BOUNDS ON THE DISPERSION OF VORTICITY IN 2D
INCOMPRESSIBLE, INVISCID FLOWS WITH A PRIORI
UNBOUNDED VELOCITY

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Abstract. We consider approximate solution sequences of the 2D incompressible Euler equations
obtained by mollifying compactly supported initial vorticities in $L^p$, $1 \leq p \leq 2$, or bounded measures
in $H^{-1}_{loc}$ and exactly solving the equations. For these solution sequences we obtain uniform estimates
on the evolution of the mass of vorticity and on the measure of the support of vorticity outside a ball
of radius $R$. If the initial vorticity is in $L^p$, $1 \leq p \leq 2$, these uniform estimates imply certain a priori
estimates for weak solutions which are weak limits of these approximations. In the case of nonnegative
vorticities, we obtain results that extend, in a natural way, the cubic-root growth of the diameter of
the support of vorticity proved first by C. Marchioro for bounded initial vorticities [Comm. Math.

Key words. incompressible flow, ideal flow, vorticity, irregular transport

AMS subject classifications. 35Q35, 76C05

PII. S0036141098337503

Introduction. The main object of this work is the behavior of weak solutions
of the 2D Euler equations, modeling the flow of incompressible, inviscid ideal fluids
in two space dimensions. We will be concerned with flows of fluids that are assumed
to fully occupy the 2D Euclidean plane, with velocity vanishing at infinity. We write
the initial value problem in the form of the vorticity equation:

$$
\begin{cases}
\omega_t + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\text{div } u = 0 & \text{in } \mathbb{R}^2 \times [0, \infty), \\
\text{curl } u = \omega & \text{in } \mathbb{R}^2 \times (0, \infty), \\
\omega(x, 0) = \omega_0(x) & \text{on } \mathbb{R}^2 \times \{t = 0\}.
\end{cases}
$$

(0.1)

The velocity can be eliminated from the vorticity equation by means of the Biot–
Savart law:

$$
u(x, t) = (K * \omega(\cdot, t))(x) \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(y, t) dy.$$

The usual strategy to obtain existence of weak solutions to the problem (0.1)
is to consider a suitable approximate problem, for which existence of solutions is
known, and then to obtain enough estimates to pass to the limit in the weak form
of the equations. The standard approximation schemes used in the literature are the
following: smoothing out initial data, the vanishing viscosity limit of the Navier–
Stokes equations, and desingularized vortex methods. In this work we are specifically
concerned with weak solutions obtained by exactly solving (0.1) with smoothed-out initial data. If the initial vorticity $\omega_0$ is a function in $L^p(\mathbb{R}^2)$, $1 < p < \infty$, with compact support, the existence of a weak solution obtained as the weak limit of a sequence of approximate solutions (produced by mollifying initial data) was first proved by DiPerna and Majda in [4]. For nonnegative initial vorticities in the space of bounded Radon measures with compact support, $\mathcal{B}\mathcal{M}_c(\mathbb{R}^2)$, and in $H^{1,1}_{\text{loc}}(\mathbb{R}^2)$ a corresponding existence result was proved by Delort in [2]. Vecchi and Wu in [13] extended Delort’s proof to initial vorticities of compact support in $L^1(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$, without sign restrictions. Uniqueness, in these cases, is an outstanding open problem, as is existence for arbitrary bounded Radon measures of compact support in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$. Following DiPerna and Majda, we will refer to initial vorticities in $\mathcal{B}\mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ as vortex sheet initial data, which we will abbreviate with the acronym VSID. For bounded initial vorticities, then both existence and uniqueness of weak solutions were obtained by Yudovich in [14].

Little is known regarding the qualitative behavior of weak solutions of (0.1). The general problem we will focus on is the following: How fast can a fluid particle be displaced from its initial position and how is this displacement affected by the regularity of the subjacent flow? If the initial vorticity lies in the space $L^p_c(\mathbb{R}^2)$ (the space of compactly supported functions in $L^p$), $p > 2$, it is well known that the corresponding velocity field is bounded a priori. This means that the trajectory of almost all fluid particles is contained in a space-time cone centered at their initial positions and with aperture bounded by global conserved quantities of the flow. Since vorticity is constant along particle trajectories, this implies that the support of vorticity remains compact and its diameter grows at most linearly in time. For nonnegative bounded vorticities Marchioro [10] showed that the growth of the displacement from the initial position is at most of the order of the cubic-root of time, so that the space-time cone above can be substituted with a space-time cubic parabola. This result captures the trend that flows with single-signed vorticity have of rotating, rather than spreading particles. The result was extended by two of the authors in [9] to nonnegative initial vorticities in $L^p_c(\mathbb{R}^2)$, $p > 2$. However, the estimate on the aperture of the cubic parabola obtained is lost when $p \to 2^+$. If the initial vorticity is in $L^p_c(\mathbb{R}^2)$, $1 \leq p \leq 2$, or in $\mathcal{B}\mathcal{M}_c(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$, it is not known whether the flow preserves the compactness of the support of vorticity. This problem was the initial motivation for the present work. The results we obtain here address the rate of dispersion of vorticity (or, equivalently, of material domains) in time. We will prove that the pictures obtained for more regular flows, i.e., linear cones in space-time for general vorticities and cubic parabolas for nonnegative vorticities, remain substantially true for even the most irregular cases. More precisely, we will show that, for any approximate solution sequence, given an initial disk in the plane and any $\varepsilon > 0$, there exists an aperture for a space-time cone (and for a cubic parabola in the case of nonnegative vorticity), uniform in the sequence, for which the set of particles in the initial disk whose trajectories leave the cone (respectively, the cubic parabola) has Lebesgue measure less than $\varepsilon$.

The remainder of this paper is organized in three sections: the first on flows without sign restriction on the vorticity, the second on flows with nonnegative vorticity, and the third containing extensions and conclusions. In the first one, we obtain estimates resembling Chebyshev inequalities for the Lagrangian maps that are applicable to any linear transport equation with a divergence-free smooth velocity field bounded in $L^q(\mathbb{R}^2)$. These results can be better understood in the context of the transport
theory by vector fields with Sobolev space regularity by DiPerna and Lions [3]. In the specific context of the 2D vorticity equation, we also obtain a result of the same nature in the physically relevant situation where the velocity is only $L^2_{\text{loc}}(\mathbb{R}^2)$. The second section begins with a simplified proof of an exponential decay estimate on the mass of vorticity near infinity due to Marchioro (this is the heart of the proof of Theorem 2.1 in [10]). We apply Marchioro’s result and the Chebyshev inequalities obtained in the first section to get results on the smallness of the mass and of the Lebesgue measure of the support of vorticity outside a suitable cubic parabola. All our results are proved for a smooth approximate solution sequence generated by regularizing initial data, with estimates independent of the regularization parameter.

Some remarks regarding notation are in order. We denote by $B(p;R)$ the open ball centered at $p$ with radius $R$ in the plane. The Lebesgue measure of the set $E$ is denoted by $|E|$ and the complement of $E$ is denoted by $E^c$. If $z = (z_1, z_2)$ is a point in the plane, then $z^\perp = (-z_2, z_1)$. We denote the Lebesgue conjugate exponent of $p$ by $p' = p/(p-1)$. Finally, we will use $\text{supp} \, \omega$ to denote the support of the function $\omega$.

**1. Chebyshev inequalities.** We begin with a result which applies to a general flow by a divergence-free, time-dependent vector field $u$. Consider a bounded, divergence-free, smooth vector field $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$.

Let $X = X(\alpha,t)$ denote an orbit associated with the flow by $u$:

$$
\begin{aligned}
\frac{dX}{dt} &= u(X,t), \quad 0 < t < T, \\
X(\alpha,0) &= \alpha \in \mathbb{R}^n.
\end{aligned}
$$

We use $X(D,t) \equiv \{X(\alpha,t) \mid \alpha \in D\}$ to denote the flow of a set $D \subseteq \mathbb{R}^n$ under the vector field $u$. We often refer to the family of diffeomorphisms $\alpha \mapsto X(\alpha,t)$ as Lagrangian maps.

The first result of this section will be referred to as the filtering theorem.

**Theorem 1.1.** Let $0 < R_1 < R_2$ and define the annulus $A = \{x \in \mathbb{R}^n \mid R_1 < |x| < R_2\}$, $\Sigma(R_1,R_2,t) \equiv \{\alpha \in B(0;R_1) \mid |X(\alpha,t)| > R_2\}$ and let $q \geq 1$. Then

$$
|\Sigma(R_1,R_2,t)| \leq \left( t \sup_{0 \leq \tau \leq T} \|u(\cdot,t)\|_{L^q(A)} \right)^q \frac{R_2 - R_1}{R_2 - R_1}.
$$

**Proof.** Fix $t > 0$. In this proof we will abbreviate $\Sigma(R_1,R_2,t)$ by $\Sigma$. Let us introduce the material cylinder $C$ defined by

$$
C \equiv \bigcup_{0 \leq s \leq t} X(\Sigma,s).
$$

The proof consists of integrating and estimating the radial component of velocity on the set $(A \times [0,t]) \cap C$. Let $\chi_A = \chi_A(x)$ denote the characteristic function of the annulus $A$. Then, by incompressibility we have

$$
\int_C \chi_A(x)u(x,s) \cdot \frac{x}{|x|} \, dx \, ds = \int_0^t \int_{\Sigma} \chi_A(X(\alpha,s)) \frac{d|X(\alpha,s)|}{ds} \, ds \, ds.
$$

**Claim.** For any $\alpha \in \Sigma$, we have

$$
\int_0^t \chi_A(X(\alpha,s)) \frac{d|X(\alpha,s)|}{ds} \, ds = R_2 - R_1.
$$
To see that, consider $\Gamma \equiv \{ 0 < s < t \mid X(\alpha, s) \in A \}$. Since $\Gamma$ is open, it can be written as a countable union of disjoint open intervals:

$$\Gamma = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

Therefore,

$$\int_{t_0}^{t} \chi_A(X(\alpha, s)) \frac{d|X(\alpha, s)|}{ds} ds = \sum_{i=1}^{\infty} \int_{a_i}^{b_i} \frac{d|X(\alpha, s)|}{ds} ds$$

$$= \sum_{i=1}^{\infty} (|X(\alpha, b_i)| - |X(\alpha, a_i)|).$$

By the continuity of the trajectories $X(\alpha, \cdot)$, each of these numbers $|X(\alpha, b_i)|$ and $|X(\alpha, a_i)|$ is either $R_1$ or $R_2$. The curve $s \mapsto X(\alpha, s)$ has finite total length, and hence the summation above has a finite number of nonzero terms, which correspond precisely to the time intervals during which the curve completely traverses the annulus. Since $|X(\alpha, 0)| = |\alpha| < R_1$ and $|X(\alpha, t)| > R_2$,

$$\sum_{i=1}^{\infty} (|X(\alpha, b_i)| - |X(\alpha, a_i)|) = R_2 - R_1,$$

and the claim is proved. \(\Box\)

Hence, in view of (1.1),

$$\int_C \chi_A(x)u(x, s) \cdot \frac{x}{|x|} dx ds = (R_2 - R_1)|\Sigma|. \tag{1.2}$$

On the other hand, we also have

$$\int_C \chi_A(x)u(x, s) \cdot \frac{x}{|x|} dx ds \leq \int_0^{t} \int_{X(\Sigma, s)} |\chi_A(x)u(x, s)| dx ds$$

$$\leq \int_0^{t} \|\chi_A(\cdot)u(\cdot, s)\|_{L^q(X(\Sigma, s))} |\Sigma|^{(q-1)/q} ds$$

$$\leq t \sup_{0 \leq s \leq t} \|u(\cdot, s)\|_{L^q(A)} |\Sigma|^{(q-1)/q}.$$

Putting together the identity (1.2) and the inequality above we obtain the estimate we wished. \(\Box\)

This result can be understood in the context of the linear transport theory developed by DiPerna and Lions in [3]. What we achieve is control over the local transport in terms of weak local control over the transporting vector fields that can be applied in situations where the flow is very singular. A theorem of this nature can also be proved for vector fields with bounded divergence, which is the context of [3]. However, in this work we are interested in the incompressible situation.
In [3], DiPerna and Lions observed that if the vector field \( u \in L^q(\mathbb{R}^n) \cap W^{1,1}_\text{loc}(\mathbb{R}^n) \) has bounded divergence, then the Lagrangian maps are \( L^q_{\text{loc}}(\mathbb{R}^n) \). Let \( X(\cdot, t) \) be the unique renormalized flow associated with \( u \), with \( X(\cdot, t) \in L^q_{\text{loc}}(\mathbb{R}^n) \). In order to compare the estimate in Theorem 1.1 with results obtained by DiPerna and Lions, first recall the classical Chebyshev inequality, which states that if \( \Omega \subset \mathbb{R}^n \) and if \( f \) in \( L^q(\Omega) \), then for any \( \lambda > 0 \),

\[
|\{ x \in \Omega : |f(x)| > \lambda \} | \leq \frac{\| f \|_{L^q(\Omega)}^q}{\lambda^q}.
\]

Note that we have

\[
|\{ \alpha \in B(0; R_1) | |X(\alpha, t)| > R_2 \} | \leq |\{ \alpha \in B(0; R_1) | |X(\alpha, t) - \alpha| > R_2 - R_1 \} |
\]

\[
\leq \frac{\| X(\alpha, t) - \alpha \|_{L^q(B(0; R_1))}^q}{(R_2 - R_1)^q} = (1.1),
\]

where the last inequality follows from the Chebyshev inequality applied to \( X(\alpha, t) - \alpha \in L^q(B(0, R_1)) \),

\[
(1.1) = \frac{\| \int_0^t u(X(\alpha, s), s)ds \|^q_{L^q(B(0; R_1))}}{(R_2 - R_1)^q} \leq \frac{t^q \sup_{0 \leq s \leq t} \| u(\cdot, s) \|^q_{L^q(\mathbb{R}^n)}}{(R_2 - R_1)^q},
\]

where the final inequality was deduced from the generalized Minkowski inequality; see [12]. The estimate in Theorem 1.1 is a generalization of this conclusion mainly because our estimate is local, in the sense that it depends only on the \( L^q \)-norm of \( u \) in the annulus \( A \) and not on a global \( L^q \) bound.

In the next result we single out a special case of the filtering theorem, which is more in the nature of a Chebyshev inequality for the Lagrangian maps, and which will be useful in the applications to 2D incompressible flow. Once again, we assume that the flow \( u \) is smooth.

**Corollary 1.2.** Let \( S_0 \subseteq B(0; R_0) \) and \( t > 0 \). Then, for every \( R > R_0 \), we have

\[
|X(S_0, t) \cap B(0; R)^c| \leq \left( \frac{t \sup_{0 \leq s \leq t} \| u(\cdot, s) \|^q_{L^q(\mathbb{R}^n)}}{R - R_0} \right)^q.
\]

**Proof.** Since

\[
|X(S_0, t) \cap B(0; R)^c| = |\{ \alpha \in S_0 | X(\alpha, t) > R \}|,
\]

we have, by incompressibility, that this is equal to

\[
|\{ \alpha \in S_0 | X(\alpha, t) > R \}| \leq |\Sigma(R_0, R, t)|,
\]

and the result follows from Theorem 1.1. \( \square \)

This estimate applies to incompressible flows of ideal fluids in a number of instances. First, the case \( q = 2 \) applies to \( n \)-dimensional incompressible flows as long as the flow exists and the initial data has globally bounded kinetic energy. In the remainder of this article we will develop applications to 2D Euler flows.
For any $1 \leq p \leq \infty$, the $L^p$-norm of vorticity is a conserved quantity as long as the flow is smooth. We first assume that $\omega_0 \in L^p_c$, and we are interested in the cases $1 < p \leq 2$. Our concern is the propagation of the support of vorticity, which is a material domain. In order to apply Corollary 1.2, we need to know the appropriate a priori estimate for velocity. This is given in the next result, which is an analogue of the Sobolev embedding $W^{1,p} \hookrightarrow L^{p^*}$, with $p^* = \frac{2p}{2 - p}$.

Proposition 1.3. If $\omega \in L^p(\mathbb{R}^2)$, for some $1 < p < 2$, then $u = K * \omega \in L^{p^*}(\mathbb{R}^2)$, where $p^*$ is the critical Sobolev exponent introduced above. Moreover, we have the estimate

$$\|u\|_{L^{p^*}} \leq \frac{C}{\sqrt{2 - p}} \|\omega\|_{L^p},$$

for some $C = C(p)$, which blows up as $p \to 1$ and remains bounded as $p \to 2$.

Proof. Let $I_1$ be the first-order Riesz potential, so that, for $f$ in the Schwarz space $S(\mathbb{R}^n)$

$$(\overline{I_1f})(\xi) = \frac{\hat{f}(\xi)}{2\pi|\xi|}.$$ We consider also the Riesz transforms $R_j$ in $\mathbb{R}^2$, $j = 1, 2$ so that, for a function $f \in S(\mathbb{R}^n)$

$$(\overline{R_jf})(\xi) = i\xi_j \hat{f}(\xi) / |\xi|,$$

where $i = \sqrt{-1}$ and $\xi = (\xi_1, \xi_2)$.

The Riesz transforms are bounded in $L^q(\mathbb{R}^2)$, for any $1 < q < \infty$ with the operator norm continuous with respect to $q$, blowing up as $q \to 1$; see [12]. By the Hardy–Littlewood–Sobolev theorem, the Riesz potential maps $L^p(\mathbb{R}^2)$ continuously into $L^{p^*}(\mathbb{R}^2)$. We observe that the Biot–Savart law can be rewritten (up to a constant factor) as

$$u = I_1 R^\perp \omega,$$

where $R^\perp = (-R_2, R_1)$.

To see this, we first note that $I_1 R^\perp$ maps $L^p$ continuously into $L^{p^*}$. For a function in $f \in S(\mathbb{R}^2)$ we have that $I_1 R^\perp f = K_1 * f$, where $K_1 \in S'(\mathbb{R}^2)$ is such that its Fourier transform is $\hat{K_1} = i\xi^\perp / (2\pi|\xi|^2)$, for $\xi \neq 0$.

Let $\omega$ be a vorticity in the Schwarz space $S(\mathbb{R}^2)$, and consider both $u_1 = K_1 * \omega$ and $u_2 = K * \omega$. We will show they are the same. Observe that $u_1$ and $u_2$ are tempered distributions. The vector field $u_2$ is the unique solution to the elliptic system:

$$\begin{cases}
\text{div } u = 0, \\
\text{curl } u = \omega, \\
|u| \to 0 \text{ as } |x| \to \infty.
\end{cases}$$

We can pass the Fourier transform on the system above, and invert the resulting linear system for $\xi \neq 0$, to find that the Fourier transforms of $u_1$ and $u_2$ coincide. In particular, by varying $\omega$, one may conclude that $\hat{K} = \hat{K_1}$, for $\xi \neq 0$, which then
implies, by Theorem 3.2.3 of [5] (since \( \hat{K} \) and \( \hat{K}_1 \) are homogeneous of degree \(-1 > -2\)), that \( K_1 = K \) and hence that \( u_1 = u_2 \).

The proposition is proved, except for the asymptotic behavior, as \( p \to 2 \) of the operator norm of \( I_1 \).

To prove the asymptotic estimate, we begin by following the proof of Proposition 3.1.2 in [1], which gives a pointwise estimate of \( I_1f \) for \( f \in L^p(\mathbb{R}^2) \) in terms of the maximal function \( M[f] \). Tracking the constants, one arrives at the following estimate:

\[
|I_1f(x)| \leq C 2\pi + (2\pi)^{(p-1)/p} \left(\frac{p-1}{2-p}\right)^{(p-1)/p} \|f\|_{L^p(\mathbb{R}^2)}^{p/2} (M[f](x))^{p/2}.
\]

Using the Hardy–Littlewood maximal theorem, this implies that

\[
\|I_1f\|_{L^p(\mathbb{R}^2)} \leq A_p \|f\|_{L^p(\mathbb{R}^2)},
\]

with \( A_p = C[2\pi + (2\pi)^{(p-1)/p} (\frac{p-1}{2-p})^{(p-1)/p}] \). Since \( 1 < p < 2 \), \( A_p \) can be bounded from above by \( C/\sqrt{2-p} \).

Estimates for the propagation of support of vorticity can now be proved as a further corollary of Theorem 1.1 and Proposition 1.3.

**Corollary 1.4.** Assume that \( \omega_0 \) is a smooth function such that \( \text{supp} \ (\omega_0) \subseteq B(0; R_0) \). If \( 1 < p < 2 \), then there exists \( C_p > 0 \) such that, for any \( R > R_0 \),

\[
|\text{supp} \ \omega(\cdot, t) \cap B(0; R)| \leq C_p \frac{t}{R - R_0} \|\omega_0\|_{L^p}^{p/2}.
\]

In addition, there exist constants \( C > 0 \) and \( \eta > 0 \) such that if \( t/(R - R_0) < \eta \), then

\[
|\text{supp} \ \omega(\cdot, t) \cap B(0; R)| \leq |\text{supp} \ \omega_0| \exp \left(-C \frac{(R - R_0)^2}{t^2} \|\omega_0\|_{L^2}^2\right).
\]

**Proof.** The first part is a trivial consequence of Corollary 1.2 and Proposition 1.3 together with the fact that the support of vorticity is a material domain: \( \text{supp} \ \omega(\cdot, t) = X(\text{supp} \ \omega_0, t) \).

We now consider the second part. We use the fact that a compactly supported function in \( L^2 \) is also in \( L^p \) for any \( p < 2 \). More precisely, we have

\[
\|\omega_0\|_{L^p} \leq |\text{supp} \ \omega_0|^{1/p} \|\omega_0\|_{L^2}.
\]

Hence, from Proposition 1.3, we know that

\[
|\text{supp} \ \omega(\cdot, t) \cap B(0; R)| \leq |\text{supp} \ \omega_0| \left(C t \|\omega_0\|_{L^2} \right)^{p/2},
\]

for any \( 1 < p < 2 \). We optimize the estimate above in \( p \). Since we are interested only in the behavior for \( p \) near 2, we restrict ourselves to searching for minima in the range \( 7/4 < p < 2 \). We find that if

\[
\frac{t}{R - R_0} < \frac{1}{2C \|\omega_0\|_{L^2}^{7/16}} \equiv \eta,
\]

then there exists another constant \( \tilde{C} > 0 \) such that

\[
|\text{supp} \ \omega(\cdot, t) \cap B(0; R)| \leq |\text{supp} \ \omega_0| \exp \left(-\tilde{C} \frac{(R - R_0)^2}{t^2} \|\omega_0\|_{L^2}^2\right).
\]
as we wanted. □

The critical estimate in terms of the $L^2$-norm obtained above can be understood as a Trudinger–Moser inequality for the Lagrangian maps. The proof we presented is a variation on the standard proofs in this context.

If the initial vorticity has compact support and it belongs to $L^1(\mathbb{R}^2)$ or it is a bounded Radon measure in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$, then $\mathcal{S}V$ID, the associated velocity belongs to $L^2_{\text{loc}}(\mathbb{R}^2)$ for each fixed time. It is well known that velocity belongs to $L^2(\mathbb{R}^2)$ only if the vorticity has vanishing integral over all of $\mathbb{R}^2$. Consequently, for initial vorticities of compact support in $L^1(\mathbb{R}^2)$ or VSID, with integral zero, we also have an estimate, valid for smooth approximate solution sequences, of the form

$$|\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq C \left( \frac{t\|u_0\|_{L^2}}{R - R_0} \right)^2.$$  

However, flows with locally bounded kinetic energy are of physical interest. Furthermore, the only rigorous existence result for weak solutions with VSID requires that the initial vorticities have a distinguished sign. We can still prove an estimate for the Lebesgue measure of the support of vorticity lying outside a ball of radius $R$ in this setting.

**Theorem 1.5.** Let $\omega_0$ be a smooth function and let $T > 0$. Assume the support of $\omega_0$ is contained in the ball $B(0; R_0)$. Then there exists a constant $C = C(T, R_0) > 0$ such that for all $R > R_0$ and $0 \leq t \leq T$ we have

$$|\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq C \left( \frac{t}{R - R_0} \right)^2.$$

**Proof.** Fix $R > R_0$ and $0 \leq t \leq T$. Recall

$$\Sigma = \Sigma(R_0, R, t) \equiv \{ \alpha \in B(0; R_0) | |X(\alpha, t)| > R \}.$$  

Observe that

$$|\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq |\Sigma|.$$  

Hence, it is enough to estimate $|\Sigma|$. We have

$$|\Sigma| \leq \frac{1}{R^2} \int_{\Sigma} |X(\alpha, t)|^2 d\alpha$$  

$$\leq \frac{2}{R^2} \left( \int_{\Sigma} |X(\alpha, t) - \alpha|^2 d\alpha + \int_{\Sigma} |\alpha|^2 d\alpha \right)$$  

$$\equiv \frac{2}{R^2} (I_1 + I_2).$$

Note that $X(\alpha, t) - \alpha = \int_0^t u(X(\alpha, s), s) ds$. We will need to make use of the DiPerna–Majda decomposition of an $L^2_{\text{loc}}$ velocity (see [4]). To do this, choose a circularly symmetric, smooth, and compactly supported function $\tilde{\omega} = \tilde{\omega}(|x|)$ such that $\int_{\mathbb{R}^2} \tilde{\omega}(|x|) dx = \int_{\mathbb{R}^2} \omega_0(x) dx$. Let $\tilde{u} \equiv K * \tilde{\omega}(|\cdot|)$. The stationary velocity field $\tilde{u}$ is
smooth and decays as $1/|x|$, as $|x| \to \infty$. Let $\hat{\omega}(x, s) \equiv \omega(x, s) - \hat{\omega}(|x|)$ and $\hat{u} \equiv K \ast \hat{\omega}$.

It was shown in [4] that
\[
\|\hat{u}\|^2_{L^2(\mathbb{R}^2)} \leq K(T) = \|\hat{u}(\cdot, 0)\|^2_{L^2(\mathbb{R}^2)} e^{cT}.
\]

Let $\hat{\omega}(x, s) \equiv \omega(x, s) - \bar{\omega}(x)$ and $\hat{u} \equiv K \ast \hat{\omega}$.

Let $\alpha \in \Sigma$. Using the decomposition $u = \bar{u} + \hat{u}$ we have
\[
|X(\alpha, t) - \alpha|^2 \leq \left( \int_0^t |\bar{u}(X(\alpha, s))| ds + \int_0^t |\hat{u}(X(\alpha, s), s)| ds \right)^2
\]
\[
\leq \frac{Ct^2}{R^2} + Ct \int_0^t |\hat{u}(X(\alpha, s), s)|^2 ds.
\]

We can now estimate $I_1$:
\[
I_1 \leq \frac{Ct^2}{R^2} |\Sigma| + Ct \int_0^t \int_\Sigma |\hat{u}(X(\alpha, s), s)|^2 d\alpha ds
\]
\[
\leq \frac{Ct^2}{R^2} + Ct^2 K(T).
\]

In order to estimate $I_2$ note that if $\alpha \in \Sigma$, then
\[
|X(\alpha, t) - \alpha| \geq R - R_0 \geq \frac{|\alpha|(R - R_0)}{R_0}.
\]

Therefore,
\[
|\alpha| \leq \frac{R_0}{R - R_0} |X(\alpha, t) - \alpha|.
\]

We hence obtain
\[
I_2 \leq \left( \frac{R_0}{R - R_0} \right)^2 I_1 \leq \left( \frac{R_0}{R - R_0} \right)^2 \left( \frac{Ct^2}{R^2} + Ct^2 K(T) \right).
\]

Collecting these estimates, we finally get
\[
|\Sigma| \leq \frac{2}{R^2} \left( \frac{Ct^2}{R^2} + Ct^2 K(T) \right) \left( 1 + \left( \frac{R_0}{R - R_0} \right)^2 \right)
\]
\[
\leq \frac{Ct^2}{(R - R_0)^2},
\]

since, at the same time, $R > R_0$ and $R > R - R_0$.

There is one essential difference between this result and that of Corollary 1.4, which is the exponential growth of the constant $C$ in $T$, while the constant in Corollary 1.4 did not depend on $T$.

For flows with no restrictions on the sign of vorticity, there is a well-known trend for paired eddies with vorticities of opposite sign to move off to infinity with constant speed. Since more than one such pair of eddies may be present in a given flow,
with different average speeds, one expects that in some situations the diameter of the support of vorticity may grow linearly in time. We offer an explicit example from vortex dynamics to illustrate this behavior.

We consider the point-vortex evolution of four vortices, with initial configuration occupying the four vertices of the rectangle \([-a_0, a_0] \times [-b_0, b_0]\), with vorticity strength \(+\omega\) at \((a_0, b_0)\) and \((-a_0, -b_0)\) and with vorticity strength \(-\omega\) at \((a_0, -b_0)\) and \((-a_0, b_0)\). This configuration is called a vortex quadrupole. The evolution preserves the quadrupole structure and is determined by a \(2 \times 2\) system of ordinary differential equations for \((a(t), b(t))\), which is the position of the point vortex in the first quadrant. (In fact, by the reflexion method, the evolution of a vortex quadrupole is precisely the evolution of a single vortex in the first quadrant, regarded as a domain with boundary.) This \(2 \times 2\) system is explicitly integrable, and the solution is given by the formulas

\[
a(t) = \sqrt{\frac{1}{2} \left( q(t)^2 + 4k^2 + q(t)\sqrt{q(t)^2 + 4k^2} \right)},
\]

\[
b(t) = \frac{ka(t)}{\sqrt{a(t)^2 - k^2}},
\]

where

\[
c_0 = \sqrt{a_0^2 + b_0^2}, \quad k = a_0b_0/c_0, \quad q(t) = \frac{\omega t}{4\pi k} - \frac{b_0^2 - a_0^2}{c_0}.
\]

From these formulas it can be seen that the diameter of the support of vorticity grows linearly in time.

Since this article was first distributed in preprint form, a continuous version of this example was obtained by Iftimie, Sideris, and Gamblin in [6].

2. Flows with vorticity of distinguished sign. In this section we will concentrate on 2D flows with nonnegative vorticity. Our objective is to derive results for flows with a priori unbounded velocity that capture the \(O(t^{1/3})\) growth on the diameter of the support of vorticity proved by Marchioro in [10] for bounded vorticities.

Our results rely heavily on an exponential decay estimate on the mass of vorticity far from the center of motion. Although originally proved for flows with bounded vorticities in [10], this estimate actually applies, with negligible changes in the original proof, to very singular vorticities such as weak solutions of (0.1) with VSID, obtained as limits of approximate solution sequences generated by regularizing initial data. This exponential decay estimate was derived by Marchioro in the course of proving Theorem 2.1 in [10] and has never been stated as an independent result. We will do so here and we will offer a simplified proof, in part for the sake of completeness, in which we avoid the use of dyadic decompositions. We note that an even simpler and more elegant proof of Marchioro’s exponential decay estimate has been derived independently by Iftimie and Sideris [7].

We begin with an elementary technical lemma and then proceed to Marchioro’s result.

**Lemma 2.1.** Let \(\phi = \phi(r) \geq 0\) be a function such that

\[
\int_0^\infty \phi(r)r^2\,dr \equiv L < \infty.
\]
Let $0 < \lambda < 1$ and $a > 0$. Then
\[
\int_0^{\lambda a} \frac{r}{a(a-r)} \phi(r) r dr \leq \frac{L}{a^2(1-\lambda)^2}.
\]

Proof. Set $F(r) = \int_r^{\lambda a} \phi(s) ds$. Then
\[
\int_0^{\lambda a} \frac{r}{a(a-r)} \phi(r) r dr = - \int_0^{\lambda a} \frac{r}{a(a-r)} F'(r) dr = \int_0^{\lambda a} \frac{1}{(a-r)^2} F(r) dr
\]
\[
\leq \frac{1}{a^2(1-\lambda)^2} \int_0^{\lambda a} F(r) dr \leq \frac{1}{a^2(1-\lambda)^2} \int_0^{\lambda a} \int_r^{\lambda a} \phi(s) s ds dr
\]
\[
= \frac{1}{a^2(1-\lambda)^2} \int_0^{\lambda a} \phi(s) s^2 ds \leq \frac{L}{a^2(1-\lambda)^2}
\]
as we wanted. \(\square\)

**Theorem 2.2** (see Marchioro [10]). Let $\omega_0$ be a smooth nonnegative function with support contained in $B(0; R_0)$. Let $\omega = \omega(x,t)$ be the unique smooth solution of the vorticity equation (0.1) with initial vorticity $\omega_0$. For $R > 0$ define

\[
m_t(R) \equiv \int_{|x| > R} \omega(x,t) dx.
\]

Then there exists a constant $C > 0$, depending only on $\int_{\mathbb{R}^2} \omega_0(x) dx$, on the moment of inertia $\int_{\mathbb{R}^2} |x|^2 \omega_0(x) dx$ and on $R_0$, such that for any $n \in \mathbb{N}$ and any $R > 0$ satisfying $nR_0 < R \leq (n+1)R_0$, we have

\[
m_t(R + R_0) \leq \left( \frac{Ct}{(R - R_0)^3} \right)^n.
\]

Proof. Let $W = W(r)$ be a nondecreasing smooth function such that $W(r) = 0$, if $r \leq R_0$ and $W(r) = 1$ if $r \geq 2R_0$. Let $R > R_0$. Set $\varphi = \varphi(y) \equiv W(|y| - (R - R_0))$ for $y \in \mathbb{R}^2$. Clearly, if $|y| > R_0 + R$ or $|y| < R$, $\varphi$ is constant, and hence its first derivatives vanish. We will need the fact that the second derivatives of $\varphi$ are uniformly bounded, independently of $R \geq R_0$. Indeed,

\[
\frac{\partial^2 \varphi}{\partial y_i \partial y_j} = W''(|y| - (R - R_0)) \frac{y_i y_j}{|y|^2} + W'(|y| - (R - R_0)) \left( \frac{\delta_{ij}}{|y|} - \frac{y_i y_j}{|y|^3} \right).
\]

Therefore, this second derivative vanishes outside $R < |y| < R + R_0$ and is bounded by $C(1 + 1/R) < C(1 + 1/R_0)$.

Following Marchioro, we introduce the smoothed-out version of $m_t(R)$:

\[
\tilde{m}_t(R) = \int_{\mathbb{R}^2} \varphi(y) \omega(y,t) dy.
\]

Then
\[
\frac{d}{dt} \tilde{m}_t(R) = \int_{\mathbb{R}^2} \varphi(y) \omega_t(y,t) dy = \int_{\mathbb{R}^2} \nabla \varphi(y) u(y,t) \omega(y,t) dy
\]

We divide $\mathbb{R}^2$ into three regions: $O_1 = B(0, R/2)$, $O_2 = \{R/2 \leq |x| < R\}$, and $O_3 = \{|x| \geq R\}$, and divide $\mathbb{R}^2 \times \mathbb{R}^2$ into the nine disjoint regions $O_{ij} = O_i \times O_j$. We first observe that

$$\int_{O_{ij}} (\nabla \varphi(y) - \nabla \varphi(z)) K(y-z) \omega(y, t) \omega(z, t) dy dz = 0$$

if both $i$ and $j$ are at most 2.

We begin by estimating the integral on $O_{13}$:

$$\frac{1}{2} \int_{O_{13}} (\nabla \varphi(y) - \nabla \varphi(z)) K(y-z) \omega(y, t) \omega(z, t) dy dz$$

$$= -\frac{1}{2} \int_{|y|<R/2} \int_{|z|\geq R} W'(|z| - (R - R_0)) \frac{z}{|z|} K(y-z) \omega(y, t) \omega(z, t) dy dz$$

$$\leq C \left( \sup_{|z|\geq R} \left| \int_{|y|<R/2} \frac{z}{|z|} K(y-z) \omega(y, t) dy \right| \right) \int_{|z|\geq R} \omega(z, t) dz.$$ 

We will now make use of Lemma 2.1. Let $\phi(r, t) \equiv \int_0^{2\pi} \omega(r \cos \theta, \sin \theta) d\theta$. Then,

$$\int_0^\infty \phi(r, t) r^2 dr = \int_{\mathbb{R}^2} |x| \omega(x, t) dx \leq \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|^2) \omega(x, t) dx$$

$$= C \int_{\mathbb{R}^2} (1 + |x|^2) \omega_0(x) dx = L.$$ 

Note that, for $|z| \geq R$, we have

$$\left(2.2\right) \quad \left| \int_{|y|<R/2} \frac{z}{|z|} K(y-z) \omega(y, t) dy \right|$$

$$= \left| \int_{|y|<R/2} \frac{y}{|z|} \frac{(y-z)\perp}{2\pi|y-z|^2} \omega(y, t) dy \right| \leq \int_{|y|<|z|/2} \frac{|y|}{|z|(|z| - |y|)} \omega(y, t) dy$$

$$\left(2.3\right) \quad \int_0^{|z|/2} r \frac{r}{|z|(|z| - r)} \phi(r, t) rdr \leq \frac{4L}{|z|^2}$$

using Lemma 2.1 with $a = |z|$ and $\lambda = 1/2$.

We conclude the estimate on $O_{13}$ obtaining

$$\frac{1}{2} \int_{O_{13}} (\nabla \varphi(y) - \nabla \varphi(z)) K(y-z) \omega(y, t) \omega(z, t) dy dz \leq C \frac{m_4(R)}{R^2}.$$
Similarly, on $O_{31}$

$$\frac{1}{2} \int_{O_{31}} (\nabla \varphi(y) - \nabla \varphi(z)) K(y - z) \omega(y, t) \omega(z, t) dydz \leq C \frac{m_t(R)}{R^2}.$$ 

Next observe that since the moment of inertia is conserved, $m_t(r) \leq C/r^2$. We now estimate the integral on $O_{23} \cup O_{33}$. We have

\begin{align*}
\frac{1}{2} \int_{O_{23} \cup O_{33}} & (\nabla \varphi(y) - \nabla \varphi(z)) K(y - z) \omega(y, t) \omega(z, t) dydz \\
& \leq C \left( \sup_{y, z \in \mathbb{R}^2} |(\nabla \varphi(y) - \nabla \varphi(z)) K(y - z)| \right) \int_{|y| \geq R/2} \omega(y, t) dy \int_{|z| \geq R} \omega(z, t) dz \\
& \leq C \frac{m_t(R)}{R^2},
\end{align*}

similarly for $O_{32} \cup O_{33}$. Finally, observe that

\begin{align*}
\frac{1}{2} \int_{O_{33}} & (\nabla \varphi(y) - \nabla \varphi(z)) K(y - z) \omega(y, t) \omega(z, t) dydz \\
& \leq C(m_t(R))^2 \leq C \frac{m_t(R)}{R^2},
\end{align*}

since the second derivatives of $\varphi$ are bounded.

We have therefore shown that

$$\frac{d}{dt} \tilde{m}_t(R) \leq C \frac{m_t(R)}{R^2},$$

that is,

$$\tilde{m}_t(R) \leq \frac{C}{R^2} \int_0^t m_s(R) ds,$$

since $\tilde{m}_0(R) = 0$. Now we repeat the backwards induction argument of Marchioro. Note that

$$\tilde{m}_t(R) \leq m_t(R) \leq \tilde{m}_t(R - R_0).$$

We now fix $n \in \mathbb{N}$ and $R$ such that $nR_0 < R \leq (n+1)R_0$. By iterating backwards in time and in $R$ we get

\begin{align*}
\tilde{m}_t(R) & \leq C^n \frac{n!}{\prod_{i=0}^{n-1} (R - iR_0)^2} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} m_{s_n}(R - (n - 1)R_0) ds_n \ldots ds_2 ds_1 \\
& \leq \frac{C^n t^n}{(n!)^3} \leq \frac{C^n t^n e^{3n}}{n^{3n}} \equiv \left( \frac{Ct}{n^3} \right)^n \leq \left( \frac{Ct}{(R - R_0)^3} \right)^n.
\end{align*}

Since $\tilde{m}_t(R) \geq m_t(R + R_0)$, the conclusion follows. \qed
The result above offers no control on the mass of vorticity contained in the annulus \( \{ R_0 \leq |x| \leq 2R_0 \} \). The proof could be modified, by suitably changing the definition of \( W \), so that this absence of control would occur only on the annulus \( \{ R_0 \leq |x| \leq R_0 + \varepsilon \} \), with \( \varepsilon \) arbitrary. However, the constant \( C \) would blow up as \( \varepsilon \to 0 \). To obtain uniform control over the mass of vorticity outside a ball of radius \( R_0 + \varepsilon \), we need to use the Chebyshev-type inequalities proved in the first section.

The following results are extensions of the statement of Theorem 2.1 in [10] to much more singular flows. We will continue using the notation \( m_t(R) \) as in (2.1).

**Proposition 2.3.** Let \( \omega_0 \) be a smooth nonnegative function, with support contained in \( B(0; R_0) \). Let \( 1 < p \leq 2 \) and \( \| \omega_0 \|_{L^p(B)} \leq K \). Then, for every \( \delta > 0 \), there exists \( b = b(K, \delta) > 0 \) such that for any \( t > 0 \)

\[
m_t((R_0^3 + bt)^{1/3}) < \delta.
\]

**Proof.** Fix \( 0 < \delta < 1 \). We start with the trivial observation that, for any \( R > R_0 \),

\[
m_t(R) \leq \| \omega_0 \|_{L^p} |\text{supp } \omega(\cdot, t) \cap B(0; R)|^{1/p'}.
\]

From this and from Corollary 1.4 it follows that there exists \( b_1 = b_1(K, \delta) > 0 \) such that \( m_t(R_0 + b_1 t) < \delta \) for all \( t > 0 \).

From Theorem 2.2, there exists \( b_2 = b_2(K, \delta) > 0 \) such that \( m_t(2R_0 + (b_2 t)^{1/3}) < \delta \), again for every \( t > 0 \).

It is easy to see that one can choose \( b = b(K, \delta) \) such that

\[
\min \left\{ R_0 + b_1 t, 2R_0 + (b_2 t)^{1/3} \right\} \leq (R_0^3 + bt)^{1/3},
\]

and this concludes the proof. \( \square \)

A similar result is still true in \( L^1 \); however, the constant \( b \) obtained does not depend uniformly on the \( L^1 \)-norm of vorticity.

**Proposition 2.4.** Let \( \{ \omega_0^\varepsilon \} \) be a uniformly integrable family of nonnegative smooth functions, with support contained in \( B(0; R_0) \). Then, for every \( \delta > 0 \), there exists \( b = b(\delta) \) such that for any \( \varepsilon \)

\[
\int_{|x| > (R_0^3 + bt)^{1/3}} \omega^\varepsilon(x, t)dx < \delta.
\]

**Proof.** Fix \( \delta > 0 \). By the definition of uniform integrability, there exists \( \eta > 0 \) such that for any \( E \subseteq \mathbb{R}^2 \), with \( |E| < \eta \) and for any \( \varepsilon \),

\[
\int_E \omega_0^\varepsilon(\alpha)d\alpha < \delta.
\]

Recall, from the proof of Theorem 1.5, that

\[
|\Sigma(R_0, R, t)| = |\{ \alpha \in B(0; R_0) \mid |X^\varepsilon(\alpha, t)| > R \}| \leq C \frac{t^2}{(R - R_0)^2},
\]

where \( X^\varepsilon \) is the trajectory associated with the velocity field induced by \( \omega^\varepsilon \). Note from the proof of Theorem 1.5 that \( C \) does not depend on \( \varepsilon \).

It is then possible to choose \( b_1 \) such that \( |\Sigma(R_0, R_0 + b_1 t, t)| < \eta \). Next note that

\[
\int_{|x| > R_0 + b_1 t} \omega^\varepsilon(x, t)dx = \int_{\Sigma(R_0, R_0 + b_1 t, t)} \omega_0^\varepsilon(\alpha)d\alpha < \delta.
\]
The remainder of the argument follows precisely as in the proof of Proposition 2.3. □

Let \( \omega_0 \in L^1_0(\mathbb{R}^2) \) be nonnegative. Consider any weak solution \( \omega \), obtained by mollifying the initial data \( \omega_0 \), in such a way as to keep the support of the regularized vorticities inside \( B(0; R_0) \). Such a weak solution was first shown to exist by Delort in [2]. Then Proposition 2.4 implies that, for any \( \delta > 0 \), there exists \( b \) such that \( m_t((R_0^3 + bt)^{1/3}) < \delta \). Of course, Proposition 2.3 implies the same estimate for weak solutions obtained by regularizing initial vorticities in \( L^p_0(\mathbb{R}^2) \). The subtle difference is that, for \( 1 < p \leq 2 \), \( b \) depends on \( \omega_0 \) through its \( L^p \)-norm. The dependence of \( b \) on \( \omega_0 \) in the \( L^1 \) case is more delicate. (It depends on the modulus of continuity of \( \omega_0 \), regarded as a measure.)

For \( \omega_0 \) a nonnegative, compactly supported bounded measure in \( H^1_{loc}(\mathbb{R}^2) \), we cannot prove a result of this nature for the approximate solution sequences obtained by mollifying \( \omega_0 \). Theorem 2.2 remains valid in this situation, enabling the choice of \( b_2 \). However, we have no tools to choose \( b_1 \), i.e., to estimate the mass of vorticity outside \( B(0; R) \), with \( R \) close to \( R_0 \). Proposition 2.4 cannot be used since, by the Dunford–Pettis theorem, regularizing \( \omega_0 \) does not produce a uniformly integrable sequence. The best result we can obtain along these lines retains the asymptotic behavior as \( t \to \infty \). It is a trivial consequence of Theorem 2.2 that, for every \( \delta > 0 \), there exists \( b > 0 \) such that

\[
m_t(2R_0 + (bt)^{1/3}) < \delta.
\]

In a sense, the results above, controlling the dispersion of the mass of vorticity, are unsatisfactory. We set out to study how much the fluid particles can get displaced by irregular fluid flow, and the control on the dispersion of the mass of vorticity, at first glance, does not give information in that respect. The next two results address this issue more precisely, demonstrating that the control on the dispersion of the mass of vorticity achieved so far, plus the techniques and results of the first section, do indeed control the dispersion of material domains in general. We cast the results in terms of the measure of the set of vorticity-bearing particles flung far from their initial positions by the flow, a very particular material domain, but this restriction is not essential. We prove two results: one showing precisely how the control on the dispersion of the mass of vorticity implies control on particle trajectories and the second one, giving a less precise, but more elegant description, in which we bring out explicitly the cubic parabola behavior of the dispersion discovered by Marchioro.

**Theorem 2.5.** Let \( \omega_0 \) be a nonnegative smooth function, with support contained in \( B(0; R_0) \). Suppose that \( \|\omega_0\|_{L^1(\mathbb{R}^2)} \leq K \). Then there exist \( C_1 = C_1(K, R_0) > 0 \) and \( C_2 = C_2(K, R_0) > 0 \) such that if \( R > 2R_0 \) and \( 0 < t < C_2R^3 \), then

\[
|\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq \frac{C_1 t R^3}{C_2 R^3 - t} m_t(R/4).
\]

**Proof.** Fix \( R > 2R_0 \) and \( t > 0 \). Let \( \Sigma(R/2, R, t) \equiv \{ \alpha \in B(0; R/2) \mid |X(\alpha, t)| > R \} \), which we abbreviate \( \Sigma_R \). Then, \( |\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq |\Sigma_R| \).

We intend to estimate the velocity in the annulus \( A_R \equiv \{ R/2 < |x| < R \} \). We decompose the velocity \( u = K * \omega(\cdot, s) \), for \( 0 \leq s \leq t \), into a near-field and a far-field velocity in the following way:

\[
u_N(x, s) = \int_{|y| > R/4} K(x - y) \omega(y, s) dy; \quad u^F = u - u^N.
\]
Consider the cylinder \( C \), defined by \( C \equiv \bigcup_{0 \leq s \leq t} X(\Sigma_R, s) \). Next, observe that

\[
\int_C \chi_{A_R}(x) u^N(x, s) \cdot \frac{x}{|x|} \, dx ds
\]

\[
= \int_C \chi_{A_R}(x) u(x, s) \cdot \frac{x}{|x|} \, dx ds - \int_C \chi_{A_R}(x) u^F(x, s) \cdot \frac{x}{|x|} \, dx ds
\]

\[
\geq (R - R/2)|\Sigma_R| - t|\Sigma_R| \sup_{(x, s) \in C} \left| u^F(x, s) \cdot \frac{x}{|x|} \chi_{A_R}(x) \right|
\]

\[
\geq \frac{R}{2}|\Sigma_R| - t|\Sigma_R| \frac{C}{R^2} = \left( \frac{R}{2} - \frac{Ct}{R^2} \right)|\Sigma_R|,
\]

where the latter inequality follows from the same reasoning as in (2.2)–(2.3), with \( C = C(K, R_0) \), and the former inequality is a consequence of (1.1). On the other hand, using Hölder’s inequality we have

\[
\int_C \chi_{A_R}(x) u^N(x, s) \cdot \frac{x}{|x|} \, dx ds \leq t \sup_{0 \leq s \leq t} \| u^N(\cdot, s) \|_{L^1(A_R)}
\]

\[
\leq \tilde{C} t \sup_{0 \leq s \leq t} \int_{|y| > R/4} \omega(y, s) \int_{A_R} \frac{1}{|x - y|} \, dx \, dy
\]

\[
\leq \tilde{C} t R \tilde{m}(R/4),
\]

since, due to the monotonicity of \( 1/r \), the integral of \( |x|^{-1} \) on any set of measure \( 3\pi R^2/4 \) is maximized by taking the integrating set to be the ball with this measure centered at 0, and hence it is bounded by \( \sqrt{3}\pi R \). Taking \( C_2 = (2C)^{-1}, C_1 = \tilde{C}/C \), and assuming that \( t < C_2 R^2 \) we obtain the desired conclusion.

**Proposition 2.6.** Let \( \omega_0 \) be a nonnegative smooth function with support contained in \( B(0; R_0) \). Suppose that \( \| \omega_0 \|_{L^1(B^2)} \leq K \). Then for every \( \delta > 0 \), there exists \( b = b(K, R_0, \delta) > 0 \) such that for every \( t > 0 \)

\[
|\supp \omega(\cdot, t) \cap B(0; (R_0^3 + bt)^{1/3})| < \delta.
\]

**Proof.** The proof follows the reasoning of the proofs of Propositions 2.3 and 2.4, but it is more intricate.

We begin by choosing \( b_1 > 0 \) such that for \( 0 \leq t \leq 1 \)

\[
|\supp \omega(\cdot, t) \cap B(0; R_0 + b_1 t)| < \delta.
\]

This is done using Theorem 1.5, with \( T = 1 \).

Next we choose \( b_2 > 0 \) such that for \( t \geq 1 \)

\[
|\supp \omega(\cdot, t) \cap B(0; 8R_0 + (b_2 t)^{1/3})| < \delta.
\]

This is accomplished using Theorems 2.5 and 2.2, as we shall describe below.
Denote $R(t) = 8R_0 + (b_2t)^{1/3}$ for some $b_2$ to be determined.

First we choose $b_2$ large enough so as to guarantee that, for some $C_3 > 0$, we have $t < t + C_3 < C_2(R(t))^3$, where $C_2$ comes from Theorem 2.5. Next, we invoke the estimate on $m_4(R(t)/4)$, given by Theorem 2.2, together with Theorem 2.5. After a number of straightforward estimates, we observe it is enough to find $b_2$ large enough, so that

$$C_4b_2(1 + t^2) \left( \frac{C_5}{b_2} \right) C_6(b_2t)^{1/3} < \delta$$

for certain constants $C_4$, $C_5$, and $C_6$. Then it is enough to note that if $b_2$ is large enough, the left-hand side of the inequality above is monotone decreasing as a function of $t$ and its value at $t = 1$ converges to zero as $b_2 \to \infty$.

Finally, it is easy to see that we can choose $b$ so that

$$R_0 + b_1t \leq (R_0^3 + bt)^{1/3}, \quad 0 \leq t \leq 1,$$

$$8R_0 + (b_2t)^{1/3} \leq (R_0^3 + bt)^{1/3}, \quad 1 \leq t < \infty.$$ 

This concludes the proof. □

3. Concluding remarks. We have proved several results concerning the amount of vorticity near infinity, arising from flow with compactly supported initial vorticity, explicitly formulated as uniform estimates on approximate solution sequences. Let us now consider a weak solution $u$ of the incompressible 2D Euler equations, obtained as the weak limit of an approximate solution sequence $u^\varepsilon$. Let $\omega = \text{curl } u$, $\omega^\varepsilon = \text{curl } u^\varepsilon$. We restrict ourselves to approximate solution sequences obtained by mollifying the initial data and exactly solving the equations.

First we assume that $\omega_0 = \omega(\cdot, 0) \in L^p_0(\mathbb{R}^2)$, $1 \leq p \leq 2$. If $p = 1$, we have to assume in addition that $u_0 = u(\cdot, 0) \in L^p_{\text{loc}}(\mathbb{R}^2)$. The estimates obtained in Proposition 2.6 for nonnegative vorticities, in Corollary 1.4 for vorticities without sign restriction and $p > 1$, and in Theorem 1.5 if $p = 1$, which are all estimates on the size of the support of vorticity near infinity do not extend in any obvious way to the weak limit, because the measure of the support of an $L^p$ function is not even weakly lower semicontinuous. These uniform estimates on the size of the support of vorticity imply estimates on the $p$th power integral of vorticity near infinity, which remain valid for the weak limit because the $p$th power integral is a weakly lower semicontinuous functional over $L^p$.

Let us be more precise. We will detail the argument in the case of a nonnegative initial vorticity $\omega_0 \in L^p_0(\mathbb{R}^2)$, $1 \leq p \leq 2$, supported in $B(0; R_0)$. We begin by observing that the mollified initial data $\omega_0^\varepsilon$ is uniformly $p$th power integrable, by the Dunford–Pettis theorem, since $|\omega_0^\varepsilon|^p$ converges strongly in $L^1$ to $|\omega_0|^p$. This means that, for every $\eta > 0$, there exists $\delta > 0$, independent of $\varepsilon$, such that

$$\int_E |\omega_0^\varepsilon|^p dx \leq \eta,$$

for any measurable set $E$ with Lebesgue measure less than $\delta$. Fix $\eta > 0$ and consider the corresponding $\delta$. We use Proposition 2.6 to obtain $b > 0$, depending only on $\|\omega_0\|_{L^1}$, on $R_0$, and on $\delta$ such that

$$|\text{supp } \omega^\varepsilon(\cdot, t) \cap B(0; (R_0^3 + bt)^{1/3})^c| < \delta.$$
Hence, if $E^\varepsilon$ is the backwards flow through $u^\varepsilon$ of supp $\omega^\varepsilon(\cdot, t) \cap B(0; (R_0^3 + bt)^{1/3}^\varepsilon)$, then

$$\int_{|x|>({R_0^3} + bt)^{1/3}^\varepsilon}|\omega^\varepsilon|^p(x, t)dx = \int_{E^\varepsilon}\omega_0^\varepsilon(x)dx \leq \eta,$$

since $|E^\varepsilon| < \delta$. Clearly, by the weak lower semicontinuity,

$$\int_{|x|>({R_0^3} + bt)^{1/3}^\varepsilon}|\omega|^p(x, t)dx \leq \eta.$$

Analogous results for vorticity without sign restrictions follow from Corollary 1.4 and Theorem 1.5 in the same manner.

For VSID, the only result obtained that extends, in the sense above, to an estimate on the weak limits is Theorem 2.2, again because the total variation of measures is weak- lower semicontinuous.

We have mentioned that there is no obvious way to pass the weak limit in the estimates on the size of the support of vorticity near infinity. We will show now that $\omega^\varepsilon$ is a renormalized solution of (3.1). Then, for almost every $t > 0$ and every $R > 0$

$$|\text{supp } \omega(\cdot, t) \cap B(0; R)| \leq \limsup_{\varepsilon \to 0} |\text{supp } \omega^\varepsilon(\cdot, t) \cap B(0; R)|.$$

**Proof.** Fix $T > 0$ and let $R_0 > 0$ be such that $B(0; R_0)$ contains the support of $\omega_0^\varepsilon$ for every $\varepsilon$.

We begin by observing that $\omega = \text{curl } u$ is the unique renormalized solution of the linear transport equation:

$$\begin{cases} f_t + u \cdot \nabla f = 0, \\
f(x, 0) = \omega_0, \end{cases}$$

as defined by DiPerna and Lions; see [3].

To see this, first note that the restriction to $p > 1$ is needed to ensure that $u \in L^1([0, T]; W^{1,1}_{\text{loc}}(\mathbb{R}^2))$. Of course, $\text{div } u = 0$. We also have that

$$u \frac{1}{1 + |x|} \in L^1([0, T]; L^2(\mathbb{R}^2)) + L^1([0, T]; L^\infty(\mathbb{R}^2)).$$

(See Remark 1.1 and Section 1.C of [4].) In order to show that $\omega$ is a renormalized solution of (3.1), we will make use of the uniqueness result, Theorem II.3, and the stability result, Theorem II.4, in [3]. Both of these results require $u$ as above except that $u/(1 + |x|)$ has to belong to $L^1([0, T]; L^2(\mathbb{R}^2)) + L^1([0, T]; L^\infty(\mathbb{R}^2))$. However, it is easy to see that one can substitute this condition with (3.2) and still prove uniqueness and stability.

Next we check the hypothesis of the stability result. We know that $\omega$ is the weak- limit in $L^\infty([0, T]; L^p(\mathbb{R}^2))$ of the sequence $\{\omega^\varepsilon\}$, which is a smooth solution (and hence a renormalized solution) of (3.1) with $u$ replaced by $u^\varepsilon$ and $\omega_0$ replaced
by \( \omega_0 \). Additionally, the initial data \( \omega_0 \) converge strongly to \( \omega_0 \) in \( L^p \). Therefore, by Theorem II.4 of [3], \( \omega \) is the unique renormalized solution of (3.1).

Let \( X^\varepsilon \) be the Lagrangian map associated with the flow \( u^\varepsilon \) and \( X \) be the unique renormalized Lagrangian map associated with \( u \), by Theorem III.2 of [3]. Then, by a time-dependent version of the stability of Lagrangian maps, Corollary III.1 in [3], we conclude that \( X^\varepsilon \to X \) locally uniformly in time and locally in measure in space. Therefore, for every \( \eta > 0 \), there exists \( \varepsilon_0 \) independent of \( t \in [0, T] \) such that for \( \varepsilon < \varepsilon_0 \),

\[
|\{ \alpha \in B(0; R_0) | X^\varepsilon(\alpha, t) - X(\alpha, t) > \eta \}| \leq \eta.
\]

Fix \( \eta > 0 \) and choose \( \varepsilon_0 = \varepsilon_0(\eta) \) as above. Let \( R > 0 \). Then, for any \( \varepsilon < \varepsilon_0 \), we have that

\[
|\{ \alpha \in B(0; R_0) | X(\alpha, t) > R \}| \leq |\{ \alpha \in B(0; R_0) | X^\varepsilon(\alpha, t) - X(\alpha, t) > \eta \}|
\]

\[
+ |\{ \alpha \in B(0; R_0) | X^\varepsilon(\alpha, t) > R - \eta \}|
\]

\[
\leq \eta + |\{ \alpha \in B(0; R_0) | X^\varepsilon(\alpha, t) > R - \eta \}|
\]

\[
\leq \eta + |\{ \alpha \in B(0; R_0) | R - \eta < |X^\varepsilon(\alpha, t)| \leq R \}| + |\{ \alpha \in B(0; R_0) | |X^\varepsilon(\alpha, t)| > R \}|
\]

\[
\leq \eta + (2\pi R \eta - \eta^2) + |\{ \alpha \in B(0; R_0) | |X^\varepsilon(\alpha, t)| > R \}|
\]

where the last inequality follows from the fact that \( X^\varepsilon \) is area-preserving. Therefore, taking \( \limsup_{\eta \to 0} \), we have

\[
|\{ \alpha \in B(0; R_0) | X(\alpha, t) > R \}| \leq \limsup_{\varepsilon \to 0} |\{ \alpha \in B(0; R_0) | X^\varepsilon(\alpha, t) > R \}|
\]

since we may assume that \( \varepsilon_0(\eta) \to 0 \) as \( \eta \to 0 \).

We conclude by observing that the renormalized Lagrangian map \( X \) is area-preserving and \( \omega(X(\alpha, t), t) = \omega_0(\alpha) \) (see Theorem III.2 in [3]); hence

\[
|\{ \alpha \in B(0; R_0) | X(\alpha, t) > R \}| = |\text{supp } \omega(\cdot, t) \cap B(0; R)^c|.
\]

Let \( u \) be a weak solution of the incompressible 2D Euler equations with vorticity \( \omega = \text{curl } u \in L^\infty([0, T]; L^p(\mathbb{R}^2)) \), \( p > 1 \). It was mentioned in the proof of Theorem 4.1 in [8] that \( \omega \) is the unique renormalized solution of the vorticity equation, regarded as a linear transport equation. We included an outline of the proof of this fact in the sake of completeness. Of course, given Theorem 3.1 above, the uniform estimates derived in Corollary 1.4 and in Proposition 2.6 remain valid for the weak solution, if the initial vorticity belongs to \( L^p \), \( p > 1 \).

One natural question at this point is how close these estimates are to being optimal. In this respect, we do not have anything to add to the comments made by Marchioro in [10] and we refer the reader to the discussion contained there. We are left with no answer to the question we started with, i.e., whether the support of vorticity remains compact as time evolves. We can say only that the knowledge developed here does not appear to be enough to answer this question.

Finally, we note that since this article was first distributed in preprint form, a significant improvement of Marchioro’s cubic-root estimate was obtained by Serfati; see [11]. A similar, if slightly weaker improvement was obtained independently by Iftimie and Sideris in [6]. By using the conservation of the center of vorticity, they observe that Marchioro’s space-time cubic-root parabola can almost be improved to a fourth-root parabola. It would be possible to rewrite section 2 of our work reflecting these improvements, in a straightforward manner.
REFERENCES


