DOI: 10.1002/zamm.201700251

Received: 31 August 2017 | Revised: 27 September 2018 | Accepted: 12 November 2018
DOI: 10.1002/zamm.201700251

ORIGINAL PAPER

ZAMM

Delta shock waves for a system of Keyfitz-Kranzer type

Richard De la cruz[1] | Marcelo Santos[2]

[1]School of Mathematics and Statistics, Universidad Pedagógica y Tecnológica de Colombia, Av. Central del Norte 39-115, Tunja, Boyacá, Colombia

[2]Institute of Mathematics, Statistics and Scientific Computing – IMECC, State University of Campinas – UNICAMP, Rua Sérgio Buarque de Holanda 651, Campinas, SP, Brazil

Correspondence
Richard De la cruz, School of Mathematics and Statistics, Universidad Pedagógica y Tecnológica de Colombia, Av. Central del Norte 39-115, Tunja, Boyacá, Colombia.
Email: richard.delacruz@uptc.edu.co

Funding information
Fundação de Amparo à Pesquisa do Estado de São Paulo, Grant/Award Number: 2016/19502-4

This paper is concerned with a hyperbolic system of conservation laws of Keyfitz-Kranzer type. We show the existence of a delta shock wave solution using the vanishing viscosity method. The obtained solution satisfies the generalized Rankine-Hugoniot relation and the entropy condition. As a consequence, we have also a uniqueness result for the obtained solution.

KEYWORDS
delta shock wave, keyfitz–kranzer type, vanishing viscosity method

1 | INTRODUCTION

Many natural phenomena are modeled by systems of Keyfitz-Kranzer type

$$\begin{cases} \rho_t + (\rho\phi(\rho,u))_x = 0, \\ (\rho u)_t + (\rho u\phi(\rho,u))_x = 0, \end{cases} \tag{KK}$$

where $\phi(\rho,u) = f(u) - P(\rho)$. For example, when $f(u) = u$ the system (KK) was introduced as a macroscopic model for traffic flow by Aw and Rascle[1] where ρ and u are, respectively, the density and the velocity of cars on a roadway, and P is a smooth and strictly increasing function satisfying

$$\rho P''(\rho) + 2P'(\rho) > 0 \text{ for } \rho > 0.$$

Another important study on the system (KK) is due to Blake Temple. In 10, he consider the case when $\rho\phi(\rho,u)$ is not a convex function (in this case, the system describes how the addition of a polymer affects the flow of water and oil in a reservoir) and prove the existence of a global weak solution to the Cauchy problem. Lu[8] showed the existence of a global weak solution of the Cauchy problem for (KK) when f is a non negative convex function and P satisfies the following condition:

$$P(\rho) \le 0 \text{ for } \rho > 0,\ P(0) = 0, \lim_{\rho\to 0} \rho P'(\rho) = 0, \lim_{\rho\to\infty} P(\rho) = \infty,$$

$$\rho P''(\rho) + 2P'(\rho) < 0 \quad \text{for } \rho > 0.$$

Using delta-shock waves, H. Cheng[3] solved the Riemann problem to the non symmetric system of Keyfitz-Kranzer type (KK) when $P(\rho)$ is the function $P(\rho) = -\frac{1}{\rho}$, $\rho > 0$, and $f(u)$ is the indentity function, i.e. $f(u) = u$ for all $u \in \mathbb{R}$. In this case, P satisfies the condition $\rho P''(\rho) + 2P'(\rho) = 0$ and it is not possible to solve the Riemann problem to (KK) using only classical waves. In this paper, we generalize the result of 3 for an arbitrary strictly increasing function $f(u)$. More precisely, we solve the Riemann problem for the following system,

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + \left(\rho u \left(f(u) + \frac{1}{\rho}\right)\right)_x = 0, \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R})$, with $f'(u) > 0$ for all $u \in \mathbb{R}$, and $\rho > 0$.

The concept of delta shock wave is a generalization of a classical shock wave. This generalization was introdduced by Korchinski in the year of 1977 in his PhD thesis.[6] Motivated by some numerical results, he constructed the unique Riemann solution using generalized delta functions to obtain singular shocks in the sense of distributions. In 1994, Tan, Zhang and Zheng established in 9 the existence, uniqueness and stability of delta shock waves for a viscous perturbation of the system studied by Korchinski. Other works dealing with delta shock waves are due to Ercole[5] who in 2000 obtained a delta shock solution as a limit of smooth solutions by the vanishing viscosity method. In 2005, Brenier[2] considered the Riemann problem for the Chaplygin gas system and he showed the existence of solutions with concentration. Some more recent works on delta shocks for general hyperbolic conservation laws are due to Danilov and Shelkovich,[4] where they described delta shock wave generation from continuous initial data by using smooth approximations in the weak sense.

Using the vanishing viscosity method, and following works by Tan, Zhang and Zheng,[9] Li, Zhang and Yang,[7] and Yang,[11] we show the existence of a delta shock wave solution for the system (1). The main difficulty in applying the vanishing viscosity method is to choose a suitable Banach space and a bounded convex closed subset to use the Schauder fixed point theorem and, thus, obtain the solution of the viscous system (7). Passing to the limit when the viscosity tends to zero in solutions of the system (7), we obtain the existence of a delta shock wave for the system (1). Moreover, adding an entropy condition, we establish the uniqueness of our solution.

The remainder of this paper is organized as follows. In Section 2, we present the classical solutions and a short description of the delta shock wave solution for the Riemann problem for the system of Keyfitz-Kranzer type (1). In Section 3, we solve the viscous system (7). Also, in this Section, we obtain the existence and uniqueness of a solution for the system (1) involving delta shock waves.

2 | RIEMANN PROBLEM

In this section, we consider the system (1) with an initial data

$$(\rho(x,0), u(x,0)) = \begin{cases} (\rho_-, u_-), & \text{if } x < 0 \\ (\rho_+, u_+), & \text{if } x > 0 \end{cases} \tag{2}$$

for arbitrary constant states $(\rho_\pm, u_\pm)$.

The eigenvalues of the system (1), with respective right eigenvectors, are given by

$$\lambda_1(\rho, u) = f(u),\ \mathbf{r}_1 = (1, 0) \quad \text{and} \quad \lambda_2(\rho, u) = f(u) + \frac{1}{\rho},\ \mathbf{r}_2 = (f'(u), 1/\rho^2).$$

Notice that both characteristics are linearly degenerate and thus the system is in the Temple class. More specifically, shock wave curves coincide with rarefaction wave curves in the phase plane (ρ, u), corresponding to the so called contact discontinuities in the (x, t) plane. Since our system is a 2×2 system, we have two families of such curves in the (ρ, u) plane. In our case, these families through a point (ρ_-, u_-) are given by

Family 1: it is the vertical line $u = u_-$, $\rho > 0$;

Family 2: it consists of all (ρ, u) such that $f(u) + \frac{1}{\rho} = f(u_+) + \frac{1}{\rho_+}$, $\rho > 0$.

We shall denote these families, respectively, by $J_1 \equiv J_1(\rho_-, u_-)$ and $J_2 \equiv J_2(\rho_-, u_-)$. It is a straightforward computation to verify that the shock and rarefactions wave curves for the system (1) through a point (ρ_-, u_-), which coincide, are given by J_1 and J_2, and so we omit this computation here.

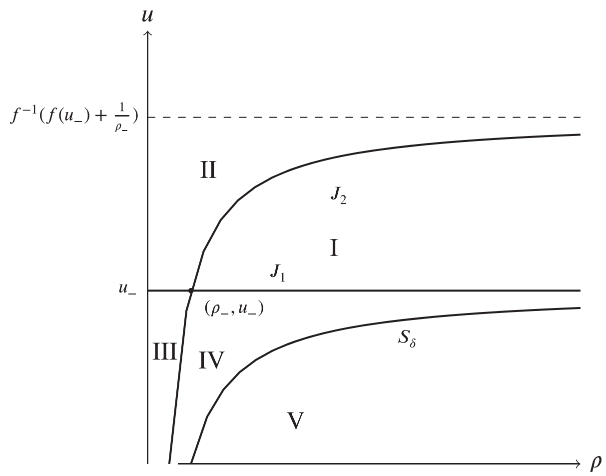

FIGURE 1 Phase plane $\rho - u$. Regions I, II, III, IV and V. J_1, J_2: contact discontinuites; J_2: $f(u)+\frac{1}{\rho}=f(u_-)+\frac{1}{\rho_-}$. S_δ: $f(u)+\frac{1}{\rho}=f(u_-)$

2.1 | Classical Riemann problem

Consider the left constant state (ρ_-, u_-). The phase plane is divided into five regions as follows (see Figure 1)

- $I = \{(\rho,u) : u_- < u < f^{-1}(f(u_-)+\frac{1}{\rho_-}), \frac{1}{f(u_-)+\frac{1}{\rho_-}-f(u)} < \rho < \infty\}$
- $\mathrm{II} = \{(\rho,u) : u_- < u < f^{-1}(f(u_-)+\frac{1}{\rho_-}), 0 < \rho < \frac{1}{f(u_-)+\frac{1}{\rho_-}-f(u)}\} \cup \{(\rho,u) : f^{-1}(f(u_-)+\frac{1}{\rho_-}) \leq u < \infty, 0 < \rho < \infty\}$
- $\mathrm{III} = \{(\rho,u) : -\infty < u < u_-, 0 < \rho < \frac{1}{f(u_-)+\frac{1}{\rho_-}-f(u)}\}$
- $\mathrm{IV} = \{(\rho,u) : -\infty < u < u_-, \frac{1}{f(u_-)+\frac{1}{\rho_-}-f(u)} < \rho < \frac{1}{f(u_-)-f(u)}\}$
- $\mathrm{V} = \{(\rho,u) : -\infty < u < u_-, \frac{1}{f(u_-)-f(u)} < \rho < \infty\}$

For fixed (ρ_-, u_-), we consider the family of curves

$$\mathcal{F} = \{J_2(\rho_*, u_*) : (\rho_*, u_*) \in J_1(\rho_-, u_-)\}.$$

Then the solution to the Riemann problem is given when we connect (ρ_*, u_*) to (ρ_-, u_-) by a 1-contact discontinuity, denoted by J_1, and we connect (ρ_+, u_+) to (ρ_*, u_*) by a 2-contact discontinuity, J_2. When (ρ_+, u_+) belongs to I,II, III or IV, one can get the solution consisting of two different (or just one) contact discontinuities (see Figure 2). The intermediate state (ρ_*, u_*) connecting two contact discontinuities satisfies

$$u_* = u_- \quad \text{and} \quad f(u_*) + \frac{1}{\rho_*} = f(u_+) + \frac{1}{\rho_+}.$$

In this case we have that $f(u_-) < f(u_+) + \frac{1}{\rho_+}$ and the solution to the Riemann problem (1)–(2) is given by

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & \text{if } x < \lambda_1(\rho_-, u_-)t, \\ (\rho_*, u_*), & \text{if } \lambda_1(\rho_-, u_-)t < x < \lambda_2(\rho_+, u_+)t, \\ (\rho_+, u_+), & \text{if } x > \lambda_2(\rho_+, u_+)t, \end{cases}$$

where

$$u_* = u_- \quad \text{and} \quad f(u_*) + \frac{1}{\rho_*} = f(u_+) + \frac{1}{\rho_+}.$$

We shall denote the common boundary between regions IV and V by S_δ. The reason for this notation is that for right states (ρ_+, u_+) in region V we cannot connect (ρ_-, u_-) and (ρ_+, u_+) by the classical shock waves J_1 and J_2. Thus, it is for right states

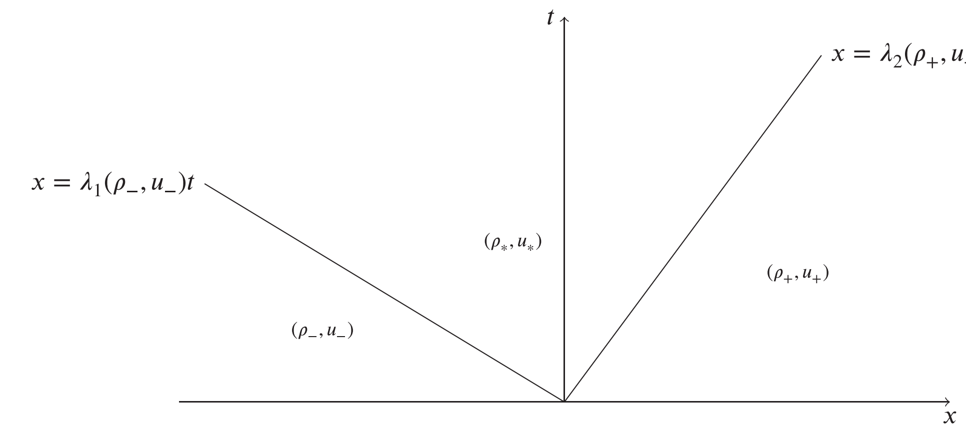

FIGURE 2 $\lambda_1(\rho_-, u_-) < \lambda_2(\rho_+, u_+)$ and Riemann solution involving two contact discontinuities

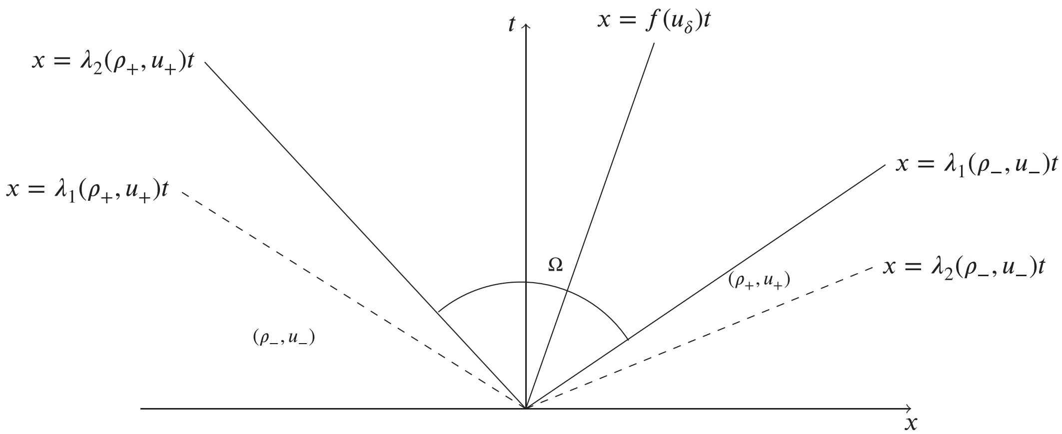

FIGURE 3 Characteristic analysis of the delta shock wave

in region V that we must use delta shock waves to solve the Riemman problem for the system (1). Notice that the curve S_δ has the line $u = u_-$ as its asymptotic line. When $(\rho_+, u_+) \in V$, we have that $f(u_+) + \frac{1}{\rho_+} = \lambda_2(\rho_+, u_+) \leq \lambda_1(\rho_-, u_-) = f(u_-)$. The characteristic lines from initial data will overlap in a domain $\Omega = \{(x, t) \; : \; \lambda_2(\rho_+, u_+)t \leq x \leq \lambda_2(\rho_+, u_+)t, t > 0\}$ shown in Figure 3. So singularity must happen in Ω.

Suppose $(\rho_+, u_+) \in V$. Let (ρ_*, u_*) be any point on $J_1(\rho_-, u_-)$. For the point (ρ_+, u') on $J_2(\rho_*, u_*)$, we have

$$f(u') + \frac{1}{\rho_+} = f(u_*) + \frac{1}{\rho_*} = f(u_-) + \frac{1}{\rho_*} \geq f(u_+) + \frac{1}{\rho_+} + \frac{1}{\rho_*}.$$

Thus,

$$f(u') \geq f(u_+) + \frac{1}{\rho_*}, \quad (\rho_* > 0),$$

and (ρ_+, u_+) cannot lie on any curve in $\mathcal{F}$, the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot relation is not satisfied on the bounded jump.

2.2 | Delta shock solution

In this section, we discuss the case when $\lambda_1(\rho_-, u_-) \geq \lambda_2(\rho_+, u_+)$. If $\lambda_1(\rho_-, u_-) \geq \lambda_2(\rho_+, u_+)$, then $(\rho_+, u_+) \in V$. The characteristic lines from the initial data will overlap in a domain $\Omega = \{(x, t) \; : \; f(u_+) + \frac{1}{\rho_+} \leq x/t \leq f(u_-), t > 0\}$. Hence, the singularity will develop in Ω, while this singularity cannot be a jump with a finite amplitude. To analyze the singularity in Ω for the special case $\lambda_2(\rho_+, u_+) = \lambda_1(\rho_-, u_-)$, let us consider the limit of the solution $\rho(\xi)$ and $u(\xi)$ when ρ_-, u_- and ρ_+ are fixed,

$u_+ \to f^{-1}(f(u_-) - \frac{1}{\rho_+}) + 0$ and the right state (ρ_+, u_+) belongs to IV (in this situation the solution is given by $J_1 + J_2$). The intermediate state (ρ_*, u_*) determined by the intersection point of J_1 and J_2 satisfies

$$f(u_-) = f(u_*) \text{ and } f(u_+) + \frac{1}{\rho_+} = f(u_*) + \frac{1}{\rho_*}.$$

Notice that defining $a = f^{-1}(f(u_-) - \frac{1}{\rho_+})$, we have that

$$\lim_{u_+ \to a+0} \rho_* = \infty \text{ and } \lim_{u_+ \to a+0} u_* = u_- < \infty.$$

Now let us calculate the total quantities of ρ and u between J_1 and J_2 as ρ_-, u_- and ρ_+ are fixed, $u_+ \to f^{-1}(f(u_-) - \frac{1}{\rho_+}) + 0$. From the first Equation of 1, with $\xi = x/t$, it follows that

$$0 = \int_{\xi=f(u_-)-0}^{\xi=f(u_+)+\frac{1}{\rho_+}+0} (-\xi d\rho + d(\rho f(u))) = (-\xi\rho)\Big|_{\xi=f(u_-)-0}^{\xi=f(u_+)+\frac{1}{\rho_+}+0} + \int_{\xi=f(u_-)-0}^{\xi=f(u_+)+\frac{1}{\rho_+}+0} \rho d\xi + (\rho f(u))\Big|_{\xi=f(u_-)-0}^{\xi=f(u_+)+\frac{1}{\rho_+}+0}$$

so

$$\lim_{u_+ \to a+0} \int_{\xi=f(u_-)-0}^{\xi=f(u_+)+\frac{1}{\rho_+}+0} \rho(\xi) d\xi = \int_{f(u_-)-0}^{f(u_-)+0} \rho(\xi) d\xi \neq 0.$$

The last equality shows that $\rho(\xi)$ has the same singularity as a weighted Dirac delta function at $\xi = f(u_-)$. We shall call this type of nonlinear hyperbolic waves a *delta shock wave*. For the case $\lambda_2(\rho_+, u_+) < \lambda_1(\rho_-, u_-)$, we suggest that the solution of the Riemann problem is also a delta shock wave defined by the speed σ satisfying

$$\lambda_2(\rho_+, u_+) \le \sigma \le \lambda_1(\rho_-, u_-). \tag{3}$$

Next, we recall the following definition:

Definition 1. A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(x(s), t(s)) \,:\, a < s < b\}$, for $w \in L^1((a, b))$, is defined as

$$\langle w(\cdot)\delta_L, \phi(\cdot, \cdot)\rangle = \int_a^b w(s)\phi(x(s), t(s))\, ds, \quad \phi \in C_0^\infty(\mathbb{R} \times [0, \infty)).$$

Now, we define a delta shock wave solution for the system (1) with initial data (2).

Definition 2. A distribution pair (ρ, u) is a *delta shock wave solution* of (1) and (2) in the sense of distribution if there exist a smooth curve L and a function $w \in C^1(L)$ such that ρ and u are represented in the following form

$$\rho = \widetilde{\rho}(x, t) + w\delta_L \text{ and } u = \widetilde{u}(x, t),$$

$\widetilde{\rho}, \widetilde{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R})$ and

$$\begin{cases} \langle \rho, \varphi_t \rangle + \langle \rho f(u), \varphi_x \rangle = 0, \\ \langle \rho u, \varphi_t \rangle + \langle \rho u f(u), \varphi_x \rangle + \int_0^\infty \int_{\mathbb{R}} u\phi_x dx dt = 0, \end{cases} \tag{4}$$

for all the test functions $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, where $u|_L = u_\delta(t)$ and

$$\langle \rho, \varphi \rangle = \int_0^\infty \int_{\mathbb{R}} \widetilde{\rho}\varphi\, dx dt + \langle w\delta_L, \varphi \rangle,$$

$$\langle \rho G(u), \varphi \rangle = \int_0^\infty \int_{\mathbb{R}} \widetilde{\rho} G(\widetilde{u})\varphi\, dx dt + \langle wG(u_\delta)\delta_L, \varphi \rangle.$$

With the previous definitions, we are going to find a solution with discontinuity $x = x(t)$ for (1) of the form

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-(x,t), u_-(x,t)), & \text{if } x < x(t), \\ (w(t)\delta_L, u_\delta(t)), & \text{if } x = x(t), \\ (\rho_+(x,t), u_+(x,t)), & \text{if } x > x(t), \end{cases} \tag{5}$$

where $\rho_\pm(x,t)$, $u_\pm(x,t)$ are piecewise smooth solutions of system (1), $\delta(\cdot)$ is the Dirac measure supported on the curve $x(t) \in C^1$, and $x(t)$, $w(t)$ and $u_\delta(t)$ are to be determined.

We define

$$\frac{1}{\rho} := \begin{cases} \frac{1}{\rho_-}, & \text{if } x < x(t) \\ 0, & \text{if } x = x(t) \\ \frac{1}{\rho_+}, & \text{if } x > x(t). \end{cases}$$

Now, we have the following theorem:

Theorem 1. *If the curves $x(t)$, $w(t)$ and $u_\delta(t)$ solve*

$$\begin{cases} \dfrac{dx(t)}{dt} = f(u_\delta(t)), \\ \dfrac{dw(t)}{dt} = -f(u_\delta(t))[\rho] + [\rho f(u)], \\ \dfrac{d(w(t)u_\delta(t))}{dt} = -f(u_\delta(t))[\rho u] + \left[\rho u\left(f(u) + \dfrac{1}{\rho}\right)\right] \end{cases} \tag{6}$$

then the solution $(\rho(x,t), u(x,t))$ defined in (5) *satisfies* (1) *in the sense of distributions.*

The relations given by (6) are called the *generalized Rankine-Hugoniot conditions*.

Proof. If Equations 6 holds, then, for any test functions $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, we obtain

$$\begin{aligned}
\langle \rho u, \varphi_t\rangle + \langle \rho u f(u), \varphi_x\rangle + \int_0^\infty \int_{\mathbb{R}} u\phi_x dxdt &= \int_0^\infty \int_{\mathbb{R}} (\rho u\varphi_t + \rho u\left(f(u) + \frac{1}{\rho}\right)\varphi_x) dxdt \\
&\quad + \int_0^\infty (w(t)u_\delta(t)\varphi_t + w(t)u_\delta(t)f(u_\delta(t))\varphi_x)dt \\
&= \int_0^\infty \int_{-\infty}^{x(t)} (\rho_-u_-\varphi_t + \rho_-u_-\left(f(u_-) + \frac{1}{\rho_-}\right)\varphi_x) dxdt \\
&\quad + \int_0^\infty \int_{x(t)}^\infty (\rho_+u_+\varphi_t + \rho_+u_+\left(f(u_+) + \frac{1}{\rho_+}\right)\varphi_x) dxdt \\
&\quad + \int_0^\infty w(t)u_\delta(t)(\varphi_t + f(u_\delta(t))\varphi_x)dt \\
&= -\oint -\left(\rho_-u_-\left(f(u_-) + \frac{1}{\rho_-}\right)\varphi\right)dt + (\rho_-u_-\varphi)dx
\end{aligned}$$

$$+\oint -\left(\rho_+ u_+\left(f(u_+)+\frac{1}{\rho_+}\right)\varphi\right)dt+(\rho_+u_+\varphi)dx-\int_0^\infty \varphi\frac{d(w(t)u_\delta(t))}{dt}dt$$

$$=\int_0^\infty \varphi\left(-f(u_\delta)[\rho u]+[\rho u\phi(\rho,u)]-\frac{d(w(t)u_\delta(t))}{dt}\right)dt=0$$

which implies the second Equation of 4. A completely similar argument leads to the first Equation of 4. □

Thereby, the Riemann problem is reduced to find $x(t)$, $w(t)$ and $u_\delta(t)$ such that they solve (6) with the initial data (at $t=0$):

$$x(0)=0,\qquad w(0)=0,\qquad u_\delta(0)=0.$$

Observe that if $u_\delta(t)$ is a constant, say u_δ, then by (6) we have that $x(t)$ and $w(t)$ are linear functions of t. In the next section, we will utilize the vanishing viscosity method to show that $u_\delta(t)$ is a constant and therefore we obtain existence of delta shock waves. Moreover, we show uniqueness of delta shock wave under the entropy condition (3).

3 | VANISHING VISCOSITY METHOD AND DELTA SHOCK WAVE

In this section we only focus on the study of the case $f(u_-)>f(u_+)+1/\rho_+$. We consider the following problem

$$\begin{cases}\rho_t+(\rho f(u))_x=0,\\(\rho u)_t+(\rho u(f(u)+\frac{1}{\rho})_x=\varepsilon t u_{xx},\end{cases}\tag{7}$$

with initial data

$$(\rho(x,0),u(x,0))=\begin{cases}(\rho_-,u_-), & \text{if } x<0,\\(\rho_+,u_+), & \text{if } x>0.\end{cases}\tag{8}$$

By the self-similar transformation $\xi=x/t$, we have

$$\begin{cases}-\xi\rho_\xi+(\rho f(u))_\xi=0,\\-\xi(\rho u)_\xi+(\rho u(f(u)+\frac{1}{\rho}))_\xi=\varepsilon u_{\xi\xi}\end{cases}\tag{9}$$

with data at infinity given by

$$(\rho(\pm\infty),u(\pm\infty))=(\rho_\pm,u_\pm).\tag{10}$$

3.1 | Existence of solutions to the viscous system (9)–(10)

Let R_1 and R_2 be positive numbers such that $R_1\ge|f(u_-)|$ and $R_2\ge|f(u_+)+\frac{1}{\rho_+}|$. Then, for $R\ge\max\{R_1,R_2,|u_-|,|u_+|\}$ we consider the Banach space $C([-R,R])$, endowed with the supremum norm, and we take the set K given by

$$K=\{U\in C([-R,R])\,|\,U \text{ is monotone decreasing with } U(-R)=u_- \text{ and } U(R)=u_+\}$$

which is bounded and a convex closed set in $C([-R,R])$.

Lemma 1. *Suppose $U\in K\cap C^1([-R,R])$. Let*

$$\rho(\xi)=\begin{cases}\rho_1(\xi), & \textit{if } -R\le\xi<\xi_\sigma,\\\rho_2(\xi), & \textit{if } \xi_\sigma<\xi\le R,\end{cases}\tag{11}$$

where ξ_σ is the unique solution of the equation $f(U(\xi_\sigma)) = \xi_\sigma$ (which solution exists because $f(u_-) > f(u_+)$ and R is big enough),

$$\rho_1(\xi) := \rho_- \frac{f(u_-) + R}{f(U(\xi)) - \xi} \exp\left(-\int_{-R}^{\xi} \frac{ds}{f(U(s)) - s} \right) \tag{12}$$

and

$$\rho_2(\xi) := \rho_+ \frac{R - f(u_+)}{\xi - f(U(\xi))} \exp\left(\int_{\xi}^{R} \frac{ds}{f(U(s)) - s} \right). \tag{13}$$

Then $\rho \in L^1([-R, R])$, ρ is continuous in $[-R, \xi_\sigma) \cup (\xi_\sigma, R]$ and it is a weak solution for

$$-\xi \rho_\xi + (\rho f(U))_\xi = 0, \tag{14}$$

and $\rho(\pm R) = \rho_\pm$.

Proof. The Equation 14 can be rewritten as

$$(f(U(\xi)) - \xi)\rho' + \rho(f(U(\xi)))' = 0. \tag{15}$$

Integrating (15) on $[-R, \xi]$ for $-R < \xi < \xi_\sigma$, we get

$$(f(U(\xi)) - \xi)\rho_1(\xi) - (f(u_-) + R)\rho_- + \int_{-R}^{\xi} \rho_1(s)ds = 0. \tag{16}$$

Let

$$p(\xi) = \int_{-R}^{\xi} \rho_1(s)ds, \quad A_1 = (f(u_-) + R)\rho_- \quad \text{and } a(\xi) = (f(U(\xi)) - \xi).$$

Then (16) can be written as

$$\begin{cases} a(\xi)p'(\xi) + p(\xi) = A_1, \\ p(-R) = 0. \end{cases}$$

It follows that

$$p(\xi) = A_1 \left\{ 1 - \exp\left(-\int_{-R}^{\xi} \frac{ds}{a(s)} \right) \right\}$$

Noting that $a(\xi) > 0$ and $a(\xi) = O(|\xi - \xi_\sigma|)$ as $\xi \to \xi_\sigma -$, we obtain

$$\lim_{\xi \to \xi_\sigma -} \int_{-R}^{\xi} \rho_1(s)ds = \lim_{\xi \to \xi_\sigma -} p(\xi) = A_1. \tag{17}$$

Hence

$$\lim_{\xi \to \xi_\sigma -} (f(U(\xi)) - \xi)\rho_1(\xi) = 0. \tag{18}$$

Similarly, one can get

$$\lim_{\xi\to\xi_\sigma+}\int_{\xi}^{R}\rho_2(s)ds = A_2, \tag{19}$$

$$\lim_{\xi\to\xi_\sigma+}(f(U(\xi))-\xi)\rho_2(\xi) = 0,$$

where $A_2 = (f(u_+) - R)\rho_+$. The equalities (17) and (19) imply that $\rho(\xi) \in L^1([-R, R])$.

Now, for arbitrary $\phi \in C_0^\infty([-R, R])$, we verify that

$$I \equiv -\int_{-R}^{R}(f(U(\xi))-\xi)\rho(\xi)\phi'(\xi)d\xi + \int_{-R}^{R}\rho(\xi)\phi(\xi)d\xi = 0.$$

For any ξ_1, ξ_2, such that $-R < \xi_1 < \xi_\sigma < \xi_2 < R$ we can write $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{-R}^{\xi_1}(-(f(U(\xi))-\xi)\rho(\xi)\phi'(\xi)+\rho(\xi)\phi(\xi))d\xi,$$

$$I_2 = \int_{\xi_1}^{\xi_2}(-(f(U(\xi))-\xi)\rho(\xi)\phi'(\xi)+\rho(\xi)\phi(\xi))d\xi \text{ and}$$

$$I_3 = \int_{\xi_2}^{R}(-(f(U(\xi))-\xi)\rho(\xi)\phi'(\xi)+\rho(\xi)\phi(\xi))d\xi.$$

Observe that

$$\begin{aligned}|I_1| &= \left|-(f(U(\xi_1))-\xi_1)\rho_1(\xi_1)\phi(\xi_1) + \int_{-R}^{\xi_1}(((f(U(\xi))-\xi)\rho(\xi))'\phi(\xi)+\rho(\xi)\phi(\xi)))d\xi\right| \\ &= \left|(f(U(\xi_1))-\xi_1)\rho_1(\xi_1)\phi(\xi_1)\right|\end{aligned}$$

By (18), we have that

$$\lim_{\xi_1\to\xi_\sigma-}|I_1| = \lim_{\xi_1\to\xi_\sigma-}\left|(f(U(\xi_1))-\xi_1)\rho_1(\xi_1)\phi(\xi_1)\right| = 0.$$

In similar way, we show that

$$\lim_{\xi_2\to\xi_\sigma+}|I_3| = \lim_{\xi_2\to\xi_\sigma+}\left|(f(U(\xi_2))-\xi_2)\rho_2(\xi_2)\phi(\xi_2)\right| = 0.$$

Since $\rho \in L^1([-R, R])$,

$$|I_2| \le \int_{\xi_1}^{\xi_2}|-(f(U(\xi))-\xi)\phi'(\xi)+\phi(\xi)||\rho(\xi)|d\xi \to 0, \quad \text{as } \xi_1 \to \xi_\sigma-, \xi_2 \to \xi_\sigma+.$$

But I is independent of ξ_1 and ξ_2, so $I = 0$. Therefore, ρ defined in (11) is a weak solution. □

Define an operator $T : K \to C^2([-R,R])$ as follows: for any $U \in K$, $u = TU$ is the unique solution of the boundary value problem

$$\begin{cases} \varepsilon u'' = (\rho(U,\xi)(f(U(\xi)) - \xi) + 1)\, u' \\ u(\pm R) = u_\pm \end{cases} \tag{20}$$

where $\rho(U,\xi) \equiv \rho(\xi)$ is defined in (12) or (13). In fact, the solution to this problem can be found explicitly and it is given by

$$u(\xi) = u_- + \frac{(u_+ - u_-)\int_{-R}^{\xi} \exp\left(\int_{-R}^{r} \frac{\rho(U,s)(f(U(s))-s)+1}{\varepsilon} ds\right) dr}{\int_{-R}^{R} \exp\left(\int_{-R}^{r} \frac{\rho(U,s)(f(U(s))-s)+1}{\varepsilon} ds\right) dr}. \tag{21}$$

Lemma 2. *$T : K \to K$ is a continuous operator.*

Proof. Choose $\{U_n\}$ in K such that $U_n \to U$. As U belongs to K, then each $u_n = TU_n$ and $u = TU$ satisfy the problem (20). Now, we have the following problem

$$\begin{cases} \varepsilon(u_n - u)'' = (\rho(U_n,\xi)(f(U_n(\xi)) - \xi) + 1)(u_n - u)' + (\rho(U_n,\xi)(f(U_n(\xi)) - \xi) - \rho(U,\xi)(f(U(\xi)) - \xi))u' \\ (u_n - u)(\pm R) = 0. \end{cases} \tag{22}$$

Setting $p_n(\xi) = \rho(U_n,\xi)(f(U_n(\xi)) - \xi)$ and $q_n(\xi) = (\rho(U_n,\xi)(f(U_n(\xi)) - \xi) - \rho(U,\xi)(f(U(\xi)) - \xi))u'$, from problem (22) we have

$$(u_n - u)'(\xi) = -\frac{\int_{-R}^{R}\int_{-R}^{y} \frac{q_n(r)}{\varepsilon} \exp\left(\int_{r}^{y} \frac{p_n(s)+1}{\varepsilon} ds\right) dr dy}{\int_{-R}^{R} \exp\left(\int_{-R}^{r} \frac{p_n(s)+1}{\varepsilon} ds\right) dr} \exp\left(\int_{-R}^{\xi} \frac{p_n(s)+1}{\varepsilon} ds\right) + \int_{-R}^{\xi} \frac{q_n(r)}{\varepsilon} \exp\left(\int_{-r}^{\xi} \frac{p_n(s)+1}{\varepsilon} ds\right) dr \tag{23}$$

$$(u_n - u)(\xi) = -\frac{\int_{-R}^{R}\int_{-R}^{y} \frac{q_n(r)}{\varepsilon} \exp\left(\int_{r}^{y} \frac{p_n(s)+1}{\varepsilon} ds\right) dr dy}{\int_{-R}^{R} \exp\left(\int_{-R}^{r} \frac{p_n(s)+1}{\varepsilon} ds\right) dr} \int_{-R}^{\xi} \exp\left(\int_{-R}^{r} \frac{p_n(s)+1}{\varepsilon} ds\right) dr + \int_{-R}^{\xi}\int_{-R}^{y} \frac{q_n(r)}{\varepsilon} \exp\left(\int_{-r}^{y} \frac{p_n(s)+1}{\varepsilon} ds\right) dr dy \tag{24}$$

From (14), we have

$$((f(U(\xi)) - \xi)\rho(\xi))' = -\rho(\xi) < 0,$$
$$((f(U_n(\xi)) - \xi)\rho_n(\xi))' = -\rho_n(\xi) < 0 \qquad \text{for } n = 1, 2, \dots.$$

Then, $\rho(f(U) - \xi)$ and $\rho_n(f(U_n) - \xi)$, $n = 1, 2, \dots$, are monotone decreasing and continuous functions. Because the sequence of monotone functions which converges to a continuous function must converge uniformly, we get that $q_n(\xi)$ converges to zero uniformly. Then, from (22), (23) and (24) it follows that

$$u_n \to u \text{ in } C^2([-R,R]), \text{ as } n \to \infty.$$

Therefore $T : K \to C^2([-R,R])$ is continuous. In addition, from (21), we have

$$u'(\xi) = \frac{(u_+ - u_-)\exp\left(\int_{-R}^{\xi} \frac{\rho(U,s)(f(U(s))-s)+1}{\varepsilon} ds\right)}{\int_{-R}^{R} \exp\left(\int_{-R}^{r} \frac{\rho(U,s)(f(U(s))-s)+1}{\varepsilon} ds\right) dr}$$

which implies that $u = TU$ is monotone. So we get $TK \subset K$. □

Lemma 3. *TK is a bounded set in $C^2([-R, R])$.*

Proof. For any $U \in K$, if $s < \xi_\sigma$, we have

$$0 < \rho(U, s)(f(U(s)) - s) = \rho_-(f(u_-) + R) - \int_{-R}^{s} \rho(r)dr < \rho_-(f(u_-) + R) \tag{25}$$

and if $s > \xi_\sigma$,

$$0 > \rho(U, s)(f(U(s)) - s) = \rho_+(f(u_+) - R) + \int_{s}^{R} \rho(r)dr > \rho_+(f(u_+) - R). \tag{26}$$

From (20), we can deduce that

$$u''(\xi) < 0, \quad \xi \in [-R, \xi_\sigma).$$

Then, $u'(\xi) \le u'(-R) < 0$, $\xi \in [-R, \xi_\sigma)$, and

$$u_- - u_+ > u(-R) - u(\xi_\sigma) = u'(\zeta)(-R - \xi_\sigma) > u'(\zeta)(-R - f(u_+)), \qquad \zeta \in (-R, \xi_\sigma).$$

Thus,

$$0 > u'(-R) > u'(\zeta) > -\frac{u_- - u_+}{R + f(u_+)}.$$

Also, from (20) we have

$$u'(\xi) = u'(-R) \exp\left(\int_{-R}^{\xi} \frac{\rho(f(U) - s) + 1}{\varepsilon} ds \right)$$

and by (25) and (26), we conclude that u' is uniformly bounded. Consequently, u'' is also uniformly bounded.

So, TK is a bounded set in $C^2([-R, R])$. □

Lemma 4. *TK is precompact in $C([-R, R])$.*

Proof. This is a consequence of the compact embedding $C^2([-R, R]) \hookrightarrow C([-R, R])$. □

From the above lemmas, by virtue of Schauder fixed point theorem, we get the following result.

Theorem 2. *For each $R \ge \max\{R_1, R_2, |u_-|, |u_+|\}$, there exists a weak solution*

$$(\rho_R, u_R) \in L^1([-R, R]) \times C^2([-R, R])$$

for the system (9) with boundary value $(\rho_R(\pm R), u_R(\pm R)) = (\rho_\pm, u_\pm)$, and, in addition, being u_R a decreasing function.

The next step is to obtain from this family of solutions a sequence $R_k \to \infty$ such that (ρ_{R_k}, u_{R_k}) converges to a weak solution of (9)–(10). To this end, we need the following lemma.

Lemma 5.

1. *$u_R(\xi)$, $u'_R(\xi)$ and $u''_R(\xi)$ are uniformly bounded, with respect to R and $\xi \in [-R, R]$.*
2. *There exist a sequence $R_k \to \infty$ and a decreasing function $u \in C^1(\mathbb{R})$ such that u_{R_k} converges to u in $C^1([-M, M])$, for each positive number M (i.e. u_{R_k}, u'_{R_k} converge uniformly in compact sets of $\mathbb{R}$ to u, u', respectively).*
3. *$\rho_{R_k}(u_{R_k}, \xi)$ converges to $\rho(u, \xi)$, as $R_k \to \infty$, for each $\xi \in \mathbb{R} \setminus \{\xi_\sigma\}$, where $\rho_{R_k}(u_{R_k}, \xi)$, $\rho(u, \xi)$ are defined accordingly with (11), (12) and (13), being $R = \infty$ for $\rho(u, \xi)$, and ξ_σ satisfies $f(u(\xi_\sigma)) = \xi_\sigma$.*

Proof.

1. To simplify the notation in this proof, we shall use u, u' and u'' instead of u_R, u'_R and u''_R.
Observe that $f(u_+) + \frac{1}{\rho_+} < \xi_\sigma < f(u_-)$. We choose ξ_1 such that $-R < \xi_1 < f(u_+) + \frac{1}{\rho_+}$. From (9) it follows that

$$u'(\xi) = u'(\xi_1)\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds\right).$$

As $u''(\xi) < 0$ for $\xi \in (-R, \xi_\sigma)$, then we have that $u'(\xi) < u'(\xi_1) < 0$, $\xi \in (\xi_1, \xi_\sigma)$. Since

$$u_- - u_+ > u(\xi_1) - u(\xi_\sigma) = u'(\zeta)(\xi_1 - \xi_\sigma) > u'(\zeta)(\xi_1 - f(u_+)),$$

where $\zeta \in (\xi_1, \xi_\sigma)$, we get

$$u'(\zeta) > \frac{u_- - u_+}{\xi_1 - f(u_+)}, \qquad \zeta \in (\xi_1, \xi_\sigma).$$

It follows that

$$0 > u'(\xi_1) > \frac{u_- - u_+}{\xi_1 - f(u_+)}.$$

When $\xi < \xi_1$,

$$\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds\right) < 1.$$

When $\xi_1 < \xi < \xi_\sigma$, observe that

$$\begin{aligned}\rho_1(\xi_1) &= \rho_- \frac{f(u_-) + R}{f(u(\xi_1)) - \xi_1} \exp\left(-\int_{-R}^{\xi_1} \frac{ds}{f(u(s)) - s}\right)\\ &\leq \rho_- \frac{f(u_-) + R}{f(u(\xi_1)) - \xi_1} \exp\left(-\int_{-R}^{\xi_1} \frac{ds}{f(u_-) - s}\right) = \rho_- \frac{f(u_-) - \xi_1}{f(u(\xi_1)) - \xi_1}\end{aligned}$$

and

$$\begin{aligned}\rho(\xi)(f(u(\xi)) - \xi) + 1 &= \rho(\xi_1)(f(u(\xi_1)) - \xi_1) + 1 - \int_{\xi_1}^{\xi} \rho(s) ds\\ &\leq \rho(\xi_1)(f(u(\xi_1)) - \xi_1) + 1 \leq \rho_-(f(u_-) - \xi_1) + 1,\end{aligned}$$

and we obtain

$$\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds\right) \leq \exp\left(\frac{(\rho_-(f(u_-) - \xi_1) + 1)(f(u_-) - \xi_1)}{\varepsilon}\right).$$

When $\xi > \xi_\sigma$, we have

$$\int_{\xi_1}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds = \int_{\xi_1}^{\xi_\sigma} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds + \int_{\xi_\sigma}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds$$

$$< \int_{\xi_1}^{\xi_\sigma} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds$$

or

$$\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds \right) < \exp\left(\int_{\xi_1}^{\xi_\sigma} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds \right).$$

Therefore, $u'(\xi)$ and $u(\xi)$ are uniformly bounded. From (20) it follows that $u_R''(\xi)$ is also uniformly bounded, with respect to R and $\xi \in [-R, R]$.

2. Fixing $M > 0$, we consider $R >> M$ and apply the Arzelá-Ascoli theorem to obtain a sequence (u_{R_k}) converging in $C^1([-M, M])$ to a decreasing function u. Then, by a diagonalization process, we obtain a sequence (u_{R_k}), which we do not relabel, such that (u_{R_k}) converges to a decreasing function $u \in C^1(\mathbb{R})$, uniformly in compact sets in $\mathbb{R}$, (u'_{R_k}) also converges uniformly in compact sets in $\mathbb{R}$ to u', and $u(-\infty) = u_-$, $u(\infty) = u_+$.
3. Claim 3 is obtained from (12), (13) by passing to the limit as $R_k \to \infty$, for each fixed $\xi \neq \xi_\sigma$, noting that, up to a subsequence, we can assume that $\xi_\sigma^{R_k}$, defined by $f(u_{R_k}(\xi_\sigma^{R_k})) = \xi_\sigma^{R_k}$, converges to ξ_σ (where ξ_σ is defined by $f(u(\xi_\sigma)) = \xi_\sigma$). □

Theorem 3. *Let u be the function obtained in Lemma 5. Then, for each $\varepsilon > 0$, u satisfies*

$$\begin{cases} \varepsilon u'' = (\rho(u, \xi)(f(u) - \xi) + 1)u', \\ u(\pm\infty) = u_\pm, \end{cases}$$

and

$$\rho(\xi) = \begin{cases} \rho_1(\xi), & \text{if } -\infty < \xi < \xi_\sigma, \\ \rho_2(\xi), & \text{if } \xi_\sigma < \xi < \infty, \end{cases}$$

where ξ_σ satisfies $f(u(\xi_\sigma)) = \xi_\sigma$,

$$\rho_1(\xi) = \rho_- \exp\left(- \int_{-\infty}^{\xi} \frac{(f(u(s)))'}{f(u(s)) - s} ds \right) \text{ and } \rho_2(\xi) = \rho_+ \exp\left(\int_{\xi}^{+\infty} \frac{(f(u(s)))'}{f(u(s)) - s} ds \right).$$

Proof. Denote by $(\rho_R(\xi), u_R(\xi))$ the solution of the problem (9) with boundary value $(\rho(\pm R), u(\pm R)) = (\rho_\pm, u_\pm)$. Fixing ξ_2 and integrating (20) from ξ_2 to ξ, we obtain

$$\varepsilon(u_R'(\xi) - u_R'(\xi_2)) = (\rho_R(\xi)(f(u_R(\xi)) - \xi) + 1)u_R(\xi) - (\rho_R(\xi_2)(f(u_R(\xi_2)) - \xi_2) + 1)u_R(\xi_2) + \int_{\xi_2}^{\xi} \rho_R(s)u_R(s)ds$$

(independently of whether ξ_σ is between ξ_2 and ξ). Letting $R \to +\infty$, by the Lebesgue Convergence Theorem it follows that

$$\varepsilon(u'(\xi) - u'(\xi_2)) = (\rho(\xi)(f(u(\xi)) - \xi) + 1)u(\xi) - (\rho(\xi_2)(f(u(\xi_2)) - \xi_2) + 1)u(\xi_2) + \int_{\xi_2}^{\xi} \rho(s)u(s)ds. \tag{27}$$

Differentiating (27) with respect to ξ, we obtain

$$\varepsilon u'' = (\rho(f(u) - \xi) + 1)u',$$

and from (21) we have $u(\pm\infty) = u_{\pm}$. □

Theorem 4. *There exists a weak solution* $(\rho, u) \in L^1_{loc}((-\infty, +\infty)) \times C^2((-\infty, +\infty))$ *for the boundary value problem* (9)–(10).

Proof. Let (ρ, u) be defined in Theorem 3. By Lemma 5 we know that u is decreasing and of class C^1 in $\mathbb{R}$. Then ρ is of class C^1 in $(-\infty, \xi_\sigma) \cup (\xi_\sigma, \infty)$. In addition, it is also bounded, hence, locally integrable. From (27) it follows that u is of class C^2. The first equation in (9) comes by differentiating ρ_1 and ρ_2, and the second is equivalent to the first and the equation stated in Theorem 3. □

3.2 | The limit solutions of (7)–(8) as viscosity vanishes

We continue this section studying the case when $f(u_-) > f(u_+) + \frac{1}{\rho_+}$ and we are interested in analyzing the behavior of the solutions $(\rho^\varepsilon, u^\varepsilon)$ of (9)–(10) as $\varepsilon \to 0+$.

Lemma 6. *Let ξ^ε_σ be the unique point satisfying $f(u^\varepsilon(\xi^\varepsilon_\sigma)) = \xi^\varepsilon_\sigma$, and let ξ_σ be the limit*

$$\xi_\sigma = \lim_{\varepsilon\to 0+} \xi^\varepsilon_\sigma$$

(passing to a subsequence if necessary). Then for any $\eta > 0$,

$$\lim_{\varepsilon\to 0+} u^\varepsilon_\xi(\xi) = 0, \qquad \text{for } |\xi - \xi_\sigma| \geq \eta,$$

$$\lim_{\varepsilon\to 0+} u^\varepsilon(\xi) = \begin{cases} u_-, & \text{if } \xi \leq \xi_\sigma - \eta, \\ u_+, & \text{if } \xi \geq \xi_\sigma + \eta, \end{cases}$$

uniformly in the above intervals.

Proof. To simplify the notation in this proof, we shall use ρ, u instead of ρ^ε, u^ε.

Take $\xi_3 = \xi_\sigma + \eta/2$, and let ε be so small such that $\xi^\varepsilon_\sigma < \xi_3 - \eta/4$. For $\xi > \xi_\sigma$,

$$\begin{aligned} \rho(\xi) = \rho_+ \exp\left(\int_\xi^{+\infty} \frac{(f(u(s)))'}{f(u(s)) - s} ds\right) &= \lim_{R\to+\infty} \rho_+ \frac{R - f(u_+)}{\xi - f(u(\xi))} \exp\left(\int_\xi^R \frac{ds}{f(u(s)) - s}\right) \\ &\leq \lim_{R\to+\infty} \rho_+ \frac{R - f(u_+)}{\xi - f(u(\xi))} \exp\left(\int_\xi^R \frac{ds}{f(u_+) - s}\right) = \lim_{R\to+\infty} \rho_+ \frac{R - f(u_+)}{\xi - f(u(\xi))} \frac{\xi - f(u_+)}{R - f(u_+)} \\ &= \rho_+ \frac{\xi - f(u_+)}{\xi - f(u(\xi))}, \end{aligned}$$

and we have

$$\rho(\xi)(f(u(\xi)) - \xi) + 1 \geq \rho_+(f(u_+) - \xi) + 1, \qquad \xi \in (\xi_\sigma, +\infty).$$

Now, integrating the second Equation of 9 twice on $[\xi_3, \xi]$, we get

$$u(\xi_3) - u(\xi) = -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{r} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds \right) dr$$

$$\geq -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{r} \frac{\rho_+(f(u_+) - s) + 1}{\varepsilon} ds \right) dr$$

$$= -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\frac{\rho_+}{\varepsilon} \left(\left(f(u_+) + \frac{1}{\rho_+} - \xi_3 \right) (r - \xi_3) - \frac{1}{2}(r - \xi_3)^2 \right) \right) dr$$

$$= -u'(\xi_3) \int_{0}^{\xi - \xi_3} \exp\left(\frac{\rho_+}{\varepsilon} \left(\left(f(u_+) + \frac{1}{\rho_+} - \xi_3 \right) r - \frac{1}{2} r^2 \right) \right) dr.$$

Letting $\xi \to +\infty$, we get

$$u_- - u_+ \geq -u'(\xi_3) \int_{0}^{+\infty} \exp\left(\frac{\rho_+}{\varepsilon} \left(\left(f(u_+) + \frac{1}{\rho_+} - \xi_3 \right) r - \frac{1}{2} r^2 \right) \right) dr$$

$$\geq -u'(\xi_3) \int_{0}^{2\varepsilon} \exp\left(\frac{\rho_+}{\varepsilon} \left(\left(f(u_+) + \frac{1}{\rho_+} - \xi_3 \right) r - \frac{1}{2} r^2 \right) \right) dr$$

$$\geq -u'(\xi_3) \sqrt{\varepsilon} A_3$$

for $0 \leq \varepsilon \leq 1$, where A_3 is a constant independent of ε. Thus

$$|u'(\xi_3)| \leq \frac{u_- - u_+}{\sqrt{\varepsilon} A_3}.$$

So

$$|u'(\xi)| \leq \frac{u_- - u_+}{\sqrt{\varepsilon} A_3} \exp\left(\int_{\xi_3}^{\xi} \frac{\rho(s)(f(u(s)) - s) + 1}{\varepsilon} ds \right). \tag{28}$$

For $\xi > \xi_3$,

$$\rho(\xi) = \lim_{R \to +\infty} \rho_+ \frac{R - f(u_+)}{\xi - f(u(\xi))} \exp\left(\int_{\xi}^{R} \frac{ds}{f(u(s)) - s} \right)$$

$$\geq \lim_{R \to +\infty} \rho_+ \frac{R - f(u_+)}{\xi - f(u(\xi))} \exp\left(\int_{\xi}^{R} \frac{ds}{f(u(\xi_3)) - s} \right)$$

$$= \rho_+ \frac{\xi - f(u(\xi_3))}{\xi - f(u(\xi))} \lim_{R \to +\infty} \frac{R - f(u_+)}{R - f(u(\xi_3))}$$

$$= \rho_+ \frac{\xi - f(u(\xi_3))}{\xi - f(u(\xi))}$$

and we have

$$\rho(\xi)(f(u(\xi))-\xi)+1\leq\rho_{+}(f(u(\xi_3))-\xi)+1,\qquad \xi>\xi_3. \tag{29}$$

From (28) and (29) we have

$$|u'(\xi)|\leq\frac{u_{-}-u_{+}}{\sqrt{\varepsilon}A_3}\exp\left(-\frac{\rho_{+}}{\varepsilon}\int_{\xi_3}^{\xi}\left(s-\left(f(u(\xi_3))+\frac{1}{\rho_{+}}\right)\right)ds\right)$$

which implies that

$$\lim_{\varepsilon\to0+}u^{\varepsilon}_{\xi}(\xi)=0,\qquad \text{uniformly for } \xi\geq\xi_{\sigma}+\eta.$$

Now, we choose ξ and ξ_4 such that $\xi>\xi_4\geq\xi_{\beta}+\eta$. From

$$u(\xi_4)-u(\xi)=-u'(\xi_4)\int_{\xi_4}^{\xi}\exp\left(\int_{\xi_4}^{r}\frac{\rho(s)(f(u(s))-s)+1}{\varepsilon}ds\right)dr,$$

we get

$$|u(\xi_4)-u(\xi)|\leq|u'(\xi_4)|\int_{\xi_4}^{\xi}\exp\left(-\frac{A_4}{\varepsilon}(r-\xi_4)\right)dr\leq\frac{\varepsilon}{A_4}|u'(\xi_4)|\left(1-\exp\left(\frac{A_4}{\varepsilon}(\xi_4-\xi)\right)\right),$$

where $A_4=\rho_{+}(\xi_4-(f(u(\xi_4))+\frac{1}{\rho_{+}}))$. When $\xi\to+\infty$, we obtain

$$|u(\xi_4)-u_{+}|\leq\frac{\varepsilon}{A_4}|u'(\xi_4)|,$$

which implies that

$$\lim_{\varepsilon\to0+}u^{\varepsilon}(\xi)=u_{+},\qquad \text{uniformly for } \xi\geq\xi_{\sigma}+\eta.$$

The results for $\xi<\xi_{\sigma}-\eta$ can be obtained analogously. □

Lemma 7. *For any $\eta>0$,*

$$\lim_{\varepsilon\to0+}\rho^{\varepsilon}(\xi)=\begin{cases}\rho_{-}, & \text{if } \xi<\xi_{\sigma}-\eta,\\ \rho_{+}, & \text{if } \xi>\xi_{\sigma}+\eta,\end{cases}$$

uniformly, with respect to ξ.

Proof. Take $\varepsilon_0>0$ so small such that $|\xi^{\varepsilon}_{\sigma}-\xi_{\sigma}|<\frac{\eta}{2}$ whenever $0<\varepsilon<\varepsilon_0$. For any $\xi>\xi_{\sigma}+\eta$ and $\varepsilon<\varepsilon_0$, we have

$$\xi>\xi^{\varepsilon}_{\sigma}+\frac{\eta}{2}$$

and

$$\rho^{\varepsilon}(\xi)=\rho_{+}\exp\left(\int_{\xi}^{\infty}\frac{(f(u^{\varepsilon}(s)))'}{f(u^{\varepsilon}(s))-s}ds\right).$$

For any $s \in [\xi, +\infty)$, we have

$$f(u^\varepsilon(s)) - s < f(u^\varepsilon(\xi)) - \xi = (1 - (f(u^\varepsilon(\zeta)))')(\xi_\sigma^\varepsilon - \xi) \\ \leq -\frac{\eta}{2}.$$

As $f' > 0$ and u is decreasing, we have that $(f(u(\xi)))' = f'(u(\xi))u'(\xi) < 0$, and

$$\frac{(f(u(s)))'}{f(u^\varepsilon(s)) - s} < -\frac{2}{\eta}(f(u(s)))', \qquad \text{for any } s \in [\xi, +\infty).$$

Now, in the last inequality, integrating on $[\xi, +\infty)$ we have

$$0 \leq \int_\xi^\infty \frac{(f(u^\varepsilon(s)))'}{f(u^\varepsilon(s)) - s} ds \leq -\frac{2}{\eta} \int_\xi^\infty (f(u^\varepsilon(s)))' ds = -\frac{2}{\eta}(f(u_+) - f(u^\varepsilon(\xi))),$$

so

$$1 \leq \exp\left(\int_\xi^\infty \frac{(f(u^\varepsilon(s)))'}{f(u^\varepsilon(s)) - s} ds \right) \leq \exp\left(-\frac{2}{\eta}(f(u_+) - f(u^\varepsilon(\xi))) \right). \tag{30}$$

By Lemma 6 we have that $\lim_{\varepsilon \to 0+} u^\varepsilon(\xi) = u_+$, and from (30) we have

$$\lim_{\varepsilon \to 0+} \exp\left(\int_\xi^\infty \frac{(f(u^\varepsilon(s)))'}{f(u^\varepsilon(s)) - s} ds \right) = 1$$

and

$$\lim_{\varepsilon \to 0+} \rho^\varepsilon(\xi) = \lim_{\varepsilon \to 0+} \rho_+ \exp\left(\int_\xi^\infty \frac{(f(u^\varepsilon(s)))'}{f(u^\varepsilon(s)) - s} ds \right) = \rho_+, \quad \text{uniformly for } \xi > \xi_\sigma + \eta.$$

Similarly, we obtain also $\lim_{\varepsilon \to 0} \rho^\varepsilon(\xi) = \rho_-$, uniformly for $\xi < \xi_\sigma - \eta$. □

Now, we study the limit behavior of $(\rho^\varepsilon, u^\varepsilon)$ in the neighborhood of ξ_σ as $\varepsilon \to 0+$.

Theorem 5. *Denote*

$$\sigma = \xi_\sigma = \lim_{\varepsilon \to 0+} \xi_\sigma^\varepsilon = \lim_{\varepsilon \to 0+} f(u^\varepsilon(\xi_\sigma^\varepsilon)) = f(u(\sigma)). \tag{31}$$

Then

$$\lim_{\varepsilon \to 0+} (\rho^\varepsilon(\xi), u^\varepsilon(\xi)) = \begin{cases} (\rho_-, u_-), & \text{if } \xi < \sigma, \\ (w_0 \cdot \delta, u_\delta), & \text{if } \xi = \sigma, \\ (\rho_+, u_+), & \text{if } \xi > \sigma, \end{cases}$$

where $\rho^\varepsilon(\xi)$ converges in the sense of the distributions to the sum of a step function and a Dirac measure δ with weight $w_0 = -\sigma(\rho_- - \rho_+) + (\rho_- f(u_-) - \rho_+ f(u_+))$.

Proof. As $\sigma = \xi_\sigma = \lim_{\varepsilon \to 0+} f(u^\varepsilon(\xi_\sigma^\varepsilon)) = f(u(\sigma))$, then we have

$$f(u_+) + \frac{1}{\rho_+} < \sigma < f(u_-). \tag{32}$$

Let ξ_1 and ξ_2 be real numbers such that $\xi_1 < \sigma < \xi_2$ and $\phi \in C_0^\infty([\xi_1,\xi_2])$ such that $\phi(\xi) \equiv \phi(\sigma)$ for ξ in a neighborhood Ω of σ, $\Omega \subset (\xi_1,\xi_2)$ [1]. Then $\xi_\sigma^\varepsilon \in \Omega$ whenever $0 < \varepsilon < \varepsilon_0$. From (9) we have

$$-\int_{\xi_1}^{\xi_2} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon \phi d\xi = 0. \tag{33}$$

For $\alpha_1, \alpha_2 \in \Omega$, α_1, α_2 near σ such that $\alpha_1 < \sigma < \alpha_2$, we write

$$\int_{\xi_1}^{\xi_2} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi = \int_{\xi_1}^{\alpha_1} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi + \int_{\alpha_2}^{\xi_2} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi,$$

and from Lemmas 6 and 7, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\xi_1}^{\xi_2} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi &= \int_{\xi_1}^{\alpha_1} \rho_- (f(u_-) - \xi)\phi' d\xi + \int_{\alpha_2}^{\xi_2} \rho_+ (f(u_+) - \xi)\phi' d\xi \\ &= \left(\rho_- f(u_-) - \rho_+ f(u_+) - \rho_-\alpha_1 + \rho_+\alpha_2\right)\phi(\sigma) + \int_{\xi_1}^{\alpha_1} \rho_-\phi(\xi) d\xi + \int_{\alpha_2}^{\xi_2} \rho_+\phi(\xi) d\xi\end{aligned}$$

Then taking $\alpha_1 \to \sigma-$, $\alpha_2 \to \sigma+$, we arrive at

$$\lim_{\varepsilon\to 0+}\int_{\xi_1}^{\xi_2} \rho^\varepsilon (f(u^\varepsilon) - \xi)\phi' d\xi = (-[\rho]\sigma + [\rho f(u)])\,\phi(\sigma) + \int_{\xi_1}^{\xi_2} J(\xi - \sigma)\phi(\xi) d\xi \tag{34}$$

where $[q] = q_- - q_+$ and

$$J(x) = \begin{cases} \rho_-, & \text{if } x < 0, \\ \rho_+, & \text{if } x > 0. \end{cases}$$

From (33) and (34), we get

$$\lim_{\varepsilon\to 0+}\int_{\xi_1}^{\xi_2} (\rho^\varepsilon - J(\xi - \sigma))\phi(\xi) d\xi = (-[\rho]\sigma + [\rho f(u)])\,\phi(\sigma).$$

for all sloping test functions $\phi \in C_0^\infty([\xi_1,\xi_2])$.

For an arbitrary $\psi \in C_0^\infty([\xi_1,\xi_2])$, we take a sloping test function ϕ, such that $\phi(\sigma) = \psi(\sigma)$ and

$$\max_{[\xi_1,\xi_2]} |\psi - \phi| < \mu,$$

for a sufficiently small $\mu > 0$. As $\rho^\varepsilon \in L^1([\xi_1,\xi_2)$ uniformly, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\xi_1}^{\xi_2} (\rho^\varepsilon - J(\xi - \sigma))\psi(\xi) d\xi &= \lim_{\varepsilon\to 0+}\int_{\xi_1}^{\xi_2} (\rho^\varepsilon - J(\xi - \sigma))\phi(\xi) d\xi + O(\mu) \\ &= (-[\rho]\sigma + [\rho f(u)])\,\phi(\sigma) + O(\mu) = (-[\rho]\sigma + [\rho f(u)])\,\psi(\sigma) + O(\mu).\end{aligned}$$

[1] The function ϕ is called a *sloping test function.*[7,9]

Then, when $\mu \to 0+$, we find that

$$\lim_{\varepsilon \to 0+} \int_{\xi_1}^{\xi_2} (\rho^\varepsilon - J(\xi - \sigma))\psi(\xi) d\xi = (-[\rho]\sigma + [\rho f(u)])\, \psi(\sigma) \tag{35}$$

holds for all test functions $\psi \in C_0^\infty([\xi_1, \xi_2])$. Thus, ρ^ε converges in the sense of the distributions to the sum of a step function and a Dirac delta function with strength $-[\rho]\sigma + [\rho f(u)]$. In a similar way, from

$$-\int_{\xi_1}^{\xi_2} (\rho^\varepsilon (f(u^\varepsilon) - \xi) + 1)\, u^\varepsilon \phi' d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon u^\varepsilon \phi d\xi = \varepsilon \int_{\xi_1}^{\xi_2} (u^\varepsilon)'' \phi d\xi \tag{36}$$

we can obtain

$$\lim_{\varepsilon \to 0+} \int_{\xi_1}^{\xi_2} (\rho^\varepsilon u^\varepsilon - \widetilde{J}(\xi - \sigma))\phi(\xi) d\xi = \left(-[\rho u]\sigma + \left[\rho u \left(f(u) + \frac{1}{\rho} \right) \right] \right) \phi(\sigma)$$

for all test functions $\phi \in C_0^\infty([\xi_1, \xi_2])$, where

$$\widetilde{J}(x) = \begin{cases} \rho_- u_-, & \text{if } x < 0, \\ \rho_+ u_+, & \text{if } x > 0. \end{cases}$$

Thus ρu also converges in the sense of the distributions to the sum of a step function and a Dirac delta function with strength $-[\rho u]\sigma + [\rho u(f(u) + \frac{1}{\rho})]$.

If we take the test function in (36) as $\frac{\psi}{\widetilde{u}^\varepsilon + \nu}$, $\nu > 0$, where $\widetilde{u}^\varepsilon$ is a modified function satisfying $u^\varepsilon(\sigma)$ in Ω and u^ε outside Ω, and let $\nu \to 0+$, we find

$$\lim_{\varepsilon \to 0+} \int_{\xi_1}^{\xi_2} (\rho^\varepsilon - J(\xi - \sigma))\psi d\xi \cdot u(\sigma) = \left(-[\rho u]\sigma + \left[\rho u \left(f(u) + \frac{1}{\rho} \right) \right] \right) \psi(\sigma) \tag{37}$$

for all test functions $\psi \in C_0^\infty([\xi_1, \xi_2])$. Let w_0 be the strength of the Dirac delta function in ρ, and denote

$$u_\delta = \lim_{\varepsilon \to 0+} u^\varepsilon(\xi_\sigma^\varepsilon) = u(\sigma).$$

From (31), (35) and (37) it follows that

$$\begin{cases} \sigma = f(u_\delta), \\ w_0 = -\sigma[\rho] + [\rho f(u)], \\ w_0 u_\delta = -\sigma[\rho u] + \left[\rho u \left(f(u) + \frac{1}{\rho} \right) \right]. \end{cases} \tag{38}$$

Under the entropy condition (32) the system (38) admits a unique solution (σ, w_0, u_δ). □

Then we get the following theorem.

Theorem 6. *Suppose* $f(u_-) > f(u_+) + \frac{1}{\rho_+}$. *Let* $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ *be the self-similar solution of* (7)–(8). *Then the limit*

$$\lim_{\varepsilon \to 0+} (\rho^\varepsilon(x,t), u^\varepsilon(x,t)) = (\rho(x,t), u(x,t))$$

exists in the measure sense and (ρ, u) *solves* (1)–(2). *Moreover,*

$$(\rho(x,t),u(x,t)) = \begin{cases} (\rho_-, u_-), & \text{if } x < \sigma t, \\ (w_0 t\delta(x-\sigma t), u_\delta), & \text{if } x = \sigma t, \\ (\rho_+, u_+), & \text{if } x > \sigma t, \end{cases}$$

where σ, w_0, *and* u_δ *are determined uniquely by the entropy condition* $f(u_+) + \frac{1}{\rho_+} < \sigma < f(u_-)$ *and*

$$\begin{cases} \sigma = f(u_\delta), \\ w_0 = -\sigma[\rho] + [\rho f(u)], \\ w_0 u_\delta = -\sigma[\rho u] + \left[\rho u \left(f(u) + \frac{1}{\rho}\right)\right]. \end{cases}$$

Finally, from the previous theorem we can conclude that the function $u_\delta(t)$ is a constant which we denote by u_δ, $x(t) = f(u_\delta)t$ and $w(t) = (-f(u_\delta)[\rho] + [\rho f(u)])t$. Moreover by the above results, the uniqueness of delta shock wave is guaranteed by entropy condition (3) (see also (32)).

Theorem 7. *Assume that* $f(u)$ *is a smooth and strictly monotone function and* $f(u_-) > f(u_+) + \frac{1}{\rho_+}$. *Then the Riemann problem* (1)–(2) *admits one and only one measure solution of the form*

$$(\rho(x,t),u(x,t)) = \begin{cases} (\rho_-, u_-), & \text{if } x < \sigma t, \\ (w_0 t\delta(x-\sigma t), u_\delta), & \text{if } x = \sigma t, \\ (\rho_+, u_+), & \text{if } x > \sigma t, \end{cases}$$

where σ, w_0, *and* u_δ *are determined uniquely by the entropy condition* $f(u_+) + \frac{1}{\rho_+} < \sigma < f(u_-)$ *and*

$$\begin{cases} \sigma = f(u_\delta), \\ w_0 = -\sigma[\rho] + [\rho f(u)], \\ w_0 u_\delta = -\sigma[\rho u] + \left[\rho u \left(f(u) + \frac{1}{\rho}\right)\right]. \end{cases}$$

ACKNOWLEDGMENTS

This research was supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo, Brazil) under the Grant #2016/19502-4.

ORCID

Richard De la cruz https://orcid.org/0000-0003-1342-7946

REFERENCES

[1] A. Aw, M. Rascle, Resurrection of "second order" models of traffic flow, *SIAM J. Appl. Math.* **2000**, *60*, 916.

[2] Y. Brenier, Solutions with concentration to the Riemann problem for one-dimensional Chaplygin gas dynamics, *J. Math. Fluid Mech.* **2005**, *7*, S326.

[3] H. Cheng, Delta shock waves for a linearly degenerate hyperbolic system of conservation laws of Keyfitz-Kranzer type, *Adv. Math. Phys.* **2013**, *2013* Article ID 958120, 10 pages.

[4] V. G. Danilov, V. M. Shelkovich, Delta-shock wave type solution of hyperbolic systems of conservation laws, *Q. Appl. Math.* **2005**, *63*, 401.

[5] G. Ercole, Delta-shock waves as self-similar viscosity limits, *Q. Appl. Math.* **2000**, *LVIII*, 177.

[6] D. Korchinski, *Solution of a Riemann problem for a* 2×2 *system of conservation laws possessing no classical weak solution*, Ph.D. thesis, Adelphi University, **1977**.

[7] J. Li, T. Zhang, S. Yang, *The two-dimensional Riemann problem in gas dynamics*, Pitman monographs and surveys in Pure and Applied mathematics 98, Addison Wesley Longman Limited, **1998**.

[8] Y.-G. Lu, Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz–Kranzer type, *J. Funct. Anal.* **2011**, *261*, 2797.

[9] D. Tan, T. Zhang, Y. Zheng, Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws, *J. Differ. Equ.* **1994**, *112*, 1.

[10] B. Temple, Global solution of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws, *J. Adv. in Appl. Math.* **1982**, *3*, 335.

[11] H. Yang, Riemann problems for a class of coupled hyperbolic systems of conservation laws, *J. Differ. Equ.* **1999**, *159*, 447.

How to cite this article: De la cruz R, Santos M. Delta shock waves for a system of Keyfitz-Kranzer type. *Z Angew Math Mech*. 2019;99:e201700251. https://doi.org/10.1002/zamm.201700251