Lagrangian-Hamiltonian formulation of paraxial optics and applications: Study of gauge symmetries and the optical spin Hall effect

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In the context of the paraxial regime, usually valid for optical frequencies and also in the microwave spectrum of guided waves, the propagation of electromagnetic fields can be analyzed through a paraxial wave equation, which is analogous to the nonrelativistic Schrödinger equation of quantum mechanics but replacing time \( t \) with spatial coordinate \( z \). Considering that, here it is shown that for lossless media in optical frequencies it is possible to construct a Lagrangian operator with an one-to-one correspondence with nonrelativistic quantum mechanics, which allows someone to use the same mathematical methods and techniques for solving problems.

To demonstrate that, we explore a few applications in optics with increasing levels of complexity. In the spirit of a Hamiltonian formulation, the ray-tracing trajectories of geometric optics in paraxial regime are obtained in a clear manner. Following that, the gauge symmetries of the optical-field Lagrangian density is discussed in a detailed way, leading to the general form of the interaction Hamiltonian. Through the use of perturbation theory, we discuss a classical analog for a quantum NOT gate, making use of mode coupling in an isotropic chiral medium.

At last, we explore the optical spin Hall effect and its possible applications using an effective geometric optics equation derived from an interaction Hamiltonian for the optical fields. We also predict within the framework of paraxial optics a spin Hall effect of light induced by gravitational fields.

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1. INTRODUCTION

It is well known that optics and quantum mechanics have a lot in common. From a historical point of view, it was an old formulation based on optics by Hamilton and Jacobi that inspired Erwin Schrödinger to put forward his wave-mechanical version of quantum mechanics \[1\]. Hamilton and Jacobi were searching for an identification of a particle’s trajectory with the gradient of constant phase surfaces and Jacobi were searching for an identification of a particle’s wave mechanics, with the role of time coordinate played by the longitudinal coordinate \( z \) in the paraxial equation. Although in the paraxial regime an analysis of electromagnetic fields is greatly simplified, the paraxial optics is still widely used to describe different phenomena in optical communication systems without lack of generality, especially in those where propagation on fiber optics are considered \[12\]. Beyond that, in some situations, especially those involving optical or microwave frequencies, the paraxial regime can also be used to simplify analysis of electromagnetic fields in different waveguides, meaning that in such cases the full vectorial electromagnetic field behavior is replaced by a scalar one and the results obtained can be considered as a good first approximation. As a last point, for monochromatic waves, the paraxial regime corresponds to the analysis of the propagation of the superposition of waves with slightly different wave vectors \( \mathbf{k} \), but all nearly parallel to the \( z \) axis, which can still be reasonably present in some integrated optical waveguide analyses.

In this paper, it is our aim to show that for lossless media, a Lagrangian version of optics is in perfect analogy with quantum mechanics, allowing someone to precisely use the same mathematical methods and techniques for solving and discussing optical problems. We explore the gauge symmetries equations assume a Dirac-like form, making the study of paraxial propagation easier in a weakly inhomogeneous medium \[11\].
of the optical fields and use the perturbation theory to describe mode coupling in a chiral medium as a warming-up exercise in order to explore possible applications of the optical spin Hall effect (OSHE) [13–16], which is a deviation of polarized light from the trajectory predicted by ordinary geometrical optics [17], using an effective geometric optics equation obtained from the interaction Hamiltonian for the optical fields. The OSHE depends on the spin or helicity of the optical field. The content of this article can be described as follows. In the next section we put forward the theory of an optical paraxial regime where at first the equivalence between optics and quantum mechanics is shown and, later, exemplified through the ray-tracing-trajectories equation of geometric optics. In Sec. III, we study the meaning of what we have called “gauge” symmetries of the optical field. In fact, we observe that an effective scalar potential is related to the refractive index of the medium under analysis. In Sec. IV we use perturbation theory to describe mode coupling in a waveguide filled with a chiral medium and the wave function $\psi$. The symmetries of the optical field. In fact, we observe that an effective scalar potential is related to the refractive index of the medium under analysis. In Sec. IV we use perturbation theory to describe mode coupling in a waveguide filled with a lossless isotropic medium and discuss the possible analogy between a quantum computer based on spin and such a classical microwave system. As a last application, in Sec. V, the present-day theme of OSHE is discussed in some important situations, showing that the procedure here introduced can be used in many different fields of optics, no matter the level of difficulty presented. To finish, in the last section, a few conclusions and remarks are added.

II. LAGRANGIAN AND HAMILTONIAN VERSIONS OF PARAXIAL OPTICS

Let us start this section with a brief review of the main aspects of nonrelativistic Schrödinger wave mechanics under the point of view of a Lagrangian density function. In order to get the correct nonrelativistic Schrödinger equation [18],

$$\text{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi,$$  \hspace{1cm} (1)

where $\psi$ being the quantum-mechanical complex wave function, $\hbar = h / (2\pi)$ the Planck’s constant, $m$ the particle mass, and $V$ the effective scalar potential, we must introduce a Lagrangian density $\mathcal{L}$ of the form [19]

$$\mathcal{L} = \text{i}\hbar \psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \nabla \psi^\dagger \cdot \nabla \psi - V \psi \psi^\dagger \psi.$$  \hspace{1cm} (2)

Defining the action integral by the expression $A = \int_V \int_T \mathcal{L}(\psi, \nabla \psi, \partial \psi / \partial t) d^3 x dt$, where $d^3 x$ and $dt$ are differentials of volume and time, respectively, and applying the principle of the least action, $\delta A = 0$, meaning that the variation $\delta A$ of the action functional $\mathcal{A}$ must vanish to make it an extremum, one is led to the so-called Euler-Lagrange equations for $\psi$ (the same equation applies for $\psi^\dagger$), and one gets

$$\frac{\partial \mathcal{L}}{\partial \psi} - \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} \right) = 0,$$

which, applied to Eq. (2), leads to the nonrelativistic Schrödinger equation (1).

Therefore, if one is interested in solving the hydrogen atom, the potential energy must be specified to be $V(r) = -e^2 / (4\pi \epsilon_0 r)$ and the required boundary conditions are $\psi(r \rightarrow \infty) \rightarrow 0$ and $\int_V \psi^\dagger \psi d^3 x = 1$, leading to the knowledge of an orthonormal set of basis functions which allows expansion of any quantum-mechanical field $\psi$ in terms of such a basis,

$$\psi = \sum_n c_n(t) \psi_n,$$

$c_n(t)$ being time-varying complex coefficients, and, in principle, $\int_V \psi_n \psi_d^\dagger d^3 x = \delta_{mn}$, where $\delta_{mn}$ is the Kronecker $\delta$ function. It is important to notice that $\sum_n |c_n(t)|^2 = 1$, which means that the norm of the abstract vector $\psi$ in a complex vector space is preserved, known as a Hilbert space. The inclusion of a perturbation into the system in the form of a time-varying potential $v(t)$ can be easily treated using time-dependent perturbation theory [18].

After those considerations on quantum mechanics, let us turn the attention to the wave equation in frequency domain, the so-called Helmholtz equation:

$$\left( \nabla^2 + k^2 \right) \psi(x, y, z) = 0,$$  \hspace{1cm} (3)

where $k^2 = n^2 c^2 / c^2$, $n(x, y, z, \omega)$ is the refractive index of the medium and the wave function $\psi$ is used to merge the time harmonic electric and magnetic fields, $E$ and $H$, into a single entity, as follows:

$$\psi = \left( E \frac{Z}{ZH} \right) e^{-i\omega t},$$  \hspace{1cm} (4)

$Z = \sqrt{\mu / \varepsilon}$ being the characteristic impedance of a medium with magnetic permeability $\mu$ and dielectric permittivity $\varepsilon$. Next, we write $\Psi(x, y, z) = \psi(x, y, z)e^{i\beta z}$, removing the rapid variations of the wave function along the $z$ axis, which we assume to be the longitudinal coordinate. Neglecting second-order derivatives of $\psi(x, y, z)$ with respect to $z$ (paraxial regime) we get

$$\text{i} \frac{\partial \psi}{\partial z} = -\frac{1}{2\beta} \left[ \nabla^2 \psi + (k^2 - \beta^2) \psi \right],$$  \hspace{1cm} (5)

where $\nabla^2 \psi = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the transverse Laplacian operator and $\beta$ is the propagation constant along the longitudinal axis. The above expression is the so-called paraxial wave equation. The reader must have in mind that $\psi(x, y, z)$ is not a scalar. As a matter of fact, the function $\psi$ can be regarded as some kind of spinor describing a spin-1 field.

Looking at Eqs. (1) and (5), the enormous similarities between them are quite obvious, with the role of time coordinate $t$ in (1) being played by the longitudinal coordinate $z$ in (5), the full Laplacian operator $\nabla^2$ in the quantum mechanical Schrödinger equation is replaced by its transverse version $\nabla^2 \parallel$ in the paraxial wave equation and the potential energy $V(x, y, z, t)$ corresponding to $[k^2(x, y, z) - \beta^2] / (2\beta)$. Indeed, it is straightforward to write down a Lagrangian density for the paraxial field $\psi(x, y, z)$ in close resemblance with (2), yielding the following expression:

$$\mathcal{L} = \text{i} \psi^\dagger \frac{\partial \psi}{\partial z} - \frac{1}{2\beta} \nabla^2 \parallel \psi^\dagger \cdot \nabla \psi + \frac{1}{2\beta} (k^2 - \beta^2) \psi \psi^\dagger \psi.$$  \hspace{1cm} (6)

An important conclusion can be drawn from the above scenario: Since the paraxial regime is analogous to quantum mechanics, the paraxial theory of electromagnetic wave propagation can be described by a set of basis functions pertaining to a Hilbert space. We can obtain a Hamiltonian density by
means of a Legendre transformation, provided that we define the conjugate momenta to $\psi$ by $\pi = \partial \mathcal{L}/\partial (\partial \psi/\partial z) = i\psi$. The Hamiltonian density $\mathcal{H} = \pi \psi - \mathcal{L}$ is given by

$$\mathcal{H} = \frac{1}{2\beta} \nabla_\perp \psi^\dagger \cdot \nabla_\perp \psi - \frac{1}{2\beta} (k^2 - \beta^2) \psi^\dagger \psi,$$

(7)

$\psi = \partial \psi/\partial z$ being in the context of the paraxial regime. In analogy with the quantum mechanical momentum operator, $p = -i\hbar \nabla$, we are able to define a momentum operator of the form $p_\perp = -i\nabla_\perp$.

Thus, the wave-particle duality of quantum mechanics corresponds to the wave propagation-geometric optics duality here. In other words, we are mentioning here that using the Lagrangian and Hamiltonian found before one can easily obtain, for example, an equation describing ray trajectories in geometric optics. Therefore, to show that we notice the Hamiltonian density (7) is obtained from an expression of the form $\mathcal{H} = \psi^\dagger H \psi$, being the single “particle” Hamiltonian

$$H = \frac{1}{2\beta} p_\perp^2 + V(x,y,z),$$

(8)

where $p = (p_x, p_y) = \beta(\partial x/\partial z, \partial y/\partial z)$ and $V(x,y,z) = -\frac{1}{2\beta}(k^2 - \beta^2)$ is the potential energy. Making $k(x,y,z) = k_0 n(x,y,z)$ a position-dependent function, $n(x,y,z) = n_0 + \delta n(x,y,z)$, where $n_0$ is a constant and $\delta n(x,y,z)$ is a small perturbation in the refractive index, and choosing $\beta = k_0 n_0$ we use Hamilton’s equations of motion (written below for the sake of completeness),

$$\frac{\partial x_i}{\partial z} = \frac{\partial H}{\partial p_i},$$

(9)

$$\frac{\partial p_i}{\partial z} = -\frac{\partial H}{\partial x_i},$$

(10)

where $i = 1,2$ and $x_\perp = (x_1, x_2) = (x,y)$, to get the ray-tracing equations of motion,

$$\frac{\partial^2 x}{\partial z^2} = \frac{1}{n} \frac{\partial n}{\partial x},$$

(11)

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{n} \frac{\partial n}{\partial y},$$

(12)

which resemble Newton’s second law. The above equations reduce to a single one in the axis symmetric situation, for which $\rho = \sqrt{x^2 + y^2}$:

$$\frac{\partial^2 \rho}{\partial z^2} = \frac{1}{n(\rho,z)} \frac{\partial n(\rho,z)}{\partial \rho}.$$

(13)

For instance, the above equation is largely used to obtain the ray tracing in a multimode fiber when geometric optics is used, providing valuable information about dispersion phenomena and limiting bit rate in a fiber-optic communication system [12].

In what follows, we extend our analysis of Lagrangian and Hamiltonian to discuss some aspects related to the so-called gauge symmetries for the field $\psi$, following the paradigmatic methods of quantum field theories [19].

### III. “GAUGE” SYMMETRIES OF THE OPTICAL FIELDS

In quantum mechanics, gauge symmetries are related to internal degrees of freedom of the field $\psi$ and always lead to some conserved quantity of the theory. The local phase transformations of the wave functions in the form $\psi(x,t) = e^{-i\Lambda(x,t)} \psi(x,t)$ lead to electric charge conservation and the introduction of the electromagnetic potentials in order to make the whole theory invariant under the symmetry transformation [18–20]. The group symmetry of local phase transformations is known formally as the U(1) gauge group.

Here we follow closely the paradigm of gauge invariance, often used in quantum mechanics [19,20]. Starting with a free Lagrangian density for the optical field $\mathcal{L}$,

$$\mathcal{L} = i \psi^\dagger \frac{\partial \psi}{\partial z} - \frac{1}{2\beta} \nabla_\perp \psi^\dagger \cdot \nabla_\perp \psi,$$

(14)

which is the same as (6) without the term $\frac{1}{2\beta}(k^2 - \beta^2) \psi^\dagger \psi$. In fact, we want to produce a phase transformation to $\psi$, as follows:

$$\psi' = e^{-i\Lambda} \psi,$$

(15)

with $\Lambda(x,y,z)$ a scalar function depending on spatial coordinates. Clearly, the resulting Lagrangian density no longer has the same form of (14). Explicitly written, we obtain

$$\mathcal{L}' = i \psi'^\dagger \frac{\partial \psi'}{\partial z} - \frac{1}{2\beta} \nabla_\perp \psi'^\dagger \cdot \nabla_\perp \psi'$$

$$+ \psi'^\dagger \frac{\partial \Lambda}{\partial z} \psi' - \frac{i}{2\beta} \psi'^\dagger (\nabla_\perp \Lambda) \cdot \nabla_\perp \psi'$$

$$+ \frac{i}{2\beta} \nabla_\perp \psi'^\dagger \cdot (\nabla_\perp \Lambda) \psi' - \frac{1}{2\beta} |\nabla_\perp \Lambda|^2 \psi'^\dagger \psi'.$$

(16)

Observe that the first two terms on the right-hand side of the above equation have the same form as (14) but terms including $\Lambda$ and its derivatives destroy invariance and alter the content of the theory. In such a case we are left with two options: (i) Accept the fact that the theory is not invariant under phase transformations of the form (15) or (ii) build an invariant theory modifying the original free Lagrangian in order to make the local phase transformations a symmetry of the whole theory. We are tempted to try the second possibility.

Following the standard procedure of a gauge field theory, we must replace the ordinary derivatives appearing in (14) with covariant ones in such a way that the theory will remain invariant under the phase transformations, provided that we make the following substitutions:

$$\frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial z} + i A_0,$$

(17)

$$\nabla_\perp \rightarrow \nabla_\perp - i A_\perp,$$

(18)

$A_0$ and $A_\perp$ being the scalar and vector gauge potentials of the theory, respectively. The complete Lagrangian density of the theory can be written as

$$\mathcal{L} = i \psi^\dagger \frac{\partial \psi}{\partial z} - \frac{1}{2\beta} (\nabla_\perp + i A_\perp) \psi^\dagger \cdot (\nabla_\perp - i A_\perp) \psi$$

$$- \psi^\dagger A_0 \psi.$$

(19)
which is now invariant under the transformation (15), provided that a gauge change in the potentials obey the following rule:

\[ A'_0 = A_0 - \frac{\delta \Lambda}{\delta z}, \]

\[ A'_\perp = A_\perp + \nabla_\perp \Lambda. \]  

(20)  

(21)  

We point out that the gauge fields \( A_0 \) and \( A_\perp \) have direct connection with the refractive index \( n(x,y,z) \) of the medium. To get the correct theory we set \( A_0 = -\frac{1}{2\beta}(k^2 - \beta^2) \) while \( A_\perp \) is some kind of “magnetic” vector potential which can be usually set to zero. An “electric” field can be obtained through the definition \( \vec{E} = -\nabla_\perp A_0 \), making Eqs. (11) and (12) resemble Newton’s second law for a “charged” particle, as follows:

\[ \beta \frac{\partial^2 \vec{x}_\perp}{\partial z^2} = \vec{F}. \]  

(22)  

The potential \( A_\perp \) defines a “magnetic” field through the relation \( \vec{B} = \nabla_\perp \times A_\perp \), introducing an additional “force” field of the form \( (\partial \vec{x}_\perp / \partial z) \times \vec{B} \) into the above equation.

Going further, postulating gauge invariance of the theory under gauge groups of increasing complexity makes it possible to include anisotropy [21,22], chirality [23–33], and other effects into the theory. The interested reader will find a comprehensive treatment of anisotropy and fully bianisotropic phenomena in Refs. [34] and [35]. A gauge theory of rotations can be readily constructed, producing the following transformation:

\[ \psi' = \begin{pmatrix} E' \\ Z' \end{pmatrix} = S \begin{pmatrix} E \\ Z \end{pmatrix}, \]  

(23)  

where \( S \) is a unitary operator defined as

\[ S = \begin{pmatrix} R_E(z) & R_R(z) \\ R_R^*(z) & R_E^*(z) \end{pmatrix}, \]  

(24)  

\( R_E, R_H, \) and \( R_c \) being the SO(3) rotation group matrices with \( z \)-dependent angle of rotation. The matrices \( R_E, R_H, \) and \( R_c \) are related to anisotropy, while \( R_R \) is related to chiral rotations. By demanding invariance of the Lagrangian density under (23), one must replace ordinary derivatives \( \partial / \partial z \) by covariant ones in accordance with Eq. (17) to compensate for terms of the form \( \psi' (\partial S / \partial z) \psi. \) The correct physical theory for fully bianisotropic nonreciprocal lossless media, where all matrices \( R_E, R_H, \) and \( R_c \) are nonvanishing, is obtained identifying the gauge potential as follows:

\[ A_0 = -\frac{k_0^2}{2\beta} \begin{pmatrix} \delta \vec{\kappa} \\ \delta \vec{\kappa} \end{pmatrix}, \]  

where \( \delta \vec{\kappa} \) and \( \delta \vec{\mu} \) are \( 3 \times 3 \) matrices related to the small perturbation of the dielectric permittivity and magnetic permeability tensors of the medium, \( \delta \vec{\kappa} \) is known as the reciprocity matrix, and \( \delta \vec{k} \) is the chirality matrix. This way, we get the interaction Hamiltonian

\[ \hat{H}_I = -\frac{k_0^2}{2\beta} \begin{pmatrix} \delta \vec{\kappa} \\ \delta \vec{\kappa} \end{pmatrix}, \]  

In the next section we give an illustrative example using perturbation theory to obtain analytical results for an isotropic chiral medium.

IV. STUDY OF AN ISOTROPIC CHIRAL MEDIUM THROUGH PERTURBATION THEORY

To get complete analogy with quantum mechanics, consider that the Hamiltonian of the unperturbed problem is given by

\[ \hat{H}_0 = \int d^2x \left[ \frac{1}{2\beta} \nabla_\perp \psi^\dagger \cdot \nabla_\perp \psi - \frac{1}{2\beta} (k^2 - \beta^2) \psi^\dagger \psi \right], \]  

(26)  

whose eigenstates form a complete set of orthonormal basis functions \( \{ \psi_1, \psi_2, \ldots, \psi_n, \ldots \} \) in a Hilbert space. Next, a perturbation can be included in such a way that \( \hat{H} = \hat{H}_0 + \hat{H}_I \), \( \hat{H}_I \) being the so-called interaction Hamiltonian, which can be obtained by imposing some gauge symmetry to the theory or including some effective phenomenological term. The perturbed fields can be expanded in terms of the unperturbed basis functions, as follows:

\[ \psi = \sum_m c_m(z) \psi_m(x,y,z), \]  

(27)  

where \( \psi_m(x,y,z) = \psi(x,y) e^{i\beta_m z} \) are the eigenstates of \( \hat{H}_0 \), as previously mentioned. From

\[ i \frac{dc_n}{dz} = \sum_m c_m e^{i(\beta_n - \beta_m)z} \langle n | \hat{H}_I | m \rangle, \]  

(29)  

where we have defined \( \langle n | \hat{H}_I | m \rangle \) through the following equation:

\[ \langle n | \hat{H}_I | m \rangle = \int d^2x \psi_n^\dagger(x,y) \hat{H}_I \psi_m(x,y). \]  

(30)  

In the general case of a fully bianisotropic media [34,35], the above formula can be written in terms of the electromagnetic fields explicitly, taking the form

\[ \langle n | \hat{H}_I | m \rangle = -\frac{k_0^2}{2\beta} \int d^2x \left[ E_n^* \cdot \delta \vec{\kappa} \cdot E_m + Z^2 H_n^* \cdot \delta \vec{\mu} \cdot H_m + Z E_n^* \cdot (\delta \vec{\kappa} + i\delta \vec{k}) \cdot H_m + Z H_n^* \cdot (\delta \vec{\kappa} - i\delta \vec{k}) \cdot E_m \right]. \]  

(31)  

The matrices \( \delta \vec{\kappa} \) and \( \delta \vec{\mu} \) are the anisotropic perturbations of the dielectric permittivity and magnetic permeability of the medium, respectively, allowing usual mode coupling of electromagnetic fields, but \( \delta \vec{\kappa} \) and \( \delta \vec{k} \) couple the electric field of one mode to the magnetic field of the other. Observe that non-diagonal elements in \( \delta \vec{\kappa}, \delta \vec{\mu} \) couple modes with orthogonal polarizations, allowing for polarization mode rotation.
Isotropic nonchiral media forbid rotation of the polarization mode. Looking at (31) an important conclusion can be drawn: The only way to couple orthogonal polarization modes in reciprocal and nonchiral media is the existence of dielectric and/or magnetic anisotropy. By contrast, in isotropic chiral media the effect of polarization mode rotation is allowed, as will be shown.

For the sake of simplicity, we take the case of a microwave metallic waveguide having square cross section of length \( a \), partially filled over a distance \( L \) with an isotropic chiral medium, characterized by microstructures lacking mirror symmetry, leading to the property of handedness and can be achieved in practice by the use of a metamaterial composed of small metallic helices equally distributed in a paraffin dielectric medium, with the helix orientation axes randomly distributed [26], as illustrated in Fig. 1. The helices are resonant at a particular frequency, leading to the existence of an attenuation parameter \( \alpha \), a feature that can be inferred through the Kramers-Kronig relations. In Ref. [36] an experimental method for the extraction of permittivity, permeability, and chirality parameters in helix-loaded chiral composites is discussed and it is pointed out that, in practice, chirality cannot be varied independently of the other parameters. Indeed, an important contribution for the attenuation constant \( \alpha \) comes from the fractional volume occupied by the metallic helices, which is in the range of 1%–3% of the total volume of the chiral medium. It is shown that for microwave domain in the band of 6–12 GHz, the attenuation reaches its maximum value near the resonant frequency of the chirality parameter [37]. Another important feature of a waveguide fully filled with chiral material, known as a chirowaveguide, is that individual transverse electric (TE), transverse magnetic (TM), or transverse electromagnetic (TEM) modes cannot be supported [38] and only hybrid modes are allowed.

In view of the above discussion, the use of perturbation theory is plausible only under the following assumptions: (i) The operating frequency is away from resonance and the chirality parameter is small enough to allow the use of the normalized modes of the hollow metallic waveguide as the nonperturbed basis functions and (ii) the distance \( L \), over which the waveguide is filled with chiral medium, is lower than the characteristic penetration depth \( \delta = 1/\alpha \) of the chiral medium, in such a way that attenuation effects can be neglected to a first-order approximation.

For an isotropic chiral medium the interaction Hamiltonian (25) takes the simple form

\[
\hat{H}_l = i \frac{k_0^2 \kappa}{2\beta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

where \( 0 \) and \( 1 \) are the null and identity \( 3 \times 3 \) matrices. The limited region of the waveguide filled with a chiral medium allows for the coupling between the fields \( \mathbf{E} \) and \( \mathbf{H} \) of different modes of propagation. If the operating frequency \( \omega \) is well chosen, only the fundamental modes \( \text{TE}_{10} \) and \( \text{TE}_{01} \) are not cut off. The cutoff frequency for these two modes is \( \omega_c = c\pi/\alpha \). The normalized electromagnetic fields of the \( \text{TE}_{10} \) propagating mode is given as

\[
E_z^{10} = 0,
\]

\[
H_z^{10} = -\frac{2\pi i}{a^2 \sqrt{2\omega \mu \beta}} \cos \left( \frac{\pi x}{a} \right) e^{-i(\omega t - \beta z)},
\]

\[
E_\perp^{10} = \frac{2\omega \mu}{\beta a^2} \sin \left( \frac{\pi x}{a} \right) e^{-i(\omega t - \beta z)} \hat{a}_y,
\]

\[
H_\perp^{10} = -\frac{2\beta}{\omega \mu a^2} \sin \left( \frac{\pi x}{a} \right) e^{-i(\omega t - \beta z)} \hat{a}_x,
\]

while for the \( \text{TE}_{01} \) mode we have

\[
E_z^{01} = 0,
\]

\[
H_z^{01} = \frac{2\pi i}{a^2 \sqrt{2\omega \mu \beta}} \cos \left( \frac{\pi y}{a} \right) e^{-i(\omega t - \beta z)},
\]

\[
E_\perp^{01} = \frac{2\omega \mu}{\beta a^2} \sin \left( \frac{\pi y}{a} \right) e^{-i(\omega t - \beta z)} \hat{a}_x,
\]

\[
H_\perp^{01} = -\frac{2\beta}{\omega \mu a^2} \sin \left( \frac{\pi y}{a} \right) e^{-i(\omega t - \beta z)} \hat{a}_y,
\]

where \( \beta = \sqrt{\omega^2 - \pi^2 / a^2} / c \) [39]. The coefficients \( \langle n | \hat{H}_l | m \rangle \) are given by

\[
\langle n | \hat{H}_l | m \rangle = i \frac{Z k_0^2 \kappa}{2\beta} \int d^2 x \{ \mathbf{E}_n^* \cdot \mathbf{H}_m - \mathbf{H}_n^* \cdot \mathbf{E}_m \}. \]

Identifying \( \psi_1 \) with \( \text{TE}_{10} \) and \( \psi_2 \) with \( \text{TE}_{01} \), it is a straightforward matter to show that

\[
\langle 1 | \hat{H}_l | 1 \rangle = \langle 2 | \hat{H}_l | 2 \rangle = 0,
\]

\[
\langle 1 | \hat{H}_l | 2 \rangle = \langle 2 | \hat{H}_l | 1 \rangle^* = -i\Omega,
\]

where \( \Omega = 8k \frac{Z k_0^2}{(\pi^2 \beta)} \). The equations of motion for the coefficients \( c_n(z) \), corresponding to the amplitudes of the \( n \)th mode at a distance \( z \) are given by

\[
\frac{dc_1}{dz} = \Omega c_2,
\]

\[
\frac{dc_2}{dz} = -\Omega c_1.
\]
which are the optical analogs of the two-level quantum system exhibiting Rabi oscillations [18]. The physical meaning of the above system of equations is that a piece of chiral material placed inside the waveguide couples the modes TE_{10} and TE_{01} corresponding to two linear orthogonal polarizations in such a way that the energy transported by one polarization mode flows to the other as the wave propagates through the chiral medium. The above situation can be directly applied to the construction of a classical analog for a NOT quantum gate. The spin of electrons can play the role of qubits in a quantum computer, where the spin up corresponds to the qubit |0⟩ and spin down to the qubit |1⟩, but a physical state can be prepared in a superposition of the form |ψ⟩ = a|0⟩ + b|1⟩. A usual NOT quantum gate would be obtained applying time-varying magnetic fields to switch the spin. By analogy, in our above example using a waveguide filled with chiral medium the two orthogonal polarization modes play the role of the qubits while the chirality parameter corresponds to the applied magnetic fields. A microwave field with mixed polarization can be initially prepared and evolves as a function of the propagation distance z, in close analogy with the state |ψ⟩, which evolves in time in the presence of an external magnetic field. It must be pointed out that even in theory, no lossless isotropic chiral material exists because optical rotation and circular dichroism together satisfy the Kramers-Kronig relations. That issue applies not only to isotropic chiral metamaterials but to all isotropic chiral materials. In our toy model, losses can be taken into account postulating that Ω is a complex number, whose imaginary part is related to attenuation in the wave propagation, but only in the case of low-lossy medium. Making a naive analysis, in practice, for a helix-loaded chiral composite, the metallic helices, which are the main source of attenuation in the chiral region, must be a small fraction of the total volume of the chiral medium. The quantum mechanical counterpart is the decoherence process. The results obtained in this way are valid even for lossy media, provided that the characteristic attenuation length L_a = 1/α is one or two orders of magnitude lower than the wavelength λ.

V. THE OPTICAL SPIN HALL EFFECT

In this section we analyze the OSHE. Such effect corresponds to a subtle and usually undetectable deviation of light from the trajectory predicted by the ordinary geometrical optics, which depends on the helicity state of the electromagnetic field [17]. Here we focus on two cases of particular interest: (i) propagation in a lossless anisotropic dielectric and (ii) the OSHE induced by gravity effects.

First consider a lossless anisotropic dielectric, for which we have δ̂k = δk̂ = δ̂μ = 0 and the interaction Hamiltonian (25) reduces to H_I = −k_β ξ δ̂k̂, where [34,35]

\[
\begin{pmatrix}
\frac{n_1^2}{2} & i\xi & 0 \\
-i\xi & n_2^2 & 0 \\
0 & 0 & n_3^2
\end{pmatrix}
\]

(46)
in an appropriate reference frame. For circularly polarized light the electric field polarization vector is written as

\[
E_± = \frac{E_0}{\sqrt{2}}(\hat{a}_x ± i\hat{a}_y),
\]

(47)

allowing a direct analogy between light and spin-1/2 particles by identifying the right-handed circular polarization (RHCP) with spin σ = +1 and the left-handed circular polarization (LHCP) with spin σ = −1. Assuming n_1^2 = n_2^2 = n_3^2 = ω_0(x,y,z), ξ = ξ(x,y,z), and δ = k_0, the geometric optics equations of motion follow straightforwardly from the interaction Hamiltonian, yielding

\[
\frac{d^2x^\sigma}{dz^2} = \frac{1}{n} \nabla_⊥ n - \frac{1}{2} σ ξ_⊥.
\]

(48)

Looking at second term on right-hand side of the preceding equation, one can notice that it is equivalent to a spin-dependent “force” acting on the light ray. Indeed, the light polarization or helicity is the analog of the spin magnetic moment and the anisotropy parameter ξ can be interpreted as the projection of a “magnetic” field along the spin quantization axis, in a manner that $\frac{1}{2} σ ξ_⊥$ is the corresponding term to $μ_0 B_⊥$, for an electron. For a general situation of lossless anisotropic dielectric we can introduce a “magnetic” field $b = -i(ε_{yz}, ε_{xz}, ε_{xy})$ and generalize the expression $\frac{1}{2} σ ξ_⊥$ to $\frac{1}{2} σ \cdot b$, where $σ = (σ_x, σ_y, σ_z)$ are the Pauli spin matrices.

Furthermore, we propose to construct an optical device based on the OSHE acting as a field sensor, since the separation angle θ between the RHCP and the LHCP light beams, after propagating a distance L, depends on the parameter ξ, usually controllable applying dc magnetic or electric fields or mechanical stress or strain. Equation (48) is greatly simplified supposing the parameters n and ξ depend only on x and z,

\[
\frac{d^2x^\sigma(z)}{dz^2} = \frac{∂}{∂x} \ln n(x) - \frac{1}{2} σ ξ(x),
\]

(49)
yielding the following result for the separation angle:

\[
θ = \int_0^L \frac{∂ξ(x)}{∂x} dz.
\]

(50)

For a lossless electronic gas subjected to an applied dc magnetic field we have the following anisotropy parameter [40]:

\[
ξ = -\frac{ω_p^2 \omega_B}{ω(ω^2 - ω_p^2)},
\]

(51)

where ω is the angular frequency of the propagating wave, $ω_p = \sqrt{Nq^2/e_0}\$ is the plasma frequency of the medium, $ω_B = qB/m$ is the so-called cyclotron frequency, $N = N(x,z)$ is the electronic density, q is the electronic charge, m is the effective electron mass, $e_0$ is the vacuum permittivity constant, and $B = B(x,z)$ is the magnitude of the magnetic field applied along the z axis. For $ω_B ≪ ω$ it can be straightforwardly shown that

\[
θ ≈ -\frac{ω_p^2 \omega_B}{ω^3} \int_0^L \frac{∂ ln(NB)}{∂x} dz.
\]

(52)

The above equation suggests an immediate application as a noninvasive sensor for the determination of electric currents.
transported by high-power-conducting wires. The proposed apparatus is illustrated in Fig. 2. The magnetic field generated by a long wire transporting a current $I$ at a distance $\rho = x + a$ from the wire is given by $B(x) = \mu_0 I / (\pi \rho)$, which can be approximated by $B(x) \approx B_0 (1 - x/a)$ with $B_0 = \mu_0 I / (2\pi a)$. The distance $a$ is the reference distance between the wire and the sensor. In this way we have

$$\theta = \frac{\omega^2 \omega_B L}{a}. \quad (53)$$

Figure 3 illustrates the splitting of an initially unpolarized beam or an eventually linearly polarized one, into two circularly polarized beams with separation angle $\theta$ after a propagated distance $L$. Notice that $\theta$ is a linear function of the current intensity $I$ through the cyclotron frequency $\omega_B = q B_0 / m = \left[q \mu_0 (2\pi a) I \right] / m$. In practice, the parameters $\omega$ and $L/a$ can be adjusted to get $\theta$ in the range of $\pi/180$ rad, which is easily measurable within current technology. As a matter of fact, an experimental sensor device would take advantage of one of the two parameters which can be measured to determine the current $I$: one is the angle $\theta$ itself, and the other way is to set a reference angle $\theta$ and vary the frequency $\omega$ until $\theta$ is reached for a given current. The sensitivity of the proposed sensor strongly depends on the precision of angle determination. Typical values of current $I$ in high-power systems are in the range of $10 - 100$ A but can be as high as thousands of amperes in aluminum production industrial process. The proposed device can measure dc electric current through the magnetic field it produces, since ammeters based on Faraday’s law of induction cannot be used to measure dc currents. To obtain an anisotropic dielectric permittivity tensor of the form (46) in an experimental way, we propose to use a semiconductor plasma waveguide (SPW). Following Ref. [40], for a SPW we can set the values of the main parameters appearing in (53) to be $\omega_p = 10^{12} \text{ rad/s}$, $\omega = 10^{11} \text{ rad/s}$, and $m = 0.01 m_e$, where $m_e = 9.11 \times 10^{-31}$ kg, which yields $\omega_p (I_0) = 3.5 \times 10^7 \text{ rad/s}$ at $I_0 = 1$ A. Typical dimensions which can be used in practice are $a = 10$ cm and $L = 2$ cm. In this way we get $\theta = 0.007 (I/I_0)$ rad and the resulting sensitivity parameter $S = d\theta/d I$ corresponds to 0.47/A. For a current $I = 10$ A, the separation angle goes to $\theta = 0.07$ rad (or 4.5'), which is easily detectable within current angle-measurement technology.

Now we turn our attention to the problem of light propagation under the influence of the gravitational fields, which is also of great interest. The study of the spin Hall effect for spin-1/2 particles under the influence of gravity was already done in Ref. [41]. It can be shown that spinning particles slightly deviate from geodetic motion predicted by general relativity [42,43]. Quantum-mechanically, light is described by a spin-1 field, but our aim is to show that under the correct assumptions paraxial optics can also be used, giving insightful results. It is a well known fact that gravity bends the light trajectory as it passes near the source of a gravitational field. The light propagation direction coincide with that of Poynting’s vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, which slightly changes as the light beam bends. Clearly, the fields $\mathbf{E}$ and $\mathbf{H}$ must rotate in function of propagated distance, in order to give the correct direction of $\mathbf{S}$ as the light beam passes near the gravitational source. We can accommodate such a situation into our theory requiring gauge invariance under $(x,y,z)$-dependent rotations, translations, and phase transformations. To do that, a gauge potential must be introduced, which in the paraxial regime is to be identified with specific components of the metric tensor of the general relativity in the weak gravitational-field regime. In other words, the effects of gravity can be taken into account by effective fully bianisotropic refractive index tensors. The gravitational effects are correctly described in the lowest-order approximation by the Lagrangian density,

$$\mathcal{L} = i \psi^\dagger \frac{\partial \psi}{\partial z} - \frac{1}{2\beta} \nabla \psi^\dagger \cdot \nabla \psi + 2\beta \Phi_\phi \psi^\dagger \psi, \quad (54)$$

$\Phi_\phi = -GM/(r c^2)$ being the Newtonian gravitational potential, $G$ the Newton gravitational constant, $M$ the mass of the gravitational source, and $r$ the distance from the source. The ray trajectory under the influence of a gravitational field is illustrated in Fig. 4, where $r = (x + b)^2 + y^2 + z^2$ and $b$ is known as the impact parameter. The relevant geometric optics equation of motion can be derived from (54), resulting in the following expression:

$$\frac{\partial^2 x(z)}{\partial z^2} = 2 \frac{\partial \Phi_\phi}{\partial x}. \quad (55)$$
The deflection angle $\alpha$ is easily obtained through integration of the above equation, yielding

$$\alpha = 2 \left[ \int_{-\infty}^{\infty} \frac{\partial \Phi_x^2}{\partial z} \bigg|_{x=0, y=0} \right] = \frac{4GM}{bc^3}. \quad (56)$$

The result obtained in this way is the same as the prediction of general relativity in a weak gravitational field [42,43]. Encouraged by the previous success we go one step further and introduce a phenomenological term responding for the separation angle between LHCP and RHCP beams will be given by (56). For an initially unpolarized or linearly polarized beam passing near the gravitational source, the separation angle between LHCP and RHCP beams will be given by $\Delta \gamma_y = 32G^2M^2/(\beta c^4 b^3)$. For weak gravitational fields such value is far beyond precision limits of the current angle-measurement technologies. For instance, taking into account the values for the Sun’s mass and radius, $M_{\text{Sun}} \approx 1.99 \times 10^{30}$ kg, $b = 6.96 \times 10^8$ m, respectively, we get $\Delta \gamma_y \approx 4.3 \times 10^{-16}$ s of arc, which is undetectable in practice. However, we speculate that the higher gravitational fields produced by massive stars or black holes can produce observable effects. As a last and illustrative example we considered a neutron star, for which we have $M = 1.44M_{\text{Sun}}$ and $b \approx 10$ km [43], yielding $\Delta \gamma_y \sim 1$ s of arc in the microwave domain, which is in the same order of magnitude of light deflection by the Sun (1.75 s of arc [42,43]), being surely detectable within current technology.

VI. CONCLUSION

In summary, in this paper we demonstrated the analogy between quantum mechanics and the electromagnetic field propagation in the paraxial regime. We put forward complete Lagrangian and corresponding Hamiltonian formulations well suited for the use of perturbation theory in an even closer resemblance with quantum mechanics. Also, we have studied the possible gauge symmetries of the theory and showed that the gauge fields are connected to the spatially varying refractive index of the medium. In the geometric optics approximation, equation of motion for ray trajectory can be put in close correspondence to Newton’s second law. As a further step, we have used the perturbation theory in an illustrative example of a chiral medium, leading to the analogs of the two-level quantum system and Rabi oscillations. Within the framework developed here, we suggest that it is possible to make classical analog computation of quantum systems, hardly feasible in practice through the use of current technology. For instance, we pointed out the possible analogy between the qubits used in quantum computation and the polarization modes in a waveguide filled with chiral media. Finally, we have proposed a mechanism for electric current measurement in high-power systems using the OSHE in an anisotropic medium and introduced a phenomenological potential energy to study the OSHE induced by gravitational fields.

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