Solution of the fractional Langevin equation and the Mittag–Leffler functions

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We introduce the fractional generalized Langevin equation in the absence of a deterministic field, with two deterministic conditions for a particle with unitary mass, i.e., an initial condition and an initial velocity are considered. For a particular correlation function, that characterizes the physical process, and using the methodology of the Laplace transform, we obtain the solution in terms of the three-parameter Mittag–Leffler function. As particular cases, some recent results are also presented. © 2009 American Institute of Physics. [DOI: 10.1063/1.3152608]

I. INTRODUCTION

In the last ten years processes involving anomalous diffusion phenomena are growing up. Those processes appear in several areas of the knowledge, particularly, in biology7,14,20 and physical sciences.1,17,18

In a recent paper30 a type of Mittag–Leffler function was introduced in the study of kinetic equation, random walks, and anomalous diffusion.19 Particularly, anomalous diffusion has been the subject of numerous researches, in several areas of knowledge, for example, we mention11 where the authors discuss anomalous diffusion in quantum Brownian motion with colored noise. On the other hand, another recent paper27 shows an exact solution of the generalized Langevin equation (GLE) for harmonically bounded particle. The same authors28 presented and discussed the anomalous diffusion induced by a Mittag–Leffler correlated noise, where the authors present some particular cases that characterizes the noise. Moreover, according to the value of the anomalous diffusion exponent, they distinguish subdiffusion and superdiffusion and mention some papers where the theory was successfully applied. We also mention that the anomalous diffusion plays an important role in the study of the wave reaction-diffusion systems.10,19

A natural way to discuss anomalous diffusion is by means of the GLE.22 In that paper the authors show how the long-time behavior, of the mean square displacement for systems described by GLEs, depends on the properties of the correlation function and of the memory kernel.

This paper deals with the fractional version of the generalized Langevin differintegral equation in the absence of a deterministic field. Our notation follows Ref. 28 and for the corresponding physical dimensions of the terms which appear in the Langevin differintegral equation we refer the reader to the Inizan’s paper.12

The paper is organized as follows: Section II presents some results associated with the classical GLE including also the methodology of Laplace transform which is used to obtain its solution in terms of a general relaxation function. Section III presents a FGLE where the deriva-
tives are taken in the Caputo’s sense, all results of Sec. II are also discussed. Section IV, using a particular correlation function, the relaxation function is obtained. In Sec. V, the main results are presented, where the kernels in terms of the generalized Mittag–Leffler function are explicitly calculated. Finally, the concluding remarks are presented. An Appendix where we present some relations involving the Mittag–Leffler functions, close the paper.

II. THE GLE

This section presents the classical GLE and its solution in terms of a relaxation function that characterizes the physical process obtained by using the Laplace integral transform.

The classical approach known as the Ornstein–Uhlenbeck (Einstein–Ornstein–Uhlenbeck, also) theory of Brownian motion was first presented by Langevin, while the normal diffusion and Brownian motion are associated with Langevin equation. The classical and the GLEs were revis-ited and a fractional treatment was proposed some years ago by Mainardi–Pironi.16 In that paper the authors discussed the Langevin equation associated only with the velocity. To discuss the Langevin equation associated with the displacement, one must calculate an integration with respect to time variable.

The GLE, in the absence of a deterministic field, is given by

\[ D_0^2 x(t) + \int_0^t \mu(t - \xi) D_0 x(\xi) d \xi = F(t), \tag{1} \]

with \( D_0 = d/dy \) where \( y = t, \xi \), and \( \mu(t) \) is the dissipative memory kernel and \( F(t) \) is a random force. We consider Eq. (1) with two deterministic conditions, i.e., \( x(0) = x_0 \), initial condition, and \( \dot{x}(0) = v_0 \), initial velocity, of a particle with unitary mass.

To solve Eq. (1) we introduce the Laplace transform. From the convolution theorem we can write the solution in the form

\[ [s^2 + s\hat{\mu}(s)]\hat{x}(s) = x_0[s + \hat{\mu}(s)] + v_0 + \hat{F}(s), \tag{2} \]

where \( \hat{\mu}(s), \dot{x}(s), \) and \( \hat{F}(s) \) are the Laplace transforms of \( \mu(t), x(t), \) and \( F(t), \) respectively, and \( s \) is the parameter of the Laplace transform.

Introducing the relaxation function \( H(t) \) as the inverse Laplace transform of the function, i.e.,

\[ \hat{H}(s) = \frac{1}{s^2 + s\hat{\mu}(s)}, \tag{3} \]

we can write

\[ \hat{x}(s) = \hat{H}(s)[x_0[s + \hat{\mu}(s)] + v_0 + \hat{F}(s)], \]

which the respective inverse transform gives

\[ x(t) = x_0 + v_0H(t) + \int_0^t H(t - \xi)F(\xi)d\xi \]

and can be written in the final form

\[ x(t) = \langle x(t) \rangle + \int_0^t H(t - \xi)F(\xi)d\xi, \tag{4} \]

where we have considered \( \langle x(t) \rangle = x_0 + v_0H(t). \)

The first derivative of Eq. (4) is
\[ \dot{x}(t) = \langle \dot{x}(t) \rangle + \int_0^t h(t - \xi) F(\xi) d\xi, \]  

(5)

where \( \langle \dot{x}(t) \rangle = v_0 h(t) \) and the relaxation function \( h(t) \) is the derivative of the relaxation function \( H(t) \), i.e., \( h(t) = \dot{H}(t) \). Thus, we have

\[ \dot{h}(s) = \frac{1}{s + \hat{\mu}(s)} = s \hat{H}(s), \]

with the conditions \( H(0) = 0 \) and \( h(0) = 1 \), that were obtained by Eqs. (4) and (5).

With the help of Eqs. (4) and (5), we can discuss the explicit equations for the variances of process associated with Eq. (1), particularly anomalous diffusion and probability distribution.\textsuperscript{22}

III. THE FRACTIONAL GLE

This section presents a discussion of the so-called fractional GLE (FGLE) considering the Laplace transform methodology in a way similar to the one presented for the GLE case in Sec. II. It is interesting to note that, in a recent paper\textsuperscript{15} was proposed a Langevin equation with two fractional orders based on Weyl and Riemann–Liouville fractional derivatives.

The FGLE, a fractional version of the GLE, is defined as follows:

\[ D_\alpha^n x(t) + \int_0^t \mu(t - \xi) D_\xi^\beta x(\xi) d\xi = F(t), \]

(6)

with \( 1 < \alpha \leq 2 \) and \( 0 < \beta \leq 1 \). The operator \( D \) denotes the fractional derivative in the Caputo’s sense, given by

\[ D_\alpha^n x(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t d\xi \frac{x(\xi)}{(t - \xi)^{\alpha-n+1}}, \]

with \( n-1 < \alpha \leq n \) and \( D_\alpha^n = d^n / d^n t \) being the ordinary derivative.\textsuperscript{6}

Here \( x(t) \) represents the position of a particle, the function \( \mu(t-t') \) is the dissipative memory kernel which is taken as the same used in Ref. \textsuperscript{28}, i.e., in terms of a Mittag–Leffler function (The Mittag–Leffler function is a generalization of the exponential function. Since the stationary driving noise is exponentially correlated, it seems to be natural to propose a correlation function modeled by a Mittag–Leffler function,) and \( F(t) \) is a zero-centered Gaussian and stationary random force. Taking the limits \( \alpha \rightarrow 2 \) and \( \beta \rightarrow 1 \), the results associated with the classical GLE are obtained.

On the other hand, in a recent paper\textsuperscript{24} was presented a type of GLE with a fractional derivative, also, in the Caputo’s sense. The author presents a discussion of a system involving diffusion processes with a correlation function of two types: the exponential and power law and also with terms associated with the deterministic field equal to zero, i.e., Eq. (6) for the case, where \( 0 < \alpha < 1 \) and \( \beta = 1 \). The same author\textsuperscript{25} presents a discussion involving a fractional Langevin equation with the fractional derivative being considered in the Riemann–Liouville sense. He compares the results with those results obtained in Ref. \textsuperscript{24}, considering the deterministic term is equal to zero. Recently, the anomalous diffusion induced by a Mittag–Leffler correlation noise was discussed.\textsuperscript{28} In that paper, the authors obtain the exact expressions for the mean values, variances, and diffusion coefficients for a free particle in terms of two-parameter Mittag–Leffler functions and its derivatives. We mention that the equation discussed in that paper is a GLE conversely the equation presented in Ref. \textsuperscript{25} which is a FGLE.

The analytical solutions of Eq. (6) by means of the Laplace integral transform are considered. Thus, taking the Laplace transform of Eq. (6) and using the relation involving the Laplace transform of the derivatives,\textsuperscript{8} we have
\[ [s^\alpha + s^\beta \dot{\mu}(s)] \dot{x}(s) = s^{\alpha-1} x_0 + s^{\alpha-2} v_0 + x_0 s^{\beta-1} \dot{\mu}(s) + \hat{F}(s), \]

which for \( \alpha=2 \) and \( \beta=1 \) gives Eq. (2).

Using the inverse Laplace transform, denoted by \( \mathcal{L}^{-1}[\cdot] \), we introduce the relaxation function \( H(t) \) defined by

\[ H(t) = \mathcal{L}^{-1}[\hat{H}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha + s^\beta \dot{\mu}(s)}\right], \]

with \( \dot{\mu}(s) = \mathcal{L}[\mu(t)] \); the above equation can be written in the following form:

\[ \dot{x}(s) = \hat{H}(s) \left\{ \frac{x_0}{s} [s^\alpha + s^\beta \dot{\mu}(s)] + v_0 s^{\alpha-2} + \hat{F}(s) \right\}. \]

Thus, applying the corresponding inverse Laplace transform we get

\[ x(t) = x_0 + v_0 \mathcal{L}^{-1}[\hat{H}(s)s^{\alpha-2}] + \mathcal{L}^{-1}[\hat{H}(s)\hat{F}(s)]. \]

Then, by means of the convolution theorem associated with the Laplace transform, we get for solution

\[ x(t) = x_0 + v_0 \mathcal{L}^{-1}[\hat{H}(s)s^{\alpha-2}] + \int_0^t H(t-\xi)F(\xi)d\xi, \]

where \( \hat{H}(s) \) is the Laplace transform of the function \( H(t) \). On the other hand, to calculate the variances associated with a physical process, for example, one must introduce the derivative of the relaxation function, \( h(t) = \dot{H}(t) \), and the respective integration, \( \dot{h}(t) = \dot{H}(t) \), as we will see in Sec. IV. From here one must distinguish a particular relaxation function.

As already said, for \( \alpha=2 \) and \( \beta=1 \) the results associated with Eq. (1) as well as its consequences showed in Ref. 24 are presented. These results are generalized in the sense that we consider the fractional version of GLE and taking the correlation function modeled with a convenient classical Mittag–Leffler function, i.e., the Mittag–Leffler function with only one parameter, as considered in Ref. 28.

IV. A PARTICULAR CORRELATION FUNCTION

In this section we explicit the calculation of the kernels \( H(t) \) and \( h(t) \), that appear in the expressions involving variances,

\[ \sigma_{xx}(t) = k_B T [2I(t) - H^2(t)], \]

\[ \sigma_{vv}(t) = k_B T [1 - \dot{h}^2(t)], \]

\[ \sigma_{vw}(t) = k_B T H(t)[1 - h(t)], \]

(7)

where

\[ I(t) = \int_0^t d\xi H(\xi) \]

and \( k_B \) is the Boltzmann constant, and \( T \) is the absolute temperature of the environment.

Equation (1) represents a system in the equilibrium state, the functions \( \mu(t) \), the dissipative memory kernel, and \( C(t) \), the correlation function, are related to each other by means of the so-called fluctuation-dissipation theorem.\(^{22}\)
\( C(t) = k_B T \mu(t). \)

Hereafter, we consider the case where the dissipative memory kernel can be given by \(^2\)

\[ \mu(t) = \gamma \lambda \frac{s^{\lambda-1}}{1+s^\lambda \tau^\lambda}, \quad (8) \]

with \(0 < \lambda < 2\). The \( \mu(t) \) must be determined by the dynamical mechanism of the physical process. \( E_\lambda(\cdot) \) is the one-parameter Mittag–Leffler function (The one-parameter Mittag–Leffler function for \(0 < \lambda < 1\) is a completely monotone function and for \(1 < \lambda < 2\) can be decomposed in a completely monotone function plus an oscillatory contribution.\(^26\), \( \gamma \lambda \) is a constant that depends on \( \lambda \) but it is time independent and \( \tau \) is a characteristic memory time. The particular case for \( \lambda = 1 \) is associated with the standard Ornstein–Uhlenbeck process.\(^2\) Using Eq. (8) the correlation function can be written as

\[ C(t) = C_0(\lambda) \tau^\lambda E_\lambda[-(t/\tau)^\lambda]. \]

where \( \tau \) acts as a characteristic memory time and \( C_0(\lambda) = \gamma \lambda k_B T \) is a proportionality coefficient depending on the parameter \( \lambda \) but does not depend on the time.

Proceeding as in Sec. II, using the dissipative memory kernel given by Eq. (8), the respective Laplace transform can be written as

\[ \hat{\mu}(s) = \gamma \lambda \frac{s^{\lambda-1}}{1+s^\lambda \tau^\lambda}, \quad (9) \]

which gives the corresponding Laplace transform of the relaxation function,

\[ \hat{H}(s) = \hat{H}_0(s) + \hat{H}_1(s), \quad (10) \]

with \( \hat{H}_1(s) = \tau^\lambda s^{-\lambda} \hat{H}_0(s) \), where \( \hat{H}_0(s) \) is given by the following expression:

\[ \hat{H}_0(s) = \frac{s^{\lambda-\alpha}}{s^\lambda + (\gamma \lambda \tau^\lambda)^{\lambda+\beta-\alpha-1} + (1/\lambda)} \quad (11) \]

which for \( \alpha = 2 = 2 \beta \) gives the results obtained in Ref. 28.

V. THE KERNELS AND THE GENERALIZED MITTAG–LEFFLER FUNCTIONS

In this section, using Eq. (11) we obtain the kernels needed to calculate the variances. All the results are presented in terms of the generalized Mittag–Leffler functions as introduced by Prabhakar,\(^23\) conversely the notation used in Ref. 28 where the authors present the results in terms of the derivatives of the two-parameter Mittag–Leffler function (see Appendix).

Thus, taking the inverse Laplace transform of Eq. (11), we obtain (See Appendix.)

\[ H_0(t) = \mathcal{L}^{-1}[\hat{H}_0(s)] = t^{\alpha-1} \sum_{r=0}^{\infty} \left( -\frac{\gamma \lambda}{\tau^\lambda} \right)^r t^{\rho r} E^\rho_{\lambda,\alpha+\rho}[-(t/\tau)^\lambda], \quad (12) \]

where the parameter \( \nu = \alpha - \beta + 1 \) and \( E^\rho_{\alpha,\beta}(\cdot) \) is the generalized Mittag–Leffler function.\(^13\) This function is a generalization of the two-parameter Mittag–Leffler function since for \( \rho = 1 \) we have \( E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) \) which, as we already known, is the exponential function, with \( \alpha, \beta, \rho = 1 \), i.e., \( E^1_{1,1}(x) = e^x \).

Following the same procedure as above we have

\[ H_1(t) = \mathcal{L}^{-1}[\hat{H}_1(s)] = \frac{t^{\lambda} \gamma \lambda}{(\tau^\lambda)^{\lambda+\beta-\alpha-1} + (1/\lambda)} \sum_{r=0}^{\infty} \left( -\frac{\gamma \lambda}{\tau^\lambda} \right)^r t^{\rho r} E^\rho_{\lambda,\alpha+\rho}[-(t/\tau)^\lambda]. \quad (13) \]

On the other hand, to obtain the kernel \( h(t) \) the inverse Laplace transforms are calculated as
\[ h_0(t) = \mathcal{L}^{-1}[s\hat{H}_0(s)] \quad \text{and} \quad h_1(t) = \mathcal{L}^{-1}[s\hat{H}_1(s)], \]

and then we get \( h(t) = h_0(t) + h_1(t) \). Similarly we can write

\[ h_0(t) = t^{\alpha-2} \sum_{r=0}^{\infty} \left( -\frac{r}{\tau^r} \right)^r t^{\nu} E_{\alpha,\nu+1+r}[-(t/\tau)^\lambda] \quad (14) \]

and

\[ h_1(t) = (t/\tau)^\lambda t^\nu \sum_{r=0}^{\infty} \left( -\frac{r}{\tau^r} \right)^r t^{\nu} E_{\lambda,\nu+1+r}[-(t/\tau)^\lambda], \quad (15) \]

where \( \nu = \alpha - \beta + 1 \).

Finally, by integration, we obtain \( I(t) = I_0(t) + I_1(t) \), where

\[ I_0(t) = \int_0^t d\xi H_0(\xi) = t^\nu \sum_{r=0}^{\infty} \left( -\frac{r}{\tau^r} \right)^r t^{\nu} E_{\alpha,\nu+1+r}[-(t/\tau)^\lambda] \quad (16) \]

and

\[ I_1(t) = (t/\tau)^\lambda t^\nu \sum_{r=0}^{\infty} \left( -\frac{r}{\tau^r} \right)^r t^{\nu} E_{\lambda,\nu+1+r}[-(t/\tau)^\lambda], \quad (17) \]

with \( \nu = \alpha - \beta + 1 \).

Equations (12)–(17), which generalize the results obtained in Ref. 28, are our main results. With those kernels we can obtain the temporal evolution of the mean values of the position, \( \langle x(t) \rangle = x_0 + v_0 t H(t) \), and the velocity, \( \langle \dot{x}(t) \rangle = v_0 H(t) \), as well as the variances of the process, as in Eqs. (7).

VI. CONCLUDING REMARKS

This paper presented an extension to the classical GLE using the fractional derivatives in the Caputo’s sense instead of the integer order, the FGLE. For a particular correlation function we obtained the solution of the FGLE in terms of the three-parameter Mittag–Leffler function.

In the study of a particular case of relaxation functions, interesting calculations can be discussed, for example, the asymptotic behavior related with a particular anomalous diffusive process since we have two fractional parameters, \( \alpha \) and \( \beta \) as in Eq. (6), and then one can make a comparison between the results presented in Ref. 28 which can be seen as a particular case of our main result.

We mentioned that Ref. 21 presented a fractional Langevin equation as a dynamical model of the observed memory in financial time series. In this approach the authors consider the fractional derivative in the Riemann–Liouville’s sense. A similar treatment involving the fractional derivative in the Caputo’s sense will be presented in a forthcoming paper.4

A natural continuation of this paper is to discuss the same FGLE associated with a particle under the influence of a random force modeled as Gaussian colored noise and an external field proportional to the displacement,5 similar to the paper presented by Viñales et al.20 where an interesting study on the Mittag–Leffler noise is discussed.

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APPENDIX: MITTAG–LEFFLER FUNCTIONS AND THE LAPLACE TRANSFORM

In this appendix the definition and the corresponding Laplace transform of the generalized Mittag–Leffler function are presented. We also present the relation between this generalized Mittag–Leffler function with the integer derivative of two-parameter Mittag–Leffler function.

1. The generalized Mittag–Leffler function

The generalized Mittag–Leffler function, denoted by $E_{\alpha,\beta}(z)$, was introduced by means of the following convergent series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta) k!} z^k,$$

where $z \in \mathbb{C}$, Re$(\alpha) > 0$, Re$(\beta) > 0$, and Re$(\rho) > 0$ and $(\rho)_k$ is the Pochhammer symbol defined as

$$(\rho)_k = \frac{\Gamma(\rho + k)}{\Gamma(\rho)}$$

and $\Gamma(\rho)$ is the Euler gamma function.

This Mittag–Leffler function is a generalization of the two-parameter Mittag–Leffler function $E_{\alpha,\beta}(x)$, which is recovered for $\rho = 1$, i.e.,

$$E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x).$$

2. The Laplace transform

Using the definition of the Laplace transform we can write

$$\mathcal{L}[t^{\alpha-1}E_{\alpha,\beta}(at^\alpha)] = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha + \beta) k!} \int_0^{\infty} e^{-st} t^{\alpha k + \beta - 1} dt,$$

with the last step, the change of the sum with the integral, being possible because the series and the integral are uniformly convergent.

From the resulting integral and using the definition of the geometric series we obtain

$$\mathcal{L}[t^{\alpha-1}E_{\alpha,\beta}(at^\alpha)] = \frac{s^\alpha - \beta}{(s^\alpha - a)^\beta},$$

where Re$(s) > 0$, Re$(\beta) > 0$, $a \in \mathbb{C}$, and $|as^{-\alpha}| < 1$, whose corresponding inverse Laplace transform is given by

$$\mathcal{L}^{-1} \left[ \frac{s^{\alpha-\beta}}{(s^\alpha - a)^\beta} \right] = t^{\alpha-1}E_{\alpha,\beta}(at^\alpha).$$

We mention also that the relation

$$\mathcal{L}^{-1} \left[ \frac{s^{\alpha-1}}{s^\alpha + As^\beta + B} \right] = t^{\alpha-1} \sum_{r=0}^{\infty} (-A)^r (\alpha - \beta)_r E_{\alpha,\alpha+1-\rho(\alpha-\beta)}(-Bt^\rho),$$

with Re$(\alpha) >$ Re$(\beta) > 0$, which can be obtained also by means of the geometric series.


We mentioned the relation between a generalized Mittag–Leffler function and an integer derivative of two-parameter Mittag–Leffler function, denoted by $E_{\alpha,\beta}^{(k)}(x)$, with $k \in \mathbb{N}$.
Using the definition of the $k$-derivative of two-parameter Mittag–Leffler function,
\[ E^{(k)}_{\alpha,\beta}(x) = \frac{d^k}{dx^k} E_{\alpha,\beta}(x) = \sum_{r=0}^{\infty} \frac{(k+r)!}{\Gamma(\alpha r+\alpha k+\nu) k!} x^r, \]
and Eq. (A1), we can write
\[ E^{(k)}_{\alpha,\beta}(x) = k! E^{k+1}_{\alpha,\beta}(x), \]
with $k \in \mathbb{N}$.