FRW cosmologies between chaos and integrability

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(Received 12 November 2001; published 21 February 2002)

A recent paper by Castagnino, Giacomini and Lara concludes that there is no chaos in a conformally coupled closed Friedmann-Robertson-Walker universe, which is in apparent contradiction with previous works. We point out that although nonchaotic the quoted system is nonintegrable.

DOI: 10.1103/PhysRevD.65.068502 PACS number(s): 04.62.+v, 98.80.−k

Castagnino, Giacomini and Lara (CGL) [1] analyzed the dynamical evolution of the spatially closed Friedmann-Robertson-Walker (FRW) cosmology conformally coupled to a massive, real, scalar field. Employing the cosmological time as the time parameter, they show that for arbitrary initial conditions the universe will collapse in a finite time and with a divergent rate of contraction. Their conclusion is that there is no chaos in the model.

The same system had been previously considered by Calzetta and El Hasi [2], who presented evidence of chaotic behavior subsequently confirmed by Bombelli, Lombardo and Castagnino [3]. Since these authors employed the conformal time in their analyses, CGL argue that the discrepancy of the results would rely on the different time parameters used—the origin of an intensive debate in the paradigmatic mixmaster model [4]. The aim of this Comment is to show that this is not the case and that all these fine works are in complete agreement in spite of the different choices of coordinates.

The classical theory of dynamical systems regards the study of systems of the form

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}), \tag{1}$$

for a fixed choice of the time parameter t. An invariant set of the phase space $\mathbf{x}$ is chaotic if it presents a *sensitive dependence on initial conditions and mixing*. This characterization is invariant under space diffeomorphisms: $\mathbf{y}=\psi(\mathbf{x})$. In general relativity, the absence of an absolute time forces us to consider system (1) under space-*time* diffeomorphisms: $\mathbf{y}=\psi(\mathbf{x},t)$, $d\tau=\lambda(\mathbf{x},t)dt$. Usually, the dynamical variables in this context are either functions of the space-time coordinates x^{μ} (possibly together with spins, Euler angles, etc.), when we study motions in a given background geometry, or functions of the metric $g_{\mu\nu}$ (possibly together with other fields), when we consider the evolution of the geometry itself. The study of chaos in general relativity faces both conceptual and technical difficulties. The former are associated with the dependence of classical indicators of chaos on the choice of the time parameter. This problem has been intensively discussed in the literature since Francisco and Matsas [5] showed the coordinate dependence of Lyapunov exponents. The latter difficulties are related with characteristic properties of relativistic systems. In cosmology, for example, we often meet high dimensionality, noncompacity, nonpositive kinetic energy, non-normalizable measure, nonexistence of global coordinates, nontrivial topology, singularities, etc. These properties strongly restrict the practical use of standard indicators of chaos, even if the system is treated as a classical one. The second class of difficulties is in fact the origin of most of the problems concerning chaos in cosmology, including the one discussed in this communication.

In terms of the conformal time η the system considered by CGL [1] is modeled by the $H=0$ energy surface of the Hamiltonian

$$2H=-p_a^2+p_\phi^2-a^2+\phi^2+a^2\phi^2, \tag{2}$$

where a is the radius of the universe and ϕ is the reparametrized scalar field [6]. CGL proved that the dynamics is nonchaotic *if* formulated in terms of the cosmological time. Our first point is that this is also true when the dynamics is formulated in terms of the conformal time as well as in terms of any other well defined time parameter. Translating the CGL work into conformal time it follows that for every physical initial condition, $a>0$ at $\eta=0$, there will be a finite time η_1 such that $a\to 0$ and $a'\to C$ for $\eta\to\eta_1$, where the prime denotes $d/d\eta$ and C is a nonzero, negative, finite constant. The absence of physical meaning for a negative universe radius, $a<0$, prevents us from extending the solutions beyond the big crunch. Since chaos is a concept associated with an infinite number of recurrences, it is clear that Eq. (2) regarded as a cosmological system is nonchaotic. To see the generality of this statement we can consider the Ricci scalar

$$\frac{R}{6}=m^2\left(\frac{\phi}{a}\right)^2, \tag{3}$$

where m is the mass of the coupled field. From Eq. (2) we have $\phi'^2+\phi^2\to C^2$ when $a\to 0$ and $a'\to C$, which leads to two possibilities: $R\to 6m^2$ if $\phi\to 0$, and $R\to\infty$ if $\phi\to\phi_1\neq 0$. It is easy to see from a Poincaré section defined by $a=0$, $a'<0$ that ϕ is typically nonzero at the big crunch, resulting in a divergent behavior for R. Such singularity is coordinate invariant and forbids physical extensions of the solutions throughout the big crunch, whatever coordinates we use.

Since there is no chaos the next question is whether or not the system is integrable. The answer comes from Refs. [2,3], where it is shown that the universe would present chaotic

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regions in the phase space (p_a, p_ϕ, a, ϕ) for an *infinite* sequence of contractions and expansions [7]. This extension through $a<0$ has no physical meaning, as mentioned before, but is an ingenious mathematical trick to obtain properties in the physical region from properties in the extended, unphysical domain. For example, chaos in the extended domain implies nonintegrability. In particular, it implies nonintegrability in the physical region defined by the first half-cycle $a>0$, between a big bang and the following big crunch. We stress that, even though the extension of the dynamics beyond the big crunch can be performed in the conformal time formulation and not in the cosmological time approach (because of the divergence of the contraction rate), the result concerning nonintegrability in the physical region is invariant under coordinate changes. In fact, the relation between the cosmological time and the conformal time in the physical domain is $dT=a\,d\eta$, which is a particular case of an *autonomous* transformation of the form

$$\mathbf{y}=\psi(\mathbf{x}), \tag{4}$$

$$d\tau=\lambda(\mathbf{x})dt, \tag{5}$$

where λ is a positive function and ψ is a diffeomorphism. The integrability is coordinate invariant under this class of transformations because if $\{I_1, I_2, \ldots\}$ are independent integrals of motion in the original variables $(\mathbf{x},t)$ (i.e., $\mathbf{F}\cdot\nabla I_i=0$ and $\{\nabla I_1, \nabla I_2, \ldots\}$ are linearly independent) then $\{I_1\circ\psi^{-1}, I_2\circ\psi^{-1}, \ldots\}$ are independent integrals of motion in the variables $(\mathbf{y},\tau)$. Therefore we can say that the conformally coupled Friedmann-Robertson-Walker model (2), is non-integrable in a meaningful sense. Accordingly, we cannot hope to find exact solutions for arbitrary initial conditions.

The same idea can be used to study the integrability of others FRW cosmologies, as long as we can find nonsingular coordinates to mathematically extend the solutions. In particular, this procedure works in the spatially closed cosmology conformally coupled to a scalar field, with both mass and cosmological constant terms, considered in Ref. [8]. Since it was shown that this model is chaotic in the extended domain, it follows that the system is nonintegrable in the physical region. We observe that methods based on extensions to unphysical values have been used for a long time to study integrability in cosmology. Perhaps the best known of them is the one based on the Painlevé theory of differential equations [9]. In the Painlevé analysis we look for necessary conditions for integrability (equivalently, sufficient conditions for nonintegrability) by studying critical points in the complex plane of time.

Summarizing, the dynamics of the model studied by CGL is chaotic when analyzed for an unphysical sequence of expansions and contractions of the universe, and is nonchaotic when considered for the period of time limited by a big bang and a big crunch. Nevertheless, it does not mean that the dynamics is simple since the onset of chaos in the extended domain implies nonintegrability in the physical region, which poses obstructions to the study of exact solutions. This characterization is coordinate invariant and does not rely on *natural* or *physical* choices of the time parameter, consistent with the covariant principle of general relativity.

A.E.M. thanks Y.-C. Lai for his kind hospitality at Arizona State University. This work was supported by Fapesp and CNPq.

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[7] The analysis of Hamiltonian (2) is highly non-trivial because the kinetic energy $2K=-p_a^2+p_\phi^2$ is not positive definite. We cannot find confined regions by analyzing the potential $2V=-a^2+\phi^2+a^2\phi^2$, what makes difficult the use of standard methods for chaos like Poincaré section and Lyapunov exponent methods. The authors of Refs. [2,3] managed to apply Melnikov methods to study the effect of the coupled field. For small values of a and ϕ, the coupling term $a^2\phi^2/2$ is a small perturbation and only a small fraction of the Kolmogorov-Arnold-Moser (KAM) tori are destroyed. For large values of a and/or ϕ, the coupling term becomes significant and most of the KAM surfaces are destroyed, leading their orbits to access a higher dimensional region.

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