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Research article

L^p-solutions of the Navier-Stokes equation with fractional Brownian noise

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Abstract: We study the Navier-Stokes equations on a smooth bounded domain $D \subset \mathbb{R}^d$ ($d = 2$ or 3), under the effect of an additive fractional Brownian noise. We show local existence and uniqueness of a mild L^p-solution for $p > d$.

Keywords: stochastic partial differential equations; Navier-Stokes equations; mild solution; fractional Brownian motion
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1. Introduction

The Navier-Stokes equations have been derived, more than one century ago, by the engineer C. L. Navier to describe the motion of an incompressible Newtonian fluid. Later, they have been reformulated by the mathematician-physicist G. H. Stokes. Since that time, these equations continue to attract a great deal of attention due to their mathematical and physical importance. In a seminal paper [16], Leray proved the global existence of a weak solution with finite energy. It is well known that weak solutions are unique and regular in two spatial dimensions. In three dimensions, however, the question of regularity and uniqueness of weak solutions is an outstanding open problem in mathematical fluid mechanics, we refer to excellent monographs [15], [17] and [20].

More recently, stochastic versions of the Navier-Stokes equations have been considered in the literature; first by introducing a stochastic forcing term which comes from a Brownian motion (see, e.g., the first results in [2, 22]). The addition of the white noise driven term to the basic governing equations is natural for both practical and theoretical applications to take into account for numerical and empirical uncertainties, and have been proposed as a model for turbulence. Later on other kinds of noises have been studied.

In this paper we consider the the Navier-Stokes equations with a stochastic forcing term modelled by a fractional Brownian motion

$$\begin{cases} \partial_t u(t,x) = (\nu\Delta u(t,x) - [u(t,x)\cdot\nabla]u(t,x) - \nabla\pi(t,x))dt + \Phi\partial_t W^{\mathcal{H}}(t,x) \\ \text{div } u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

We fix a smooth bounded domain $D \subset \mathbb{R}^d$ ($d = 2$ or 3) and consider the homogeneous Dirichlet boundary condition for the velocity. In the equation above, $u(t,x) \in \mathbb{R}^d$ denotes the vector velocity field at time t and position $x \in D$, $\pi(t,x)$ denotes the pressure field, $\nu > 0$ is the viscosity coefficient. In the random forcing term there appears a Hilbert space-valued cylindrical fractional Brownian motion $W^{\mathcal{H}}$ with Hurst parameter $\mathcal{H} \in (0,1)$ and a linear operator Φ to characterize the spatial covariance of the noise.

When $\mathcal{H} = \frac{1}{2}$, i.e. $W^{\frac{1}{2}}$ is the Wiener process, there is a large amount of literature on the stochastic Navier-Stokes equation (1.1) and its abstract setting. For an overview of the known results, recent developments, as well as further references, we refer to [1], [8], [14] and [22]. On the other hand, when $0 < \mathcal{H} < 1$ there are results by Fang, Sundar and Viens; in [6] they prove when $d = 2$ the existence of a unique global solution which is L^4 in time and in space by assuming that the Hurst parameter $\mathcal{H}$ satisfies a condition involving the regularity of Φ.

Our aim is to deal with L^p-solutions of the Navier-Stokes systems (1.1) for $p > d$. Our approach to study L^p-solutions is based on the concept of mild solution as in [6]; but we deal with dimension $d = 2$ as well as with $d = 3$ and any $p > d$.

We shall prove a local existence and uniqueness result. Some remarks on global solutions will also be given. Let us recall also that results on the local existence of mild L^p-solutions in the deterministic setting were established in the papers [9–13, 23].

In more details, in Section 2 we shall introduce the mathematical setting, in Section 3 we shall deal with the linear problem and in Section 4 we shall prove our main result.

2. Functional setting

In this section we introduce the functional setting to rewrite system (1.1) in abstract form.

2.1. The functional spaces

Let D be a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with smooth boundary ∂D. For $1 \leq p < \infty$ we denote

$$L^p_\sigma = \text{ the closure in } [L^p(D)]^d \text{ of } \{u \in [C_0^\infty(D)]^d,\ div\ u = 0\}$$

and

$$G^p = \{\nabla q, q \in W^{1,p}(D)\}.$$

We then have the following Helmholtz decomposition

$$[L^p(D)]^d = L^p_\sigma \oplus G^p,$$

where the notation $\oplus$ stands for the direct sum. In the case $p = 2$ the sum above reduces to the orthogonal decomposition and L^2_σ is a separable Hilbert space, whose scalar product is denoted by $(\cdot,\cdot)$.

2.2. The Stokes operator

Let us recall some results on the Stokes operator (see, e.g., [20]).

Now we fix p. Let P be the continuous projection from $[L^p(D)]^d$ onto L^p_σ and let Δ be the Laplace operator in L^p with zero boundary condition, so that $D(\Delta) = \{u \in [W^{2,p}(D)]^d : u|_{\partial D} = 0\}$.

Now, we define the Stokes operator A in L^p_σ by $A = -P\Delta$ with domain $H^{2,p} := L^p_\sigma \cap D(\Delta)$. The operator $-A$ generates a bounded analytic semigroup $\{S(t)\}_{t\geq 0}$ of class C_0 in L^p_σ.

In particular, for $p = 2$ we set $H^2 = H^{2,2}$ and the Stokes operator $A : H^2 \to L^2_\sigma$ is an isomorphism, the inverse operator A^{-1} is self-adjoint and compact in L^2_σ. Thus, there exists an orthonormal basis $\{e_j\}_{j=1}^\infty \subset H^2$ of L^2_σ consisting of the eigenfunctions of A^{-1} and such that the sequence of eigenvalues $\{\lambda_j^{-1}\}_{j=1}^\infty$, with $\lambda_j > 0$, converges to zero as $j \to \infty$. In particular, λ_j behaves as $j^{\frac{2}{d}}$ for $j \to \infty$. Then, $\{e_j\}_j$ is also the sequence of eingenfunctions of A corresponding to the eigenvalues $\{\lambda_j\}_j$. Moreover A a is positive, selfadjoint and densely defined operator in L^2_σ. Using the spectral decomposition, we construct positive and negative fractional power operators A^β, $\beta \in \mathbb{R}$. For $\beta \geq 0$ we have the following representation for $(A^\beta, D(A^\beta))$ as a linear operator in L^2_σ

$$D(A^\beta) = \{v \in L^2_\sigma : \ \|v\|^2_{D(A^\beta)} = \sum_{j=1}^\infty \lambda_j^{2\beta} |(v, e_j)|^2 < \infty\},$$

$$A^\beta v = \sum_{j=1}^\infty \lambda_j^\beta (v, e_j) e_j.$$

For negative exponents, we get the dual space: $D(A^{-\beta}) = (D(A^\beta))'$. We set $H^s = D(A^{\frac{s}{2}})$. Let us point out that the operator $A^{-\beta}$ is an Hilbert-Schmidt operator in L^2_σ for any $\beta > \frac{d}{4}$; indeed, denoting by $\|\cdot\|_{\gamma(L^2_\sigma, L^2_\sigma)}$ the Hilbert-Schmidt norm, we have

$$\|A^{-\beta}\|^2_{\gamma(L^2_\sigma, L^2_\sigma)} := \sum_{j=1}^\infty \|A^{-\beta} e_j\|^2_{L^2_\sigma} = \sum_{j=1}^\infty \lambda_j^{-2\beta} \sim \sum_{j=1}^\infty j^{-2\beta\frac{2}{d}}$$

and the latter series in convergent for $2\beta\frac{2}{d} > 1$.

We also recall (see, e.g., [23]) that for any $t > 0$ we have

$$\|S(t)u\|_{L^p_\sigma} \leq \frac{M}{t^{\frac{d}{2}(\frac{1}{r}-\frac{1}{p})}} \|u\|_{L^r_\sigma} \ \text{ for } \ 1 < r \leq p < \infty \tag{2.1}$$

$$\|A^\alpha S(t)u\|_{L^r_\sigma} \leq \frac{M}{t^\alpha} \|u\|_{L^r_\sigma} \ \text{ for } \ 1 < r < \infty, \ \alpha > 0 \tag{2.2}$$

for any $u \in L^r_\sigma$, where M denotes different constants depending on the parameters. Moreover we have the following result on the Hilbert-Schmidt norm of the semigroup, that we shall use later on. What is important is the behaviour for t close to 0, let us say for $t \in (0, 1)$.

Lemma 2.1. *We have*

$$\|S(t)\|_{\gamma(H^{\frac{d}{2}}; L^2_\sigma)} \leq M(2 - \ln t) \qquad \forall t \in (0, 1)$$

and for $q < \frac{d}{2}$

$$\|S(t)\|_{\gamma(H^q; L^2_\sigma)} \leq \frac{M}{t^{\frac{d}{4}-\frac{q}{2}}} \qquad \forall t > 0 \tag{2.3}$$

Proof. The Hilbert-Schmidt norm of the semigroup can be computed. Recall that $\{\frac{e_j}{\lambda_j^{q/2}}\}_j$ is an orthonormal basis of H^q. Thus

$$\|S(t)\|^2_{\gamma(H^q,L^2_\sigma)} = \sum_{j=1}^{\infty} \|S(t)\frac{e_j}{\lambda_j^{q/2}}\|^2_{L^2_\sigma} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^q}\|e^{-\lambda_j t}e_j\|^2_{L^2_\sigma} = \sum_{j=1}^{\infty} \frac{e^{-2\lambda_j t}}{\lambda_j^q}.$$

Since $\lambda_j \sim j^{\frac{2}{d}}$ as $j \to \infty$, we estimate

$$\|S(t)\|^2_{\gamma(H^q,L^2_\sigma)} \leq C\sum_{j=1}^{\infty} \frac{e^{-2j^{\frac{2}{d}}t}}{j^{\frac{2q}{d}}}.$$

Therefore we analyse the series $s_q(t) = \sum_{j=1}^{\infty} \frac{e^{-2j^{\frac{2}{d}}t}}{j^{\frac{2q}{d}}}$. Let us consider different values of the parameter q.

• When $q = \frac{d}{2}$ the series becomes

$$s_{\frac{d}{2}}(t) = \sum_{j=1}^{\infty} j^{-1}e^{-2j^{\frac{2}{d}}t} = e^{-2t} + \sum_{j=2}^{\infty} j^{-1}e^{-2j^{\frac{2}{d}}t} \leq e^{-2t} + \int_1^{\infty} \frac{1}{x}e^{-2x^{\frac{2}{d}}t}dx.$$

The integral is computed by means of the change of variable $x = y^d t^{-\frac{d}{2}}$ so to get

$$\int_1^{\infty} \frac{1}{x}e^{-2x^{\frac{2}{d}}t}\mathrm{d}x = \int_{\sqrt{t}}^{\infty} \frac{d}{y}e^{-2y^2}\mathrm{d}y.$$

Hence, for $t \in (0,1)$ we get

$$s_{\frac{d}{2}}(t) \leq e^{-2t} + d\int_{\sqrt{t}}^{1} \frac{1}{y}\mathrm{d}y + \int_1^{\infty} e^{-2y^2}\mathrm{d}y \leq 1 - \frac{d}{2}\ln t + C.$$

• When $0 \leq q < \frac{d}{2}$ then the sequence of the addends is monotone decreasing and therefore we estimate the series by an integral:

$$\sum_{j=1}^{\infty} \frac{e^{-2j^{\frac{2}{d}}t}}{j^{\frac{2q}{d}}} \leq \int_0^{\infty} \frac{e^{-2x^{\frac{2}{d}}t}}{x^{\frac{2q}{d}}}dx.$$

Again, by the change of variable $x = y^d t^{-\frac{d}{2}}$ we calculate the integral and get

$$\sum_{j=1}^{\infty} \frac{e^{-2j^{\frac{2}{d}}t}}{j^{\frac{2q}{d}}} \leq t^{q-\frac{d}{2}}d\int_0^{\infty} y^{d-2q-1}e^{-2y^2}\mathrm{d}y.$$

The latter integral is convergent since $d - 2q - 1 > -1$ by the assumption that $q < \frac{d}{2}$. Hence we get the bound (2.3) for the Hilbert-Schmidt norm of $S(t)$.

• When $q < 0$ the sequence of the addends in the series $s_q(t)$ is first increasing and then decreasing. Let us notice that $t \mapsto s_q(t)$ (defined for $t > 0$) is a continuous decreasing positive function converging to 0 as $t \to +\infty$. Hence to estimate it for $t \to 0^+$ it is enough to get an estimate over a sequence $t_n \to 0^+$. We choose this sequence in such a way that the maximal value of the function $a_t(x) := x^{-\frac{2q}{d}} e^{-2x^{\frac{2}{d}} t}$ (defined for $x > 0$) is attained at the integer value $n = (-\frac{q}{2t_n})^{\frac{d}{2}} \in \mathbb{N}$. In this way we can estimate the series by means of an integral:

$$\begin{aligned} s_q(t_n) \equiv \sum_{j=1}^{\infty} a_{t_n}(j) &\leq \int_1^n a_{t_n}(x)\mathrm{d}x + a_{t_n}(n) + \int_n^{\infty} a_{t_n}(x)\mathrm{d}x \\ &= \int_1^{\infty} x^{-\frac{2q}{d}} e^{-2x^{\frac{2}{d}} t_n}\mathrm{d}x + n^{-\frac{2q}{d}} e^{-2n^{\frac{2}{d}} t_n} \\ &\leq d\Big(\int_0^{\infty} y^{d-1-2q} e^{-2y^2}\mathrm{d}y \Big)\, t_n^{q-\frac{d}{2}} + C_q t_n^q \end{aligned}$$

where we have computed the integral by means of the change of variable $x = y^d t_n^{-\frac{d}{2}}$ as before. Hence, we get that

$$s_q(t_n) \leq \tilde{C} t_n^{q-\frac{d}{2}} \quad \text{for any } n$$

and therefore for $t \to 0^+$

$$s_q(t) \leq \frac{C}{t^{\frac{d}{2}-q}}.$$

This proves (2.3) when $q < 0$. □

2.3. The bilinear term

Let us define the nonlinear term by $B(u, v) = -P[(u \cdot \nabla)v]$. Following [20], this is first defined on smooth divergence free vectors fields with compact support and one proves by integration by parts that

$$\langle B(u, v), z\rangle = -\langle B(u, z), v\rangle, \qquad \langle B(u, v), v\rangle = 0 \tag{2.4}$$

Then one specifies that B is continuous with respect to suitable topologies. In particular, Hölder inequality provides

$$\|B(u, v)\|_{H^{-1}} \leq \|u\|_{L^4_\sigma} \|v\|_{L^4_\sigma}$$

and thus $B : L^4_\sigma \times L^4_\sigma \to H^{-1}$ is continuous.

Since u is a divergence free vector field, we also have the representation $B(u, v) = -P[\mathrm{div}\,(u \otimes v)]$ which will be useful later on (again this holds for smooth entries and then is extended for u and v suitably regular).

For short we shall write $B(u)$ instead of $B(u, u)$.

2.4. Fractional Brownian motion

First, we recall that a real fractional Brownian motion (fBm) $\{B^{\mathcal{H}}(t)\}_{t\geq 0}$ with Hurst parameter $\mathcal{H} \in (0, 1)$ is a centered Gaussian process with covariance function

$$\mathbb{E}[B^{\mathcal{H}}(t)B^{\mathcal{H}}(s)] := R_{\mathcal{H}}(t, s) = \frac{1}{2}(t^{2\mathcal{H}} + s^{2\mathcal{H}} - |t - s|^{2\mathcal{H}}), \quad s, t \geq 0 \tag{2.5}$$

For more details see [18].

We are interested in the infinite dimensional fractional Brownian motion. We consider the separable Hilbert space L^2_σ and its orthonormal basis $\{e_j\}_{j=1}^\infty$. Then we define

$$W^{\mathcal{H}}(t) = \sum_{j=1}^{\infty} e_j \beta_j^{\mathcal{H}}(t) \tag{2.6}$$

where $\{\beta_j^{\mathcal{H}}\}_j$ is a family of independent real fBm's defined on a complete filtered probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_t, \mathbb{P})$. This is the so called L^2_σ-cylindrical fractional Brownian motion. Moreover we consider a linear operator Φ defined in L^2_σ. Notice that the series in (2.6) does not converge in L^2_σ.

We need to define the integral of the form $\int_0^t S(t-s)\Phi dW^{\mathcal{H}}(s)$, appearing in the definition of mild solution; we will analyze this stochastic integral in Section 3.

2.5. *Abstract equation*

Applying the projection operator P to (1.1) we get rid of the pressure term; setting $\nu = 1$, equation (1.1) becomes

$$\begin{cases} du(t) + Au(t)\, dt = B(u(t))\, dt + \Phi \mathrm{d}W^{\mathcal{H}}(t), & t > 0 \\ u(0) = u_0 \end{cases} \tag{2.7}$$

We consider its mild solution on the time interval $[0, T]$ (for any finite T).

Definition 2.2. A measurable function $u : \Omega \times [0, T] \to L^p_\sigma$ is a mild L^p-solution of equation (2.7) if

- $u \in C([0, T]; L^p_\sigma)$, $\mathbb{P}$-a.s.
- for all $t \in (0, T]$, we have

$$u(t) = S(t)u_0 + \int_0^t S(t-s)B(u(s))\, ds + \int_0^t S(t-s)\Phi \mathrm{d}W^{\mathcal{H}}(s) \tag{2.8}$$

$\mathbb{P}$-a.s.

3. The linear equation

Now we consider the linear problem associated to the Navier-Stokes equation (2.7), that is

$$dz(t) + Az(t)\, dt = \Phi dW^{\mathcal{H}}(t) \tag{3.1}$$

When the initial condition is $z(0) = 0$, its mild solution is the stochastic convolution

$$z(t) = \int_0^t S(t-s)\Phi\, dW^{\mathcal{H}}(s). \tag{3.2}$$

To analyze its regularity we appeal to the following result.

Proposition 1. *Let* $0 < \mathcal{H} < 1$.
If there exist $\lambda, \alpha \geq 0$ *such that*

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \leq \frac{C}{t^\lambda} \qquad \forall t > 0 \tag{3.3}$$

and

$$\lambda + \frac{\alpha}{2} < \mathcal{H} \tag{3.4}$$

then z has a version which belongs to $C([0,T]; H^\alpha)$.

Proof. This is a well known result for $\mathcal{H} = \frac{1}{2}$. Moreover, the case $\mathcal{H} < \frac{1}{2}$ is proved in Theorem 11.11 of [19] and the case $\mathcal{H} > \frac{1}{2}$ in Corollary 3.1 of [5], by assuming that the semigroup $\{S(t)\}_t$ is analytic. □

Now we use this result with $\alpha = d(\frac{1}{2} - \frac{1}{p})$ for $p > 2$; by means of the Sobolev embedding $H^{d(\frac{1}{2}-\frac{1}{p})}(D) \subset L^p(D)$, this provides that z has a version which belongs to $C([0,T]; L^p_\sigma)$.

We have our regularity result for the stochastic convolution by assuming that $\Phi \in \mathcal{L}(L^2_\sigma; H^q)$ for some $q \in \mathbb{R}$, as e.g. when $\Phi = A^{-\frac{q}{2}}$.

Proposition 2. *Let $0 < \mathcal{H} < 1$, $2 < p < \infty$ and $\Phi \in \mathcal{L}(L^2_\sigma, H^q)$ for some $q \in \mathbb{R}$. If the parameters fulfil*

$$\frac{d}{2}(1 - \frac{1}{p}) - \frac{q}{2} < \mathcal{H} \tag{3.5}$$

then the process z given by (3.2) has a version which belongs to $C([0,T]; H^{d(\frac{1}{2}-\frac{1}{p})})$. By Sobolev embedding this version is in $C([0,T]; L^p_\sigma)$ too.

Proof. According to Proposition 1 we have to estimate the Hilbert-Schmidt norm of the operator $S(t)\Phi$. We recall that the product of two linear operators is Hilbert-Schmidt if at least one of them is of Hilbert-Schmidt type.

Bearing in mind Lemma 2.1, when $q < \frac{d}{2}$ we get

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \le \|S(t)\|_{\gamma(H^q, L^2_\sigma)} \|\Phi\|_{\mathcal{L}(L^2_\sigma, H^q)} \le \frac{C}{t^{\frac{d}{4}-\frac{q}{2}}} \tag{3.6}$$

and when $q = \frac{d}{2}$ we get

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \le \|S(t)\|_{\gamma(H^{\frac{d}{2}}, L^2_\sigma)} \|\Phi\|_{\mathcal{L}(L^2_\sigma, H^{\frac{d}{2}})} \le \frac{C}{t^a} \tag{3.7}$$

for any $a > 0$ (here the constant depends also on a). Therefore when $q < \frac{d}{2}$ we choose $\lambda = \frac{d}{4} - \frac{q}{2}$, $\alpha = d(\frac{1}{2} - \frac{1}{p})$ and condition $\lambda + \frac{\alpha}{2} < \mathcal{H}$ becomes (3.5); when $q = \frac{d}{2}$ we choose $\lambda = a$, $\alpha = d(\frac{1}{2} - \frac{1}{p})$ and since a is arbitrarily small we get again (3.5).

Otherwise, when $q > \frac{d}{2}$ we have that Φ is a Hilbert-Schmidt operator in L^2_σ (since $\|\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \le \|A^{-\frac{q}{2}}\|_{\gamma(L^2_\sigma, L^2_\sigma)} \|A^{\frac{q}{2}}\Phi\|_{\mathcal{L}(L^2_\sigma, L^2_\sigma)}$) and we estimate

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \le \|S(t)\|_{\mathcal{L}(L^2_\sigma, L^2_\sigma)} \|\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} \le C \tag{3.8}$$

for all $t \ge 0$. Actually we can prove something more; we write $A^{\frac{1}{2}(q-\frac{d}{2})} = A^\varepsilon A^{-\frac{d}{4}-\varepsilon} A^{\frac{q}{2}}$ and for any $\varepsilon > 0$ we have

$$\begin{aligned}
\|S(t)A^{\frac{1}{2}(q-\frac{d}{2})}\Phi\|_{\gamma(L^2_\sigma, L^2_\sigma)} &\le \|A^\varepsilon S(t)\|_{\mathcal{L}(L^2_\sigma, L^2_\sigma)} \|A^{-\frac{d}{4}-\varepsilon}\|_{\gamma(L^2_\sigma, L^2_\sigma)} \|A^{\frac{q}{2}}\|_{\mathcal{L}(H^q, L^2_\sigma)} \|\Phi\|_{\mathcal{L}(L^2_\sigma; H^q)} \\
&\le \frac{M}{t^\varepsilon}
\end{aligned}$$

According to Proposition 1, choosing $\gamma = \varepsilon$ and $\alpha = d(\frac{1}{2} - \frac{1}{p}) - (q - \frac{d}{2})$ we obtain that the process

$$\int_0^t S(t-s)A^{\frac{1}{2}(q-\frac{d}{2})}\Phi\, dW^{\mathcal{H}}(s), \quad t \in [0,T]$$

has a $C([0,T]; H^{d(\frac{1}{2}-\frac{1}{p})-(q-\frac{d}{2})})$-valued version if

$$\varepsilon + \frac{1}{2}[d(\frac{1}{2} - \frac{1}{p}) - (q - \frac{d}{2})] < \mathcal{H} < 1$$

i.e. choosing ε very small, if

$$\frac{d}{2}(1 - \frac{1}{p}) - \frac{q}{2} < \mathcal{H} < 1.$$

Since $S(t)$ and $A^{\frac{1}{2}(q-\frac{d}{2})}$ commute, we get as usual that the result holds for the process $A^{\frac{1}{2}(q-\frac{d}{2})}z$. Therefore z has a $C([0,T]; H^{d(\frac{1}{2}-\frac{1}{p})})$-version. Actually this holds when $\alpha = d(\frac{1}{2} - \frac{1}{p}) - (q - \frac{d}{2}) \geq 0$, that is when $q \leq d(1 - \frac{1}{p})$. For larger values of q the regularising effect of the operator Φ is even better and the result holds true for any $0 < \mathcal{H} < 1$. □

Remark 1. Instead of appealing to the Sobolev embedding $H^{d(\frac{1}{2}-\frac{1}{p})} \subset L^p_\sigma$, we could look directly for an L^p-mild solution z, that is a process with $\mathbb{P}$-a.e. path in $C([0,T]; L^p_\sigma)$. Let us check if this approach would be better.

There are results providing the regularity in Banach spaces; see e.g. Corollary 4.4. in the paper [4] by Čoupek, Maslowski, and Ondreját. They involve the γ-radonifying norm instead of the Hilbert-Schmidt norm (see, e.g., [21] for the definition of these norms). However the estimate of the γ-radonifying norm of the operator $S(t)\Phi$ is not trivial. The estimates involved lead anyway to work in a Hilbert space setting. Let us provide some details about this fact.

According to [4], assuming $\frac{1}{2} < \mathcal{H} < 1$ and $1 \leq p\mathcal{H} < \infty$ one should verify that there exists $\lambda \in [0, \mathcal{H})$ such that

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^p_\sigma)} \leq \frac{C}{t^\lambda} \qquad \forall t > 0$$

Given $\Phi \in \mathcal{L}(L^2; H^q)$ we just have to estimate the $\gamma(H^q, L^p_\sigma)$-norm of $S(t)$, since

$$\|S(t)\Phi\|_{\gamma(L^2_\sigma, L^p_\sigma)} \leq \|\Phi\|_{\mathcal{L}(L^2_\sigma, H^q)} \|S(t)\|_{\gamma(H^q, L^p_\sigma)}.$$

The $\gamma(H^q, L^p_\sigma)$-norm of $S(t)$ is equivalent to

$$\left[\int_D \Big(\sum_{j=1}^\infty |S(t)\frac{e_j(x)}{\lambda_j^{q/2}}|^2\Big)^{\frac{p}{2}} dx\right]^{1/p}$$

since $\{\frac{e_j}{\lambda_j^{q/2}}\}_j$ is an orthonormal basis of H^q.

Therefore, we estimate the integral. Let us do it for $p \in 2\mathbb{N}$. We have

$$\begin{aligned}\int_D \Big(\sum_{j=1}^\infty |S(t)\frac{e_j(x)}{\lambda_j^{q/2}}|^2\Big)^{\frac{p}{2}} dx &= \int_D \Big(\sum_{j=1}^\infty \lambda_j^{-q} e^{-2\lambda_j t}|e_j(x)|^2\Big)^{\frac{p}{2}} dx \\ &= \int_D \Pi_{n=1}^{p/2}(\sum_{j_n=1}^\infty \lambda_{j_n}^{-q} e^{-2\lambda_{j_n} t}|e_{j_n}(x)|^2) dx\end{aligned}$$

Using the Hölder inequality, we get

$$\int_D |e_{j_1}(x)|^2 |e_{j_2}(x)|^2 \cdots |e_{j_{p/2}}(x)|^2 dx \le \|e_{j_1}\|^2_{L^p} \|e_{j_2}\|^2_{L^p} \cdots \|e_{j_{p/2}}\|^2_{L^p}$$

Hence

$$\int_D \Big(\sum_{j=1}^\infty |S(t) \frac{e_j(x)}{\lambda_j^{q/2}}|^2 \Big)^{\frac{p}{2}} dx \le \left(\sum_{j=1}^\infty \lambda_j^{-q} e^{-2\lambda_j t} \|e_j\|^2_{L^p} \right)^{p/2}$$

How to estimate $\|e_j\|_{L^p}$? Again using the Sobolev embedding $H^{d(\frac{1}{2}-\frac{1}{p})} \subset L^p$. Actually we are back again to Hilbert spaces and we obtain nothing different with respect to our procedure which started in the Hilbert spaces since the beginning. We leave the details to the reader.

Finally, let us point out that for $0 < \mathcal{H} < \frac{1}{2}$, an L^p-mild solution z can be obtained in the Banach setting by means of Theorem 5.5 in [3]; this requires the operator Φ to be a γ-radonifying operator from L^2_σ to L^p_σ, which is a quite strong assumption. Our method exploits the properties of the semigroup $S(t)$ so to allow weaker assumptions on the operator Φ.

4. Existence and uniqueness results

In this section we study the Navier-Stokes initial problem (2.7) in the space L^p_σ. We prove first the local existence result and then the pathwise uniqueness.

4.1. Local existence

Following [7], we set $v = u - z$, where z is the mild solution of the linear equation (3.1). Therefore

$$\begin{cases} \frac{dv}{dt}(t) + Av(t) = B(v(t) + z(t)), & t > 0 \\ v(0) = u_0 \end{cases} \tag{4.1}$$

and we get an existence result for u by looking for an existence result for v. This is given in the following theorem.

Theorem 4.1. *Let $0 < \mathcal{H} < 1$, $d < p < \infty$ and $\Phi \in \mathcal{L}(L^2_\sigma, H^q)$ for some $q \in \mathbb{R}$. Given $u_0 \in L^p_\sigma$, if the parameters fulfil*

$$\frac{d}{2}(1 - \frac{1}{p}) - \frac{q}{2} < \mathcal{H} \tag{4.2}$$

then there exists a local mild L^p-solution to equation (2.7).

Proof. From Proposition 2 we know that z has a version which belongs to $C([0,T]; L^p_\sigma)$.

Now we observe that to find a mild solution (2.8) to equation (2.7) is equivalent to find a mild solution

$$v(t) = S(t)u_0 + \int_0^t S(t-s)B(v(s) + z(s))ds$$

to equation (4.1).

We work pathwise and define a sequence by iterations: first $v^0 = u_0$ and inductively

$$v^{j+1}(t) = S(t)u_0 + \int_0^t S(t-s)B(z(s) + v^j(s))\,ds, \quad t \in [0,T]$$

for $j = 0, 1, 2, \ldots$.

Let us denote by K_0 the random constant

$$K_0 = \max\left(\|u_0\|_{L^p_\sigma}, \sup_{t\in[0,T]} \|z(t)\|_{L^p_\sigma}\right).$$

We shall show that there exists a random time $\tau > 0$ such that $\sup_{t\in[0,\tau]} \|v^j(t)\|_{L^p_\sigma} \le 2K_0$ for all $j \ge 1$. We have

$$\|v^{j+1}(t)\|_{L^p_\sigma} \le \|S(t)u_0\|_{L^p_\sigma} + \int_0^t \|S(t-s)B(v^j(s) + z(s))\|_{L^p_\sigma}\,ds$$

We observe that from (2.1) and (2.2) we get

$$\|S(t)u_0\|_{L^p_\sigma} \le \|u_0\|_{L^p_\sigma} \tag{4.3}$$

and

$$\begin{aligned}
&\int_0^t \|S(t-s)B((v^j(s) + z(s))\|_{L^p_\sigma} ds \\
&\quad\le \int_0^t \|A^{\frac12} S(t-s)A^{-\frac12} P \operatorname{div}((v^j(s)+z(s)) \otimes (v^j(s)+z(s)))\|_{L^p_\sigma}\,ds, \\
&\quad\le \int_0^t \frac{1}{(t-s)^{\frac12}} \|S(t-s)A^{-\frac12} P \operatorname{div}((v^j(s)+z(s)) \otimes (v^j(s)+z(s)))\|_{L^p_\sigma}\,ds \\
&\quad\le \int_0^t \frac{M}{(t-s)^{\frac12+\frac{d}{2p}}} \|A^{-\frac12} P \operatorname{div}((v^j(s)+z(s)) \otimes (v^j(s)+z(s)))\|_{L^{p/2}_\sigma}\,ds \\
&\quad\le \int_0^t \frac{M}{(t-s)^{\frac12+\frac{d}{2p}}} \|(v^j(s)+z(s)) \otimes (v^j(s)+z(s))\|_{L^{p/2}_\sigma}\,ds \\
&\quad\le \int_0^t \frac{M}{(t-s)^{\frac12+\frac{d}{2p}}} \|v^j(s)+z(s)\|^2_{L^p_\sigma}\,ds
\end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we deduce that

$$\begin{aligned}
\|v^{j+1}(t)\|_{L^p_\sigma} &\le K_0 + \int_0^t \frac{M}{(t-s)^{\frac12+\frac{d}{2p}}} \|v^j(s)+z(s)\|^2_{L^p_\sigma}\,ds \\
&\le K_0 + \int_0^t \frac{2M}{(t-s)^{\frac12+\frac{d}{2p}}} \|z(s)\|^2_{L^p_\sigma}\,ds + \int_0^t \frac{2M}{(t-s)^{\frac12+\frac{d}{2p}}} \|v^j(s)\|^2_{L^p_\sigma}\,ds
\end{aligned}$$

Thus, when $\frac12 + \frac{d}{2p} < 1$ (i.e. $p > d$) we get

$$\begin{aligned}
\sup_{t\in[0,T]} \|v^{j+1}(t)\|_{L^p_\sigma} &\le K_0 + 2M \frac{T^{\frac12-\frac{d}{2p}}}{\frac12 - \frac{d}{2p}} \sup_{t\in[0,T]} \|z(t)\|^2_{L^p_\sigma} + 2M \frac{T^{\frac12-\frac{d}{2p}}}{\frac12 - \frac{d}{2p}} (\sup_{t\in[0,T]} \|v^j(t)\|_{L^p_\sigma})^2 \\
&\le K_0 + \frac{4pM}{p-d} T^{\frac12-\frac{d}{2p}} K_0^2 + \frac{4pM}{p-d} T^{\frac12-\frac{d}{2p}} (\sup_{t\in[0,T]} \|v^j(t)\|_{L^p_\sigma})^2
\end{aligned}$$

Now we show that if $\sup_{t\in[0,T]} \|v^j(t)\|_{L^p_\sigma} \leq 2K_0$, then $\sup_{t\in[0,T]} \|v^{j+1}(t)\|_{L^p_\sigma} \leq 2K_0$ on a suitable time interval. Indeed, from the latter relationship we get

$$\begin{aligned}\sup_{t\in[0,T]} \|v^{j+1}(t)\|_{L^p_\sigma} &\leq K_0 + \frac{4pM}{p-d} T^{\frac{1}{2}-\frac{d}{2p}} K_0^2 + \frac{4pM}{p-d} T^{\frac{1}{2}-\frac{d}{2p}} 4K_0^2 \\ &= 2K_0 \left(\frac{1}{2} + \frac{1}{2} \frac{20pM}{p-d} T^{\frac{1}{2}-\frac{d}{2p}} K_0 \right).\end{aligned}$$

Hence, when T is such that

$$\frac{20pM}{p-d} T^{\frac{1}{2}-\frac{d}{2p}} K_0 \leq 1$$

we obtain the required bound. Therefore we define the stopping time

$$\tau = \min\left\{T, \left(\frac{p-d}{20pMK_0}\right)^{\frac{2p}{p-d}}\right\} \tag{4.5}$$

so that

$$\frac{20pM}{p-d} \tau^{\frac{1}{2}-\frac{d}{2p}} K_0 \leq 1 \tag{4.6}$$

and obtain that

$$\sup_{t\in[0,\tau]} \|v^j(t)\|_{L^p_\sigma} \leq 2K_0 \qquad \forall j. \tag{4.7}$$

Now, we shall show the convergence of the sequence v^j. First, notice that

$$\begin{aligned}&B(v^{j+1} + z) - B(v^j + z) \\ &\quad = -P\mathrm{div}\,((v^{j+1} - v^j) \otimes v^{j+1} + v^j \otimes (v^{j+1} - v^j) + (v^{j+1} - v^j) \otimes z + z \otimes (v^{j+1} - v^j)).\end{aligned}$$

We proceed as in (4.4) and get

$$\begin{aligned}\|v^{j+2}(t)&-v^{j+1}(t)\|_{L^p_\sigma} \\ &\leq \int_0^t \|S(t-s)(B(v^{j+1}(s) + z(s)) - B(v^j(s) + z(s)))\|_{L^p_\sigma} ds \\ &\leq \int_0^t \frac{M}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}} (\|v^{j+1}(s)\|_{L^p_\sigma} + \|v^j(s)\|_{L^p_\sigma} + 2\|z(s)\|_{L^p_\sigma})\, \|v^{j+1}(s) - v^j(s)\|_{L^p_\sigma} ds\end{aligned}$$

Hence, using (4.7) we get

$$\begin{aligned}\sup_{t\in[0,\tau]} \|v^{j+2}(t) - v^{j+1}(t)\|_{L^p_\sigma} &\leq \int_0^\tau \frac{M6K_0}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}} ds \left(\sup_{s\in[0,\tau]} \|v^{j+1}(s) - v^j(s)\|_{L^p_\sigma} \right) \\ &= \frac{12pMK_0}{p-d}\, \tau^{\frac{1}{2}-\frac{d}{2p}} \left(\sup_{t\in[0,\tau]} \|v^{j+1}(t) - v^j(t)\|_{L^p_\sigma} \right)\end{aligned}$$

Setting $C_0 = \frac{12pMK_0}{p-d}\, \tau^{\frac{1}{2}-\frac{d}{2p}}$, from (4.5)-(4.6) we obtain that $C_0 < 1$. Moreover

$$\begin{aligned}\sup_{t\in[0,\tau]} \|v^{j+2}(t) - v^{j+1}(t)\|_{L^p_\sigma} &\leq C_0 \sup_{t\in[0,\tau]} \|v^{j+1}(t) - v^j(t)\|_{L^p_\sigma} \\ &\leq C_0^{j+1} \sup_{t\in[0,\tau]} \|v^1(t) - v^0(t)\|_{L^p_\sigma}\end{aligned}$$

Therefore $\{v^j\}_j$ is a Cauchy sequence; hence it converges, that is there exists $v \in C([0,\tau]; L^p_\sigma)$ such that $v^j \to v$ in $C([0,\tau]; L^p_\sigma)$. This proves the existence of a unique local mild L^p-solution v for equation (4.1).

Since $u = v + z$, we have got a local mild L^p-solution u for equation (2.7). □

Remark 2. We briefly discuss the case of cylindrical noise, i.e. $\Phi = Id$. Bearing in mind Theorem 4.1, the parameters fulfil

$$\frac{d}{2}(1 - \frac{1}{p}) < \mathcal{H} < 1. \tag{4.8}$$

When $2 = d < p$, this means that p and $\mathcal{H}$ must be chosen in such a way that

$$1 - \frac{1}{p} < \mathcal{H} < 1 \tag{4.9}$$

This means that $\mathcal{H}$ must be at least larger than $\frac{1}{2}$. On the other hand, when $3 = d < p$ we cannot apply our procedure, since $\frac{d}{2}(1 - \frac{1}{p}) > 1$ and therefore the set of conditions (4.8) is void.

4.2. Uniqueness

Now we show pathwise uniqueness of the solution given in Theorem 4.1.

Theorem 4.2. *Let* $0 < \mathcal{H} < 1$, $d < p < \infty$ *and* $\Phi \in \mathcal{L}(L^2_\sigma, H^q)$ *for some* $q \in \mathbb{R}$.
Given $u_0 \in L^p_\sigma$, *if the parameters fulfil*

$$\frac{d}{2}(1 - \frac{1}{p}) - \frac{q}{2} < \mathcal{H}$$

then the local mild L^p*-solution to equation* (2.7) *given in Theorem 4.1 is pathwise unique.*

Proof. Let u and $\tilde{u}$ be two mild solutions of equation (2.7) with the same fBm and the same initial velocity. Their difference satisfies an equation where the noise has disappeared. Hence we work pathwise. We get

$$u(t) - \tilde{u}(t) = \int_0^t S(t-s)\big(B(u(s)) - B(\tilde{u}(s))\big)\, ds.$$

Writing $B(u) - B(\tilde{u}) = B(u - \tilde{u}, u) + B(\tilde{u}, u - \tilde{u})$, by classical estimations as before we have

$$\begin{aligned} \|u(t) - \tilde{u}(t)\|_{L^p_\sigma} &\leq \int_0^t \|S(t-s)\big(B(u(s)) - B(\tilde{u}(s))\big)\|_{L^p_\sigma}\, ds \\ &\leq \int_0^t \frac{M}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}} (\|u(s)\|_{L^p_\sigma} + \|\tilde{u}(s)\|_{L^p_\sigma}) \|u(s) - \tilde{u}(s)\|_{L^p_\sigma}\, ds \end{aligned}$$

Thus

$$\sup_{[0,\tau]} \|u(t) - \tilde{u}(t)\|_{L^p_\sigma} \leq 4K_0 M \frac{\tau^{\frac{1}{2}-\frac{d}{2p}}}{\frac{1}{2} - \frac{d}{2p}} \sup_{t\in[0,\tau]} \|u(t) - \tilde{u}(t)\|_{L^p_\sigma}.$$

Keeping in mind the definition (4.5) of τ and (4.6) we get

$$\sup_{[0,\tau]} \|u(t) - \tilde{u}(t)\|_{L^p_\sigma} \leq \frac{2}{5} \sup_{[0,\tau]} \|u(t) - \tilde{u}(t)\|_{L^p_\sigma}$$

which implies $u(t) = \tilde{u}(t)$ for any $t \in [0,\tau]$. □

4.3. Global existence

Let us recall that [6] proved global existence an uniqueness of an $L^4((0,T)\times D)$-valued solution. A similar result of global existence for a less regular (in time) solution holds in our setting.

Let us begin with the case $d = 2$ and consider a process solving equation (2.7) whose paths are in $L^{\frac{2p}{p-2}}(0,T;L^p_\sigma)$. Its local existence comes from the previous results. However we can prove an a priori bound leading to global existence.

Let us multiply equation (4.1) by v in L^2_σ; we obtain by classical techniques (see Lemma 4.1 of [8])

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2_\sigma} + \|\nabla v(t)\|^2_{L^2} &= \langle B(v(t)+z(t),z(t)),v(t)\rangle \\ &\le \|v(t)+z(t)\|_{L^4_\sigma}\|z(t)\|_{L^4_\sigma}\|\nabla v(t)\|_{L^2} \\ &\le \frac{1}{2}\|\nabla v(t)\|^2_{L^2} + \frac{C}{2}\|z(t)\|^4_{L^4_\sigma}\|v(t)\|^2_{L^2_\sigma} + \frac{C}{2}\|z(t)\|^4_{L^4_\sigma}\end{aligned}$$

Hence

$$\frac{d}{dt}\|v(t)\|^2_{L^2_\sigma} \le C\|z(t)\|^4_{L^4_\sigma}\|v(t)\|^2_{L^2_\sigma} + C\|z(t)\|^4_{L^4_\sigma}.$$

As soon as z is a $C([0,T];L^4_\sigma)$-valued process we get by means of Gronwall lemma that $v \in L^\infty(0,T;L^2_\sigma)$. And integrating in time the first inequality we also obtain that $v \in L^2(0,T;H^1)$. By interpolation $L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1) \subset L^{\frac{2p}{p-2}}(0,T;H^{1-\frac{2}{p}})$ for $2 < p < \infty$. Using the Sobolev embedding $H^{1-\frac{2}{p}} \subset L^p_\sigma$, we have the a priori estimate for v in the $L^{\frac{2p}{p-2}}(0,T;L^p_\sigma)$ norm, which provides the global existence of v and hence of u. This holds for $d = 2$ and $4 \le p < \infty$, since the global estimate holds when z is $C([0,T];L^4_\sigma)$-valued at least.
Notice that for $d = 2$ and $p = 4$ we obtain the same result as by Fang, Sundar and Viens (see Corollary 4.3 in [6]).

Similarly one proceeds when $d = 3$. The change is in the Sobolev embedding, which depends on the spatial dimension. Thus from $v \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1)$ we get by interpolation that $v \in L^{\frac{4p}{3(p-2)}}(0,T;H^{3\frac{p-2}{2p}})$ for $2 < p \le 6$. Using the Sobolev embedding $H^{3\frac{p-2}{2p}} \subset L^p_\sigma$ we conclude that the $L^{\frac{4p}{3(p-2)}}(0,T;L^p_\sigma)$-norm of v is bounded. Hence the global existence of a solution $v \in L^{\frac{4p}{3(p-2)}}(0,T;L^p_\sigma)$ for $4 \le p \le 6$ as well as of a solution $u \in L^{\frac{4p}{3(p-2)}}(0,T;L^p_\sigma)$.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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