ref. 18. In that paper the authors present several results involving the Laplace transform associated with the generalized Mittag–Leffler function. These systems were also discussed by the same authors[27] where they calculated Green's function associated with a fractional partial differential equation with constant coefficients. Particular cases were also presented.

[a)]Electronic mail: rubens@ime.unicamp.br.
[b)]Electronic mail: charnet@ime.unicamp.br.
[c)]Electronic mail: capelas@ime.unicamp.br.

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As we have already said, there are many fractional differential equations in the literature that model several physical problems, specifically diffusion-wave equations[17] and reaction-diffusion systems.[20,27] Some nonlinear waves in reaction-diffusion systems[18] and a unified fractional reaction-diffusion system[19] were also recently presented. We remember that Mathai *et al.*[19] discussed the basic reaction-diffusion equation, i.e., the derivatives are of integer orders.

Well, the goal of this article is to study a particular equation, a generalized fractional differential equation, that permits us to treat several different cases in a unified way. The solution of our generalized fractional partial differential equation which coefficients are constants, which can also be equal to zero, is presented. The generalized differential equation is considered and, as particular cases, we obtain the classical harmonic oscillator, the damped harmonic oscillator, relaxation equation, diffusion equation, and telegraph equation, all in the fractional versions. The integer order equation is also presented. The so-called Fisher–Kolmogorov equation[4,10,14,20] and Ginzburg–Landau equation,[1,20] which are related to reaction kinetic and represent the nonlinearity term, are also mentioned.

The paper is organized as follows. Section II introduces notations and presents the fractional partial differential equation that will be studied. Section III recovers some results involving the fractional derivative in the Caputo sense, and Sec. IV presents the solution of the fractional differential equation using the juxtaposition of the integral transforms. Section V presents the so-called fractional Green's function, also the propagator, and Sec. VI discusses some particular cases. Finally, concluding remarks are presented.

II. WAVE REACTION-DIFFUSION EQUATION

In this section we introduce the notations and the main fractional partial differential equation[1] associated with the wave reaction-diffusion systems. Let $\Phi(x,t)$ be the concentration of a substance distributed in space (one dimensional space) and $\varphi(x,t,\Phi)$ be a nonlinear function, where t is the time variable and x is the space variable. It is important to note that the function $\varphi(x,t,\Phi)$ can represent a term of the so-called Fisher–Kolmogorov equation[4,10,14,20] or a term of the Ginzburg–Landau equation,[1,20] which appear in field theory and superconductivity. In the first case, the nonlinear term is $\varphi(x,t,\Phi)=A\Phi(x,t)[1-\Phi(x,t)]$ with A a constant, whereas in the second case we have $\varphi(x,t,\Phi)=B\Phi(x,t)[1-\Phi^2(x,t)]$ with B another constant.[27] Here we discuss only the case $\varphi(x,t,\Phi)\equiv\varphi(x,t)$, i.e., a linear case, as in Eq. (1).

The following fractional partial differential equation, the so-called generalized wave reaction-diffusion equation, is considered:

$$a\ {}_0\mathbf{D}_t^{2\alpha}\,\Phi(x,t) + b\ {}_0\mathbf{D}_t^{\alpha}\,\Phi(x,t) = c\ {}_{-\infty}\mathbf{D}_x^{2\beta}\,\Phi(x,t) - \nu^2\Phi(x,t) + \varphi(x,t), \tag{1}$$

where $t>0$ and $-\infty<x<\infty$, with $0<\alpha\leq 1$, $0<\beta\leq 1$, and a, b, c, and ν^2 are real constants. The fractional derivative is considered in Caputo's sense in the time variable and in the Riesz sense in the space variable.[7,13,23] Here we just consider the memory in the spatial variable, in the sense that the space variable goes from $-\infty$ to $+\infty$, and for the physical dimensions we refer to Ref. 12. Assuming the initial conditions

$$\Phi(x,0)=f(x) \quad \text{and} \quad \Phi_t(x,0)=0$$

for $x\in\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty}\Phi(x,t)=0=\lim_{x\to\infty}\Phi(x,t)$$

for $t>0$.

[1]Reaction-diffusion equation is an equation that describes how the concentration of a substance is distributed in space change under a chemical reaction and diffusion.

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Equation (1) is considered as a generalized case in the sense that the classical reaction-diffusion equation is obtained for $\alpha=1=\beta$ and $\nu=0=a$ and particular cases are presented in Sec. VI, among them the telegraph equation.

III. FRACTIONAL DERIVATIVES

Probably the multiple nonequivalent definitions of fractional derivatives and a nonevident geometrical interpretation contribute to the lack of utilization in large scale of the concepts of fractional calculus, which involves fractional integration and fractional derivatives.[23,26] Despite this fact, numerous important results and generalizations were obtained, thanks to fractional calculus.

There are several ways to introduce the fractional derivatives as a generalization of ordinary derivatives, particularly, the Riemann–Liouville and Caputo ones. We also mention the Weyl and Grünwald–Letnikov definitions.[23] Here, we present two definitions of fractional derivatives, the so-called Riesz–Caputo and Caputo fractional derivatives, which we believe are the most appropriate to study fractional differential equations.[23]

With the aim to introduce the Caputo–Riesz fractional derivative, we first introduce the Riesz potential or the Riesz fractional integral, which is defined in terms of the so-called left and right Riemann–Liouville fractional integrals.

For $x>a$ and $\alpha>0$, we denoted by $J_+^\alpha f(x)$ the left Riemann–Liouville integral and by $J_-^\alpha f(x)$ the right Riemann–Liouville, i.e.,

$$J_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \quad \text{and} \quad J_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt.$$

Generally, the integral limits are $a=-\infty$ and $b=+\infty$, and the definitions hold only for a special class of functions named Riesz class of functions.[2]

For $n-1<\mu\leq n$, we define the Riesz–Caputo fractional derivative of a function $f(x)$, denoted by $\mathbf{D}_{\mathrm{RC}}^\mu f(x)$, in terms of the left Caputo fractional derivative,[2,28]

$$\mathbf{D}_+^\mu f(x) = \frac{1}{\Gamma(n-\mu)}\int_a^x (x-t)^{n-\mu-1}\mathbf{D}^n f(t)dt = J_+^{n-\mu}\mathbf{D}^n f(x),$$

and the right Caputo fractional derivative,

$$\mathbf{D}_-^\mu f(x) = \frac{1}{\Gamma(n-\mu)}\int_x^b (t-x)^{n-\mu-1}(-\mathbf{D})^n f(t)dt = J_-^{n-\mu}(-1)^n\mathbf{D}^n f(x),$$

as follows:

$$\mathbf{D}_{\mathrm{RC}}^\mu f(x) \equiv {}_{-\infty}\mathbf{D}_x^\mu(x) = \frac{\mathbf{D}_+^\mu f(x) + (-1)^n \mathbf{D}_-^\mu f(x)}{2}. \tag{2}$$

We mention again that usually the integral limits are $a=-\infty$ and $b=+\infty$.

Moreover, setting ω as the parameter associated with the Fourier transform, we can write for the Fourier transform of the fractional derivative,[13]

$$\mathfrak{F}\{{}_{-\infty}\mathbf{D}_x^\mu f(x,t)\} = |\omega|^{2\mu} F(\omega,t), \tag{3}$$

where $F(\omega,t)$ is the spatial Fourier transform of the function $f(x,t)$.

On the other hand, Caputo's fractional derivatives of order μ can be introduced as follows (see Refs. 3 and 23):

[2]At this work every time we consider the integral or the derivative of Riesz of one function, we will be assuming that this function belongs to the Riesz class of functions. For more details, see Ref. 2.

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$$ {}_a\mathbf{D}_t^{\mu} f(x,t) \equiv \frac{\partial^{\mu}}{\partial t^{\mu}} f(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\mu)} \displaystyle\int_a^t \frac{f^{(n)}(x,\tau)}{(t-\tau)^{\mu+1-n}} d\tau, & n-1 < \mu \le n, \\ f^{(n)}(x,t), & \mu \equiv n \in \mathbb{N} \end{cases} $$

with $f^{(n)}(x,t)$ denoting the usual partial derivative of integer order n in relation to the variable t.

From now on, we will consider that the lower limit a is set to $-\infty$ in the spatial variable and zero at the time variable, i.e., we will ignore the so-called memory integrals in time variable. The first case is associated with Fourier transform and the second one associated with Laplace transform.[6,15]

Considering s, with $\mathrm{Re}(s) > 0$ as the parameter associated with the Laplace transform, it is known that[23]

$$ \mathfrak{L}\left\{ \frac{\partial^{\mu}}{\partial t^{\mu}} f(x,t) \right\} = s^{\mu} F(x,s) - \sum_{k=0}^{n-1} s^{\mu-1-k} f^{(k)}(x,0^+), \tag{4} $$

with $n-1 < \mu \le n$ and $n \in \mathbb{N}$. In this equation, $F(x,s)$ is the time Laplace transform of the function $f(x,t)$.

Whereas the Laplace transform of the Caputo formulation depends on initial conditions that possess a physical interpretation, the Riemann–Liouville formulation does not since no physical meaning is known (so far) for the fractional derivative ${}_a\mathbf{D}_t^{\mu-k-1} f(t)|_{t=0}$, except for some particular cases.[3] Another important difference between the Caputo and the Riemann–Liouville formulations is that, in the first case, the fractional derivative of a constant is equal to zero, while in the second formulation, it is not.[4] This justifies the use of the Caputo–Riesz and Caputo fractional derivatives instead of the Riemann–Liouville one to solve fractional differential equations.

IV. JUXTAPOSITION OF INTEGRAL TRANSFORMS

In this section we discuss Eq. (1) by means of the juxtaposition of integral transforms. The Laplace integral transform is used in the time variable and the Fourier integral transform in the space variable to turn the fractional partial differential equation into an algebraic equation.[5]

First, we introduce the Laplace integral transform in t, defined as follows:

$$ \mathfrak{L}[\Phi(x,t)] \equiv \bar{\Phi}(x,s) = \int_0^{\infty} e^{-st} \Phi(x,t) dt $$

for $t > 0$, with $\mathrm{Re}(s) > 0$ and $x \in \mathbb{R}$ which inverse is given by

$$ \mathfrak{L}^{-1}[\bar{\Phi}(x,s)] \equiv \Phi(x,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{\Phi}(x,s) ds, \tag{5} $$

in which γ is taken to the right of all singularities in order to ensure[6] $\int_0^{\infty} e^{-\gamma t} |\Phi(x,t)| dt < \infty$.

The Laplace transform in Eq. (1) is given by

$$ a\{s^{2\alpha}\bar{\Phi}(x,s) - s^{2\alpha-1} f(x)\} + b\{s^{\alpha}\bar{\Phi}(x,s) - s^{\alpha-1} f(x)\} = c\,{}_{-\infty}\mathbf{D}_x^{2\beta} \bar{\Phi}(x,s) - \nu^2 \bar{\Phi}(x,s) + \bar{\varphi}(x,s), \tag{6} $$

where $\bar{\varphi}(x,s)$ is the Laplace transform of $\varphi(x,t)$.

On the other hand, let $\hat{\Phi}(k,s)$ be the Fourier integral transform, in the space variable, defined by

[3] See Ref. 24 and references therein.

[4] Note that, as a result, the Riemann–Liouville fractional derivative cannot be interpreted as the variation rate.

[5] Since the fractional partial differential equation has constant coefficients, the auxiliary equation (transform equation) is an algebraic equation.

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$$\mathfrak{F}[\bar{\Phi}(x,s)] \equiv \hat{\Phi}(k,s) = \int_{-\infty}^{\infty} e^{ikx}\bar{\Phi}(x,s)dx,$$

and the corresponding inverse be given by

$$\bar{\Phi}(x,s) \equiv \mathfrak{F}^{-1}[\hat{\Phi}(k,s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{\Phi}(k,s)dk. \tag{7}$$

Taking the Fourier transform in Eq. (6) we get

$$a\{s^{2\alpha}\hat{\Phi}(k,s) - s^{2\alpha-1}\hat{f}(k)\} + b\{s^{\alpha}\hat{\Phi}(k,s) - s^{\alpha-1}\hat{f}(k)\} = c|k|^{2\beta}\hat{\Phi}(k,s) - \nu^2\hat{\Phi}(k,s) + \hat{\varphi}(k,s), \tag{8}$$

where the Laplace transform and the Fourier transform of the fractional derivative[7] are given by Eqs. (4) and (3), respectively.

The functions $\hat{f}(k)$ and $\hat{\varphi}(k,s)$ which appear in Eq. (8) are the Fourier transforms of $f(x)$ and $\bar{\varphi}(x,s)$, respectively.

As mentioned earlier, Eq. (8) is an algebraic equation and the solution can be written as

$$\hat{\Phi}(k,s) = \frac{(as^{2\alpha-1} + bs^{\alpha-1})\hat{f}(k) + \hat{\varphi}(k,s)}{as^{2\alpha} + bs^{\alpha} + \Lambda(k)}, \tag{9}$$

where we have put $\Lambda(k) = \nu^2 - c|k|^{2\beta}$.

The generalized Mittag–Leffler function is defined by[25]

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n}{\Gamma(\alpha n + \beta)}\frac{z^n}{n!},$$

with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $(\rho)_n$ is the Pochhammer symbol, defined by

$$(\rho)_n = \frac{\Gamma(\rho + n)}{\Gamma(\rho)},$$

with $\rho \in \mathbb{C}$ and $n = 0,1,2,\ldots$.

For the pair of Laplace transforms we have[13]

$$\mathfrak{L}[t^{\alpha_2-1}E^{\rho}_{\alpha_1,\alpha_2}(wt^{\alpha_1})] = \frac{s^{\rho\alpha_1-\alpha_2}}{(s^{\alpha_1} - w)^{\rho}} \Leftrightarrow \mathfrak{L}^{-1}\left[\frac{s^{\rho\alpha_1-\alpha_2}}{(s^{\alpha_1} - w)^{\rho}}\right] = t^{\alpha_2-1}E^{\rho}_{\alpha_1,\alpha_2}(wt^{\alpha_1}) \tag{10}$$

valid for $\mathrm{Re}(\alpha_1) > 0$ and $\mathrm{Re}(\alpha_2) > 0$, $\omega \in \mathbb{C}$ and $|\omega s^{-\alpha_1}| < 1$.

Using the linearity of the Laplace transform, the product of convolution and Eq. (10) into Eq. (9), we have for the inverse Laplace transform of $\hat{\Phi}(k,t)$, denoted by $\tilde{\Phi}(k,t)$,

$$\tilde{\Phi}(k,t) \equiv \mathfrak{L}^{-1}[\hat{\Phi}(k,s)] = \sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j}\left\{E^{j+1}_{2\alpha,\alpha j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right] + \frac{b}{a}t^{\alpha}E^{j+1}_{2\alpha,\alpha(j+1)+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right]\right\}\hat{f}(k)$$
$$+ \sum_{j=0}^{\infty}\frac{1}{a}\left(-\frac{b}{a}\right)^j \int_0^t \tilde{\varphi}(k,t-\xi)\xi^{\alpha j+2\alpha-1}E^{j+1}_{2\alpha,\alpha j+2\alpha}\left[-\frac{\Lambda(k)}{a}\xi^{2\alpha}\right]d\xi, \tag{11}$$

valid for $|bs^{\alpha}/(as^{2\alpha}+\Lambda(k))| < 1$ and where we have used the relation (see the Appendix A)

$$\mathfrak{L}^{-1}\left[\frac{s^{\rho-1}}{s^{\alpha_1} + As^{\alpha_2} + B}\right] = t^{\alpha_1-\rho}\sum_{j=0}^{\infty}(-A)^j t^{(\alpha_1-\alpha_2)j}E^{j+1}_{\alpha_1,(\alpha_1-\alpha_2)j+\alpha_1+1-\rho}(-Bt^{\alpha_1}). \tag{12}$$

Finally, to obtain the solution of Eq. (1), the inverse Fourier transform of Eq. (11) is calculated. Using the linearity of the Fourier transform, the convolution theorem, and Eq. (7), we obtain

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$$\Phi(x,t) \equiv \mathfrak{F}^{-1}[\tilde{\Phi}(k,t)] = \sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{2\alpha,\alpha j+1}^{j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right]\hat{f}(k)dk + \frac{b}{a}t^{\alpha}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{2\alpha,\alpha(j+1)+1}^{j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right]\hat{f}(k)dk\right\} + \frac{1}{a}\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j\int_0^t \xi^{\alpha j+2\alpha-1}\Omega(t,x,\xi,j)d\xi, \tag{13}$$

where

$$\Omega(t,x,\xi,j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{2\alpha,\alpha j+2\alpha}^{j+1}\left[-\frac{\Lambda(k)}{a}\xi^{2\alpha}\right]\tilde{\varphi}(k,t-\xi)dk \tag{14}$$

and $\Lambda(k)=\nu^2-c|k|^{2\beta}$.

V. FRACTIONAL GREEN'S FUNCTION

In this section we present the so-called fundamental solution of fractional Green's function. Two cases are discussed: (a) the classical Green's function, i.e., considering $\varphi(x,t)=\delta(x)\delta(t)$ and $f(x)=0$, and (b) the so-called propagator, i.e., $\varphi(x,t)=0$ and $f(x)=\delta(x)$. These two cases can be considered as particular cases of Eq. (13).

A. Classical Green's function

In this case the homogeneous initial condition is $f(x)=0$. Substituting this result in Eq. (13), we get

$$\Phi(x,t) \equiv G(x,t|0,0) = \frac{1}{a}\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j\int_0^t \xi^{\alpha j+2\alpha-1}\Omega(k,t,x,\xi,j)d\xi, \tag{15}$$

where $\Omega(k,t,x,\xi,j)$ is given by Eq. (14).

Thus, we need to obtain $\tilde{\varphi}(k,t-\xi)$. Calculating the Fourier transform of the delta function and substituting in equation to Green's function, we have

$$G(x,t|0,0) = \frac{1}{a}\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j+2\alpha-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{2\alpha,\alpha j+2\alpha}^{j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right]dk. \tag{16}$$

B. The propagator

Substituting $\varphi(x,t)=0$ and $f(x)=\delta(x)$ in Eq. (13), we can write

$$G_p(x,t|0,0) = \sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\Omega_\alpha(j,t,k)dk, \tag{17}$$

where

$$\Omega_\alpha(j,t,k) = E_{2\alpha,\alpha j+1}^{j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right] + \frac{b}{a}t^{\alpha}E_{2\alpha,\alpha(j+1)+1}^{j+1}\left[-\frac{\Lambda(k)}{a}t^{2\alpha}\right].$$

In Eqs. (16) and (17), we need to calculate an integral involving the generalized Mittag–Leffler function. Unfortunately, this can only be done for some particular class of functions.

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VI. PARTICULAR CASES

The particular cases, the reaction-diffusion obtained for $a=0$, temporal damped harmonic oscillator, obtained for $c=0$, the driven harmonic oscillator, and relaxation equation, obtained when $a=0=c$, both considered as fractional versions, are presented in this section.

A. Reaction-diffusion equation

Here we consider the particular case of Eq. (1) in which $a=0$, $b\neq 0$, and $c\neq 0$, i.e.,

$$b\,{}_0\mathbf{D}_T^{\alpha}\Phi(x,t) = c\,{}_{-\infty}\mathbf{D}_x^{2\beta}\Phi(x,t) - \nu^2\Phi(x,t) + \varphi(x,t). \tag{18}$$

This is a reaction-diffusion equation, without the term involving wave.[27]

Applying the Laplace transform in the time variable and the Fourier transform in the spatial variable, we obtain an algebraic equation whose solution is given by

$$\hat{\Phi}(k,s) = \frac{bs^{\alpha-1}\hat{f}(k) + \hat{\varphi}(k,s)}{bs^{\alpha} + \Lambda(k)}, \tag{19}$$

where $\Lambda(k)=\nu^2-c|k|^{2\beta}$.

Using the linearity of the inverse Laplace transform and the product of convolution, we have

$$\tilde{\Phi}(k,t) \equiv \mathfrak{L}^{-1}[\hat{\Phi}(k,s)] = \hat{f}(k)E_{\alpha}\left[-\frac{\Lambda(k)}{b}t^{\alpha}\right] + \frac{1}{b}\int_0^t \tilde{\varphi}(k,t-\xi)\xi^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{\Lambda(k)}{b}\xi^{\alpha}\right]d\xi, \tag{20}$$

where $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ are the one-parameter and two-parameter Mittag–Leffler functions, respectively.[7,23]

Finally, to obtain the solution of Eq. (18), the inverse Fourier transform of Eq. (20) is taking such that

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\frac{\Lambda(k)}{b}t^{\alpha}\right]\hat{f}(k)dk + \frac{1}{b}\int_0^t \xi^{\alpha-1}\Xi(t,x,\xi)d\xi, \tag{21}$$

where

$$\Xi(t,x,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha,\alpha}\left[-\frac{\Lambda(k)}{b}t^{\alpha}\right]\tilde{\varphi}(k,t-\xi)dk.$$

Equation (21) and the corresponding Green's function can also be obtained by using Eq. (13) with the following substitution: $b\to 0$. In this case, the unique term in the sum that contribute is the corresponding $j=0$, after this we put $2\alpha\to\alpha$ and $a\to b$ and we recover Eq. (21).

Setting $f(x)=0$ and $\varphi(x,t)=\delta(x)\delta(t)$ into the last equation, we have Green's function

$$\mathbf{G}(x,t|0,0) = \frac{t^{\alpha-1}}{b}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha,\alpha}\left[-\frac{\Lambda(k)}{b}t^{\alpha}\right]dk,$$

whereas the corresponding propagator, obtained with the substitution $\varphi(x,t)=0$ and $f(x)=\delta(x)$, is given by

$$\mathbf{G}_P(x,t|0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\alpha}\left[-\frac{\Lambda(k)}{b}t^{\alpha}\right]dk,$$

where $\Lambda(k)=\nu^2-c|k|^{2\beta}$.

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B. Damped harmonic oscillator

In this case only one variable is considering and taking the parameter $c=0$ in Eq. (1), after the convenient changing, we get

$$a_0\mathbf{D}_T^{2\alpha}\Phi(t)+b_0\mathbf{D}_t^{\alpha}\Phi(t)=-w^2\Phi(t)+\varphi(t), \tag{22}$$

with $\nu^2=w^2$, where w is the frequency of the harmonic oscillator. In the case with $\varphi(t)=0$, we get the same equation as discussed by Ref. 22.

Proceeding as above, the Laplace transform is obtained as the solution of the algebraic equation

$$\bar{\Phi}(s)=\frac{(as^{2\alpha-1}+bs^{\alpha-1})\Phi(0)+\bar{\varphi}(s)}{as^{2\alpha}+bs^{\alpha}+w^2}, \tag{23}$$

which is formally equal to Eq. (9). To obtain the solution we use Eq. (12) and calculating the inverse Laplace transform, we get

$$\Phi(t)=\Phi(0)\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j}\left\{E_{2\alpha,\alpha j+1}^{j+1}\left(-\frac{\omega^2}{a}t^{2\alpha}\right)+\frac{b}{a}t^{\alpha}E_{2\alpha,\alpha j+\alpha+1}^{j+1}\left(-\frac{\omega^2}{a}t^{2\alpha}\right)\right\}$$
$$+\frac{1}{a}\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j\int_0^t\varphi(t-y)y^{\alpha j+2\alpha-1}E_{2\alpha,\alpha j+2\alpha}^{j+1}\left(-\frac{\omega^2}{a}y^{2\alpha}\right)dy.$$

To explicit the corresponding Green's function, which is solution of Eq. (22) satisfying the condition $\Phi(0)=0$ and taking $\varphi(t)=\delta(t)$, we use the linearity of the inverse Laplace transform and Eq. (12) to obtain

$$G(t|0)\equiv\mathfrak{L}^{-1}[\bar{\Phi}(s)]=\frac{1}{a}\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j+2\alpha-1}E_{2\alpha,\alpha j+2\alpha}^{j+1}\left(-\frac{w^2}{a}t^{2\alpha}\right). \tag{24}$$

On the other hand, taking $\varphi(t)=0$, we have a homogeneous fractional differential equation whose solution is given by

$$\Phi(t)=\Phi(0)\sum_{j=0}^{\infty}\left(-\frac{b}{a}\right)^j t^{\alpha j}\left\{E_{2\alpha,\alpha j+1}^{j+1}\left(-\frac{\omega^2}{a}t^{2\alpha}\right)+\frac{b}{a}t^{\alpha}E_{2\alpha,\alpha j+\alpha+1}^{j+1}\left(-\frac{\omega^2}{a}t^{2\alpha}\right)\right\}.$$

C. The driven harmonic oscillator

Considering Eq. (22) written in the following form:

$${}_0\mathbf{D}_t^{2\alpha}\Phi(t)=-w^{2\alpha}\Phi(t)+\varphi(t),$$

with $b=0$ and $1/2<\alpha\leq 1$, the fractional differential equation associated with the driven harmonic oscillator[22] is obtained. We obtain the same result taking $b=0=c$, $a=1$, and $\nu^2=\omega^{2\alpha}$, in Eq. (1). The solution of this fractional differential equation is given by

$$\Phi(t)=\Phi(0)E_{2\alpha}(-\omega^{2\alpha}t^{2\alpha})+\int_0^t\varphi(t-y)y^{2\alpha-1}E_{2\alpha,2\alpha}(-\omega^{2\alpha}y^{2\alpha})dy.$$

The corresponding Green's function can be written as

$$G(t|0)=t^{2\alpha-1}E_{2\alpha,2\alpha}(-\omega^{2\alpha}t^{2\alpha}).$$

In the case $\alpha=1$ we have

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$$G(t|0) = tE_{2,2}(-\omega^2 t^2) = \frac{\sin \omega t}{\omega},$$

which is the classical Green's function associated with the driven harmonic oscillator. Note that, taking $a=1$, $b=0$ (also here, the unique term in the sum that contribute is $j=0$), in Eq. (24) we get exactly the last equation.

On the other hand, taking $\varphi(t)=0$, we have a homogeneous fractional differential equation whose solution is given by

$$\Phi(t) = \Phi(0)E_{2\alpha}(-\omega^{2\alpha}t^{2\alpha}).$$

As above, in the case $\alpha=1$ we get

$$\Phi(t) = \Phi(0)\cos \omega t$$

a well-known result.

D. Fractional relaxation equation

As a last particular case involving the relaxation equation, Green's function is also presented. Taking in Eq. (1) $a=0=c$ and we obtain, with the corresponding changing, the following equation:

$$b\,{}_0\mathbf{D}_T^{\alpha}\Phi(t) = -w^2\Phi(t) + \varphi(t), \tag{25}$$

where the algebraic equation has the solution given by

$$\bar{\Phi}(s) = \frac{bs^{\alpha-1}\Phi(0) + \bar{\varphi}(s)}{bs^{\alpha} + w^2}, \tag{26}$$

which can be obtained putting $a=0$ in Eq. (23).

Calculating the corresponding inverse Laplace transform, we obtain

$$\Phi(t) = \Phi(0)E_{\alpha}\left(-\frac{\omega^2}{b}t^{\alpha}\right) + \frac{1}{b}\int_0^t \varphi(t-y)y^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\omega^2}{b}y^{\alpha}\right)dy.$$

Here, we obtain explicitly Green's function, introducing $\varphi(t)=\delta(t)$ and $\Phi(0)=0$, i.e.,

$$G(t|0) = \frac{1}{b}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{w^2}{b}t^{\alpha}\right),$$

which is the same result recently obtained in Ref. 16.

Note that this solution can also be obtained with the same substitution as before, i.e., $b=0$ (the unique term in the sum that contribute is $j=0$), $a\to b$, $2\alpha\to\alpha$, in Eq. (24). Finally, in the integer case, $\alpha=1$, we get the classical Green's function

$$G(t|0) = \frac{1}{b}\exp\left(-\frac{w^2}{b}t\right).$$

On the other hand, taking $\varphi(t)=0$, we have a homogeneous fractional differential equation whose solution is given by

$$\Phi(t) = \Phi(0)E_{\alpha}\left(-\frac{\omega^2}{b}t^{\alpha}\right).$$

Finally, the case $\alpha=1$ furnishes the result,

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$$\Phi(t) = \Phi(0)\exp\left(-\frac{\omega^2}{b}t\right),$$

which is the same result associated with the respective Green's function, as above.

VII. CONCLUDING REMARKS

We discussed the wave reaction-diffusion fractional differential equation by means of the methodology of integral transforms, the Laplace transform in time variable, and the Fourier transform in space variable. Some particular cases were also presented, particularly, involving the reaction-diffusion equation and some versions of the harmonic oscillator.

As we have already said, the fractional telegraph differential equation can also be obtained, i.e., taking $a=1=c$, $b=2\lambda$, $\nu^2=0$, and $\varphi(x,t)=\delta(x)\delta(t)$ in Eq. (1), we get exactly a recent result.[7] We remember that, to see this fact (see Appendix B), one must perform the inverse Laplace transform as in Eq. (10), with $\lambda_1=1$ and $\lambda_2=0$, of Ref. 7.

Finally, in Eq. (1), the case where x is a positive random variable with gamma density, which can be used in an input model, as in Ref. 19 is being studied and will be presented in Ref. 8. Another natural continuation of this study is to discuss the so-called generalized fractional Lange in equation.[9]

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APPENDIX A: INVERSE LAPLACE TRANSFORM

In this appendix Eq. (12) is obtained. Let A and B be real numbers and set $\mathrm{Re}(\alpha_1)>\mathrm{Re}(\alpha_2)>0$. The quotient in Eq. (12) can be written as follows:

$$\frac{s^{\rho-1}}{s^{\alpha_1}+As^{\alpha_2}+B} = \frac{s^{\rho-1}}{s^{\alpha_1}+B}\left(1+\frac{As^{\alpha_2}}{s^{\alpha_1}+B}\right)^{-1}.$$

In the case where $|As^{\alpha_2}/(s^{\alpha_1}+B)|<1$, we use the geometric series to obtain

$$\frac{s^{\rho-1}}{s^{\alpha_1}+As^{\alpha_2}+B} = \frac{s^{\rho-1}}{s^{\alpha_1}+B}\sum_{j=0}^{\infty}(-A)^j\left(\frac{s^{\alpha_2}}{s^{\alpha_1}+B}\right)^j = \sum_{j=0}^{\infty}(-A)^j\frac{s^{\alpha_2 j+\rho-1}}{(s^{\alpha_1}+B)^{j+1}}.$$

Taking the inverse Laplace transform in both members of the last equation, we have, from the linearity of the inverse Laplace transform and from the uniformly convergence of the geometric series, that

$$\mathfrak{L}^{-1}\left[\frac{s^{\rho-1}}{s^{\alpha_1}+As^{\alpha_2}+B}\right] = \sum_{j=0}^{\infty}(-A)^j\mathfrak{L}^{-1}\left[\frac{s^{\alpha_2 j+\rho-1}}{(s^{\alpha_1}+B)^{j+1}}\right],$$

and using Eq. (10), we can write

$$\mathfrak{L}^{-1}\left[\frac{s^{\rho-1}}{s^{\alpha_1}+As^{\alpha_2}+B}\right] = \sum_{j=0}^{\infty}(-A)^j t^{(\alpha_1-\alpha_2)j+\alpha_1-\rho}E^{j+1}_{\alpha_1,(\alpha_1-\alpha_2)j+\alpha_1+1-\rho}(-Bt^{\alpha_1}),$$

which is exactly Eq. (12).

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APPENDIX B: FRACTIONAL TELEGRAPH EQUATION

Here we discuss the particular case associated with the fractional telegraph equation as recently obtained.[7] We present explicitly only the case associated with the following problem: the fractional differential equation,

$$ {}_0\mathbf{D}_t^{2\alpha}\Phi(x,t) + 2\lambda\,{}_0\mathbf{D}_t^{\alpha}\Phi(x,t) = {}_{-\infty}\mathbf{D}_x^{2\beta}\Phi(x,t) + \delta(x)\delta(t), \tag{B1} $$

where $t>0$ and $-\infty<x<\infty$, with $1/2<\alpha\leq 1$ and $0<\beta\leq 1$, which is obtained from Eq. (1) taking $a=1=c$, $\nu^2=0$, and $b=2\lambda$. The initial conditions are taken homogeneous, i.e., $\Phi(x,0)=0=\Phi_t(x,0)$ and the boundary conditions are the same imposed to solve Eq. (1).

Taking the Laplace transform in t and the Fourier transform in x, in Eq. (B1) we get the algebraic equation

$$ \hat{\Phi}(k,s) = \frac{1}{s^{2\alpha} + 2\lambda s^{\alpha} + \Lambda(k)}, \tag{B2} $$

with $\Lambda(k)=-|k|^{2\beta}$, where k and s are the parameters associated with the Fourier transform and the Laplace transform, respectively.

This equation, Eq. (B2), can be written in the form

$$ \hat{\Phi}(k,s) = \sum_{j=0}^{\infty} [-\Lambda(k)]^j \frac{s^{-\alpha j-\alpha}}{(s^{\alpha}+2\lambda)^{j+1}}, $$

with $|\Lambda(k)s^{-\alpha}/(s^{\alpha}+2\lambda)|<1$. Taking the inverse Laplace transform, we have

$$ \tilde{\Phi}(k,t) \equiv \mathfrak{L}^{-1}[\hat{\Phi}(k,s)] = \sum_{j=0}^{\infty} [-\Lambda(k)]^j t^{2\alpha j+2\alpha-1} E_{\alpha,2\alpha j+2\alpha}^{j+1}(-2\lambda t^{\alpha}), \tag{B3} $$

where $E_{\alpha,\beta}^{\rho}(\cdot)$ is the generalized Mittag–Leffler function. Equation (B3) is the same expression obtained by Eq. (10) in Ref. 7 in the case $\alpha=\beta$.

On the other hand, using Eq. (11) above, with $a=1=c$, $\nu^2=0$, and $b=2\lambda$, we get

$$ \tilde{\Phi}(k,t) = \sum_{j=0}^{\infty} (-2\lambda)^j t^{\alpha j+2\alpha-1} E_{2\alpha,\alpha j+2\alpha}^{j+1}(|k|^{2\beta}t^{2\alpha}). \tag{B4} $$

To show that two last equations are equal, we use the definition of the generalized Mittag–Leffler function[25] and the uniform convergence of the series, i.e., Eq. (B4) can be rewritten as follows:

$$ \tilde{\Phi}(k,t) = \sum_{n=0}^{\infty} \frac{(|k|^{2\beta}t^{2\alpha})^n}{n!} \sum_{j=0}^{\infty} \frac{(j+1)_n}{\Gamma(2\alpha n+\alpha j+2\alpha)} (-2\lambda)^j t^{\alpha j+2\alpha-1}. $$

Using the relation involving the Pochhammer symbol

$$ (j+1)_n = \frac{\Gamma(n+1)}{\Gamma(j+1)}(n+1)_j, $$

we can write

$$ \tilde{\Phi}(k,t) = \sum_{n=0}^{\infty} |k|^{2\beta n} t^{2\alpha n+2\alpha-1} E_{\alpha,2\alpha n+2\alpha}^{n+1}(-2\lambda t^{\alpha}), $$

which is equal to Eq. (B3) because $|k|^{2\beta}=-\Lambda(k)$.

[1] Aranson, I. S., "The world of the complex Ginzburg-Landau equation," Rev. Mod. Phys. **74**, 99 (2002).
[2] Agrawal, O. P., "Fractional variational calculus in terms of Riesz fractional derivatives," J. Phys. A: Math. Theor. **40**,

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6287 (2007).
[3] Caputo, M., "Vibrations of an infinite viscoelastic layer with a dissipative memory," J. Acoust. Soc. Am. **56**, 897 (1974).
[4] del-Castillo-Negrete, D., Carreras, B. A., and Lynch, V. E., "Front dynamics in reaction-diffusion systems with Levy flight: A fractional diffusion approach," Phys. Rev. Lett. **91**, 018302 (2003).
[5] Chen, Y., Xue, D., and Dou, H., First IEEE International Conference on Robotics and Biomimetics (RoBiol.04), China, 2004 (unpublished).
[6] Figueiredo Camargo, R., "Cauchy theorem and the Cagniard's method," MS thesis, UNICAMP, 2005 (in Portuguese).
[7] Figueiredo Camargo, R., Chiacchio, A. O., and Capelas de Oliveira, E., "Differentiation to fractional orders and the fractional telegraph equation," J. Math. Phys. **49**, 033505 (2008).
[8] Figueiredo Camargo, R., Charnet, R., and Capelas de Oliveira, E., "Some fractional Green's function with applications to the reaction-diffusion equations" (unpublished).
[9] Figueiredo Camargo, R., Chiacchio, A. O., Charnet, R., and Capelas de Oliveira, E., "On the fractional Langevin equation" (submitted).
[10] Fisher, R. A., "The wave of advance of advantageous genes," Ann. Eugen. **7**, 355 (1936).
[11] Gorenflo, R. and Mainardi, F., in *Anomalous Transport: Foundations and Applications*, edited by Klages, R., Radons, G., and Sokolov, I. M. (Wiley-VCH, Weinheim, 2008), pp. 93–127.
[12] Inizan, P., "Homogeneous fractional embedding," J. Math. Phys. **49**, 082901 (2008).
[13] Kilbas, A. A., Srivastava, H. S., and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
[14] Kolmogorov, A., Petrovsky, I., and Piscounov, N., "Etude de l'equation de la diffusion avec croissance de la quantité de matière et son application a un problem biologique," Moscow Univ. Math. Bull. **1**, 1 (1937).
[15] Lorenzo, C. F. and Hartley, T. T., "Initialized fractional calculus," Int. J. Appl. Math. **3**, 249 (2000).
[16] Luisa Soubhia, A., "Integral equations, Green's function and the fractional oscillator (in Portuguese)," Technical Report 22/08, Imecc-Unicamp, Campinas (October 29, 2008).
[17] Mainardi, F., "The fundamental solutions for the fractional diffusion-wave equation," Appl. Math. Lett. **9**, 23 (1996).
[18] Manne, K. K., Hurd, A. J., and Kenkre, V. M., "Nonlinear waves in reaction-diffusion systems: The effect of transport memory," Phys. Rev. E **61**, 4177 (2000).
[19] Mathai, A. M., Saxena, R. K., and Haubold, H. J., "A certain class of Laplace transforms with applications to reaction and reaction-diffusion equations," Astrophys. Space Sci. **305**, 283 (2006).
[20] Metzler, R. and Klafter, J., "The random walk's guide to anomalous diffusion: A fractional dynamics approach," Phys. Rep. **339**, 1 (2000).
[21] Mura, A. and Pagnini, G., "Characterizations and simulations of a class of stochastic processes to model anomalous diffusion," J. Phys. A: Math. Theor. **41**, 285003 (2008).
[22] Narahari Achar, B. N., Hanneken, J. W., and Clarke, T., "Response characteristics of a fractional oscillator," Physica A **309**, 275 (2002).
[23] Podlubny, I., *Fractional Differential Equations, Mathematics in Science and Engineering* (Academic, San Diego, 1999), Vol. 198.
[24] Podlubny, I., "Geometric and physical interpretation of fractional integration and fractional differentiation," Fractional Calculus Appl. Anal. **5**, 367 (2002).
[25] Prabhakar, T. R., "A singular integral equation with generalized Mittag-Leffler function in the kernel," Yokohama Math. J. **19**, 7 (1971).
[26] Samko, S. G., Kilbas, A. A., and Marichev, O. I., *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, New York, 1993).
[27] Saxena, R. K., Mathai, A. M., and Haubold, H. J., "Solutions of fractional reaction-diffusion equations in terms of Mittag-Leffler functions," Int. J. Sci. Res. **15**, 1 (2006).
[28] Shen, S., Liu, F., Anh, V., and Turner, I., "The fundamental solution and numerical solution of the Riesz fractional advection-dispersion equation," IMA J. Appl. Math. **73**, 850 (2008).
[29] Solteiro Pires, E. J., Tenreiro Machado, J. A., and de Moura Oliveira, P. B., "Fractional order dynamics in a GA planner," Signal Process. **83**, 2377 (2003).
[30] Tenreiro Machado, J. A., Barbosa, R. S., Jesus, I. S., Silva, M. F., Figueiredo, L. B., Galhano, A. F., Duarte, F. B., Lima, M. L., Pires, E. S., and Fonseca, N. M., Dincon-2007, Sixth Brazilian Conference on Dynamical, Control and Their Applications, Rio Preto, 2007 (unpublished).

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