ON THE OBSERVABILITY AND DETECTABILITY OF CONTINUOUS-TIME MARKOV JUMP LINEAR SYSTEMS

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Abstract. The paper introduces a new detectability concept for continuous-time Markov jump linear systems with finite Markov space that generalizes previous concepts found in the literature. The detectability in the weak sense is characterized as mean square detectability of a certain related stochastic system, making both detectability senses directly comparable. The concept can also ensure that the solution of the coupled algebraic Riccati equation associated to the quadratic control problem is unique and stabilizing, making other concepts redundant. The paper also obtains a set of matrices that plays the role of the observability matrix for deterministic linear systems, and it allows geometric and qualitative properties. Tests for weak observability and detectability of a system are provided, the first consisting of a simple rank test, similar to the usual observability test for deterministic linear systems.

Key words. Markov jump systems, detectability and observability of stochastic systems, optimal control, stochastic systems, quadratic control

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1. Introduction. The concepts of observability and detectability play an important role in the theory of dynamic systems. For instance, in optimal control problems, these concepts provide a connection between closed-loop stability and finiteness of the cost functional, and they ensure uniqueness of the solution to the algebraic Riccati equation. This is the scenario in the theory of deterministic time-invariant linear systems (see [14]), deterministic linear time-varying systems (see [1], [2] or [11]), and to some extent, in Markov jump linear systems (MJLS) (see [7], [9], [13], [16], and [17]).

Thanks to those developments, a number of well-established results concerning detectability and the good behavior of solutions of filtering and control problems exist today which can be found in a literature that spans more than four decades. Among the results we refer to concerning linear time-invariant deterministic systems are the following: (I) invariance of nonobserved trajectories, (II) existence of a simple rank-test condition for observability, (III) correspondence between nonobserved trajectories and stable modes of detectable systems, and (IV) relationship between observability and detectability. However, it was not known to this date how properties (I)–(IV) extend to MJLS.

Consider the continuous-time MJLS written as

\[
\Phi: \begin{cases}
\dot{x}(t) = A_{\theta(t)}x(t), & t \geq 0, \\
y(t) = C_{\theta(t)}x(t), & x(0) = x_0, \quad \theta(0) = \theta_0,
\end{cases}
\]

defined in a fundamental probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), where \(\mathcal{F}_t\) denotes the \(\sigma\)-field generated by \(\{x(s), \theta(s), 0 \leq s \leq t\}\). The variables \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^q\) are the
continuous state and the output, respectively; \( x_0 \) is a second order random variable. The mode \( \theta \) is the state of an underlying continuous-time homogeneous Markov chain \( \Theta = \{ \theta(t); t \geq 0 \} \) having \( S = \{ 1, \ldots, N \} \) as state space and \( \Lambda = [ \lambda_{ij} ], i, j = 1, \ldots, N \), as the transition rate matrix. The initial distribution of \( \Theta \) is determined by \( \mu_0 = P(\theta_0 = i), i = 1, \ldots, N \). Matrices \( A_i \) and \( C_i, 1 \leq i \leq N \), belong to the collections of \( N \) real matrices: \( A = (A_1, \ldots, A_N), \dim(A_i) = n \times n \), and \( C = (C_1, \ldots, C_N) \), \( \dim(C_i) = q \times n \). Consider also the functional

\[
W^t(x, \theta) = E \left\{ \int_0^t x(\tau)' C_{\theta(\tau)}' C_{\theta(\tau)} x(\tau) d\tau \right\}
\]

defined for \( x(0) = x \) and \( \theta(0) = \theta \). Here we consider the following concept of observability, which is drawn from the observability concept for time-variant MJLS that appears, for instance, in [16]. The concept is more general than other observability concepts for MJLS, like the ones in [13].

**Definition 1** (W-observability). We say that \((A, C, \Lambda)\) is weakly (W-) observable when there exist scalars \( t_d \geq 0 \) and \( \gamma > 0 \) such that \( W^{t_d}(x, \theta) \geq \gamma |x|^2 \) for each \( x \in \mathbb{R}^n \) and \( \theta \in S \).

In this paper, for the time-invariant system \( \Phi \), we present a collection of matrices \( \mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_N) \) associated to the W-observability concept that resembles observability matrices of deterministic linear systems. Then we provide extensions of properties (I) and (II) mentioned above, respectively: we show that nonobserved trajectories are invariant in the sense that, if \( x(s) \) is in the kernel of \( \mathcal{O}_{\theta(s)} \) for some \( s \geq 0 \), then \( x(t) \) is in the kernel of \( \mathcal{O}_{\theta(t)} \) for any \( t \geq s \) (see Corollary 15), and we show that \((A, C, \Lambda)\) is W-observable if and only if each of the matrices of the set \( \mathcal{O} \) is of full rank. We also demonstrate that the largest attainable dimensionality of \( \mathcal{O} \) is constrained by the system dimensions \( n \) and \( N \) (see Lemma 12) in a similar manner to observability matrices of deterministic systems.

**Definition 2** (MS-stability). We say that \((A, \Lambda)\) is MS-stable if, for system \( \Phi \) and for each \( x_0 \in \mathbb{R}^n \) and \( \theta_0 \in S \),

\[
\lim_{t \to \infty} E\{|x(t)|^2\} = 0.
\]

**Definition 3** (MS-detectability). We say that \((A, C, \Lambda)\) is MS-detectable when there exists \( G = \{ G_1, \ldots, G_N \} \) of appropriate dimension for which \((A - GC, \Lambda)\) is MS-stable.

In connection with the MS-detectability concept, we have that none of the well-known properties (III) and (IV) mentioned above hold. Moreover, W-observability is not comparable to MS-detectability. In Example 2, we present a system that is W-observable but is not MS-detectable. It is also simple to provide a converse example: if one takes \((A, \Lambda)\) as MS-stable and \( C = 0 \), one has that \((A, C, \Lambda)\) is MS-detectable but is not W-observable. This lack of structure sometimes compels authors to consider either a detectability or an observability hypothesis (see, for example, [9] and [16]), where these conditions appear as sufficient conditions for uniqueness of solutions to coupled algebraic Riccati equations (CAREs) arising in the optimal linear quadratic problem.
In this paper, we develop the following associate concept of W-detectability from the W-observability concept. We mention that it is analogous to a concept for time-varying systems that appears in [1].

**Definition 4 (W-detectability).** We say that $(A, C, \Lambda)$ is W-detectable if there exist scalars $t_d, s_d \geq 0$, $\gamma > 0$, and $0 \leq \delta < 1$ such that $W^t_i(x_0, \theta_0) \geq \gamma|x_0|^2$ whenever $E\{|x(s_d)|^2\} \geq \delta|x_0|^2$.

We show that W-detectability generalizes and can retrieve each of the properties (III) and (IV), respectively: for every nonobserved trajectory, a contraction condition holds, ensuring that the trajectory converges in the MS sense (see Lemma 20); W-detectability generalizes W-observability. Moreover, in one of the main results of this paper, we characterize W-detectability by means of MS-detectability as follows: $(A, C, \Lambda)$ is W-detectable if and only if $(A, O, \Lambda)$ is MS-detectable (see Theorem 24). This result allows us to clarify the conservativeness of MS-detectability when compared with W-detectability, and, at same time, it provides a testable condition for W-detectability; see section 4.1.

For the controlled MJLS, we show that the W-detectability concept ensures that finite cost implies stable trajectories in the MS sense and, in particular, that the solution to the CARE arising in optimal control problems is unique and stabilizing; see section 5. This result generalizes previous characterizations in [9], [13], [16], and [17].

The paper is organized as follows. In section 2 basic results and relevant definitions are introduced, and, in section 3, we introduce the observability matrices and related properties. In section 4, some characterizations of W-detectability are presented, and, in section 4.1, it is shown that W-detectability generalizes MS-detectability. In section 5, we set up the link between W-detectability and stabilizing quadratic control.

2. Notation, concepts, and basic results. Let $\mathbb{R}^n$ be the $n$th dimensional Euclidean space. Let $\mathbb{R}^{n,q}$ (respectively, $\mathbb{R}^n$) represent the normed linear space formed by all $n \times q$ (respectively, $n \times n$) real matrices and $\mathbb{R}^{0,n}$ ($\mathbb{R}^{n,n}$) the closed convex cone $\{U \in \mathbb{R}^n : U = U^t \geq 0\}$ (the open cone $\{U \in \mathbb{R}^n : U = U^t > 0\}$), where $U^t$ denotes the transpose of $U$: $U \geq V$ ($U > V$) signifies that $U - V$ is positive semidefinite (definite). For $U \in \mathbb{R}^{n,q}$, $N\{U\}$ and $\mathcal{R}\{U\}$ represent the kernel and the range of $U$, respectively.

Let $\mathcal{M}^{n,q}$ denote the linear space formed by a number $N$ of matrices such that $\mathcal{M}^{n,q} = \{U = (U_1, \ldots, U_N) : U_i \in \mathbb{R}^{n,q}, i = 1, \ldots, N\}$; also, $\mathcal{M}^n \equiv \mathcal{M}^{n,n}$. We denote by $\mathcal{M}^{0,n}$ ($\mathcal{M}^{n,n}$) the set $\mathcal{M}^n$ when it is made up by some $U_i \in \mathcal{R}^{0,n}$ ($U_i \in \mathcal{R}^{n,n}$) for all $i = 1, \ldots, N$. Analogously, for $U, V \in \mathcal{M}^{n,n}$ $U \geq V$ ($U > V$) signifies that $U - V \in \mathcal{M}^{0,n}$ ($U - V \in \mathcal{M}^{n,n}$). It is known that $\mathcal{M}^{n,q}$ equipped with the inner product

$$\langle U, V \rangle = \sum_{j=1}^N \text{tr}\{U_j^tV_j\}$$

forms a Hilbert space. Let us define the norm $||U|| = \langle U, I \rangle$ on $\mathcal{M}^{n,0}$.

Consider system $\Phi$ in (1). For $i = 1, \ldots, N$, we define

$$X_i(t) = E\{x(t)x(t)^t1_{\{\theta(t) = i\}}|\mathcal{F}_0\}, \quad t \geq 0.$$  

(3)

With this notation, we can write, for instance, $E\{|x(t)|^2|\mathcal{F}_0\} = \langle X(t), I \rangle = ||X(t)||$.

Now let us introduce the operators $\mathcal{L} : \mathcal{M}^n \to \mathcal{M}^n$ and their adjoint in the inner
product sense $T: \mathcal{M}^n \rightarrow \mathcal{M}^n$ as

\[(4a)\]
\[L_i(U) = A_i U_i + U_i A_i + \sum_{j=1}^{N} \lambda_{ij} U_j,\]

\[(4b)\]
\[T_i(U) = A_i U_i + U_i A_i + \sum_{j=1}^{N} \lambda_{ji} U_j, \quad i = 1, \ldots, N.\]

Let also $L(t)$ and $U(t), t \geq 0$, be defined by the matrix linear differential equations

\[(5a)\]
\[\dot{L}_i(t) := L_i(L(t)) + C'_i C_i, \quad L(0) = 0, \quad t \geq 0,\]

\[(5b)\]
\[\dot{U}_i(t) := T_i(U(t)), \quad U(0) = U \in \mathcal{M}^{n_0},\]

for each $i = 1, \ldots, N$. The operators $L$ and $T$ are linear, and $L(t)$ and $U(t)$ defined by (5) are unique. The following results are adapted from [6] and [13]; the proof is omitted.

**Proposition 5.** The following assertions hold:

(i) \[\dot{X}_i(t) = T_i(X(t)), \quad t \geq 0, \quad i = 1, \ldots, N,\]

for $X(0) \in \mathcal{M}^{n_0}$, such that $X_i(0) = x_0 x_0' 1_{\theta(0) = i}, i = 1, \ldots, N$;

(ii) \[W^t(x, i) = \int_{0}^{t} \langle X(\tau), C' C \rangle d\tau = \langle X(0), L(t) \rangle.\]

Consider the corresponding generalization of (7)

\[(8)\]
\[W^t(U) = \int_{0}^{t} \langle U(\tau), C' C \rangle d\tau = \langle U, L(t) \rangle,\]

where $L(\cdot)$ and $U(\cdot)$ are given by (5).

**Lemma 6.** $U(\cdot) \in \mathcal{M}^{n_0}$ and $L(s) \geq L(t)$ whenever $s \geq t$.

**Proof.** Notice that, for any $U \in \mathcal{M}^{n_0}$, one can adopt the following representation (cf. Theorem 7.5.2 of [12]):

\[U_i = x_1^i x_1^{i'} + \cdots + x_r^i x_r^{i'},\]

where $x_k^i \in \mathbb{R}^n, k = 1, \ldots, r$ and $r_i = \text{rank}(U_i) \leq n$. In connection, we can define $X_{i}^{j,k}(\cdot)$ as the solution of (6) with $X_{i}^{j,k}(0) = x_k^i x_k^{i'}$; it is clear from the second moment definition in (3) that $X_{i}^{j,k}(\cdot) \in \mathcal{M}^{n_0}$. Also, from the linearity of the operator $T$, we have that

\[U_i(t) = \sum_{j=1}^{N} \sum_{k=1}^{r_i} X_{i}^{j,k}(t), \quad t \geq 0,\]
and \( U(\cdot) \in \mathcal{M}^{n0} \), which proves the first assertion.

From the expression (8) and the first assertion, it is simple to check the result for \( L(\cdot) \). In fact, whenever \( s \geq t \), one has that \( W^s(U) \geq W^t(U) \) for each \( U \in \mathcal{M}^{n0} \), and thus \( \langle U, L(s) - L(t) \rangle \geq 0 \). The assertion follows from the Fejer’s trace theorem; cf. [12].

It is well known that the MS-stability of \( A \) is equivalent to the requirement that \( \text{Re}\{\lambda(T)\} < 0 \); see, for instance, [6]. Then we can rewrite the MS-stability concept as follows.

**Definition 7 (MS-stability).** We say that \((A, \Lambda)\) is MS-stable if
\[
\lim_{t \to \infty} \|X(t)\| = 0 \quad \forall X \in \mathcal{M}^{n0}.
\]

**Remark 1.** Feng et al. in [10] have shown that the MS-stability concept is equivalent to other second moment stability concepts, such as exponential stability. Thus the system is MS-stable if and only if there exist \( 0 < \xi < 1 \) and \( \alpha \geq 1 \) such that \( \|X(t)\| \leq \alpha \xi^t \|X(0)\| \) for every \( X(0) \in \mathcal{M}^{n0} \). It is also known that, if \((A, \Lambda)\) is not MS-stable, then there exists \( X(0) \in \mathcal{M}^{n0} \) such that \( \|X(t)\| \geq \beta \xi^t \|X(0)\| \) for some \( \zeta \geq 1 \) and \( 0 < \beta \leq 1 \).

### 3. W-observability and observability matrices.

In this section, we introduce a collection of observability matrices, and, in one of the main results, we derive a test for observability based on the rank of these matrices, in a parallel with the observability test for deterministic linear systems. We also derive a counterpart for MJLS for the well-known result for linear deterministic systems that nonobserved trajectories are invariant. An illustrative example is also provided.

Let us introduce the matrices \( \mathcal{O}_i \in \mathbb{R}^{n(n^2N,n)} \), defined for each \( i = 1, \ldots, N \), as
\[
\mathcal{O}_i := [O_i(0) \, O_i(1) \, \cdots \, O_i(n^2N - 1)]',
\]
where each matrix \( O_i(\cdot) \) belongs to the sequence of matrices on \( \mathcal{M}^{n0} \) defined as
\[
O_i(k) := L_i(O(k - 1)), \quad k > 0,
\]
with \( O_i(0) := C_i'C_i \), for each \( i = 1, \ldots, N \). Notice by inspection of (5a) that
\[
O_i(k) = \frac{d^{k+1}L}{dt^{k+1}}(0).
\]

The collection of matrices \( \mathcal{O} \in \mathcal{M}^{n0} \) is called the set of observability matrices of system \( \Phi \). In fact, \( \mathcal{O} \) resembles the observability matrices of linear deterministic systems in many aspects, as we shall see in this section. We can mention in passing that, for an isolated Markov state \( i \), namely, \( \lambda_{ji} = 0 \), \( j = 1, \ldots, N \), a direct equivalence is retrieved: the pair \((A_i, C_i)\) is observable in the deterministic sense if and only if \( \mathcal{O}_i \) is a full rank matrix.

Next we present some preliminary results.

For \( V \in \mathbb{R}^n \), let us identify the columns of \( V = [v_1; v_2; \cdots; v_n] \). For \( U = (U_1, \ldots, U_N) \) and following [5], we introduce the linear and invertible operator \( \tilde{\phi} : \mathcal{M}^{n0} \to \mathbb{R}^{n^2N} \) as
\[
\tilde{\phi}(U) = \begin{bmatrix} \varphi(U_1) \\ \vdots \\ \varphi(U_N) \end{bmatrix}, \quad \text{where} \quad \varphi(V) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.
\]
Let $V \otimes Z$ represent the Kronecker tensor product of matrices $V$ and $Z$. From (4a), using basic properties of the Kronecker product [3], we obtain
\[
\varphi(L(U)) = (I_n \otimes A'_i)\varphi(U_i) + (A'_i \otimes I_n)\varphi(U_i) + \sum_{j=1}^{N} \lambda_{ij}\varphi(U_j),
\]
and one can check that
\[
\varphi(L(U)) = \mathcal{A}\hat{\varphi}(U),
\]
where $\mathcal{A} \in \mathbb{R}^{n^2N}$ is the matrix defined by
\[
\begin{bmatrix}
\hat{A}_1 + \lambda_{11}I_{n^2} & \lambda_{12}I_{n^2} & \cdots & \lambda_{1N}I_{n^2} \\
\lambda_{21}I_{n^2} & \hat{A}_2 + \lambda_{22}I_{n^2} & \cdots & \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N1}I_{n^2} & \cdots & \cdots & \hat{A}_N + \lambda_{NN}I_{n^2}
\end{bmatrix}
\]
and $\hat{A}_i = (I_n \otimes A'_i + A'_i \otimes I_n)$. Applying the operator $\hat{\varphi}$ in (5a) and employing (12), we obtain
\[
\hat{\ell}(t) = \hat{\varphi}[C'C + L(L(t))] = q + \mathcal{A}\hat{\varphi}(L(t)) = q + \mathcal{A}\ell(t), \quad t \geq 0,
\]
where $\ell(t) \in \mathbb{R}^{n^2N}$ and $q \in \mathbb{R}^{n^2N}$ are defined by
\[
\ell(t) = \hat{\varphi}(L(t)), \quad q = \hat{\varphi}(C'C).
\]
Notice by inspection of (13) that
\[
\frac{d^k\ell(0)}{dt^k} = \mathcal{A}^k q.
\]
We also introduce the following representation for the expression $\langle U, L(t) \rangle$:
\[
\langle U, L(t) \rangle = \hat{\varphi}(U)'\ell(t).
\]

**Lemma 8.** Consider $x \in \mathbb{R}^n$ and $i \in S$; define $X \in \mathcal{M}^{n0}$ as $X_i = xx'$ and $X_j = 0$ for all $j \neq i$. Set $w \in \mathbb{R}^{n^2N}$ as $w = \hat{\varphi}(X)$. The following assertions are equivalent:
(i) $x' L_i(x) x = 0$ or, equivalently, $w' \ell(s) = 0$ for some $s > 0$;
(ii) $w' d^m \ell / dt^m(0) = 0$ for $m = 1, \ldots, n^2N$;
(iii) $w' A^{m-1} q = 0$ for $m = 1, \ldots, n^2N$;
(iv) $x \in \mathcal{N}(L_i(t))$ or, equivalently, $w' \ell(t) = 0$ for all $t \geq 0$;
(v) $x \in \mathcal{N}(O_i)$.

**Proof.** (i) $\Rightarrow$ (ii): From Lemma 6, $L(t) \leq L(s)$ for $t \leq s$; from (15), we evaluate
\[
w' \ell(t) = \langle \hat{\varphi}^{-1}(w), L(t) \rangle \leq \langle \hat{\varphi}^{-1}(w), L(s) \rangle = w' \ell(s) = 0, \quad t \leq s.
\]
In addition, noticing that $w' \ell(t) \geq 0$ and recalling that $L(0) = 0$, we can write $w' \ell(t) = 0$ for all $0 \leq t \leq s$, which leads to
\[
w' \frac{d^m \ell}{dt^m}(0) = 0 \quad \forall m \geq 0.
\]
(ii) ⇒ (iii): The result follows immediately from (14).

(iii) ⇒ (iv): For the linear deterministic system in (13), we can write for any \( t \geq 0 \)

\[
\ell(t) = \int_0^t e^{A(t-\tau)} \, q \, d\tau = \int_0^t \sum_{m=1}^{n^2 N} \alpha_m(\tau) A^{m-1} \, q \, d\tau
\]

\[
= \sum_{m=1}^{n^2 N} A^{m-1} q \int_0^t \alpha_m(\tau) \, d\tau = \sum_{m=1}^{n^2 N} \hat{\alpha}_m(t) A^{m-1} q,
\]

where \( \alpha_m \) and \( \hat{\alpha}_m \) are scalar functions. Then we get that

\[
w'(\ell(t)) = \sum_{m=1}^{n^2 N} \hat{\alpha}_m(t) w' A^{m-1} q = 0.
\]

(iv) ⇒ (i): This part of the proof is trivial.

(ii) ⇔ (v): Employing (15), we write

\[
w' \frac{d^m}{dt^m}(0) = 0 \Leftrightarrow \left\langle X, \frac{d^m L}{dt^m}(0) \right\rangle = 0 \Leftrightarrow \frac{d^m L}{dt^m}(0) x = 0
\]

for \( m = 1, \ldots, n^2 N \). The proof is easily completed by noticing from (11) that

\[
O_i = \begin{bmatrix} O_i(0) \\ \vdots \\ O_i(n^2 N - 1) \end{bmatrix} = \begin{bmatrix} \frac{d^1 L_i}{dt}(0) \\ \vdots \\ \frac{d^{n^2 N} L_i}{dt}(0) \end{bmatrix}.
\]

The next corollary restates some assertions in Lemma 8 for further use.

**Corollary 9.** The following assertions are equivalent:

(i) \( x \in \mathcal{N}(O_i) \);

(ii) \( W^s(x,i) = 0 \) for some \( s > 0 \);

(iii) \( W^t(x,i) = 0 \) for all \( t \geq 0 \).

**Remark 2.** Notice from Corollary 9 that if the conditions in Definitions 1 or 4 hold for some \( t_d \geq 0 \), then they hold for all \( t \geq 0 \).

The next theorem provides a rank test on the set of observability matrices \( O \).

**Definition 10 (W-observability).** We say that \( (A,C,\Lambda) \) is W-observable when there exist scalars \( t_d \geq 0 \) and \( \gamma > 0 \) such that \( W^{t_d}(X) \geq \gamma \|X\| \) for each initial condition \( X \).

**Theorem 11.** Consider system \( \Phi \). \( (A,C,\Lambda) \) is W-observable if and only if \( O_i \) has full rank for each \( i = 1, \ldots, N \).

**Proof.** From (8), we can write the condition in Definition 1 equivalently as

\[
\langle X, L(t_d) \rangle \geq \gamma \|X\| \quad \forall X \in \mathcal{M}^{n^2}.
\]

This is equivalent to requiring that \( L_i(t_d) \) be positive definite for each \( i = 1, \ldots, N \).

The equivalencies (i) and (v) of Lemma 8 complete the proof.

**Example 1.** Let \( N = 2 \), \( n = 2 \), and set

\[
A_1 = I_2; \ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \ C_1 = [1 \ 0]; \ C_2 = 0; \ \Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.
\]
From (9), one evaluates rank(\(O_1\)) = rank(\(O_2\)) = 2, and Theorem 11 ensures that \((A,C,\Lambda)\) is W-observable.

Remark 3. It is known that \((A,C,\Lambda)\) is W-observable if each pair \((A_i,C_i)\), \(i = 1, \ldots, N\), is observable; see, e.g., [16]. However, this condition is not necessary; for instance, in Example 1, none of the pairs \((A_i,C_i)\) are observable.

3.1. Properties of the observability matrices and pathwise invariance of nonobserved trajectories. The next lemma establishes a counterpart for the well-known result about the largest attainable dimensionality of observability matrices.

**Lemma 12.**

\[ N\{O_i\} = N\{[O_i(0); \ldots; O_i(k)]^T\} \quad \forall k \geq n^2N - 1. \]

**Proof.** For \(x \in N(O_j)\), let \(X_t = x^\prime\) and \(X_j = 0\) for all \(j \neq i\), and let \(w = \hat{\varphi}(X)\). From Lemma 8 (iii) and (v), we have that \(O_jx = 0\) is equivalent to \(w^\prime A^{r-1}q = 0\), \(r = 1, \ldots, n^2N\). From the Cayley–Hamilton lemma, \(A^m = \sum_{r=0}^{n^2N-1} \alpha_r A^r\) for each \(m \geq 0\), and we obtain

\[
\begin{align*}
\begin{cases}
w'q &= 0, \\
w'Aq &= 0, \\
& \quad \vdots \\
w'A^m q &= 0,
\end{cases}
\end{align*}
\]

which, from (14), is equivalent to \(w'd^m\ell(0)/dt^m = 0\), \(m \geq 0\). Finally, applying (16) for a generic \(m \geq 0\), we obtain \(d^mL_i(0)/dt^m x = 0\), and from (11) we write \(O_i(m)x = d^{m+1}L_i(0)/dt^{m+1} x = 0\) for \(m \geq 0\) and, in particular, for \(m \geq n^2N - 1\). Thus \(N\{O_i\} \subset N\{[O_i(0); \ldots; O_i(k)]^T\}\) for all \(k \geq n^2N - 1\); the opposite relation holds trivially. \(\Box\)

Next we present a relation between the null spaces of the observability matrices which will be useful in what follows. The following preliminary result is needed.

**Proposition 13.** For each scalar \(M > 0\), there exists \(t_M > 0\) for which \(|x(t) - x_0| \leq M|x_0|\) almost surely (a.s.) for all \(t \leq t_M\).

**Lemma 14.** Assume that the Markov state \(j\) is accessible from the state \(i\). Then \(N\{O_i\} \subset N\{O_j\}\).

**Proof.** Let us deny the assertion of the lemma; that is, we assume that there exist a scalar \(m > 0\) and \(x_0 \in \mathbb{R}^n\) such that

\[(18) \quad x_0 \in N\{O_i\}\]

for which \(|x_0 - x| \geq m\), for all \(x \in N\{O_j\}\). Notice that \(x_0 \neq 0\), and let \(x_0\) and \(\theta_0 = i\) be initial conditions.

We start the proof by setting \(M = m/(2|x_0|)\) in Proposition 13 to obtain that there exists \(t_M\) for which \(x(t) \in B_{m/2}(x_0), t \leq t_M\), where \(B_{m/2}(x_0) = \{x : |x - x_0| \leq m/2\}\). Let \(\hat{x}(t_M)\) be the orthogonal projection of \(x(t_M)\) on \(N\{O_j\}\) and \(R\{O_j\}\), respectively. Notice that \(\hat{x}(t_M) \perp \ddot{x}(t_M)\) and \(|\ddot{x}(t_M)| \geq m/2\); see Figure 1.

From Lemma 8 (i) and (v), one has that \(N\{O_j\} = N\{L_j(s)\}\) for \(s \geq 0\), and thus \(R\{O_j\} = R\{L_j(s)\}\). In this situation, we can write

\[x(t_M)'L_j(s)x(t_M) = \hat{x}(t_M)'L_j(s)\hat{x}(t_M) \geq \mu|\hat{x}(t_M)|^2 \geq \mu(m/2)^2,
\]
where $\mu$ is the smallest strictly positive eigenvalue of $L_j(s)$, and Proposition 5 leads to

$$W^*(x(t_M), j) \geq \frac{\mu m^2}{4} \text{ a.s.}$$

Now we evaluate

$$E\{W^*(x(t_M), \theta(t_M)) | F_0\} \geq E\{1_{\{\theta(t_M) = j\}} | F_0\} \geq \frac{\mu m^2}{4} E\{1_{\{\theta(t_M) = j\}} | F_0\} > 0,$$

where the last inequality comes from the assumption of the lemma. Finally, we can write that

$$W^{*+t_M}(x_0, \theta_0) = E\{W^*(x(t_M), \theta(t_M)) | F_0\} \geq E\{1_{\{\theta(t_M) = j\}} | F_0\} > 0,$$

and, from Corollary 9 (i) and (iii), it follows that $x_0 \not\in \mathcal{N}(O_{\theta_0})$, which is a contradiction in view of (18).

The next corollary establishes that nonobserved trajectories are pathwise invariant.

**Corollary 15.** If $x(t) \in \mathcal{N}(O_{\theta(t)})$, then $x(s) \in \mathcal{N}(O_{\theta(s)})$ a.s. for all $s \geq t$.

**4. W-detectability.** Let us start this section by rewriting the concept of W-detectability in terms of the notation introduced in section 2.

**Definition 16 (W-detectability).** We say that $(A, C, \Lambda)$ is W-detectable if there exist scalars $t^d, s^d \geq 0$, $\gamma > 0$, and $0 \leq \delta < 1$ such that $W^{t^d}(X) \geq \gamma \|X\|$ whenever $\|X(s^d)\| \geq \delta \|X\|$, with $X(0) = X$.

Notice that W-detectability requires positivity of $W^{t^d}()$ only when the condition $\|X(s^d)\| \geq \delta \|X\|$, related to stability of the system, is satisfied. The next result is immediate; the proof is omitted.

**Lemma 17.** If $(A, C, \Lambda)$ is W-observable, then $(A, C, \Lambda)$ is W-detectable.
The concept of W-detectability resembles standard concepts of detectability for linear discrete time-varying systems; see, e.g., [1] or [11]. As we shall see in Lemma 19, the concept retrieves the idea that every nonobserved state corresponds to stable modes of the system. Notice that every MS-stable MJLS is W-detectable with \( t_d \) and \( \gamma \) arbitrary and \( \delta \) and \( s_d \) such that \( \delta = \alpha \xi ^{s_d} < 1 \), where \( \alpha \) and \( \xi \) are as in Remark 1.

In what follows, basic properties of W-detectability are derived. We start with some properties of the functional \( W^t(X) \). In this section, the initial condition \( X(0) \in \mathcal{M}^{n_0} \) is denoted by \( X \); \( W^t(X(s)) = W^t(X) \) whenever \( X(s) = X \).

**Lemma 18.** Let \( T > 0 \). The following assertions hold:

(i) \( W^t(X) \) is continuous on \( X \).

(ii) Assume that \( W^T(X) = 0 \) for some \( T \geq 0 \); then \( W^t(X(s)) = 0 \) for all \( t, s \geq 0 \).

Proof. (i) The assertion follows immediately from the representation in (8), \( W^t(X) = \langle X, L(t) \rangle \), and the continuity of the inner product.

(ii) Since \( W^T(X) = 0 \) for some \( T > 0 \), from Corollary 9 (ii) and (iii), we conclude that \( W^{s+t}(X) = 0 \). Now let us define \( U(0) = U = X(s) \); since \( U(t) \) is defined by \( U_i(t) = T_i(U(t)) \) and, from Proposition 5 (i), it holds that \( X_i(t) = T_i(X(t)) \), we have that \( U(\tau) = X(\tau + s) \) for \( 0 \leq \tau \leq t \). Then, from the definition of \( W \) in (8), we can write that

\[
W^t(X(s)) = \int_0^t (U(\tau), C'C) d\tau \\
= \int_s^{s+t} \langle X(\tau), C'C \rangle d\tau \\
\leq \int_0^{s+t} \langle X(\tau), C'C \rangle d\tau = W^{s+t}(X) = 0. \quad \square
\]

The result in the next lemma parallels the known result in deterministic linear systems theory that every nonobserved trajectory corresponds to stable modes of the system. The proof is a counterpart of the discrete-time case presented in [4, Lemma 8].

**Lemma 19.** Consider system \( \Phi \), and let \( T > 0 \). \( (A, C, \Lambda) \) is W-detectable if and only if \( \|X(t)\| \to 0 \) as \( t \to \infty \) whenever \( W^T(X) = 0 \).

Proof. Sufficiency. Let us consider the set

\[
Z = \{ Z : \|Z\| = 1, W^T(Z) = 0 \},
\]

and let us denote as \( Z(t) \) the trajectory corresponding to an initial condition \( Z \in Z \). By hypothesis, \( \|Z(t)\| \to 0 \) as \( t \to \infty \), and we can write as in Remark 1 that there exist \( 0 < \xi < 1 \) and \( \alpha \geq 1 \) such that \( \|Z(t)\| \leq \alpha \xi ^t \). Consequently, there exist \( s_d \geq 0 \) and \( 0 \leq \delta < 1 \) such that \( \|Z(s_d)\| < \delta \) for all \( Z \in Z \), and we can write

\[
Z \subset \mathcal{C} = \{ Z : \|Z\| = 1, \|Z(s_d)\| < \delta \}.
\]

In this proof, we shall demonstrate that there exists \( \gamma > 0 \) such that, whenever \( \|X(s_d)\| \geq \delta \), then \( W^T(X) \geq \gamma \|X\| \), and consequently \( (A, C, \Lambda) \) is W-detectable. Let us deny the assertion and suppose that, for each \( \gamma > 0 \), there exists \( X \), \( \|X\| = 1 \) such that \( W^T(X) < \gamma \) and \( \|X(s_d)\| \geq \delta \), i.e., \( X \in \mathcal{C} \), where

\[
\mathcal{C} = \{ X : \|X\| = 1, \|X(s_d)\| \geq \delta \}.
\]

Notice that, since \( X(s_d) \) is solution of the differential equation (6), \( X(s_d) \) is continuous on the initial condition \( X \), and hence the set \( \mathcal{C} \) is a compact set. Then we can take a
sequence $X_n \in \mathcal{C}$ with $\gamma_n \to 0$ as $n \to \infty$ in such a manner that, from the compactness of $\mathcal{C}$, there exists a subsequence $X_{m}$, which converges to some $\hat{X} \in \mathcal{C}$, and, from the continuity of $W^T$ (see Lemma 18),

$$\lim_{m \to \infty} W^T(X_m) = W^T(\hat{X}) = 0.$$  

In view of (19), $\hat{X} \in \mathbb{Z} \subset \mathcal{C}$, which completes the proof by contradiction.

**Necessity.** We shall show that, under W-detectability of $(A,C,\Lambda)$, $\|X(t)\| \to 0$ as $t \to \infty$ when $W^T(X) = 0$. Since $W^T(X) = 0$, from Lemma 18, we have that $W^T(X(t)) = 0$ for all $t \geq 0$. Then, in view of the W-detectability of $(A,C,\Lambda)$, we have that $\|X(t + s_d)\| < \delta \|X(t)\|$ for all $t \geq 0$ and some $s_d \geq 0$ and $0 \leq \delta < 1$; consequently, $\|X(t + ns_d)\| < \delta^n \|X(t)\|$, and hence

$$\lim_{n \to \infty} \sup_{0 \leq t \leq s_d - 1} \|X(t + ns_d)\| \leq \lim_{n \to \infty} \delta^n \sup_{0 \leq t \leq s_d - 1} \|X(t)\| = 0,$$

and the result follows in a straightforward manner. \qed

The next lemma presents a second version of the previous result, coined here in terms of the set of observability matrices $\mathcal{O}$.

**Lemma 20.** $(A,C,\Lambda)$ is W-detectable if and only if $\lim_{t \to \infty} E\{|x(t)|^2\} = 0$ whenever $x_0 \in \mathcal{N}(\mathcal{O}_{\theta_0})$.

**Proof.** **Necessity.** We show that $\lim_{t \to \infty} \|X(t)\| = 0$ whenever $x_0 \in \mathcal{N}(\mathcal{O}_{\theta_0})$, provided $(A,C,\Lambda)$ is W-detectable. For the initial condition $x_0, \theta_0$, we have that $X_j(0) = 0$, $j \neq \theta_0$, and $X_{\theta_0}(0) = x_0 x_0'$, and since $x_0 \in \mathcal{N}(\mathcal{O}_{\theta_0})$, Corollary 9 yields $W^T(X(0)) = 0$; Lemma 19 completes the proof.

**Sufficiency.** Let us assume that $W^T(X) = 0$. Any such $X \in \mathcal{M}^{n_0}$ can be written in the following form (see Theorem 7.5.2 of [12]):

$$X_i = x_i^{1} x_i^{1'} + \cdots + x_i^{r_i} x_i^{r_i'},$$

where $x_i^{k} \in \mathbb{R}^n$, $k = 1, \ldots, r_i$, and $r_i = \text{rank}(X_i) \leq n$. From (8), we have that $\langle X, L(T) \rangle = W^T(X) = 0$, and we can write that $W^T(x_i^{k}, i) \leq \langle X, L(T) \rangle = 0$ for any $i$ and $k$. Thus (i) and (ii) of Corollary 9 provide

$$x_i^{k} \in \mathcal{N}(\mathcal{O}_i), \quad i = 1, \ldots, N, \quad k = 1, \ldots, r_i.$$  

Now let $x_i^{k}(0) = x_i^{k} \in \mathcal{N}(\mathcal{O}_i)$. Let $v_i^{k}(t) \in \mathbb{R}^n$, $k = 1, \ldots, r_i$, be given by the differential equation $\dot{v}_i^{k}(t) = A_{\theta(t)} v_i^{k}(t)$, $\theta(0) = i$. Since $x_i^{k}(0) \in \mathcal{N}(\mathcal{O}_{\theta(0)})$, from the assumption of the lemma, we have that

$$\lim_{t \to \infty} E\{|v_i^{k}(t)|^2\} = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, r_i.$$  

Let $X_i^{j}(t) \in \mathcal{M}^{n_0}$ be the second moment matrix $X_i^{j}(t) = E\{v_i^{k}(t)v_i^{k}(t)1_{\{\theta(t) = j\}}\}$, $j = 1, \ldots, N$. Notice that $X_i^{j}(0) = x_i^{k} x_i^{k'}$ and $X_i^{j}(0) = 0$ for $j \neq i$; in view of (20), we can write $X_i = \sum_{j=1}^{N} \sum_{k=1}^{r_i} X_i^{j,k}$. Then, from (6) and the linearity of the operator $T$, we have that

$$X_i(t) = \sum_{j=1}^{N} \sum_{k=1}^{r_i} X_i^{j,k}(t),$$
and from (21) we evaluate
\[ \lim_{t \to \infty} \|X(t)\| \leq \sum_{j=1}^{N} \sum_{k=1}^{r_j} \lim_{t \to \infty} \|X^{j,k}(t)\| = \sum_{j=1}^{N} \sum_{k=1}^{r_j} \lim_{t \to \infty} E[\|v^{j,k}(t)\|^2] = 0. \]

We have shown, under the assumption of the lemma, that \( \|X(t)\| \to 0 \) as \( t \to \infty \) for each \( X \in M^{n_0} \) such that \( W^T(X) = 0 \); Lemma 19 provides that \((A,C,\Lambda)\) is W-detectable.

**Corollary 21.** If the triplet \((A,C,\Lambda)\) is not W-detectable, then there exist \( i \in S \) and \( x_0 \in N(\mathcal{O}_i) \) such that \( \lim_{t \to \infty} E[\|x(t)\|^2] \neq 0 \), for the initial condition \( x(0) = x_0 \) and \( \theta(0) = 0 \).

4.1. W-detectability and MS-detectability. This section deals with the relation between the concepts of MS-detectability and W-detectability. In the main result of this section, we show that W-detectability of \((A,C,\Lambda)\) is equivalent to MS-detectability of \((A,\mathcal{O},\Lambda)\). From the main result, we also derive a computational test for W-observability.

We start by dealing with the following closed-loop version of the MJLS:

\[ \Phi_o : \dot{x}(t) = (A_{\theta(t)} + G_{\theta(t)} \mathcal{O}_{\theta(t)})x(t), \quad x(0) = x_0, \ \theta(0) = \theta_0. \]

For each \( i = 1, \ldots, N \), we set
\[ G_i = (-A_i - I)\mathcal{O}_i^+, \]
where \( \mathcal{O}_i^+ \) denotes the pseudoinverse of \( \mathcal{O}_i \).

Let us present some properties of system \( \Phi_o \) with \( G \) given in (23). First, one has that \( \mathcal{O}_i^+ \mathcal{O}_i x \) is the orthogonal projection of \( x \) onto \( \mathcal{R}\{\mathcal{O}_i^+\} \) and \( I - \mathcal{O}_i^+ \mathcal{O}_i \) is the projection onto \( \mathcal{N}\{\mathcal{O}_i\} \). Notice that we can write \( x(t) = \hat{x}(t) + \tilde{x}(t) \), where \( \hat{x}(t) = \mathcal{O}_{\theta(t)}^+ \mathcal{O}_{\theta(t)} x(t) \) and \( \tilde{x}(t) = (I - \mathcal{O}_{\theta(t)}^+ \mathcal{O}_{\theta(t)}) x(t) \), and one can easily check that
\[ \tilde{x}(t) \perp \hat{x}(t). \]

In what follows, we study each component \( \hat{x} \) and \( \tilde{x} \) separately. For ease of notation, we denote \( \mathcal{O}_{\theta(\cdot)} = \lim_{s \to t} \mathcal{O}_{\theta(s)} \) and similarly for \( \hat{x}(\cdot) \) and \( \tilde{x}(\cdot) \). Let us define the sequence of jump times \( t_1, t_2, \ldots, \) as
\[ \left\{ \begin{array}{l}
t_0 = 0, \\
t_{m+1} = \inf\{t > t_m : \mathcal{N}\{\mathcal{O}_{\theta(\cdot)}\} \neq \mathcal{N}\{\mathcal{O}_{\theta(t)}\}\}, \quad m \geq 0.
\end{array} \right. \]

**Lemma 22.** Consider system \( \Phi_o \) with \( G \) given in (23). Then \( |\hat{x}(t)| \leq e^{-t} |\hat{x}(0)| \) a.s.

**Proof.** From (23) and (22), it is a simple matter to check that, for \( t_{m-1} \leq t < t_m \), \( \hat{x}(t) = -\hat{x}(t) \) with a given condition \( \hat{x}(t_{m-1}) = \hat{x}(t_{m}) \) due to the strong Markov property of MJLS [8] and the linearity of \( \Phi_o \); this means that
\[ \hat{x}(t) = e^{-(t-t_{m-1})} \hat{x}(t_{m-1}), \quad t_{m-1} \leq t < t_m \text{ a.s.} \]

Regarding the sequence of jump times, from (25) and Lemma 14, we have that \( \mathcal{N}\{\theta(t_m)\} \subset \mathcal{N}\{\theta(t_m)\} \) holds strictly, which yields that the projection of \( \hat{x}(t_m) \) onto \( \mathcal{R}\{\mathcal{O}_{\theta(t_m)}\} \) is null and \( \hat{x}(t_m) \) is simply the result of the orthogonal projection of \( \hat{x}(t_m) \).
onto $R\{O'_{\theta(t_m)}\}$. Then the value of the Euclidean norm of $\dot{x}(\cdot)$ decreases at the sequence of jump times,

$$
|\dot{x}(t_m)| \leq |\dot{x}(t_m^n)| \text{ a.s.}
$$

The result follows from (26) and (27). $\square$

**Lemma 23.** Consider system $\Phi$, with $G$ given in (23), and assume that $(A, C, \Lambda)$ is $W$-detectable. Then $E\{|\dot{x}(t)|^2\} \to 0$ as $t \to \infty$.

**Proof.** In this proof, $P_{t_m} = (I - O^+_{\theta(t_m)}O_{\theta(t_m)})$ stands for the orthogonal projection onto $N\{O_{\theta(t_m)}\}$. We start showing inductively that $E\{|\dot{x}(t)|^2\} < \infty$. For $m = 0$, the result is immediate since $E\{|\dot{x}(0)|^2\} \leq |x_0|^2$. Now we assume that $E\{|\dot{x}(t_m-1)|^2\} < \infty$. At time instant $t_m$, the orthogonal projection of $\dot{x}(t_m)$ onto $N\{O_{\theta(t_m)}\}$ is added to $\dot{x}$; that is,

$$
\dot{x}(t_m) = \dot{x}(t_m) + P_{t_m}\dot{x}(t_m).
$$

Notice that $P_{t_m-1}P_{t_m} = P_{t_m}$ since $N\{O_{\theta(t_m-1)}\} \subset N\{O_{\theta(t_m)}\}$, and we write $P_{t_m-1}P_{t_m}\dot{x}(t_m) = P_{t_m}\dot{x}(t_m) = 0$; notice that $P_{t_m}\dot{x}(t_m) \in R\{O'_{\theta(t_m)}\}$ or, equivalently,

$$
\dot{x}(t_m) \perp P_{t_m}\dot{x}(t_m).
$$

On the other hand, from (22) and (23), it is easy to check that, for $t_m-1 \leq t < t_m$, $\dot{x}(t) = A_{\theta(t)}\dot{x}(t)$ with given condition $\dot{x}(t_m-1)$ due to the strong Markov property of MJLS [8] and the linearity of $\Phi$. Recalling that $\dot{x}(t) \in N\{O_{\theta(t)}\}$ for $t_m \leq t < t_{m+1}$, Lemma 20, Remark 1, and the strong Markov property yield

$$
E\{|\dot{x}(t)|^2\}1_{\{t_m \leq t < t_{m+1}\}} \leq \alpha E\{\xi^{t-t_m}|\dot{x}(t_m)|^2\}1_{\{t_m \leq t < t_{m+1}\}},
$$

where $0 < \xi < 1$ and $\alpha \geq 1$. Then, from (28), by employing (29) and (30) and from Lemma 22, we evaluate

$$
E\{|\dot{x}(t)|^2\} = E\{|\dot{x}(t_m) + P_{t_m}\dot{x}(t_m)|^2\} = E\{|\dot{x}(t_m)|^2\} + E\{|P_{t_m}\dot{x}(t_m)|^2\}
\leq \alpha E\{\xi^{t-t_m}|\dot{x}(t_m)|^2\} + E\{|\dot{x}(t_m)|^2\}
< \alpha E\{|\dot{x}(t_m)|^2\} + E\{|\dot{x}(t_m)|^2\} < \infty,
$$

and the induction is complete. From (30) and (31), we can find $o(t) > 0$ for which

$$
E\{|\dot{x}(t)|^2\}1_{\{t_m \leq t < t_{m+1}\}} \leq \alpha E\{\xi^{t-t_m}|\dot{x}(t_m)|^2\}1_{\{t_m \leq t < t_{m+1}\}} \leq o(t)
$$

holds for each interval $t_m-1 < t < t_m$, where $o(t) \to 0$ as $t \to \infty$. Then we can write

$$
E\{|\dot{x}(t)|^2\} = E\{|\dot{x}(t)|^2\}1_{\{t_0 \leq t < t_1\}} + E\{|\dot{x}(t)|^2\}1_{\{t_1 \leq t < t_2\}} + \cdots
\leq \alpha E\{\xi^{t-t_0}|\dot{x}(t_0)|^2\}1_{\{t_0 \leq t < t_1\}} + \alpha E\{\xi^{t-t_1}|\dot{x}(t_1)|^2\}1_{\{t_1 \leq t < t_2\}} + \cdots
\leq o(t) + o(t) + \cdots.
$$

Finally, it can be checked that (32) has at most $n$ elements. Indeed, from (25) and Lemma 14, it is simple to check that $N\{\theta(t_0)\} \subset N\{\theta(t_1)\} \subset \cdots \subset N\{\theta(t_m)\}$ strictly, which yields $m \leq \dim N\{O_{\theta(t_m)}\} \leq n$, where the limit comes from the fact that $O_i \in R^{n \times N_i}$ for all $i$. Hence

$$
\lim_{t \to \infty} E\{|\dot{x}(t)|^2\} \leq n \lim_{t \to \infty} o(t) = 0. \quad \square
$$
Now we are ready to present the main result of the section.

**Theorem 24.** The triplet \((A, C, \Lambda)\) is W-detectable if and only if the triplet \((A, O, \Lambda)\) is MS-detectable.

**Proof.** Necessity. Consider system \(\Phi_o\), let \(G\) be defined as in (23), and assume that \((A, C, \Lambda)\) is W-detectable. From (24) and Lemmas 22 and 23, we evaluate
\[
\lim_{t \to \infty} E\{|x(t)|^2\} = \lim_{t \to \infty} E\{||\hat{x}(t)||^2\} + \lim_{t \to \infty} E\{|\hat{x}(t)|^2\} \\
\leq \lim_{t \to \infty} e^{-2t}E\{|x(0)|^2\} + \lim_{t \to \infty} E\{|\hat{x}(t)|^2\} = 0.
\]
Thus \((A + GO, O)\) is MS-stable, which implies that \((A, O, \Lambda)\) is MS-detectable.

Sufficiency. We show that \((A, O, \Lambda)\) is not MS-detectable provided the triplet \((A, C, \Lambda)\) is not W-detectable. Consider \(i \in S\) and \(x_0 \in \mathcal{N}(O_i)\) as in Corollary 21. For the initial condition \(x(0) = x_0\) and \(\theta(0) = i\), we have from Corollary 15 that \(G_{\theta(i)}x(t) = 0, t \geq 0\). Then the term \(G_{\theta(i)}O_{\theta(i)}x(t)\) vanishes in (22), and \(x(t)\) evolves according to \(\dot{x}(t) = A_{\theta(i)}x(t)\) for any \(G \in \mathcal{M}^n\) in such a manner that the system \(\Phi_o\) behaves as its open-loop version \(\Phi\) no matter how \(G\) is chosen. Finally, from Corollary 21, we have that \(\lim_{t \to \infty} E\{|x(t)|^2\} \neq 0\), and we conclude that there is no \(G \in \mathcal{M}^n\) such that \(A + GO\) is MS-stable; hence \((A, O, \Lambda)\) is not MS-detectable. \(\Box\)

The relationship between MS-detectability and W-detectability is established as follows.

**Theorem 25.** If \((A, C, \Lambda)\) is MS-detectable, then \((A, C, \Lambda)\) is W-detectable.

**Proof.** It follows from the definition of \(O\) that \(\mathcal{N}\{O_i\} \subset \mathcal{N}\{C_i\}\). In view of this fact, it is simple to check that, given \(K \in \mathcal{M}^{n,q}\), there always exists \(G \in \mathcal{M}^{n,n} \otimes \mathbb{R}^q\) such that \(G_iO_i = K_iC_i, i = 1, \ldots, N\); hence we have that MS-detectability of \((A, C, \Lambda)\) implies MS-detectability of \((A, O, \Lambda)\). Theorem 24 completes the proof. \(\Box\)

Notice that the converse of Theorem 25 does not hold in general since it is a simple matter to find situations for which \(\mathcal{N}\{C_i\} \subset \mathcal{N}\{O_i\}\) strictly.

**Remark 4.** Theorem 24 allows one to test the W-detectability of the triplet \((A, C, \Lambda)\) by checking the MS-detectability of the triplet \((A, O, \Lambda)\). For a downsizing in the dimensionality, one can check alternatively if the triplet \((A, O' O, \Lambda)\) is MS-detectable. The following computational form for the MS-detectability test, posed in terms of linear matrix inequalities, is an adaptation of the results in [17]: the MS-detectability of \((A, C, \Lambda)\) is equivalent to the feasibility of the set
\[
A_i'X_i + X_iA_i + C_i' L_i + L_i C_i + \mathcal{E}_i(X) < 0, \quad i = 1, \ldots, N,
\]
in the unknowns \(X_i \in \mathbb{R}^{m_i}\) and \(L_i\) of appropriate dimensions.

**Example 2.** Let \(N = 2, n = 1\), and set
\[
A_1 = -2; \quad A_2 = 2; \quad C_1 = 1; \quad C_2 = 0; \quad \Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.
\]
From (9), we evaluate \(O_1 = [1 - 5]'\) and \(O_2 = [0 1]'\), and Theorem 11 ensures that \((A, C, \Lambda)\) is W-observable. Notice also that the condition in Lemma 20 is trivially satisfied. On the other hand, one can check by employing the result in Remark 4 that \((A, C, \Lambda)\) is not MS-detectable.

**Remark 5.** It can be shown that matrix \(O_i\) is full rank if the pair \((A_i, C_i)\) is observable. From this result and the result in Lemma 14, we conclude that a sufficient condition for W-observability of \((A, C, \Lambda)\) is that the pair \((A_i, C_i)\) is observable and \(\lambda_{ji} > 0\) for all \(j \neq i\). For instance, this is the scenario in Example 2.
Remark 6. For the deterministic linear system described by the pair \((A_i, C_i)\), let \(N_i\) stand for the observability matrix \(N_i = [C_i; A_i C_i; \cdots; A_i^{n-1} C_i]\). It is widely known that the detectability of \((A_i, N_i)\) is equivalent to the detectability of \((A_i, C_i)\). This property is not mirrored by the MS-detectability concept since Theorem 24 states that the MS-detectability of \((A, O, \Lambda)\) is equivalent to the W-detectability of \((A, C, \Lambda)\), which is more general than the MS-detectability of \((A, C, \Lambda)\). The scenario of MJLS with W-detectability is as follows:

W-detectable \((A, O, \Lambda)\) \(\Leftrightarrow\) W-detectable \((A, C, \Lambda)\) \(\Leftrightarrow\) MS-detectable \((A, O, \Lambda)\),

where the first equivalence follows from Lemma 12.

Remark 7. For the degenerate case \(\Lambda = 0\), we can show that W-detectability is equivalent to MS-detectability. This relation is exposed by the following equivalencies; most of them are simple to verify and are stated without further reference. We use a concise but self-evident notation; as in the remark above, \(N_i\) denotes the observability matrix of the system described by \((A_i, C_i)\).

(i) \(\text{MS-detec}(A, C, \Lambda) \Leftrightarrow \text{detec}(A_i, C_i)\) for all \(i\);
(ii) \(N(N_i) \equiv N(O_i)\) for all \(i\);
(iii) \(\text{detec}(A_i, N_i) \Leftrightarrow \text{detec}(A_i, O_i)\);
(iv) \(\text{detec}(A_i, O_i)\) for all \(i\) \(\Leftrightarrow\) MS-detec\((A, O, \Lambda)\) \(\Leftrightarrow\) W-detec\((A, C, \Lambda)\) (Theorem 24).

5. W-detectability and the LQ problem. In this section, we consider the linear quadratic control for system \(\Phi\). Under the W-detectability assumption, we show that the closed-loop system is MS-stable if a set of coupled algebraic matrix equations associated with the closed-loop system has a solution. In particular, we conclude that the solution to the CARE arising in the LQ problem is a unique stabilizing solution. Thus W-detectability not only generalizes MS-detectability but also plays the same role of MS-detectability in optimal LQ problems; see [7] and [9].

We start with some preliminary results for the open-loop system \(\Phi\). Consider the cost functional

\[
J^T(X) = \int_0^T \langle U(\tau), S \rangle d\tau = E \left\{ \int_0^T x(\tau)' S_{\theta(\tau)} x(\tau) d\tau | F_0 \right\}
\]

(33)
defined whenever \(U(0) = X\), where \(S \in M^{n_0}\). Notice that the functionals \(J^T\) and \(W^T\) are closely related. In fact, it is easy to check that, when \(C = S^{1/2}\), \(W^T(X)\) and \(J^T(X)\) coincide.

Let us consider the following coupled equation in the unknown \(P \in M^n\):

\[
0 = L_i(P) + S_i, \quad i = 1, \ldots, N,
\]

(34)
with \(S \in M^{n_0}\). The following results are derived from [6] and [13].

**Proposition 26.** Consider system \(\Phi\) and the set of equations (34). Then the following assertions hold:

(i) If there exists \(P \in M^{n_0}\) satisfying (34), then

\[
J^\infty(X) = \lim_{T \to \infty} J^T(X) \leq \langle X, P \rangle.
\]

(35)

(ii) Assume that \(A\) is MS-stable. Then there exists a unique \(P\) satisfying (34) and \(P \in M^{n_0}\); moreover,

\[
J^\infty(X) = \langle X, P \rangle.
\]
Lemma 27. Assume that \((A, S^{1/2}, \Lambda)\) is \(W\)-detectable and that there exists \(P \in \mathcal{M}^{n_0}\) such that \(J^\infty(X) < \langle X, P \rangle\). Then \(A\) is MS-stable.

Proof. Let \(t_d, s_d, \delta, \gamma\) and \(\beta\) be as in Definition 16. Let us assume that \(A\) is not MS-stable; in this situation, there exists \(X(0) \neq 0\) such that

\[
\|X(t)\| \geq \beta \zeta^t \|X(0)\|
\]

for some \(0 < \beta \leq 1\) and \(\zeta \geq 1\); see Remark 1. Let us define the sequence \(\mathcal{N} = \{n_0, n_1, \ldots\}\), where \(n_0 = 0\) and each \(n_m, m = 1, 2, \ldots\) is the smallest integer such that \(n_m > n_{m-1}\)

\[
\|X((n_m+1)s_d)\| \geq \delta \|X(ns_d)\|
\]

hold. It is easy to check that, if the number of elements of \(\mathcal{N}\) is finite, then

\[
\lim_{m \to \infty} \|X(ns_d)\| = 0,
\]

which contradicts (36) and we conclude that \(\mathcal{N}\) has infinitely many elements. Hence we can take a subsequence with infinitely many elements \(\mathcal{N}' = \{n_{m_0}, n_{m_1}, \ldots\}\), where \(n_{m_0} = m_0 = 0\) and each \(m_k, k = 1, 2, \ldots\) is the smallest integer, such that \(n_{m_k} \geq n_{m_{k-1}} + \max\{1, t_d/s_d\}\). In view of the \(W\)-detectability, we can evaluate

\[
J^T(X) = \int_0^T \langle X(\tau), S \rangle d\tau \geq \sum_{k=0}^{k'} \int_{n_{m_k}s_d}^{n_{m_k}s_d+t_d} \langle X(\tau), S \rangle d\tau
\]

\[
= \sum_{k=0}^{k'} W^\tau d(X(n_{m_k}s_d)) \geq \sum_{k=0}^{k'} \gamma \|X(n_{m_k}s_d)\|
\]

\[
\geq \gamma \sum_{k=0}^{k'} \beta \zeta^{(n_{m_k}s_d)} \|X(0)\| \geq \gamma \beta \zeta^{(n_{m_0}s_d)} (k' + 1) \|X(0)\|,
\]

where \(k'\) is the largest integer for which \(n_{m_k}, s_d + t_d \leq T\), in such a manner that \(k' \to \infty\) as \(T \to \infty\), and we conclude that \(J^\infty(X) = \infty\), which, from Proposition 26 (i), contradicts the hypothesis of the lemma. \(\square\)

Now we consider the system \(\Phi\) in closed-loop form. Recall that we assumed in section 1 that both the state \(x\) and the jump variable \(\theta\) are accessible for control. In this situation, it is well known that the optimal control is in linear state feedback form; see, e.g., [13]. Then we consider the following closed-loop version of system \(\Phi\):

\[
\Phi_c : \dot{x}(t) = (A_{\theta(t)} + B_{\theta(t)}G_{\theta(t)})x(t),
\]

where \(B \in \mathcal{M}^{n,r}\) is given and \(G \in \mathcal{M}^{r,n}\) can be regarded as a linear state feedback control. The associated infinite horizon cost functional is

\[
J^\infty(X) = \lim_{t \to \infty} \int_0^T \langle U(\tau), Q + G'RG \rangle d\tau,
\]

where \(Q \in \mathcal{M}^{n,0}\) and \(R \in \mathcal{M}^{r,+}\), defined whenever \(U(0) = X\). The system \(\Phi_c\) is said to be MS-stabilizable when there exists \(G \in \mathcal{M}^{r,n}\) such that \(A + BG\) is MS-stable. In what follows, \(\mathcal{L}_G\) refers to the operator \(\mathcal{L}\) associated to the closed-loop system with gain \(G\); namely,

\[
\mathcal{L}_G(U) = \bar{A}_i U_i + U_i \bar{A}_i + \sum_{j=1}^N \lambda_{ij} U_j,
\]
where $\hat{A}_i = A_i + B_iG_i$ for each $i$. The same notation applies to $\mathcal{T}_G$.

A question that arises is whether a $W$-detectable open-loop triplet $(A,C,\Lambda)$ can turn into a non-$W$-detectable closed-loop triplet $((A+BG),C,\Lambda)$. The next lemma gives an answer to this conjecture.

**Lemma 28.** If $(A,Q^{1/2},\Lambda)$ is $W$-detectable, then $(A+BG, (Q+G'RG)^{1/2}, \Lambda)$ is $W$-detectable for any $G \in \mathcal{M}^r$ and $R \in \mathcal{M}^+$. 

**Proof.** In this proof, $T$ and $W$ refer to the system $\Phi$, and $\mathcal{T}_G$ and $W_G$ refer to $\Phi_c$: $X(\cdot)$ represents the trajectory of system $\Phi_c$. We show that $\|X(t)\| \to 0$ as $t \to \infty$ whenever $W_G^T(X) = 0$. From Lemma 18, we can write for all $t \geq 0$

\begin{equation}
W_G^T(X) = \int_0^t \langle X(\tau), (Q + G'RG)^{1/2}(Q + G'RG)^{1/2}\rangle d\tau = \int_0^t \langle X(\tau), (Q + G'RG)\rangle d\tau.
\end{equation}

From the continuity of $X(t)$, we can evaluate

\[ \langle X(t), G'RG \rangle = \langle R^{1/2}GX(t)^{1/2}, R^{1/2}GX(t)^{1/2} \rangle = 0 \]

for all $t \geq 0$, and, since $R_i > 0$, we get that $G_iX(t) = 0$ for all $t \geq 0$ and $i$. Then we can write

\[ \mathcal{T}_{G_i}(X(t)) = \hat{A}_iX_i(t) + X_i\hat{A}_i' + \sum_{j=1}^N \lambda_{ji}X_j \]

\[ = A_iX_i(t) + X_iA_i' + \sum_{j=1}^N \lambda_{ji}X_j = T_i(X(t)) \]

for all $t \geq 0$, and, in view of Proposition 5 with $\hat{A}_i = (A_i + B_iG_i)$ for $i = 1, \ldots, N$, we have that such trajectories of systems $\Phi_c$ and $\Phi$ coincide whenever the initial conditions coincide. We set $C = Q^{1/2}$ in system $\Phi$ to conclude, similarly to (40), that

\[ W^T(X) = \int_0^T \langle X(\tau), C'C \rangle d\tau = \int_0^T \langle X(\tau), Q \rangle d\tau \leq W_G^T(X) = 0 \]

for any $s > 0$, and the detectability of $(A,Q^{1/2},\Lambda)$ ensures that $\|X(t)\| \to 0$ as $t \to \infty$. $\square$

**Theorem 29.** Consider the closed-loop system $\Phi_c$ with a linear state feedback control $G \in \mathcal{M}^r$. Assume that $(A,Q^{1/2},\Lambda)$ is $W$-detectable. If there exists a solution $P \in \mathcal{M}^{nn}$ of

\begin{equation}
\mathcal{L}_{G_i}(P) + Q_i + G'_iR_iG_i = 0, \quad i = 1, \ldots, N,
\end{equation}

then $A + BG$ is MS-stable.

**Proof.** Set $S = Q + G'RG$, and notice, from Lemma 28 and the assumption in the theorem, that $(A,S^{1/2},\Lambda)$ is W-detectable, where $\hat{A} = (A + BG)$. Then, from Proposition 26 (i), we have that $J^G_0(X) \leq \langle X, P \rangle$, and Lemma 27 states that $\hat{A}$ is MS-stable. $\square$

**Theorem 30.** Consider the system

\begin{equation}
\dot{x}(t) = A_0x(t) + B_0u(t),
\end{equation}
the associated infinite-horizon linear quadratic cost \( J^\infty(X) \), and the CARE in the unknown \( P \in \mathcal{M}^{n0} \):

\[
A_i'P_i + P_iA_i + \sum_{j=1}^{N} \lambda_{ij} P_j - P_iB_iR_i^{-1}B_i'P_i + Q_i = 0.
\]  

(43)

Assume that \((A, Q^{1/2}, \Lambda)\) is W-detectable. Then the following assertions hold:

(i) There exists a solution \( P \in \mathcal{M}^{n0} \) of (43) if and only if the system is MS-stabilizable;

(ii) If \( P \) is a solution of (43), then it is unique. The optimal state feedback control

\[
u(t) = -R_i^{-1}B_i'P_i x(t) \quad \text{whenever} \quad \theta(t) = i
\]

is such that

\[
\lim_{t \to \infty} E[|x(t)|^2] = 0.
\]

Proof. The sufficiency part of assertion (i) is a well-known result; see, e.g., [17, Theorem 3.1]. Notice that, if we denote \( G_i = -R_i^{-1}B_i'P_i \), we can write (43) equivalently as

\[
\mathcal{L}_{G_i}(P) + G_i'R_iG_i + Q_i = 0, \quad i = 1, \ldots, N,
\]

(45) and, from Theorem 29, we have that \( A + BG \) is MS-stable. This argument completes the proof of assertion (i) and also part of the assertion (ii) regarding the MS-stability of the closed-loop system defined by (44). Let us show now that \( P \) is unique. Suppose that \( \bar{P} \in \mathcal{M}^{n0} \) is a solution of (43). In a similar fashion to (45), we can write

\[
\mathcal{L}_{G_i}(\bar{P}) + G_i'R_i\bar{G}_i + Q_i = 0, \quad i = 1, \ldots, N,
\]

(46) where \( \bar{G}_i = -R_i^{-1}B_i'\bar{P}_i \); notice that, from Theorem 29, the system with gain \( \bar{G} \) is also MS-stable. Subtracting (46) from (45), we get, after some manipulations, that

\[
\mathcal{L}_{G_i}(P - \bar{P}) + (G_i - \bar{G}_i)'R_i(G_i - \bar{G}_i) = 0, \quad i = 1, \ldots, N,
\]

(47) and we identify \( S_i = (G_i - \bar{G}_i)'R_i(G_i - \bar{G}_i) \) in (34) to get, from Proposition 26 (ii), that \( P - \bar{P} \in \mathcal{M}^{n0} \); that is, \( P_i - \bar{P}_i \geq 0, \quad i = 1, \ldots, N \). Now, subtracting (45) from (46), we get similarly that \( \bar{P}_i - P_i \geq 0, \quad i = 1, \ldots, N \), and we conclude that \( \bar{P} = P \). It remains only to show that the feedback control (44) is optimal. Let us suppose that there exist \( X \in \mathcal{M}^{n0} \) and \( \bar{G} \in \mathcal{M}^{r,n} \) such that \( J_G^\infty(X) < J_{\bar{G}}^\infty(X) \). From Proposition 26 (ii), we have that \( J_G^\infty(X) < J_{\bar{G}}^\infty(X) = \langle X, P \rangle \), and, since \((A + BG, Q + 2G'R\bar{G})\) is W-detectable (see Lemma 28), Lemma 27 ensures that the closed-loop system with gain \( \bar{G} \) is MS-stable. Then, from Proposition 26 (ii), we have that there exists a unique solution \( \bar{P} \) to (46) and \( J_G^\infty(X) = \langle X, \bar{P} \rangle \). Once again, subtracting (45) from (46), we obtain \( \bar{P} \geq P \), and we conclude that \( J_G^\infty(X) = \langle X, \bar{P} \rangle \geq \langle X, P \rangle = J_{\bar{G}}^\infty(X) \), which denies the initial hypothesis, and hence

\[
J_G^\infty(X) \leq J_{\bar{G}}^\infty(X) \quad \forall K \in \mathcal{M}^{r,n}.
\]

The fact that the optimal control action is in linear state feedback form is a well-established result which comes from dynamic programming arguments and from the fact that the system is Markovian; see, for instance, [13] and [15].
6. Conclusions. This paper introduces the concept of W-detectability and the set of observability matrices $O$ that is related to the concept of W-observability for continuous-time MJLS.

We show that the concepts of W-observability and W-detectability reproduce geometric and qualitative properties of the deterministic concepts within the MJLS setting. In particular, we show how the properties (I)–(IV) mentioned in section 1 extend to MJLS; respectively, we have shown that

- if $x(t) \in N(O_{\theta(t)})$, then $x(s) \in N(O_{\theta(s)})$ a.s. for all $s \geq t$;
- $(A, C, \Lambda)$ is W-observable if and only if $O_i$ has full rank for each $i = 1, \ldots, N$;
- $(A, C, \Lambda)$ is W-detectable if and only if $\lim_{t \to \infty} E\{|x(t)|^2\} = 0$ whenever $x_0 \in N(O_{\theta_0})$; and
- $(A, C, \Lambda)$ is W-detectable provided $(A, C, \Lambda)$ is W-observable.

We also show that those concepts generalize the previous concepts encountered in the literature and that they play the same role in the quest for stabilizing solutions of quadratic control problems. Regarding the concept of MS-detectability, in one of the main results of this paper, we show that $(A, C, \Lambda)$ is W-detectable if and only if $(A, O, \Lambda)$ is MS-detectable. The result provides a testable condition for W-detectability. Moreover, the kernel of $O$ is in general smaller than that of the original matrices $C$, henceforth making W-detectability and MS-detectability directly comparable.

Testable conditions for the concept of W-observability is also developed in terms of the set of observability matrices $O$. The test of W-observability in Theorem 11 for MJLS and the observability test for $N$ deterministic time-invariant linear systems, each with dimension $n$, are alike.

REFERENCES


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