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TRANSMISSION LINE MODELING: A CIRCUIT THEORY APPROACH*

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Abstract. As shown in this paper, the classical transmission line equations for the distributed parameters model can be obtained from standard two-port network matrices avoiding the explicit use of partial differential equations. This is performed through a lemma derived directly from the Cayley–Hamilton theorem. The main advantage of this approach is that the modeling arises naturally in the frequency domain, allowing the consideration of frequency-dependent parameters (as, for instance, the resistance and inductance variations caused by the skin effect), normally not taken into account in time domain models.

Key words. two-port network, transmission lines, circuit theory

AMS subject classifications. 94-01, 94C05, 15A15

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1. Introduction. The main operational advantage of applying Fourier or Laplace transform to linear circuit networks is to allow, for instance, the transfer function computation without the explicit use of differential equations. In other words, only algebraic manipulations are needed, and any impedance is dealt with as being mere resistors.

At the end of a basic electric circuit course, a typical student is capable of performing conventional two-port network calculation involving any linear elements.

The purpose of this paper is to provide the students with the ability to extend this methodology to transmission line models.

Another point is that this paper addresses again the study of transmission lines, which has nowadays gained special importance due to the development of high-speed digital electronics. As transient times become faster, the transmission line behavior of electrical interconnects significantly affects the waveforms, and accurate modeling of all components becomes essential [5], [9].

In the traditional (even recent) literature concerning transmission line models, the departing point is the partial differential equations (named telegrapher's equations) [1], [2], [4], [6], [8]. Then, applying the Laplace transform to these equations and taking the current and voltage at the input and the output terminals, the **ABCD** matrix is obtained (see the appendix for a concise description of this approach).

Here, the starting point is the **ABCD** matrix of a small section of the line. Then, the transmission line **ABCD** model is obtained from the product of the matrices associated with these sections when they become of infinitesimal length. Similar to [7], the application of the Cayley–Hamilton theorem is the core of the proposal of this note, also yielding a simple way to compute the nth power of a matrix. To

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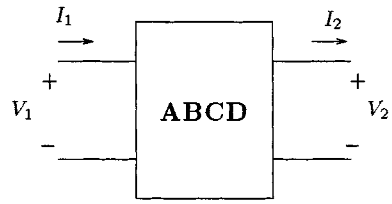

FIG. 1. *Two-port linear network.*

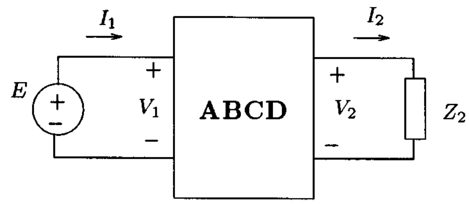

FIG. 2. *Transmission line inserted between a sinusoidal source E and an impedance Z_2.*

illustrate this procedure, three different circuit configurations are used, providing the same expected result.

The technique presented in this paper can be viewed as an alternative method to the approach based on image parameters, i.e., image impedance and image transfer constant, used in some transmission line books [8].

2. Preliminaries. Consider a general network having *two pairs of terminals*, one labeled the *input terminals* and the other the *output terminals*. A pair of terminals at which a signal may enter or leave a network is called a *port*, and the currents in the two leads making up each port must be equal. Each port can be connected only to a port of another network [10].

A general two-port network with terminal voltages and currents as specified in Figure 1 can be described by an **ABCD** matrix if it is composed only by linear elements (zero initial conditions), possibly including dependent sources, but not containing any independent sources.

The **ABCD** parameters satisfy the linear relationship

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}, \tag{1}$$

where V_i and I_i, $i = 1, 2$, represent either the Fourier transforms of $v_i(t)$ and $i_i(t)$, $i = 1, 2$, respectively, or the associated phasors. As is well known [10] for reciprocal (that is, passive, linear, and bilateral) two-port networks, the determinant of the associated **ABCD** matrix is

$$\mathbf{AD} - \mathbf{BC} = 1. \tag{2}$$

Furthermore, if the two-port network is symmetric, $\mathbf{D} = \mathbf{A}$.

Figure 2 shows a typical application of the **ABCD** matrix, yielding after simple computation

$$I_2 = \frac{E}{\mathbf{A}Z_2 + \mathbf{B}}. \tag{3}$$

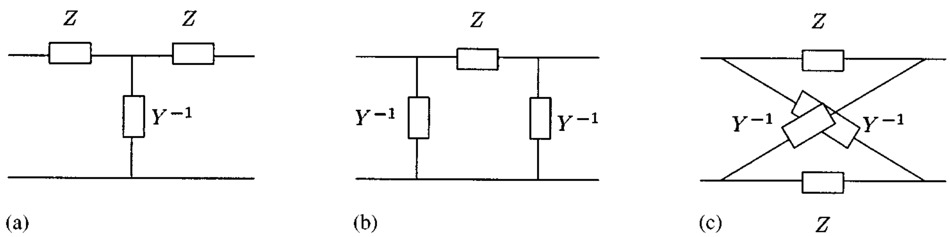

FIG. 3. *Three symmetric two-port network configurations.* (a) *T circuit;* (b) Π *circuit;* (c) *trellis circuit.*

To illustrate, Figure 3 shows three different configurations for symmetric two-port networks.

The associated **ABCD** are given by

$$\text{(a)}\quad \begin{bmatrix} 1+ZY & Z(2+ZY) \\ Y & 1+ZY \end{bmatrix}; \qquad \text{(b)}\quad \begin{bmatrix} 1+ZY & Z \\ Y(2+ZY) & 1+ZY \end{bmatrix};$$
$$\text{(c)}\quad \frac{1}{1-ZY}\begin{bmatrix} 1+ZY & 2Z \\ 2Y & 1+ZY \end{bmatrix}. \tag{4}$$

Note that the determinants are all equal to 1 (the circuits are reciprocal) and that the elements on the main diagonal are identical (the networks are symmetric).

The main property of **ABCD** matrices is that the **ABCD** matrix of the cascade association of N two-port networks is equal to the product of the individual matrices. The use of matrix algebra allows the proposition of the following statement.

LEMMA 2.1. *For any nonzero complex numbers* $\mathbf{b}$ *and* $\mathbf{c}$ *and positive integer* N*, define*

$$[\,\mathbf{t}\,] \triangleq \begin{bmatrix} \sqrt{1+\mathbf{bc}} & \mathbf{b} \\ \mathbf{c} & \sqrt{1+\mathbf{bc}} \end{bmatrix}. \tag{5}$$

Then

$$[\,\mathbf{t}\,]^N = \begin{bmatrix} \dfrac{\lambda^N+\lambda^{-N}}{2} & \dfrac{\lambda^N-\lambda^{-N}}{2}\sqrt{\dfrac{\mathbf{b}}{\mathbf{c}}} \\ \dfrac{\lambda^N-\lambda^{-N}}{2}\sqrt{\dfrac{\mathbf{c}}{\mathbf{b}}} & \dfrac{\lambda^N+\lambda^{-N}}{2} \end{bmatrix} \tag{6}$$

with $\lambda = \sqrt{1+\mathbf{bc}} + \sqrt{\mathbf{bc}}$*, provided that*

$$\sqrt{z} \triangleq \sqrt{\rho}\exp(j\theta/2), \ \text{where} \ z = \rho\exp(j\theta), \quad \theta \in (-\pi,\pi], \quad \rho \geq 0. \tag{7}$$

Proof. The eigenvalues of matrix $[\,\mathbf{t}\,]$ are λ (as above) and λ^{-1}, since $\det[\,\mathbf{t}\,] = 1$. From Cayley–Hamilton's theorem [3],

$$[\,\mathbf{t}\,]^N = \alpha_0\mathbf{I} + \alpha_1[\,\mathbf{t}\,], \tag{8}$$

$\mathbf{I}$ being the identity matrix and

$$\alpha_0 = \frac{\lambda^{-N+1}-\lambda^{N-1}}{2\sqrt{\mathbf{bc}}}; \quad \alpha_1 = \frac{\lambda^N-\lambda^{-N}}{2\sqrt{\mathbf{bc}}}. \tag{9}$$

After some manipulations, the proof is concluded. ∎

3. Transmission line model. In this section, the usual transmission line matrix **ABCD** model is obtained from an incremental segment of the line and the result of Lemma 2.1.

Consider a transmission line with length d and frequency dependent distributed parameters l, c, r, and g. The line is divided into N equally spaced sections of length d/N, and each section can be represented by one of the symmetric configurations depicted in Figure 2.

By computing $[\ \mathbf{t}\ ]$ for the three elementary configurations, one gets

$$\text{(10)}\quad \text{(a)} \begin{bmatrix} 1+\frac{\delta^2}{2} & Z_0\left(\delta+\frac{\delta^3}{4}\right) \\ \frac{\delta}{Z_0} & 1+\frac{\delta^2}{2} \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 1+\frac{\delta^2}{2} & Z_0\delta \\ \frac{1}{Z_0}\left(\delta+\frac{\delta^3}{4}\right) & 1+\frac{\delta^2}{2} \end{bmatrix}; \quad \text{(c)}\ \frac{1}{1-\frac{\delta^2}{4}} \begin{bmatrix} 1+\frac{\delta^2}{4} & Z_0\delta \\ \frac{\delta}{Z_0} & 1+\frac{\delta^2}{4} \end{bmatrix}$$

with

$$\text{(11)}\qquad \delta \triangleq \frac{\gamma d}{N}; \quad \gamma \triangleq \sqrt{(r+j\omega l)\,(g+j\omega c)}; \quad Z_0 \triangleq \sqrt{\frac{(r+j\omega l)}{(g+j\omega c)}},$$

where γ is the propagation constant and Z_0 is the characteristic impedance of the line.

Moreover, from Lemma 2.1, λ and $\sqrt{\mathbf{b}/\mathbf{c}}$ for each case are given by

$$\text{(12)}\quad \text{(a)}\qquad \lambda = 1+\frac{\delta^2}{2}+\delta\left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2}}; \quad \sqrt{\frac{\mathbf{b}}{\mathbf{c}}} = Z_0\left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2}};$$

$$\text{(13)}\quad \text{(b)}\qquad \lambda = 1+\frac{\delta^2}{2}+\delta\left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2}}; \quad \sqrt{\frac{\mathbf{b}}{\mathbf{c}}} = Z_0\left(1+\frac{\delta^2}{4}\right)^{-\frac{1}{2}};$$

$$\text{(14)}\quad \text{(c)}\qquad \lambda = \left(1+\delta+\frac{\delta^2}{4}\right)\left(1-\frac{\delta^2}{4}\right)^{-1}; \quad \sqrt{\frac{\mathbf{b}}{\mathbf{c}}} = Z_0.$$

Using Taylor's series, the three configurations provide

$$\text{(15)}\qquad \lambda = 1+\delta+\mathcal{O}(\delta); \quad \sqrt{\frac{\mathbf{b}}{\mathbf{c}}} = Z_0\left(1+\mathcal{O}(\delta)\right),$$

where $\mathcal{O}(\delta)$ denotes any polynomial on δ with order greater or equal to 2.

Now, as $N \to \infty$,

$$\text{(16)}\qquad \left(1+\frac{\gamma d}{N}+\mathcal{O}\left(\frac{\gamma d}{N}\right)\right)^N \ \to\ \exp(\gamma d); \qquad \sqrt{\frac{\mathbf{b}}{\mathbf{c}}}\ \to\ Z_0,$$

implying that

$$\text{(17)}\qquad [\ \mathbf{T}\ ] \ \triangleq\ \lim_{N\to\infty} [\ \mathbf{t}\ ]^N \ =\ \begin{bmatrix} \cosh(\gamma d) & Z_0\sinh(\gamma d) \\ \sinh(\gamma d)/Z_0 & \cosh(\gamma d) \end{bmatrix}.$$

As is well known, equation (17) is the usual expression for the **ABCD** matrix of a transmission line. Note that the eigenvalues of $[\,\mathbf{T}\,]$ are given by

$$\exp(-\gamma d); \quad \exp(\gamma d), \tag{18}$$

which represent the direct or the inverse transfer functions of a matched line.

4. Conclusion. The paper has presented a new way to obtain the **ABCD** matrix model for a distributed parameter transmission line using circuit theory concepts.

Appendix A. Consider a transmission line, where the distributed parameters are g, shunt conductance; c, shunt capacitance; r, series resistance; and l, series inductance.

Writing the voltage and current Kirchhoff's laws for a section Δx of the line and taking the limit as $\Delta x \to 0$, one gets the partial differential equations [8]

$$\frac{\partial v}{\partial x} + ri + l\frac{\partial i}{\partial t} = 0, \tag{19}$$

$$\frac{\partial i}{\partial x} + gv + c\frac{\partial v}{\partial t} = 0, \tag{20}$$

where $v = v(x,t)$ is the voltage between the two wires and $i = i(x,t)$ is the current through the line. At the input of the line $x = 0$, and $x = d$ at the line output.

Since the equations are linear, $v = V\exp(j\omega t)$ implies $i = I\exp(j\omega t)$, with $V = V(x,\omega)$ and $I = I(x,\omega)$ being the solution of the linear ordinary differential equations

$$\frac{dV}{dx} = -zI; \quad \frac{dI}{dx} = -yV, \tag{21}$$

where

$$z = r + j\omega l; \quad y = g + j\omega c. \tag{22}$$

Defining the matrix

$$M \triangleq \begin{bmatrix} 0 & -z \\ -y & 0 \end{bmatrix}, \tag{23}$$

the equations (21) can be rewritten as

$$\frac{d}{dx}\begin{bmatrix} V \\ I \end{bmatrix} = M\begin{bmatrix} V \\ I \end{bmatrix}. \tag{24}$$

Defining

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V \\ I \end{bmatrix}\Bigg|_{x=0}; \quad \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ I \end{bmatrix}\Bigg|_{x=d}, \tag{25}$$

the solution of (24) for $x = d$ is given by

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \exp(Md)\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}, \tag{26}$$

implying that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \exp(-Md) = \sum_{n=0}^{+\infty}\frac{1}{n!}\begin{bmatrix} 0 & zd \\ yd & 0 \end{bmatrix}^n. \tag{27}$$

On the other hand, for $n = 0, 1, 2, \dots,$

(28)
$$\begin{bmatrix} 0 & zd \\ yd & 0 \end{bmatrix}^{2n} = \begin{bmatrix} (zyd^2)^n & 0 \\ 0 & (zyd^2)^n \end{bmatrix}$$

and

(29)
$$\begin{bmatrix} 0 & zd \\ yd & 0 \end{bmatrix}^{2n+1} = \begin{bmatrix} 0 & zd(zyd^2)^n \\ yd(zyd^2)^n & 0 \end{bmatrix},$$

implying that

(30)
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \sum_{n=0}^{+\infty} \begin{bmatrix} \dfrac{(\gamma d)^{2n}}{2n!} & Z_0 \dfrac{(\gamma d)^{2n+1}}{(2n+1)!} \\ \dfrac{1}{Z_0} \dfrac{(\gamma d)^{2n+1}}{(2n+1)!} & \dfrac{(\gamma d)^{2n}}{2n!} \end{bmatrix} = \begin{bmatrix} \cosh(\gamma d) & Z_0 \sinh(\gamma d) \\ \sinh(\gamma d)/Z_0 & \cosh(\gamma d) \end{bmatrix}.$$

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