STABILITY AND STABILIZATION OF CONTINUOUS-TIME
SWITCHED LINEAR SYSTEMS

JOSÉ C. GEROMEL† AND PATRIZIO COLANERI‡

Abstract. This paper addresses two strategies for the stabilization of continuous-time, switched linear systems. The first one is of open loop nature (trajectory independent) and is based on the determination of a minimum dwell time by means of a family of quadratic Lyapunov functions. The relevant point on dwell time calculation is that the proposed stability condition does not require the Lyapunov function to be uniformly decreasing at every switching time. The second one is of closed loop nature (trajectory dependent) and is designed from the solution of what we call Lyapunov–Metzler inequalities from which the stability condition (including chattering) is expressed. Being nonconvex, a more conservative but simpler-to-solve version of the Lyapunov–Metzler inequalities is provided. The theoretical results are illustrated by means of examples.

Key words. switched systems, continuous-time systems, Lyapunov–Metzler inequalities, linear matrix inequality

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1. Introduction. This paper aims to provide new results on stability analysis and stabilizing control synthesis for a continuous-time, switched linear system of the general form

\[ \dot{x}(t) = A_{\sigma(t)}x(t) , \quad x(0) = x_0, \]

defined for all \( t \geq 0 \), where \( x(t) \in \mathbb{R}^n \) is the state, \( \sigma(t) \) is the switching rule, and \( x_0 \) is the initial condition. Considering a set of matrices \( A_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N \), be given, the switching rule \( \sigma(t) \), for each \( t \geq 0 \), is such that

\[ A_{\sigma(t)} \in \{A_1, \ldots, A_N\}, \]

where it is clear that this model naturally imposes a discontinuity on \( A_{\sigma(t)} \) since this matrix must jump instantaneously from \( A_i \) to \( A_j \) for some \( i \neq j = 1, \ldots, N \) once switching occurs. In other words, \( A_{\sigma(t)} \) is constrained to jump among the \( N \) vertices of the matrix polytope \( \{A_1, \ldots, A_N\} \).

Stability of continuous-time switched linear systems has been addressed by several authors [3], [5], [11], [12], [14], [16], [17], [18], [20] and [23]. While the survey papers [5] and [17] give a complete and detailed description on the problems arising in this area, the recent paper [11], dealing with extensions of LaSalle’s invariance principle, provides an interesting discussion on a collection of results on uniform stability of switched systems. The themes dealt with in the present paper have their roots in the conference paper [9].
Generally speaking, when $\sigma(\cdot)$ is state independent, that is, when it is an a priori piecewise constant signal, the reported stability conditions are obtained using a family of symmetric and positive definite matrices $\{P_1, \ldots, P_N\}$, each one associated to the correspondent matrix of the set $\{A_1, \ldots, A_N\}$ such that a Lyapunov function $v(x(t))$ is nonincreasing for all $t \geq 0$; see [11]. In this paper, for minimum dwell time design preserving global stability it is assumed that each matrix of the set $\{A_1, \ldots, A_N\}$ is asymptotically stable but the above mentioned nonincreasing condition on the Lyapunov function is relaxed. It is replaced by the weaker condition that at every switching time $t_k$ the sequence $v(x(t_k))$, for $k = 0, \ldots, \infty$, converges uniformly to zero. In some instances, our design procedure for the determination of the minimum dwell time, based on a quadratic guaranteed cost, is related to the results of [22] assuming further that the switching rule is not a priori given but can be taken arbitrarily, among the feasible ones; see [8]. For comparison purposes, a simple second-order example is solved, and it is shown that the estimation of the minimum dwell time provided in this paper is sensibly better than the one obtained from the classical result of [18] and the well-known average dwell time stability condition provided in [16], [10], and the references therein. The results obtained in this context have some resemblance with those achieved in [21], where the characterization of the exponential growth rate of switched systems is provided. However, much work is needed to establish the possible links between these two approaches.

For switched systems with $\sigma(\cdot)$ being state dependent, the stability condition is expressed in terms of a set of inequalities that we call Lyapunov–Metzler inequalities because the variables involved are a set of symmetric and positive definite matrices $\{P_1, \ldots, P_N\}$ and a Metzler matrix $\Pi$. The point to be noticed is that our asymptotic stability condition does not require any stability property associated to each individual matrix of the set $\{A_1, \ldots, A_N\}$ and it contains as a special case the quadratic stability condition. An important point of our main result is that it includes the stability of possible sliding modes, a fact that in the particular case $N = 2$ was observed in [16]. It is also important to stress that in [20] we can find some stability results related to the same problem (without the analysis of sliding modes) but restricted to the special case $N = 2$, which does not require the formalism based on the Lyapunov–Metzler inequalities introduced here. In our general case, the price to be paid, however, is the nonconvex nature of the the Lyapunov–Metzler inequalities being thus difficult to solve numerically. From this previous result, a more conservative but easier-to-solve asymptotic stability condition is proposed.

As a final remark, notice that the dwell time calculation provided in the first part of the paper also suggests a way to solve the state-feedback stabilization problem for a input-driven switched system characterized by the pairs $(A_i, B_i)$. Under mild assumptions it is possible to design matrices $K_i$ so as to stabilize the closed-loop systems $A_i + B_iK_i$. Hence, one can compute an upper bound of the dwell time to establish the time duration of the associated piecewise-constant control law. Notice that the general problem of minimization of the dwell time as a function of the design local control laws $K_i$ is still open.

The paper is organized as follows. In section 2, time-switching control is analyzed. The switching rule $\sigma(t)$ is considered piecewise constant and a minimum dwell time preserving stability is determined. Section 3 is entirely devoted to state-switching control where the goal is to design a function $u(\cdot)$ such that the system (1) is globally asymptotically stable with the switching rule $\sigma(t) = u(x(t))$. Section 4 concludes the paper.
The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors, and small Greek letters denote scalars. For matrices or vectors, (T) indicates transpose. For symmetric matrices, \( X > 0 \) (\( \geq 0 \)) indicates that \( X \) is positive definite (positive semidefinite), and \( \lambda_{\min}(X), \lambda_{\max}(X) \) denote its minimum and maximum eigenvalue, respectively. The sets of real and natural numbers are denoted by \( \mathbb{R} \) and \( \mathbb{N} \), respectively. The \( L_2 \) squared norm of \( x(t) \in \mathbb{R}^n \) defined for all \( t \geq 0 \) equals \( \|x(t)\|_2^2 = \int_0^\infty x(t)'x(t)dt \); see [4].

2. Time-switching control. This section is entirely dedicated to the design of a time-switching control law for the switched linear system defined by the model (1) and (2) where it is assumed that each matrix of the set \( \{A_1, \ldots, A_N\} \) is asymptotically stable. The problem under consideration can be stated as follows. Determine a minimum dwell time \( T_* > 0 \) such that the equilibrium point \( x = 0 \) of the system (1) is globally asymptotically stable with the time switching control

\[
\sigma(t) = i \in \{1, \ldots, N\}, \quad t \in [t_k, t_{k+1}),
\]

where \( t_k \) and \( t_{k+1} \) are successive switching times satisfying \( t_{k+1} - t_k \geq T_* \) for all \( k \in \mathbb{N} \) and the index \( i \in \{1, \ldots, N\} \) selected at each instant of time \( t \geq 0 \) is arbitrary. Hence, asymptotic stability is preserved whenever \( \sigma(t) \) remains unchanged for a period of time greater or equal to the minimum dwell time \( T_* \). The next theorem provides the theoretical basis toward a possible solution of this problem by characterizing an upper bound for \( T_* \). It uses the concept of multiple Lyapunov function with the innovation that the classical nonincreasing assumption at switching times is no longer needed.

**Theorem 1.** Assume that for some \( T > 0 \), there exists a collection of positive definite matrices \( \{P_1, \ldots, P_N\} \) of compatible dimensions such that

\[
A_i'P_i + P_iA_i < 0 \quad \forall \ i = 1, \ldots, N
\]

and

\[
e^{A_i'T}P_je^{A_j'T} - P_i < 0 \quad \forall \ i \neq j = 1, \ldots, N.
\]

The time-switching control (3) with \( t_{k+1} - t_k \geq T \) makes the equilibrium solution \( x = 0 \) of (1) globally asymptotically stable.

**Proof.** Consider, in accordance to (3), that \( \sigma(t) = i \in \{1, \ldots, N\} \) for all \( t \in [t_k, t_{k+1}) \), where \( t_{k+1} = t_k + T_k \) with \( T_k \geq T > 0 \), and that at \( t = t_{k+1} \) the time-switching control jumps to \( \sigma(t) = j \in \{1, \ldots, N\} \); otherwise the result trivially follows. From (4), it is seen that for all \( t \in [t_k, t_{k+1}) \), the time derivative of the Lyapunov function \( v(x(t)) = x(t)'P_{\sigma(t)}x(t) \) along an arbitrary trajectory of (1) satisfies

\[
\dot{v}(x(t)) = x(t)'(A_i'P_i + P_iA_i)x(t)
\]

\[
< 0,
\]

which enables us to conclude that there exist scalars \( \alpha > 0 \) and \( \beta > 0 \) such that

\[
\|x(t)\|^2 \leq \beta e^{-\alpha(t-t_k)}v(x(t_k)) \quad \forall t \in [t_k, t_{k+1}).
\]
On the other hand, using the inequalities (5) we have
\[
v(x(t_{k+1})) = x(t_{k+1})' P_{j} x(t_{k+1}) \\
= x(t_{k})' e^{A_{i} T_{k}} P_{j} e^{A_{i} T_{k}} x(t_{k}) \\
< x(t_{k})' e^{A_{i} (T_{k} - T)} P_{i} e^{A_{i} (T_{k} - T)} x(t_{k}) \\
< x(t_{k})' P_{i} x(t_{k}) \\
< v(x(t_{k})),
\]
where the second inequality holds from the fact that for every \( \tau = T_{k} - T \geq 0 \) it is true that \( e^{A_{i} \tau} P_{i} e^{A_{i} \tau} \leq P_{i} \). The consequence is that there exists \( \mu \in (0, 1) \) such that
\[
v(x(t_{k})) \leq \mu v(x_{0}) \quad \forall k \in \mathbb{N},
\]
which together with (7) implies that the equilibrium solution \( x = 0 \) of (1) is globally asymptotically stable.

This result deserves some comments. First, it is simple to determine the scalars \( \alpha, \beta \) and \( \mu \) such that (7) and (9) hold. Indeed, assuming that \( \{P_{1}, \ldots, P_{N}\} \) satisfy the conditions of Theorem 1, then from (4) there exists \( \epsilon > 0 \) such that \( A_{i}' P_{i} + P_{i} A_{i}' \leq -\epsilon I \) for all \( i = 1, \ldots, N \) yielding \( \alpha = \epsilon / \max_{i} \lambda_{\max}(P_{i}) > 0 \) and \( \beta = 1 / \min_{i} \lambda_{\min}(P_{i}) > 0 \). Furthermore, from (5) there exists \( 0 < \mu < 1 \) such that \( e^{A_{i} T_{k}} P_{i} e^{A_{i} T_{k}} \leq \mu P_{i} \) for all \( i \neq j = 1, \ldots, N \) leading to \( v(x(t_{k+1})) \leq \mu v(x(t_{k})) \) and consequently (9). Second, since all matrices of the set \( \{A_{1}, \ldots, A_{N}\} \) are supposed to be asymptotically stable, the constraints (4) are always feasible and the constraints (5) are satisfied when \( T > 0 \) is taken large enough. Third, assuming that matrices \( A_{1}, \ldots, A_{N} \) are quadratically stable, which is the same as saying that they share a positive definite matrix \( P \) such that
\[
A_{i}' P + PA_{i} < 0 \quad \forall i = 1, \ldots, N,
\]
then the inequality (5) is satisfied for \( P_{1} = \cdots = P_{N} = P \) for any \( T > 0 \), meaning that even in the case the switching policy (3) jumps from \( i \) to \( j \) arbitrarily fast, the asymptotic stability is preserved. Hence, Theorem 1 contains, as a particular case, the quadratic stability condition. Finally, with \( T > 0 \) fixed it is always possible to define a time-switching control strategy (3) such that \( A_{\sigma(t)} \) is periodic. As a consequence, a necessary condition for the feasibility of constraints (4) and (5) is
\[
\theta(T) := \max_{q=1, \ldots, N} \left| \lambda_{q} \left( \prod_{p=1}^{N} e^{B_{p} T} \right) \right| < 1,
\]
where \( \lambda_{q}(\cdot) \) denotes a generic eigenvalue of \( (\cdot) \) and \( \{B_{1}, \ldots, B_{N}\} \) are matrices corresponding to any permutation among those of the set \( \{A_{1}, \ldots, A_{N}\} \). However, since (3) may produce nonperiodic policies as well, the necessary condition (11) for the existence of a feasible solution to inequalities (4)–(5) generally does not meet sufficiency. In what follows, this aspect will be illustrated by means of an example.

**Remark 1** (multiple Lyapunov functions). The important concept of multiple Lyapunov functions introduced in [3] is largely used in switching systems analysis; see also [11, 16]. In the present context, consider the family of Lyapunov functions \( V_{i}(x(t)) = x(t)' P_{i} x(t) \) valid in each time interval such that \( \sigma(t) = i \) for all \( i = 1, \ldots, N \). From Theorem 3.1 of [16], the switched system (1) is globally asymptotically
stable whenever each Lyapunov function satisfies $V_i(x(t_p)) < V_i(x(t_q))$, where $\sigma(t_q) = \sigma(t_p) = i$ for every switching times $t_p < t_q$. For dwell time calculation, the use of this stability condition is not simple because it is difficult to impose the decreasing property for each function $V_i(\cdot)$ at nonsuccessive switching times. In Theorem 1 this constraint is replaced by $V_j(x(t_{k+1})) < V_i(x(t_k))$, where $t_k$ and $t_{k+1}$ are successive switching times such that $\sigma(t_k) = i$ and $\sigma(t_{k+1}) = j$. See [3] for a general discussion of these conditions applied to nonlinear switching systems. In [21], a similar stability condition is obtained based on the characterization of the exponential growth rate of switched systems.

From the previous result, an upper bound for the minimum dwell time $T_*$ can be computed by taking the minimum value of $T$ satisfying the conditions of Theorem 1. Hence, it can be calculated with no great difficulty from the optimal solution of the optimization problem

$$\min_{T>0, P_1>0, \ldots, P_N>0} \{ T : (4) \text{ and } (5) \}$$

which, for each $T > 0$ fixed, reduces to a convex programming problem with linear matrix inequality constraints that can be handled by any LMI solver available in the literature to date; see [2] for an important study on systems and LMIs. A line search procedure is then used to deal with the scalar variable $T > 0$.

Finally, it is possible to generalize the result of Theorem 1 to define a guaranteed cost-to-go from an arbitrary initial point to the origin, associated to the stabilizing time-switching rule (3) with $t_{k+1} - t_k \geq T$ for any fixed $T > 0$. To this end, we make the assumption that $T > 0$ is known such that $t_{k+1} - t_k \leq T$ for all $k \in \mathbb{N}$. Clearly, these quantities are related through $T \geq T \geq T_*$, where the second inequality assures global stability.

**Theorem 2.** Let $Q \geq 0 \in \mathbb{R}^{n \times n}$ and $T \geq T > 0$ be given. Define the set of symmetric, nonnegative definite matrices

$$R_i := \int_0^T e^{A_i^T}Qe^{A_i}dt, \quad i = 1, \ldots, N.$$  

Assume that there exists a collection of positive definite matrices $\{P_1, \ldots, P_N\}$ of compatible dimensions such that

$$A_i^T P_i + P_i A_i + Q < 0 \quad \forall \ i = 1, \ldots, N$$

and

$$e^{A_i^T} P_j e^{A_i T} - P_i + R_i < 0 \quad \forall \ i \neq j = 1, \ldots, N.$$  

The time-switching control (3) with $T \geq t_{k+1} - t_k \geq T$ makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable and

$$\int_0^\infty x(t)^T Q x(t) dt < x_0^T P_{\sigma(0)} x_0.$$  

**Proof.** Since for $Q \geq 0$ and $T \geq T > 0$ given, each matrix $R_i$ defined in (13) is positive semidefinite and inequalities (14)–(15) are satisfied, then inequalities (4)–(5)
are also satisfied. As a consequence, asymptotic stability follows from Theorem 1. On the other hand, using (13) together with the inequalities (14) and (15) we have that $P_i > R_i$ and

$$A_i'(P_i - R_i) + (P_i - R_i)A_i < -Q - A_i'R_i - R_iA_i < 0$$

(17)

for all $i = 1, \ldots, N$. The important consequence of this calculation is that for each $i = 1, \ldots, N$ the inequality $e^{A_i \tau}(P_i - R_i)e^{A_i \tau} \leq (P_i - R_i)$ holds for any $\tau \geq 0$. Using this property, taking into account the switching strategy (3) with $t_{k+1} - t_k = T_k \geq T$ and the inequalities (15) one obtains

$$v(x(t_{k+1})) = x(t_{k+1})'P_1x(t_{k+1}) < x(t_k)'e^{A_i'(T_k - T)}(P_i - R_i)e^{A_i(T_k - T)}x(t_k)$$

$$< x(t_k)'(P_i - R_i)x(t_k) < v(x(t_k)) - x(t_k)'R_{\sigma(t_k)}x(t_k),$$

(18)

which summing up for all $k \in \mathbb{N}$ and taking into account that $T \geq t_{k+1} - t_k$ allows us to write

$$\int_0^\infty x(t)'Qx(t)dt = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} x(t_k)'e^{A_i'(t - t_k)}Qe^{A_i(t - t_k)}x(t_k)dt$$

$$\leq \sum_{k=0}^\infty x(t_k)'R_{\sigma(t_k)}x(t_k)$$

(19)

which proves the proposed theorem.

It is interesting to observe that the conditions of Theorem 2 are feasible if and only if $T \geq T \geq T_*$, and from (16) it is seen that a more accurate guaranteed cost is obtained whenever the value of $T$ is chosen as small as possible. In addition, the choice $T = +\infty$ enables us to conclude that the proposed time-switching rule (3) with $t_{k+1} - t_k \geq T_*$, makes the trajectory $y(t) = Q^{1/2}x(t)$, $t \geq 0$, quadratically integrable. Theorem 2 admits the extreme situation $T = T = +\infty$ for which no jump occurs and inequalities (14)–(15) are verified for

$$P_i = \int_0^{\infty} e^{A_i t}(Q + \varepsilon I)e^{A_i t}dt > R_i \geq 0$$

(20)

with $\varepsilon > 0$ arbitrary. When $\varepsilon > 0$ goes to zero, $P_i$ goes to $R_i$ and (16) becomes a well-known result. On the other hand, for $T > 0$ arbitrarily small and any $T \geq T$, feasibility holds whenever the set of matrices $\{A_1, \ldots, A_N\}$ admits a common Lyapunov function.

Example 1. For illustration purposes of the theoretical results obtained so far, let us consider the example with $N = 2$ and matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix},$$

(21)
which are not quadratically stable. First, from problem (12), we have calculated an upper bound for the minimum dwell time as being $T_* \leq 2.76$. To give an idea of its conservativeness we have calculated from the plot of Figure 1 the value $T_{per} = 2.71$ corresponding to the necessary condition for stability (11), arising from linear periodic systems. Both being very close indicates, for this simple example, a good precision on the determination of $T_*$. On the other hand, for comparison we have applied the classical result of [18] for the determination of an alternative upper bound for the minimum dwell time $T_*$ given by $T_* \leq \max_{i=1,...,N}\{T_i\}$, where

$$T_i = \inf_{\alpha>0,\beta>0} \left\{ \frac{\alpha}{\beta} : \|e^{A_i t}\| < e^{(\alpha-\beta)t} \quad \forall t \geq 0 \right\}.$$  

For the matrices in (21), we have numerically determined $T_1 = 2.33$ and $T_2 = 6.66$, yielding an estimation for the minimum dwell time as being $T_* \leq 6.66$. Hence, in this particular example, the result provided by the solution of problem (12) is much more precise but at the expense of a more expressive computational effort. Moreover, using the method in [10] we have obtained the upper bound of the so-called average dwell time as being 16.5554, which is obviously greater than that obtained by our method.

Figure 2 has been constructed by simulation of system (1) with the time-switching rule (3), $t_{k+1} - t_k = 3.0$, initial conditions $x_0 = [1 \ 1]'$, $\sigma(0) = 2$, and $Q = I$. The family of Lyapunov functions has been calculated from the optimal solution of the convex programming problem

$$\min_{P_1 > 0,...,P_N > 0} \max_{i=1,...,N} \{ x_0' P_i x_0 : (14) \text{ and } (15) \},$$
which puts in evidence that a guaranteed cost can be determined for the worst case as far as the initial condition $\sigma(0)$ appearing in (16) is concerned. For $T = T = 3.0$, we have obtained the minimum guaranteed cost equal to $\delta^* = 100.61$, valid for both initial conditions. As noted earlier, the Lyapunov function $v(x(t)) = x(t)'P_{\sigma(t)}x(t)$ goes to zero as $t$ goes to infinity; however, it is not uniformly decreasing with respect to time. In Figure 2, due to the stability conditions of Theorem 2, the discontinuity points, marked with circles, define a globally convergent sequence $v(x(t_k))$ for all $k \in \mathbb{N}$. Solving again problem (23) but for $T = +\infty$ and $T = 3.0$, the minimum guaranteed cost increases to $\delta^* = 147.94$ as a consequence of allowing a more flexible switching rule (3) with $t_{k+1} - t_k \geq 3.0$.

The example above shows that there is a clear improvement on stability conditions, dwell time, and guaranteed cost calculations when compared to the results available in the literature to date; see [11], [18].

3. State-switching control. In this section, we consider once again the system (1) where the switching rule satisfies (2). The main difference from the previous section is that, presently, it is assumed that the state vector $x(t)$ is available for feedback for all $t \geq 0$. That is, our goal is to determine the function $u(\cdot) : \mathbb{R}^n \rightarrow \{1, \ldots, N\}$ such that

$$\sigma(t) = u(x(t))$$

makes the equilibrium point $x = 0$ of (1) asymptotically stable. In this case, we do not assume that each matrix of the set $\{A_1, \ldots, A_N\}$ is asymptotically stable. To this end, let us define the simplex

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^N \lambda_i = 1, \lambda_i \geq 0 \right\},$$

which together with the set of positive definite matrices $\{P_1, \ldots, P_N\}$ enables us to introduce the following piecewise quadratic Lyapunov function:

$$v(x) := \min_{i=1,\ldots,N} x'P_ix = \min_{\lambda \in \Lambda} \left( \sum_{i=1}^N \lambda_i x'P_ix \right).$$

As it will be clear in what follows, this Lyapunov function is crucial to our purposes; see [1] and the references therein. However, it presents some difficulties to be handled, including the fact that it is not differentiable everywhere. To analyze this aspect, the set $I(x) = \{i : v(x) = x'P_ix\}$ plays a central role since $v(x)$ fails to be differentiable on $x \in \mathbb{R}^n$ such that $I(x)$ is composed of more than one element or, in other words, when the result of the minimization indicated in (26) is not unique, [19].

Before proceeding, let us recall the class of Metzler matrices denoted by $\mathcal{M}$ and consisting of all matrices $\Pi \in \mathbb{R}^{N \times N}$ with elements $\pi_{ij}$, such that

$$\pi_{ij} \geq 0 \forall i \neq j, \sum_{i=1}^N \pi_{ij} = 0 \forall j.$$

It is clear that any $\Pi \in \mathcal{M}$ presents an eigenvalue at the origin of the complex plane since $c'\Pi = 0$, where $c' = [1 \cdots 1]$. In addition, it is well known from the Frobenius–Perron theorem that the eigenvector associated to the null eigenvalue of
Π is nonnegative, yielding the conclusion that there always exists \( \lambda_\infty \in \Lambda \) such that \( \Pi \lambda_\infty = 0 \). The next theorem summarizes the main result of this section.

**Theorem 3.** Assume that there exist a set \( \{P_1, \ldots, P_N\} \) of positive definite matrices and \( \Pi \in \mathcal{M} \) satisfying the Lyapunov–Metzler inequalities

\[
A_i'P_i + P_iA_i + \sum_{j=1}^{N} \pi_{ji}P_j < 0, \quad i = 1, \ldots, N.
\]  

The state-switching control (24) with

\[
u(x(t)) = \arg \min_{i=1, \ldots, N} x(t)'P_i x(t)
\]

makes the equilibrium solution \( x = 0 \) of (1) globally asymptotically stable.

**Proof.** It follows from the Lyapunov function (26), which, as we have said before, is not differentiable for all \( t \geq 0 \). For this reason we need to deal with the Dini derivative (see [7]):

\[
D^+ v(x(t)) = \lim_{h \to 0^+} \sup_{t \in (x(t))} \frac{v(x(t + h)) - v(x(t))}{h}.
\]  

Assume, in accordance to (29), that at an arbitrary \( t \geq 0 \), the state-switching control is given by \( \sigma(t) = u_x(t) = i \) for some \( i \in I(x(t)) \). Hence, from (30) and the system dynamic equation (1), applying the result of Theorem 1 of [15, p. 420] we have

\[
D^+ v(x(t)) = \min_{i \in I(x(t))} x(t)'(A_i'P_i + P_i A_i)x(t)
\]

where the inequality holds from the fact that \( i \in I(x(t)) \). Finally, remembering that (27) is valid for \( \Pi \in \mathcal{M} \) and that \( x(t)'P_j x(t) \geq x(t)'P_i x(t) \) for all \( j \neq i = 1, \ldots, N \) once again due to the fact that \( i \in I(x(t)) \), using the Lyapunov–Metzler inequalities (28) one obtains

\[
D^+ v(x(t)) < -x(t)'\left( \sum_{j=1}^{N} \pi_{ji}P_j \right)x(t)
\]

\[
< - \left( \sum_{j=1}^{N} \pi_{ji} \right)x(t)'P_i x(t) < 0,
\]

which proves the proposed theorem since the Lyapunov function \( v(x(t)) \) defined in (26) is radially unbounded.

It is important to observe that Theorem 3 does not require the set \( \{A_1, \ldots, A_N\} \) be exclusively composed of asymptotically stable matrices. Indeed, with \( \Pi \in \mathcal{M} \), a necessary condition for the Lyapunov–Metzler inequalities to be feasible with respect to \( \{P_1, \ldots, P_N\} \) is matrices \( A_i + (\pi_{ii}/2)I \) for all \( i = 1, \ldots, N \) be asymptotically stable. Since \( \pi_{ii} \leq 0 \) this condition does not imply the asymptotic stability of \( A_i \). However,
an interesting case occurs when all matrices \( \{A_1, \ldots, A_N\} \) are asymptotically stable, for which the choice \( \Pi = 0 \) is possible and the state-switching strategy proposed preserves stability. Furthermore, if the set \( \{A_1, \ldots, A_N\} \) is quadratically stable, then the Lyapunov–Metzler inequalities admit a solution \( P_1 = \cdots = P_N = P \) and \( I(x(t)) = \{1, \ldots, N\} \) for all \( t \geq 0 \). In this classical but particular case, at any \( t \geq 0 \), the control law \( u(x(t)) = i \in \{1, \ldots, N\} \) can be chosen arbitrarily and asymptotic stability is guaranteed. Hence, Theorem 3 contains as a particular case (since the Lyapunov–Metzler inequalities do not depend on \( \Pi \) anymore) the quadratic stability condition.

Remark 2 (chattering). Another important feature of Theorem 3 is that chattering in the switching, when it occurs, is always stable. Indeed, assume that \( x \in \mathbb{R}^n \) belongs to a certain region \( \mathcal{R} \) of the state space where the cardinality of \( I(x) \) is greater than one. From the Lyapunov function (26) and the time derivative (31) a switching from any \( i \in I(x) \) to some \( j \in I(x) \) is possible only if \( x'(A'_i P_j + P_j A_i)x \leq x'(A'_i P_i + P_i A_i)x < 0 \), where the second inequality follows from (28). Hence, with scalars \( \alpha_i \geq 0 \) \( \forall i \in I(x) \) satisfying \( \sum_{i \in I(x)} \alpha_i = 1 \), we conclude that whenever \( x \in \mathcal{R} \) the time derivative of the positive definite function \( \nu(x) = x'P_j x \) is strictly negative along the trajectories of \( \dot{x} = (\sum_{i \in I(x)} \alpha_i A_i)x \). From Filippov’s definition this implies that the switched system is asymptotically stable. In the particular case characterized by \( N = 2 \), this aspect has already been treated in [16, pp. 70]. In [20] it is noted that a Lyapunov function like (26) but with min replaced by max does not exhibit this property, in which instance the chattering must be ruled out. In this sense, the numerical procedure proposed in [13] for the determination of a switching-state dependent control has to be further qualified to prevent chattering since when it occurs instability may be observed.

In the literature, the Lyapunov–Metzler inequalities with \( \Pi \in \mathcal{M} \) fixed have been introduced to study the mean-square (MS) stability of Markov jump linear systems (MJLS). In that context, the Metzler matrix \( \Pi = \Pi_0 \in \mathcal{M} \) is given and \( \Pi_0' \) represents the infinitesimal transition matrix of a Markov chain \( \sigma(t) \) governing the dynamical system (1). In this respect, each component of the vector \( \lambda(t) \in \Lambda \) is the probability of the Markov chain to be in the \( i \)th logical state and obeys the differential equation

\[
\dot{\lambda}(t) = \Pi_0 \lambda(t) , \quad \lambda(0) = \lambda_0 \in \Lambda,
\]

where the eigenvector \( \lambda_\infty \in \Lambda \) associated to the null eigenvalue of \( \Pi_0 \) represents the stationary probability vector. Hence, using the fact that the stochastic system under consideration is said to be MS-stable if

\[
\lim_{t \to +\infty} E(||x(t)||^2) = 0
\]

for any initial state \( x(0) \) and any initial probability pattern \( \lambda_0 \in \Lambda \), it has been shown (see, e.g., [6]) that the system is MS-stable if and only if there exists a set of positive definite matrices \( \{P_1, \ldots, P_N\} \) satisfying the Lyapunov–Metzler inequalities (28) for \( \Pi = \Pi_0 \). Numerically speaking, this is a simple case, since (28) reduces to a set of linear matrix inequalities.

A relevant point to be discussed now concerns the existence of a solution of the Lyapunov–Metzler inequalities (28) with respect to the variables \( \Pi \in \mathcal{M} \) and \( \{P_1, \ldots, P_N\} \). Standard Kronecker calculus shows that for \( \Pi \in \mathcal{M} \) fixed, a solution with respect to the remaining variables exists if and only if the \( Nn^2 \)-dimensional
square matrix $J := A + BC$ is asymptotically stable, where

$$(35) \quad A = \begin{bmatrix}
A_1' \oplus A_1' & 0 & \cdots & 0 \\
0 & A_2' \oplus A_2' & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & A_N' \oplus A_N'
\end{bmatrix}$$

and

$$(36) \quad B = \Pi' \begin{bmatrix}
0_{N-1} \\
I_{N-1}
\end{bmatrix} \otimes I_{n^2}, \quad C = \begin{bmatrix}
-1_{N-1} & I_{N-1}
\end{bmatrix} \otimes I_{n^2},$$

with the symbols $\oplus$ and $\otimes$ indicating the Kronecker sum and Kronecker product respectively, $0_{N-1}$ denoting a row vector of $N-1$ zeros components, and $1_{N-1}$ denoting a column vector of $N-1$ ones components. Hence, the existence of a solution to (28) reduces to the existence of $\Pi \in M$ rendering matrix $J$ asymptotically stable. A possible approach to verify the existence of such a matrix is based on the observation that any $\alpha \geq 0$ and $\Pi \in M$ implies $\alpha \Pi \in M$, which from the introduction of this new degree of freedom makes possible to verify the existence of $\alpha \geq 0$ such that $J(\alpha) := A + \alpha BC$ is asymptotically stable. Putting aside the situation in which all matrices $\{A_1, \ldots, A_N\}$ are asymptotically stable (making it possible to set $\alpha = 0$), let us consider the other extreme situation corresponding to $\alpha \to +\infty$. Simple determinant manipulations show that a certain number of eigenvalues goes to $-\infty$ while the other ones that remain finite, coincide with the invariant zeros of the triple $(A, B, C)$.

Fortunately, these invariant zeros can be determined with no great difficulty from the definition

$$(37) \quad \begin{bmatrix}
\mu I - A & B \\
C & 0
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} = 0$$

with the key observation that matrix $C$, being constant, is independent of $\alpha$ and $\Pi$ which imposes to the solution of $C\xi = 0$ a vector of compatible dimension with the particular structure $\xi' = [x' \cdots x'], x \in \mathbb{R}^{n^2}$. In addition, taking $\lambda_{\infty} \in \Lambda$ such that $\Pi \lambda_{\infty} = 0$, multiplying each subequation above by $\lambda_{\infty}$ and summing up, it follows that

$$(38) \quad \left(\mu I - \sum_{i=1}^{N} \lambda_{\infty} A_i' \oplus A_i'\right) x = 0,$$

which can be rewritten as

$$(39) \quad \left(\mu I - A_{\lambda_{\infty}}' \oplus A_{\lambda_{\infty}}'\right) x = 0,$$

where $A_{\lambda_{\infty}} = \sum_{i=1}^{N} \lambda_{\infty} A_i$. Therefore, as $\alpha$ goes to infinity, the eigenvalues of $J(\alpha)$ that remain finite tend to the eigenvalues of $A_{\lambda_{\infty}}' \oplus A_{\lambda_{\infty}}'$ which are in the left-hand plane if and only if the eigenvalues of $A_{\lambda_{\infty}}$ are. This means that if there exists $\lambda_{\infty} \in \Lambda$.

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2While the Kronecker product is more or less standard, the sum requires a formal definition. In this respect we define the Kronecker sum of two matrices $D$ and $E$ as $D \oplus E = D \otimes I + I \otimes E$. It is important to recall that the eigenvalues of the Kronecker sum $D \oplus E$ are given by all sums of all eigenvalues of $D$ and $E$.1
such that $A_{\lambda_\infty}$ is asymptotically stable, then any $\Pi_0 \in \mathcal{M}$ satisfying $\Pi_0 \lambda_\infty = 0$ and $\alpha$ a sufficiently large positive number provide $\Pi = \alpha \Pi_0 \in \mathcal{M}$ such that the Lyapunov–Metzler inequalities are feasible with respect to the remaining variables $\{P_1, \ldots, P_N\}$.

**Example 2.** To illustrate the above point, let us consider a simple example with $N = 2$, the pair of matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 8 \end{bmatrix},$$

and

$$\Pi_0 = \begin{bmatrix} -0.51 & 0.49 \\ 0.51 & -0.49 \end{bmatrix} \in \mathcal{M}.$$  

The eigenvector associated to the null eigenvalue of $\Pi_0$ is $\lambda_\infty' = [0.49 \, 0.51]$. We have determined numerically that the Lyapunov–Metzler inequalities (28) have a solution of the form $\Pi = \alpha \Pi_0$ for all $\alpha \geq 615.7374$, in accordance to the fact that the invariant zeros of the triple $(A, B, C)$ are $-0.33, -0.33, -0.33 \pm j0.226$, which, as discussed before, can alternatively be obtained from the eigenvalues of the asymptotically stable matrix $A_{\lambda_\infty} = 0.49 A_1 + 0.51 A_2$, taking all sums.

The existence of this particular solution to the inequalities (28) meets exactly the already classical stability condition provided in [16] and more recently in [24] as a particular case of switched nonlinear systems. In our present context, let us assume that there exits $\lambda_\infty \in \Lambda$ such that $A_{\lambda_\infty}$ is asymptotically stable, making possible the determination of $P > 0$ satisfying the Lyapunov inequality $A_{\lambda_\infty}' P + P A_{\lambda_\infty} < 0$. Hence, the switching rule (24) with

$$u(x(t)) = \arg \min_{i=1, \ldots, N} x(t)' (A_i' P + P A_i) x(t)$$

makes the equilibrium point $x = 0$ of the switched system (1) globally asymptotically stable. Indeed, considering the Lyapunov function $v(x(t)) = x(t)' P x(t)$ we have

$$\dot{v}(x(t)) = x(t)' \left( A_{\sigma(t)}' P + P A_{\sigma(t)} \right) x(t)$$

$$= \min_{i=1, \ldots, N} x(t)' (A_i' P + P A_i) x(t)$$

$$= \min_{\lambda \in \Lambda} x(t)' (A_{\lambda}' P + P A_{\lambda}) x(t)$$

$$\leq x(t)' (A_{\lambda_\infty}' P + P A_{\lambda_\infty}) x(t)$$

$$< 0.$$  

(43)

It is important to keep in mind that if the set of matrices $\{A_1, \ldots, A_N\}$ does not admit $A_\lambda$ asymptotically stable for some $\lambda \in \Lambda$, then the above stabilizing switching rule cannot be determined. In addition, even if it is known that there exists $\lambda \in \Lambda$ such that $A_\lambda$ is asymptotically stable, the numerical determination of $\lambda \in \Lambda$ and $P > 0$ such that $A_\lambda' P + P A_{\lambda} < 0$ is not a simple task due to the nonlinear nature of this inequality. The Lyapunov–Metzler inequalities introduced in Theorem 3 suffer the same difficulty, but fortunately a simple numerical procedure based on line search can be settled to determine its solution. This aspect will be considered next. First, let us introduce a guaranteed quadratic cost associated to the proposed state switching control law (29).
Lemma 1. Let \( Q \geq 0 \) be given. Assume that there exist a set of positive definite matrices \( \{P_1, \ldots, P_N\} \) and \( \Pi \in \mathcal{M} \) satisfying the Lyapunov–Metzler inequalities

\[
A_i'P_i + P_iA_i + \sum_{j=1}^{N} \pi_{ji}P_j + Q < 0, \quad i = 1, \ldots, N.
\]

The state-switching control (24) with \( u(x(t)) \) given by (29) makes the equilibrium solution \( x = 0 \) of (1) globally asymptotically stable and

\[
\int_0^\infty x(t)'Qx(t)dt < \min_{i=1,\ldots,N} x_0'P_ix_0.
\]

Proof. The proof has the same pattern of the proof of Theorem 3. The Lyapunov function (26) and the Lyapunov–Metzler inequalities (44) yield

\[
D^+v(x(t)) < -x(t)'Qx(t),
\]

which after integration gives

\[
v(x(t)) - v(x(0)) = \int_0^t D^+v(x(\tau))d\tau < -\int_0^t x(\tau)'Qx(\tau)d\tau \quad \forall t \geq 0,
\]

thus proving the proposed lemma since, due to the asymptotic stability, \( v(x(t)) \) goes to zero as \( t \) goes to infinity.

The numerical determination, if any, of a solution of the Lyapunov–Metzler inequalities with respect to the variables \( (\Pi, \{P_1, \ldots, P_N\}) \) is not a simple task and certainly deserves additional attention. The main source of difficulty stems from its nonconvex nature due to the products of variables and so LMI solvers do not apply. Perhaps a point to be further investigated is that its particular structure with \( \pi_{ji} \) being scalars may help with the design of an interactive method based on relaxation.

In this paper we pursue an alternative route. The main idea is to obtain a simpler, although certainly more conservative, stability condition that can be expressed by means of LMIs and thus solvable by the machinery available in the literature to date. The next theorem shows that working with a subclass of Metzler matrices, characterized by having the same diagonal elements, this goal is accomplished.

Theorem 4. Let \( Q \geq 0 \) be given. Assume that there exist a set of positive definite matrices \( \{P_1, \ldots, P_N\} \) and a scalar \( \gamma > 0 \) satisfying the modified Lyapunov–Metzler inequalities

\[
A_i'P_i + P_iA_i + \gamma(P_j - P_i) + Q < 0, \quad j \neq i = 1, \ldots, N.
\]

The state-switching control (24) with \( u(x(t)) \) given by (29) makes the equilibrium solution \( x = 0 \) of (1) globally asymptotically stable and

\[
\int_0^\infty x(t)'Qx(t)dt < \sum_{i=1}^{N} x_0'P_ix_0.
\]

Proof. The proof follows from the choice of \( \Pi \in \mathcal{M} \) such that \( \pi_{ii} = -\gamma \) and the remaining elements satisfying

\[
\gamma^{-1} \sum_{j \neq i=1}^{N} \pi_{ji} = 1
\]
for all $i = 1, \ldots, N$. Taking into account that $\pi_{ji} \geq 0$ for all $j \neq i = 1, \ldots, N$, multiplying (48) by $\pi_{ji}$, summing up for all $j \neq i = 1, \ldots, N$, and finally multiplying the result by $\gamma^{-1} > 0$ we obtain

$$A_i'P_i + P_iA_i + Q < -\sum_{j \neq i = 1}^{N} \pi_{ji}(P_j - P_i)$$

$$< -\sum_{j = 1}^{N} \pi_{ji}P_j$$

(51)

for all $i = 1, \ldots, N$, which are the Lyapunov–Metzler inequalities (44). From Lemma 1, the upper bound (45) holds, which trivially implies that (49) is verified. The proposed theorem is thus proved.

The basic theoretical features of Theorem 3 and Lemma 1 are still present in Theorem 4. The most important is that the asymptotic stability of the set of matrices $\{A_1, \ldots, A_N\}$ is still not required. In addition, notice that the guaranteed cost (49) is clearly worse than the one provided by Lemma 1, but the former being convex makes it possible to solve the problem

$$\min_{\gamma > 0, P_1 \succ 0, \ldots, P_N \succ 0} \left\{ \sum_{i = 1}^{N} x_0'P_ix_0 : (48) \right\}$$

(52)

by using LMI solvers and line search. The next example illustrates some aspects of the theoretical results obtained so far.

Example 3. Consider the system (1) with $N = 2$ and matrices $\{A_1, A_2\}$ given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix},$$

(53)

which, as it can be easily verified by inspection, are both unstable. Considering $Q = I$ and the initial condition $x_0 = [1 \ 1]'$, problem (52) has been solved by line search fixing $\gamma$ and minimizing its objective function, denoted by $\delta(\gamma)$, with respect to the remaining variables. Figure 3 shows the behavior of the function $\delta(\gamma)$ which enables us to determine its minimum value $\delta^* = 23.56$, corresponding to $\gamma^* = 11.80$. It is important to stress that, in this particular example, the function $\delta(\gamma)$ has a unique minimum. However, we do not have any evidence that this is a generic property valid.
in all cases. Figure 4 shows the trajectories of the state variable $x(t) \in \mathbb{R}^2$ versus time for the system controlled by the state switching rule $\sigma(t) = u(x(t))$ given by (29) with the positive definite matrices

$$P_1 = \begin{bmatrix} 6.7196 & 1.6293 \\ 1.6293 & 1.0222 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6.0825 & 2.1293 \\ 2.1293 & 2.2206 \end{bmatrix}$$

obtained from the optimal solution of problem (52). As it can be seen, the proposed control strategy is very effective in stabilizing the system under consideration.

4. Conclusion. In this paper we have introduced stability conditions for switched linear systems. They have been used for control synthesis of state-independent (open loop) and state-dependent (closed loop) switching rules. In both cases, the determination of a guaranteed cost associated to each control strategy has been addressed. Special attention has been devoted to the numerical solvability of the design problems by means of methods based on linear matrix inequalities.

Two issues deserve more attention. The first is related to the development of numerical algorithms for the solution of the Lyapunov–Metzler inequalities introduced in section 3. The second is the possible generalization of the stability conditions to cope with linear control design and guaranteed quadratic cost. Taking into account the nonlinear nature of the involved stability conditions, this point constitutes a real theoretical challenge.

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