Convergence of traveltime power series for a layered VTI medium

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ABSTRACT

For a multiply reflected SH-wave or a multiply reflected and converted qP/qSV-wave in a layered VTI medium, traveltime and offset can be expressed by power series in horizontal slowness. As for a stack of isotropic layers, these can be used to express traveltime as a series of even powers of offset. The existence and behavior of this power series depend on the derivative of offset with respect to horizontal slowness, formally expressed as traveltime multiplied by NMO-velocity squared. When the NMO-velocity squared is different from zero, the power series always exists for sufficiently small offset. The NMO-velocity squared is always positive for SH-waves and for qP-waves in media with normal polarization, for which there are no triplications in traveltime as function of offset. For qSV-wave propagation in some layers, the NMO-velocity squared may be positive, zero, or negative. When it is positive, the power series exists, but there may be an off-axis triplication in traveltime. When the NMO-squared velocity is zero, the power series does not exist, and there is an incipient triplication on the vertical axis. When the NMO-velocity squared is negative, there is a triplication in traveltime on the vertical axis. The power series exists and represents the first-arrival traveltime branch. Numerical examples show that when the value of the NMO-velocity squared is small, the power series for a qSV-wave can be used only for very small values of offset. The series of traveltime squared in powers of offset always exists when the series for traveltime exists; therefore, it must have the same or larger region of convergence.

INTRODUCTION

To a large extent, routine seismic processing is based on traveltime expressions that are computed for a stack of homogeneous plane layers. The standard common-midpoint method uses the so-called normal moveout for P-waves in isotropic horizontally stratified media (Dix, 1955). The normal moveout is nothing else than the second-order Taylor expansion of traveltime squared in such media.

Many studies of P-wave isotropic power series of traveltime or traveltime squared in terms of offset are reported in the literature (Slotnick, 1959; Taner and Koehler, 1969; Brown, 1969; Al-Chalabi, 1973; Ursin, 1977; Hubral and Krey, 1980; Castle, 1994). All of these investigations assume the existence of some minimum radius of convergence. In other words, for sufficiently small offsets, the power series converges and represents the traveltime function under consideration. Papers that explicitly address the problem of the existence and estimation of the radius of convergence are much more rare, but proof of convergence can be found in Goldin (1986) and in Tygel (1994). A summary of these results is given in Appendix A.

More recent investigations have shown that, in many situations, better processing results can be obtained by considering anisotropy within the layers. The most popular case is that of transverse isotropy with a vertical symmetry axis, for which layers are referred to simply as VTI media (Thomsen, 1986; Tsvankin and Thomsen, 1994; Alkhalifah and Tsvankin, 1995; Alkhalifah, 1997; Stovas and Ursin, 2003; Ursin and Stovas, 2006). Similarly to the isotropic case, the VTI power series traveltime as a function of offset (truncated to its first two or three terms) is used for a variety of seismic processing purposes, including velocity analysis (Tsvankin and Thomsen, 1994; Alkhalifah, 1997), geometric-spreading correction (Ursin and Hokstad, 2003) and inversion (Ursin and Stovas, 2005). In particular, the first power-series coefficients are important to design alternative traveltime functions of offset. This is the case of nonhyperbolic traveltimes, shown to better approximate reflection events, especially for large offsets.
Ursin and Stovas (2006) have extended to VTI media the classical approach for isotropic media (Taner and Koehler, 1969; Hubral and Krey, 1980) of expressing traveltimess as a power series in a horizontal slowness and then using this to express traveltimess in even powers of offset. This derivation is valid for multiply reflected SH-waves or multiply reflected and converted qPqSV-waves, but no proof of convergence was given. They also did not discuss the problems which arise when there is a triplication in the qSV-traveltimess curve. Following widespread use, we adopt the simplifying terms horizontal slowness and vertical slowness to designate the horizontal and vertical components of the slowness vector, respectively. The horizontal slowness is also referred to as the ray parameter. In a VTI medium, the SH-wave has polarization which is normal to the polarization plane of the qPqSV-waves. The slowness surface is an ellipse, and the traveltimess approximations behave similarly to the isotropic case.

For the qP/qSV-waves there are two slowness surfaces. An inner surface representing the faster wave, normally a qP-wave, and an outer surface representing a slower wave, normally a qSV-wave. For certain values of the medium parameters, the polarization of the fast wave may change from qP to qSV (polarization normal to the slowness vector) depending on the direction of the slowness vector. This is called anomalous polarization by Helbig and Schoenberg (1987). See also the discussions in Dellinger (1991) and in Carcione (2001).

The inner slowness surface (qP-wave) is always convex (Dellinger, 1991) and there can be no triplication in the group-velocity surface. For the outer slowness surface (qSV-wave), there may be triplication in the group-velocity surface. Triplications can occur for qSV-waves propagating at three different angles with respect to the axis of symmetry: parallel to the symmetry axis, perpendicular to the symmetry axis, or off-axis, at an angle in between. For a reflected wave, only the on-axis and off-axis triplications can occur. Conditions on the elastic constants such that triplications for qSV-waves exist are given in Dellinger (1991) and in Thomsen and Dellinger (2003).

Triplications that can occur for qSV-wave propagation in VTI media represent a complication because the traveltimess then becomes a multivalued function of offset. It can not be fully represented by a convergent power series, which is able to describe single-valued functions only. In the same way, no truncated series approximation or polynomial can correctly describe a multivalued function. What actually happens is that the power series (or its truncation) approximates the traveltimess within the first branch (smaller traveltimes) of the triplication; at most, up to the offset where the traveltimess curve starts to backtrack.

For a VTI layered medium, we show that existence of a power-series expansion of traveltimess as a function of offset is guaranteed whenever the NMO velocity of the medium above the reflector is different from zero. Here we adopt the usual definition that the reciprocal of the NMO-velocity squared equals the zero-offset traveltimess multiplied by the second derivative of traveltimess with respect to offset, also at zero-offset. We have chosen to continue to use the term, NMO-velocity squared, even though in anisotropic media it may be positive, zero, or negative. In the case that the NMO-velocity squared is zero, the second derivative of traveltimess does not exist and, as a consequence, the traveltimess cannot be represented by a power series of offset. Only when there is a triplication on the vertical axis for qSV propagation in one or more layers, the NMO-velocity squared may become zero or negative. A vanishing NMO-velocity squared characterizes an incipient triplication on the vertical axis. Then the ray angle pauses in its forward motion but does not go backwards (backtrack) as slowness increases. The traveltimess curve, as a function of offset, has a discontinuity of the derivative at zero offset. In this case, no power series representation of traveltimess exists.

When the NMO-velocity squared is negative, a traveltimess triplication occurs on the vertical axis. In this situation, the ray angle becomes negative and then increases (backtrack) as horizontal slowness increases. The traveltimess as a function of offset becomes multi-valued. The power series of traveltimess exists, but represents the branch of traveltimess which decreases with offset (i.e., the branch with the first arrival). For a single VTI layer, the NMO-velocity squared for a qSV-reflected wave becomes negative when the parameter \( \sigma < -0.5 \) (Tsvankin and Thomsen, 1994). This is exactly the condition for a vertical, on-axis triplication given by Thomsen and Dellinger (2003).

From elementary results of complex function theory, it can be readily established that the power series of squared traveltimess exists whenever the power series of (nonsquared) traveltimess does. As a consequence, convergence properties of traveltimess-squared power series cannot be worse than corresponding ones for nonsquared traveltimess.

For a few simple models, accuracy of the traveltimess-squared approximations obtained with the truncated Taylor series is investigated for qSV-reflected waves near an incipient triplication point. Traveltimess approximations have a very small, useful offset range in this case, confirming the results of Ursin and Stovas (2006). All traveltimess approximations break down at an on-axis triplication of a qSV-wave.

**TRAVELTIME AND OFFSET FUNCTIONS OF HORIZONTAL SLOWNESS**

We consider wave propagation in a stack of horizontal homogeneous VTI layers. For a multiply reflected SH-wave or multiply reflected and converted qPqSV-wave, the traveltimess \( t \) and offset \( x \) are governed by the expressions of Ursin and Stovas (2006),

\[
t(p) = \sum_{k} \Delta z_k V_k \cos \alpha_k = \sum_{k} \Delta z_k V_k \cos \phi_k \left( 1 + \frac{p}{v_k^f} \right),
\]

and

\[
x(p) = \sum_{k} \Delta x_k \cos \beta_k = \sum_{k} \frac{v_k \Delta z_k}{\cos \phi_k} \left( p + \frac{v_k^f}{v_k^i} \right).
\]

The index \( k \) accounts for summation along the ray, so that a new term is added each time the wave passes through a layer. It may, thus, pass through the same layer several times. The quantities in the sums are computed for the proper wave mode: qP/qSV- or SH-wave. In these equations, for each layer \( k \) that is traversed by the ray, \( \Delta z_k \) represents the thickness, \( V_k \), and \( \phi_k \) the group and phase velocities, and \( \alpha_k \) and \( \beta_k \) are the group and phase angles, respectively. The group angle is the one the group velocity (namely, the ray) makes with the vertical. The phase angle is the one that the slowness vector (namely, the phase velocity or wavefront normal) makes with the vertical. Moreover, \( v_k^f \) denotes the derivative of \( v_k \) with respect to \( p \). Finally, we have used the invariance (Snell’s law) of the horizontal slowness or ray parameter, i.e.,
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**Dependence of medium parameters**

Omitting the index $k$, the vertical velocities for qP/qSV-waves, as a function of the elastic parameters in Voigt notation of the VTI medium, are given by

$$p = \sin \theta_k / v_k,$$

for all layers $k$, which is valid for a horizontally stratified medium.

The above expressions of traveltime and offset can be recast into a more convenient form by introducing the vertical slowness (i.e., the vertical component of the slowness vector)

$$q_k = \sqrt{1/v_k^2 - p^2} = \cos \theta_k / v_k,$$

where we assume the condition of precritical propagation $p^2 < 1/v_i^2$.

Substituting into equations 1 and 2 yields

$$t(p) = \sum_k \Delta z_k (q_k - p q_k^i) \quad \text{and} \quad x(p) = - \sum_k \Delta z_k q_k^i,$$

where $q_i = dq_i / dp$, and we have used

$$\frac{v_i}{v_k} = - \left( \frac{1}{2} \frac{1}{v_i} \right)^{1/2} = -(q_k q_k^i + p).$$

It is seen that the vertical slowness $q_i(p)$ determines the behavior of $t(p)$ and $x(p)$.

For qP/qSV-waves, the vertical slowness squared gets more complicated, being given by (see, e.g., Ursin and Stovas, 2006, equation B-1)

$$q^2 = \frac{1}{2} \left( \frac{1}{\alpha_0} + \frac{1}{\beta_0} \right) - p^2(1 + \sigma + \delta),$$

$$\pm \frac{1}{2} \left( \frac{1}{\beta_0} - \frac{1}{\alpha_0} \right) \sqrt{1 + bp^2 + cp^4},$$

with

$$b = \frac{4 \alpha_0^2 \beta_0^4}{\alpha_0^2 - \beta_0^2}(\sigma - \delta) \quad \text{and} \quad c = \frac{4 \alpha_0^2 \beta_0^4}{(\alpha_0^2 - \beta_0^2)^2} \left[ 2(\alpha_0^2 - \beta_0^2) \sigma + (\sigma + \delta)^2 \right].$$

In equation 13 the $-$ sign is normally associated with the qP-wave and the $+$ sign is associated with the qSV-wave.

Substitution of the vertical slowness (positive square roots) of equations 12 or 13, together with their derivatives with respect to $p$ into equation 5 provides the desired expressions for the traveltime $t(p)$ and offset $x(p)$ as a function of the VTI medium parameters.

**Power-series representations**

We want to represent the traveltime and offset functions as Taylor power series of $p$ centered at $p = 0$. To study their convergence properties, we need to treat them as complex functions of a complex variable $p$. These representations are valid within a disc, $|p| < p_m$, of the complex $p$-plane whenever $t(p)$ and $x(p)$ are analytic functions in that region (see, e.g., Churchill, 1960, Chapter 6). From equation 5, it is sufficient to show the analyticity of vertical slowness function, $q = q(p)$.

For the simple situation of an SH-wave (see equation 12), the proof of analyticity follows closely the isotropic case: We have two singularities, located at the points $p_{1,2} = \pm 1/(\beta_0^2 + 2\gamma)$. We recall that the zeros of a square-root function are singularities (points of no analyticity) in the complex plane. As singular points, these zeros must be away from the domain of analyticity of the function. We observe that, in our case, the singular points of the vertical slowness are the horizontal slownesses for which we have a horizontally propagating wave. As seen from equations 8 and 10, we have $1 + 2\gamma = c_{66}/c_{44} > 0$, so that $p_{1,2}$ are well-defined real numbers. Thus, the function $q(p)$ is analytic within the disc $|p| < 1/(\beta_0^2 + 2\gamma)$.

For the more complicated case of a qP/qSV-wave (see equation 13), the singularities of $q(p)$ are the zeros of the inner square root function $\sqrt{1 + bp^2 + cp^4}$, and the zeros of the function $q(p)$ itself (this is because $q(p)$ is also a square root function). As a consequence, $q(p)$ is a well-defined and analytic function within the disc $|p| < p_m$, where $p_m$ is the smallest of the distances of its singularities with respect to the origin $p = 0$. Note that, because $g(0)$ and $g(0)$ are (real) nonzero numbers, we find that $p_m = 0$ is not a singularity. This implies that $p_m > 0$. We now need to find a convergence disc that is common to all single-layer type functions that comprise the sums that define $t(p)$ and $x(p)$ in equation 5.

Setting $p_m$ to be the minimum of all single-layer radii $p_m = p_m^{(i)}$, we find that $t(p)$ and $x(p)$ are analytic at least within the disc $|p| < p_m$. 

This completes the proof that the traveltime and offset are analytic functions of horizontal slowness within some disc of positive radius.

We note that a more precise estimation of the radii \( p_r \), which could be provided by an analysis of the vertical slowness, is actually not needed for the purposes of this paper. It suffices to show the existence of some disc \( |p| < p_r \), where \( q(p) \) is analytic, and this has been established above.

Expanding \( q_d(p) \) in Taylor series and interchanging the order of summation gives (Ursin and Stovas, 2006)

\[
t(t) = t(0) + \frac{1}{2} t(0) v_{NMO}^2 p^2 + \frac{3}{8} t(0) \mu_4 p^4 + \cdots ,
\]

(15)

and

\[
x(t) = t(0) v_{NMO}^2 p + \frac{1}{2} t(0) \mu_4 p^3 + \cdots ,
\]

(16)

where

\[
t(0) = \sum_k \Delta t_{0,k} = \sum_k \frac{\Delta z_k}{v_{0,k}},
\]

(17)

and

\[
v_{NMO}^2 = \frac{1}{t(0)} \sum_k v_{0,k}^2 a_{0,k} \Delta t_{0,k} = \frac{1}{t(0)} \sum_k v_{0,k} a_{0,k} \Delta z_k,
\]

(18)

are the vertical traveltime and NMO-velocity squared, respectively. In these equations, \( \Delta t_{0,k} \) and \( v_{0,k} \) are the vertical traveltime and velocity in layer \( k \).

Omitting the index \( k \), the coefficient \( a_0 \), defining the NMO-velocity squared in equation 18, is given by

\[
a_0 = \begin{cases} 
1 + 2\delta & \text{(qP – wave)} \\
1 + 2\sigma & \text{(qSV – wave)} \\
1 + 2\gamma & \text{(SH – wave)}.
\end{cases}
\]

(19)

Finally, the coefficient \( \mu_4 \) is defined as Ursin and Stovas (2006),

\[
\mu_4 = \frac{1}{t_0} \sum_k v_{0,k}^4 a_{0,k} [a_{0,k}^2 + 4a_{1,k}],
\]

(20)

where, also omitting the index \( k \), the coefficient \( a_1 \) reads

\[
a_1 = \begin{cases} 
2(\epsilon - \delta) \left( 1 + \frac{2\delta a_0^2}{a_0^2 - \beta_0^2} \right) & \text{(qP – wave)} \\
- 2\delta \left( 1 + \frac{2\delta a_0^2}{a_0^2 - \beta_0^2} \right) & \text{(qSV – wave)} \\
0 & \text{(SH – wave)}.
\end{cases}
\]

(21)

From the theory of complex variables (see, e.g., Churchill, 1960, p. 181), \( x(p) \) has an analytic inverse \( p(x) \) in the neighborhood of \( x(p = 0) = 0 \) whenever \( x(p) \) has a nonvanishing derivative at \( p = 0 \). From equations 16 and 18, this condition reads

\[
x'(0) = t(0) v_{NMO}^2 = \sum_k v_{0,k} a_{0,k} \Delta z_k \neq 0,
\]

(22)

where \( a_{0,k} \) is defined by equation 19. Since \( t(0) \neq 0 \), the above condition is equivalent to the condition \( v_{NMO}^2 \neq 0 \). In the next section, we show that \( a_{0,k} > 0 \) for SH and qP propagation in layer \( k \). As a consequence, condition 22 may only fail when qSV propagation occurs at least in one layer.

We have seen that traveltime \( t(p) \) and offset \( x(p) \) are analytic functions of horizontal slowness within a disc \( |p| < p_r \). Furthermore, when \( x'(0) = t(0) v_{NMO}^2 = 0 \), the function \( x(p) \) admits an analytic inverse \( p(x) \) that is defined within some small disc \( |x| < x_0 \). Here, \( x_0 \) denotes the (unknown) positive radius of that disc. It can be observed that we have only shown that the inverse function \( p(x) \) maps the disc \( |x| < x_0 \) onto some domain within the disc \( |p| < p_r \). The actual shape of that domain is not known.

Substituting \( p = p(x) \) into the traveltime function \( t(p) \), we obtain the composite function \( T(x) = t(p(x)) \). The different notations \( t(p) \) and \( T(x) \) express the fact that traveltime, seen as a dependent variable, is given by different functional relationships that act on different independent variables, namely horizontal slowness and offset, respectively. Due to the analyticity of both \( t(p) \) and \( p(x) \), as well as to the fact that, by definition, \( p(x) \) maps the disc \( |x| < x_0 \) into the domain of analyticity \( |p| < p_r \) of \( t(p) \), it follows that \( T(x) \) is a well-defined, analytic function within \( |x| < x_0 \). As a consequence, \( T(x) \) can then be represented by a convergent power series for offsets \( |x| < x_0 \).

We consider two waves, with the same ray code, starting at the same source point and going in two opposite directions. Because of azimuthal symmetry, they have the same traveltime when the distances between the source and receivers are equal. Therefore, \( T(-x) = T(x) \) and the power series of \( T(x) \) with respect to \( x = 0 \) has only even powers of \( x \). From equations 15 and 16 it can be shown that

\[
T(x) = T(0) + \frac{x^2}{2T(0)v_{NMO}^2} - \frac{\mu_4 x^4}{8T(0)^3(v_{NMO}^2)^4} + \cdots
\]

(23)

Note that \( T(0) = t(0) \) is the zero-offset traveltime. Higher-order terms can be found in Ursin and Stovas (2006). Inspection of equation 23 readily shows that, if \( v_{NMO}^2 = 0 \), the power series \( T(x) \) does not exist. This is expected since condition 22 is violated.

## TRIPLICATIONS ON THE VERTICAL AXIS

The previous considerations show that the behavior of \( T(x) \) in the vicinity of \( x = 0 \) varies dramatically whenever the NMO-velocity squared is positive or negative. When \( v_{NMO}^2 > 0 \), we have the more intuitive situation in which traveltime and offset increase when horizontal slowness increases from zero. As a consequence, the traveltime \( T(x) \) increases when offset increases from zero. This situation is illustrated by the models C and D of Figure 1. When \( v_{NMO}^2 < 0 \), traveltime decreases and offset becomes negative when horizontal slowness increases from zero. The traveltime function \( T(x) \) experiences a triplication on the vertical axis. An illustration of this is provided by the models A and B of Figure 1.
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Single-layer case

For a reflection from a single layer with no mode conversions, equation 18 reduces to the simple expression

$$v_{NMO}^2 = v_{0}^2,$$

(24)

where \(v_0\) is the vertical velocity and \(a_0\) is given by equation 19. As \(v_0^2\) is always positive, the quantity \(a_0\) controls the sign of \(v_{NMO}^2\). With the help of equations 8 and 19, we can express \(a_0\) directly as a function of the VTI elastic parameters. After some simple algebra, we find

$$a_0 = \begin{cases} 1 + 2 \delta = \frac{c_{44}(c_{33} - c_{44}) + (c_{13} + c_{44})^2}{c_{33}(c_{33} - c_{44})} & \text{(qP - wave)} \\ 1 + 2 \sigma = \frac{c_{13}(c_{33} - c_{44}) - (c_{13} + c_{44})^2}{c_{44}(c_{33} - c_{44})} & \text{(qSV - wave)} \\ 1 + 2 \gamma = \frac{c_{44}}{c_{33}} & \text{(SH - wave)} \end{cases}$$

(25)

From the energy constraints 10, we readily see that, for the SH-wave, \(a_0 = c_{44}/c_{44} > 0\). Hence, the NMO-velocity squared is always positive for the SH-wave. Under the condition \(c_{33} > c_{44}\), which is included in the set of constraints 11 that defines a normally polarized VTI medium (Thomsen and Delleringer, 2003), we readily see that \(a_0 > 0\) for the qP-wave. This shows that, within a homogeneous, normally polarized VTI medium, the NMO-velocity squared is always positive for a qP-wave.

For a reflected qSV-wave in a single-layer VTI medium, there are no restrictions for the sign of the NMO-velocity squared. In particular, the NMO-velocity squared is zero for \(\sigma = -0.5\). In this situation, equation 25 provides the relationship

$$(c_{13} + c_{44})^2 = c_{11}(c_{33} - c_{44}),$$

(26)

referred to as the condition for incipient on-axis triplication (Delleringer, 1991; Thomsen and Delleringer, 2003). Then the power series does not exist. For \(\sigma < -0.5\) there is a triplication on the vertical axis. We have

$$(c_{13} + c_{44})^2 - c_{11}(c_{33} - c_{44}) > 0,$$

(27)

which is the condition given in Thomsen and Delleringer (2003, equation 6). In this situation, the power series exists, but the range of convergence must be less than the distance between the offsets where the traveltime starts to backtrack. Referring to Figure 1, these are the positive and negative offsets where the traveltime curve has a cusp. In fact, the Taylor expansion of traveltime in offset only describes the traveltime branch representing the first arrival.

Multilayer case

For a multiply reflected SH-wave or a multiply reflected and converted qP/qSV-wave, we have at \(p = 0\),

$$x(0) = 0 \text{ and } x'(0) = t(0)v_{NMO}^2 = t''(0),$$

(28)

where the derivatives are taken with respect to \(p\). When \(p\) is increasing from zero and approaches the inverse of the maximum horizontal velocity, the offset will become very large since the wave propagation is almost horizontal. This means that the total offset will be large positive. From equation 18 and the previous discussion, it is seen that the contribution to \(v_{NMO}^2\) in equation 18 for the SH- or qP-wave is always positive, but the contribution for the qSV-wave may be positive, zero, or negative.

For small values of horizontal slowness there are three cases:

1) \(v_{NMO}^2 > 0\): Both \(x(p)\) and \(t(p)\) increase when \(p\) increases from zero, and there is no triplication. This is always the case for SH- and qP-waves. If there is qSV-wave propagation in one or more layers, there may still be an off-axis triplication.

2) \(v_{NMO}^2 = 0\): This is the case of on-axis incipient triplication. The traveltime curve as a function of offset has a discontinuity of the derivative at zero offset, and no power-series representation exists. It is necessary that qSV-wave propagation occurs in at least one layer.

3) \(v_{NMO}^2 < 0\): Both \(x(p)\) and \(t(p)\) increase when \(p\) increases from zero and there is a triplication on the vertical axis. Offset initially becomes negative before the traveltime curve starts to increase and later become positive. Also here it is necessary that qSV-wave propagation occurs in at least one layer.

The radius of convergence must be limited by the offset where traveltime starts to backtrack. Both for vertical on-axis and off-axis triplication this occurs when

$$x'(p) = -\sum_k \Delta x_k q_k''(p) = 0,$$

(29)

where equation 5 has been used.

The coefficients in the power series for \(T(x)\) in equation 23 become large when \(v_{NMO}^2\) is small (positive or negative), and we expect that the radius of convergence will be small.

**TRAVELTIME SQUARED**

The power series for traveltime squared can also be obtained from equations 15 and 16 (Taner and Koehler, 1969; Hubral and Krey, 1980; and Ursin and Stovas, 2006). In order to obtain the first few terms, it is easier to square equation 23 which gives

$$T(x)^2 = T(0)^2 + \frac{x^2}{v_{NMO}^2} \left[ \frac{\mu_4}{(v_{NMO}^2)^2} - 1 \right] \frac{x^4}{4T(0)^2v_{NMO}^4} + \cdots.$$
For two-terms power series, nonsquared traveltime is approximated by a parabolic function, and traveltime squared is approximated by a hyperbolic function. For a stack of isotropic layers, Ursin (1977) has shown that the standard hyperbolic approximation differs less from the exact traveltime than the corresponding parabolic approximation. Numerical evidence suggests that this is the case also for VTI layers, but no studies that support this conjecture seems to be available.

**NUMERICAL RESULTS**

The behavior of the functions \( x(p), t(p), \) and \( T(x) \), as well the accuracy of the truncated Taylor series for \( T(x) \), are illustrated for reflected qSV-waves in simple two-layer models. The first layer consists of medium I with negative squared NMO-velocity. The second layer consists of medium II with positive squared NMO-velocity. The elastic parameters of the two media, as given in Table 1, are identical except for the slightly different values of \( \epsilon \). Five different models, all with total thickness equal to 1000 m, are described in Table 2. Model A is medium I with \( v_{NMO}^2 = -0.2 \), model B is 750 m of medium I over 250 m of medium II with \( v_{NMO}^2 = -0.1 \), model C is 500 m of medium I over 500 m of medium II with \( v_{NMO}^2 = 0 \), model D is 250 m of medium I over 750 m medium II with \( v_{NMO}^2 = 0.1 \), and model E is medium II with \( v_{NMO}^2 = 0.2 \).

Figure 1 shows the functions \( x(p), t(p), \) and \( T(x) \) for all models. We clearly see the different behavior for negative, zero, and positive NMO-velocity squared. Triplication and backtracking of the traveltime curve occurs for the traveled times of models A and B in Figure 1. The traveltime curve of model A has the largest triplication domain. The middle curve refers to model C, in which \( v_{NMO}^2 = 0 \) and the power-series expansion does not exist. This results in a discontinuity of the derivative of the traveltime curve at zero offset. For models D and E there is no triplication in the traveltime curves.

Figure 2 shows the errors of nonsquared traveltime approximation \( T(x) \) in equation 23 truncated after two and three terms. As expected, these approximations are only valid for offsets less than where the traveltime curve starts to backtrack (at \( dx/dp = 0 \)). The approximations obtained by taking the square root of the truncated series for \( T(x)^2 \) in equation 30 are extremely similar, and the plots look almost identical to the ones in Figure 2. It is seen that the range of validity of the approximations for \( T(x) \) for qSV-reflection is very limited when \( |v_{NMO}^2| \) is small. For a reflector at

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<th>Table 2. Model composition and traveltime parameters.</th>
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</thead>
<tbody>
<tr>
<td>Models</td>
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<tr>
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<tr>
<td>Medium I (m)</td>
</tr>
<tr>
<td>Medium II (m)</td>
</tr>
<tr>
<td>( v_{NMO}^2 ) (km²/s²)</td>
</tr>
<tr>
<td>( \mu_4 ) (km³/s⁴)</td>
</tr>
</tbody>
</table>
time and offset are expressed as parametric functions of horizontal traveltime squared power series around the origin for sufficient second-order and fourth-order approximations computed for a qSV-wave in models A, B, D, and E. The power-series expansion for model C does not exist. The difference now is that the velocity \( v_x \) does not depend on \( p \). Then \( q_k = -p/q_x \), which, after substitution into equations 5, leads to

\[
\frac{\Delta z_k}{v_x^2 q_k} \quad \text{and} \quad x(p) = p \sum_k \frac{\Delta z_k}{q_k}. 
\]

The function \( q_k(p) \) is analytic when \( |p| < 1/v_x \), so that \( t(p) \) and \( x(p) \) are analytic in the disc

\[
|p| < 1/v_{\text{max}},
\]

where \( v_{\text{max}} \) is the maximum velocity encountered along the ray. As before, \( x(p) \) has an analytic inverse in the vicinity of the origin \( x(0) = 0 \) when

\[
x'(0) = t(0) v_x^2 = \sum_k v_x \Delta z_k \neq 0,
\]

which is trivially the case. This concludes the proof of existence of a power series \( T(x) \) with a certain radius of convergence \( x \), around \( x = 0 \). Goldin (1986) has obtained an upper bound, and Tygel (1994) has obtained a lower bound for this radius. Their results can be combined in the form

\[
\sum_k \frac{\Delta z_k}{\sqrt{1 + (v_{\text{max}}/v_k)^2}} \leq x \leq \sum_k \Delta z_k.
\]
For a nonconverted reflected wave on a single-layer model, the exact traveltime is given in the closed form

\[ T(x) = \frac{2\Delta z}{v} \sqrt{1 + \left( \frac{x}{2\Delta z} \right)^2}, \]  

(A-6)

where \( v \) and \( \Delta z \) are the velocity and thickness of the single layer. The square root can be expanded in a convergent power series when \( |x| < 2\Delta z \). Taking into account that the single layer is specified by \( k = 1, 2 \) and \( \Delta z_1 = \Delta z_2 = \Delta z \), this corresponds exactly to the upper bound, as predicted by the inequality A-5.

Squaring both sides of equation A-6, we see that \( T^2(x) \) is exactly expressed as a second-order polynomial, namely a power series that converges for all offsets. The significant increase of the convergence radius that is observed in the single-layer situation when \( T(x) \) is replaced by \( T^2(x) \) leads one to conjecture that this should be the case also for multilayered models. So far, no proof of this conjecture has been given.

REFERENCES


Hubral, P., and T. Krey, 1980, Interval velocities from seismic reflection time measurements: SEG.

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