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On the completeness of the ADHM construction

Sobre a completude da construção ADHM

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Resumo

Consideramos as soluções anti-auto-duais da equação de Yang-Mills em S^4, conhecidas na literatura como *instantons*. O objetivo deste trabalho é estabelecer a completude da construção ADHM, isto é, que todo instanton em S^4 pode ser construído utilizando este método. A construção ADHM é um procedimento para construir fibrados holomorfos no espaço projetivo complexo $\mathbb{P}^3$, estes sendo relacionados a instantons em S^4 via a correspondência twistor. Verificamos que fibrados holomorfos E correspondentes a instantons satisfazem $H^1(E(-2)) = 0$, uma hipótese fundamental para o teorema de Barth, que afirma a completude da construção ADHM.

Abstract

We consider the anti-self-dual solutions to the Yang-Mills equation on S^4, known in the literature as *instantons*. The aim of this work is to establish the completeness of the ADHM construction, that is, that every instanton on S^4 can be constructed using this technique. The ADHM construction is a procedure for constructing holomorphic bundles on the complex projective space $\mathbb{P}^3$; these being related to instantons on S^4 via the twistor correspondence. It is verified that holomorphic bundles E corresponding to instantons satisfy $H^1(E(-2)) = 0$, a fundamental hypothesis for a theorem of Barth, which asserts completeness of the ADHM construction.

Contents

1 Introduction

This dissertation concerns the construction described in (ATIYAH et al., 1978) of (anti-)self-dual solutions to the Yang-Mills equation on $\mathbb{R}^4$ with respect to gauge groups $G = Sp(n)$, $SU(n)$ and $O(n)$ satisfying appropriate asymptotic conditions. These solutions are known as *G-instantons*. Moreover, we show to a reasonable degree of detail that this construction yields all G-instantons.

We shall first sketch a brief introduction to Yang-Mills theory and the (anti-)self-duality condition, referring to (FADEL, 2016) or (WARD; WELLS, 1990) for a more comprehensive treatment.

The Yang-Mills equation is derived from a variational principle in much the same manner as the Euler-Lagrange equations. The setting however, is not that of ordinary euclidean space. We suppose $E \to S^4$ is a vector bundle (cf. Chapter 2) equipped with a connection ∇ whose curvature we call F (cf. Chapter 4). Then, the the action for this theory is given by

$$\mathcal{A} = -\int_{\mathbb{R}^4} \operatorname{tr}(F \wedge *F) d^4x, \tag{1.1}$$

which is in fact the square of the L^2-norm $\|F\|$ of the curvature in an appropriate context. Notice that it is entirely possible that the integral above attains infinite values, and to prevent such an event we impose appropriate decay conditions on the curvature. The standard approach is to consider instead Yang-Mills theory on the sphere S^4, which is the one-point compactification of $\mathbb{R}^4$; from compactness, the action is then always finite. We can project stereographically onto $\mathbb{R}^4$ and then transpose the Yang-Mills equation on S^4 to $\mathbb{R}^4$. This approach is entirely satisfactory, since by a theorem of K. Uhlenbeck (UHLENBECK, 1982), all finite-action solutions to the Yang-Mills equations arise in this way. With this in mind, following the standard procedure of looking for solutions that are extremal points of the action functional, we obtain the Yang-Mills equation

$$\nabla * F = 0,$$

where $*$ is the Hodge star operator. Since we are working in four dimensions, the Hodge star for 2-forms (such as the curvature F) has the property $*^2 = 1$; as such, $*$ then has eigenvalues $+1$ and -1. We call 2-forms ω satisfying $*\omega = \omega$ *self-dual* (SD) and those satisfying $*\omega = -\omega$ *anti-self-dual* (ASD). It is straightforward to verify that the space of 2-forms decomposes orthogonally as

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-. \tag{1.2}$$

We note that any (A)SD curvature 2-form automatically satisfies the Yang-Mills equation by Bianchi's identity (Proposition 14):

$$\nabla F = 0,$$

and we call such special solutions *(anti-)instantons.*

Vector bundles $E \to S^4$ with structure group G are classified topologically by an element in $\pi_3(G)$; for the groups $G = Sp(n)$, $SU(n)$ or $O(n)$, it follows that $\pi_3(G) = \mathbb{Z}$, and the integer invariant classifying each bundle is its second Chern class $c_2(E) = k$ (also called *instanton number* or *topological charge*), which for $SU(2)$ takes the form

$$c_2(E) = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F \wedge F) d^4x.$$

The expression for the other groups is essentially the same, with k replaced by a suitable multiple (cf. (ATIYAH; HITCHIN; SINGER, 1978) for more details). Taking advantage of decomposition (1.2), we can write the above expression as

$$8\pi^2 k = \|F^+\|^2 - \|F^-\|^2,$$

and (1.1) as

$$\|F\|^2 = \|F^+\|^2 + \|F^-\|^2,$$

giving us

$$\begin{aligned} F^+ &= 0, \text{ if } k < 0, \\ F &= 0, \text{ if } k = 0, \\ F^- &= 0, \text{ if } k > 0. \end{aligned}$$

This shows that (anti-)instantons correspond to absolute *minima* of the Yang-Mills action functional. This argument is due to Belavin in (BELAVIN et al., 1975), who also showed the existence of $SU(2)$-instantons of topological charge $k = \pm 1$.

Atiyah *et al.* showed in (ATIYAH; HITCHIN; SINGER, 1977) that the moduli space of $SU(2)$-instantons of topological charge k is an $8|k| + 3$-dimensional manifold (for $k = 1$ it is hyperbolic 5-space), and a method of constructing instanton solutions was described in (ATIYAH; WARD, 1977) using methods of algebraic geometry. More specifically, through a procedure called *twistor transform* (cf. (PENROSE, 1977)), they established a correspondence between instantons on S^4 and holomorphic vector bundles over $\mathbb{P}^3$ where the techniques of algebraic geometry apply. However, the assertion that every k-instanton arose from this construction was still speculative, with a proof appearing in (ATIYAH et al., 1978) along with a more polished version of the construction, which came to be known as the *ADHM construction*. The proof outlined in (ATIYAH et al., 1978) was possible due to the work of Barth *et al.* in (BARTH; HULEK, 1978) on vector bundles over projective spaces.

2 Vector Bundles

The twistor transform establishes a 1-1 correspondence between instantons and certain vector bundles over twistor spaces. We review here the necessary facts regarding vector bundles we shall use. For proofs of the propositions in this chapter, we refer to (WARD; WELLS, 1990) or (LEE, 2012).

2.1 Basic facts

Definition 1. *Let X, E be smooth manifolds, $p : E \to X$ a smooth submersion, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that (p, E, X) is a* vector bundle of rank k *over X if*

1. *Each fiber $E_x := p^{-1}(x)$ is a k-dimensional vector space,*
2. *For every $x \in X$ there is a neighborhood $U \subset X$ of x such that $\Phi_U : E|_U := p^{-1}(U) \to U \times \mathbb{K}^k$ is a diffeomorphism which is linear when restricted to each fiber.*

We shall usually refer to $p : E \to X$ as a vector bundle, as this string of characters contains the data necessary to define one and also happens to be mathematically meaningful. The maps Φ_U are called *local trivializations* of E over U, with each U being a *trivializing open set*[1]. Moreover, when $\mathbb{K} = \mathbb{C}$, we say that E is a *complex vector bundle*, and when $\mathbb{K} = \mathbb{R}$, we say that E is a *real vector bundle.*

Example 1 (Trivial bundle)**.** *If X is any smooth n-manifold, $E = X \times \mathbb{K}^k$ and $p : E \to X$ is given by $p(x, v) = x$, then E is a vector bundle of rank k, called the trivial bundle. In this case, the fibers are*

$$E_x = \{x\} \times \mathbb{K}^k$$

with the obvious vector space structure inherited from $\mathbb{K}^k$. It is clear that p is smooth, since, in local coordinates $(x_1, \ldots, x_n, v_1, \ldots, v_k)$, it is given by projection onto the first n factors.

The following example will be important in what follows.

Example 2 (Tautological bundle)**.** *If $X = \mathbb{CP}^n$,*

$$E = \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid x \in v\},$$

and $p : E \to X$ is given by $p(x, v) = x$, then E is a complex vector bundle of rank 1, called the tautological bundle and we denote it by $\mathcal{O}(-1)$.

[1] We can alternatively state property 2 in Definition 1 as saying that a vector bundle is locally isomorphic to a trivial bundle, or more concisely, *locally trivial.*

Vector bundles of rank 1 are also called *line bundles.*

Definition 2. *Let $p : E \to X$ and $q : F \to Y$ be vector bundles. A* vector bundle map *from E to F is a pair $(f, \tilde{f})$ of smooth maps $f : E \to F$ and $\tilde{f} : X \to Y$ such that*

1. *The following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\tilde{f}} & Y \end{array}$$

2. *Each restriction $f|_x : E_x \to F_{\tilde{f}(x)}$ is a linear map.*

In the above setting, we also say that f *covers* $\tilde{f}$. Furthermore, a vector bundle map is called an *isomorphism* if it is bijective and its inverse is also smooth.

Definition 3. *Given a vector bundle $E \to X$, then a subset $F \subset E$ is said to be a* subbundle *of E if*

1. *F is a submanifold of E.*

2. *$F \cap E_x \subset E_x$ is a vector subspace for every $x \in X$,*

Suppose we are given vector bundles E and F over X. If $f : E \to F$ is a bundle map covering the identity, then we say f has *constant rank* if $f|_x$ has constant rank for every $x \in X$. This allows us to state

Proposition 1. *If $f : E \to F$ is a bundle map between vector bundles over X, then*

$$\ker f = \coprod_{x \in X} \ker f_x, \quad \operatorname{im} f = \coprod_{x \in X} \operatorname{im} f_x$$

are subbundles of E and F, respectively.

Suppose now that $\{U_\alpha\}$ is an open cover of a manifold X such that each U_α is a trivializing open set for a vector bundle $p : E \to X$, i.e. for each α there exists an isomorphism $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{K}^k$. Pick now two open sets from this covering that are not disjoint, say, U_α and U_β. The composition

$$\Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{K}^k \to U_\alpha \cap U_\beta \times \mathbb{K}^k$$

is a diffeomorphism such that $\pi_1 \circ \Phi_\alpha \circ \Phi_\beta^{-1} = \pi_1$, where $\pi_1 : U_\alpha \cap U_\beta \times \mathbb{K}^n \to U_\alpha \cap U_\beta$ is the projection onto the first factor; this is to say $\Phi_\alpha \circ \Phi_\beta^{-1}$ has the form

$$(x, v) \mapsto (x, \phi_{\alpha\beta}(x, v)),$$

for some smooth map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \times \mathbb{K}^n \to \mathbb{K}^n$. In addition, since both Φ_α and Φ_β^{-1} are linear fiberwise, $\phi_{\alpha\beta}(x, v) = g_{\alpha\beta}(x)v$ for some smooth map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{K}).$$

The maps comprising the set $\{g_{\alpha\beta}\}$ are called *transition functions* relative to the cover $\{U_\alpha\}$. It is clear from their construction that they satisfy

$$g_{\beta\gamma} g_{\gamma\alpha} g_{\alpha\beta} = 1 \tag{2.1}$$

where the concatenation on the left indicates matrix multiplication.

Somewhat surprisingly, a set of transition functions (satisfying (2.1), of course) are in fact sufficient to define a vector bundle; more specifically, we have

Proposition 2. *Let $\{U_\alpha\}$ be an open cover of a manifold X and $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{K})\}$ a collection of smooth maps satisfying (2.1). Then there exists a vector bundle $p : E \to X$ with $\{g_{\alpha\beta}\}$ as transition functions.*

Furthermore, vector bundles with the same transition functions are isomorphic, as shown by

Proposition 3. *Let $p : E \to X$ and $q : F \to X$ be vector bundles with the same set of transition functions $\{g_{\alpha\beta}\}$ relative to an open cover $\{U_\alpha\}$ of X. Then $E \cong F$.*

The situation is actually much more general. First, notice that if we have two bundles E and F over X, their transition functions might be defined relative to *different* covers. However, taking intersections, we can always assume transition functions to be defined relative to the same cover. With this in mind, we have

Proposition 4. *Let $p : E \to X$ and $q : F \to X$ be vector bundles with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$, respectively, relative to an open covering U_α of X. Suppose there are smooth maps $\lambda_\alpha : U_\alpha \to GL(k, \mathbb{K})$ such that*

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} g_{\alpha\beta} \lambda_\beta.$$

Then $E \cong F$.

In what follows, we shall make no distinction between a vector bundle and its isomorphism class. This means, in particular, that a vector bundle is uniquely specified by a collection of transition functions.

Example 3 (Tangent bundle)**.** *If X is a smooth manifold with an atlas $\{(U_\alpha, \phi_\alpha)\}$, then the maps*

$$g_{\alpha\beta} := d(\phi_\alpha \circ \phi_\beta^{-1})$$

are transition functions for what is called the tangent bundle.

We can also consider the tangent bundle to a given vector bundle. This gives rise to

Definition 4. *Let TE be the tangent bundle to $p : E \to X$. Then $dp : TE \to TX$ and we call $VE := \ker dp$ the* vertical bundle *to E.*

The fibers of VE can be thought of as tangent spaces to the fibers of E.

Definition 5. *Let $p : E \to X$ be a vector bundle. A* section *of E is a smooth map $s : X \to E$ such that $p \circ s = 1_X$. The set of all smooth sections of E is denoted by $\Gamma(E)$.*

We can use local trivializations to see what sections look like locally:

$$\begin{aligned} \Phi_U \circ s : U &\to U \times \mathbb{K}^k \\ x &\mapsto (x, s_U(x)), \end{aligned}$$

where $s_U : U \to \mathbb{K}^k$ is a smooth function.

Proposition 5. *Let $E \to X$ be a vector bundle of rank k with transition functions $\{g_{\alpha\beta}\}$ relative to an open cover $\{U_\alpha\}$ of X. Suppose that we are given a collection of smooth maps*

$$s_\alpha : U_\alpha \to \mathbb{K}^k,$$

satisfying on each intersection $U_\alpha \cap U_\beta$

$$s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x).$$

Then $\{s_\alpha\}$ determines a unique section $s \in \Gamma(E)$, and all sections of E arise this way.

Suppose $E \to X$ is a vector bundle of rank k. Choose a trivialization Φ_U, and let

$$s_i(x) := \Phi_U^{-1}(x, e_i) \quad \text{for every } x \in U,$$

where $\{e_i\}$ is the canonical basis for R^k. We see that $\{s_i(x)\}$ is a basis for E_x. More generally,

Definition 6. *Let $E \to X$ be a vector bundle of rank k. Suppose we are given k local sections s_i, $1 \leq i \leq k$, such that $\{s_i(x)\}$ is a basis for E_x, for every $x \in X$. We say that $\{s_i\}$ is a* local frame *for E.*

2.2 New bundles from old

Vector spaces are often combined together with other vector spaces in order to form new ones. One can form direct sums, tensor products, exterior products, and so on. All of these operations are carried over to the context of vector bundles. The idea behind the construction of these new bundles is that, given vector spaces V, W and linear maps $f : V \to V$, $g : W \to W$, there are natural maps defined on whatever combination of vector spaces we are working on. For example, on $V \oplus W$ we have the map

$$\begin{aligned} f \oplus g : V \oplus W &\to V \oplus W \\ (v, w) &\mapsto (f(v), g(w)). \end{aligned}$$

For, say, the top exterior power of V, $\Lambda^n V$, we have a map given by

$$e_1 \wedge \cdots \wedge e_n \mapsto f(e_1) \wedge \cdots \wedge f(e_n) = (\det f) e_1 \wedge \cdots \wedge e_n.$$

The maps at which we shall be looking are, of course, the transition functions for a given bundle. Let us see some examples in that direction. Let $p : E \to X$ and $q : F \to X$ be vector bundles with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, respectively.

Example 4 (Dual bundle)**.** *The vector bundle defined by the transition functions* $g'_{\alpha\beta} = (g_{\alpha\beta}^{-1})^T$ *is called the dual bundle*[2] *to* E *and is denoted by* E^**.*

As a special case of the above construction, the dual bundle to the tangent bundle is called the *cotangent bundle.*

Example 5 (Direct sum bundle)**.** *The vector bundle defined by the transition functions*

$$k_{\alpha\beta} := g_{\alpha\beta} \oplus h_{\alpha\beta} = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{bmatrix}$$

is called the direct sum bundle and is denoted by $E \oplus F$*.*

Example 6 (Tensor product bundle)**.** *The vector bundle defined by the transition functions* $g_{\alpha\beta} \otimes h_{\alpha\beta}$ *is called the tensor product bundle and is denoted by* $E \otimes F$*.*

Example 7 (Pullback bundle)**.** *If* $f : X \to Y$ *is a map between manifolds, then the vector bundle over* Y *defined by the transition functions* $g'_{\alpha\beta} = g_{\alpha\beta} \circ f$ *is called the pullback bundle of* E *and is denoted by* f^*E*.*

Example 8 (Quotient bundle)**.** *Suppose* $F \subset E$ *is a subbundle of rank* k*, and assume* E *has rank* n*. The submanifold condition is equivalent to saying that for every* $x \in X$ *there exists a neighborhood* U *of* $x \in X$ *and a trivialization*

$$\Phi_U : E|_U \to U \times \mathbb{C}^n$$

such that

$$\Phi_U|_{F_U} : F_U \to U \times \mathbb{C}^k \subset U \times \mathbb{C}^n,$$

[2] This is, of course, due to an elementary result in linear algebra which states that the change of coordinates matrix for the dual space is the transposed inverse to that of the original space.

with the inclusion $\mathbb{C}^k \subset \mathbb{C}^n$ being the natural inclusion in the first k coordinates. This means that the transition functions $g_{\alpha\beta}$ with respect to these trivializations are given by

$$g_{\alpha\beta} = \begin{bmatrix} h_{\alpha\beta} & k_{\alpha\beta} \\ 0 & j_{\alpha\beta} \end{bmatrix},$$

where $h_{\alpha\beta}$ are transition functions for F and $j_{\alpha\beta}$ are transition functions for the quotient bundle

$$E/F := \coprod_{x \in X} E_x/F_x.$$

As a particular example of a quotient bundle, we have

Definition 7. *Let X be a manifold and $Y \subset X$ a submanifold. Then there is a natural inclusion $TY \subset TX$, and the quotient bundle*

$$N = (TX)|_Y / TY$$

is called the normal bundle *to Y in X.*

The dual to the normal bundle of a given submanifold is called the *conormal bundle.*

3 Sheaf theory

3.1 Presheaves and sheaves

Definition 8. *A* presheaf $\mathcal{F}$ *over a topological space* X *is defined by the following data*

1. *An assignment of a set* $\mathcal{F}(U)$ *to each open set* $U \subset X$*,*
2. *A* restriction map
$$r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$
for each pair of open sets $V \subset U$*, having the properties*
   a) $r_U^U = 1_U$*,*
   b) $r_W^U = r_W^V \circ r_V^U$ *whenever* $U \subset V \subset W$*.*

If $\mathcal{F}$ *and* $\mathcal{G}$ *are presheaves over* X *with restriction maps* r_V^U *and* ρ_V^U*, respectively, then a* morphism *of presheaves is defined to be a collection of maps*

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U),$$

where U *runs over the open sets* $U \subset X$*, such that* $\phi_V \circ r_V^U = \rho_V^U \circ \phi_U$*. Moreover,* $\mathcal{F}$ *is said to be a* subpresheaf *of* $\mathcal{G}$ *if the maps* ϕ_U *are inclusions.*

We will usually work with presheaves having an additional algebraic structure; $\mathcal{F}(U)$ will often be an abelian group for each open set $U \subset X$. Naturally, we then require restriction maps and morphisms of presheaves to be homomorphisms in the appropriate category.

Regarding (abuse of) notation, we shall frequently denote the restriction of $s \in \mathcal{F}(U)$ to some open set $V \subset U$ by $s|_V$ instead of the onerous $r_V^U(s)$.

Definition 9. *A presheaf* $\mathcal{F}$ *over* X *is called a* sheaf *if for every open set* $U \subset X$ *and every collection* $\{U_i\}$ *of open sets of* X *such that* $U = \cup U_i$*,*

1. *(Identity) Given* $s, t \in \mathcal{F}(U)$*, if* $r_{U_i}^U(s) = r_{U_i}^U(t)$ *for each* i*, then* $s = t$*.*
2. *(Gluability) If* $s_i \in \mathcal{F}(U_i)$ *for every* i *and*
$$r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_j)$$
holds true whenever $U_i \cap U_j \neq \emptyset$*, then there exists* $s \in \mathcal{F}(U)$ *such that* $r_{U_i}^U(s) = s_i$ *for every* i*.*

A morphism of sheaves $\phi = \phi_U$ is given by a morphism between the underlying presheaves, and we say that ϕ is an *isomorphism* if every ϕ_U is an isomorphism in the appropriate category. Moreover, when a subpresheaf of a sheaf $\mathcal{F}$ is also a sheaf, it is said to be a *subsheaf* of $\mathcal{F}$.

Notice that for a presheaf to admit a sheaf structure it is necessary that $\mathcal{F}(\emptyset)$ be a singleton ($\{0\}$ in case $\mathcal{F}$ is a sheaf of abelian groups). To see this, notice that $\emptyset$ itself is an open cover for $\emptyset$. Thus, if $s, t \in \mathcal{F}(\emptyset)$, it is vacuously true that $r^U_{U_i}(s) = r^U_{U_i}(t)$ for each $U_i \in \emptyset$, which means that $s = t$ by the identity axiom. A presheaf with $\mathcal{F}(\emptyset)$ not a singleton is then an example of a presheaf that is not a sheaf, failing to obey the identity axiom.

Most of the sheaves we will see here are subsheaves of the following sheaf:

Example 9. *Let X and Y be topological spaces. Then for each open set $U \subset X$, set*

$$\mathcal{C}_{X,Y}(U) := \{f : U \to Y \mid f \text{ is continuous}\},$$

and for every $f \in \mathcal{C}_{X,Y}(U)$, set

$$r^U_V(f) := f|_V$$

as being the ordinary restriction as a function.

Example 10. *Let X be a (topological, smooth, complex) manifold. Then*

$$\mathcal{C}_X(U) := \{f : U \to \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ is continuous}\}$$
$$\mathcal{E}_X(U) := \{f : U \to \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ is smooth}\}$$
$$\mathcal{O}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$$

define sheaves over X, each instance being called the structure sheaf *of X.*

We shall frequently omit reference to the base manifold when writing down its structure sheaf in order to lighten notation, writing only, e.g. $\mathcal{O}$ for the sheaf of holomorphic functions with the base manifold implicitly understood.

An example of a presheaf that does not obey the gluability axiom is

Example 11. *Let $X = \mathbb{C}$ and $\mathcal{B}(U)$ be the algebra of bounded holomorphic functions on the open set $U \subset X$. Set $U_i = \{z \in X \mid |z| < i\}$. It's clear that $X = \cup U_i$. Moreover, by Liouville's theorem, $\mathcal{B}(X) \cong \mathbb{C}$. Thus, if $f_i \in \mathcal{B}(U_i)$ is given by $f_i(z) = z$, there is no global $f \in \mathcal{B}(X)$ such that $f|_{U_i} = f_i$ for every i.*

The following sheaves play an important role in this work:

Definition 10. *Let $\mathcal{R}$ be a presheaf of commutative rings and $\mathcal{M}$ be a presheaf of abelian groups, both over the same topological space X. Suppose that $\mathcal{M}(U)$ has the structure of* *an $\mathcal{R}(U)$-module for every open set $U \subset X$ that is compatible with the restriction maps; this is to say that*

$$r_V^U(\alpha f) = \rho_V^U(\alpha) r_V^U(f), \quad \alpha \in \mathcal{R}(U) \text{ and } f \in \mathcal{M}(U),$$

where r_V^U and ρ_V^U are the restriction maps for $\mathcal{R}$ and $\mathcal{M}$, respectively. Then $\mathcal{M}$ is said to be a presheaf of $\mathcal{R}$-modules. *If, in addition, $\mathcal{M}$ is a sheaf, it is said to be a* sheaf of $\mathcal{R}$-modules, *or simply an* $\mathcal{R}$-module.

Notice that, *a priori*, there's a different ring acting on each module.

Example 12. *Let $E \to X$ be a (topological, smooth, holomorphic) vector bundle over a (topological, smooth, complex) manifold. Then the assignment of the set of sections of E over U, $\Gamma(U, E)$, to each open set $U \subset X$ together with the natural restrictions define a sheaf of ($\mathcal{C}_X$, $\mathcal{E}_X$, $\mathcal{O}_X$)-modules over X, called the* sheaf of (continuous, smooth, holomorphic) sections of E.

And we will also have plenty of opportunity to see in action

Example 13. *If $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $p \in \mathbb{C}$, define the* sheaf of holomorphic functions vanishing at p *by $\mathcal{I}(U) = \mathcal{O}(U)$ if $p \notin U$ and $\mathcal{I}(U) = \{f \in \mathcal{O}(U) \mid f(p) = 0\}$ if $p \in U$.*

3.2 Sheafification

We have seen examples of presheaves that fail to be sheaves. There is a device, called *sheafification*, which turns every presheaf into a sheaf and which we now begin to describe.

Definition 11. *An* étalé space *over a topological space X is a topological space Y together with a continuous surjection $p : Y \to X$ which is also a local homeomorphism. A* section *of such an étalé space over an open set $U \subset X$ is a continuous map $f : U \to Y$ such that $\pi \circ f = 1_U$.*

We shall denote the sections of Y over $U \subset X$ by $\Gamma(U, Y)$, which is clearly a subsheaf of $\mathcal{C}_{X,Y}$. Context will distinguish whether we're referring to sections of an étalé space or a vector bundle.

Definition 12. *Let $\mathcal{F}$ be a presheaf over X, and define the* stalk *of $\mathcal{F}$ at $x \in X$ to be*

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

with respect to the restriction maps r_V^U. As such, for every $x \in X$ there is a natural map $r_x^U : \mathcal{F}(U) \to \mathcal{F}_x$ taking an element in $\mathcal{F}(U)$ to its equivalence class. Further, if $s \in \mathcal{F}(U)$, *then* $s_x := r_x^U(s)$ *is called the* germ *of* s *at* x*, with* s *being the* representative *of the germ* s_x*.*

Any sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ induces a morphism of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$, since the map $(s, U) \mapsto (\phi_U(s), U)$, where (s, U) represents $s \in \mathcal{F}(U)$, respects the equivalence classes: if $(s, U) \sim (t, V)$, i.e. if there exists an open set $W \subset U \cap V$ containing x on which $s|_W = t|_W$, then $(\phi_U(s)|_W, W) = (\phi_W(s|_W), W) = (\phi_W(t|_W), W) = (\phi_V(t)|_W, W)$, and thus $(\phi_U(s), U) \sim (\phi_V(t), V)$.

Now if

$$\mathrm{et}(\mathcal{F}) := \coprod_{x \in X} \mathcal{F}_x$$

and $p : \mathrm{et}(\mathcal{F}) \to X$ is the natural projection, we need only a suitable topology to turn $\mathrm{et}(\mathcal{F})$ into an étalé space. For every $s \in \mathcal{F}(U)$, let

$$\begin{aligned} \tilde{s} : U &\to \mathrm{et}(\mathcal{F}) \\ x &\mapsto s_x. \end{aligned}$$

Clearly $p \circ \tilde{s} = 1_U$. We then equip $\mathrm{et}(\mathcal{F})$ with the final topology with respect to the functions $\tilde{s} : U \to \mathrm{et}(\mathcal{F})$; this is to say

$$\{\tilde{s}(U) \subset \mathrm{et}(\mathcal{F}) \mid U \subset X \text{ open}, s \in \mathcal{F}(U)\}$$

is a basis for the above topology. We see that the functions $\tilde{s}$ are trivially continuous and, in addition, if $U \subset X$ is an open set,

$$p^{-1}(U) = \bigcup \tilde{s}(V),$$

where the union runs through all open sets $V \subset X$ such that $V \cap U \neq \emptyset$ and $s \in \mathcal{F}(V)$. Thus $p^{-1}(U)$ is an open set, which makes p continuous. It is also a local homeomorphism: $\tilde{s} \circ (p|_{\tilde{s}(U)})(s_x) = \tilde{s}(x) = s_x$ and $p|_{\tilde{s}(U)} \circ \tilde{s} = 1_U$.

We have therefore associated to any given presheaf $\mathcal{F}$ an étalé space $\mathrm{et}(\mathcal{F})$. In fact, we have associated a *sheaf* to any given presheaf: the sheaf of sections of the associated étalé space. Moreover,

Proposition 6. *If* $\mathcal{F}$ *is a sheaf, then*

$$\begin{aligned} \tau_U : \mathcal{F}(U) &\to \Gamma(U, \mathrm{et}(\mathcal{F})) \\ s &\mapsto \tilde{s} \end{aligned}$$

is an isomorphism of sheaves.

Proof. We will prove that τ_U is bijective for each open set $U \subset X$, since this suffices for sheaves of abelian groups.

To show injectivity, suppose that $s, s' \in \mathcal{F}(U)$ and $\tau_U(s) = \tau_U(s')$. Then

$$r_x^U(s) = \tau_U(s)(x) = \tau_U(s')(x) = r_x^U(s') \quad \text{for every } x \in U,$$

and, consequently, there must be a neighborhood $V \subset U$ of x such that $s|_V = s'|_V$. Since x is arbitrary, we can cover U with open sets U_i such that

$$s|_{U_i} = s'|_{U_i} \quad \text{for all } i.$$

Since $\mathcal{F}$ is a sheaf, the identity axiom implies $s = s'$.

To show surjectivity, suppose $\sigma \in \Gamma(U, \mathrm{et}(\mathcal{F}))$. Then for every $x \in U$ there are a neighborhood $V \subset U$ of x and $s \in \mathcal{F}(V)$ such that

$$\sigma(x) = s_x = \tau_v(s)(x).$$

But sections of an étalé space are local inverses for π: any two sections which agree at some point agree in some neighborhood of such a point. Thus, for some neighborhood $W \subset V$ of X, we have

$$\sigma|_W = \tau_V(s)|_W = \tau_W(s|_W).$$

Since $x \in U$ is arbitrary, we can cover U with open sets U_i such that there are $s_i \in \mathcal{F}(U_i)$ satisfying

$$\tau_{U_i}(s_i) = \tau_{U_j}(s_j) \text{ on } U_i \cap U_j.$$

By injectivity, it follows that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Since $U = \cup U_i$ and $\mathcal{F}$ is a sheaf, it follows by the gluability axiom that there exists $s \in \mathcal{F}(U)$ such that

$$s|_{U_i} = s_i.$$

Thus

$$\tau_U(s)|_{U_i} = \tau_{U_i}(s|_{U_i}) = \tau_{U_i}(s_i) = \sigma|_{U_i},$$

and, again, by the identity axiom, it follows that $\tau_U(s) = \sigma$. □

Because of this correspondence, we shall frequently refer to elements of a sheaf as *sections.*

Example 14. *The naive assignment of the same abelian group G to every open set U of a topological space X with the identity as restriction maps does* not *define a sheaf in general, only a presheaf. This is easily seen by assigning, say, two distinct elements $g_1, g_2 \in G$ to two disjoint open sets U_1, U_2 of X. There is no $g \in G(U_1 \cup U_2)$ such that $g|_{U_1} = g_1$ and $g|_{U_2} = g_2$. Thus, we call the sheafification of the presheaf $G(U) = G$ the* constant sheaf *and denote it by G.*

3.3 Vector bundles as locally free sheaves

If $\mathcal{F}$ is a sheaf over X and $U \subset X$ is an open set, we naturally define the restriction $\mathcal{F}|_U$ to be the sheaf given by $\mathcal{F}|_U(V) = \mathcal{F}(V)$, where V is open in U (and thus open in X).

Definition 13. *Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X, and define the* direct sum *of $\mathcal{R}$ to be the sheaf of $\mathcal{R}$-modules given by*

$$\mathcal{R}^n(U) = \underbrace{\mathcal{R}(U) \oplus \cdots \oplus \mathcal{R}(U)}_{n \text{ times}}.$$

Further, if $\mathcal{M}$ is an $\mathcal{R}$-module such that $\mathcal{M} \cong \mathcal{R}^n$ for some $n > 0$, then $\mathcal{M}$ is said to be a free sheaf*. Finally, if $\mathcal{M}$ is an $\mathcal{R}$-module such that each $x \in X$ has a neighborhood U on which $\mathcal{M}|_U$ is free, then $\mathcal{M}$ is said to be a* locally free sheaf*.*

Theorem 1. *Let X be a connected (topological, smooth, complex) manifold. Then there exists a one-to-one correspondence between isomorphism classes of (topological, smooth, holomorphic) bundles over X and isomorphism classes of locally free sheaves of ($\mathcal{C}_X$, $\mathcal{E}_X$, $\mathcal{O}_X$)-modules.*

Proof. Let $\mathcal{S} = \mathcal{C}_X$ or $\mathcal{E}_X$ or $\mathcal{O}_X$. Then to a given vector bundle $E \to X$, we associate the sheaf of sections $\mathcal{S}(E)$. Since E is locally trivial, for every $x \in X$ there is an open set $U \subset X$ containing x and a bundle isomorphism

$$\Phi : E|_U \to U \times \mathbb{K}^n,$$

which induces the isomorphism of sheaves $\mathcal{S}(E)|_U \cong \mathcal{S}(U \times \mathbb{K}^n)$ given by the maps

$$\begin{aligned} h_V : \mathcal{S}(E)|_U(V) &\to \mathcal{S}(U \times \mathbb{K}^n)(V) \\ s &\mapsto \Phi \circ s. \end{aligned}$$

In addition, any $f \in \mathcal{S}(U \times \mathbb{K}^n)(V)$ has the form $f(x) = (x, g(x))$, with $g : V \to \mathbb{K}^n$; that is, $g = (g_1, \dots, g_n)$ with each $g_i \in \mathcal{S}(V)$. Therefore, the maps

$$\begin{aligned} H_V : \mathcal{S}(U \times \mathbb{K}^n)(V) &\to \mathcal{S}(V) \oplus \cdots \oplus \mathcal{S}(V) \\ f &\mapsto (g_1, \dots, g_n) \end{aligned}$$

define an isomorphism of sheaves, meaning $\mathcal{S}(E)$ is a locally free sheaf of $\mathcal{S}$-modules.

Conversely, suppose $\mathcal{L}$ is a locally free sheaf of $\mathcal{S}$-modules. Then there exists an open cover $\{U_i\}$ of X such that

$$g_i : \mathcal{L}|_{U_i} \to \mathcal{S}^n|_{U_i}$$

is an isomorphism for some $n > 0$. Since X is connected, n is independent of i. Now define the isomorphisms

$$g_{ij} : \mathcal{S}^n|_{U_i \cap U_j} \to \mathcal{S}^n|_{U_i \cap U_j}$$

by $g_{ij} = g_i \circ g_j^{-1}$ (this composition is, of course, evaluated for each open set $V \subset U_i \cap U_j$, seeing each g_i represents a family of maps $(g_i)_V$). Then the map $(g_{ij})_{U_i \cap U_j}$ which for the sake of notation we purposefully write as

$$g_{ij} : \mathcal{S}^n(U_i \cap U_j) \to \mathcal{S}^n(U_i \cap U_j),$$

is nothing but an $n \times n$ invertible matrix whose entries are elements of $\mathcal{S}(U_i \cap U_j)$; this is to say it defines a (continuous, smooth, holomorphic) map (again, abusing notation)

$$g_{ij} : U_i \cap U_j \to GL(n, \mathbb{K}),$$

and the collection of all such maps, which by their definition satisfy $g_{ij}g_{jk}g_{ki} = 1$ on $U_i \cap U_j \cap U_k$, are the familiar transition functions defining a vector bundle E by Proposition 2. Moreover, it's clear that the assignments $E \mapsto \mathcal{S}(E)$ and $\mathcal{L} \mapsto E$ are inverses to one another.

We need now show that isomorphism classes are preserved under this correspondence. First, suppose $F : E \to \tilde{E}$ is an isomorphism of vector bundles. Then $S(E) \cong S(\tilde{E})$ via the map $s \mapsto F \circ s$. Now, suppose $G : \mathcal{L} \to \tilde{\mathcal{L}}$ is an isomorphism of sheaves. Passing to a refinement if necessary, we can assume that they trivialize over the same open cover. But

$$g_{ij} = \underbrace{(g_i \circ G^{-1} \circ \tilde{g}_i^{-1})}_{=:\lambda_i} \circ \tilde{g}_{ij} \circ \underbrace{(\tilde{g}_j \circ G \circ g_j^{-1})}_{=:\lambda_j^{-1}},$$

where $\lambda_i : \mathcal{S}^n(U_i) \to \mathcal{S}^n(U_i)$ is an isomorphism of sheaves for each i, which we can view as (continuous, smooth, holomorphic) maps $\lambda_i : U_i \to GL(n, \mathbb{K})$. The result then follows by Proposition 4. □

3.4 Exactness and quotients

We haven't yet defined what it means for a morphism of sheaves to be injective or surjective.

Definition 14. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over a topological spaces X. A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is said to be* injective *(resp.* surjective*) when $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective) for all $x \in X$.*

It is straightforward to show that ϕ is injective if and only if ϕ_U is injective for every open set $U \subset X$, but the same doesn't hold true for surjectivity. The classical example that explores this issue starts with the following complex of sheaves over the complex plane $\mathbb{C}$:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where the first map is inclusion (as locally constant integer valued functions) and the second map is exponentiation of functions: $f \mapsto e^{2\pi i f}$, taking values in the (multiplicative) sheaf of non-vanishing holomorphic functions $\mathcal{O}^*$ over $\mathbb{C}$. By the non-existence of a global logarithm, the exponential mapping is not surjective. However, if $z \in \mathbb{C}$ and $U \ni z$ is a simply connected open set, then any $g \in \mathcal{O}^*(U)$ admits a logarithm. Thus $f = (1/2\pi i) \log g \in \mathcal{O}(U)$ is such that $e^{2\pi i f} = g$, which shows that the stalk map is surjective. Moreover, it's not difficult to show that this sequence is also exact at $\mathcal{O}$ and is therefore an example of a *short exact sequence of sheaves.*

Definition 15. *If $\mathcal{F}$ is a sheaf of abelian groups over a topological space X and $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, then the sheafification of the presheaf $\mathcal{F}(U)/\mathcal{G}(U)$ is called the* quotient sheaf *of $\mathcal{F}$ bt $\mathcal{G}$ and is denoted by $\mathcal{F}/\mathcal{G}$.*

Proposition 7. *If $f \in \Gamma(X, \mathcal{F}/\mathcal{G})$ is a global section, then there exists an open covering $\{U_i\}$ of X and $f_i := f|_{U_i} \in \mathcal{F}(U_i)$ such that $f_i - f_j \in \mathcal{G}(U_i \cap U_j)$. Conversely, the data given by an open covering $\{U_i\}$ and $f_i \in \mathcal{F}(U_i)$ such that $f_i - f_j \in \mathcal{G}(U_i \cap U_j)$ specifies a global section $f \in \Gamma(X, \mathcal{F}/\mathcal{G})$.*

Proof. Just follow the definitions. □

Definition 16. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{R}$-modules over a topological space X. The sheafification of the presheaf $\mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$ is called the* tensor product *of $\mathcal{F}$ and $\mathcal{F}$ and is denoted by $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$.*

It is straightforward to show that $(\mathcal{F} \otimes_{\mathbb{R}} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x$. Moreover, we have

Proposition 8. *If E is a locally free sheaf of $\mathcal{R}$-modules and*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a short exact sequence of sheaves, then

$$0 \to \mathcal{A} \otimes_{\mathcal{R}} E \to \mathcal{B} \otimes_{\mathcal{R}} E \to \mathcal{C} \otimes_{\mathcal{R}} E \to 0$$

is also exact.

Proof. Again, a matter of unwinding definitions and using the observation above. □

3.5 Cohomology of sheaves

Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. Fix an open cover $\{U_\alpha\}$ of X.

Definition 17. *A q*-simplex *is a $(q+1)$-tuple of sets of $\{U_\alpha\}$ with non-empty intersection, which we denote by $\sigma = (U_0, \dots, U_q)$. We call the intersection $|\sigma| := U_0 \cap \dots \cap U_q$ the* support *of σ.*

We also write $|\sigma_i| := U_0 \cap \dots \hat{U}_i \dots \cap U_q$ for the intersection of every set in the support but the i-th one.

Definition 18. *A q*-cochain *of $\{U_\alpha\}$ with coefficients in $\mathcal{F}$ is a map f associating a q-simplex $\sigma = (U_0, \dots, U_q)$ to a section $f(\sigma) \in \mathcal{F}(|\sigma|)$.*

We denote the set of q-cochains with coefficients in $\mathcal{F}$ by $C^q(\{U_\alpha\}, \mathcal{F})$ and pointwise addition turns it into an abelian group.

Definition 19. *The* coboundary operator *is the homomorphism $\delta : C^q(\{U_\alpha\}, \mathcal{F}) \to C^{q+1}(\{U_\alpha\}, \mathcal{F})$ defined as follows: given $f \in C^q(\{U_\alpha\}, \mathcal{F})$ and $\sigma = (U_0, \dots, U_{q+1})$, set*

$$\delta f(\sigma) = \sum_{i=1}^{q+1} (-1)^i r_{|\sigma|}^{|\sigma_i|} f(\sigma_i),$$

where $r_{|\sigma|}^{|\sigma_i|}$ is the sheaf restriction map of $\mathcal{F}$.

It is straightforward to show that $\delta^2 = 0$, which gives us a cochain complex

$$C^*(\{U_\alpha\}, \mathcal{F}) := C^0(\{U_\alpha\}, \mathcal{F}) \to \cdots C^q(\{U_\alpha\}, \mathcal{F}) \to \cdots .$$

As usual, we can form the cohomology groups of this complex: if

$$Z^q(\{U_\alpha\}, \mathcal{F}) = \ker \delta : C^q(\{U_\alpha\}, \mathcal{F}) \to C^{q+1}(\{U_\alpha\}, \mathcal{F}),$$
$$B^q(\{U_\alpha\}, \mathcal{F}) = \operatorname{im} \delta : C^{q-1}(\{U_\alpha\}, \mathcal{F}) \to C^q(\{U_\alpha\}, \mathcal{F}),$$

we define

$$H^q(\{U_\alpha\}, \mathcal{F}) := H^q(C^*(\{U_\alpha\}, \mathcal{F})) = Z^q(\{U_\alpha\}, \mathcal{F}) / B^q(\{U_\alpha\}, \mathcal{F}),$$

the *q-th cohomology group of the complex* $C^*(\{U_\alpha\}, \mathcal{F})$. This theory of cohomology is called the *Čech cohomology* of X with coefficients in $\mathcal{F}$ with respect to the covering $\{U_\alpha\}$.

To get rid of the dependence on the covering, notice that if $\{V_\beta\}$ is a refinement of $\{U_\alpha\}$, then there is a natural group homomorphism induced by the restriction maps of $\mathcal{F}$

$$h : H^q(\{U_\alpha\}, \mathcal{F}) \to H^q(\{V_\beta\}, \mathcal{F}).$$

This allows us to define

$$H^q(X, \mathcal{F}) := \varinjlim H^q(\{U_\alpha\}, \mathcal{F})$$

with respect to the partially ordered set of all coverings of X, which we call the *sheaf cohomology* of X of degree q with coefficients in $\mathcal{F}$. As usual with direct limits, we have a natural homomorphism

$$H^q(\{U_\alpha\}, \mathcal{F}) \to H^q(X, \mathcal{F}).$$

If $\{U_\alpha\}$ is an open cover of X such that the map above is an isomorphism, then we call $\{U_\alpha\}$ a *Leray cover.*

Proposition 9. *Let $\{U_\alpha\}$ be an open cover of X such that for all $q = 1, 2, \dots$ and all q-simplices σ we have*

$$H^q(|\sigma|, \mathcal{F}) = 0.$$

Then $\{U_\alpha\}$ is a Leray cover.

Proof. See, for instance, (HIRZEBRUCH; BOREL; SCHWARZENBERGER, 1966). □

The following theorem collects the necessary facts to work with sheaf cohomology. They hold in a much more general setting, that of abelian categories, and we refer to (GROTHENDIECK, 1957) and (ROTMAN, 2008) for such a point of view.

Theorem 2. *Let X be a paracompact Hausdorff space. Then*

1. *For any sheaf $\mathcal{F}$ over X, $H^0(X, \mathcal{F}) = \mathcal{F}(X)$, the global sections of $\mathcal{F}$.*
2. *For any morphism of sheaves $h : \mathcal{F} \to \mathcal{G}$, there is, for each $q \geq 0$, a group homomorphism*

$$h^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G})$$

such that

   a) $h^0 = h_X : \mathcal{F}(X) \to \mathcal{G}(X)$,
   b) For every $q \geq 0$, $h^q = 1$ if $h = 1$.
   c) $g^q \circ h^q = (g \circ h)^q$ for all $q \geq 0$ if g is a second morphism of sheaves.

3. *For each short exact sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

there is a group homomorphism, called the Bockstein homomorphism*,*

$$\delta^q : H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A})$$

for all $q \geq 0$ such that

a) The induced sequence

$$\begin{aligned} 0 \to H^0(X,\mathcal{A}) \to H^0(X,\mathcal{B}) \to H^0(X,\mathcal{C}) \to H^1(X,\mathcal{A}) \to \cdots \\ \cdots \to H^q(X,\mathcal{A}) \to H^q(X,\mathcal{B}) \to H^q(X,\mathcal{C}) \to H^{q+1}(X,\mathcal{A}) \to \cdots \end{aligned}$$

is exact,

b) A commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{C}' & \longrightarrow & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(X,\mathcal{A}) & \longrightarrow & H^0(X,\mathcal{B}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(X,\mathcal{A}') & \longrightarrow & H^0(X,\mathcal{B}') \end{array}$$

$$\begin{array}{cccccc} \longrightarrow & H^0(X,\mathcal{C}) & \longrightarrow & H^1(X,\mathcal{A}) & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \\ \longrightarrow & H^0(X,\mathcal{C}') & \longrightarrow & H^1(X,\mathcal{A}') & \longrightarrow & \cdots . \end{array}$$

Proof. (WARD; WELLS, 1990), p. 180. □

We shall use the following result extensively in Chapter 7:

Proposition 10. *The cohomology groups of the line bundles $\mathcal{O}(m)$ over complex projective spaces satisfy*

$$H^q(\mathbb{P}^n,\mathcal{O}(m)) = \begin{cases} 0 \text{ if } 0 < q < n, \\ 0 \text{ if } q = 0, m < 0, \\ 0 \text{ if } q = n, m > -n-1. \end{cases}$$

Proof. (HUYBRECHTS, 2006), p. 241. □

We also shall make use of the notion of an *extension* of a sheaf $\mathcal{A}$. Consider a short exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0. \tag{3.1}$$

We say that $\mathcal{B}$ (or the whole sequence) is an *extension* of $\mathcal{A}$ by C. In addition, if

$$0 \to \mathcal{A} \to \mathcal{B}' \to \mathcal{C} \to 0 \tag{3.2}$$

is another extension of $\mathcal{A}$ by $\mathcal{C}$, then (3.1) and (3.2) are said to be *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{C} & \longrightarrow & 0, \end{array}$$

where the maps $\mathcal{A} \to \mathcal{A}$ and $\mathcal{C} \to \mathcal{C}$ are the identity and $\mathcal{B} \to \mathcal{B}'$ is an isomorphism.

If E and F are vector bundles over X, there is an isomorphism between $H^1(X, F^* \otimes E)$ and the group of extensions of E by F. The details of this correspondence go far beyond the scope of this work, and we refer to Chapter 4 of (GROTHENDIECK, 1957) for a detailed treatment.

4 Complex geometry

4.1 Complex linear algebra

Unless otherwise stated, V will always denote a $2n$-dimensional real vector space in what follows.

Definition 20. *A* complex structure *on V is a linear mapping $J : V \to V$ such that $J^2 = -1$.*

One might think of introducing a complex structure on a real vector space of odd dimension, say, $2n+1$. However, it is then easy to see that we would have $(\det J)^2 = (-1)^{2n+1} = -1$, which is impossible.

Example 15. *Take $\mathbb{R}^{2n}$ together with $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by*

$$J_0(x_1, \dots, x_n, y_1, \dots, y_n) = (-y_1, \dots, -y_n, x_1, \dots, x_n),$$

acting on the canonical basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ as

$$J(e_j) = f_j, \quad J(f_j) = -e_j; \quad j = 1, \dots, n.$$

J_0 is often called the canonical complex structure on $\mathbb{R}^{2n}$.

We note that V equipped with a complex structure J has a natural structure of a complex vector space: the $\mathbb{C}$-action is given by

$$(a + ib) \cdot v := av + bJ(v); \quad a, b \in \mathbb{R} \text{ and } v \in V.$$

Moreover, if $\{e_1, \dots, e_n\}$ is a basis for V as a complex vector space, then it should be clear that $\{e_1, \dots, e_n, J(e_1), \dots, J(e_n)\}$ is a basis for V as a real vector space, by analogy with the usual isomorphism $\mathbb{R}^{2n} \simeq_{\mathbb{R}} \mathbb{C}^n$.

It should also be evident that, since J corresponds to multiplication by i, a linear map $T : V \to V$ is complex linear precisely when $TJ = JT$. In particular, this allows us to identify $GL(n, \mathbb{C})$ as the subgroup of $GL(2n, \mathbb{R})$ commuting with J_0. In fact, the set of complex structures of $\mathbb{R}^{2n}$ is naturally given by the coset space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, as can be viewed introducing the map $[A] \mapsto A^{-1} J_0 A$.

There is another procedure for turning a real vector space into a complex one. Namely, if W is a real vector space of dimension n, we say $\mathbb{C}W := W \otimes_{\mathbb{R}} \mathbb{C}$ is the *complexification* of W. This is, at first, a real vector space of dimension $2n$, but there's a natural $\mathbb{C}$-action given by

$$a \cdot (v \otimes b) := v \otimes (ab); \quad a, b \in \mathbb{C} \text{ and } v \in W.$$

Thus $\mathbb{C}W$ is an n-dimensional complex vector space. In addition, if $\{e_j\}_1^n$ is a basis for W, it is also a basis for $\mathbb{C}W$ as a complex vector space. Notice that the complexification makes no restrictions as to the dimension of the original real vector space, as opposed to the introduction of a complex structure.

Complexifying V, we obtain a $2n$-dimensional complex vector space $\mathbb{C}V$. We can extend J to act on $\mathbb{C}V$ as

$$J(v \otimes a) = J(v) \otimes a; \quad v \in V, a \in \mathbb{C}.$$

It is clear that we still have $J^2 = -1$, and as such, we see that J now has precisely two eigenvalues, $+i$ and $-i$. Let's denote the $+i$-eigenspace (resp. $-i$-eigenspace) by $V^{1,0}$ (resp. $V^{0,1}$). Then

$$\mathbb{C}V = V^{1,0} \oplus V^{0,1}. \tag{4.1}$$

The conjugation map

$$\begin{aligned} Q : \mathbb{C}V &\to \mathbb{C}V \\ v \otimes a &\mapsto v \otimes \bar{a} \end{aligned}$$

gives an isomorphism $V^{1,0} \cong_{\mathbb{R}} V^{0,1}$.

A straightforward computation shows that

$$V^{1,0} = \{v - iJ(v) \in \mathbb{C}V \mid v \in V\}, \quad V^{0,1} = \{v + iJ(v) \in \mathbb{C}V \mid v \in V\}. \tag{4.2}$$

Having this in mind, we remark that if $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a basis of V for which J is the canonical complex structure, that is, $J(e_j) = f_j$ and $J(f_j) = -e_j$, then

$$w_j := \frac{1}{2}(e_j - if_j), \quad \bar{w}_j := \frac{1}{2}(e_j + if_j); \quad j = 1, \dots, n, \tag{4.3}$$

form bases for $V^{1,0}$ and $V^{0,1}$, respectively.

We can, of course, carry over the decomposition (4.1) to the exterior algebra of $\mathbb{C}V$. That is, there are natural injections

$$\Lambda(V^{1,0}) \to \Lambda(\mathbb{C}V) \text{ and } \Lambda(V^{0,1}) \to \Lambda(\mathbb{C}V),$$

and denoting by $\Lambda^{p,q}V$ the subspace of $\Lambda(\mathbb{C}V)$ spanned by elements of the form $v \wedge w$, with $v \in \Lambda^p(V^{1,0})$ and $w \in \Lambda^q(V^{0,1})$, a routine counting exercise shows that

$$\Lambda(\mathbb{C}V) = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \Lambda^{p,q}V. \tag{4.4}$$

It goes without saying that all of the above applies to the dual space V^*, with the induced complex structure J^*. Care must be taken, since if $\{e^j, f^j\}_1^n$ is the dual basis to the one previously fixed for V, then

$$J^*(e^j) = -f^j, \quad J^*(f^j) = e^j; \quad j = 1, \dots, n. \quad \text{(compare with example 15)}$$

Further, it should be noted that $(V^*)_\mathbb{C} = \mathrm{Hom}_\mathbb{R}(V, \mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$, whereas $(V_\mathbb{C})^* = \mathrm{Hom}_\mathbb{C}(V_\mathbb{C}, \mathbb{C})$. It is, nevertheless, reassuring to know that they are naturally isomorphic, as shown by the map $f \otimes a \mapsto af$. Finally, the dual basis to (4.3) is given by

$$w^j := e^j + if^j, \quad \bar{w}^j := e^j - if^j. \tag{4.5}$$

Do notice that, had we not introduced the 1/2 factor in (4.3), it would have to appear here by duality. This is the more standard choice and likely due to the fact that the exterior product of dual vectors appears more often than that of ordinary vectors.

4.2 Almost complex manifolds

The notion of a complex manifold is very close to that of a smooth manifold; the only difference being that we require charts to take their values in an open set of $\mathbb{C}^n$ and that coordinate changes must be holomorphic instead of merely smooth. However, since the real and imaginary parts of a holomorphic map are smooth (real analytic even), then an n-dimensional complex manifold M gives rise to a $2n$-dimensional smooth manifold $M_\mathbb{R}$. Moreover, given $x \in M$, T_xM is a complex vector space, and as such, the map $v \mapsto iv$ makes sense. This map induces a complex structure on $T_xM_\mathbb{R}$ given by

$$J(\partial/\partial x_j) = \partial/\partial y_j, \quad J(\partial/\partial y_j) = -\partial/\partial x_j,$$

when $(x_1, \dots, x_n, y_1, \dots, y_n)$ are (real) coordinates near x. It is natural to ask whether an almost complex structure on each tangent space of an even-dimensional smooth manifold is sufficient to turn it into a complex manifold. The answer is in the negative, which motivates the following

Definition 21. *An* almost complex manifold M *is an even-dimensional smooth manifold together with an* almost complex structure*; that is, an endomorphism* $J : TM \to TM$ *such that* $J^2 = -1$.

There is a deep theorem giving the exact conditions under which an almost complex manifold can be given the structure of a complex manifold. We are in no position to offer a proof of this theorem, but we can state it here for completeness. Given an almost complex manifold, we can compute the *torsion* of its almost complex structure. This is the tensor field given by

$$N(X, Y) = 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]); \quad X, Y \in \Gamma(TM).$$

Theorem 3 (Newlander-Nirenberg)**.** *An almost complex manifold can be given the structure of a complex manifold if and only if its almost complex structure has no torsion.*

4.3 Complex-valued differential forms

Let (X, J) be a $2n$-dimensional almost complex manifold. Denote by $\mathbb{C}TX = TX \otimes_{\mathbb{R}} \mathbb{C}$ (resp. $\mathbb{C}T^*X = T^*X \otimes_{\mathbb{R}} \mathbb{C}$) the complexification of the tangent (resp. cotangent) bundle to X.

Definition 22. *A* complex-valued differential form of total degree r *on* X *is a section of the bundle* $\Lambda^r(\mathbb{C}T^*X)$*.*

We shall denote the space of such sections by $\mathbb{C}\Omega^r(X)$. These differ from regular (real-valued) differential forms, which we denote $\Omega^r(X)$ in that they are locally like

$$\phi(x) = \sum_I \phi_I(x) dx^I,$$

using multi-index notation, where each ϕ_I is a *complex-valued* smooth function. We can extend d naturally to complex-valued forms by letting it act on the vector part, leaving the complex scalar part unaltered. It is clear that $d^2 = 0$.

Carrying over the results of section 4.1 to this context, the vector spaces $T_xX^{1,0}$ and $T_xX^{0,1}$ form subbundles of the complexified tangent bundle $\mathbb{C}TX$, since if f and g are bundle endomorphisms of $TX_{\mathbb{C}}$ given by $f(v) = v + iJv$ and $g(v) = v - iJv$, then

$$TX^{1,0} = \ker f, \quad TX^{0,1} = \ker g,$$

and the assertion follows by Proposition 1. Moreover,

$$\mathbb{C}TX = TX^{1,0} \oplus TX^{0,1}$$

by a dimension count. Further, the almost complex structure J on X induces a bundle endomorphism J^* of $\mathbb{C}T^*X$ by duality. This endows each *cotangent* space with a complex structure, and thus we also get a bundle decomposition

$$\mathbb{C}T^*X = T^*X^{1,0} \oplus T^*X^{0,1}.$$

The bundle whose fibers are $\Lambda^{p,q}\mathbb{C}T_x^*X$, here denoted by $\Lambda^{p,q}T^*X$, is of interest to us. More specifically, we are interested in the space of sections

$$\Omega^{p,q}(X) := \Gamma(\Lambda^{p,q}T^*X),$$

which we call *complex-valued differential forms of type (p,q)*. As in the vector space case, the r-th exterior bundle also decomposes as

$$\Lambda^r(\mathbb{C}T^*X) = \bigoplus_{p+q=r} \Lambda^{p,q}T^*X,$$

which induces the decomposition (as $\mathcal{E}_X$-modules)

$$\mathbb{C}\Omega^r(X) = \bigoplus_{p+q=r} \Omega^{p,q}(X). \tag{4.6}$$

Since $T^*X^{1,0} \cong_{\mathbb{R}} T^*X^{0,1}$ by conjugation, we can express $s \in \mathbb{C}\Omega^{p,q}(X)$ locally as

$$s = \sum_{|I|=p,|J|=q} f_{IJ} w^I \wedge \bar{w}^J; \quad f_{IJ} \in \mathcal{E}_X(U),$$

with $\{w^1, \ldots, w^n\}$ being a local frame for $T^*X^{1,0}$. The action of d, locally, is then given by

$$ds = \sum_{|I|=p,|J|=q} df_{IJ} \wedge w^I \wedge \bar{w}^J + f_{IJ} d(\omega^I \wedge \bar{w}^J). \tag{4.7}$$

The second term on the right above doesn't automatically vanish because, as a function of local coordinates on X, w_i might not be a constant as is the case with the *coordinate* frame for T^*X, $\{dx^j, dy^j\}_1^n$, dual to the frame $\{\partial/\partial x^j, \partial/\partial y^j\}_1^n$ for TX.

Following (4.3) and (4.5), we see that

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \quad \frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right); \quad j = 1, \ldots, n,$$

and

$$dz^j := dx^j + idy^j, \quad d\bar{z}^j := dx^j - idy^j; \quad j = 1, \ldots, n$$

are (constant) frames for $\mathbb{C}TX$ and $\mathbb{C}T^*X$, although they are *not* coordinate frames, since $(z^j, \bar{z}^j)$ are not local coordinates for X.

From (4.6), there are natural projection maps

$$\pi_{p,q} : \mathbb{C}\Omega^r(X) \to \Omega^{p,q}(X).$$

It is clear from (4.7) that $d(\Omega^{p,q}) \subset \Omega^{p+q+1}$, which in turn decomposes as

$$\Omega^{p+q+1}(X) = \bigoplus_{r+s=p+q+1} \Omega^{r,s}(X),$$

due to which we define the following operators

$$\partial = \pi_{p+1,q} \circ d,$$
$$\bar{\partial} = \pi_{p,q+1} \circ d.$$

Much like the exterior derivatives, we extend these to all forms by linearity.

4.4 Connections and curvature

Let $E \to X$ be a vector bundle.

Definition 23. *A morphism of* $\mathbb{K}$*-sheaves* $\nabla : \Omega^0(E) \to \Omega^1(E)$ *is called a* connection *if it satisfies the Leibniz rule*

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for every $f \in \Omega^0(M)$ *and every* $s \in \Omega^0(E)$.

Equip E with a connection ∇. If $e = \{e_1, \ldots, e_n\}$ is a frame for E over some open set $U \subset X$, then we can write

$$\nabla e_i = \sum_j A_{ij} \otimes e_j,$$

where $A := (A_{ij})$ is a matrix of 1-forms, called the *connection matrix* of ∇ with respect to the frame e. It so happens that it suffices to specify a frame e and a matrix A of 1-forms to obtain a connection on every connected component of X. To see this, suppose $\sigma \in \Omega^0(E)(U)$. We can then write σ using the e frame as

$$\sigma = \sum_i \sigma_i e_i,$$

with $\sigma_i \in \Omega^0(U)$, and thus

$$\begin{aligned} \nabla\sigma &= \sum_i d\sigma_i \otimes e_i + \sum_i \sigma_i \nabla e_i \\ &= \sum_j d\sigma_j \otimes e_j + \sum_i \sigma_i \left(\sum_j A_{ij} \otimes e_j \right) \\ &= \sum_j \left(d\sigma_j + \sum_i \sigma_i A_{ij} \right) \otimes e_j. \end{aligned} \tag{4.8}$$

If $e' = \{e'_1, \ldots, e'_n\}$ is another frame over some open set $V \subset X$ intersecting U, say

$$e'_i(z) = \sum_j g_{ij}(z) e_j(z),$$

then from (4.8) we get

$$\begin{aligned} \nabla e'_i &= \sum_j \left(dg_{ij} + \sum_k g_{ik} A_{kj} \right) \otimes e_j \\ &= \sum_{j,l} \left(dg_{ij} + \sum_k g_{ik} A_{kj} \right) \otimes g^{-1}_{jl} e'_l \\ &= \sum_j \left(\sum_l dg_{ij} g^{-1}_{jl} + \sum_{k,l} g_{ik} A_{kj} g^{-1}_{jl} \right) \otimes e'_l, \end{aligned}$$

which we can write compactly as

$$A' = dg \cdot g^{-1} + g \cdot A \cdot g^{-1}.$$

Suppose now X is a complex manifold and $E \to X$ is a holomorphic vector bundle. From the decomposition $T^*X = T^*X^{1,0} \oplus T^*X^{0,1}$, we can write $\nabla = \nabla^{1,0} + \nabla^{0,1}$, with

$$\begin{aligned}\nabla^{1,0} &: \mathbb{C}\Omega^0(E) \to \mathbb{C}\Omega^{1,0}(E),\\ \nabla^{0,1} &: \mathbb{C}\Omega^0(E) \to \mathbb{C}\Omega^{0,1}(E).\end{aligned}$$

We note that $\nabla^{0,1}$ acts on a given section $\sigma = \sum \sigma_i e_i$ under a local frame e as

$$\nabla^{0,1}\sigma = \bar{\partial}\sigma_i \otimes e_i + \sigma_i \nabla^{0,1} e_i.$$

Proposition 11. *Let $E \to X$ be a holomorphic vector bundle with a connection $\nabla = \nabla^{1,0}$. If $s \in \Gamma(E)$ is covariantly constant, then s is holomorphic.*

Proof. In local coordinates, we have

$$\nabla s = \sum_j \left(ds_j + \sum_i A_{ij} s_i \right) \otimes e_j,$$

with A being a matrix of $(1,0)$ forms. Since $\nabla s = 0$, we have

$$ds_j = -\sum s_i A_{ij} \in \Omega^{1,0}(E),$$

and thus

$$\frac{\partial s_j}{\partial \bar{z}^k} = 0$$

for all k and j. □

Definition 24. *A connection ∇ on a holomorphic vector bundle $E \to X$ is said to be* compatible with the holomorphic structure *on E when $\nabla^{0,1} = \bar{\partial}$.*

Definition 25. *A connection ∇ on a complex vector bundle $E \to X$ equipped with a hermitian metric h is said to be* compatible with the metric h *when*

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2).$$

Proposition 12. *Suppose E is a holomorphic vector bundle equiped with a hermitian metric h. Then there is a unique connection ∇ compatible with both the holomorphic structure and the hermitian metric.*

Proof. Let $e = \{e_1, \ldots, e_n\}$ be a holomorphic frame for E and set $h_{ij} = h(e_i, e_j)$. If we are given such a connection ∇, then the associated connection matrix A with respect to e is of type $(1,1)$, since e is a holomorphic frame, and thus

$$\begin{aligned} dh_{ij} &= h(\nabla e_i, e_j) + h(e_i, \nabla e_j)\\ &= \sum_k h(A_{ik} e_k, e_j) + \sum_k h(e_i, A_{jk} e_k)\\ &= \underbrace{\sum_k A_{ik} h_{kj}}_{\text{type (1,0)}} + \underbrace{\sum_k \bar{A}_{jk} h_{ik}}_{\text{type (0,1)}}, \end{aligned}$$

which gives us

$$\partial h_{ij} = \sum_k A_{ik} h_{kj}$$
$$\bar{\partial} h_{ij} = \sum_k \bar{A}_{jk} h_{ik},$$

and it is easy to see that these (they are in fact equivalent) force $A = \partial h \cdot h^{-1}$. □

We call ∇ as above the *Chern connection* of E.

When $\nabla s = 0$, we say that s is *covariantly constant*.

We collect here for further reference the induced connections on associated bundles.

Proposition 13. *Let $E_1, E_2 \to X$ be vector bundles equipped with connections ∇_1, ∇_2, respectively. Suppose $s_1 \in \Omega^0(E_1)$ and $s_2 \in \Omega^0(E_2)$. Then the following expressions define (natural) connections on the specified bundles*

1. $\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2)$ *on* $E_1 \oplus E_2$,
2. $\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$ *on* $E_1 \otimes E_2$,
3. $\nabla(f)(s_1) = \nabla_2(f(s_1)) - f(\nabla_1(s_1))$ *on* $\mathrm{Hom}(E_1, E_2)$, *where f is a local homomorphism $E_1 \to E_2$.*
4. $f^*\nabla|_{f^{-1}(U_i)} = d + f^*A_i$ *on f^*E, where $E \to Y$ is a vector bundle, $f : X \to Y$ is a smooth map. and $\{U_i\}$ is a trivializing open cover of N.*

There is a unique linear extension of ∇

$$\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$$

such that

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla s,$$

where α is a local p-form and s a local section of E. The Leibniz rule for this extension takes the form

$$\nabla(\beta \wedge t) = d\beta \wedge t + (-1)^p \beta \wedge \nabla t,$$

where β is a p-form and t is a section of $\Omega^q(E)$.

Definition 26. *If ∇ is a connection on a vector bundle E, we call the composition*

$$F := \nabla \circ \nabla : \Omega^0(E) \to \Omega^2(E)$$

the curvature *of* ∇.

It is easy to see that F is $\Omega^0(E)$-linear and thus can be considered to be a global section of $\Omega^2(\mathrm{End}(E))$, i.e. a 2-form taking values in endomorphisms of E.

If $e = \{e_1, \ldots, e_n\}$ is a frame for E, then with respect to the frame $\{e^i \otimes e_j\}$ for $E^* \otimes E \cong \mathrm{End}\, E$, we can write

$$Fe_i = \sum_j F_{ij} \otimes e_j,$$

and we call $F = (F_{ij})$ the *curvature matrix* of ∇ with respect to the frame e. Moreover, if $e' = \{e'_1, \ldots, e'_n\}$ is another local frame, we have

$$\begin{aligned} Fe'_i &= F\left(\sum_j g_{ij} e_j\right) \\ &= \sum_j g_{ij} F e_j \\ &= \sum_j g_{ij} \left(\sum_k F_{jk} \otimes e_k\right) \\ &= \sum_j g_{ij} F_{jk} g_{kl}^{-1} e'_l, \end{aligned}$$

that is, $F' = gFg^{-1}$.

We can also obtain an expression for F in terms of the connection matrix of ∇:

$$\begin{aligned} Fe_i &= \nabla\left(\sum_j A_{ij} e_j\right) \\ &= \sum_{j,k} dA_{ij} \otimes e_j - A_{ij} \wedge A_{jk} \otimes e_k \\ &= \sum_{j,k} (dA_{ij} - A_{ik} \wedge A_{kj}) \otimes e_j, \end{aligned}$$

giving us

$$F = dA - A \wedge A,$$

known as *Cartan's structure equation.*

One could ponder if composing ∇ iteratively always gives rise to a new operator. The answer to this is in

Proposition 14 (Bianchi identity)**.** *If F is the curvature of a connection ∇ on E, then*

$$\nabla F = 0.$$

Proof. (HUYBRECHTS, 2006), p. 183. □

4.5 Divisors and line bundles

We'll now explore a correspondence between line bundles and certain objects, to be defined, called *divisors*.

The tensor product and the dual functor provide the set of isomorphism classes of holomorphic line bundles on a complex manifold X with the structure of an abelian group, the zero element being the isomorphism class of the trivial bundle. This group is called the *Picard group* of X and is denoted by $\mathrm{Pic}(X)$. From Proposition 4 and the fact that transition functions satisfy the cocycle condition, it is clear that $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

Definition 27. *An* (analytic) hypersurface *of X is a subset $Y \subset X$ with the property that for every $p \in Y$ there is an open set $U \subset X$ and a not identically zero holomorphic function f_U such that $U \cap Y$ is the zero locus of f_U. We call these f_U* local defining functions *for Y.*

We note that the zeros of, e.g. $f(z,w) = z^2 - w^3$ in $\mathbb{C}^2$ define a hypersurface that is *not* a complex manifold; it is not locally euclidean around the origin. However, we will not discuss singular points, as we will only deal with smooth analytic hypersurfaces in what follows. In this case, they are also complex submanifolds and we will say that a hypersurface is *irreducible* when it is connected. This is not the standard definition, and for more details we refer to (GRIFFITHS; HARRIS, 2014).

Definition 28. *A* divisor *D on X is an element of the free abelian group $\mathrm{Div}(X)$ generated by the irreducible hypersurfaces of X. If $Y_i \subset X$ are hypersurfaces, we write*

$$D = \sum a_i [Y_i].$$

When all the $a_i \geq 0$ we say that D is effective.

The proposition below, found on (GRIFFITHS; HARRIS, 2014), is necessary for some of what follows:

Proposition 15. *If f and g are relatively prime in $\mathcal{O}_0$, then for sufficiently small $\varepsilon > 0$, f and g are relatively prime in $\mathcal{O}_z$ when $|z| < \varepsilon$.*

Definition 29. *If $Y \subset X$ is an irreducible hypersurface and f is a holomorphic function defined on a neighborhood of $x \in Y$, then the* order of f along Y at x, *denoted $\mathrm{ord}_{Y,x}(f)$, is the largest integer k such that $f = g^k \cdot h$ in $\mathcal{O}_{Y,x}$.*

From Proposition 15 and the irreducibility (i.e. connectedness) of Y, we see that $\mathrm{ord}_{Y,x}(f)$ is independent of x, and thus we can speak of the order of f along Y, $\mathrm{ord}_Y(f)$. Moreover, if g is another holomorphic function defined near $x \in Y$, then

$$\mathrm{ord}_Y(fg) = \mathrm{ord}_Y(f) + \mathrm{ord}_Y(g). \tag{4.9}$$

Definition 30. *A* meromorphic function f *on an open set* $U \subset X$ *is the following data:*

1. *an open covering* $\{U_i\}$ *of* U,
2. *for each* i, *a pair of relatively prime holomorphic functions* $g_i, h_i \in \mathcal{O}_X(U_i)$ *such that*

$$g_i h_j = g_j h_i \text{ on } U_i \cap U_j. \tag{4.10}$$

We evaluate f *at a given point* z *by*

$$f(z) = \frac{g_i(z)}{h_i(z)} \text{ if } z \in U_i,$$

which is well defined because of (4.10).

When both $g_i(z) = h_i(z) = 0$, we leave it undefined. Thus, f is, strictly speaking, not a function on U.

Suppose now f is a meromorphic function near $x \in V$. Then there exists a pair of holomorphic functions g, h such that $f = g/h$. We then define the order of f along a hypersurface Y as

$$\operatorname{ord}_Y(f) = \operatorname{ord}_Y(g) - \operatorname{ord}_Y(h).$$

We can also associate to a meromorphic function f a divisor:

$$(f) = \sum_{Y \subset X \text{ irr.}} \operatorname{ord}_Y(f)[Y]$$

Divisors associated to meromorphic functions are called *principal*.

The meromorphic functions on open sets of X form a (multiplicative) sheaf, which we denote by $\mathcal{K}_X$. We will make frequent use of a subsheaf of $\mathcal{K}_X$, namely, the subsheaf $\mathcal{K}_X^*$ of meromorphic functions on X that are not identically zero. Moreover, it is clear that $\mathcal{O}_X^*$ is a subsheaf of $\mathcal{K}_X^*$ in a canonical way. We will now see that divisors also have a sheaf-theoretic interpretation, which offers an often fruitful point of view.

Consider the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. From Proposition 7, we see that a global section $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ of this sheaf is given by an open covering $\{U_i\}$ of X and meromorphic functions $f_i \in \mathcal{K}_X^*/\mathcal{O}_X^*$ such that

$$\frac{f_i}{f_j} \in \mathcal{O}_X^*(U_i \cap U_j).$$

Thus for any irreducible hypersurface $Y \subset X$ with $Y \cap U_i \cap U_j \neq \emptyset$, we have $\operatorname{ord}_Y(f_i) = \operatorname{ord}_Y(f_j)$, which allows us to speak of $\operatorname{ord}_Y(f)$. Further,

$$(f) = \sum_{Y \subset X} \operatorname{ord}_Y(f)[Y] \in \operatorname{Div}(X)$$

is a divisor associated to the global section $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Notice that (4.9) makes this assignment a homomorphism, which has an inverse described as follows: if $D = \sum a_i[Y_i]$, there is an open cover $\{U_i\}$ of X such that $Y_i \cap U_j$ is the zero locus of $g_{ij} \in \mathcal{O}(U_j)$, unique up to elements of $\mathcal{O}^*(U_j)$. Setting

$$f_j := \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j), \tag{4.11}$$

we obtain a global section $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. We have thus proved that

Proposition 16. *As abelian groups,* $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \mathrm{Div}(X)$.

In standard algebraic geometry terminology, elements of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ are called *Cartier divisors*, whereas elements of $\mathrm{Div}(X)$ are referred to as *Weil divisors*.

We now associate, in a natural way, a line bundle to a given divisor D. Let's use the Cartier interpretation. Then D corresponds to a collection of $f_i \in \mathcal{K}_X^*(U_i)$ for an open covering $\{U_i\}$ of X. Setting

$$g_{ij} = \frac{f_i}{f_j} \in H^0(U_i \cap U_j, \mathcal{O}^*(X)),$$

then $\{g_{ij}\}$ clearly satisfies the cocycle condition and thus defines a line bundle $\mathcal{O}(D)$. Moreover, if D' is another divisor given by $\{f_i'\}$, $D + D'$ corresponds to (passing to a refinement if necessary) $\{f_i \cdot f_j\}$, essentially by (4.11). Therefore $D+D'$ corresponds to the line bundle with transition functions $\{g_{ij} \cdot g_{ij}'\}$; this is to say that $\mathcal{O}(D+D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$. The zero divisor, without loss of generality, is given by $f_i \equiv 1 \in \mathcal{K}_X^*(U_i)$, meaning that $\mathcal{O}(0) \cong \mathcal{O}_X$. Finally, $-D$ is given, also because of (4.11), by $\{1/f_i\}$, which gives us $\mathcal{O}(-D) \cong \mathcal{O}(D)^*$. In short, we have a homomorphism

$$\begin{aligned} \mathrm{Div}(X) &\to \mathrm{Pic}(X) \\ \mathcal{O} &\mapsto \mathcal{O}(D) \end{aligned} \tag{4.12}$$

Example 16. *Since $\mathbb{P}^3$ is the twistor space associated to S^4, we will give an example of a line bundle associated to a divisor given by the hyperplane $H \subset \mathbb{P}^3$ defined by $z_0 = 0$, even though what follows applies to any hyperplane $H \subset \mathbb{P}^n$. We take as open cover $\{U_i\}_{i=0}^3$ the standard one, and the local defining functions are, in local coordinates,*

$$\begin{aligned} f_0(u, v, w) &\equiv 1, \\ f_i(u, v, w) &= u, \quad i = 1, 2, 3. \end{aligned}$$

Then, making sure both functions are defined in the same local coordinates, we get

$$g_{ij} = \frac{f_i}{f_j} = \frac{z_j}{z_i},$$

meaning that $\mathcal{O}(H) \cong \mathcal{O}(1)$, which is one of the reasons why the dual to the canonical bundle is frequently called the "hyperplane bundle".

Let's refine our map $D \mapsto \mathcal{O}(D)$.

Lemma 1. *$D \in \mathrm{Div}(X)$ is principal if and only if $\mathcal{O}(D) \cong \mathcal{O}_X$.*

Proof. If $D = (f)$ with f a meromorphic function on X, then D is represented as the Cartier divisor given by the natural projection $H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. As such, we get $f_i = f_j$ on the intersection $U_i \cap U_j$ of an open cover of X, showing that $\mathcal{O}(D) \cong \mathcal{O}_X$.

Suppose now that $\mathcal{O}(D)$ is trivial. Then there is an open cover $\{U_i\}$ of X such that D is given by $f_i \in \mathcal{K}_X^*(U_i)$ and, by Proposition 4,

$$g_{ij} = \frac{f_i}{f_j} = \frac{h_i}{h_j},$$

with $h_k \in \mathcal{O}_X^*(U_k)$. Thus

$$\frac{f_i}{h_i} = \frac{f_j}{h_j} \text{ on } U_i \cap U_j,$$

meaning that these glue to a global meromorphic function f defined locally by $f|_{U_i} = f_i/g_i$. It follows that $D = (f)$. □

Definition 31. *We say D and D' are* linearly equivalent*, writing $D \sim D'$, when $D - D'$ is a principal divisor.*

From the above lemma, the natural map $D \mapsto \mathcal{O}(D)$ factors through the equivalence classes to an injective map $\mathrm{Div}(X)/\sim \to \mathrm{Pic}(X)$. It also happens that if a line bundle admits a non-trivial global section, then it arises from a divisor; this is to say that the natural map above is surjective. To see this, we first define a canonical map

$$H^0(X, L) \setminus \{0\} \to \mathrm{Div}(X),$$

where L is a line bundle over X. Suppose $\{\Phi_i\}$ are local trivializations associated to an open covering $\{U_i\}$ of X. Then L is given by the cocycle $\{(U_i, \Phi_{ij} := \Phi_i \circ \Phi_j^{-1})\}$. Take $s \in H^0(X, L) \setminus \{0\}$ and define $f_i := \Phi_i \circ s|_{U_i} \in \mathcal{O}(U_i)$. By holomorphicity, it follows that f_i is not identically zero, and thus $f_i \in \mathcal{K}_X^*$. In addition,

$$\Phi_i \circ s|_{U_i \cap U_j} = \Phi_{ij} \cdot \Phi_j \circ s|_{U_i \cap U_j},$$

which gives us

$$\frac{f_i}{f_j} = \frac{\Phi_i \circ s|_{U_i \cap U_j}}{\Phi_j \circ s|_{U_i \cap U_j}} = \Phi_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_X^*),$$

and finally, we see that $f := \{(U_i, f_i)\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is a (Cartier) divisor associated to s. We will denote the image of the map just described by $Z(s)$. Now, we are ready to prove

Proposition 17. *1. Given $s \in H^0(X, L) \setminus \{0\}$, then $\mathcal{O}(Z(s)) \cong L$.*

2. *For any effective divisor D, there is a section $s \in H^0(X, \mathcal{O}(D)) \setminus \{0\}$ with $Z(s) = D$.*

Proof. 1. By our definition of the natural map $D \mapsto \mathcal{O}(D)$ and using the notation in our discussion above, $\mathcal{O}(Z(s))$ is given by the cocycles $\{(U_i, f_i/f_j = \Phi_{ij})\}$.

2. Suppose D, given by $\{(U_i, f_i \in \mathcal{K}_X^*(U_i))\}$ is effective. Then $f_i \in \mathcal{O}(U_i)$, which is easy to see looking at (4.11). Since $\mathcal{O}(D)$ is given by the transition functions $g_{ij} := f_i/f_j$, which we can rewrite as

$$f_i = g_{ij} \cdot f_j,$$

we see that the collection $\{f_i\}$ defines a global section, say, $s \in H^0(X, \mathcal{O}(D))$ satisfying $Z(s) = D$. □

Using Lemma 1, we deduce immediately

Corollary 1. *Any two sections $s_1 \in H^0(X, L_1) \setminus \{0\}$ and $s_2 \in H^0(X, L_2) \setminus \{0\}$ define linearly equivalent divisors $Z(s_1) \sim Z(s_2)$ if and only if $L_1 \cong L_2$.*

Corollary 2. *The image of the natural map $\mathcal{O} \mapsto \mathcal{O}(D)$ is generated by the line bundles L with $H^0(X, L) \neq \{0\}$, in the sense that $\mathcal{O}(D)$ is isomorphic to a combination of tensor products and duals of line bundles admitting global sections.*

Proof. Proposition 17 shows that any line bundle L admitting a global section s is isomorphic to $\mathcal{O}(Z(s))$, meaning that L is in the image of the natural map.

Conversely, if $D = \sum a_i[Y_i]$ is a divisor, we can split it as

$$D = \sum a_i^+[Y_i] - \sum a_j^-[Y_j]$$

with all of the $a_k \geq 0$. Hence, $\mathcal{O}(\sum a_i^+[Y_i])$ and $\mathcal{O}(\sum a_j^-[Y_j])$ are line bundles associated to effective divisors, which by Proposition 17 admit global sections. In addition,

$$\mathcal{O}(D) \cong \mathcal{O}\left(\sum a_i^+[Y_i]\right) \otimes \mathcal{O}\left(\sum a_j^-[Y_j]\right)^*.$$ □

We need not restrict to holomorphic sections.

Definition 32. *A* meromorphic section *of a line bundle L with transition functions g_{ij} is given by an open covering $\{U_i\}$ of X and a collection of meromorphic functions $f_i \in \mathcal{K}_X(U_i)$ such that $f_i = g_{ij} \cdot f_j$.*

Repeating the arguments in Proposition 17, we see that there is a canonical divisor, not necessarily effective, associated to any meromorphic section of a line bundle.

Let $Y \subset X$ be an irreducible hypersurface. Then, any non-trivial global section s of $\mathcal{O}([Y])$, which exists seeing $[Y]$ is an effective divisor, gives rise to a sheaf morphism

$\phi : \mathcal{O}_X \to \mathcal{O}(Y)$ defined by $f \mapsto s|_U \cdot f$. The dual map $\mathcal{O}(-Y) \to \mathcal{O}_X$, or equivalently $\phi \otimes_{\mathcal{O}_X} 1$, can be read off

$$\begin{aligned} \mathcal{O}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(-Y)(U) &\to \mathcal{O}(Y)(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(-Y)(U) \\ g \otimes t &\mapsto (g \cdot s|_U) \otimes t = g(s|_U \otimes t) \end{aligned}$$

as being

$$\begin{aligned} \mathcal{O}(-Y)(U) &\to \mathcal{O}_X \\ t &\mapsto s|_U \otimes t. \end{aligned} \tag{4.13}$$

Proposition 18. *The morphism of sheaves $\mathcal{O}(-Y) \to \mathcal{O}_X$ is injective and its image is the ideal sheaf $\mathcal{O}_Y$ of Y.*

Proof. Everything needs to be done at the stalk level, which means that it suffices to pick an open covering $\{U_i\}$ of X and work with morphisms of the restrictions of the sheaves to the open sets of this covering. Notice that the $s|_{U_i}$ are local defining functions for Y, where s is defined as in the discussion above. This follows from the construction of the natural map (4.12). We choose a cover $\{U_i\}$ compatible with trivializations for both $\mathcal{O}(Y)$ and $\mathcal{O}(-Y)$, which allows us to identify a given section of either with a map in $\mathcal{O}(U_i)$. Then, from (4.13), the induced map is given by ordinary multiplication by the local defining function for Y, $t_i \mapsto s_i \cdot t_i$, where $s_i \in \mathcal{O}(U_i)$ corresponds to s_{U_i}. It should now be apparent that the image is the ideal sheaf $\mathcal{O}_Y$. □

The above generalizes naturally to any effective divisor D, which gives us

Corollary 3. *The following sequence of sheaves is exact:*

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

5 The twistor correspondence

5.1 Twistor space

In what follows, we shall use the identification $S^4 \cong \mathbb{HP}^1$, where multiplication by (quaternionic) scalars is done on the *left* so that (q_1, q_2) and $(\lambda q_1, \lambda q_2)$ represent the same point in $\mathbb{HP}^1$.

Consider the 3-dimensional complex projective space $\mathbb{P}^3$. Every $\ell \in \mathbb{P}^3$ is, of course, a complex line passing through the origin in $\mathbb{C}^4$. Moreover, we can identify $\mathbb{C}^4 \cong \mathbb{H}^2$ via the map

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 + z_2 j, z_3 + z_4 j).$$

Now, given any $\ell \in \mathbb{P}^3$, we can associate to it the *quaternionic* line passing through the origin in $\mathbb{H}^2$,

$$\mathbb{H}\ell = \{qz \mid q \in \mathbb{H}, z \in \ell \subset \mathbb{C}^4 \simeq \mathbb{H}^2\},$$

which is an element of $\mathbb{HP}^1$. Thus we get a map

$$\begin{aligned} \pi : \mathbb{P}^3 &\to \mathbb{HP}^1 \\ \ell &\mapsto \mathbb{H}\ell \end{aligned}$$

Notice that any given quaternionic line $L \in \mathbb{HP}^1$ passing through the origin in $\mathbb{H}^2$ is a copy of $\mathbb{C}^2$, which allows us to identify $\pi^{-1}(L)$ with the set of all lines through the origin in this copy of $\mathbb{C}^2$, that is, $\pi^{-1}(L) \simeq \mathbb{P}^1$. It follows that π is a fibration with fiber $\mathbb{P}^1$. We call $\pi : \mathbb{P}^3 \to S^4$ the *twistor fibration*.

Notice that $(z_1 + z_2 j, z_3 + z_4 j)$ left multiplied by j corresponds to

$$(-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3) \in \mathbb{C}^4.$$

This map factors through $\mathbb{P}^3$ and is also well defined as a map to $\mathbb{P}^3$. Let us denote it by $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$. In homogeneous coordinates,

$$\sigma(z_1, z_2, z_3, z_4) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3). \tag{5.1}$$

Furthermore, it is clear that σ is anti-holomorphic and that it acts trivially on S^4, which means that $\pi \circ \sigma$ is also a fibration, with σ acting on the fibers $\mathbb{P}^1 \simeq S^2$ as the antipodal map. A straightforward computation also shows that σ has no fixed points. However, being the antipodal map on each fiber, σ has invariant *lines*, which we call *real lines*.

There is no natural choice of complex structure on $\mathbb{R}^4$ compatible with a given metric and orientation, although if we pick a specific complex structure J, then conjugating by elements of $SO(4)$ produces all the other choices. Since a $U(2)$ action preserves the complex structure, the space of all complex structures on $\mathbb{R}^4$ is identified with

$$\frac{SO(4)}{U(2)} \cong S^2 \cong \mathbb{P}^1. \tag{5.2}$$

Therefore, in order to be able to use complex variable methods in $\mathbb{R}^4$ we need three complex variables (u, z_1^u, z_2^u), with u being a non-homogeneous local coordinate for $\mathbb{P}^1$, and z_1^u, z_2^u the complex coordinates for $\mathbb{R}^4$. This means that for each $u \in \mathbb{P}^1$ there is an $\mathbb{R}$-linear isomorphism $I_u \in SO(4)$ between $\mathbb{R}^4$ and $\mathbb{C}^2$ (the latter being essentially $\mathbb{R}^4$ with the standard complex structure J) modulo an element in $U(2)$, which defines a complex structure $J_u = I_u^{-1} \circ J \circ I_u$ on $\mathbb{R}^4$. Have in mind that it is meaningless to consider any two distinct $(\mathbb{R}^4, u)$ being $\mathbb{C}$-isomorphic to each other, as they are not complex vector spaces.

This construction will be made clearer in the next section.

5.2 Anti-self-duality and complex structures

The tangent space to $x \in S^4$ acquires a complex structure[1] as follows: given $u \in \mathbb{P}^3$ such that $x = \pi(u)$, we can locally form the quotient of complex vector spaces $T_u\mathbb{P}^3/V_u$, where $V_u := \ker d\pi_u \subset T_u\mathbb{P}^3$ denotes the subspace of vectors tangent to the $\mathbb{P}^1$ fiber over $\pi(u)$. The differential of the twistor fibration map then descends to an $\mathbb{R}$-isomorphism, which, by abuse of notation, we also write as $d\pi_u : T_u\mathbb{P}^3/V_u \to T_xS^4$. From the discussion in the previous section, we then see that T_xS^4 inherits a complex structure $J_u = d\pi_u \circ \tilde{J}_u \circ (d\pi_u)^{-1}$, $\tilde{J}_u$ being the complex structure on $T_u\mathbb{P}^3/V_u$.

Let V be a 4-dimensional real vector space with an inner product $(,)$. Suppose that we equip V with a complex structure J that is compatible with the inner product, and suppose further that we endow V with the natural orientation coming from J; this is to say if $\{e_1, e_2\}$ is a *complex* basis for (V, J), then $\{e_1, Je_1, e_2, Je_2\}$ is an oriented *real* basis for V. We know from (4.4) that we can write $\omega \in \Lambda^2(\mathbb{C}V^*)$ as

$$\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}.$$

We can also decompose ω into SD and ASD parts:

$$\omega = \omega^+ + \omega^-.$$

These two decompositions are related by

Lemma 2. *An alternating bilinear map on an oriented* 4*-dimensional real inner product space is ASD if and only if it is of type* $(1,1)$ *for all complex structures compatible with the given inner product and orientation.*

[1] This pointwise complex structure is *not* an almost complex structure on S^4, let alone an integrable one. It is known (BOREL; SERRE, 1953) that S^n admits an almost complex structure only when $n = 2, 6$. It is not known, however, whether S^6 has an integrable complex structure.

Proof. Without loss of generality, we assume $V = \mathbb{C}^2$ with coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$. Consider now the 2-form given by

$$\omega(v, w) = -(v, Jw) = (Jv, w).$$

A quick computation shows that

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \tag{5.3}$$

meaning $\omega \in \Lambda^{1,1}(\mathbb{C}V^*)$. In fact, this is true for every complex structure J on V: recall from Section 4.1 that there is an induced complex structure $\mathbf{J}$ on $\Lambda^2(\mathbb{C}V^*)$. Thus,

$$(\mathbf{J}\omega)(v, w) = \omega(Jv, Jw) = -(Jv, J^2 w) = (Jv, w),$$

which shows $\Lambda^{1,1}(\mathbb{C}V^*)$.

Let us now decompose $\Lambda^{1,1}(\mathbb{C}V^*)$ as

$$\Lambda^{1,1}(\mathbb{C}V^*) = \mathbb{C}\omega \oplus \Lambda_0^{1,1},$$

where $\mathbb{C}\omega$ is the 1-dimensional subspace spanned by ω, and $\Lambda_0^{1,1}$ its orthogonal complement. This allows us to write

$$\Lambda^2(\mathbb{C}V^*) = \Lambda^{2,0} \oplus \mathbb{C}\omega \oplus \Lambda_0^{1,1} \oplus \Lambda^{2,0}.$$

It is clear from (5.3) that $\omega \in \Lambda_+^2(\mathbb{C}V^*)$. Moreover, $\Lambda^{2,0}(\mathbb{C}V^*)$ is spanned by

$$dz_1 \wedge dz_2 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4 + i(dx_2 \wedge dx_3 + dx_1 \wedge dx_4),$$

which is clearly SD. An entirely analogous computation shows that forms of type $(0, 2)$ are also SD. Thus, we conclude that

$$\Lambda_-^2(\mathbb{C}V^*) = \Lambda_0^{1,1},$$

that is, every ASD 2-form is of type $(1, 1)$ for every complex structure on V.

Conversely, denote by $\Lambda_u^{1,1}$ the space of 2-forms on V of type $(1, 1)$ with respect to the complex structure parameterized by $u \in \mathbb{P}^1$, according to (5.2). We have already shown that $\Lambda_-^2(\mathbb{C}V^*) \subset \Lambda_u^{1,1}$ for every $u \in \mathbb{P}^1$. Then

$$\Lambda_-^2(\mathbb{C}V^*) \subset S = \bigcap_{u \in \mathbb{P}^1} \Lambda_u^{1,1}.$$

However, S is clearly invariant under $SO(4)$ and is not the whole space. We must then have

$$S = \Lambda_-^2(\mathbb{C}V^*). \qquad \square$$

Now, suppose ω is a 2-form on S^4. Let $\tilde{\omega} := \pi^*\omega$ be the lift of ω to $\mathbb{P}^3$. Since $V_u = \ker d\pi_u$ for every $u \in \mathbb{P}^3$, it follows that $\tilde{\omega}$ vanishes when at least one of its two entries is in V_u. With this in mind, given $x \in S^4$ and $u \in \mathbb{P}^3$ such that $x = \pi(u)$, the alternating bilinear map $\tilde{\omega}_u$ descends to an alternating bilinear map on $T_u/L_u \cong T_x S^4$. We then apply Lemma 2 pointwise to obtain

Lemma 3. *A 2-form on S^4 is ASD if and only if its lift to twistor space is of type $(1,1)$.*

Suppose now E is a $U(n)$-vector bundle on S^4 equipped with a metric connection. We now lift E to a bundle $\tilde{E}$ over twistor space; that is, we let $\tilde{E} := \pi^* E$ and pull-back the connection and metric from E. This allows us to apply the above lemma to the matrix entries of the lifted curvature, which gives us

Proposition 19. *A $U(n)$-bundle with a metric-compatible connection on S^4 has ASD curvature if and only if the lifted bundle on twistor space has curvature of type $(1,1)$.*

5.3 Complex structures as subbundles

We now wish to give the lifted bundle $\tilde{E}$ a natural holomorphic structure. For that, we need a new interpretation of complex structures.

Definition 33. *A subbundle $V \subset T^*X$ (resp. $V \subset \mathbb{C}T^*X$), where X is a smooth manifold, is said to be* involutive *if*

$$d\Gamma(V) = \Gamma(V_2), \tag{5.4}$$

where $V_2 \subset \Omega^2(X)$ (resp. $V_2 \subset \mathbb{C}\Omega^2(X)$) is the image of V under exterior multiplication by elements of $\Omega^1(X)$ (resp. $\mathbb{C}\Omega^1(X)$).

Let X be a $2n$-dimensional smooth manifold. Suppose $V \subset \mathbb{C}T^*X$ is a (complex) subbundle of the complexified tangent bundle to X such that

$$\mathbb{C}T^*X = V \oplus \overline{V}, \tag{5.5}$$

where $\overline{V}$ is the image of V under the conjugation map $v \otimes \alpha \to v \otimes \bar{\alpha}$. We note that the rank of both V and $\overline{V}$ is n. Such a subbundle allows us to endow X with a natural almost complex structure J. To see this, notice that we can define a map $J^* : \mathbb{C}T^*X \to \mathbb{C}T^*X$ by letting

$$J_x^*(v) = iv \text{ for every } v \in V,$$
$$J_x^*(v) = -iv \text{ for every } v \in \overline{V},$$

and extending by $\mathbb{C}$-linearity. This map is also smooth: since $V \subset \mathbb{C}T^*X$ (and also $\overline{V} \subset \mathbb{C}T^*X$) is a subbundle, there are local coordinates in which we can write J^* as

$$(x_j, a_k, b_l) \mapsto (x_j, ia_k, -ib_l),$$

and thus J^* is a bundle endomorphism of $\mathbb{C}T^*X$ satisfying $(J^*)^2 = -1$. By duality, we get a bundle endomorphism

$$J : \mathbb{C}TX \to \mathbb{C}TX$$

also satisfying $J^2 = -1$, i.e. an almost complex structure on X. If $\{w^j\}$ is a local frame for $V \subset \mathbb{C}T^*X$ and $\{w_j\}$ the corresponding dual frame for $V^* \subset \mathbb{C}TX$, then J is given locally by

$$\langle Jw_j, w^k\rangle = \langle w_j, J^*w^k\rangle = i\delta_j^k,$$
$$\langle J\bar{w}_j, \bar{w}^k\rangle = \langle \bar{w}_j, J^*\bar{w}^k\rangle = -i\delta_j^k,$$

and we see that $Jw_j = iw_j$ and $J\bar{w}_j = -i\bar{w}_j$ for all j. Thus

$$TX^{1,0} = V^* \quad \text{and} \quad TX^{0,1} = \overline{V}^*.$$

Of course, we also have

$$T^*X_{1,0} = V \quad \text{and} \quad T^*X^{0,1} = \overline{V}$$

Proposition 20. *Suppose X is a $2n$-dimensional smooth manifold and $V \subset \mathbb{C}T^*X$ is an involutive subbundle satisfying (5.5). Then the induced almost complex structure J has zero torsion.*

Proof. Suppose $\{w^j\}$ is a smooth local frame for V. If V is involutive, then given local vector fields $Z, W \in \Gamma(TX^{1,0})$,

$$dw(Z,W) = \sum_{j=1}^{n}(w^i \wedge f^j)(Z,W) = 0$$

for an arbitrary local section $w \in \Gamma(\overline{V})$. Since

$$2dw(Z,W) = Zw(W) - Ww(Z) - w([Z,W]),$$

it follows that $w([Z,W]) = 0$ and thus $[Z,W] \in \Gamma(TX^{1,0})$.

Now set $Z = [X - iJX, Y - iJY]$ for arbitrary real vector fields $X, Y \in \Gamma(TX)$. From (4.2) and the preceding discussion we see that Z is of type $(1,0)$. One can show that

$$2(Z + iJZ) = -N(X,Y) - iJ(N(X,Y)),$$

and since $Z + iJZ = 0$ if and only if Z is of type $(1,0)$, it follows that $N(X,Y) = 0$. □

Corollary 4. *If X is a $2n$-dimensional smooth manifold and $V \subset \mathbb{C}T^*X$ is an involutive subbundle satisfying (5.5), then X has a natural complex manifold structure.*

Proof. It follows immediately from Theorem 3 and Proposition 20. □

5.4 An integrability result

Let $E \to X$ be a real vector bundle of rank k over an n-dimensional smooth manifold.

Definition 34. *Any section $s \in \Gamma(E)$ gives rise to a function $s^\vee : E^* \to \mathbb{R}$ given by*

$$s^\vee(\varepsilon_x) = \langle s(x), \varepsilon_x \rangle.$$

Now pick a local frame $\{e_1, \dots, e_k\}$ over the open set $U \subset X$ for E and the corresponding dual frame $\{e^1, \dots, e^k\}$ for E^*. We parametrize $E^*|_U$ by

$$(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) \mapsto \sum_{i=1}^{k} \lambda_i e^i(x), \tag{5.6}$$

where x_i are local coordinates for X and λ_i are linear coordinates for E^*. Writing s over U as

$$s = \sum_{i=1}^{k} f_i e_i,$$

with each $f_i : U \subset X \to \mathbb{R}$ a smooth function, then

$$s^\vee(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) = \sum_{i=1}^{k} \lambda_i f_i(x)$$

and

$$ds^\vee = \sum_{i=1}^{k} d\lambda_i f_i + \sum_{i=1}^{k} \lambda_i df_i, \tag{5.7}$$

where each df_i is to be interpreted as lifted to E^* via the bundle projection $p : E^* \to X$.[2]

Let us now consider a *linear first order differential operator* $\bar{D} : \Gamma(E) \to \Gamma(F)$, where $F \to X$ is a vector bundle of rank m. This is a bundle map which under local trivializations for E and F over an open set $U \subset X$, having in mind that local sections for E and F get identified with functions taking values in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, is given by

$$\bar{D} = \sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} + B(x), \tag{5.8}$$

where each $A_i(x)$ and $B(x)$ are $m \times k$ matrices.

Let

$$I_x = \{\bar{D}(s)_x \in F_x \mid s \in \Gamma_p(E)\}.$$

We say that $\bar{D}$ is of *constant rank* whenever $\dim I_x$ does not depend on x. In this case, we can associate a vector bundle $V(\bar{D}) \to E^* \setminus 0$ (the complement of the zero section) that is a subbundle of $T^*(E \setminus 0)$. Specifically, the fiber over $\varepsilon_x \in E^*_x$ is given by

$$V(\bar{D})_{\varepsilon_x} = \{(ds^\vee)_{\varepsilon_x} \in T^*(E^* \setminus 0) \mid s \in \Gamma(E|_U), U \ni x, \bar{D}(s)_x = 0\}. \tag{5.9}$$

[2] We are not making a distinction between df_i and $p^* df_i$.

Example 17. *Let us see what $V(\bar{D})$ looks like when $\bar{D}$ is a connection $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$ on E: near $\varepsilon_x \in E_x^*$ we can represent a given section $s \in \Gamma(E)$ by the k-vector $(f_1, \ldots, f_k)$. Setting*

$$\nabla e_i = \sum_j \omega_{ij} \otimes e_j,$$

then

$$\nabla s = \sum_j \left(df_j + \sum_i f_i \omega_{ij} \right) \otimes e_j,$$

or more explicitly,

$$\nabla s = \sum_{i,j} \left(\frac{\partial f_j}{\partial x_i} + \sum_l f_l \omega_{lj}^i \right) dx_i \otimes e_j,$$

which translates to

$$\begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \mapsto \begin{bmatrix} \frac{\partial f_1}{\partial x_1} + \sum_l f_l \omega_{l1}^1 \\ \vdots \\ \frac{\partial f_1}{\partial x_n} + \sum_l f_l \omega_{l1}^n \\ \frac{\partial f_2}{\partial x_1} + \sum_l f_l \omega_{l2}^1 \\ \vdots \\ \frac{\partial f_2}{\partial x_n} + \sum_l f_l \omega_{l2}^n \\ \vdots \\ \frac{\partial f_k}{\partial x_1} + \sum_l f_l \omega_{lk}^1 \\ \vdots \\ \frac{\partial f_k}{\partial x_n} + \sum_l f_l \omega_{lk}^n \end{bmatrix}$$

when viewing ∇ as a differential operator. Either way, the condition that $\bar{D}(s)_x = 0$ in (5.9) *means that*

$$df_j + \sum_i f_i \omega_{ij} = 0 \text{ for } j = 1, \ldots, k,$$

and thus

$$\begin{aligned} ds^\vee &= \sum d\lambda_i f_i + \sum \lambda_i df_i \\ &= \sum d\lambda_i f_i - \sum_{i,j} \lambda_j f_i \omega_{ij} \\ &= \sum_i f_i \left(d\lambda_i - \sum_j \lambda_j \omega_{ij} \right), \end{aligned}$$

which shows that $V(\nabla)$ is (locally) spanned by the 1-forms

$$\theta_i = d\lambda_i - \sum_j \lambda_j \omega_{ij} \text{ for } i = 1, \ldots, k. \tag{5.10}$$

An alternative and more illuminating perspective is given by the jet bundle approach (see Appendix A). In this setting, a first order differential operator is defined to be a bundle map

$$\bar{D} : J^1(E) \to F \tag{5.11}$$

which is given fiberwise by an $m \times (k + kn)$ matrix $A(x)$. Notice that a given differential operator as in (5.8) corresponds to the bundle map given (locally) by

$$A(x) = \begin{bmatrix} B & & & & \\ & A_1 & & & \\ & & A_2 & & \\ & & & \ddots & \\ & & & & A_n \end{bmatrix}.$$

We now lift $J^1(E)$ to E^* via the bundle projection $p : E^* \to X$ to define a map between bundles over E^*

$$V : p^* J^1(E) \to T^* E^*$$
$$j^1_{p(\varepsilon_x)} s \mapsto (ds^*)_{\varepsilon_x},$$

and the bundle $V(\bar{D})$ defined by (5.9) is then seen to be equal to $V(p^* R)$, where R is the kernel of the bundle map (5.11). When $\bar{D}$ is of constant rank, R is a bundle by Proposition 1.

Example 18. *In the case $\bar{D} = \nabla$, we have $R \cong E$ under the natural projection $J^1(E) \to E$ given by $j^1_x s \mapsto s(x)$. More explicitly, $\bar{D} j^1_x s = 0$ implies that, when s is locally identified with the k-vector $(f_1, \ldots, f_k)$,*

$$df^i = -\sum_j f^j \omega^i_j, \quad i = 1, \ldots, k,$$

meaning that the derivative coordinates u^α_i (following the notation in the appendix) are uniquely specified by the linear coordinates u^α of the section at a given point.

Lemma 4. *The map V restricted to $p^*(E \otimes \Omega^1) \subset p^* J^1(E)$ (cf. Proposition 26) satisfies*

$$V((e_x \otimes \omega_x)_{\varepsilon_x}) = \langle e_x, \varepsilon_x \rangle p^* \omega_x \in (T^* E^*)_{\varepsilon_x}.$$

Proof. It suffices to show that

$$V(e_i \otimes dx_j)_{\varepsilon_x} = \langle e_i, \varepsilon_x \rangle dx_j, \quad i = 1, \ldots, k \text{ and } j = 1, \ldots, n$$

whenever $\{e_i\}$ is a local frame for E.[3] Using local coordinates for E^* as in (5.6), it follows that

$$\langle e_i, \varepsilon_x \rangle dx_j = \lambda_i dx_j,$$

while $e_i \otimes dx_j$ corresponds to the 1-jet $j^1_x s$ of a local section s of E such that $s^\alpha(x) = 0$ for $\alpha = 1, ..., k$ and $s^\alpha_l(x) = 0$ for all α and l, with the exception of $s^i_j = 1$. For such a local section, we have from (5.7) that

$$V(j^1_{p(\varepsilon_x)} s) = ds^\vee = \sum \lambda_i df_i = \lambda_i dx_j,$$

which proves the lemma. □

[3] Note that dx_j is to be interpreted as $p^* dx_j$.

Lemma 5. *The bundle map*

$$V : p^* J^1(E) \to T^* E^*$$

is surjective away from the zero section of E^**.*

Proof. Using local coordinates as in (5.6), the zero section is characterized by $\lambda_i = 0$ for $i = 1, \dots, k$. Further, T^*E^* is locally spanned by $\{dx_i, d\lambda_j\}$. Now, if we take a local section s of E given by a k-vector $(f_1, \dots, f_k)$, then from (5.7) we have

$$ds^\vee = \sum d\lambda_i f_i + \sum \lambda_i df_i = \sum d\lambda_i f_i + \sum \lambda_i \frac{\partial f_i}{\partial x_j} dx_j,$$

meaning V is surjective unless $\lambda_i = 0$, in which case the image of V lies in the subspace of T^*E^* spanned by $d\lambda_i$ for $i = 1, \dots, k$. □

Consider now a general first order differential operator $\bar{D} = \sigma\nabla$, where $\sigma : E \otimes T^*X \to F$ is the *symbol* of $\bar{D}$. Let $S_1 := \ker \sigma$ and let $S_2 \subset E \otimes (\Lambda^2 T^*X)$ be the image of S_1 under exterior multiplication by 1-forms on X. In this decomposition, we see that

$$R = E \oplus S_1 \subset E \oplus (E \otimes T^*X) \cong J^1(E), \tag{5.12}$$

where we used that $\ker \nabla = E$ as discussed in Example 18. Moreover, we have

Lemma 6. *Suppose* S_1 *is of rank* m *and that it is locally spanned by*

$$\sigma_i = \sum_{j,l} s_{ijl} e_j \otimes dx_l, \quad i = 1, \dots, m.$$

Then $V(p^*S_1)$ *is locally spanned by*

$$\sigma_i^\vee = \sum_{j,l} s_{ijl} \lambda_j dx_l, \quad i = 1, \dots, m.$$

Proof. A straightforward computation using Lemma 4. □

Proposition 21. *If* $\bar{D} = \sigma\nabla$*, then* $V(\bar{D}) \subset T^*(E^* \setminus 0)$ *is involutive if and only if*

1. $\nabla_1 \Gamma(S_1) \subset \Gamma(S_2)$,
2. $F_\nabla \Gamma(E) \subset \Gamma(S_2)$,

where $\nabla_1 : \Omega^1(E) \to \Omega^2(E)$ *is the extended connection.*

Proof. We would like to show that $d\Gamma(V(\bar{D})) \subset \Gamma(V(\bar{D})_2)$, following (5.4). From (5.12), we see that $V(\bar{D}) = V(p^*(E \oplus S_1))$ and, consequently, $V(\bar{D})$ is locally spanned by the 1-forms

$$\begin{aligned} \theta_i &= d\lambda_i - \sum \omega_{ij} \lambda_j \text{ in } V(p^*E) \text{ for } i = 1, \dots, k, \\ \sigma_i^\vee &= \sum_{j,l} s_{ijl} \lambda_j dx_l \text{ in } V(p^*S_1) \text{ for } i = 1, \dots, m, \end{aligned}$$

using (5.10) and Lemma 6. From Lemma 5, it follows that $\{\theta_i, dx_j\}$ is a local basis for $T^*(E^* \setminus 0)$, and thus V_2 is spanned by $\theta_i \wedge \theta_j$, $\theta_i \wedge dx_j$ and $\sigma_i^\vee \wedge dx_j$.[4] Therefore,

$$\begin{aligned} d\theta_i &= -\sum \lambda_j d\omega_{ij} - \sum d\lambda_j \wedge \omega_{ij} \\ &= -\sum \lambda_j d\omega_{ij} - \sum \omega_{jk} \wedge \omega_{ij}\lambda_k - \sum \theta_j \wedge \omega_{ij} \text{ using (5.10)} \\ &= -\sum \lambda_k \Omega_{ik} - \sum \theta_j \wedge \omega_{ij}, \end{aligned}$$

this being a section of V_2 if and only if $\mathrm{F}_\nabla \Gamma(E) \subset \Gamma(S_2)$. Lastly,

$$\begin{aligned} d\sigma_i^\vee &= \sum s_{ijk} d\lambda_j \wedge dx_k + \sum \lambda_j ds_{ijk} \wedge dx_k \\ &= \sum s_{ijk}\theta_j \wedge dx_k + \sum s_{ijk}\omega_{mj}\lambda_m \wedge dx_k + \sum \lambda_j ds_{ijk} \wedge dx_k \\ &= \sum s_{ijk}\theta_j \wedge dx_k + (\nabla_1 \sigma_i)^\vee, \end{aligned}$$

which is a section of V_2 if and only if $\nabla_1 \Gamma(S_1) \subset \Gamma(S_2)$. □

As a final remark, we note that in the event E is a *complex* vector bundle, then the proposition above still holds provided ∇ and σ commute with the almost complex structure on E induced by multiplication by i.

5.5 Instantons and holomorphic bundles

An application of Proposition 21 is

Theorem 4. *Let X be a complex manifold and $E \to X$ a smooth hermitian vector bundle with connection ∇ whose curvature F is of type $(1,1)$. Then E has a natural holomorphic structure such that ∇ is the Chern connection of E.*

Proof. We consider the differential operator

$$\bar{D} : \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*X_{\mathbb{C}}) \xrightarrow{\sigma} \Gamma(E \otimes \Lambda^{1,0}T^*X),$$

with $S_1 := \ker \sigma = E \otimes \Lambda^{1,0}T^*X$. Suppose $z_1, \ldots, z_n$ are local coordinates for X. From Lemma 6, noting that S_1 has $\{e_i \otimes dz_j\}$ as a local basis, we see that $V(p^*S_1)$ is spanned by

$$\sigma_{ij}^\vee = \lambda_i dz_j, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n,$$

which shows that dz_j is a local *basis* for $V(p^*S_1)$. From (5.10), we then conclude that a local basis for $V(\bar{D})$ is given by

$$d\lambda_i - \sum \omega_{ij}\lambda_j, \quad i = 1, \ldots, k, \quad \text{and} \quad dz_j, \quad j = 1, \ldots, n. \tag{5.13}$$

This shows $\dim V(\bar{D}) = \dim X + \dim E$ and that $V(\bar{D}) \cap \overline{V(\bar{D})} = 0$, i.e. that

$$T^*(E^* \setminus 0) = V(\bar{D}) \oplus \overline{V(\bar{D})},$$

[4] Elements in the span of $\sigma_i^\vee \wedge \theta_j$ already lie in the span of $\theta_i \wedge dx_j$

so that we have an almost complex structure on $E^* \setminus 0$. To apply Proposition 21, we must verify the integrability conditions:

1. Given $e \otimes dz \in \Gamma(S_1)$, then
$$\nabla_1(e \otimes dz) = \sum_i \nabla_i^{1,0} e \otimes dz_i \wedge dz + \sum_i \nabla_i^{0,1} e \otimes d\bar{z}_i \wedge dz,$$
which is in $\Omega^{2,0}(E) + \Omega^{1,1}(E) \subset \Gamma(S_2)$.

2. Given $F_\nabla \in \Omega^{1,1}(\operatorname{End} E)$, then $F_\nabla \Gamma(E) \subset \Omega^{1,1}(E) \subset \Gamma(S_2)$.[5]

Thus Proposition 21 gives us an involutive almost complex structure on $E^* \setminus 0$. It is clear from the local basis (5.13) we described for $V(\bar{D})$ that this subbundle defining the almost complex structure extends naturally to the whole of E^*. Therefore, by Proposition 20, E^* is a complex manifold. Furthermore, the zero section Z is a *complex* submanifold and consequently its normal bundle $N \to Z$ is holomorphic, which is clearly seen using local coordinates. Since $Z \cong X$, N can be thought of as a bundle over X; in addition, it is isomorphic to E^* as a bundle over X, again following directly by using local coordinates. It then follows that both E^* and $E \cong N^*$ are holomorphic vector bundles.

We explicitly describe the isomorphism $E \cong N^*$: given $s \in \Gamma(E)$, the 1-form $ds^\vee|_Z \in (T^*E^*)|_Z$ restricted to the zero section is a section of N^*. Indeed, if $\{e_i\}$ is a local frame for E, then this map is given locally by

$$e_i \mapsto d\lambda_i|_Z, \tag{5.14}$$

with $d\lambda_i|_Z$ being a section of N^* because it can be seen as a map to $\mathbb{C}$ from $(VE^*)|_Z$, the latter being isomorphic to $E^* \cong N$ as a bundle over X.

Since $Z \subset E^*$ is a complex submanifold, it is locally given as the zero locus of k *holomorphic* functions on E^*, say, $u_1, \dots u_k = 0$. Being holomorphic, these functions must each satisfy $du_i \in V(\bar{D}) = (T^*E^*)^{1,0}$; this is to say

$$du_i = \sum A_{ij}(d\lambda_j - \omega_{jk}\lambda_k) + \sum B_{ij} dz_j. \tag{5.15}$$

By abuse of notation, let us denote by $du_i|_Z$ the local *holomorphic* section of N^* given by

$$du_i|_Z = \sum A_{ij} d\lambda_j|_Z.$$

Then under the isomorphism (5.14), we apply the covariant derivative in the $\partial/\partial\bar{z}^l$ direction to find that

$$\nabla_{\bar{l}}(du_i|_Z) = \sum \frac{\partial A_{ij}}{\partial \bar{z}_l} d\lambda_j|_Z + \sum A_{ij}\omega_{jk}^{\bar{l}} d\lambda_k|_Z, \tag{5.16}$$

[5] It should be clear here that it is only necessary to require that the curvature has no (0,2) component; however, by hypothesis, ∇ is hermitian, which means the (2,0) component vanishes automatically.

From (5.15), we deduce

$$A_{ij} = \frac{\partial u_i}{\partial \lambda_j} \quad \text{and} \quad \frac{\partial w_i}{\partial \bar{z}_l} = -\sum A_{ij}\omega^{\bar{l}}_{jk}\lambda_k,$$

and thus

$$\frac{\partial A_{ij}}{\partial \bar{z}_l} = \frac{\partial^2 w_i}{\partial \bar{z}_l \partial \lambda_j} = -\sum A_{ik}\omega^{\bar{l}}_{kj},$$

and plugging this into (5.16) gives us

$$\nabla_{\bar{l}}(du_i|_Z) = 0,$$

meaning that ∇ is a $(1,0)$ connection. □

Theorem 5. *Let E be a hermitian vector bundle with an ASD connection over S^4, and let $F = \pi^*E$, where $\pi : \mathbb{P}^3 \to S^4$ is the twistor fibration. Then*

1. *F is holomorphic on $\mathbb{P}^3$,*
2. *F is holomorphically trivial on each fiber,*
3. *There is a holomorphic isomorphism $\tau : \sigma^*\overline{F} \to F^*$, with σ as in (5.1), such that τ induces a hermitian inner product on $H^0(P_x, F)$.*

Conversely, every such bundle $F \to \mathbb{P}^3$ is given by $F = \pi^*E$ for some bundle $E \to S^4$ with ASD connection.

Proof. 1. This follows directly from Proposition 19 and Theorem 4.

2. The restriction $F|_{\mathbb{P}^1}$ of F to a given fiber of the twistor fibration is clearly trivial as a *topological* bundle. Moreover, the pulled-back connection on $F|_{\mathbb{P}^1}$ is trivial, since the connection form is horizontal and thus lifts to an identically zero connection form on the restricted bundle. This means that any basis for E_x (which lifts to a global frame for the restricted bundle $F|_{\mathbb{P}^1}$, this $\mathbb{P}^1$ being the fiber at x) is covariantly constant. But since ∇ is a $(1,0)$ connection, every covariantly constant section is holomorphic by Proposition 11, meaning that any basis of E_x lifts to a global holomorphic frame for $F|_{\mathbb{P}^1}$.

3. Suppose $\{\varepsilon_1, \ldots, \varepsilon_k\}$ is a local *unitary* frame for E^*, which we lift to F^* via the twistor fibration so as to obtain a unitary frame relative to the pulled-back metric. Then, as in Theorem 4, the complex structure on F^* is defined by the forms

$$\theta_i = d\lambda_i - \sum \omega_{ij}\lambda_j \quad \text{and} \quad dz_j.$$[6]

[6] As usual, we are abusing notation as follows: θ_i is in fact $p^*\theta_i$, where $p : E^* \to X$, ω_{ij} is $\pi^*p^*\omega_{ij}$ and dz_j is q^*dz_j, where $q : F^* \to \mathbb{P}^3$.

The pullback nature of the θ_i implies that

$$(\theta_i)_{(z,\lambda)} = (\theta_i)_{(\pi(z),\lambda)},$$

where the LHS is a form on F^* and the RHS is a form on E^*. Thus, applying σ to the base space $\mathbb{P}^3$ leaves θ_i unchanged (recall that the fibers of the twistor fibration are invariant under σ)[7], and we get a complex structure on $\overline{F}^*$ defined by

$$d\bar{\lambda}_i - \sum \bar{\omega}_{ij}\bar{\lambda}_j \quad \text{and} \quad dz_j.$$

Since the connection is compatible with the metric and we are working under a local unitary frame, it follows that $\bar{\omega}_{ij} = -\omega_{ji}$, and thus the complex structure on $\overline{F}^*$ is given by

$$d\bar{\lambda}_i + \sum \omega_{ji}\bar{\lambda}_j \quad \text{and} \quad dz_j.$$

Since the hermitian metric gives us an isomorphism $\overline{F}^* \cong F$, we get an isomorphism

$$\tau : \sigma^*\overline{F} \cong F^*.$$

Conversely, suppose $F \to \mathbb{P}^3$ is a bundle satisfying 1, 2 and 3. Since F is holomorphically trivial on each fiber, we get a natural isomorphism

$$H^0(P_x, F) \to F_z,$$

where $P_x = \pi^{-1}(x)$ and $\pi(z) = x$, given by the evaluation of a section at a given point z in the fiber. This is due to the compacity of the fibers and the fact that holomorphic functions on compact spaces are constant. In addition, since holomorphic sections s, are the solutions of the equation

$$\bar{\partial} s = 0,$$

it follows that

$$E := \coprod_{x \in S^4} H^0(P_x, F)$$

is a vector bundle over S^4. Moreover, $F \cong \pi^* E$.

It is clear that σ induces a hermitian metric on each E_x, and consequently, σ induces a hermitian metric on E and F.

Consider now the Chern connection on F with respect to the hermitian structure induced by σ. One can prove using the technique of formal neighborhoods that this connection is in fact the pullback of an ASD connection on S^4. The machinery required for a satisfactory proof of this result is rather elaborate, and we refer to (HITCHIN, 1980) for a detailed treatment. □

[7] It is necessary to apply σ, which is *anti-holomorphic*, since otherwise $\overline{F}^*$ would not have a holomorphic structure: the base space would be a complex manifold with the *opposite* complex structure to that of the fibers of $\overline{F}^*$.

The isomorphism $\tau : \sigma^*\overline{F} \to F^*$ is called a *positive real form*. We can also think of τ as an *anti-linear* bundle map $\tau : F \to F^*$ covering σ on $\mathbb{P}^3$. Moreover, we say two holomorphic vector bundles V and W with positive real forms τ and τ', respectively, are isomorphic if there is a holomorphic isomorphism $T : V \to W$ such that $T \circ \tau = \tau' \circ T$.

We refer to (WARD; WELLS, 1990), (ATIYAH; WARD, 1977) for a a more algebraic derivation of this correspondence. Moreover, it should be noted that the instanton number and the second Chern class of the holomorphic bundle are equal (cf. (ATIYAH; HITCHIN; SINGER, 1978)).

It is easy to see that the correspondence for $O(n)$-instantons is similar to the $U(n)$ case, with the exception that the positive real form τ is now required to be holomorphic and to satisfy $\langle u, \tau v\rangle = \langle v, \tau u\rangle$, thus inducing a *symmetric* form on the space of holomorphic sections of the bundle restricted to real lines.

The theorem above is also true for instantons with the compact symplectic group $Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$ as gauge group; this being the subgroup of $U(2n)$ preserving the standard symplectic structure of $\mathbb{C}^{2n}$ given by $\omega(u, jv) = u \cdot v$. This means an $Sp(n)$-instanton is given by a complex vector bundle $E \to S^4$ of rank $2n$ with a metric-compatible connection equipped with an isomorphism $\alpha : E \to E^*$ giving rise to a symplectic form ω, i.e. α satisfies

$$\langle u, \alpha v\rangle = -\langle v, \alpha u\rangle,$$

which allows us to set $\omega(u, v) = \langle u, \alpha v\rangle$. Naturally, we require that α takes the connection on E to the induced connection on the dual bundle E^*.

Given then an $Sp(n)$-instanton on S^4, Theorem 5 applies since $Sp(n) \subset U(2n)$ is a subgroup and it follows that the lifted bundle $\tilde{E}$ is holomorphic and has a positive real form τ. Furthermore, α lifts to a holomorphic isomorphism $\tilde{\alpha} : \tilde{E} \to \tilde{E}$, and the compatibility conditions of α gives us that $\tilde{\sigma} := \tilde{\alpha}^{-1} \circ \tau : \tilde{E} \to \tilde{E}$ is an anti-linear isomorphism with the following properties:

$$\tilde{\sigma} \text{ covers } \sigma \text{ on } \mathbb{P}^3, \tag{5.17}$$

$$\tilde{\sigma}^2 = -1, \tag{5.18}$$

$$\omega(\tilde{\sigma}u, \tilde{\sigma}v) = \overline{\omega(u, v)}. \tag{5.19}$$

$$h(u, v) := \omega(u, \tilde{\sigma}v) \text{ is a hermitian inner product.} \tag{5.20}$$

We can then restate Theorem 5 for $Sp(n)$-instantons as

Theorem 6. *There is a natural bijection between*

1. *$Sp(n)$-instanton bundles over S^4 (modulo gauge transformation),*
2. *Holomorphic vector bundles of rank $2n$ over $\mathbb{P}^3$ together with a symplectic form ω and an anti-linear isomorphism $\sigma : \tilde{E} \to \tilde{E}$ satisfying (5.17), (5.18), (5.19) and (5.20).*

As a final remark, going over the steps of our construction should make it clear that the correspondence is valid not just for the whole of S^4, but for any open set $U \subset S^4$. For instance, we could consider $\mathbb{R}^4 \subset S^4$, which would correspond to $\mathbb{P}^3 \setminus \mathbb{P}^1$, with S^4 viewed as the one-point compactification of $\mathbb{R}^4$.

6 The ADHM construction

6.1 Construction of bundles

We shall now describe in detail the construction of bundles that correspond to instantons via Theorem 5, outlined in (ATIYAH et al., 1978). This construction came to be known as the ADHM construction.

Let V and W be complex vector spaces with $\dim V = 2k+2$ and $\dim W = k$. Suppose V is equipped with a symplectic form ω. Let $A_i : W \to V$ $(i = 1, \ldots, 4)$ be linear maps and put $A(z) = \sum_i A_i z_i$. Let $B_z = A(z)W \subset V$ and assume that for all $z \neq 0$, $\dim B_z = k$ (i.e. $A(z)$ is injective for all $z \neq 0$) and that B_z is isotropic under ω. Notice that $B_z = B_{\lambda z}$ for $\lambda \in \mathbb{C}^*$, meaning that the vector spaces B_z are more naturally parametrized by points of $\mathbb{P}^3$. We shall purposefully use both $B_{(z)}$ and B_z to denote the vector space associated to the point $(z) \in \mathbb{P}^3$. Moreover, notice that due to isotropy, $B_z \subset B_z^0$, the latter being the symplectic complement of B_z. Thus $E_z := B_z^0 / B_z$ is a vector space of dimension 2 for all $(z) \in \mathbb{P}^3$ and

$$E = \coprod_{(z)\in\mathbb{P}^3} E_z$$

is a vector bundle over $\mathbb{P}^3$. It is easy to see that also due to isotropy, the symplectic form ω descends to E and thus E admits a reduction to $SL(2, \mathbb{C})$.

Proposition 22. *Given distinct points $(x), (y) \in \mathbb{P}^3$, $B_x \cap B_y = \{0\}$.*

Proof. First, notice that

$$B := \coprod_{(z)\in\mathbb{P}^3} B_{(z)}$$

is isomorphic to k copies of the tautological line bundle $\oplus_1^k L$. To see this, suppose $\{U_i\}$ is the standard open covering of $\mathbb{P}^3$. Then the trivialization over, say, U_0 is given by $\Phi_0 : B|_{U_0} \to U_0 \times \mathbb{C}^k$, where

$$\Phi_0\left((z), v^1 A\left(1, \frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right) e_1, \ldots, v^k A\left(1, \frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right) e_k\right) = ((z), v^1, \ldots, v^k),$$

assuming $\{e_i\}_1^k$ is a basis for W. The trivializations over the other open sets U_i are built similarly by changing the entries of A to the natural coordinates over them. It is then a tedious verification that the transition functions are the same as for $\oplus_1^k L$.

With this in mind, if $v \in B_x \cap B_y$, then it determines a holomorphic section of the restricted bundle $B|_\ell$, where ℓ is the line passing through (x) and (y) in $\mathbb{P}^3$. This is because ℓ is compact, and by Liouville's theorem, all sections are constant. But the only holomorphic section of L is the zero section, and since B is a direct sum of k copies of L, it follows that $v = 0$. □

We shall now characterize the lines $\ell \subset \mathbb{P}^3$ such that the restricted bundle $E|_\ell$ is trivial. First, a few lemmas:

Lemma 7. *Given distinct points $(x), (y) \in \mathbb{P}^3$, then $B_x^0 \cap B_y = 0$ if and only if $B_y^0 \cap B_x = 0$.*

Proof. First, notice that, since $B_x \cap B_y = \{0\}$, there's a 2-dimensional complement to $B_x \oplus B_y$ in V, which we shall denote by R; that is, $V = B_x \oplus R \oplus B_y$. Suppose now that $\{e_i\}_1^k$, $\{f_i\}_1^2$ and $\{g_i\}_1^k$ are bases for B_x, R and B_y, respectively. To say that $B_x^0 \cap B_y = \{0\}$ is equivalent to saying that

$$(\cap_{i=1}^k \ker \omega(e_i, \cdot)) \cap (\cap_{i=1}^2 \ker f^i) \cap (\cap_{i=1}^k \ker e^i) = \{0\}, \tag{6.1}$$

which in turn is equivalent to saying that

$$\Xi = \begin{bmatrix} I_{k+2} & 0 \\ * & \Omega \end{bmatrix},$$

where $\Omega = (\Omega_{ij})_{i,j=1}^k = (\omega(e_i, f_j))_{i,j=1}^k$, is invertible. This is because the matrix above can be seen as a linear operator $\Xi : V \to V$ in the basis $\{e_i\} \oplus \{f_i\} \oplus \{g_i\}$, and thus $\ker \Xi$ is precisely the left hand side of (6.1). Similarly, $B_y^0 \cap B_x = \{0\}$ is equivalent to saying that

$$\Xi' = \begin{bmatrix} -\Omega^T & * \\ 0 & I_{k+2} \end{bmatrix}$$

is invertible. But $\det \Xi' = \det(-\Omega^T) = (-1)^k \det \Omega = (-1)^k \det \Xi$, so both matrices are invertible whenever one of them is. □

Suppose now that (x) and (y) are distinct points in $\mathbb{P}^3$ such that $B_x^0 \cap B_y = \{0\}$. From the proof of the above lemma, we can conclude that the 2-dimensional complement to $B_x \oplus B_y$ in V can be written as $R = B_x^0 \cap B_y^0$. Furthermore, $R \subset B_z^0$ for every (z) in the line ℓ passing through (x) and (y), since for such (z), $B_z \subset B_x \oplus B_y$, and R consists of vectors that annihilate both B_x and B_y under ω.

Lemma 8. *For every $(z) \in \ell$ we have $B_z^0 = R \oplus B_z$.*

Proof. We need only check that $R \cap B_z = \{0\}$. But if $v \in R \cap B_z$, then there exist complex numbers α, β and $w \in W$ such that

$$v = \alpha A(x)w + \beta A(y)w.$$

We can suppose both $\alpha, \beta \neq 0$ without loss of generality. Moreover, $\omega(v, u) = \omega(v, u') = 0$ for all $u \in B_x$ and all $u' \in B_y$. In particular,

$$\alpha\omega(A(x)w, u) + \beta\omega(A(y)w, u) = 0 \quad \forall u \in B_x.$$

Due to isotropy, the first term on the left is zero, and since $\beta \neq 0$, we find that

$$\omega(A(y)w, u) = 0 \quad \forall u \in B_x.$$

This means that $A(y)w \in B_x^0 \cap B_y = \{0\}$, and from injectivity of $A(y)$, we find that $w = 0$. Thus, $v = 0$. □

As a result, we can decompose V as

$$V = B_a \oplus R \oplus B_b$$

for every $(a), (b) \in \ell$, the line passing through (x) and (y).

Corollary 5. *If ℓ is a line passing through (x) and (y) such that $U_x^o \cap U_y = \{0\}$, then $U_a^o \cap U_b = \{0\}$ for every pair of distinct points $(a), (b) \in \ell$.*

The proposition below finally states that the lines $\ell \subset \mathbb{P}^3$ on which the restriction $E|_\ell$ is trivial are precisely the ℓ such that $U_x^o \cap U_y \neq \{0\}$ for some (and thus every, from the above corollary) pair of distinct points (x) and (y) belonging to ℓ.

Proposition 23. *Suppose ℓ is a (projective) line joining (x) and (y). Then $B_x^o \cap B_y = \{0\}$ if and only if $E|_\ell$ is a trivial bundle.*

Proof. Assuming $B_x^o \cap B_y = \{0\}$, then from Lemma 8 we can conclude that $B_z^o / B_z = R$ for every $(z) \in \ell$, and thus $E|_\ell$ is the trivial bundle $\ell \times R$.

Further, if $B_x^o \cap B_y \neq \{0\}$, then R intersects both B_x and B_y. Suppose $v \in R \cap B_x$ is non-zero. Then v defines a holomorphic section

$$\begin{aligned} s : \ell &\to E \\ p &\mapsto v \end{aligned}$$

such that $s(x) = 0$, as v is then quotiented out to zero, being in B_x, and $s(y) \neq 0$, since from Proposition 22, $B_x \cap B_y = \{0\}$. From Liouville's theorem, if $E|_\ell$ were trivial, then every holomorphic section of $E|_\ell$ should be constant. Since this is not the case, it follows that $E|_\ell$ is not trivial. □

We now introduce the structure necessary for our bundles over $\mathbb{P}^3$ to be trivial along real lines, i.e. the fibers of the twistor fibration.

We assume we have anti-linear maps (we use the same letter σ to denote both maps)

$$\sigma : W \to W \text{ such that } \sigma^2 = 1,$$
$$\sigma : V \to V \text{ such that } \sigma^2 = -1.$$

Notice that an anti-linear map σ squaring to -1 on a complex vector space turns it into a quaternionic vector space, with σ playing the role of j. We also require that σ preserve the symplectic form ω on V in the sense that

$$\omega(\sigma u, \sigma v) = \overline{\omega(u, v)}$$

and that

$$h(u, v) := \omega(u, \sigma v)$$

is a positive hermitian form, which by non-degeneracy of ω then defines a hermitian inner product on V.

We want to make sure that E is trivial on the real lines of $\mathbb{P}^3$, and the condition that guarantees this is

$$\sigma A(z)w = A(\sigma z)\sigma w \quad \forall w \in W, \tag{6.2}$$

with σz being left multiplication by j in $\mathbb{P}^3$, as usual. This compatibility condition will be made clearer in the next section. For now, notice that an immediate consequence of this reality condition is that

$$B_{\sigma z} = \sigma B_z. \tag{6.3}$$

In addition, by the definition of h, we also have $B_z^0 = B_{\sigma z}^{\perp}$. We then get the *orthogonal* decomposition

$$V = \underbrace{B_z \oplus R_x}_{B_z^0} \oplus B_{\sigma z},$$

where $R_x = B_z^0 \cap B_{\sigma z}^0$ depends only on $x \in S^4$, since real lines (in this case, the one passing through (z) and (σz)) are in 1-1 correspondence with points of S^4 via the twistor fibration. It readily follows by Proposition 23 that E is trivial along real lines. We also conclude from the decomposition that σ on V descends to our bundle E since we can then identify $E_z = B_z^0 / B_z \cong R_x$.

The bundle $E \to \mathbb{P}^3$ we have constructed together with the map σ on E descending from V then satisfies the conditions of Theorem 6 and therefore corresponds to an $Sp(1)$-instanton on S^4. We note that $Sp(1) \cong SU(2)$, and so this construction suffices for the $n = 2$ case of the special unitary group. For $Sp(n)$, we need only V to have dimension $2k + 2n$.

For $SU(n)$ (also for $O(n)$ with the obvious modifications), we still need a positive real form to satisfy the hypotheses of Theorem 5; this is naturally given by

$$\begin{aligned} \tau : E &\to E^* \\ v &\mapsto h(\cdot, v), \end{aligned}$$

with $\tau(v)u = h(u, v)$ for $v \in E_z$ and $u \in E_{\sigma z}$. Since h is an inner product on V and not on E, we must check that this is well defined. That is, given $u, u' \in B_{\sigma z}^0$ and $v, v' \in B_z^0$

such that $u - u' \in B_{\sigma z}$ and $v - v' \in B_z$, we must have $h(u,v) = h(u',v')$. But

$$h(u - u', v - v') = \omega(u - u', \sigma(v - v')),$$

and $\sigma(v - v') \in B_{\sigma z}$ from (6.3). The result then follows from the skewness of ω.

As a final remark, we note that the instanton number in all cases is $-k$. For details, cf. (ATIYAH, 1979).

6.2 The t'Hooft solution

The (complex) vector space V used in the construction above of dimension $2k + 2n$ can be thought of as a left quaternion vector space (technically, a left module over the *ring* of quaternions) of dimension $k + n$ with σ playing the role of multiplication by j.

Turning our attention to the k-dimensional vector space W, we see that since $\sigma^2 = +1$, we can decompose W as the direct sum of the eigenspaces of σ. The $+1$-eigenspace, which we call $W_{\mathbb{R}}$, is then a k-dimensional *real* vector space, and this splitting allows us to canonically identify $W \cong W_{\mathbb{R}} \otimes \mathbb{C}$ by the map

$$(w_+, w_-) \to w_+ \otimes 1 + w_+ \otimes i.$$

Then using the identification $\mathbb{C}^4 \cong \mathbb{H}^2$ established in Section 5.1, we see that $\mathbb{C}^4 \otimes_{\mathbb{C}} W \cong_{\mathbb{H}} \mathbb{H}^2 \otimes_{\mathbb{R}} W_{\mathbb{R}}$.

We can now look at our map $A(z) : W \to V$ as a homomorphism of left quaternion vector spaces

$$A : \mathbb{H}^2 \otimes_{\mathbb{R}} W_{\mathbb{R}} \to V,$$

where $\mathbb{H}$-linearity corresponds to the reality condition (6.2).

Suppose now $\{e_i\}_1^k$ is a (real) basis for $W_{\mathbb{R}}$ and $\{f_i\}_1^{k+n}$ is a (quaternionic) basis for V. Define matrices $C = (c_{ij})$ and $D = (d_{ij})$ by

$$A((1,0) \otimes e_i) = \sum_{j=1}^{k+n} c_{ij} f_j, \quad A((0,1) \otimes e_i) = \sum_{j=1}^{k+n} d_{ij} f_j,$$

that is, the *rows* of C and D are given by the images of $(1,0) \otimes e_i$ and $(0,1) \otimes e_i$ under A, respectively. Then we see that

$$A(x,y) = xC + yD.$$

By way of example, taking $n = 1$, the $(k+1) \times (k+1)$ matrix $A(x,y)$ given by

$$A(x,y) = \begin{bmatrix} \lambda_1 x & \cdots & \lambda_k x \\ p_1 x - y & & \vdots \\ & \ddots & p_k x - y \end{bmatrix}$$

corresponds to the classical solution to the self-dual equations given by t'Hooft. For more details, we refer to (CHRIST; WEINBERG; STANTON, 1994), (ATIYAH, 1979), (WARD; WELLS, 1990).

7 Completeness

In this chapter we shall compute some of the cohomology groups associated to bundles E arising from the ADHM construction. This will allow us to tell whether a given holomorphic bundle E over $\mathbb{P}^3$ is in the isomorphism class of some bundle constructed by the ADHM procedure; this is the content of Barth's theorem.

Let $E \to \mathbb{P}^3$ be a holomorphic vector bundle. Write $E(n) := E \otimes \mathcal{O}(n)$ to denote E twisted n times[1]. A key result is that if E is lifted from a bundle on S^4 with ASD connection, then $H^1(E(-2)) = 0$, which is *not* obtained using traditional methods from algebraic geometry. The proof of this fact relies on the machinery of Dirac equations and formal neighborhoods, with the main reference being (HITCHIN, 1980).

7.1 The vanishing of $H^1(E(-2))$

We have seen that solutions to the anti-self-dual equations on S^4 correspond to holomorphic bundles on $\mathbb{P}^3$ via the twistor fibration – this transformation of data is called the *twistor transform*. In the event that E corresponds to to a $U(1)$-instanton bundle on $\mathbb{R}^4$, i.e. a solution to Maxwell's equations in Euclidean space, then we can apply the twistor transform and obtain a holomorphic line bundle over $\mathbb{P}^3 \setminus \mathbb{P}^1$. These line bundles are classified by $H^1(\mathbb{P}^3 \setminus \mathbb{P}^1, \mathcal{O})$ and so we obtain a correspondence between solutions to Maxwell's equations on $\mathbb{R}^4$ and elements of $H^1(\mathbb{P}^3 \setminus \mathbb{P}^1)$.

The case that is of interest to us is the differential equation whose solution space can be identified with $H^1(\mathbb{P}^3, E(-2))$ when E is the lifted bundle of an instanton bundle over S^4. This is the equation

$$(\nabla^*\nabla + R/6)u = 0,$$

where $\nabla^*\nabla$ is the Bochner Laplacian on S^4 and R is the scalar curvature of the sphere. Since this operator is positive (cf. (HITCHIN, 1980), p. 186), the only (global) solution to the equation is $u = 0$, which means that $H^1(\mathbb{P}^3, E(-2)) = 0$. We note once more that this is *not* a standard result of algebraic geometry, as there are vector bundles over $\mathbb{P}^3$ such that when twisted by $\mathcal{O}(-2)$ do not have zero first cohomology (cf. (HARTSHORNE, 1978)).

[1] $\mathcal{O}(-1)$ is often called *Serre's twisting sheaf.*

7.2 A theorem of Barth

The map $A(z)$ introduced in Section 6.1 induces a homomorphism of vector bundles over $\mathbb{P}^3$ given by

$$\begin{aligned} A : W(-1) &\to V \\ w \otimes \lambda(z) &\mapsto \lambda A(z) w, \end{aligned}$$

where W and V denote the trivial bundles with fiber W and V, respectively. From injectivity of $A(z)$, it follows that $W(-1) \cong B$, the latter being the bundle introduced in Section 6.1. That is, we have a short exact sequence

$$0 \to W(-1) \to V \to Q \to 0,$$

where $Q := V/B$. Dualizing (recall that the dual functor is exact) the standard exact sequence defining E

$$0 \to W(-1) \to B^0 \to E \to 0,$$

and using the isomorphism $E \cong E^*$ provided by the symplectic form ω, we obtain that

$$0 \to E \to Q \to W^*(1) \to 0$$

is exact, noting that $Q^* = B^0$. Putting together all these sequences and their duals, we obtain the diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W(-1) & \longrightarrow & Q^* & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W(-1) & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & W^*(1) & = & W^*(1) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{7.1}$$

We note that the second Chern class of E is computable from such a diagram, and we get $c_2(E) = \dim W$ (cf. (ATIYAH, 1979)).

Proposition 24. *1.* $H^0(Q) \cong V$*.*

2. $H^0(Q(-n)) = 0$ *for* $n \geq 1$*.*

3. $H^1(Q(n)) = 0$ *for every* n*.*

4. $H^0(E(-n)) = 0$ *for* $n \geq 1$.

5. $H^1(E(-n)) = 0$ *for* $n \geq 2$.

6. $H^1(E(-1)) \cong W^*$.

Proof. Take the row

$$0 \to W(-1) \to V \to Q \to 0 \tag{7.2}$$

from (7.1). Then we extract the following exact sequence from the long exact sequence induced by taking cohomology

$$H^0(W(-1)) \to H^0(V) \to H^0(Q) \to H^1(W(-1)).$$

Since W is trivial, then $H^0(W(-1)) = H^1(W(-1)) = 0$ from Proposition 10. The triviality of V and Liouville's theorem gives us $H^0(V) \cong V$ (the vector space), and thus $H^0(Q) \cong V$. Moreover, since W and V are trivial bundles, tensoring (7.2) by $\mathcal{O}(n)$ and taking cohomology gives us items 2 and 3, also by Proposition 10.

Tensoring the last column of (7.1) by $\mathcal{O}(-n)$ and taking cohomology, we get

$$0 \to H^0(E(-n)) \to H^0(Q(-n)) \to H^0(W^*(-n+1)) \to H^1(E(-n)) \to H^1(Q(-n)).$$

Using that $H^0(Q(-n)) = 0$ for $n \geq 1$, we have item 4. Moreover, since $H^0(W^*(-n+1)) = 0$ for all $n \geq 2$, item 5 follows. Finally, for $n = 1$, $H^0(W^*(-n+1)) = H^0(W^*) \cong W^*$ (the vector space), and we get item 6. □

From the long exact cohomology sequence induced by the last column of (7.1) and the vanishing of $H^1(Q(n))$, the map

$$H^0(W^*(n+1)) \to H^1(E(n))$$

is surjective for $n \geq -1$. We can look at $H^0(W^*(n+1))$ as the space of global sections of $\mathcal{O}(n+1)$ taking values in W^*, essentially by Liouville's theorem. From Proposition 24, $H^1(E(-1)) \cong W^*$. Thus we have a surjective map

$$H^1(E(-1)) \otimes H^0(\mathcal{O}(n+1)) \to H^1(E(n)).$$

This introduces an action of the ring of homogenous polynomials in the variables z_1, z_2, z_3, z_4 of degree $n+1$ on $H^1(E(n))$. We have then a graded module over the ring of homogeneous polynomials $M = \oplus M_n$, where $M_n := H^1(E(n))$. In addition,

M1. $M_n = 0$ for $n \leq -2$ and large n,

M2. $\dim M_{-1} = k$,

M3. M_{-1} generates M,

with item 1 following from Proposition 24 and Serre's vanishing theorem (see below).

We shall need a couple of standard results in complex geometry before we can proceed with the statement and proof of Barth's theorem.

Proposition 25. *The canonical bundle $K_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}(-n-1)$.*

Proof. (HUYBRECHTS, 2006), p. 92. □

Theorem 7 (Serre's vanishing theorem)**.** *Let X be a compact complex manifold and E a holomorphic bundle over X. Then there is an integer m_0 such that for all $m \geq m_0$ $H^q(X, E(m)) = 0$ and $q > 0$.*

Proof. (HUYBRECHTS, 2006), p. 243. □

Theorem 8 (Serre duality)**.** *Let X be a compact complex manifold and E a holomorphic bundle over X. Then there is an isomorphism $H^q(X, E) \cong H^{n-q}(X, K_X \otimes E^*)^*$, where K_X is the canonical bundle of X.*

Proof. (HUYBRECHTS, 2006), p. 171. □

Theorem 9 (Barth)**.** *Let E be a holomorphic vector bundle on $\mathbb{P}^3$ with a symplectic form ω such that*

1. *For some $\ell \subset \mathbb{P}^3$, the restriction $E|_\ell$ is trivial,*
2. $H^1(E(-2)) = 0$.

Then E is in the isomorphism class of the bundle generated from some ADHM data (A, W, V).

Proof. We shall show that diagram (7.1) can be constructed for any such E, which suffices as it contains the data of the map $A(z)$, the vector spaces W, V and the skew-form on V. We start by showing that the graded module

$$M = \bigoplus_n H^1(E(n))$$

satisfies conditions M1 to M3 above.

From hypothesis 1, it follows that E is trivial on the general line of $\mathbb{P}^3$; this is to say that the lines on which E is not trivial are contained in a subvariety of strictly smaller dimension. Taking any plane $\mathbb{P}^2$ containing the line ℓ on which $E|_\ell$ is trivial, it follows that E is also trivial on the general line of this plane, and thus $E(n)|_{\mathbb{P}^2}$ has no sections other than the zero section when $n < 0$. Now from Corollary 3,

$$0 \to E(-1) \to E \to E|_{\mathbb{P}^2} \to 0$$

is exact. Tensoring with $\mathcal{O}(n)$ and taking cohomology, we get the exact sequence

$$H^0(\mathbb{P}^2, E(n)) \to H^1(\mathbb{P}^3, E(n-1)) \to H^1(\mathbb{P}^3, E(n)).$$

Since $H^0(\mathbb{P}^2, E(n)) = 0$ for all $n < 0$, we fix $n = -2$ to find that $H^1(\mathbb{P}^3, E(-3)) = 0$. Repeating this procedure for all $n \leq -3$, we conclude $H^1(\mathbb{P}^3, E(n)) = 0$ for all $n \leq -2$. Together with Serre's vanishing theorem, we have then established property M1.

Property M2 follows from an application of the Hirzebruch-Riemann-Roch theorem, which allows the computation of the *holomorphic Euler characteristic* of E,

$$\chi(X, E) = \sum_{q=0}^{3} (-1)^q \dim H^q(E(-1))$$

in terms of the Chern classes of E, which gives us precisely k. For more information, we refer to (ATIYAH, 1979) and (HIRZEBRUCH; BOREL; SCHWARZENBERGER, 1966), but we comment that this is made possible by the fact that $H^q(E(-1)) = 0$ when $q \neq 1$; for $q = 0$, the cohomology group vanishes from triviality on general lines, and from Serre duality

$$H^3(E(-1)) \cong H^0(E(-3))^*,$$

where we have used the symplectic isomorphism $E \cong E^*$. Thus, also from triviality on general lines, we find that $H^3(E(-1)) = 0$. Finally, for $q = 2$, we can also use Serre duality to identify

$$H^2(E(-1)) \cong H^1(E(-3))^* = 0,$$

where the last equality follows from property 1.

For property M3, take local coordinates in $\mathbb{P}^3$ such that ℓ is given by $z_1 = z_2 = 0$. Then we can resolve $\mathcal{O}_\ell$ as

$$0 \to \mathcal{O}(-2) \xrightarrow{\alpha} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\beta} \mathcal{O} \to \mathcal{O}_\ell \to 0,$$

where $\alpha : s \mapsto (z_2 s, z_1 s)$, and $\beta : (s_1, s_2) \mapsto z_1 s_1 - z_2 s_2$ (cf. (HORI et al., 2003), p. 34). It should be clear that the image of β is the ideal sheaf $\mathcal{I}$ of holomorphic functions vanishing on ℓ, giving us exactness of

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{I} \to 0.$$

Tensoring with $E(n)$ and taking cohomology, we find that

$$H^1(E(n-1)) \oplus H^1(E(n-1)) \to H^1(\mathcal{I}(n)) \to H^2(E(n-2)) \tag{7.3}$$

is exact. To find out more about $H^1(\mathcal{I}(n))$, we tensor with $E(n)$ and take cohomology of the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}_\ell \to 0,$$

which gives us the exact sequence

$$H^1(\mathcal{I}(n)) \to H^1(E(n)) \to H^1(\ell, E(n)) \to 0.$$

From triviality on ℓ, $H^1(\ell, E(n)) = 0$, which shows that $H^1(\mathcal{I}(n)) \cong H^1(E(n))$. Moreover, by Serre duality,

$$H^2(E(n-2)) \cong H^1(E(-n-2))^* = 0,$$

the last equality being valid for $n \geq 0$ as a result of property 1. Back to (7.3), these two results show that

$$\beta : M_{n-1} \oplus M_{n-1} \to M_n$$

is surjective for $n \geq 0$. Inductively, we then see that M_{-1} generates M.

We now begin the construction of diagram (7.1). Setting $W := H^1(E(-1))$, and recalling the extension interpretation of $H^1(W \otimes E(-1))$, we pick the extension of E by $W^*(1)$

$$0 \to E \to Q \to W^*(1) \to 0 \tag{7.4}$$

corresponding to the identity in $H^1(W \otimes E(-1))$. Moreover, this cohomology group can be thought of as elements of $H^1(E(-1))$ with coefficients in W by Liouville's theorem; this is to say

$$H^1(W \otimes E(-1)) \cong W \otimes W^* \cong \mathrm{End}(W).$$

Thus, tensoring (7.4) with $\mathcal{O}(-1)$ and taking cohomology,

$$H^0(W^*) \xrightarrow{\delta} H^1(E(-1)) \to H^1(Q(-1)) \to H^1(W^*),$$

it follows that the Bockstein homomorphism δ is the identity, noting that $H^0(W^*) = W^*$ (the vector space) by Liouville's theorem. From $H^1(W^*) = 0$ and exactness, we have $H^1(Q(-1)) = 0$. Tensoring (7.4) with $\mathcal{O}(n)$ and taking cohomology, we get

$$H^0(W^*(n+1)) \to H^1(E(n)) \to H^1(Q(n)) \to H^1(W^*(n+1)).$$

Since $H^1(W^*(n+1)) = 0$ for all n and $H^0(W^*(n+1)) = 0$ for $n \geq -2$, we obtain

$$H^1(Q(n)) \cong H^1(E(n)) = 0 \text{ for } n \leq -2,$$

whereas for $n \geq 0$,

$$W^* \otimes H^0(\mathcal{O}(n+1)) \xrightarrow{\delta} H^1(E(n)) \to H^1(Q(n)) \to 0,$$

where we interpret $H^0(W^*(n+1))$ as global sections of $\mathcal{O}(n+1)$ taking values in W^*. We identify δ now with multiplication from M_{-1} to M_n, and from property 3 of M, δ is then surjective. This gives us

$$H^1(Q(n)) = 0 \text{ for all } n.$$

Dualizing (7.4), we get

$$0 \to W(-1) \to Q^* \to E \to 0.$$

Tensoring with $\mathcal{O}(n)$ and taking cohomology, we find that

$$H^1(Q^*(n)) \cong H^1(E(n)) \text{ for all } n.$$

From the above result, it follows that $H^1(W \otimes Q^*(1)) \cong W \otimes H^1(Q^*(-1)) \cong \text{End}(W)$, and we see that there is an extension of Q^* by $W^*(1)$

$$0 \to Q^* \to V \to W^*(1) \to 0$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q^* & \longrightarrow & V & \longrightarrow & W^*(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & W^*(1) & \longrightarrow & 0. \end{array} \tag{7.5}$$

Notice that dualizing (7.5) gives us

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(-1) & \longrightarrow & Q^* & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W(-1) & \longrightarrow & V^* & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

There are now only two things left to verify: that V is a trivial bundle and that it inherits a symplectic form, which will allow us to identify $V \cong V^*$. The triviality of V is necessary for the cohomologies to agree with Proposition 24. We refer the reader to (ATIYAH, 1979) and (BARTH; HULEK, 1978) for the verification of these assertions. □

We have shown in Section 6.1 that to a triple (A, V, W) together with reality conditions (the various maps denoted σ together with compatibility conditions) we associate an $Sp(n)$-instanton over S^4. The converse is also true. To see this, recall that an $Sp(n)$-instanton corresponds to a holomorphic bundle $\tilde{E}$ over $\mathbb{P}^3$ with a symplectic form and an anti-linear isomorphism σ covering σ on $\mathbb{P}^3$ such that $\sigma^2 = -1$. From Barth's theorem, then $\tilde{E}$ corresponds uniquely to a triple (A, V, W). It remains to show that V and W possess real structures, and for that we refer to (ATIYAH, 1979), p. 89. We mention in passing that the case for orthogonal and unitary bundles is very similar, again referring to (ATIYAH, 1979) for more details.

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APPENDIX A – First-order jet bundles

Let X be an n-dimensional smooth manifold and $E \to X$ a rank k vector bundle. The bundle of k-jets of a given vector bundle E is, roughly speaking, a coordinate-free manner of defining Taylor expansions up to the kth order of sections of E.

In what follows, we shall restrict ourselves to first-order jet bundles. For the sake of brevity, we present only definitions and a few assertions, referring to (SAUNDERS, 1989), on which this appendix is based, for a complete exposition.

Definition 35. *Let $E \to X$ be a vector bundle, and let $x \in X$. Define local sections $s, t \in \Gamma(E)$ to be* 1-equivalent at x *if $s(x) = t(x)$ and if, in some bundle chart (x^i, u^α) around $s(x)$,*

$$\frac{\partial s^\alpha}{\partial x^i}(x) = \frac{\partial t^\alpha}{\partial x^i}(x)$$

for $1 \leq i \leq n$ and $1 \leq \alpha \leq k$. The equivalence class containing s is called the 1-jet of s at x *and is denoted $j^1_x s$.*

Definition 36. *The* first jet manifold of E *is the set*

$$J^1(E) := \{j^1_x s \mid x \in X, s \in \Gamma(E)\}.$$

Notice that there are two natural projections one can define on $J^1(E)$:

$$\begin{aligned} \pi_1 : J^1(E) &\to X \\ j^1_x s &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \pi_{1,0} : J^1(E) &\to E \\ j^1_x s &\mapsto s(x) \end{aligned}$$

One can show that $J^1(E)$ is a smooth manifold and a bundle with respect to each of the above projections. We will make frequent use of coordinate systems such as

Definition 37. *Let (U, u) be a bundle chart for E, where $u = (x^i, u^\alpha)$. The* induced coordinate system (U^1, u^1) *on $J^1(E)$ is defined by*

$$\begin{aligned} U^1 &= \{j^1_x s \mid s(x) \in U\} \\ u^1 &= (x^i, u^\alpha, u^\alpha_i), \end{aligned}$$

where $x^i(j^1_x s) = x^i(x)$, $u^\alpha(j^1_x s) = u^\alpha(s(x))$, and the nk new functions

$$u^\alpha_i : U^1 \to \mathbb{R}$$

are given by

$$u_i^\alpha(j_x^1 s) = \frac{\partial s^\alpha}{\partial x^i}(x).$$

Suppose now that $E \to X$ is the trivial bundle $X \times \mathbb{K}^k$. Then sections s of E correspond to vector-valued functions $s : X \to \mathbb{K}^k$, and thus we have the isomorphism

$$\begin{aligned} J^1(E) &\cong E \oplus (E \otimes T^*X) \\ j_x^1 s &\mapsto (s(x), ds(x)). \end{aligned}$$

When E is not the trivial bundle, we need a connection ∇ on E to have a similar map:

$$\begin{aligned} J^1(E) &\cong E \oplus (E \otimes T^*X) \\ j_x^1 s &\mapsto (s(x), \nabla s(x)), \end{aligned}$$

This isomorphism clearly depends on the choice of ∇. However, restricting to jets $j_x^1 s$ such that $s(x) = 0$, we have, in local coordinates,

$$\nabla s(x) = \sum_j \left(ds^j + \sum_i A_{ij} s^i \right) \otimes e_j = ds(x),$$

which is independent of the connection we pick. Thus

Proposition 26. *If $E \to X$ is a vector bundle, then we have a natural inclusion*

$$\{0\} \oplus (E \otimes T^*X) \cong E \otimes T^*X \subset J^1(E).$$