ROBUST FILTERING OF DISCRETE-TIME LINEAR SYSTEMS WITH PARAMETER DEPENDENT LYAPUNOV FUNCTIONS*

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Abstract. Robust filtering of linear time-invariant discrete-time uncertain systems is investigated through a new parameter dependent Lyapunov matrix procedure. Its main interest relies on the fact that the Lyapunov matrix used in stability checking does not appear in any multiplicative term with the uncertain matrices of the dynamic model. We show how to use such an approach to determine high performance $H_2$ robust filters by solving a linear problem constrained by linear matrix inequalities (LMIs). The results encompass the previous works in the quadratic Lyapunov setting. Numerical examples illustrate the theoretical results.

Key words. linear systems, discrete-time systems, parameter uncertainty, filtering, parameter dependent Lyapunov functions, linear matrix inequalities

AMS subject classifications. 93C05, 93C55, 93E11, 93E25

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1. Introduction. Filtering is a very important issue in systems diagnosis, surveillance, and control. The problem, which amounts to extracting the information from the measured output to provide an estimate of the state (or a linear combination of the state), has been addressed in the stochastic as well as in the deterministic framework. Seminal works in this domain are the ones of Kalman and Luenberger (see [1, 10] for a complete discussion).

As a dual of the control design problem, the development of robust filters closely follows the same design steps. For linear uncertain systems, the problem can be stated as the minimization of an appropriate bound on a transfer function between an exogenous noise and the estimation error. There have been several contributions using $H_2$ and $H_\infty$ norms as criteria for filter determination under parameter uncertainty. In the unstructured norm bounded uncertainty case, one can cite [9, 11, 13]. The structured case, which is a bit more complex, has also received some attention in both continuous-time [5, 8] and discrete-time [6, 7] contexts. These approaches are based on the quadratic stability concept, where a single Lyapunov matrix is used for the estimation error norm evaluation over the whole uncertainty domain. If we consider time-invariant uncertain systems, this assumption reveals to be, in fact, a hard constraint, implying a significant degree of sufficiency to these results. In fact, all results based on quadratic stability can also be applied to arbitrarily fast time-variant uncertainties.

In this paper, we use a new stability condition for discrete-time uncertain systems which enables the determination of parameter dependent Lyapunov matrices. This stability condition, which was first introduced in [3], provides results which go beyond the ones attainable by the quadratic approach for time-invariant parameter uncertainty. It is expressed as a linear matrix inequality (LMI) and exhibits a kind

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of separation property between the Lyapunov matrices and the uncertain dynamic matrices. Here we show how the given condition can be extended to provide optimal performance in terms of an $H_2$ norm guaranteed cost problem. Furthermore, we show how to parametrize linear filters so that the synthesis of robust filters can be cast as an LMI optimization problem. We prove that our results encompass the results obtained on the quadratic stability framework [6, 7] and, consequently, reduce to the classical Kalman filter in the absence of uncertainty. The filtering results are generalized to cope with structure constraints such as decentralization.

The outline of the paper is as follows. In section 2, we formulate the problem to be solved. In section 3, we summarize the existent results on robust stability and robust filtering. Then we introduce the new stability and performance conditions in section 4 and illustrate its features with a numerical example. The robust $H_2$ filtering problem is developed in section 5 and extended to cope with structural constraints in section 6. Several numerical examples illustrate the results in section 7, enlightening the efficiency of the proposed approach by comparing the given results to the existent ones. The paper finishes with some concluding words.

The notation used throughout is as follows. Capital letters denote matrices, and small letters denote vectors. For scalars, we use small Greek letters. For matrices or vectors, $(\cdot)'$ indicates transposition. For symmetric matrices, $X > 0 \ (\geq 0)$ indicates that $X$ is positive definite (nonnegative definite). For square matrices, trace$(X)$ denotes the trace function of $X$ being equal to the sum of its eigenvalues. For a transfer function $T(\zeta)$ analytic outside the unit circle, $\|T(\zeta)\|_2$ denotes the standard $H_2$ norm. Finally, for the sake of easing the notation of partitioned symmetric matrices, the symbol $(\cdot)$ generically denotes each of its symmetric blocks.

2. Problem statement and definitions. Let us consider the linear time-invariant discrete-time system

\begin{align}
  x(k+1) &= Ax(k) + Bw(k), \\
  y(k) &= Cx(k) + Dw(k), \\
  z(k) &= Lx(k),
\end{align}

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is a white noise input with zero mean and identity covariance matrix, $y \in \mathbb{R}^r$ is the measured output, and $z \in \mathbb{R}^s$ is the vector to be estimated. All matrices are of compatible dimension. We assume that matrix $L$ is known and that the time-invariant parameters gathered in the matrix

\begin{equation}
  M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\end{equation}

are unknown but belong to the convex polyhedron

\begin{equation}
  \mathcal{M} := \left\{ M(\xi) = \sum_{i=1}^{N} \xi_i M_i, \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0 \right\}.
\end{equation}

The robust $H_2$ filtering problem considered here is to design an estimate $\hat{z}$ of $z$ given by $\hat{z} = \mathcal{F} \cdot y$. The filter $\mathcal{F}$ is supposed to be a linear, finite dimensional, and causal operator. We characterize $\mathcal{F}$ by the generic element given in the form of a linear time-invariant operator with minimum state space realization

\begin{align}
  \hat{x}(k+1) &= A_f \hat{x}(k) + B_f y(k), \\
  \hat{z}(k) &= C_f \hat{x}(k),
\end{align}
where the matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times r}$, and $C_f \in \mathbb{R}^{r \times n}$ are to be determined and define the filter transfer function

$$T_f(\zeta) = C_f(\zeta I - A_f)^{-1} B_f.$$  

Moreover, it is considered that the initial condition of system (1)–(3) as well as the initial condition of the filter (6)–(7) are both zero.

The connection of the filter and the system yields, for each element in the set $\mathcal{M}$, a linear system described by the transfer function from the noise input $w$ to the estimation error $e := z - \hat{z}$,

$$T_M(\zeta) := \tilde{C}(\zeta I - \tilde{A})^{-1} \tilde{B},$$

where matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ of compatible dimensions are given by

$$\tilde{A} := \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} B \\ B_f D \end{bmatrix}, \quad \tilde{C} := \begin{bmatrix} L & -C_f \end{bmatrix}.$$  

With respect to the transfer function $T_M(\zeta)$, it is possible to determine the quantity $\|T_M(\zeta)\|_2$, called the $H_2$ norm of $T_M(\zeta)$, that represents a measure of the energy appearing in the output due to the noisy input. Our aim is to solve the problem

$$\inf_{\mathcal{F}} \sup_{M \in \mathcal{M}} \|T_M(\zeta)\|_2^2.$$  

Since this problem is very hard to solve, many authors proceed by minimizing an available upper bound to the indicated supremum. In [6, 7], this problem is solved using the concept of quadratic stability to be discussed in the next section.

3. Previous results on robust stability and filtering. As stated before, the robust filtering problem (11) is very difficult, and many authors address the filtering problem by replacing the supremum over $\mathcal{M}$ by an appropriate upper bound. One of the most used upper bounds is based on the concept of quadratic stability, which we briefly review in the next paragraphs.

Consider the following linear time-invariant discrete-time system defined by its transfer function

$$T(\zeta) := C(\zeta I - A)^{-1} B,$$

where the triplet $(A, B, C)$ is composed by matrices of compatible and known dimensions. We are interested in the study of the stability of linear time-invariant systems in the form (12), where the matrices $A$ and $B$ are uncertain. More specifically, we are interested in systems whose uncertain parameters, gathered in the matrix

$$M := \begin{bmatrix} A & B \end{bmatrix},$$

belong to the convex polyhedron $\mathcal{M}$ previously defined in (5). Our objective is to characterize whenever the set $\mathcal{M}$ defines only stable systems, in which case we say $\mathcal{M}$ is Schur, and to determine whether, for a given $\mu > 0$, it is true that the upper bound

$$\sup_{M \in \mathcal{M}} \|T_M(\zeta)\|_2^2 < \mu$$

holds.
holds. As in the previous section, the subscript included in the notation of the transfer function defined in (12) indicates the dependence of \( T(\zeta) \) on \( M \in \mathcal{M} \). The next lemma provides an answer to these questions expressed in terms of well-known sufficient conditions for robust (quadratic) stability and performance.

**Lemma 3.1.** The following statements are true:

(a) The set \( \mathcal{M} \) is Schur if there exists a symmetric matrix \( P \) of compatible dimension satisfying the LMI

\[
\begin{bmatrix}
  P & A_i P \\
  (\bullet)' & P \\
\end{bmatrix} > 0
\]

for all \( i = 1, \ldots, N \).

(b) For any given \( \mu > 0 \), the inequality \( \|T_M(\zeta)\|_2^2 < \mu \) holds for all \( M \in \mathcal{M} \) if there exist symmetric matrices \( P \) and \( W \) of compatible dimensions satisfying the LMI

\[
\begin{bmatrix}
  W & CP \\
  (\bullet)' & P \\
\end{bmatrix} > 0,
\begin{bmatrix}
  P & A_i P & B_i \\
  (\bullet)' & P & 0 \\
  (\bullet)' & (\bullet)' & I \\
\end{bmatrix} > 0
\]

for all \( i = 1, \ldots, N \).

*Proof.* See [7]. \( \square \)

The above lemma deserves some comments. First, keeping \( P \) constant and independent of the index \( i \) is essential to obtain the results. This is on the origin of the quadratic stability [2] concept, largely used in robust stability studies of uncertain systems. The main drawback associated with this fact is that a single Lyapunov matrix \( P \) must work for all matrices in the uncertain domain \( \mathcal{M} \), which ensures the stability of all time-variant systems in the domain. This condition is often too conservative if used with time-invariant systems. The same reasoning can be used for the robust performance provided in part (b). Indeed, it is expected that the use of a single matrix \( P \) introduces a significant degree of conservativeness to the estimation of the worst performance attained for some \( M \in \mathcal{M} \). A measure of this gap may be obtained by calculating the minimum value of \( \mu \) given by the optimal solution to the convex programming problem

\[
\mu_q := \min \{ \mu : \text{s.t. (16)} \}
\]

as compared to \( \sup_{M \in \mathcal{M}} \|T_M(\zeta)\|_2^2 \). This fact will be illustrated in the examples.

Using Lemma 3.1, the following set of robust filters with guaranteed quadratic performance has been established in [6, 7] as the discrete-time counterpart of the continuous-time robust filter design introduced in [5].

**Lemma 3.2.** Let \( \mu > 0 \) be given. The estimation error transfer function satisfies the inequality \( \|T_M(\zeta)\|_2^2 < \mu \) for all \( M \in \mathcal{M} \) provided that the robust filter transfer function is given by

\[
T_f(\zeta) := HR^{-1} (\zeta I - QR^{-1})^{-1} F,
\]

where \( R := Z - Y \) and matrices \( Q, H, \) and \( F \) and the symmetric matrices \( Y, Z, \) and
$W$ satisfy the LMI

\begin{equation}
\text{trace}(W) < \mu,
\end{equation}

\begin{equation}
\begin{bmatrix}
W & L - H & L \\
(\star)' & Z & Z \\
(\star)' & (\star)' & Y
\end{bmatrix} > 0,
\end{equation}

\begin{equation}
\begin{bmatrix}
Z & Z & ZA_i \\
(\star)' & Y & YA_i + FC_i + Q \\
(\star)' & (\star)' & Z
\end{bmatrix}
\begin{bmatrix}
ZA_i \\
Z \\
Y
\end{bmatrix}
\begin{bmatrix}
ZB_i \\
0 \\
I
\end{bmatrix} > 0
\end{equation}

for all $i = 1, \ldots, N$.

Proof. See [7].

From this theorem, it is clear that a near optimal solution to the design problem (11) is readily calculated from the optimal solution to the convex programming problem

\begin{equation}
\mu_Q := \min \{ \mu : \text{s.t. (19) - (21)} \},
\end{equation}

which provides the best filter when a quadratic guaranteed upper bound to the worst error performance is adopted. It is worth mentioning that, for $N = 1$, the system under consideration is completely known, and, in this case, (22) generates the celebrated Kalman filter (see [7] for details).

4. Parameter dependent robust stability. This section presents the main results of this paper related to robust stability and performance of uncertain discrete-time systems. The following theorem constitutes an extension of the stability test recently introduced in [3] and will be used as a basis for the development of the new filter design procedure to be introduced in section 5.

Theorem 4.1. The following statements are true:

(a) The set $\mathcal{M}$ is Schur if there exist symmetric matrices $P_i, i = 1, \ldots, N$, and a matrix $G$ of compatible dimensions satisfying the LMI

\begin{equation}
\begin{bmatrix}
P_i \\
(\star)'
\end{bmatrix}
\begin{bmatrix}
A_iG \\
G + G' - P_i
\end{bmatrix} > 0
\end{equation}

for all $i = 1, \ldots, N$.

(b) For any given $\mu > 0$, the inequality $\|T_M(\zeta)\|_2^2 < \mu$ holds for all $M \in \mathcal{M}$ if there exist symmetric matrices $P_i, W_i, i = 1, \ldots, N$, and a matrix $G$ of compatible dimensions satisfying the LMI

\begin{equation}
\text{trace}(W_i) < \mu,
\end{equation}

\begin{equation}
\begin{bmatrix}
W_i & CG \\
(\star)' & G + G' - P_i \\
(\star)' & (\star)'
\end{bmatrix}
\begin{bmatrix}
P_i \\
A_iG \\
B_i
\end{bmatrix}
\end{equation}

for all $i = 1, \ldots, N$.

Proof. We prove part (a) by assuming that (23) holds for all $i = 1, \ldots, N$ and calculating the convex combination of inequality (23). That is, we first multiply each
inequality in (23) by the uncertain parameter \( \xi_i > 0 \) and then evaluate the sum from \( i = 1, \ldots, N \) so as to obtain
\[
\begin{bmatrix}
P(\xi) & A(\xi)G \\
(\bullet)' & G + G' - P(\xi)
\end{bmatrix} > 0.
\]

From this inequality, we first conclude that \( G + G' > P(\xi) > 0 \), where \( P(\xi) := \sum_{i=1}^{N} \xi_i P_i \). Since \( P(\xi) > 0 \), the inequality \( (P(\xi) - G)'(P(\xi) - G)^{-1}(P(\xi) - G) \geq 0 \) is true for all values of the uncertain parameter \( \xi \) so that \( G'P(\xi)^{-1}G \geq G + G' - P(\xi) \). Replacing this in the above inequality, we get
\[
\begin{bmatrix}
P(\xi) & A(\xi)G \\
(\bullet)' & G'P(\xi)^{-1}G
\end{bmatrix} \geq
\begin{bmatrix}
P(\xi) & A(\xi)G \\
(\bullet)' & G + G' - P(\xi)
\end{bmatrix} > 0.
\]

Finally, if we multiply the first inequality above by \( T(\xi) = \text{diag} [I, G^{-1}P(\xi)] \) on the right and by \( T(\xi)' \) on the left, we recover
\[
\begin{bmatrix}
P(\xi) & A(\xi)P(\xi) \\
(\bullet)' & P(\xi)
\end{bmatrix} > 0,
\]
which lets us conclude that the set \( \mathcal{M} \) is Schur.

In order to prove part (b), we manipulate the third inequality in (24) following the same steps as in the proof of part (a) so as to obtain
\[
\begin{bmatrix}
P(\xi) & A(\xi)P(\xi) \\
(\bullet)' & P(\xi)
\end{bmatrix} > 0,
\]
which lets us conclude that
\[
(25) \quad \|T_M(\xi)\|^2_2 \leq \text{trace} (CP(\xi)C') \quad \forall M \in \mathcal{M}.
\]

Then, taking the convex combination of the first and the second inequalities in (24) with respect to the uncertain parameters, we get
\[
\text{trace} (CP(\xi)C') = \text{trace} \left[ CG' (G'P(\xi)^{-1}G)^{-1} G'C' \right]
\leq \text{trace} \left[ CG' (G + G' - P(\xi)^{-1}) G'C' \right]
\leq \text{trace} \left( \sum_{i=1}^{N} \xi_i W_i \right)
\leq \max_{i=1,\ldots,N} \text{trace} (W_i)
< \mu,
\]
which, together with (25), concludes the proof of part (b).

Part (a) of the above theorem first appeared in [3]. Theorem 4.1 represents some important contributions. First, it contains the quadratic stability result as a particular case. Notice that, if we aggregate to the LMI (23) and (24) the additional linear constraints
\[
(26) \quad G = G', \quad P_1 = P, \quad i = 1, \ldots, N,
\]
then we exactly recover Lemma 3.1. Second, it generalizes the concept of quadratic stability and quadratic robust performance of uncertain systems to cope with parameter dependent Lyapunov functions. As indicated in the proof, the stability of the family of matrices $M(\xi) = \sum_{i=1}^{N} \xi_i M_i$ is tested by the parameter dependent Lyapunov function

\begin{equation}
 v(x) = x' \left( \sum_{i=1}^{N} \xi_i P_i \right) x.
\end{equation}

From part (b), it is possible to determine an upper bound to the $H_2$ norm of $T_M(\xi)$ by solving the following convex programming problem:

\begin{equation}
 \mu_p := \min \{ \mu : \text{s.t. (24)} \}.
\end{equation}

Notice that $\mu_p \leq \mu_q$ since the inequality (24) reduces to (16) with the additional constraints (26) and $W_i = W$, $i = 1, \ldots, N$. In practice, the value of $\mu_p$ is much less that $\mu_q$ since the number of free variables in the problem (28) is much bigger than the number of free variables in the quadratic robust performance design problem (17). This behavior will be illustrated by the numerical examples provided in section 7.

The fact that $\mu_p \leq \mu_q$ will guarantee that the robust filters to be designed in the next section always perform better (no worse) than the ones obtained by the filtering design procedures based on the concept of quadratic stability [7]. Unfortunately, it is hard to quantify how much improvement can be obtained. Nevertheless, the introduced analysis conditions can indeed coincide with the actual robust stability analysis in some examples (see [3] and [4]).

5. A new robust filtering procedure. At this point, after the analysis results presented in the last section, we turn to the following question: Is it possible to provide a numerically attractive procedure to synthesize a filter that takes advantage of the new robust stability and performance conditions provided in Theorem 4.1? This is the goal of this section. Applying the result given in Theorem 4.1 to the estimation error transfer function (9), we have that, for a given $\mu > 0$, the robust filter under consideration is such that $\|T_M(\xi)\|^2 < \mu$, provided that the inequalities

\begin{equation}
 \text{trace}(W_i) < \mu, \quad \begin{bmatrix} W_i & \bar{C}_i \bar{G} \\
 \bar{C} + \bar{G}' - \bar{P}_i & 0 \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{P}_i & \bar{A}_i \bar{G} \\
 \bar{G} + \bar{G}' - \bar{P}_i & 0 \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{B}_i \\
 0 \\
 I \end{bmatrix} > 0
\end{equation}

hold for all $i = 1, \ldots, N$. More precisely, we are interested in investigating whether it is possible to convert the nonlinear matrix inequality in terms of the filter parameters in (29) into an LMI. If this goal is accomplished, the robust $H_2$ filter design problem turns out to be a convex programming problem which can be solved by efficient numerical algorithms.

To this end, we proceed by partitioning $\bar{G}$ and its inverse as

\begin{equation}
 \bar{G} := \begin{bmatrix} Z^{-1} & ? \\
 U & ? \end{bmatrix}, \quad \bar{G}^{-1} := \begin{bmatrix} Y & ? \\
 V & ? \end{bmatrix},
\end{equation}

where $Z, U, Y, V,$ and "?" denote matrices in $R^{n \times n}$. Notice that, given the quadruple $(Z, U, Y, V)$, we can always calculate blocks "?" in order to have $\bar{G} \bar{G}^{-1} = I$. Also
notice that no additional constraints like symmetry or definiteness are present. From this partition of matrix $\tilde{G}$, we introduce the one-to-one change of variables

$$
\begin{bmatrix}
A_f & B_f \\
C_f & 0
\end{bmatrix} :=
\begin{bmatrix}
V' & 0 \\
0 & I
\end{bmatrix}^{-1}
\begin{bmatrix}
Q & F \\
H & 0
\end{bmatrix}
\begin{bmatrix}
UZ & 0 \\
0 & I
\end{bmatrix}^{-1},
$$

where the existence of the indicated inverses will be proven in what follows. Denoting $R := V'UZ$, the next theorem gives a solution, expressed in terms of an LMI, to the robust $H_2$ filtering problem previously stated.

**Theorem 5.1.** Let $\mu > 0$ be given. The estimation error transfer function satisfies the inequality $\|T_M(\zeta)\|^2 < \mu$ for all $M \in \mathcal{M}$, provided that the robust filter transfer function is given by

$$
T_f(\zeta) := HR^{-1} (\zeta I - QR^{-1})^{-1} F,
$$

where matrices $Q, H, F, R, Z, Y$ and $W_i = W'_i$, $P_i = P'_i$, $S_i = S'_i$, $J_i, i = 1, \ldots, N$, satisfy the LMI

$$
\text{trace}(W_i) < \mu_i,
$$

$$
\begin{bmatrix}
W_i & L - H \\
\bullet' & Z + Z' - P_i
\end{bmatrix} > 0,
$$

$$
\begin{bmatrix}
P_i & J_i \\
\bullet' & Z' A_i
\end{bmatrix} > 0,
$$

$$
\begin{bmatrix}
Z A_i & \bullet' \\
\bullet' & Z' A_i
\end{bmatrix} > 0, \quad
\begin{bmatrix}
Z'B_i \\
\bullet'
\end{bmatrix} > 0.
$$

Furthermore, (29) holds for some filter if and only if the inequalities (33)–(35) are feasible.

**Proof.** Let us first suppose that (29) is feasible. We can partition matrix $\tilde{G}$ as indicated in (30) and assume that matrices $U$ and $V$ are nonsingular. We can do that because, given singular matrices $U$ and $V$, we can always slightly perturb them, keeping feasibility due to the fact that all inequalities are strict. Furthermore, $\tilde{G} + \tilde{G}' > \bar{P}_i > 0$ ensures that $\tilde{G}^{-1}$ and $Z$ exist, and, consequently, relation (31) defines a one-to-one transformation. So, defining the square and nonsingular matrices

$$
\tilde{T} :=
\begin{bmatrix}
Z & Y \\
0 & V
\end{bmatrix},
$$

$$
\tilde{T}' \bar{P}_i \tilde{T} :=
\begin{bmatrix}
P_i \\
\bullet'
\end{bmatrix} J_i
$$

it can be verified that the second inequality in (29), multiplied on the left by the full rank matrix $T' := \text{diag}[I, \tilde{T}']$ and to the right by $T$, provides the LMI (34). Furthermore, doing the same to the third inequality in (29) with matrix $T := \text{diag}[\tilde{T}, \tilde{T}, I]$, we get the LMI (35), which, together with the first inequality in (29), implies that all inequalities (33)–(35) are feasible. In addition, since $R$ is also a nonsingular matrix, we get

$$
T_f(\zeta) = C_f (\zeta I - A_f)^{-1} B_f
$$

$$
= HR^{-1} (\zeta I - V^{-T} Q Z^{-1} U^{-1})^{-1} V^{-T} F
$$

$$
= HR^{-1} (\zeta I - QR^{-1})^{-1} F.
$$
For the converse, let us suppose that the LMIs (33)–(35) are feasible. First, notice that
\[
\begin{bmatrix}
Z + Z' & Z' + Y + R' \\
(\circ)' & Y + Y'
\end{bmatrix} >
\begin{bmatrix}
P_i & J_i \\
(\circ)' & S_i
\end{bmatrix} > 0,
\]
which, multiplied on the left by \( T = \begin{bmatrix} I & -I \end{bmatrix} \) and on the right by \( T' \), implies that \( R + R' > 0 \) so that \( R \) is a nonsingular matrix. The same inequality implies that \( Z + Z' > 0 \) and, consequently, that \( Z \) is also nonsingular. From the definition \( R = V'UZ \), the regularity of matrices \( U \) and \( V \) holds. Consequently, transformation (31) provides a filter satisfying (29).

It is important to compare the result of Theorem 5.1 with that of Lemma 3.2. The optimal guaranteed \( H_2 \) cost robust filter, provided using a single Lyapunov function, i.e., the quadratic optimal filter, is recovered by imposing on the inequalities (33)–(35) the additional constraints
\[
Z = Z', \quad Y = Y', \quad R = Z - Y,
\]
\[
W_i = W_i \begin{bmatrix} P_i & J_i \\
(\circ)' & S_i
\end{bmatrix} = \begin{bmatrix} Z & Z \\
(\circ)' & Y
\end{bmatrix}, \quad i = 1, \ldots, N,
\]
as a consequence, and, following the same steps as in [7], it is worth noticing that the previous result also contains as a particular case the celebrated Kalman filter when \( N = 1 \). It is important to remark (see the illustrative examples) that the main issue of this paper is to provide a way to relax the constraints (36)–(37). As illustrated in the previous section, this fact enables us to get smaller guaranteed costs when compared with all other available design procedures based on a single and hence parameter independent Lyapunov function.

Finally, a suboptimal robust filter is readily obtained from
\[
\mu_P := \min \{ \mu : \text{s.t. (33) – (35)} \},
\]
which is still an LMI optimal filtering problem. Notice that it is possible to show that \( \mu_P \leq \mu_Q \) holds, where \( \mu_Q \) is given by (22).

6. Decentralized filtering. As in [6, 7], another interesting point of the design procedure provided in this paper concerns the filter structure. In signal and systems estimation, when the overall system is described by a number of units coupled together by means of an interconnection network, it is of interest to know whether it is possible to connect local filters in order to estimate the local state variables [12]. The model is given by (1)–(3), where \( B, C, D, \) and \( L \) are block diagonal matrices. The goal is to determine a filter as (6)–(7) with a state space representation (see (32)), where
\[
C_f = HR^{-1}, \quad A_f = QR^{-1}, \quad B_f = F
\]
are block diagonal matrices of compatible dimensions. If possible, the filter can be split into a set of local filters acting on each subsystem level. Recalling that the inverse of a block diagonal matrix is also a block diagonal matrix and that the product of block diagonal matrices is a block diagonal matrix, (39) reveals that our goal is accomplished, provided that we include in the \( H_2 \) filtering design problem (34)–(35) the following additional constraints: \( Matrices H, R, Q, \) and \( F \) are block diagonal. Fortunately, this corresponds to constrain some entries of those matrices to be equal to zero, and so convexity is preserved. Also notice that, on the contrary to what
happens in [6, 7], the Lyapunov matrices \( P_i, S_i, \) and \( J_i \) must not present a block diagonal structure. Another surprising feature is that none of the submatrices of \( \hat{G} \) and its inverse given in (30) must be block diagonal—a direct consequence of the extra degrees of freedom introduced with the new stability condition.

7. **Illustrative examples.** In this section, we solve the proposed robust filter design problem for several systems in the form (1)–(3). This example is taken from [7], and the results are compared with the ones given by the design procedures in [11] and in [7]. We have a discrete-time system with matrices

\[
B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & \sqrt{2} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 1 \end{bmatrix}
\]

and a nominal matrix \( A = A_0 \) given by

\[
A_0 = \begin{bmatrix} 0.9 & 0.1 \\ 0.01 & 0.9 \end{bmatrix}.
\]

For this nominal system, the Kalman optimal filter \( \mathcal{F}_K \) is associated with the minimum \( H_2 \) cost equal to 8.0759 and is given by the minimal state space realization

\[
A_K = \begin{bmatrix} 0.4427 & 0.1000 \\ -0.1615 & 0.9000 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0.4573 \\ 0.1715 \end{bmatrix}, \quad C_K = \begin{bmatrix} 1 & 1 \end{bmatrix}.
\]

The first robust filter design we propose copes with the structured uncertainty defined through \( A = A_0 + \Delta A \) for

\[
\Delta A = \begin{bmatrix} 0 & 0.06 \alpha \\ 0.05 \beta & 0 \end{bmatrix} = \begin{bmatrix} 0.06 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix},
\]

where \(|\alpha| \leq 1\) and \(|\beta| \leq 1\). This is a two-block structured uncertainty which can be exactly described by the set \( \mathcal{M} \). Although the filter design procedure given in [11] cannot be directly applied to this problem without introducing some conservativeness, we take the best solution it provides without imposing the diagonal structure on the uncertainty parameters for the sake of comparison. This solution is obtained for a parameter \( \varepsilon = 1.5264 e - 04 \) and provides a suboptimal guaranteed cost \( H_2 \) filter \( \mathcal{F}_S \) with minimal state space realization

\[
A_S = \begin{bmatrix} 0.0335 & 0.1014 \\ -0.2551 & 0.9117 \end{bmatrix}, \quad B_S = \begin{bmatrix} 0.8667 \\ 0.2652 \end{bmatrix}, \quad C_S = \begin{bmatrix} 1 & 1 \end{bmatrix}.
\]

Using the result of [7] (Lemma 3.2) with \( N = 4 \) matrices corresponding to the extreme points of the uncertain domain, we get the optimal quadratic guaranteed cost \( H_2 \) filter \( \mathcal{F}_Q \) given by

\[
A_Q = \begin{bmatrix} 0.0826 & -0.0768 \\ -0.0002 & 0.8543 \end{bmatrix}, \quad B_Q = \begin{bmatrix} -0.0413 \\ 0.0001 \end{bmatrix}, \quad C_Q = \begin{bmatrix} -29.8415 & -70.1868 \end{bmatrix}.
\]

Finally, for the same set of vertices, Theorem 5.1 provides the optimal filter \( \mathcal{F}_P \):

\[
A_P = \begin{bmatrix} -0.1312 & 0.0842 \\ -0.0073 & 0.8352 \end{bmatrix}, \quad B_P = \begin{bmatrix} -0.1151 \\ -0.0007 \end{bmatrix}, \quad C_P = \begin{bmatrix} -14.7625 & -41.3592 \end{bmatrix}.
\]

Table 1 shows, for each filter, the value of the \( H_2 \) guaranteed \( H_2 \) cost \( \mu \) as well as the supremum of \( \|T_M(\zeta)\|^2_2 \) with respect to the matrix \( M \in \mathcal{M} \), calculated by brute
Table 1
Filter performance: Multiblock uncertainty.

<table>
<thead>
<tr>
<th>Filter</th>
<th>( \mathcal{F}_K )</th>
<th>( \mathcal{F}_S )</th>
<th>( \mathcal{F}_Q )</th>
<th>( \mathcal{F}_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>129.7915</td>
<td>100.0278</td>
<td>44.0039</td>
<td></td>
</tr>
<tr>
<td>( \sup_{M \in \mathcal{M}} |T_M(\zeta)|_2^2 )</td>
<td>49.4994</td>
<td>38.2183</td>
<td>30.0664</td>
<td>15.4506</td>
</tr>
</tbody>
</table>

Table 2
Filter performance: One-block uncertainty.

<table>
<thead>
<tr>
<th>Filter</th>
<th>( \mathcal{F}_K )</th>
<th>( \mathcal{F}_O )</th>
<th>( \mathcal{F}_Q )</th>
<th>( \mathcal{F}_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 2 )</td>
<td>—</td>
<td>—</td>
<td>9.6796</td>
<td>8.8499</td>
</tr>
<tr>
<td>( N = 4 )</td>
<td>—</td>
<td>—</td>
<td>13.0219</td>
<td>11.5307</td>
</tr>
<tr>
<td>( N = 8 )</td>
<td>—</td>
<td>—</td>
<td>13.0446</td>
<td>11.6053</td>
</tr>
<tr>
<td>( \mu )</td>
<td>13.0446</td>
<td>13.0446</td>
<td>11.6053</td>
<td></td>
</tr>
<tr>
<td>( \sup_{M \in \mathcal{M}} |T_M(\zeta)|_2^2 )</td>
<td>13.0036</td>
<td>11.8655</td>
<td>11.8655</td>
<td>11.5980</td>
</tr>
</tbody>
</table>

force. As in [7], the Kalman filter, which is optimal for the nominal system, is the worst under parametric uncertainty. The filters of [11] and [7] are both suboptimal with respect to the guaranteed cost—the first one because of the structure of the uncertainty and the second one because of the quadratic stability assumption. It is interesting to observe that the filter determined from Theorem 5.1 is approximately 50% better than the best obtained by the existent procedures with respect to guaranteed \( H_2 \) cost as well as with respect to the true worst case value of the \( H_2 \) estimation cost.

As a second design, we consider the same uncertain system given before, but we change the uncertainty description to

\[
\Delta A = \begin{bmatrix}
0 & 0.06\alpha \\
0 & 0.05\beta
\end{bmatrix} = \begin{bmatrix}
0.06 & 0 \\
0 & 0.05
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}\begin{bmatrix}
0 \\
1
\end{bmatrix},
\]

where the uncertain parameters are such that \( \alpha^2 + \beta^2 \leq 1 \). With respect to this one-block unstructured uncertainty, the results of [11] provide the optimal quadratic guaranteed \( H_2 \) cost filter \( \mathcal{F}_O \):

\[
A_O = \begin{bmatrix}
0.3521 & 0.1069 \\
-0.2211 & 0.9400
\end{bmatrix}, \quad B_O = \begin{bmatrix}
0.5479 \\
0.2311
\end{bmatrix}, \quad C_O = \begin{bmatrix}
1 & 1
\end{bmatrix}.
\]

Although this uncertainty domain cannot be exactly represented by the polytopic domain \( \mathcal{M} \), we proceed as in [7] by approximating the ellipsoidal uncertainty domain by the polyhedron associated with the extreme matrices

\[
\begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix} = \begin{bmatrix}
\cos(2\pi i/N) \\
\sin(2\pi i/N)
\end{bmatrix}, \quad i = 1, \ldots, N.
\]

Table 2 shows that, with \( N = 8 \), the quadratic filter \( \mathcal{F}_Q \) given by Lemma 3.2 is associated with the same guaranteed cost as \( \mathcal{F}_O \). Applying Theorem 5.1, it is possible to go even further. Notice that the optimal parameter dependent filter \( \mathcal{F}_P \),

\[
A_P = \begin{bmatrix}
0.4491 & 0.0758 \\
0.0006 & 0.9008
\end{bmatrix}, \quad B_P = \begin{bmatrix}
-0.2360 \\
-0.0013
\end{bmatrix}, \quad C_P = \begin{bmatrix}
-3.2370 & -8.5027
\end{bmatrix},
\]

is associated with a guaranteed cost which virtually matches the actual worst case performance.
8. Conclusion. The robust filtering problem for linear time-invariant discrete-time uncertain systems has been addressed in this paper using parameter dependent Lyapunov functions when convex polytopic uncertainty is present on the dynamic, input, and output matrices. The work is based on a new robust stability condition, which presents a separation between the Lyapunov matrix and the matrices of the dynamic model.

We have shown how to determine optimal $H_2$ guaranteed cost filters by solving a linear problem constrained by an LMI. The results encompass most of the results available in the literature to date which are based on the quadratic stability framework. We have also shown how to extend the results to cope with decentralized filtering without assuming a block diagonal Lyapunov matrix structure. Some numerical examples have been solved, illustrating the superiority of the results for the design of filters for time-invariant uncertain systems.

REFERENCES


