Lucas Henrique Calixto

Representations of map superalgebras

Representações de superalgebras de funções

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LUCAS HENRIQUE CALIXTO

REPRESENTATIONS OF MAP SUPERALGEBRAS

Representações de superalgebras de funções

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Orientador: Adriano Adrega de Moura
Coorientador: Alistair Rowland John Savage

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Resumo

Os problemas discutidos nesta tese estão no âmbito da teoria de representações de superalgebras de Lie de funções. Considere uma superalgebra de Lie da forma $\mathfrak{g} \otimes A$, onde A é uma $\mathbb{C}$-álgebra associativa, comutativa e com unidade, e $\mathfrak{g}$ é uma superalgebra de Lie. Dada uma ação de um grupo finito Γ em A e $\mathfrak{g}$, por automofismos, nós consideramos agora a subalgebra de $\mathfrak{g} \otimes A$ formada por todos os elementos que são invariantes com respeito a ação associada de Γ. Tal álgebra é chamada de uma superalgebra de funções equivariantes. Na primeira parte desta tese, classificaremos todas as representações irredutíveis e de dimensão finita de uma superalgebra de funções equivariantes de tipo queer (i.e. quando a superalgebra de Lie $\mathfrak{g}$ é $\mathfrak{q}(n)$, $n \geq 2$) para o caso em que Γ é abeliano e age livremente em MaxSpec(A). Mostraremos que as classes de isomorfismo de tais representações são parametrizadas por um certo conjunto de funções Γ-equivariantes de suporte finito de MaxSpec(A) no conjunto das classes de isomorfismo das representações irredutíveis de dimensão finita de $\mathfrak{q}(n)$. No caso particular em que A é o anel de coordenadas do toro, obteremos a classificação das representações irredutíveis de dimensão finita das superálgebras de laços de tipo queer torcidas. Na segunda parte da tese, introduzimos os módulos de Weyl globais e locais para $\mathfrak{g} \otimes A$, onde $\mathfrak{g}$ é uma superalgebra de Lie básica ou $\mathfrak{sl}(n,n)$, $n \geq 2$. Sob certas condições, provaremos que tais módulos satisfazem certas propriedades universais, os módulos locais tem dimensão finita e que podem ser isomorfos a produtos tensoriais de módulos de Weyl com pesos máximos menores. Também definimos os super-funtores de Weyl e provamos varias propriedades que são semelhantes àquelas satisteitas pelos funtores de Weyl no contexto de álgebras de Lie. Além disso, apontaremos alguns fatos que são novos no super contexto.

Palavras-chave: superalgebras de Lie, queer superalgebra de Lie, superalgebras de Lie básicas, superalgebras de funções equivariantes, representação de dimensão finita, módulos de Weyl.

Abstract

This thesis is concerned with the representation theory of map Lie superalgebras. We consider a Lie superalgebra of the form $\mathfrak{g} \otimes A$, where A is an associative commutative unital $\mathbb{C}$-algebra and $\mathfrak{g}$ is Lie superalgebra. Given actions of a finite group Γ on both A and $\mathfrak{g}$, by automorphisms, we also consider the subalgebra of $\mathfrak{g} \otimes A$ of points fixed by the associated action of Γ, which will be called an equivariant map superalgebra. In the first part of the thesis we classify all irreducible finite-dimensional representations of the equivariant map queer Lie superalgebras (i.e. when the Lie superalgebra $\mathfrak{g}$ is $\mathfrak{q}(n)$, $n \geq 2$) under the assumption that Γ is abelian and acts freely on $\mathrm{MaxSpec}(A)$. We show that the isomorphism classes of such representations are parametrized by a set of Γ-equivariant finitely supported maps from $\mathrm{MaxSpec}(A)$ to the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{g}$. In the special case that A is the coordinate ring of the torus, we obtain a classification of all irreducible finite-dimensional representations of the twisted loop queer superalgebra. In the second part of the thesis, we define global and local Weyl modules for $\mathfrak{g} \otimes A$ with $\mathfrak{g}$ a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. Under some mild assumptions, we prove universality, finite-dimensionality, and tensor product decomposition properties for these modules. We define super-Weyl functors for these Lie superalgebras and we prove several properties that are analogues of those of Weyl functors in the non-super setting. We also point out some features that are new in the super case.

Keywords: Lie superalgebra, queer Lie superalgebra, basic Lie superalgebra, equivariant map superalgebra, finite-dimensional representation, Weyl module.

Contents

Introduction

Equivariant map algebras can be viewed as a generalization of (twisted) current algebras and loop algebras. Namely, let X be an algebraic variety (or, more generally, a scheme) and let $\mathfrak{g}$ be a finite-dimensional Lie algebra, both defined over the field of complex numbers. Furthermore, suppose that a finite group Γ acts on both X and $\mathfrak{g}$ by automorphisms. The equivariant map algebra $M(X, \mathfrak{g})^\Gamma$ is defined to be the Lie algebra of Γ-equivariant regular maps from X to $\mathfrak{g}$. Equivalently, consider the induced action of Γ on the coordinate ring A of X. Then $M(X, \mathfrak{g})^\Gamma$ is isomorphic to $(\mathfrak{g} \otimes A)^\Gamma$, the Lie algebra of fixed points of the diagonal action of Γ on $\mathfrak{g} \otimes A$. In the case that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and A is the algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ (or equivalently, when X is the torus), the representations of $\mathfrak{g} \otimes A$ were investigated by V. Chary and A. Pressley in [Cha86, CP86]. Recently, the representation theory of equivariant map algebras, either in full generality or in special cases, has been the subject of much research. We refer the reader to the survey [NS13] for an overview.

Lie superalgebras are generalizations of Lie algebras and are an important tool for physicists in the study of supersymmetries. The finite-dimensional simple complex Lie superalgebras were classified by Victor Kac in [Kac77], and the irreducible finite-dimensional representations of the so-called basic classical Lie superalgebras were classified in [Kac77] and [Kac78]. It is thus natural to consider equivariant map superalgebras, where the target Lie algebra $\mathfrak{g}$ mentioned above is replaced by a finite-dimensional Lie superalgebra. In [Sav14], the irreducible finite-dimensional representations of $M(X, \mathfrak{g})^\Gamma$ when $\mathfrak{g}$ is a basic classical Lie superalgebra, X has a finitely-generated coordinate ring, and Γ is an abelian group acting freely on the set of rational points of X, were classified. These assumptions make much of the theory parallel to the non-super setting. The first part of this thesis (Chapter 2) was submitted and accepted for publication in the Canadian Journal of Mathematics (see [CMS15]). There we move beyond the setting of basic classical Lie superalgebras. In particular, we address the case where $\mathfrak{g}$ is the so-called *queer Lie superalgebra*. In this case, almost nothing is known about the representation theory of the equivariant map Lie superalgebra, even when Γ is trivial or X is the affine plane or torus (the current and loop cases, respectively), although the representations of the corresponding affine Lie superalgebra have been studied in [GS08].

The queer Lie superalgebra $\mathfrak{q}(n)$ was introduced by Victor Kac in [Kac77]. It is a simple subquotient of the Lie superalegbra of endomorphisms of $\mathbb{C}^{n|n}$ that commute with an odd involution (see Remark 1.4.6). It is closely related to the Lie algebra $\mathfrak{sl}(n+1)$, in the sense that $\mathfrak{q}(n)$ is a direct sum of one even and one odd copy of $\mathfrak{sl}(n+1)$. Although the queer Lie superalgebra is classical, its properties are quite different from those of the other classical Lie superalgebras. In particular, the Cartan subalgebra of $\mathfrak{q}(n)$ is not abelian. (Here, and throughout the paper, we use the term *subalgebra* even in the super setting, and avoid the use of the cumbersome term *subsuperalgebra*.) For this reason, the corresponding theory of weight modules is much more complicated. The theory requires Clifford algebra methods, since the highest weight space of an irreducible highest weight $\mathfrak{q}(n)$-module has a Clifford module structure. Nevertheless, the theory of finite-dimensional $\mathfrak{q}(n)$-modules is well developed (see, for example, [Pen86, PS97, Gor06]).

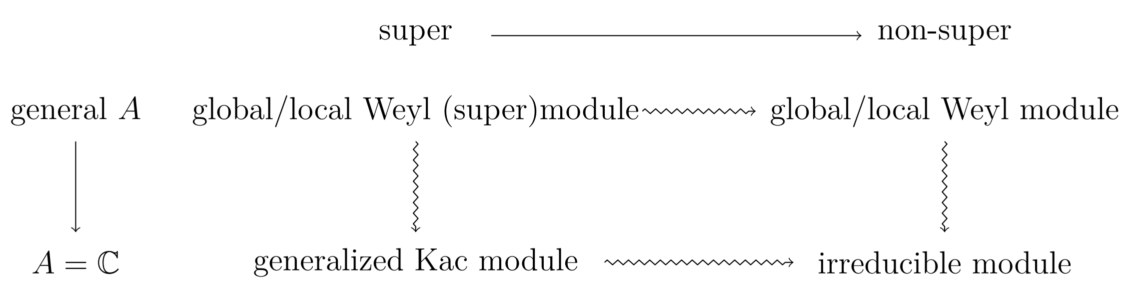
To investigate the representation theory of the Lie superalgebra $\mathfrak{q}(n) \otimes A$, where A is a commutative unital associative algebra, the first step is to understand the irreducible finite-dimensional representations of its Cartan subalgebra $\mathfrak{h} \otimes A$, where $\mathfrak{h}$ is the standard Cartan subalgebra of $\mathfrak{q}(n)$. Therefore, we first give a characterization of the irreducible finite-dimensional $\mathfrak{h} \otimes A$-modules (Theorem 2.2.3). Next, we give a characterization of *quasifinite* irreducible highest weight $\mathfrak{q}(n) \otimes A$-modules in Theorem 2.3.6. Using these results, we are able to give a complete classification of the irreducible finite-dimensional representations of the equivariant map queer Lie superalgebra in the case that the algebra A is finitely generated and the group Γ is abelian and acts freely on $\operatorname{MaxSpec}(A)$. Our main result in this direction, Theorem 2.5.4, states that the irreducible finite-dimensional modules are parameterized by a certain set of Γ-equivariant finitely supported maps defined on $\operatorname{MaxSpec}(A)$. In the special cases that X is the torus or affine line, our results yield a classification of the irreducible finite-dimensional representations of the twisted loop queer Lie superalgebra and twisted current queer Lie superalgebra, respectively.

In the second part of the thesis (Chapters 3 and 4), we are interested in the study of Weyl modules and super-Weyl functors. The content of Chapter 3 was already submitted for publication (see [CLS]).

The global and local Weyl modules are universal objects with respect to certain highest weight properties. The local Weyl modules are finite-dimensional but not, in general, irreducible. They were first defined, in the loop case, in [CP01] and extended to the map case in [FL04]. These modules had not been defined in the super setting, except for a quantum analogue in the loop case for $\mathfrak{g} = \mathfrak{sl}(m, n)$ considered in [Zha14]. In [CLS], we initiate the study of Weyl modules for Lie superalgebras. In particular, we define global and local Weyl modules for Lie superalgebras of the form $\mathfrak{g} \otimes A$, where A is an associative commutative unital $\mathbb{C}$-algebra and $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n, n)$, $n \geq 2$. After defining global Weyl modules in the super setting (Definition 3.3.6), we give a presentation in terms of generators and relations (Proposition 3.3.7) and prove that these modules are universal highest weight objects in a certain category (Proposition 3.3.8). We then define local Weyl modules (Definition 3.4.1), prove that they are finite-dimensional (Theorem 3.4.13), and that they also satisfy a certain univeral property with respect to so-called highest *map-weight* modules (Proposition 3.4.14). Finally, we show that the local Weyl modules satisfy a nice tensor product property (Theorem 3.5.1).

Once we have defined global Weyl modules in the super setting, we are able to define the super-Weyl functors (Definition 4.3.3). These are functors from the category of modules for a certain commutative algebra acting naturally on the global Weyl module to the category of modules for the map Lie superalgebra $\mathfrak{g} \otimes A$. The super-Weyl functors are the super version of the functors defined in [CFK10]. The final part of the thesis is dedicated to the study of such functors. We show that they satisfy nice homological properties (Theorems 4.3.7 and 4.3.8). Under some assumptions, we prove that images of finitely-generated (finite-dimensional) modules under super-Weyl functors are finitely-generated (finite-dimensional) modules (Corollary 4.4.4). We also prove that is possible to recover the local Weyl modules via these functors by showing that the images of one-dimensional irreducible modules under the super-Weyl functors are isomorphic to local Weyl modules (Theorem 4.6.1).

The above-mentioned results show that the Weyl modules and the Weyl functors defined in this Thesis satisfy many of the properties that their non-super analogues do. However, there are some important differences. First of all, the Borel subalgebras of basic Lie superalgebras are not all conjugate under the action of the Weyl group, in contrast to the situation for finite-dimensional simple Lie algebras. For this reason, our definitions of Weyl modules depend on a choice of system of simple roots. Second, the category of finite-dimensional modules for a basic Lie superalgebra is not semisimple in general, again in contrast to the non-super setting. For this reason, the so-called *Kac modules* play an important role in the representation theory. These are maximal finite-dimensional modules of a given highest weight. The Weyl modules defined in the current Thesis can be viewed as a unification of several types modules in the following sense. If $\mathfrak{g}$ is a simple Lie algebra, then our definitions reduce to the usual ones. Thus, the Weyl modules defined here are generalizations of the Weyl modules in the non-super case. On the other hand, if $A = \mathbb{C}$, then the global and local Weyl modules are equal and coincide with the (generalized) Kac module, which, if $\mathfrak{g}$ is a simple Lie algebra, is the irreducible module (of a given highest weight). These relationships can be summarized in the following diagram:



The text is organized in five chapters. In the first chapter we briefly review some results on commutative algebras, associative superalgebras (in particular Clifford algebras), Lie superalgebras (especially basic Lie superalgebras and the queer Lie superalgebra), and its representations. We also recall the definition of equivariant map Lie superalgebras and the classification of its irreducible finite-dimensional representations. In the second chapter we state and prove our main results involving irreducible finite-dimensional representations of $\mathfrak{q}(n) \otimes A$ and $(\mathfrak{q}(n) \otimes A)^\Gamma$. We show that the isomorphism classes of such representations are parametrized by a certain set of equivariant maps with finite support. In particular, we obtain the classification of all irreducible finite-dimensional representations of the twisted loop queer Lie superalgebras. In the third chapter we introduce global and local Weyl modules for map Lie superalgebras of basic type. Under some mild assumptions, we prove that they satisfy certain universal properties, that the local Weyl modules are finite dimensional, and that they might be isomorphic to tensor products of Weyl modules with smaller highest weights. These properties are similar to those of Weyl modules for equivariant map Lie algebras. In the fourth chapter we define super-Weyl functors. Via such functors, we recover the local Weyl modules. We also prove several properties that are analogues of those of Weyl functors in the non-super setting. Finally, in Chapter 5 we conclude this work by listing a number of directions of possible further research.

Chapter 1

Background

In this chapter, we collect the basic definitions and some important results that will be used throughout the thesis.

Notation

We let $\mathbb{Z}$ be the ring of integers, $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the quotient ring $\mathbb{Z}/2\mathbb{Z}$. Vector spaces, algebras, tensor products, etc. are defined over the field of complex numbers $\mathbb{C}$ unless otherwise stated. Whenever we refer to the dimension of an algebra or ideal, we refer to its dimension over $\mathbb{C}$.

1.1 Extentions and projectives

Let $\mathcal{M}$ be an abelian category. An object P in this category is projective if and only if $\mathrm{Hom}_{\mathcal{M}}(P, -)$ is an exact functor. We say the category $\mathcal{M}$ has enough projectives if any object in $\mathcal{M}$ is a quotient of a projective object.

Assume that $\mathcal{M}$ has enough projectives and for $M \in \mathcal{M}$ consider a projective resolution

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \to 0 \tag{1.1.1}$$

of M. For $N \in \mathcal{M}$ we apply the functor $\mathrm{Hom}_{\mathcal{M}}(-, N)$ to the complex obtained by deleting M from the resolution (1.1.1) to obtain the new complex

$$\mathrm{Hom}_{\mathcal{M}}(P_0, N) \xrightarrow{\delta_1^*} \mathrm{Hom}_{\mathcal{M}}(P_1, N) \xrightarrow{\delta_2^*} \mathrm{Hom}_{\mathcal{M}}(P_2, N) \xrightarrow{\delta_3^*} \cdots . \tag{1.1.2}$$

For all $n \in \mathbb{N}$, we define $\mathrm{Ext}^n_{\mathcal{M}}(M, N) = \mathrm{H}^n(\mathrm{Hom}_{\mathcal{M}}(P_\bullet, N))$ (the nth cohomology of the complex (1.1.2)). It can be shown that this is independent of the choice of the projective resolution (see [Rot09, Th. 6.9]).

Now we list some properties of $\mathrm{Ext}^\bullet_{\mathcal{M}}$ that will be used in the sequel.

a. The functor $\mathrm{Ext}^n_{\mathcal{M}}(-, N)\colon \mathcal{M} \to \mathbf{Ab}$ is additive and contravariant for all $n \geq 0$.

b. The functors $\mathrm{Ext}^0_{\mathcal{M}}(-,N)$ and $\mathrm{Hom}_{\mathcal{M}}(-,N)$ are naturally equivalent for all $N \in \mathrm{Ob}\,\mathcal{M}$.

c. For any projective object $P \in \mathrm{Ob}\,\mathcal{M}$ and any $N \in \mathrm{Ob}\,\mathcal{M}$ we have $\mathrm{Ext}^n_{\mathcal{M}}(P,N) = 0$ for all $n \geq 1$.

d. For any short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

and any $N \in \mathrm{Ob}\,\mathcal{M}$, there is a long exact sequence given by

$$\begin{gathered} 0 \to \mathrm{Hom}_{\mathcal{M}}(M_2,N) \to \mathrm{Hom}_{\mathcal{M}}(M,N) \to \mathrm{Hom}_{\mathcal{M}}(M_1,N) \to \\ \to \mathrm{Ext}^1_{\mathcal{M}}(M_2,N) \to \mathrm{Ext}^1_{\mathcal{M}}(M,N) \to \mathrm{Ext}^1_{\mathcal{M}}(M_1,N) \to \\ \to \mathrm{Ext}^2_{\mathcal{M}}(M_2,N) \to \mathrm{Ext}^2_{\mathcal{M}}(M,N) \to \mathrm{Ext}^2_{\mathcal{M}}(M_1,N) \to \cdots \end{gathered}$$

1.2 Commutative algebras

Let A denote a commutative associative unital algebra and let $\mathrm{MaxSpec}(A)$ be the set of all maximal ideals of A.

Definition 1.2.1 ($\mathrm{Supp}(I)$)**.** The *support* of an ideal $I \subseteq A$ is defined to be the set

$$\mathrm{Supp}(I) = \{\mathsf{m} \in \mathrm{MaxSpec}(A) \mid I \subseteq \mathsf{m}\}.$$

Lemma 1.2.2. *Let I, J be ideals of A.*

a. For all positive integer n, we have $\mathrm{Supp}(I) = \mathrm{Supp}(I^n)$.

b. If I is of finite codimension, then $\mathrm{Supp}(I)$ *is finite.*

c. If A is finitely generated, then the support of I is finite if and only if I has finite codimension in A.

d. Suppose $\mathrm{Supp}(I) \cap \mathrm{Supp}(J) = \varnothing$. *Then $I + J = A$ and $IJ = I \cap J$.*

e. If A is Noetherian, then every ideal of A contains a power of its radical.

Proof. The proofs of parts (b), (c), (d) and (e) can be found in [Sav14, §2.1]. It remains to prove part (a). Fix a positive integer n. It is clear that $\mathrm{Supp}(I) \subseteq \mathrm{Supp}(I^n)$. The reverse inclusion follows from the fact that maximal ideals are prime. Indeed, suppose m is a maximal ideal containing I^N. In particular, for all $a \in I$, we have $a^n \in \mathsf{m}$. But since maximal ideals are prime ideals, this implies that $a \in \mathsf{m}$. $\square$

1.3 Associative superalgebras

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. The parity of a homogeneous element $v \in V_i$ will be denoted by $|v|= i$, $i \in \mathbb{Z}_2$. An element in $V_{\bar{0}}$ is called *even*, while an element in $V_{\bar{1}}$ is called *odd*. A *subspace* of V is a $\mathbb{Z}_2$-graded vector space $W = W_{\bar{0}} \oplus W_{\bar{1}} \subseteq V$ such that $W_i \subseteq V_i$ for $i \in \mathbb{Z}_2$. We denote by $\mathbb{C}^{m|n}$ the vector space $\mathbb{C}^m \oplus \mathbb{C}^n$, where the elements of the first (resp. second) summand are even (resp. odd).

An *associative superalgebra* A is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a bilinear associative multiplication (with unit element) such that $A_iA_j \subseteq A_{i+j}$, for $i, j \in \mathbb{Z}_2$. A homomorphism between two superalgebras A and B is a map $g\colon A \to B$ which is a homomorphism between the underlying algebras, and, in addition, $g(A_i) \subseteq B_i$ for $i \in \mathbb{Z}_2$. The tensor product $A \otimes B$ is the superalgebra whose vector space is the tensor product of the vector spaces of A and B, with the induced $\mathbb{Z}_2$-grading and multiplication defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|}a_1a_2 \otimes b_1b_2$, for homogeneous elements $a_i \in A$, and $b_i \in B$. An A-module M is always understood in the $\mathbb{Z}_2$-graded sense, that is, $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $A_iM_j \subseteq M_{i+j}$, for $i, j \in \mathbb{Z}_2$. Subalgebras and ideals of superalgebras are $\mathbb{Z}_2$-graded subalgebras and ideals. A superalgebra having no nontrivial (graded) ideal is called *simple*. A homomorphism between A-modules M and N is a linear map $f\colon M \to N$ such that $f(xm) = xf(m)$, for all $x \in A$ and $m \in M$. A homomorphism is of degree $|f|\in \mathbb{Z}_2$, if $f(M_i) \subseteq N_{i+|f|}$ for $i \in \mathbb{Z}_2$.

We denote by $M(m|n)$ the superalgebra of complex matrices in $m|n$-block form

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right),$$

whose even subspace consists of the matrices with $b = 0$ and $c = 0$, and whose odd subspace consists of the matrices with $a = 0$ and $d = 0$. If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, then the endomorphism superalgebra $\mathrm{End}(V)$ is the associative superalgebra of endomorphisms of V, where

$$\mathrm{End}(V)_i = \{T \in \mathrm{End}(V) \mid T(V_j) \subseteq V_{i+j},\ j \in \mathbb{Z}_2\}, \quad i \in \mathbb{Z}_2.$$

Note that fixing ordered bases for $V_{\bar{0}}$ and $V_{\bar{1}}$ gives an isomorphism between $\mathrm{End}(V)$ and $M(m|n)$.

For $m \geq 1$, let $P \in M(m|m)$ be the matrix

$$\left(\begin{array}{r|r} 0 & I_m \\ \hline -I_m & 0 \end{array}\right),$$

and define $Q(m)_i := \{T \in M(m|m)_i \mid TP = (-1)^i PT\}$, for $i \in \mathbb{Z}_2$. Then $Q(m) = Q(m)_{\bar{0}} \oplus Q(m)_{\bar{1}}$ is the subalgebra of $M(m|m)$ consisting of matrices of the form

$$\left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right), \tag{1.3.1}$$

where a and b are arbitrary $m \times m$ matrices.

Theorem 1.3.1 ([CW12, p. 94])**.** *Consider* $\mathbb{C}^{m|n}$ *as an* $M(m|n)$*-module via matrix multiplication. Then the unique irreducible finite-dimensional module, up to isomorphism, of* $M(m|n)$ *(resp.* $Q(m)$*) is* $\mathbb{C}^{m|n}$ *(resp.* $\mathbb{C}^{m|m}$*).*

For an associative superalgebra A, we shall denote by $|A|$ the underlying (i.e. ungraded) algebra. Denote by $Z(|A|)$ the center of $|A|$. Note that $Z(|A|) = Z(|A|)_{\bar{0}} \oplus Z(|A|)_{\bar{1}}$, where $Z(|A|)_i = Z(|A|) \cap A_i$, for $i \in \mathbb{Z}_2$.

Theorem 1.3.2 ([CW12, Th. 3.1])**.** *Let* A *be a finite-dimensional simple associative superalgebra.*

a. If $Z(|A|)_{\bar{1}} = 0$*, then* A *is isomorphic to* $M(m|n)$*, for some* m *and* n*.*

b. If $Z(|A|)_{\bar{1}} \neq 0$*, then* A *is isomorphic to* $Q(m)$*, for some* m*.*

Definition 1.3.3 (Clifford algebra)**.** Let V be a finite-dimensional vector space and $f: V \times V \to \mathbb{C}$ be a symmetric bilinear form. We call the pair (V, f) a *quadratic pair*. Let J be the ideal of the tensor algebra $T(V)$ generated by the elements

$$x \otimes x - f(x, x)1, \quad x \in V,$$

and set $C(V, f) := T(V)/J$. The algebra $C(V, f)$ is called the *Clifford algebra* of the pair (V, f) over $\mathbb{C}$.

Remark 1.3.4 ([Hus94, Ch. 12, Def. 4.1 and Th. 4.2])**.** For a quadratic pair (V, f), there exists a linear map $\theta: V \to C(V, f)$ such that the pair $(C(V, f), \theta)$ has the following universal property: For all linear maps $u: V \to A$ such that $u(v)^2 = f(v, v)1_A$ for all $v \in V$, where A is a unital algebra, there exists a unique algebra homomorphism $u': C(V, f) \to A$ such that $u'\theta = u$.

Clifford algebras also have a natural superalgebra structure. Indeed, $T(V)$ possesses a $\mathbb{Z}_2$-grading (by even and odd tensor powers) such that J is homogeneous, so the grading descends to $C(V, f)$. Thus, the resulting superalgebra $C(V, f)$ is sometimes called the *Clifford superalgebra*.

Lemma 1.3.5 ([Mus12, Th. A.3.6])**.** *Let* (V, f) *be a quadratic pair with* f *nondegenerate. Then* $C(V, f)$ *is a simple associative superalgebra.*

Remark 1.3.6. It follows from Lemma 1.3.5, together with Theorems 1.3.1 and 1.3.2, that any Clifford superalgebra associated to a nondegenerate pair (i.e. the symmetric bilinear form associated to this pair is nondegenerate) has only one irreducible finite-dimensional module up to isomorphism.

1.4 Lie superalgebras

Definition 1.4.1 (Lie superalgebra)**.** A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear multiplication $[\cdot, \cdot]$ satisfying the following axioms:

a. The multiplication respects the grading: $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}_2$.

b. Skew-supersymmetry: $[a, b] = -(-1)^{|a||b|}[b, a]$, for all homogeneous elements $a, b \in \mathfrak{g}$.

c. Super Jacobi Identity: $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$, for all homogeneous elements $a, b, c \in \mathfrak{g}$.

Example 1.4.2. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra. We can make A into a Lie superalgebra by letting $[a, b] := ab - (-1)^{|a||b|}ba$, for all homogeneous $a, b \in A$, and extending $[\cdot, \cdot]$ by linearity. We call this the *Lie superalgebra associated to A*. The Lie superalgebra associated to $\mathrm{End}(V)$ (resp. $M(m|n)$) is called the *general linear Lie superalgebra* and is denoted by $\mathfrak{gl}(V)$ (resp $\mathfrak{gl}(m|n)$).

Observe that $\mathfrak{g}_{\bar{0}}$ inherits the structure of a Lie algebra and that $\mathfrak{g}_{\bar{1}}$ inherits the structure of a $\mathfrak{g}_{\bar{0}}$-module. A Lie superalgebra $\mathfrak{g}$ is said to be *simple* if there are no nonzero proper (graded) ideals, that is, there are no nonzero proper graded subspaces $\mathfrak{i} \subseteq \mathfrak{g}$ such that $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$. A finite-dimensional simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is said to be *classical* if the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is completely reducible. Otherwise, it is said to be of *Cartan type*.

For a classical Lie superalgebra $\mathfrak{g}$, the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is either irreducible or a direct sum of two irreducible representations. In the first case, $\mathfrak{g}$ is said to be of *type II*, and in the second case, $\mathfrak{g}$ is said to be of *type I*. A classical Lie superalgebra is said to be *basic* if it admits a nondegenerate invariant bilinear form. Otherwise, it is said to be *strange*. In chapters 3 and 4 we will mostly be concerned with basic Lie superalgebras. However, the majority of our results also hold for the Lie superalgebra $\mathfrak{sl}(n, n)$, $n \geq 2$, which is a 1-dimensional central extension of the basic Lie superalgebra $A(n-1, n-1)$. (Throughout the paper we will somewhat abuse terminology by talking of the Lie superalgebras $A(m, n)$, $B(m, n)$, etc., instead of the Lie superalgebras of *type* $A(m, n)$, $B(m, n)$, etc.)

In Table 1.2 we list all of the basic classical Lie superalgebras (up to isomorphism) that are not Lie algebras, together with their even part and their type. We also include the Lie superalgebra $sl(n, n)$, $n \geq 2$, which is a 1-dimensional central extension of $A(n, n)$.

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	Type
$A(m, n)$, $m > n \geq 0$	$A_m \oplus A_n \oplus \mathbb{C}$	I
$A(n, n)$, $n \geq 1$	$A_n \oplus A_n$	I
$\mathfrak{sl}(n, n)$, $n \geq 2$	$A_{n-1} \oplus A_{n-1} \oplus \mathbb{C}$	N/A
$C(n+1)$, $n \geq 1$	$C_n \oplus \mathbb{C}$	I
$B(m, n)$, $m \geq 0$, $n \geq 1$	$B_m \oplus C_n$	II
$D(m, n)$, $m \geq 2$, $n \geq 1$	$D_m \oplus C_n$	II
$F(4)$	$A_1 \oplus B_3$	II
$G(3)$	$A_1 \oplus G_2$	II
$D(2, 1; \alpha)$, $\alpha \neq 0, -1$	$A_1 \oplus A_1 \oplus A_1$	II

Table 1.1: The basic classical Lie superalgebras that are not Lie algebras, together with their even part and their type

Given a Lie superalgebra $\mathfrak{g}$, its universal enveloping superalgebra $U(\mathfrak{g})$ is an associative superalgebra and it satisfies the graded version of the usual universal property. Then a $\mathfrak{g}$-module is the same as a left $U(\mathfrak{g})$-module. The set of $\mathfrak{g}$-homomorphisms between two $\mathfrak{g}$-modules U and V is defined to be $\mathrm{Hom}_{U(\mathfrak{g})}(U,V)_{\bar{0}}$. In other words, all morphism between $\mathfrak{g}$-modules are even. The category of $\mathfrak{g}$-modules will be denoted by $\mathfrak{g}$-mod. The set of morphism between two $\mathfrak{g}$-modules U, V is denoted by $\mathrm{Hom}_{\mathfrak{g}}(U,V)$. Let $\mathfrak{g}$-mod denote the category of $\mathfrak{g}$-modules. Such a category is abelian.

Remark 1.4.3. It is worth to notice that if we allow morphism of arbitrary degree, and not just even morphism, then the category $\mathfrak{g}$-mod would not be abelian. Indeed, suppose it is abelian and for $V, W \in \mathrm{Ob}\,\mathfrak{g}$-mod, consider $\varphi_0 \in \mathrm{Hom}_{\mathfrak{g}}(V,W)_{\bar{0}}$ and $\varphi_1 \in \mathrm{Hom}_{\mathfrak{g}}(V,W)_{\bar{1}}$. Since we are assuming $\mathfrak{g}$-mod is an abelian category, in particular $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ must be an abelian group. Thus $\varphi = \varphi_0 + \varphi_1 \in \mathrm{Hom}_{\mathfrak{g}}(V,W)$ and $\ker\varphi$ must be a ($\mathbb{Z}_2$-graded) submodule of V. Thus $\ker\varphi = \ker\varphi \cap V_0 \oplus \ker\varphi \cap V_1$. But this would imply that $\ker\varphi \subseteq \ker\varphi_0 \cap \ker\varphi_1$. It is easy to find two maps φ_0 and φ_1 that do not satisfy such a condition. As an example, consider any Lie superalgebra $\mathfrak{g}$ acting trivially on $\mathbb{C}^{1|1}$. Let $\varphi_0\colon \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$, where $\varphi_0(a,b) = (a,b)$, and $\varphi_1\colon \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$, where $\varphi_1(a,b) = (b,a)$. Obviously, $\ker\varphi_0 \cap \ker\varphi_1 = \{(0,0)\}$, but $\ker\varphi = \{(a,-a) \mid a \in \mathbb{C}\}$.

The following proposition is a special case of the well known Künneth formula. A proof can be obtained by using universal enveloping algebras and modifying [Wei94, Th. 3.6.3].

Proposition 1.4.4. *Let $\mathfrak{g}^1$, $\mathfrak{g}^2$ be finite-dimensional Lie superalgebras, U_1, V_1 be finite-dimensional $\mathfrak{g}^1$-modules and U_2, V_2 be finite-dimensional $\mathfrak{g}^2$-modules. Then*

$$\mathrm{Ext}^n_{\mathfrak{g}^1\oplus\mathfrak{g}^2}(U_1\otimes U_2, V_1\otimes V_2) \cong \bigoplus_{p+q=n} \mathrm{Ext}^p_{\mathfrak{g}^1}(U_1,V_1)\otimes \mathrm{Ext}^q_{\mathfrak{g}^2}(U_2,V_2), \quad n \geq 0. \tag{1.4.1}$$

1.4.1 Contragredient Lie superalgebras

Contragredient Lie superalgebras give a unified way to construct most of the classical Lie superalgebras as well as Kac-Moody Lie algebras and superalgebras. The construction of these Lie superalgebras proceeds as follows: let $I = \{1,\dots,n\}$, let $A = (a_{ij})_{i,j\in I}$ be a complex matrix, and let $p\colon I \to \mathbb{Z}_2$ be a set map. Fix an even vector space $\mathfrak{h}$ of dimension $2n - \operatorname{rank} A$ and linearly independent $\alpha_i \in \mathfrak{h}^*$, $i \in I$, and $H_i \in \mathfrak{h}$, $i \in I$, such that $\alpha_j(H_i) = a_{ij}$, for all $i, j \in I$. We define $\tilde{\mathfrak{g}}(A)$ to be the Lie superalgebra generated by the even vector space $\mathfrak{h}$ and elements X_i, Y_i, $i \in I$, with the parity of X_i and Y_i equal to $p(i)$, and subject to the relations

$$[X_i, Y_j] = \delta_{ij} H_i, \quad [H, H'] = 0, \quad [H, X_i] = \alpha_i(H) X_i, \quad [H, Y_i] = -\alpha_i(H) Y_i,$$

for $i, j \in I$ and $H, H' \in \mathfrak{h}$.

The *contragredient* Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ is defined to be the quotient of $\tilde{\mathfrak{g}}(A)$ by the ideal that is maximal among all the ideals that intersects $\mathfrak{h}$ trivially (see [Mus12, §5.2]). The images of the elements X_i, Y_i, H_i, $i \in I$, in $\mathfrak{g}(A)$ are denoted by the same symbols.

Since the action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable, we have a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \quad \Delta \subseteq \mathfrak{h}^*,$$

where every root space $\mathfrak{g}_\alpha$ is either purely even or purely odd. A root α is called *even* (resp. *odd*) if $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{0}}$ (resp. $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{1}}$). We denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the sets of even and odd roots respectively. A linearly independent subset $\Sigma = \{\beta_1, \ldots, \beta_n\} \subseteq \Delta$ is called a *base* if we can find $X_{\beta_i} \in \mathfrak{g}_{\beta_i}$ and $Y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$, $i = 1, \ldots, n$, such that $\{X_{\beta_i}, Y_{\beta_i} \mid i = 1, \ldots, n\} \cup \mathfrak{h}$ generates $\mathfrak{g}(A)$, and

$$[X_{\beta_i}, Y_{\beta_j}] = 0 \text{ for } i \neq j.$$

Defining $H_{\beta_i} = [X_{\beta_i}, Y_{\beta_i}]$, it follows that the elements X_{β_i}, Y_{β_i} and H_{β_i} satisfy the following relations:

$$[H_{\beta_j}, X_{\beta_i}] = \beta_i(H_{\beta_j})X_{\beta_i},\ [H_{\beta_j}, Y_{\beta_i}] = -\beta_i(H_{\beta_j})Y_{\beta_i},\ [X_{\beta_i}, Y_{\beta_j}] = \delta_{ij}H_{\beta_i},\ i, j \in \{1, \ldots, n\}. \quad (1.4.2)$$

The matrix $A_\Sigma = (b_{ij})$, where $b_{ij} = \beta_j(H_{\beta_i})$, is called the *Cartan matrix* with respect to the base Σ. The original set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is called the *standard base*. It is clear that A is the Cartan matrix associated to Π, i.e. $A = A_\Pi$. The relations (1.4.2) imply that every root is a purely positive or purely negative integer linear combination of elements in Σ. We call such a root positive or negative, respectively, and we have the decomposition $\Delta = \Delta^+(\Sigma) \sqcup \Delta^-(\Sigma)$, where $\Delta^+(\Sigma)$ and $\Delta^-(\Sigma)$ denote the set of positive and negative roots, respectively. A positive root is called *simple* if it cannot be written as a sum of two positive roots. It is clear that a root is simple if and only if it lies in Σ. Thus, Σ is a *system of simple roots* in the usual sense. We define $\Sigma_z := \Sigma \cap \Delta_z$ and $\Delta_z^\pm(\Sigma) := \Delta_z \cap \Delta^\pm(\Sigma)$ for $z \in \mathbb{Z}_2$. The triangular decomposition of $\mathfrak{g}$ induced by Σ is given by

$$\mathfrak{g} = \mathfrak{n}^-(\Sigma) \oplus \mathfrak{h} \oplus \mathfrak{n}^+(\Sigma),$$

where $\mathfrak{n}^+(\Sigma)$ (resp. $\mathfrak{n}^-(\Sigma)$) is the subalgebra generated by X_β (resp. Y_β), $\beta \in \Sigma$. The subalgebra $\mathfrak{b}(\Sigma) = \mathfrak{h} \oplus \mathfrak{n}^+(\Sigma)$ is called the *Borel subalgebra* corresponding to Σ. Note that $\Delta_{\bar{0}}^+(\Sigma)$ is a system of positive roots for the Lie algebra $\mathfrak{g}_{\bar{0}}$. We denote by $\Sigma(\mathfrak{g}_{\bar{0}})$ the set of simple roots of $\mathfrak{g}_{\bar{0}}$ with respect to this system.

Suppose that $\mathfrak{g}$ is equal to $A(m, n)$ with $m \neq n$, $\mathfrak{gl}(n, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, or $G(3)$. By [Mus12, Theorems 5.3.2, 5.3.3 and 5.3.5], we have that $\mathfrak{g}$ is a contragredient Lie superalgebra. The Lie superalgebra $\mathfrak{sl}(n, n)$ (resp. $A(n, n)$) is isomorphic to $[\mathfrak{gl}(n, n), \mathfrak{gl}(n, n)]$ (resp. $[\mathfrak{gl}(n, n), \mathfrak{gl}(n, n)]/C$, where C is a one-dimensional center). The image of $X \in \mathfrak{sl}(n, n)$ in $A(n, n)$ will be denoted by the same symbol. Fixing a base Σ of $\mathfrak{gl}(n, n)$, the triangular decomposition $\mathfrak{gl}(n, n) = \mathfrak{n}^-(\Sigma) \oplus \mathfrak{h} \oplus \mathfrak{n}^-(\Sigma)$ induces the triangular decompositions

$$\mathfrak{sl}(n, n) = \mathfrak{n}^-(\Sigma) \oplus \mathfrak{h}' \oplus \mathfrak{n}^+(\Sigma) \quad \text{and} \quad A(n, n) = \mathfrak{n}^-(\Sigma) \oplus (\mathfrak{h}'/C) \oplus \mathfrak{n}^+(\Sigma),$$

where $\mathfrak{h}'$ is the subspace of $\mathfrak{h}$ generated by H_β, $\beta \in \Sigma$ (see [Mus12, Lem. 5.2.3]). In particular, any root of $\mathfrak{sl}(n, n)$ or $A(n, n)$ is a purely positive or a purely negative integer linear combination of elements in Σ. Therefore $\Delta = \Delta^+(\Sigma) \sqcup \Delta^-(\Sigma)$ is a decomposition of the system of roots of $\mathfrak{sl}(n, n)$ and $A(n, n)$. The matrix A_Σ is also called the Cartan matrix of $\mathfrak{sl}(n, n)$ and $A(n, n)$ corresponding to Σ.

Remark 1.4.5. Assume $\mathfrak{g}$ is a basic Lie superalgebra, $\mathfrak{gl}(n, n)$ with $n \geq 2$, or $\mathfrak{sl}(n, n)$ with $n \geq 3$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if $\alpha, \beta, \alpha + \beta \in \Delta$. In particular, the parity of $\alpha + \beta$ is the sum of the parities of α and β. Moreover, if $\mathfrak{g} \neq A(1, 1)$, then $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$ (see [Mus12, Ch. 2]).

1.4.2 The remaining Lie superalgebras

The following table contains the list of the simple finite-dimensional Lie superalgebras which are not basic.

$\mathfrak{p}(n)$	if $n \geq 3$
$\mathfrak{q}(n)$	if $n \geq 3$
$W(n)$	if $n \geq 2$
$S(n)$	if $n \geq 3$
$\widetilde{S}(n)$	if n is even and $n \geq 2$
$H(n)$	if $n \geq 4$

Table 1.2: The simple Lie superalgebras that are not basic

The structure of these Lie superalgebras is very different from that of semissimple Lie algebras. For instance, they do not have any invariant non-degenerated bilinear form, neither a nice decomposition in root spaces. Most of the study for these Lie superalgebras is carried out in a case by case. The representation theory of those of Cartan type (the last four in the above table) have been studied in [BL79], [BL82], [Sha78], and [Sha87] for instance, using a very diverse array of methods. This is also the case for the algebras $\mathfrak{p}(\mathfrak{n})$ and $\mathfrak{q}(n)$ (see [Gor06], [CW12], [Ser02]).

1.4.3 The queer Lie superalgebra

Recall the superalgebra $Q(m)$ defined in Section 1.3. If $m = n+1$, then the Lie superalgebra associated to $Q(m)$ will be denoted by $\hat{\mathfrak{q}}(n)$. The derived subalgebra $\tilde{\mathfrak{q}}(n) = [\hat{\mathfrak{q}}(n), \hat{\mathfrak{q}}(n)]$ consists of matrices of the form (1.3.1), where the trace of B is zero. Note that $\tilde{\mathfrak{q}}(n)$ has a one-dimensional center spanned by the identity matrix I_{2n+2}. The *queer Lie superalgebra* is defined to be the quotient superalgebra

$$\mathfrak{q}(n) = \tilde{\mathfrak{q}}(n)/\mathbb{C}I_{2n+2}.$$

By abuse of notation, we denote the image in $\mathfrak{q}(n)$ of a matrix $X \in \tilde{\mathfrak{q}}(n)$ again by X. The Lie superalgebra $\mathfrak{q}(n)$ has even part isomorphic to $\mathfrak{sl}(n+1)$ and odd part isomorphic (as a module over the even part) to the adjoint module. One can show that $\mathfrak{q}(n)$ is simple for $n \geq 2$ (see [Mus12, §2.4.2]). From now on, $\mathfrak{q} = \mathfrak{q}(n)$ where $n \geq 2$.

Remark 1.4.6. Some references refer to $\hat{\mathfrak{q}}(n)$ as the queer Lie superalegbra. However, in this thesis, we reserve this name for the simple Lie superalgebra $\mathfrak{q}(n)$.

Denote by N^-, H, N^+ the subset of strictly lower triangular, diagonal, and strictly upper triangular matrices in $\mathfrak{sl}(n+1)$, respectively. We define

$$\mathfrak{h}_{\bar{0}} = \left\{ \left(\begin{array}{c|c} a & 0 \\ \hline 0 & a \end{array} \right) \;\middle|\; a \in H \right\}, \quad \mathfrak{h}_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & b \\ \hline b & 0 \end{array} \right) \;\middle|\; b \in H \right\},$$

$$\mathfrak{n}^{\pm}_{\bar{0}} = \left\{ \left(\begin{array}{c|c} a & 0 \\ \hline 0 & a \end{array} \right) \;\middle|\; a \in N^{\pm} \right\}, \quad \mathfrak{n}^{\pm}_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & b \\ \hline b & 0 \end{array} \right) \;\middle|\; b \in N^{\pm} \right\},$$

$$\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}, \quad \text{and} \quad \mathfrak{n}^{\pm} = \mathfrak{n}^{\pm}_{\bar{0}} \oplus \mathfrak{n}^{\pm}_{\bar{1}}.$$

Lemma 1.4.7 ([Mus12, Lem. 2.4.1]). *We have a vector space decomposition*

$$\mathfrak{q} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \tag{1.4.3}$$

such that $\mathfrak{n}^\pm$ *and* $\mathfrak{h}$ *are graded subalgebras of* $\mathfrak{q}$ *,with* $\mathfrak{n}^\pm$ *nilpotent. The subalgebra* $\mathfrak{h}$ *is called the* standard Cartan subalgebra *of* $\mathfrak{q}$*.*

We now describe the roots of $\mathfrak{q}$ with respect to $\mathfrak{h}_{\bar{0}}$. For each $i = 1, \dots, n+1$, define $\epsilon_i \in \mathfrak{h}_{\bar{0}}^*$ by

$$\epsilon_i \left(\begin{array}{c|c} h & 0 \\ \hline 0 & h \end{array} \right) = a_i,$$

where h is the diagonal matrix with entries $(a_1, \dots, a_{n+1})$. For $1 \le i, j \le n+1$, we let $E_{i,j}$ denote the $(n+1) \times (n+1)$ matrix with a 1 in position (i,j) and zeros elsewhere, and we set

$$e_{i,j} = \left(\begin{array}{c|c} E_{i,j} & 0 \\ \hline 0 & E_{i,j} \end{array} \right) \quad \text{and} \quad e'_{i,j} = \left(\begin{array}{c|c} 0 & E_{i,j} \\ \hline E_{i,j} & 0 \end{array} \right).$$

Given $\alpha \in \mathfrak{h}_{\bar{0}}^*$, let

$$\mathfrak{q}_\alpha = \{x \in \mathfrak{q} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}_{\bar{0}}\}.$$

We call α a *root* if $\alpha \neq 0$ and $\mathfrak{q}_\alpha \neq 0$. Let Δ denote the set of all roots. Note that $\mathfrak{q}_0 = \mathfrak{h}$ and, for $\alpha = \epsilon_i - \epsilon_j$, $1 \le i \neq j \le n+1$, we have

$$\mathfrak{q}_\alpha = \mathbb{C}e_{i,j} \oplus \mathbb{C}e'_{i,j}.$$

In particular,

$$\mathfrak{q} = \bigoplus_{\alpha \in \mathfrak{h}_{\bar{0}}^*} \mathfrak{q}_\alpha.$$

A root is called *positive* (resp. *negative*) if $\mathfrak{q}_\alpha \cap \mathfrak{n}^+ \neq 0$ (resp. $\mathfrak{q}_\alpha \cap \mathfrak{n}^- \neq 0$). We denote by Δ^+ (resp. Δ^-) the subset of positive (resp. negative) roots. A positive root α is called *simple* if it cannot be expressed as a sum of two positive roots. We denote by Π the set of simple roots. Thus,

$$\Delta^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n+1\}, \quad \Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n+1\},$$
$$\Delta^- = -\Delta^+, \quad \Delta = \Delta^+ \cup \Delta^-.$$

It follows that

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{q}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{q}_\alpha.$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called the *standard Borel subalgebra* of $\mathfrak{q}$.

Notice that, since $n \ge 2$, we have $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{h}_{\bar{0}}$. Indeed, for all $i, j \in \{1, \dots, n+1\}$ with $i \neq j$, we can choose $k \in \{1, \dots, n+1\}$ such that $k \neq i$, $k \neq j$, and then

$$e_{i,i} - e_{j,j} = \frac{1}{2}[e'_{i,i} - e'_{j,j}, e'_{i,i} + e'_{j,j} - 2e'_{k,k}].$$

Thus, the result follows from the fact that elements of the form $e_{i,i} - e_{j,j}$, for $i, j \in \{1, \dots, n+1\}$ and $i \neq j$, generate $\mathfrak{h}_{\bar{0}}$.

1.5 Representations of Lie superalgebras

Just as for Lie algebras, a finite-dimensional Lie superalgebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n \geq 0$, where we define inductively $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$ for $n \geq 1$.

Lemma 1.5.1 ([Kac77, Prop. 5.2.4])**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$. Then every irreducible finite-dimensional $\mathfrak{g}$-module is one-dimensional.*

Lemma 1.5.2 ([Sav14, Lem. 2.6])**.** *Suppose $\mathfrak{g}$ is a Lie superalgebra and V is an irreducible $\mathfrak{g}$-module such that $\mathfrak{J}v = 0$ for some ideal $\mathfrak{J}$ of $\mathfrak{g}$ and nonzero vector $v \in V$. Then $\mathfrak{J}V = 0$.*

The next two results are super versions of well-known results in representation theory. Namely, the Poincaré-Birkhoff-Witt Theorem (or PBW Theorem) and Schur's Lemma, respectively.

Lemma 1.5.3 ([Mus12, Th. 6.1.1])**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra and let B_0, B_1 be totally ordered bases for $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$, respectively. Then the monomials*

$$u_1 \cdots u_r v_1 \cdots v_s, \quad u_i \in B_0, \quad v_i \in B_1 \quad \text{and} \quad u_1 \leq \cdots \leq u_r,\ v_1 < \cdots < v_s,$$

form a basis of the universal enveloping superalgebra $U(\mathfrak{g})$. In particular, if $\mathfrak{g}$ is finite-dimensional and $\mathfrak{g}_{\bar{0}} = 0$, then $U(\mathfrak{g})$ is finite-dimensional.

Lemma 1.5.4 ([Kac77, Schur's Lemma, p. 18])**.** *Let $\mathfrak{g}$ be a Lie superalgebra and V be an irreducible $\mathfrak{g}$-module. Define $\operatorname{End}_{\mathfrak{g}}(V)_i := \{T \in \operatorname{End}(V)_i \mid [T, \mathfrak{g}] = 0\}$, for $i \in \mathbb{Z}_2$. Then,*

$$\operatorname{End}_{\mathfrak{g}}(V)_{\bar{0}} = \mathbb{C}\,\mathrm{id}, \quad \operatorname{End}_{\mathfrak{g}}(V)_{\bar{1}} = \mathbb{C}\varphi,$$

where $\varphi = 0$ or $\varphi^2 = -\,\mathrm{id}$.

Assume that $\mathfrak{g}$ is a basic Lie superalgebra, $\mathfrak{sl}(n,n)$, $n \geq 2$ or $\mathfrak{q} = \mathfrak{q}(n)$, $n \geq 2$. In the case $\mathfrak{g}$ is basic or $\mathfrak{sl}(n,n)$, $n \geq 2$, we fix a system of simple roots Σ, define

$$\Delta_z^+ = \Delta_z^+(\Sigma) \text{ for all } z \in \mathbb{Z}_2,$$

and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition induced by Σ, i.e. $\mathfrak{n}^\pm = \mathfrak{n}^\pm(\Sigma)$. When $\mathfrak{g}$ is $\mathfrak{sl}(n,n)$ or $A(n,n)$, we consider the triangular decomposition induced by $\mathfrak{gl}(n,n)$. For the case $\mathfrak{g} = \mathfrak{q}$, we fix the triangular decomposition of $\mathfrak{q}$ given in (1.4.3).

Lemma 1.5.5 ([Mus12, Prop. 8.2.2])**.** *a. For every $\lambda \in \mathfrak{h}_{\bar{0}}^*$, there exists a unique irreducible $\mathfrak{h}$-module $U(\lambda)$ such that $hv = \lambda(h)v$, for all $h \in \mathfrak{h}_{\bar{0}}$ and $v \in U(\lambda)$.*

b. Any irreducible finite-dimensional $\mathfrak{h}$-module is isomorphic to $U(\lambda)$ for some $\lambda \in \mathfrak{h}_{\bar{0}}^*$.

c. If $\mathfrak{g} \neq \mathfrak{q}$, then $U(\lambda)$ is one-dimensional.

Let V be an irreducible finite-dimensional $\mathfrak{g}$-module. For $\mu \in \mathfrak{h}_{\bar{0}}^*$ (recall that $\mathfrak{h} \neq \mathfrak{h}_{\bar{0}}$ only if $\mathfrak{g} = \mathfrak{q}(n)$), let

$$V_\mu = \{v \in V \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}_{\bar{0}}\} \subseteq V$$

be the μ-weight space of V. Since $\mathfrak{h}_{\bar{0}}$ is an abelian Lie algebra and the dimension of V is finite, we have $V_\mu \neq 0$ for some $\mu \in \mathfrak{h}_{\bar{0}}^*$. We also have $\mathfrak{g}_\alpha V_\mu \subseteq V_{\mu+\alpha}$, for all $\alpha \in \Delta$. Then, by the simplicity of V, we have the weight space decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}_{\bar{0}}^*} V_\mu.$$

Since V has finite dimension, there exists $\lambda \in \mathfrak{h}_{\bar{0}}^*$ such that $V_\lambda \neq 0$ and $\mathfrak{g}_\alpha V_\lambda = 0$ for all $\alpha \in \Delta^+$. Since $[\mathfrak{h}_{\bar{0}}, \mathfrak{h}] = 0$, each weight space is an $\mathfrak{h}$-submodule of V. If U is an irreducible $\mathfrak{h}$-submodule of V_λ, then $U \cong U(\lambda)$ by Lemma 1.5.5. Now, the irreducibility of V together with the PBW Theorem (Lemma 1.5.3), implies that

$$V_\lambda \cong U(\lambda) \quad \text{and} \quad U(\mathfrak{n}^-)V_\lambda = V.$$

In particular, this shows that any irreducible finite-dimensional $\mathfrak{g}$-module is a highest weight module, where the highest weight space is an irreducible $\mathfrak{h}$-module. On the other hand, given an irreducible finite-dimensional $\mathfrak{h}$-module $U(\lambda)$, we can consider the Verma type module associated to it. Namely, regard $U(\lambda)$ as a $\mathfrak{b}$-module, where $\mathfrak{n}^+ U(\lambda) = 0$, and consider the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\lambda)$. This module has a unique proper maximal submodule which we denote by $N(\lambda)$. Define $V(\lambda) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\lambda))/N(\lambda)$. Then $V(\lambda)$ is an irreducible $\mathfrak{g}$-module and every weight of $V(\lambda)$ is of the form

$$\lambda - \sum_{\alpha \in \Pi} m_\alpha \alpha, \quad m_\alpha \in \mathbb{N} \text{ for all } \alpha \in \Pi.$$

Remark 1.5.6. In order to simplify some statements concerning representation theory of the Lie superalgebra $\mathfrak{q}(n)$, we will allow homomorphism of $\mathfrak{q}(n)$-modules to be nonhomogeneous. If we were to require such homomorphisms to be purely even, the Clifford algebra associated to a nondegenerate pair could have two irreducible representations (see Remark 1.3.6) and $\mathfrak{q}(n)$ could have two irreducible highest weight representations of a given highest weight.

Let $P(\lambda) = \{\mu \in \mathfrak{h}_{\bar{0}}^* \mid V(\lambda)_\mu \neq 0\}$. We will fix the partial order on $P(\lambda)$ given by $\mu_1 \geq \mu_2$ if and only if $\mu_1 - \mu_2 \in Q^+$, where $Q^+ := \sum_{\alpha \in \Pi} \mathbb{N}\alpha$ denotes the positive root lattice of $\mathfrak{q}$.

1.6 Equivariant map superalgebras

In this section, by algebra we will mean a commutative associative unital $\mathbb{C}$-algebra, unless otherwise specified. We now introduce the main objects of the study: the map Lie superalgebras.

Definition 1.6.1 (Map superalgebra)**.** Let $\mathfrak{g}$ be a Lie superalgebra and let A be an algebra. We consider the Lie superalgebra $\mathfrak{g} \otimes A$, where the $\mathbb{Z}_2$-grading is given by $(\mathfrak{g} \otimes A)_j = \mathfrak{g}_j \otimes A, j \in \mathbb{Z}_2$, and the bracket is determined by $[x_1 \otimes a_1, x_2 \otimes a_2] = [x_1, x_2] \otimes a_1 a_2$ for $x_i \in \mathfrak{g}$, $a_i \in A$, $i \in \{1, 2\}$. We refer to a superalgebra of this form as a *map Lie superalgebra*, inspired by the case where A is the ring of regular functions on an algebraic variety. Throughout the thesis, we consider $\mathfrak{g} \subseteq \mathfrak{g} \otimes A$ as a subalgebra via the natural isomorphism $\mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{C}$.

An action of a group Γ on a Lie superalgebra $\mathfrak{g}$ and on an algebra A will always be assumed to be by Lie superalgebra automorphisms of $\mathfrak{g}$ and by algebra automorphisms of A.

Definition 1.6.2 (Equivariant map superalgebra)**.** Let Γ be a group acting on an algebra A and on a Lie superalgebra $\mathfrak{g}$ by automorphisms. Then Γ acts naturally on $\mathfrak{g} \otimes A$ by extending the map $\gamma(u \otimes f) = (\gamma u) \otimes (\gamma f)$, $\gamma \in \Gamma$, $u \in \mathfrak{g}$, $f \in A$, by linearity. We define

$$(\mathfrak{g} \otimes A)^\Gamma = \{\mu \in \mathfrak{g} \otimes A \mid \gamma\mu = \mu, \ \forall \ \gamma \in \Gamma\}$$

to be the subsuperalgebra of points fixed under this action. In other words, if A is the coordinate ring of a scheme X, then $(\mathfrak{g} \otimes A)^\Gamma$ is the subsuperalgebra of $\mathfrak{g} \otimes A$ consisting of Γ-equivariant maps from X to $\mathfrak{g}$. We call $(\mathfrak{g} \otimes A)^\Gamma$ an *equivariant map (Lie) superalgebra*. Note that if Γ is the trivial group, this definition reduces to Definition 1.6.1.

Example 1.6.3 (Multiloop superalgebras)**.** Fix positive integers $n, m_1, \dots, m_n$. Let

$$\Gamma = \langle \gamma_1, \dots, \gamma_n \mid \gamma_i^{m_i} = 1, \ \gamma_i\gamma_j = \gamma_j\gamma_i, \ \forall \ 1 \le i, j \le n\rangle \cong \mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_n\mathbb{Z},$$

and suppose that Γ acts on $\mathfrak{g}$. Note that this is equivalent to specifying commuting automorphisms σ_i, $i = 1, \dots, n$, of $\mathfrak{g}$ such that $\sigma_i^{m_i} = \mathrm{id}$. For $i = 1, \dots, n$, let ξ_i be a primitive m_i-th root of unity. Let $X = \operatorname{Spec} A$, where $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the $\mathbb{C}$-algebra of Laurent polynomials in n variables (in other words, X is the n-dimensional torus $(\mathbb{C}^\times)^n$), and define an action of Γ on X by

$$\gamma_i(z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, \xi_i z_i, z_{i+1}, \dots, z_n).$$

Then

$$M(\mathfrak{g}, \sigma_1, \dots, \sigma_n, m_1, \dots, m_n) := (\mathfrak{g} \otimes A)^\Gamma \tag{1.6.1}$$

is the *(twisted) multiloop superalgebra* of $\mathfrak{g}$ relative to $(\sigma_1, \dots, \sigma_n)$ and $(m_1, \dots, m_n)$. In the case that Γ is trivial (i.e. $m_i = 1$ for all $i = 1, \dots, n$), we call often call it an *untwisted multiloop superalgebra*. If $n = 1$, $M(\mathfrak{g}, \sigma_1, m_1)$ is simply called a (*twisted* or *untwisted*) *loop superalgebra*.

1.7 Representations of $(\mathfrak{g} \otimes A)^\Gamma$

Suppose $\mathfrak{g}$ is a reductive Lie algebra, a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$, $n \ge 2$. In this section we explain the classification of all irreducible finite-dimensional representations of $(\mathfrak{g} \otimes A)^\Gamma$.

Suppose $\mathsf{m}_1, \dots, \mathsf{m}_\ell \in \operatorname{MaxSpec}(A)$ are distinct and $n_1, \dots, n_\ell$ are positive integers. The associated generalized evaluation map is the composition

$$\operatorname{ev}_{\mathsf{m}_1^{n_1}, \dots, \mathsf{m}_\ell^{n_\ell}} : \mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A)/\left(\mathfrak{g} \otimes \Pi_{i=1}^\ell \mathsf{m}_i^{n_i}\right) \cong \bigoplus_{i=1}^\ell \left(\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i})\right).$$

Let V_i be a finite-dimensional $\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i})-$module with corresponding representation ρ_i. Then the composition

$$(\mathfrak{g} \otimes A) \xrightarrow{\operatorname{ev}_{\mathsf{m}_1^{n_1}, \dots, \mathsf{m}_\ell^{n_\ell}}} \bigoplus_{i=1}^\ell (\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i})) \xrightarrow{\otimes_{i=i}^\ell \rho_i} \operatorname{End}\left(\bigotimes_{i=1}^\ell V_i\right)$$

is called a *generalized evaluation representation* of $\mathfrak{g} \otimes A$ and is denoted $\mathrm{ev}_{\mathsf{m}_1^{n_1},\ldots,\mathsf{m}_\ell^{n_\ell}}(\rho_1, \ldots, \rho_\ell)$. If $n_1 = \cdots = n_\ell = 1$, then the above is called *evaluation representation*. We define $\mathrm{ev}^{\Gamma}_{\mathsf{m}_1^{n_1},\ldots,\mathsf{m}_\ell^{n_\ell}}(\rho_1, \ldots, \rho_\ell)$ to be the restriction of $\mathrm{ev}_{\mathsf{m}_1^{n_1},\ldots,\mathsf{m}_\ell^{n_\ell}}(\rho_1, \ldots, \rho_\ell)$ to $(\mathfrak{g} \otimes A)^\Gamma$. These are also called *(twisted) generalized evaluation representations*.

In [Sav14], it was shown that all irreducible finite-dimensional representations $(\mathfrak{g} \otimes A)^\Gamma$ are (up to isomorphism) *twisted* generalized evaluation representations. This shows that (up to isomorphism) all the irreducible finite-dimensional representations of $\mathfrak{g} \otimes A$ are tensor product of single point generalized evaluation representations. In particular, we recover [Sav14, Th. 4.16] which states that all irreducible finite-dimensional representations of $\mathfrak{g} \otimes A$ are highest weight representations.

Remark 1.7.1. For the Lie algebra case, the classification of the irreducible finite-dimensional representations of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ (the loop case) in terms of tensor products of evaluation representations was given by V. Chary and A. Pressley [Cha86, CP86]. The context of equivariant map algebras was recently considered by E. Neher, A. Savage and P. Senesi in [NSS12]. There it was shown that all such representations are tensor products of evaluation representations and one-dimensional representations. In the special case when A is the coordinate ring of the torus, the work of A. Savage [Sav14] gives a classification of all irreducible finite-dimensional representations of the multiloop superalgebras. In the untwisted case, this recovers the classification given in [ERZ04, ER13].

Chapter 2

Equivariant map queer Lie superalgebras

In this chapter we classify all irreducible finite-dimensional representations of the equivariant map queer Lie superalgebras under the assumption that Γ is abelian and acts freely on $\mathrm{MaxSpec}(A)$. We show that such representations are parameterized by a certain set of Γ-equivariant finitely supported maps from $\mathrm{MaxSpec}(A)$ to the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{q}$. In the special case where A is the coordinate ring of the torus, we obtain a classification of the irreducible finite-dimensional representations of the twisted loop queer superalgebra.

2.1 Equivariant map queer Lie superalgebras

Let A denote a commutative associative unital $\mathbb{C}$-algebra and let $\mathfrak{q} = \mathfrak{q}(n)$, with $n \geq 2$ be the queer Lie superalgebra. Recall the map Lie superalgebra $\mathfrak{q} \otimes A$, where the $\mathbb{Z}_2$-grading is given by $(\mathfrak{q} \otimes A)_j = \mathfrak{q}_j \otimes A, j \in \mathbb{Z}_2$, and the multiplication is determined by extending the bracket $[x_1 \otimes f_1, x_2 \otimes f_2] = [x_1, x_2] \otimes f_1 f_2, \quad x_i \in \mathfrak{q},\ f_i \in A,\ i \in \{1, 2\}$, by linearity. We will refer such a Lie superalgebra as a *map queer Lie superalgebra*. If Γ is a group acting on A and $\mathfrak{q}$ by automorphisms, then

$$(\mathfrak{q} \otimes A)^\Gamma = \{z \in \mathfrak{q} \otimes A \mid \gamma z = z \text{ for all } \gamma \in \Gamma\}$$

is the Lie subalgebra of $\mathfrak{q} \otimes A$ of points fixed under the diagonal action of Γ on $\mathfrak{q} \otimes A$. We call $(\mathfrak{q} \otimes A)^\Gamma$ an *equivariant map queer Lie superalgebra*. See § 1.6 for details.

Example 2.1.1 (Multiloop queer superalgebras)**.** Let $k, m_1, \dots, m_k$ be positive integers and consider the group

$$\Gamma = \langle \gamma_1, \dots, \gamma_k \mid \gamma_i^{m_i} = 1,\ \gamma_i\gamma_j = \gamma_j\gamma_i,\ \forall\ 1 \leq i, j \leq k\rangle \cong \bigoplus_{i=1}^{k} \mathbb{Z}/m_i\mathbb{Z}.$$

An action of Γ on $\mathfrak{q}$ is equivalent to a choice of commuting automorphisms σ_i of $\mathfrak{q}$ such that $\sigma_i^{m_i} = \mathrm{id}$, for all $i = 1, \dots, k$. Let $A = \mathbb{C}[t_1^{\pm}, \dots, t_k^{\pm}]$ be the algebra of Laurent polynomials in k variables and let $X = \mathrm{Spec}(A)$ (in other words, X is the k-torus $(\mathbb{C}^\times)^k$). For each $i = 1, \dots, k$, let $\xi_i \in \mathbb{C}$ be a primitive m_i-th root of unity, and define an action of Γ on X by

$$\gamma_i(z_1, \dots, z_k) = (z_1, \dots, z_{i-1}, \xi_i z_i, z_{i+1}, \dots, z_k).$$

This induces an action on A and we call

$$M(\mathfrak{q}, \sigma_1, \dots, \sigma_k, m_1, \dots, m_k) := (\mathfrak{q} \otimes A)^\Gamma$$

the (*twisted*) *multiloop queer superalgebra* relative to $(\sigma_1, \dots, \sigma_k)$ and $(m_1, \dots, m_k)$. If Γ is trivial, we call it an *untwisted multiloop queer superalgebra*. If $n = 1$, then $M(\mathfrak{q}, \sigma_1, m_1)$ is called a (*twisted* or *untwisted*) *loop queer superalgebra*. These have been classified, up to isomorphism, in [GP04, Th. 4.4]. This classification uses the fact that the outer automorphism group of $\mathfrak{q}$ is isomorphic to $\mathbb{Z}_4$ (see [Ser84, Th. 1]).

Definition 2.1.2 ($\mathrm{Ann}_A(V)$, $\mathrm{Supp}(V)$)**.** Let V be a $(\mathfrak{q} \otimes A)^\Gamma$-module. We define $\mathrm{Ann}_A(V)$ to be the sum of all Γ-invariant ideals $I \subseteq A$, such that $(\mathfrak{q} \otimes I)^\Gamma V = 0$. If ρ is the associated representation, we set $\mathrm{Ann}_A(\rho) := \mathrm{Ann}_A(V)$. We define the *support* of V to be the support of $\mathrm{Ann}_A(V)$ (see Definition 1.2.1). We say that V has *reduced support* if $\mathrm{Ann}_A(V)$ is a radical ideal.

2.2 Irreducible finite-dimensional representations of the Cartan subalgebra

In this section we study irreducible finite-dimensional $\mathfrak{h} \otimes A$-modules. The goal is to show that, for each such module, there exists a finite-codimensional ideal $I \subseteq A$, such that I is maximal with respect to the property $(\mathfrak{h} \otimes I)V = 0$. Once this is done, we can proceed using similar arguments to those used in the study of irreducible finite-dimensional $\mathfrak{h}$-modules (see [Mus12, Prop. 8.2.1] or [CW12, §1.5.4] for example).

Lemma 2.2.1. *Let V be an irreducible finite-dimensional $\mathfrak{h} \otimes A$-module and let $I \subseteq A$ be an ideal such that $(\mathfrak{h}_{\bar{0}} \otimes I)V = 0$. Then $(\mathfrak{h}_{\bar{1}} \otimes I)V = 0$.*

Proof. Let ρ be the associated representation of $\mathfrak{h} \otimes A$ on V. We must prove that $\rho(\mathfrak{h}_{\bar{1}} \otimes I) = 0$. Note that

$$[\rho(\mathfrak{h} \otimes A), \rho(\mathfrak{h}_{\bar{1}} \otimes I)] = \rho([\mathfrak{h} \otimes A, \mathfrak{h}_{\bar{1}} \otimes I]) \subseteq \rho([\mathfrak{h}, \mathfrak{h}_{\bar{1}}] \otimes I) \subseteq \rho(\mathfrak{h}_{\bar{0}} \otimes I) = 0.$$

Thus, $\rho(\mathfrak{h}_{\bar{1}} \otimes I) \subseteq \mathrm{End}_{\mathfrak{h} \otimes A}(V)_{\bar{1}}$. Suppose that $\rho(z) \neq 0$ for some $z \in \mathfrak{h}_{\bar{1}} \otimes I$. Then, possibly after multiplying z by a nonzero scalar, we may assume, by Schur's Lemma (Lemma 1.5.4), that $\rho(z)^2 = -\mathrm{id}$. But then we obtain the contradiction

$$-2\,\mathrm{id} = 2\rho(z)^2 = [\rho(z), \rho(z)] = \rho([z, z]) = 0,$$

where the last equality follows from the fact that $[z, z] \in \mathfrak{h}_{\bar{0}} \otimes I$. □

Proposition 2.2.2. *Let V be an irreducible $\mathfrak{h} \otimes A$-module. Then V is finite-dimensional if and only if there exists a finite-codimensional ideal I of A such that $(\mathfrak{h} \otimes I)V = 0$.*

Proof. Suppose V is an irreducible finite-dimensional $\mathfrak{h} \otimes A$-module, and let ρ be the associated representation. Let I be the kernel of the linear map

$$\varphi : A \to \operatorname{Hom}_{\mathbb{C}}(V \otimes \mathfrak{h}, V), \quad a \mapsto (v \otimes h \mapsto \rho(h \otimes a)v), \quad a \in A,\ v \in V,\ h \in \mathfrak{h}.$$

Since V is finite-dimensional, I is a linear subspace of A of finite-codimension. We claim that I is an ideal of A. Indeed, if $r \in A$, $a \in I$ and $v \in V$, then we have

$$\varphi(ra)(v \otimes \mathfrak{h}_{\bar{0}}) = \rho(\mathfrak{h}_{\bar{0}} \otimes ra)v = \rho([\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \otimes ra)v$$
$$\rho([\mathfrak{h}_{\bar{1}} \otimes r, \mathfrak{h}_{\bar{1}} \otimes a])v = \rho(\mathfrak{h}_{\bar{1}} \otimes r)\rho(\mathfrak{h}_{\bar{1}} \otimes a)v + \rho(\mathfrak{h}_{\bar{1}} \otimes a)\rho(\mathfrak{h}_{\bar{1}} \otimes r)v = 0.$$

Thus $\varphi(ra)(V \otimes \mathfrak{h}_{\bar{0}}) = 0$ for all $r \in A$, $a \in I$, or equivalently, $\rho(\mathfrak{h}_{\bar{0}} \otimes AI) = 0$. In particular,

$$[\rho(\mathfrak{h}_{\bar{1}} \otimes AI), \rho(\mathfrak{h} \otimes A)] \subseteq \rho(\mathfrak{h}_{\bar{0}} \otimes AI) = 0,$$

which implies that $\rho(\mathfrak{h}_{\bar{1}} \otimes AI) \subseteq \operatorname{End}_{\mathfrak{h}\otimes A}(V)_{\bar{1}}$. Suppose now that $\varphi(ra)(v \otimes h) \neq 0$ for some $v \in V$ and $h \in \mathfrak{h}_{\bar{1}}$. Then $\rho(h \otimes ra) \neq 0$, with $h \otimes ra \in \mathfrak{h}_{\bar{1}} \otimes AI$. Thus, as in the proof of Lemma 2.2.1, we are lead to the contradiction (possibly after rescaling $h \otimes ra$):

$$-2\,\mathrm{id} = 2\rho(h \otimes ra)^2 = [\rho(h \otimes ra), \rho(h \otimes ra)] \in \rho(\mathfrak{h}_{\bar{0}} \otimes (rar)a) = 0,$$

where, in the last equality, we used that $\rho(\mathfrak{h}_{\bar{0}} \otimes AI) = 0$. Since $V \otimes \mathfrak{h}_{\bar{1}}$ is spanned by simple tensors of the form $v \otimes h$, $v \in V$, $h \in \mathfrak{h}_{\bar{1}}$, it follows that $\varphi(ra)(V \otimes \mathfrak{h}_{\bar{1}}) = 0$, and so $ra \in I$. Thus I is a finite-codimensional ideal of A such that $(\mathfrak{h} \otimes I)V = 0$.

Conversely, suppose that $(\mathfrak{h} \otimes I)V = 0$ for some ideal $I \subseteq A$ of finite codimension. Then V factors to an irreducible $\mathfrak{h} \otimes A/I$-module with $(\mathfrak{h}_{\bar{0}} \otimes A/I)v \subseteq \mathbb{C}v$ for all $v \in V$ by Schur's Lemma (Lemma 1.5.4). On the other hand, let $\{x_1, \dots, x_k\}$ be a basis of $\mathfrak{h}_{\bar{1}} \otimes A/I$. Since V is irreducible, the PBW Theorem (Lemma 1.5.3) implies that

$$V = U(\mathfrak{h} \otimes A/I)v = \sum_{1 \le i_1 < \cdots < i_s \le k} x_{i_1} \cdots x_{i_s} \mathbb{C}v,$$

where $i_1, \dots, i_s \in \{1, \dots, k\}$. Hence, V is finite-dimensional. □

Let

$$\mathcal{L}(\mathfrak{h} \otimes A) = \{\psi \in (\mathfrak{h}_{\bar{0}} \otimes A)^* \mid \psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0,\ \text{for some finite-codimensional ideal } I \subseteq A\}$$

and let $\mathcal{R}(\mathfrak{h} \otimes A)$ denote the set of isomorphism classes of irreducible finite-dimensional $\mathfrak{h} \otimes A$-modules. If $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ and $S = \{I \subseteq A \mid I \text{ is an ideal, and } \psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0\}$, we set $I_\psi = \sum_{I \in S} I$.

Recall that we allow homomorphism of $\mathfrak{q}(n)$-modules to be nonhomogeneous (see Remark 1.5.6).

Theorem 2.2.3. *For any $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$, there exists a unique, up to isomorphism, irreducible finite-dimensional $\mathfrak{h} \otimes A$-module $H(\psi)$ such that $xv = \psi(x)v$, for all $x \in \mathfrak{h}_{\bar{0}} \otimes A$ and $v \in H(\psi)$. Conversely, any irreducible finite-dimensional $\mathfrak{h} \otimes A$-module is isomorphic to $H(\psi)$, for some $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$. In other words, the map*

$$\mathcal{L}(\mathfrak{h} \otimes A) \to \mathcal{R}(\mathfrak{h} \otimes A), \quad \psi \mapsto H(\psi),$$

is a bijection.

Proof. Assume first that V is an irreducible finite-dimensional $\mathfrak{h} \otimes A$-module and that $xv = 0$ for all $x \in \mathfrak{h}_{\bar{0}} \otimes A$ and $v \in V$. Then, by Lemma 2.2.1, we have $(\mathfrak{h} \otimes A)V = 0$. So we take $H(0)$ to be the trivial module.

Assume now $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ and $\psi \neq 0$. Define a symmetric bilinear form f_ψ on $\mathfrak{h}_\psi := \mathfrak{h}_{\bar{1}} \otimes A/I_\psi$ by

$$f_\psi(x, y) = \psi([x, y]), \quad x, y \in \mathfrak{h}_\psi. \tag{2.2.1}$$

Let $\mathfrak{h}_\psi^\perp = \{x \in \mathfrak{h}_\psi \mid f_\psi(x, \mathfrak{h}_\psi) = 0\}$ denote the radical of f_ψ, and set

$$\mathfrak{c}_\psi := \frac{\mathfrak{h} \otimes A/I_\psi}{(\ker \psi) \oplus \mathfrak{h}_\psi^\perp} \cong \frac{\mathfrak{h}_{\bar{0}} \otimes A/I_\psi}{\ker \psi} \oplus \frac{\mathfrak{h}_\psi}{\mathfrak{h}_\psi^\perp}.$$

We can regard ψ as a linear functional on $(\mathfrak{c}_\psi)_{\bar{0}}$. Since $\psi \neq 0$, and $\dim((\mathfrak{c}_\psi)_{\bar{0}}) = 1$, there exists a unique $z \in (\mathfrak{c}_\psi)_{\bar{0}}$ such that $\psi(z) = 1$. Define the factor algebra $A_\psi := U(\mathfrak{c}_\psi)/(z-1)$. Consider the natural linear maps $i\colon (\mathfrak{c}_\psi)_{\bar{1}} \hookrightarrow T((\mathfrak{c}_\psi)_{\bar{1}})$ and $p\colon T((\mathfrak{c}_\psi)_{\bar{1}}) \twoheadrightarrow A_\psi$. It is straightforward to check, via the universal property of Clifford algebras (see Remark 1.3.4), that the pair $(A_\psi, p \circ i)$ is isomorphic to the Clifford algebra of $((\mathfrak{c}_\psi)_{\bar{1}}, \frac{1}{2} f_\psi)$. By Remark 1.3.6, this Clifford algebra admits only one, up to isomorphism, irreducible finite-dimensional module. Let $H(\psi)$ denote such a module. We can consider an action of $\mathfrak{h} \otimes A$ on $H(\psi)$ via the map

$$\mathfrak{h} \otimes A \twoheadrightarrow \mathfrak{c}_\psi \hookrightarrow U(\mathfrak{c}_\psi) \twoheadrightarrow A_\psi.$$

Note that $H(\psi)$ is an irreducible $U(\mathfrak{c}_\psi)$-module (and thus an irreducible $\mathfrak{c}_\psi$-module). Therefore, $H(\psi)$ is an irreducible finite-dimensional $\mathfrak{h} \otimes A$-module. In particular we have that

$$xv = \psi(x)v, \text{ for all } x \in \mathfrak{h}_{\bar{0}} \otimes A \text{ and } v \in H(\psi).$$

It remains to prove the converse statement in the lemma. Let V be any irreducible finite-dimensional $\mathfrak{h} \otimes A$-module with associated representation ρ. Since $\mathfrak{h}_{\bar{0}} \otimes A$ is central in $\mathfrak{h} \otimes A$, there exists $\psi \in (\mathfrak{h}_{\bar{0}} \otimes A)^*$ such that $xv = \psi(x)v$, for all $x \in \mathfrak{h}_{\bar{0}} \otimes A$, $v \in V$. On the other hand, by Proposition 2.2.2, there exists an ideal I of A of finite-codimension such that $(\mathfrak{h} \otimes I)V = 0$. In particular, we have that $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$, so $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$, and that V factors to an irreducible $\mathfrak{h} \otimes A/I_\psi$-module. If $\mathfrak{h}_\psi^\perp$ is defined to be the radical of the bilinear form (2.2.1), then $\rho(\mathfrak{h}_\psi^\perp) \subseteq \mathfrak{gl}(V)$ is central. Since ρ is irreducible and $\rho(\mathfrak{h}_\psi^\perp)$ consists of odd elements, it follows that $\rho(\mathfrak{h}_\psi^\perp) = 0$. Hence, V is an irreducible finite-dimensional $C((\mathfrak{c}_\psi)_{\bar{1}}, \frac{1}{2} f_\psi)$-module, and so $V \cong H(\psi)$. □

2.3 Highest weight modules

In [Sav14], the category of irreducible finite-dimensional modules of an equivariant map Lie superalgebra was investigated in the case that the target Lie superalgebra is basic. In particular, it was proved that such modules are either generalized evaluation modules or quotients of analogues of Kac modules of some evaluation modules for a reductive Lie algebra. It was heavily used that the highest weight space of any irreducible finite-dimensional module is one-dimensional, and also that tensor products of irreducible modules with disjoint supports are again irreducible modules.

In Section 2.2, we saw that irreducible finite-dimensional modules for the Cartan superalgebra $\mathfrak{h} \otimes A$ are irreducible modules for certain Clifford algebras. In particular, the dimension of such modules is not necessarily equal to one. In addition, it is not true, in general, that the tensor product of irreducible modules with disjoint supports is irreducible (see Example 2.4.1). Thus, the arguments used in [Sav14] require modification.

From now on, we consider $\mathfrak{q} \subseteq \mathfrak{q} \otimes A$ as a Lie subalgebra via the natural isomorphism $\mathfrak{q} \cong \mathfrak{q} \otimes \mathbb{C}$. We also fix the triangular decomposition of $\mathfrak{q}$ given in (1.4.3).

Definition 2.3.1 (Weight module)**.** Let V be a $\mathfrak{q} \otimes A$-module. We say V is a *weight module* if it is a sum of its weight spaces, i.e.,

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad \text{where } V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}_{\bar{0}}\}.$$

If $V_\lambda \neq 0$, then $\lambda \in \mathfrak{h}_{\bar{0}}^*$ is called a *weight* of V and the nonzero elements of V_λ are called *weight vectors* of weight λ.

Definition 2.3.2 (Quasifinite module)**.** A weight $\mathfrak{q} \otimes A$-module is called *quasifinite* if all its weight spaces are finite-dimensional.

Definition 2.3.3 (Highest weight module)**.** A $\mathfrak{q} \otimes A$-module V is called a *highest weight module* if there exists a nonzero vector $v \in V$ such that

$$U(\mathfrak{q} \otimes A)v = V, \quad (\mathfrak{n}^+ \otimes A)v = 0, \quad \text{and} \quad U(\mathfrak{h}_{\bar{0}} \otimes A)v = \mathbb{C}v. \tag{2.3.1}$$

We call v a *highest weight vector.*

Proposition 2.3.4. *If V is an irreducible finite-dimensional $\mathfrak{q} \otimes A$-module, then V is a highest weight module. Moreover, the weight space associated to the highest weight is an irreducible $\mathfrak{h} \otimes A$-module.*

Proof. Since $\mathfrak{h}_{\bar{0}}$ is an abelian Lie algebra and the dimension of V is finite, $V_\mu \neq 0$ for some $\mu \in \mathfrak{h}_{\bar{0}}^*$. Also note that $(\mathfrak{q}_\alpha \otimes A)V_\mu \subseteq V_{\mu+\alpha}$, for all $\alpha \in \Delta$. Then, by the simplicity of V, it is a weight module. Since V is finite-dimensional, there exists a maximal weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$, such that $V_\lambda \neq 0$. It follows immediately that

$$(\mathfrak{n}^+ \otimes A)V_\lambda = 0.$$

Considering V_λ as an $\mathfrak{h} \otimes A$-module, we can choose an irreducible $\mathfrak{h} \otimes A$-submodule $H(\psi) \subseteq V_\lambda$. Thus $U(\mathfrak{h}_{\bar{0}} \otimes A)v \subseteq \mathbb{C}v$, for all $v \in H(\psi)$. Now by the simplicity of V, we have $U(\mathfrak{q} \otimes A)v = V$ for any $v \in H(\psi)$. In particular, the PBW Theorem (Lemma 1.5.3) implies that $V_\lambda = H(\psi)$. □

Fix $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ and define an action of $\mathfrak{b} \otimes A$ on $H(\psi)$ by declaring $\mathfrak{n}^+ \otimes A$ to act by zero. Consider the induced module

$$\bar{V}(\psi) = U(\mathfrak{q} \otimes A) \otimes_{U(\mathfrak{b} \otimes A)} H(\psi).$$

This is a highest weight module, and a submodule of $\bar{V}(\psi)$ is proper if and only if its intersection with $H(\psi)$ is zero. Moreover any $\mathfrak{q} \otimes A$-submodule of a weight module is also a weight module. Hence, if $W \subseteq \bar{V}(\psi)$ is a proper $\mathfrak{q} \otimes A$-submodule, then

$$W = \bigoplus_{\mu \neq \lambda} W_\mu,$$

where $\lambda = \psi|_{\mathfrak{h}_{\bar{0}}}$. Therefore $\bar{V}(\psi)$ has a unique maximal proper submodule $N(\psi)$ and

$$V(\psi) = \bar{V}(\psi)/N(\psi)$$

is an irreducible highest weight $\mathfrak{q} \otimes A$-module.

By Proposition 2.3.4, every irreducible finite-dimensional $\mathfrak{q} \otimes A$-module is isomorphic to $V(\psi)$ for some $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$. Notice also that the highest weight space of $V(\psi)$ is isomorphic, as an $\mathfrak{h} \otimes A$-module, to $H(\psi)$.

Lemma 2.3.5. *Let $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ and let I be an ideal of A. Then $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$ if and only if $(\mathfrak{q} \otimes I)V(\psi) = 0$.*

Proof. Suppose that $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$ and set $\lambda = \psi|_{\mathfrak{h}_{\bar{0}}}$. We know that $V(\psi)_\lambda \cong H(\psi)$ as $\mathfrak{h} \otimes A$-modules, and, by Lemma 2.2.1, we have that $(\mathfrak{h} \otimes I)V(\psi)_\lambda = 0$. Now, let v be a nonzero vector in $V(\psi)_\lambda$. By Lemma 1.5.2, to prove that $(\mathfrak{q} \otimes I)V(\psi) = 0$, it is enough to prove that $(\mathfrak{q} \otimes I)v = 0$. It is clear that $(\mathfrak{h} \otimes I)v = 0$ and, since v is a highest weight vector, we also have that $(\mathfrak{n}^+ \otimes I)v = 0$. It remains to show that $(\mathfrak{n}^- \otimes I)v = 0$.

For $\alpha = \sum_{i=1}^n a_i\alpha_i$, with $a_i \in \mathbb{N}$ and where the α_i are the simple roots of $\mathfrak{q}$, we define the *height* of α to be

$$\mathrm{ht}(\alpha) = \sum_{i=1}^n a_i.$$

By induction on the height of α, we will show that $(\mathfrak{q}_{-\alpha} \otimes I)v = 0$. We already have the result for $\mathrm{ht}(\alpha) = 0$ (since $\mathfrak{q}_0 = \mathfrak{h}$). Suppose that, for some $m \geq 0$, the results holds whenever $\mathrm{ht}(\alpha) \leq m$. Fix $\alpha \in \Delta^+$ with $\mathrm{ht}(\alpha) = m + 1$. Then

$$(\mathfrak{n}^+ \otimes A)(\mathfrak{q}_{-\alpha} \otimes I)v \subseteq [\mathfrak{n}^+ \otimes A, \mathfrak{q}_{-\alpha} \otimes I]v + (\mathfrak{q}_{-\alpha} \otimes I)(\mathfrak{n}^+ \otimes A)v = ([\mathfrak{n}^+, \mathfrak{q}_{-\alpha}] \otimes I)v = 0, \quad (2.3.2)$$

where the last equality follows from the induction hypothesis, since any element of $[\mathfrak{n}^+, \mathfrak{q}_{-\alpha}]$ is either an element of $\mathfrak{q}_{-\gamma}$, with $\mathrm{ht}(\gamma) < \mathrm{ht}(\alpha)$, or an element of $\mathfrak{n}^+$. Now suppose that there exists a nonzero vector $w \in (\mathfrak{q}_{-\alpha} \otimes I)v \subseteq V_{\lambda-\alpha}$. By (2.3.2), we have $(\mathfrak{n}^+ \otimes A)w = 0$, and, since V is irreducible, we have $V = U(\mathfrak{q} \otimes A)w$. But, by the PBW Theorem (Lemma 1.5.3), this implies that $V(\psi)_\lambda = 0$, which is a contradiction. This completes the proof of the forward implication. The reverse implication is obvious. ☐

Theorem 2.3.6. *Let $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$. The following conditions are equivalent:*

a. The module $V(\psi)$ is quasifinite.

b. There exists a finite-codimensional ideal I of A such that $(\mathfrak{q} \otimes I)V(\psi) = 0$.

c. There exists a finite-codimensional ideal I of A such that $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$.

If A is finitely generated, then the above conditions are also equivalent to:

d. The module $V(\psi)$ has finite support.

Proof. (a) $\Rightarrow$ (b): Let $\lambda = \psi|_{\mathfrak{h}_{\bar{0}}}$ be the highest weight of $V(\psi)$. Let α be a positive root of $\mathfrak{q}$ and let I_α be the kernel of the linear map

$$A \to \operatorname{Hom}_{\mathbb{C}}(V(\psi)_\lambda \otimes \mathfrak{q}_{-\alpha}, V(\psi)_{\lambda-\alpha}), \quad f \mapsto (v \otimes u \mapsto (u \otimes f)v), \quad f \in A,\ v \in V(\psi)_\lambda,\ u \in \mathfrak{q}_{-\alpha}.$$

Since $V(\psi)$ is quasifinite, I_α is a linear subspace of A of finite-codimension. We claim that I_α is, in fact, an ideal of A. Indeed, since $\alpha \neq 0$, we can choose $h \in \mathfrak{h}_{\bar{0}}$ such that $\alpha(h) \neq 0$. Then, for all $g \in A$, $f \in I_\alpha$, $v \in V(\psi)_\lambda$ and $u \in \mathfrak{q}_{-\alpha}$, we have

$$\begin{aligned} 0 &= (h \otimes g)(u \otimes f)v \\ &= [h \otimes g, u \otimes f]v + (u \otimes f)(h \otimes g)v \\ &= -\alpha(h)(u \otimes gf)v + (u \otimes f)(h \otimes g)v. \end{aligned}$$

Since $(h \otimes g)v \in V(\psi)_\lambda$ and $f \in I_\alpha$, the last term above is zero. Since we also have $\alpha(h) \neq 0$, this implies that $(u \otimes gf)v = 0$. As this holds for all $v \in V(\psi)_\lambda$ and $u \in \mathfrak{q}_{-\alpha}$, we have $gf \in I_\alpha$. Hence I_α is an ideal of A.

Let I be the intersection of all the I_α. Since $\mathfrak{q}$ is finite-dimensional (and thus has a finite number of positive roots), this intersection is finite and thus I is also an ideal of A of finite-codimension. We then have $(\mathfrak{n}^- \otimes I)V(\psi)_\lambda = 0$. Since λ is the highest weight of $V(\psi)$, we also have $(\mathfrak{n}^+ \otimes I)V(\psi)_\lambda = 0$. Then, since $\mathfrak{h} \otimes I \subseteq [\mathfrak{n}^+ \otimes A, \mathfrak{n}^- \otimes I]$, we have $(\mathfrak{h} \otimes I)V(\psi)_\lambda = 0$. Thus $(\mathfrak{q} \otimes I)V(\psi)_\lambda = 0$. Since $V(\psi)_\lambda \neq 0$, it follows from Lemma 1.5.2 that $(\mathfrak{q} \otimes I)V(\psi) = 0$.

(b) $\Rightarrow$ (c): Let v be a highest weight vector of $V(\psi)$. Then $\psi(x)v = xv = 0$, for any $x \in \mathfrak{h}_{\bar{0}} \otimes I$. Thus $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$.

(c) $\Rightarrow$ (a): If $\psi(\mathfrak{h}_{\bar{0}} \otimes I) = 0$, then, by Lemma 2.3.5, we have $(\mathfrak{q} \otimes I)V(\psi) = 0$. Then $V(\psi)$ is naturally a module for the finite-dimensional Lie superalgebra $\mathfrak{q} \otimes A/I$. By the PBW Theorem (Lemma 1.5.3), we have

$$V(\psi) = U(\mathfrak{q} \otimes A/I)V(\psi)_\lambda = U(\mathfrak{n}^- \otimes A/I)V(\psi)_\lambda,$$

and $V(\psi)_\lambda$ is finite-dimensional. Another standard application of the PBW Theorem completes the proof.

Now suppose A is finitely generated. We prove that (b) $\Leftrightarrow$ (d). By definition, $\operatorname{Supp}_A(V(\psi)) = \operatorname{Supp}(\operatorname{Ann}_A(V(\psi)))$, where $\operatorname{Ann}_A(V(\psi))$ is the largest ideal of A such that $(\mathfrak{q} \otimes I)V(\psi) = 0$. Thus (b) is true if and only of $\operatorname{Ann}_A(V(\psi))$ is of finite-codimension. Since A is finitely generated, $\operatorname{Ann}_A(V(\psi))$ is of finite-codimension if and only if it has finite support (see Lemma 1.2.2, parts (b) and (c)). ☐

Corollary 2.3.7. *Let V be an irreducible finite-dimensional $\mathfrak{q} \otimes A$-module. Then, there exists an ideal I of A of finite-codimension such that $(\mathfrak{q} \otimes I)V = 0$.*

Proof. Since finite-dimensional modules are clearly quasifinite, the result follows from Theorem 2.3.6. ☐

2.4 Evaluation representations and their irreducible products

If R and S are associative unital algebras, all irreducible finite-dimensional modules for $R \otimes S$ are of the form $V_R \otimes V_S$, where V_R and V_S are irreducible finite-dimensional modules for R and S, respectively. Furthermore, all such modules are irreducible. When R and S are allowed to be superalgebras, the situation is somewhat different. In particular, $V_R \otimes V_S$ is not necessarily irreducible, as seen in the next example.

Example 2.4.1. By Remark 1.3.6, the unique irreducible finite-dimensional $Q(1)$-module is $\mathbb{C}^{1|1}$. However, $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$ is not an irreducible $Q(1) \otimes Q(1)$-module, since $Q(1) \otimes Q(1) \cong M(1|1)$ as associative superalgebras and, again by Remark 1.3.6, the unique irreducible finite-dimensional $M(1|1)$-module is also $\mathbb{C}^{1|1}$.

In general, if $\mathfrak{g}^1$, $\mathfrak{g}^2$ are two finite-dimensional Lie superalgebras, and V^i is an irreducible finite-dimensional $\mathfrak{g}^i$-module, for $i = 1, 2$, then the $\mathfrak{g}^1 \oplus \mathfrak{g}^2$-module $V^1 \otimes V^2$ is irreducible only if $\operatorname{End}_{\mathfrak{g}^i}(V^i)_{\bar{1}} = 0$, for some $i = 1, 2$ (recall that $\dim(\operatorname{End}_{\mathfrak{g}^i}(V^i)_{\bar{1}}) \neq 0$ implies, by Schur's Lemma (Lemma 1.5.4), that $\operatorname{End}_{\mathfrak{g}^i}(V^i)_{\bar{1}} = \mathbb{C}\varphi_i$, where $\varphi_i^2 = -1$). When $\operatorname{End}_{\mathfrak{g}^i}(V^i)_{\bar{1}} = \mathbb{C}\varphi_i$, $\varphi_i^2 = -1$, for $i = 1$ and $i = 2$, we have that

$$\widehat{V} = \{v \in V^1 \otimes V^2 \mid (\tilde{\varphi}_1 \otimes \varphi_2)v = v\}, \text{ where } \tilde{\varphi}_1 = \sqrt{-1}\,\varphi_1,$$

is an irreducible $\mathfrak{g}^1 \oplus \mathfrak{g}^2$-submodule of $V^1 \otimes V^2$ such that $V^1 \otimes V^2 \cong \widehat{V} \oplus \widehat{V}$ (see [Che95, p. 27]). Set henceforth

$$V^1 \widehat{\otimes} V^2 = \begin{cases} V^1 \otimes V^2 & \text{if } V^1 \otimes V^2 \text{ is irreducible,} \\ \widehat{V} \subsetneq V^1 \otimes V^2 & \text{if } V^1 \otimes V^2 \text{ is not irreducible.} \end{cases} \tag{2.4.1}$$

In [Che95, Prop. 8.4], it is proved that every irreducible finite-dimensional $\mathfrak{g}^1 \oplus \mathfrak{g}^2$-module is isomorphic to a module of the form $V^1 \widehat{\otimes} V^2$, where V^i is an irreducible finite-dimensional $\mathfrak{g}^i$-module for $i = 1, 2$. If ρ_i denotes the representation associated to the $\mathfrak{g}^i$-module V^i, then $\rho_1 \widehat{\otimes} \rho_2$ will denote the representation associated to the $\mathfrak{g}^1 \oplus \mathfrak{g}^2$-module $V^1 \widehat{\otimes} V^2$. Inductively, we define the $\mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^k$-module

$$V^1 \widehat{\otimes} \cdots \widehat{\otimes} V^k := (V^1 \widehat{\otimes} \cdots \widehat{\otimes} V^{k-1}) \widehat{\otimes} V^k$$

with associated representation denoted by $\rho_1 \widehat{\otimes} \cdots \widehat{\otimes} \rho_k$. We will call $\widehat{\otimes}$ the *irreducible product*. As the next lemma shows, it is associative, up to isomorphism.

Lemma 2.4.2. *For $i = 1, 2, 3$, let $\mathfrak{g}^i$ be a Lie superalgebra and let V^i be an irreducible finite-dimensional $\mathfrak{g}^i$-module. Then, $(V^1 \widehat{\otimes} V^2) \widehat{\otimes} V^3 \cong V^1 \widehat{\otimes} (V^2 \widehat{\otimes} V^3)$ as $\mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3$-modules.*

Proof. By [Che95, Prop. 8.4], the unique, up to isomorphism, irreducible finite-dimensional $\mathfrak{g}^1 \oplus (\mathfrak{g}^2 \oplus \mathfrak{g}^3)$-module contained in $V^1 \otimes (V^2 \otimes V^3)$ is $V^1 \widehat{\otimes} (V^2 \widehat{\otimes} V^3)$. On the other hand, the unique irreducible finite-dimensional $(\mathfrak{g}^1 \oplus \mathfrak{g}^2) \oplus \mathfrak{g}^3$-module contained in $(V^1 \otimes V^2) \otimes V^3$ is $(V^1 \widehat{\otimes} V^2) \widehat{\otimes} V^3$. Now, since $\mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3 \cong \mathfrak{g}^1 \oplus (\mathfrak{g}^2 \oplus \mathfrak{g}^3) \cong (\mathfrak{g}^1 \oplus \mathfrak{g}^2) \oplus \mathfrak{g}^3$ as Lie superalgebras, and $(V^1 \otimes V^2) \otimes V^3 \cong V^1 \otimes (V^2 \otimes V^3)$ as $\mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3$-modules, we conclude that $(V^1 \widehat{\otimes} V^2) \widehat{\otimes} V^3 \cong V^1 \widehat{\otimes} (V^2 \widehat{\otimes} V^3)$. □

Proposition 2.4.3. *Let $V(\psi_1)$ and $V(\psi_2)$, for $\psi_1, \psi_2 \in \mathcal{L}(\mathfrak{h} \otimes A)$, be two irreducible finite-dimensional $\mathfrak{g} \otimes A$-modules with disjoint supports. Then*

$$V(\psi_1) \otimes V(\psi_2) \cong \begin{cases} V(\psi_1 + \psi_2), \text{ or} \\ V(\psi_1 + \psi_2) \oplus V(\psi_1 + \psi_2). \end{cases}$$

Proof. Let $I_i = \operatorname{Ann}_A(V(\psi_i))$ and let ρ_i be the representation corresponding to $V(\psi_i)$, for $i = 1, 2$. Then the representation $\rho_1 \otimes \rho_2$ factors through the composition

$$\mathfrak{q} \otimes A \overset{\Delta}{\hookrightarrow} (\mathfrak{q} \otimes A) \oplus (\mathfrak{q} \otimes A) \overset{\pi}{\twoheadrightarrow} (\mathfrak{q} \otimes A/I_1) \oplus (\mathfrak{q} \otimes A/I_2), \tag{2.4.2}$$

where Δ is the diagonal embedding and the second map is the obvious projection on each summand. By Lemma 1.2.2(d), we have that $I_1 \cap I_2 = I_1 I_2$, since the supports of I_1 and I_2 are disjoint. Thus $A = I_1 + I_2$, and so $A/I_1 I_2 \cong (A/I_1) \oplus (A/I_2)$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{q} \otimes A & \overset{\Delta}{\hookrightarrow} & (\mathfrak{q} \otimes A) \oplus (\mathfrak{q} \otimes A) \\ \downarrow & & \downarrow \\ \mathfrak{q} \otimes A/I_1 I_2 & \overset{\cong}{\longrightarrow} & (\mathfrak{q} \otimes A/I_1) \oplus (\mathfrak{q} \otimes A/I_2) \end{array}$$

It follows that the composition (2.4.2) is surjective. However, as a $(\mathfrak{q} \otimes A/I_1) \oplus (\mathfrak{q} \otimes A/I_2)$-module, $V(\psi_1) \otimes V(\psi_2)$ is either irreducible or is isomorphic to $\widehat{V} \oplus \widehat{V}$, where $\widehat{V} \subsetneq V(\psi_1) \otimes V(\psi_2)$ is an irreducible $(\mathfrak{q} \otimes A/I_1) \oplus (\mathfrak{q} \otimes A/I_2)$-module. Then the result follows from the fact that $V(\psi_1) \otimes V(\psi_2)$, and hence $\widehat{V} \oplus \widehat{V}$, is generated by vectors on which $\mathfrak{h} \otimes A$ acts by $\psi_1 + \psi_2$. ☐

Note that if $V(\psi_1)$ and $V(\psi_2)$ satisfy the hypothesis of Proposition 2.4.3, then

$$V(\psi_1) \widehat{\otimes} V(\psi_2) \cong V(\psi_1 + \psi_2).$$

In general, the following result follows by induction.

Corollary 2.4.4. *Suppose that $V(\psi_1), \ldots, V(\psi_k)$ are $\mathfrak{q} \otimes A$-modules with pairwise disjoint supports. Then*

$$\widehat{\bigotimes}_{i=1}^{n} V(\psi_i) \cong V\left(\sum_{i=1}^{n} \psi_i\right).$$

Now assume Γ is a finite abelian group acting on both $\mathfrak{q}$ and A by automorphisms. We also assume that A is finitely generated and that Γ acts freely on $\operatorname{MaxSpec}(A)$.

Definition 2.4.5 (Evaluation map)**.** Suppose $\mathsf{m}_1, \ldots, \mathsf{m}_k$ are pairwise distinct maximal ideals of A. The associated *evaluation map* is the composition

$$\operatorname{ev}_{\mathsf{m}_1, \ldots, \mathsf{m}_k} : \mathfrak{q} \otimes A \twoheadrightarrow (\mathfrak{q} \otimes A) \Big/ \left(\mathfrak{q} \otimes \prod_{i=1}^{k} \mathsf{m}_i\right) \cong \bigoplus_{i=1}^{k} (\mathfrak{q} \otimes A/\mathsf{m}_i).$$

We let $\operatorname{ev}^{\Gamma}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}$ denote the restriction of $\operatorname{ev}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}$ to $(\mathfrak{q} \otimes A)^{\Gamma}$.

Let $\mathsf{m}_1, \dots, \mathsf{m}_k$ be pairwise distinct maximal ideals of A, and for each $i = 1, \dots, k$, let V_i be an irreducible finite-dimensional $\mathfrak{q} \otimes A/\mathsf{m}_i$-module, with corresponding representation $\rho_i \colon \mathfrak{q} \otimes A/\mathsf{m}_i \to \mathfrak{gl}(V_i)$. Then the representation given by the composition

$$\mathfrak{q} \otimes A \xrightarrow{\mathrm{ev}_{\mathsf{m}_1,\dots,\mathsf{m}_k}} \bigoplus_{i=1}^{k} (\mathfrak{q} \otimes (A/\mathsf{m}_i)) \xrightarrow{\widehat{\bigotimes}_{i=1}^{k} \rho_i} \mathrm{End}\left(\widehat{\bigotimes}_{i=1}^{k} V_i\right)$$

is denoted by

$$\widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k) \tag{2.4.3}$$

and the corresponding module is denoted by

$$\widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(V_1, \dots, V_k). \tag{2.4.4}$$

We define $\widehat{\mathrm{ev}}^{\Gamma}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k)$ to be the restriction of $\widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k)$ to $(\mathfrak{q} \otimes A)^{\Gamma}$. The notation $\widehat{\mathrm{ev}}^{\Gamma}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(V_1, \dots, V_k)$ is defined similarly.

If we consider tensor products instead of irreducible products, then the above are called *evaluation representations* and *evaluation modules*, respectively.

Remark 2.4.6. Observe that, by definition,

$$\widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k) \cong \widehat{\bigotimes}_{i=1}^{k} \mathrm{ev}_{\mathsf{m}_i}(\rho_i).$$

Proposition 2.4.7. *An irreducible finite-dimensional representation of $\mathfrak{q} \otimes A$ is isomorphic to a representation of the form* (2.4.3) *if and only if it has finite reduced support.*

Proof. Let ρ be an irreducible finite-dimensional representation of $\mathfrak{q} \otimes A$. Assume

$$\rho \cong \widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k),$$

where $\mathsf{m}_1, \dots, \mathsf{m}_k$ are pairwise distinct maximal ideals of A and ρ_i is an irreducible representation of $\mathfrak{q} \otimes A/\mathsf{m}_i$. Let $I = \prod_{i=1}^{k} \mathsf{m}_i$. Then $\mathrm{Supp}(I) = \{\mathsf{m}_1, \dots, \mathsf{m}_k\}$ and $\rho(\mathfrak{q} \otimes I) = 0$. Thus ρ has finite support. Furthermore we have that $\sqrt{I} = \bigcap_{i=1}^{k} \mathsf{m}_i = \prod_{i=1}^{k} \mathsf{m}_i = I$ and hence I is a radical ideal. This proves the forward implication.

Suppose now that $\rho(\mathfrak{q} \otimes I) = 0$ for some radical ideal I of A of finite support. Thus $I = \sqrt{I} = \prod_{i=1}^{n} \mathsf{m}_i$ for some distinct maximal ideals $\mathsf{m}_1, \dots, \mathsf{m}_k$ of A. Hence, ρ factors through the map

$$\mathfrak{q} \otimes A \twoheadrightarrow (\mathfrak{q} \otimes A)/\left(\mathfrak{q} \otimes \prod_{i=1}^{k} \mathsf{m}_i\right) \cong \bigoplus_{i=1}^{k} (\mathfrak{q} \otimes A/\mathsf{m}_i).$$

Then, by [Che95, Prop. 8.4], there exist irreducible finite-dimensional representations ρ_i of $\mathfrak{q} \otimes A/\mathsf{m}_i$, $i = 1, \dots, k$, such that

$$\rho \cong \widehat{\bigotimes}_{i=1}^{k} \mathrm{ev}_{\mathsf{m}_i}(\rho_i) \cong \widehat{\mathrm{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\rho_1, \dots, \rho_k).$$

Thus ρ is isomorphic to a representation of the form (2.4.3). This completes the proof of the reverse implication. $\square$

Definition 2.4.8 (X_*)**.** Let X_* denote the set of finite subsets $\mathsf{M} \subseteq \operatorname{MaxSpec}(A)$ having the property that $\mathsf{m}' \notin \Gamma\mathsf{m}$ for distinct $\mathsf{m}, \mathsf{m}' \in \mathsf{M}$.

Lemma 2.4.9 ([Sav14, Lem. 5.6])**.** *If* $\{\mathsf{m}_1, \dots, \mathsf{m}_k\} \in X_*$*, then the map* $\widehat{\operatorname{ev}}^\Gamma_{\mathsf{m}_1,\dots,\mathsf{m}_k}$ *is surjective.*

Let $\mathcal{R}(\mathfrak{q})$ denote the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{q}$. Then Γ acts on $\mathcal{R}(\mathfrak{q})$ by

$$\Gamma \times \mathcal{R}(\mathfrak{q}) \to \mathcal{R}(\mathfrak{q}), \quad (\gamma, [\rho]) \mapsto \gamma[\rho] := [\rho \circ \gamma^{-1}],$$

where $[\rho] \in \mathcal{R}(\mathfrak{q})$ denotes the isomorphism class of a representation ρ of $\mathfrak{q}$.

Definition 2.4.10 ($\mathcal{E}(\mathfrak{q}, A)$, $\mathcal{E}(\mathfrak{q}, A)^\Gamma$)**.** Let $\mathcal{E}(\mathfrak{q}, A)$ denote the set of finitely supported functions $\Psi\colon \operatorname{MaxSpec}(A) \to \mathcal{R}(\mathfrak{q})$ and let $\mathcal{E}(\mathfrak{q}, A)^\Gamma$ denote the subset of $\mathcal{E}(\mathfrak{q}, A)$ consisting of those functions that are Γ-equivariant. Here the support of Ψ, denoted $\operatorname{Supp}(\Psi)$, is the set of all $\mathsf{m} \in \operatorname{MaxSpec}(A)$ for which $\Psi(\mathsf{m}) \neq 0$, where 0 denotes the isomorphism class of the trivial (one-dimensional) representation.

If ρ and ρ' are isomorphic representations of $\mathfrak{q}$, then the representations $\operatorname{ev}_\mathsf{m}(\rho)$ and $\operatorname{ev}_\mathsf{m}(\rho')$ are also isomorphic, for any $\mathsf{m} \in \operatorname{MaxSpec} A$. Therefore, for $[\rho] \in \mathcal{R}(\mathfrak{q})$, we can define $\operatorname{ev}_\mathsf{m}[\rho]$ to be the isomorphism class of $\operatorname{ev}_\mathsf{m}(\rho)$, and this is independent of the representative ρ. For $\Psi \in \mathcal{E}(\mathfrak{q}, A)$ such that $\operatorname{Supp}(\Psi) = \{\mathsf{m}_1, \dots, \mathsf{m}_k\}$, we define $\widehat{\operatorname{ev}}_\Psi$ to be the isomorphism class of $\widehat{\operatorname{ev}}_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\Psi(\mathsf{m}_1), \dots, \Psi(\mathsf{m}_k))$, which is well-defined by the above comments and Remark 2.4.6. If Ψ is the map that is identically 0 on $\operatorname{MaxSpec}(A)$, then, by definition, $\widehat{\operatorname{ev}}_\Psi$ is the isomorphism class of the trivial (one-dimensional) representation of $\mathfrak{q} \otimes A$.

Lemma 2.4.11. *Let* $\Psi \in \mathcal{E}(\mathfrak{q}, A)^\Gamma$ *and* $\mathsf{m} \in \operatorname{MaxSpec}(A)$*. Then, for all* $\gamma \in \Gamma$*,*

$$\widehat{\operatorname{ev}}_\mathsf{m}(\Psi(\mathsf{m})) = \widehat{\operatorname{ev}}_{\gamma\mathsf{m}}(\gamma\Psi(\mathsf{m})) = \widehat{\operatorname{ev}}_{\gamma\mathsf{m}}(\Psi(\gamma\mathsf{m})).$$

Proof. If $\Psi(\mathsf{m}) = \rho$, then $\Psi(\gamma\mathsf{m}) = \gamma \cdot \rho$, and so

$$\widehat{\operatorname{ev}}_{\gamma\mathsf{m}}(\Psi(\gamma\mathsf{m}))(x) = \widehat{\operatorname{ev}}_{\gamma\mathsf{m}}(\gamma \cdot \rho)(x) = \rho \circ \gamma^{-1}(x + \gamma\mathsf{m}) = \rho(\gamma^{-1}x + \mathsf{m}) = \rho(x + \mathsf{m}) = \widehat{\operatorname{ev}}_\mathsf{m}(\rho)(x),$$

for all $x \in (\mathfrak{g} \otimes A)^\Gamma$. □

Definition 2.4.12 ($\widehat{\operatorname{ev}}^\Gamma_\Psi$)**.** Let $\Psi \in \mathcal{E}(\mathfrak{q}, A)^\Gamma$ and let $\mathsf{M} = \{\mathsf{m}_1, \dots, \mathsf{m}_k\} \in X_*$ contain one element from each Γ-orbit in $\operatorname{Supp}(\Psi)$. We define $\widehat{\operatorname{ev}}^\Gamma_\Psi := \widehat{\operatorname{ev}}^\Gamma_{\mathsf{m}_1,\dots,\mathsf{m}_k}(\Psi(\mathsf{m}_1), \dots, \Psi(\mathsf{m}_k))$. By Lemma 2.4.11, $\widehat{\operatorname{ev}}^\Gamma_\Psi$ is independent of the choice of M. If $\Psi = 0$, we define $\widehat{\operatorname{ev}}^\Gamma_\Psi$ to be the isomorphism class of the trivial (one-dimensional) representation of $(\mathfrak{q} \otimes A)^\Gamma$.

Proposition 2.4.13. *The map* $\Psi \mapsto \widehat{\operatorname{ev}}_\Psi$ *from* $\mathcal{E}(\mathfrak{q}, A)$ *to the set of isomorphism classes of irreducible finite-dimensional representations of* $\mathfrak{q} \otimes A$ *is injective.*

Proof. If $\Psi \neq \Psi' \in \mathcal{E}(\mathfrak{q}, A)$, then there exists $\mathsf{m} \in \operatorname{MaxSpec}(A)$ such that $\Psi(\mathsf{m}) \neq \Psi'(\mathsf{m})$. Without loss of generality, we may assume that $\Psi(\mathsf{m}) \neq 0$. Let $\operatorname{Supp}(\Psi) \cup \operatorname{Supp}(\Psi') = \{\mathsf{m}_1, \dots, \mathsf{m}_k\}$, where $\mathsf{m} = \mathsf{m}_1$ and consider the following ideal of A:

$$I = \mathsf{m}_2 \cdots \mathsf{m}_k.$$

Note that $\mathfrak{a} = \mathfrak{q} \otimes I$ is a Lie subalgebra of $\mathfrak{q} \otimes A$ such that $\mathrm{ev}_{\mathsf{m}}(\mathfrak{a}) \cong \mathfrak{q}$ and $\mathrm{ev}_{\mathsf{m}_j}(\mathfrak{a}) = 0$ for $j = 2, \ldots, k$.

Suppose that $\widehat{\mathrm{ev}}_{\Psi} \cong \widehat{\mathrm{ev}}_{\Psi'}$, and define

$$\rho := \mathrm{ev}_{\mathsf{m}_2}(\Psi(\mathsf{m}_2)) \widehat{\otimes} \cdots \widehat{\otimes}\, \mathrm{ev}_{\mathsf{m}_k}(\Psi(\mathsf{m}_k)) \quad \text{and} \quad \rho' := \mathrm{ev}_{\mathsf{m}_2}(\Psi'(\mathsf{m}_2)) \widehat{\otimes} \cdots \widehat{\otimes}\, \mathrm{ev}_{\mathsf{m}_k}(\Psi'(\mathsf{m}_k)),$$

with associated modules V and V', respectively. Then $\rho(\mathfrak{a}) = \rho'(\mathfrak{a}) = 0$. We divide the proof into three cases.

For the first case, assume that we have isomorphisms of $\mathfrak{q} \otimes A$-modules

$$\mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho \cong \hat{\rho} \oplus \hat{\rho} \quad \text{and} \quad \mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho' \cong \hat{\rho}' \oplus \hat{\rho}',$$

where $\hat{\rho}$ and $\hat{\rho}'$ are subrepresentations of $\mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho$ and $\mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho'$, respectively. Since $\widehat{\mathrm{ev}}_{\Psi} \cong \widehat{\mathrm{ev}}_{\Psi'}$, we must have $\hat{\rho} \cong \hat{\rho}'$, and so

$$\begin{aligned} \mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1))^{\oplus \dim V} &\cong (\mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho)|_{\mathfrak{a}} \cong (\hat{\rho} \oplus \hat{\rho})|_{\mathfrak{a}} \cong (\hat{\rho}' \oplus \hat{\rho}')|_{\mathfrak{a}} \\ &\cong (\mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho')|_{\mathfrak{a}} \cong \mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1))^{\oplus \dim V'}, \end{aligned}$$

where the first isomorphism follows from the fact that $\rho(\mathfrak{a}) = 0$ and the last follows from the fact that $\rho'(\mathfrak{a}) = 0$. But this is a contradiction, since $\Psi(\mathsf{m}_1) \neq \Psi'(\mathsf{m}_1)$.

For the second case, assume

$$\mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho' \text{ is irreducible} \quad \text{and} \quad \mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho \cong \hat{\rho} \oplus \hat{\rho},$$

where $\hat{\rho} \subseteq \mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho$ is a subrepresentation. Thus $\hat{\rho} \cong \mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho'$, which implies that

$$\mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1))^{\oplus \dim V} \cong (\hat{\rho} \oplus \hat{\rho})|_{\mathfrak{a}} \cong (\mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho')^{\oplus 2}|_{\mathfrak{a}} \cong \mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1))^{\oplus 2 \dim V'}.$$

So again we have a contradiction.

The remaining case, when both $\mathrm{ev}_{\mathsf{m}_1}(\Psi'(\mathsf{m}_1)) \otimes \rho'$ and $\mathrm{ev}_{\mathsf{m}_1}(\Psi(\mathsf{m}_1)) \otimes \rho$ are irreducible $\mathfrak{q} \otimes A$-modules, is similar. □

Corollary 2.4.14. *For all $\Psi \in \mathcal{E}(\mathfrak{q}, A)^{\Gamma}$, we have that $\widehat{\mathrm{ev}}_{\Psi}^{\Gamma}$ is the isomorphism class of an irreducible finite-dimensional representation. Furthermore, the map $\Psi \mapsto \widehat{\mathrm{ev}}_{\Psi}^{\Gamma}$ from $\mathcal{E}(\mathfrak{q}, A)^{\Gamma}$ to the set of isomorphism classes of irreducible finite-dimensional representations of $(\mathfrak{q} \otimes A)^{\Gamma}$ is injective.*

Proof. The first statement follows from Lemma 2.4.9 and the definition of the irreducible product. Suppose $\Psi, \Psi' \in \mathcal{E}(\mathfrak{q}, A)^{\Gamma}$ such that $\widehat{\mathrm{ev}}_{\Psi}^{\Gamma} = \widehat{\mathrm{ev}}_{\Psi'}^{\Gamma}$. Let $\mathsf{M} = \{\mathsf{m}_1, \ldots, \mathsf{m}_k\} \in X_*$ contain one element of each Γ-orbit in $\mathrm{Supp}(\Psi) \cup \mathrm{Supp}(\Psi')$. Then $\widehat{\mathrm{ev}}_{\Psi}^{\Gamma}$ and $\widehat{\mathrm{ev}}_{\Psi'}^{\Gamma}$ are the restrictions to $(\mathfrak{g} \otimes A)^{\Gamma}$ of $\widehat{\mathrm{ev}}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}(\Psi(\mathsf{m}_1), \ldots, \Psi(\mathsf{m}_k))$ and $\widehat{\mathrm{ev}}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}(\Psi'(\mathsf{m}_1), \ldots, \Psi'(\mathsf{m}_k))$, respectively. By Lemma 2.4.9, it follows that $\widehat{\mathrm{ev}}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}(\Psi(\mathsf{m}_1), \ldots, \Psi(\mathsf{m}_k)) = \widehat{\mathrm{ev}}_{\mathsf{m}_1, \ldots, \mathsf{m}_k}(\Psi'(\mathsf{m}_1), \ldots, \Psi'(\mathsf{m}_k))$. Then, by Proposition 2.4.13, we have $\Psi(\mathsf{m}_i) = \Psi'(\mathsf{m}_i)$ for $i = 1, \ldots, k$. Thus $\Psi = \Psi'$. □

Remark 2.4.15. If the target Lie superalgebra $\mathfrak{q}$ is replaced by a Lie algebra or a basic classical Lie superalgebra $\mathfrak{g}$, then the tensor product of irreducible finite-dimensional representations with disjoint supports is always irreducible (see [NSS12, Prop. 4.9] for Lie algebras and [Sav14, Prop. 4.12] for basic classical Lie superalgebras). In particular, the evaluation representation ev_{Ψ} is an irreducible finite-dimensional representation for all $\Psi \in \mathcal{E}(\mathfrak{g}, A)$, where ev_{Ψ} is defined by replacing the irreducible product by the tensor product in the definition of $\widehat{\mathrm{ev}}_{\Psi}$.

2.5 Classification of finite-dimensional representations

In this section we present our main result for the first part of this Thesis: the classification of the irreducible finite-dimensional $\mathfrak{q} \otimes A$-modules and $(\mathfrak{q} \otimes A)^\Gamma$-modules. We assume that A is finitely generated.

Theorem 2.5.1. *The map*

$$\mathcal{E}(\mathfrak{q}, A) \to \mathcal{R}(\mathfrak{q} \otimes A), \qquad \Psi \mapsto \widehat{\mathrm{ev}}_\Psi, \tag{2.5.1}$$

is a bijection, where $\mathcal{R}(\mathfrak{q} \otimes A)$ is the set of isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{q} \otimes A$. In particular, all irreducible finite-dimensional representations are representations of the form (2.4.3)*.*

Proof. By Proposition 2.4.13, it is enough to show that all irreducible finite-dimensional representations of $\mathfrak{q} \otimes A$ are of the form (2.4.3). Thus, it suffices, by Proposition 2.4.7, to show that, for every irreducible finite-dimensional $\mathfrak{q} \otimes A$-module V, we have $(\mathfrak{q} \otimes J)V = 0$ for some radical ideal $J \subseteq A$ of finite-codimension.

By Corollary 2.3.7, we have that $(\mathfrak{q} \otimes I)V = 0$ for some ideal I of A of finite codimension. Let $J = \sqrt{I}$ be the radical of I. To prove that $(\mathfrak{q} \otimes J)V = 0$, it suffices, by Lemma 1.5.2, to show that $(\mathfrak{q} \otimes J)v = 0$ for some nonzero vector $v \in V$.

Consider now V as a $\mathfrak{q} \otimes A/I$-module. We will show that $(\mathfrak{q} \otimes (J/I))v = 0$ for some nonzero $v \in V$. Since A is finitely generated, and hence Noetherian, we have $J^k \subseteq I$ for some $k \in \mathbb{N}$, by Lemma 1.2.2(e). Hence, $(\mathfrak{q} \otimes (J/I))^{(k)} = \mathfrak{q}^{(k)} \otimes (J^k/I) = 0$, and so $\mathfrak{q} \otimes (J/I)$ is solvable. On the other hand, since $\mathfrak{q}_{\bar{0}}$ is a simple Lie algebra, we have

$$\begin{aligned}[(\mathfrak{q} \otimes (J/I))_{\bar{1}}, (\mathfrak{q} \otimes (J/I))_{\bar{1}}] &= [\mathfrak{q}_{\bar{1}}, \mathfrak{q}_{\bar{1}}] \otimes (J^2/I) \subseteq \mathfrak{q}_{\bar{0}} \otimes (J^2/I) \\ &= [\mathfrak{q}_{\bar{0}}, \mathfrak{q}_{\bar{0}}] \otimes (J^2/I) = [(\mathfrak{q} \otimes (J/I))_{\bar{0}}, (\mathfrak{q} \otimes (J/I))_{\bar{0}}].\end{aligned}$$

Then, by Lemma 1.5.1, there exists a one-dimensional $\mathfrak{q} \otimes (J/I)$-submodule of V. Thus, we have a nonzero vector $v \in V$ and $\theta \in (\mathfrak{q} \otimes J)^*$, such that

$$\mu v = \theta(\mu)v, \quad \text{for all } \mu \in \mathfrak{q} \otimes J.$$

We want to prove that $\theta = 0$. If $\mu \in \mathfrak{n}^\pm \otimes J$, then $\theta(\mu)^m v = \mu^m v = 0$ for m sufficiently large, since V is finite dimensional and hence has a finite number of nonzero weight spaces. Thus $\theta(\mathfrak{n}^\pm \otimes J) = 0$. It remains to show that $\theta(\mathfrak{h} \otimes J) = 0$. Denote by θ' the restriction of θ to $\mathfrak{q}_{\bar{0}} \otimes J$. Then θ' defines a one-dimensional representation of the Lie algebra $\mathfrak{q}_{\bar{0}} \otimes J$, and hence the kernel of θ' must be an ideal of $\mathfrak{q}_{\bar{0}} \otimes J$ of codimension at most one. Because $\mathfrak{q}_{\bar{0}}$ is a simple finite-dimensional Lie algebra, it is easy to see that this kernel must be all of $\mathfrak{q}_{\bar{0}} \otimes J$, and hence $\theta' = 0$. Since $\mathfrak{h}_{\bar{0}} \subseteq \mathfrak{q}_{\bar{0}}$, we also have that $\theta(\mathfrak{h}_{\bar{0}} \otimes J) = 0$. Therefore, Lemma 2.2.1 implies that $(\mathfrak{h} \otimes J)v = 0$. □

Now assume Γ is a finite abelian group acting on both $\mathfrak{q}$ and A by automorphisms. We also assume that Γ acts freely on $\mathrm{MaxSpec}(A)$.

Proposition 2.5.2. *Every finite-dimensional* $(\mathfrak{q} \otimes A)^\Gamma$*-module* V *is the restriction of a* $\mathfrak{q} \otimes A$*-module* V' *whose support is an element of* X_**. Furthermore,* V *is irreducible if and only if* V' *is.*

Proof. Let V be a finite dimensional $(\mathfrak{q} \otimes A)^\Gamma$-module and let $\rho : (\mathfrak{q} \otimes A)^\Gamma \to \operatorname{End} V$ denote the corresponding representation. By [Sav14, Prop. 8.1 and Lem. 8.4], the kernel of ρ is of the form $(\mathfrak{q} \otimes I)^\Gamma$ for some Γ-invariant ideal I of A with finite support. Since A is finitely generated, Lemma 1.2.2c implies that I is of finite codimension in A. The support of I is a Γ-invariant subset of X_{rat}. Let $\mathsf{M} \subseteq X_*$ contain one point from each Γ-orbit in the support of I. Then

$$I = \prod_{\mathsf{m} \in \mathsf{M},\, \gamma \in \Gamma} \gamma I_{\mathsf{m}},$$

where I_{m} is an ideal with support $\{\mathsf{m}\}$ for each $\mathsf{m} \in \mathsf{M}$,. Thus

$$\begin{aligned}
(\mathfrak{q} \otimes A)^\Gamma / (\mathfrak{q} \otimes I)^\Gamma &\cong (\mathfrak{q} \otimes A/I)^\Gamma \\
&\cong \left(\mathfrak{q} \otimes \bigoplus_{\mathsf{m} \in \mathsf{M},\, \gamma \in \Gamma} A/(\gamma I_{\mathsf{m}}) \right)^\Gamma \\
&\cong \left(\bigoplus_{\mathsf{m} \in \mathsf{M},\, \gamma \in \Gamma} (\mathfrak{q} \otimes A/(\gamma I_{\mathsf{m}})) \right)^\Gamma \\
&\cong \bigoplus_{\mathsf{m} \in \mathsf{M}} (\mathfrak{q} \otimes A/I_{\mathsf{m}}) \\
&\cong (\mathfrak{q} \otimes A)/(\mathfrak{q} \otimes J), \quad J = \prod_{\mathsf{m} \in \mathsf{M}} I_{\mathsf{m}},
\end{aligned}$$

where, in the second-to-last isomorphism, we use the fact that, for $\mathsf{m} \in \mathsf{M}$, the group Γ permutes the summands $\mathfrak{q} \otimes A/(\gamma I_{\mathsf{m}})$, $\gamma \in \Gamma$.

We now have the following commutative diagram, where τ is the above isomorphism, π is the natural projection, and $\bar{\rho}$ is the map induced by ρ.

$$\begin{array}{ccccc}
\mathfrak{q} \otimes A & \xrightarrow{\ \pi\ } & (\mathfrak{q} \otimes A)/(\mathfrak{q} \otimes J) & & \\
\uparrow & & \uparrow \tau & & \\
(\mathfrak{q} \otimes A)^\Gamma & \longrightarrow & (\mathfrak{q} \otimes A)^\Gamma/(\mathfrak{q} \otimes I)^\Gamma & \xrightarrow{\ \bar{\rho}\ } & \operatorname{End}(V)
\end{array}$$

It is clear that $\bar{\rho} \circ \tau^{-1} \circ \pi$ is a representation of $\mathfrak{q} \otimes A$ that, when restricted to $(\mathfrak{q} \otimes A)^\Gamma$, coincides with ρ. Since both representations factor through the quotient $(\mathfrak{q} \otimes A)^\Gamma/(\mathfrak{q} \otimes I)^\Gamma$, one is irreducible if and only if the other is. ☐

Remark 2.5.3. The proof of Proposition 2.5.2 is the same as the proof of [Sav14, Prop. 8.5]. Although that reference assumes that the target Lie superalgebra $\mathfrak{g}$ is basic classical, the proof of this result only requires $\mathfrak{g}$ to be a simple finite-dimensional Lie superalgebra.

Theorem 2.5.4. *Suppose A is a finitely generated unital associative $\mathbb{C}$-algebra and Γ is a finite abelian group acting on A and $\mathfrak{q}$ by automorphisms. Furthermore, suppose that the induced action of Γ on $\operatorname{MaxSpec}(A)$ is free. Then the map*

$$\mathcal{E}(\mathfrak{q}, A)^\Gamma \to \mathcal{R}(\mathfrak{q}, A)^\Gamma, \quad \Psi \mapsto \widehat{\mathrm{ev}}_\Psi^\Gamma, \tag{2.5.2}$$

is a bijection, where $\mathcal{R}(\mathfrak{q}, A)^\Gamma$ is the set of isomorphism classes of irreducible finite-dimensional representations of $(\mathfrak{q} \otimes A)^\Gamma$.

Proof. The map (2.5.2) is surjective by Proposition 2.5.2, while injectivity follows from Corollary 2.4.14. □

Chapter 3

Weyl moduldes for Lie superalgebras

In this chapter we define global and local Weyl modules for Lie superalgebras $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. Under some mild assumptions, we prove universality, finite-dimensionality, and tensor product decomposition properties for these modules. These properties are analogues of those of Weyl modules in the non-super setting.

Throughout this chapter, we assume that $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$.

3.1 A good system of simple roots I

In Section 3.4, we will be particularly interested in systems of simple roots Σ satisfying the following property:

$$\text{For all } \alpha \in \Sigma_{\bar{1}}, \text{ there exists } \alpha' \in \Delta^+_{\bar{1}}(\Sigma) \text{ such that } \alpha + \alpha' \in \Delta(\Sigma). \tag{3.1.1}$$

Note that such an element $\alpha + \alpha'$ is necessarily an even root. Our next goal is to show that a system of simple roots satisfying (3.1.1) always exists.

Let Σ be a system of simple roots and suppose that $\beta \in \Sigma$ is an odd root with $\beta(H_\beta) = 0$. (Such a root is known as an *isotropic* odd root.) Then define the *reflection* $r_\beta \colon \Sigma \to \Delta$ with respect to β by

$$\begin{aligned} r_\beta(\beta) &= -\beta, \\ r_\beta(\beta') &= \beta', \quad \text{for } \beta' \in \Sigma,\ \beta' \neq \beta,\ \beta(H_{\beta'}) = \beta'(H_\beta) = 0, \\ r_\beta(\beta') &= \beta + \beta', \quad \text{for } \beta' \in \Sigma,\ \beta' \neq \beta,\ \beta(H_{\beta'}) \neq 0 \text{ or } \beta'(H_\beta) \neq 0. \end{aligned}$$

By [CW12, Lem. 1.30], $r_\beta(\Sigma)$ is a system of simple roots, and

$$\Delta^+(r_\beta(\Sigma)) \setminus \{-\beta\} = \Delta^+(\Sigma) \setminus \{\beta\}. \tag{3.1.2}$$

(We use here the fact that $\mathfrak{gl}(n,n)$ and $\mathfrak{sl}(n,n)$ have the same system of simple roots as $A(n,n)$.)

If $\mathfrak{g}$ is a basic Lie superalgebra, $\mathfrak{gl}(n,n)$, $n \geq 2$, or $\mathfrak{sl}(n,n)$, $n \geq 2$, then $\mathfrak{g}$ admits a system of simple roots with only one odd root (see [Mus12, Tables 3.4.4 and 5.3.1]). Let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ denote such a system and let γ_s be the unique odd root that lies in Π. The system Π is often called a *distinguished* system of simple roots. When $\mathfrak{g} \neq B(0,n)$, we have that γ_s is an odd isotropic regular root. Then we can consider the odd reflection r_{γ_s} with respect to γ_s.

Proposition 3.1.1. *Let* Π *be a distinguished system of simple roots for* $\mathfrak{g}$.

a. *If* $\mathfrak{g}$ *is a basic Lie superalgebra of type II, then* Π *satisfies condition* (3.1.1).

b. *If* $\mathfrak{g}$ *is* $\mathfrak{gl}(n,n)$, $n \geq 2$, $\mathfrak{sl}(n,n)$, $n \geq 2$, *or a basic Lie superalgebra other than* $B(0,n)$, *then* $r_{\gamma_s}(\Pi)$ *satisfies condition* (3.1.1).

In particular, $\mathfrak{g}$ *admits at least one system of simple roots satisfying* (3.1.1).

Proof. Part (a) follows from direct examination of the distinguished root systems in type II. We list below (using the notation of Section 3.6) this simple odd root $\gamma_s \in \Pi_{\bar{1}}$, together with an element $\gamma' \in \Delta_{\bar{1}}^+$ such that $\gamma + \gamma' \in \Delta$.

- $\mathfrak{g} = B(m,n)$, $m \geq 0$, $n \geq 1$; $\gamma_s = \alpha_n$; $\gamma' = \alpha_n + 2\alpha_{n+1} + \cdots + 2\alpha_{n+m}$
- $\mathfrak{g} = D(m,n)$, $m \geq 2$, $n \geq 1$; $\gamma_s = \alpha_n$; $\gamma' = \alpha_n + 2\alpha_{n+1} + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m}$
- $\mathfrak{g} = F(4)$; $\gamma_s = \alpha_1$; $\gamma' = \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$
- $\mathfrak{g} = G(3)$; $\gamma_s = \alpha_1$; $\gamma' = \alpha_1 + 4\alpha_2 + 2\alpha_3$
- $\mathfrak{g} = D(2,1;\alpha)$, $\alpha \neq 0, -1$; $\gamma_s = \alpha_1$; $\gamma' = \alpha_1 + \alpha_2 + \alpha_3$

One sees that, in each case, $\gamma_s + \gamma' \in \Delta$.

Now suppose that γ_s is isotropic and let $\Pi' = r_{\gamma_s}(\Pi)$. To prove part (b), we will show that $\alpha + r_{\gamma_s}(\gamma_s) \in \Delta_{\bar{0}}^+(\Pi')$, for all odd roots $\alpha \in \Pi' \setminus \{r_{\gamma_s}(\gamma_s)\}$. First assume that $\mathfrak{g}$ is $\mathfrak{gl}(n,n)$ ($n \geq 2$), $\mathfrak{sl}(n,n)$ ($n \geq 2$), or a basic Lie superalgebra other than $B(0,n)$ or $D(2,1;\alpha)$. One can verify, by looking at each distinguished Cartan matrix, that $\gamma_s(H_{\gamma_{s\pm1}}) = -1$ (when $1 \leq s \pm 1 \leq n$) and $\gamma_s(H_{\gamma_{s\pm j}}) = 0$ when $j \geq 2$ (and $1 \leq s \pm j \leq n$). (See Section 3.6. The odd root γ_s is indicated there by an X on the corresponding node in the Dynkin diagram.) Thus

$$r_{\gamma_s}(\gamma_s) = -\gamma_s, \quad r_{\gamma_s}(\gamma_{s\pm1}) = \gamma_s + \gamma_{s\pm1} \text{ and } r_{\gamma_s}(\gamma_{s\pm j}) = \gamma_{s\pm j}, \text{ for all } j \geq 2.$$

Since the only odd root in Π is γ_s, the odd roots of Π' are precisely $r_{\gamma_s}(\gamma_{s-1}), r_{\gamma_s}(\gamma_s), r_{\gamma_s}(\gamma_{s+1})$. Now, by (3.1.2), we have $\Delta^+(\Pi) \setminus \{\gamma_s\} = \Delta^+(\Pi') \setminus \{r_{\gamma_s}(\gamma_s)\}$, which implies that $\Delta_{\bar{0}}^+(\Pi) = \Delta_{\bar{0}}^+(\Pi')$. Thus

$$r_{\gamma_s}(\gamma_{s\pm1}) + r_{\gamma_s}(\gamma_s) = \gamma_{s\pm1} \in \Delta_{\bar{0}}^+(\Pi) = \Delta_{\bar{0}}^+(\Pi').$$

Finally, assume $\mathfrak{g} = D(2,1;\alpha)$. Then $\Pi = \{\gamma_1, \gamma_2, \gamma_3\}$, where $s = 1$ and $\gamma_1(H_{\gamma_j}) = -1$, for $j = 2,3$ (see Section 3.6). Then every element of $\Pi' = \{r_{\gamma_1}(\gamma_1), r_{\gamma_1}(\gamma_2), r_{\gamma_1}(\gamma_3)\}$ is odd, and again $r_{\gamma_1}(\gamma_j) + r_{\gamma_1}(\gamma_1) = \gamma_j \in \Delta_{\bar{0}}^+(\Pi) = \Delta_{\bar{0}}^+(\Pi')$, for $j = 2,3$. $\square$

Remark 3.1.2. There exist systems of simple roots that do not satisfy (3.1.1). For instance, if $\mathfrak{g}$ is of type I, then a distinguished system Π does not satisfy (3.1.1). This follows from the fact that the induced $\mathbb{Z}$-grading is of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (see [Kac78, Prop. 1.6]).

3.2 Generalized Kac modules

Recall that $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$, and fix a system of simple roots Σ. Define

$$\Delta_z^+ = \Delta_z^+(\Sigma) \text{ for all } z \in \mathbb{Z}_2,$$

and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition induced by Σ, i.e. $\mathfrak{n}^\pm = \mathfrak{n}^\pm(\Sigma)$. In the case that $\mathfrak{g}$ is $\mathfrak{sl}(n,n)$ or $A(n,n)$, we consider the triangular decomposition induced by $\mathfrak{gl}(n,n)$. Recall that the elements X_α, Y_α, $\alpha \in \Sigma$, generate the subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively.

Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, for each even root α we can choose elements $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in \mathfrak{h}$, such that the subalgebra generated by these elements is isomorphic to $\mathfrak{sl}(2)$, with these elements satisfying the relations for the standard Chevalley generators. In this case, we say the set $\{X_\alpha, Y_\alpha, H_\alpha\}$ is an $\mathfrak{sl}(2)$-triple.

We denote the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$ by $V(\lambda)$. Define

$$\Lambda^+ = \Lambda^+(\Sigma) = \{\lambda \in \mathfrak{h}^* \mid \dim V(\lambda) < \infty\}. \tag{3.2.1}$$

Note that, for $\lambda \in \Lambda^+$, since $V(\lambda)$ is finite dimensional, we have $\lambda(H_\alpha) \in \mathbb{N}$, for all $\alpha \in \Sigma(\mathfrak{g}_{\bar{0}})$.

Definition 3.2.1 (The module $\bar{V}(\lambda)$)**.** For $\lambda \in \Lambda^+$, we define $\bar{V}(\lambda)$ to be the $\mathfrak{g}$-module generated by a vector v_λ with defining relations

$$\mathfrak{n}^+ v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda, \quad Y_\alpha^{\lambda(H_\alpha)+1} v_\lambda = 0, \quad \text{for all } h \in \mathfrak{h},\ \alpha \in \Sigma(\mathfrak{g}_{\bar{0}}). \tag{3.2.2}$$

Proposition 3.2.2. *For all $\lambda \in \Lambda^+$, the module $\bar{V}(\lambda)$ is finite-dimensional.*

Proof. Let $L(\lambda)$ be the irreducible $\mathfrak{g}_{\bar{0}}$-module of highest weight λ. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\lambda(H_\alpha) \in \mathbb{N}$, for all $\alpha \in \Sigma(\mathfrak{g}_{\bar{0}})$, we have that $L(\lambda)$ is finite dimensional. Moreover, it is well known that $L(\lambda)$ is isomorphic to the $\mathfrak{g}_{\bar{0}}$-module generated by a vector u_λ with defining relations

$$\mathfrak{n}_{\bar{0}}^+ u_\lambda = 0, \quad h u_\lambda = \lambda(h) u_\lambda, \quad Y_\alpha^{\lambda(H_\alpha)+1} u_\lambda = 0, \quad \text{for all } h \in \mathfrak{h},\ \alpha \in \Sigma(\mathfrak{g}_{\bar{0}}).$$

Let $V' = U(\mathfrak{g}_{\bar{0}}) v_\lambda \subseteq \bar{V}(\lambda)$ be the $\mathfrak{g}_{\bar{0}}$-submodule of $\bar{V}(\lambda)$ generated by v_λ. Then the map given by

$$\varphi\colon L(\lambda) \to V', \quad x u_\lambda \mapsto x v_\lambda, \quad \text{for all } x \in U(\mathfrak{g}_{\bar{0}}),$$

is a well-defined epimorphism of $\mathfrak{g}_{\bar{0}}$-modules. Thus, V' is finite dimensional. Then it follows from the PBW Theorem for Lie superalgebras (see Lemma 1.5.3) that $\bar{V}(\lambda)$ is finite dimensional. □

Lemma 3.2.3. *Suppose V is a finite-dimensional $\mathfrak{g}$-module generated by a highest weight vector of weight $\lambda \in \Lambda^+$. Then there exists an unique submodule W of $\bar{V}(\lambda)$ such that $\bar{V}(\lambda)/W \cong V$ as $\mathfrak{g}$-modules.*

Proof. Let $v \in V_\lambda$ be a highest weight vector. Then the first two relations in (3.2.2) are satisfied by v, by the definition of a highest weight vector. The fact that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and V is finite dimensional implies that v also satisfies the last relation in (3.2.2). Thus the map $\bar{V}(\lambda) \to V$ defined by extending the assignment $v_\lambda \mapsto v$ is a well-defined epimorphism of $\mathfrak{g}$-modules. Since $\dim V_\lambda = 1 = \dim \bar{V}(\lambda)_\lambda$ and homomorphisms between modules preserve weight spaces, this map is unique up to scalar multiple. Thus, the kernel W of this map is unique. □

Since every irreducible finite-dimensional $\mathfrak{g}$-module is generated by a highest weight vector of weight $\lambda \in \Lambda^+$, Lemma 3.2.3 applies to irreducible finite-dimensional $\mathfrak{g}$-modules.

Remark 3.2.4. It follows from Lemma 3.2.3 that $\bar{V}(\lambda)$ coincides with the *generalized Kac module* defined in [Cou, p. 8]. Thus, when Σ is a distinguished root system, it follows from [Cou, Lem. 3.5] that $\bar{V}(\lambda)$ is isomorphic to the usual Kac module defined in [Kac78, p. 613].

3.3 Global Weyl modules

Let $\mathfrak{g}$ be either a basic classical Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$, and let A be an associative commutative unital $\mathbb{C}$-algebra. We can then consider the map Lie superalgebra $\mathfrak{g} \otimes_{\mathbb{C}} A$ (see Section 1.6 for details). From now on, we consider $\mathfrak{g} \subseteq \mathfrak{g} \otimes A$ as a subalgebra via the natural isomorphism $\mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{C}$.

Definition 3.3.1 (The category $\mathcal{I}_A(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$)**.** Let $\mathcal{I}$ be the full subcategory of the category of $\mathfrak{g}_{\bar{0}}$-modules whose objects are those modules that are isomorphic to direct sums of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules. Note that, if $V \in \mathcal{I}$, then every element of V lies in a finite-dimensional $\mathfrak{g}_{\bar{0}}$-submodule of V. Let $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ denote the full subcategory of the category of $\mathfrak{g} \otimes A$-modules whose objects are the $\mathfrak{g} \otimes A$-modules whose restriction to $\mathfrak{g}_{\bar{0}}$ lies in $\mathcal{I}$.

If V is a $\mathfrak{g}$-module, then, by the PBW Theorem, we have an isomorphism of vector spaces

$$P_A(V) := U(\mathfrak{g} \otimes A) \otimes_{U(\mathfrak{g})} V \cong U(\mathfrak{g} \otimes A_+) \otimes_{\mathbb{C}} V, \tag{3.3.1}$$

where A_+ is a vector space complement to $\mathbb{C} \subseteq A$. We will view V as a $\mathfrak{g}$-submodule of $P_A(V)$ via the natural identification $V \cong \mathbb{C} \otimes V \subseteq P_A(V)$.

Lemma 3.3.2. *If V is a direct sum of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules, then so is the tensor algebra $T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$.*

Proof. This follows from the fact that the action of $\mathfrak{g}_{\bar{0}}$ preserves each summand $V^{\otimes n}$, which clearly has the given property. ☐

Lemma 3.3.3. *Let V be a $\mathfrak{g}$-module whose restriction to $\mathfrak{g}_{\bar{0}}$ lies in $\mathcal{I}$. Then $P_A(V) \in \mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$.*

Proof. Consider the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g} \otimes A$ given by the restriction of the adjoint action on the first factor. Since $\mathfrak{g}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module, it follows that $\mathfrak{g} \otimes A$ is a direct sum of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules. Then, by Lemma 3.3.2, we have that $T(\mathfrak{g} \otimes A)$, and hence $U(\mathfrak{g} \otimes A)$, are direct sums of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules. Since the tensor product is distributive over direct sums, $U(\mathfrak{g} \otimes A) \otimes_{\mathbb{C}} V$ is a direct sum of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules, hence so is its quotient $P_A(V)$. Thus $P_A(V) \in \operatorname{Ob} \mathcal{I}_A$. ☐

Proposition 3.3.4. *If $\lambda \in \Lambda^+$, then $P_A(\bar{V}(\lambda))$ is generated, as a $U(\mathfrak{g} \otimes A)$-module, by the element v_λ, with defining relations*

$$\mathfrak{n}^+ v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda, \quad Y_\alpha^{\lambda(H_\alpha)+1} v_\lambda = 0, \quad \textit{for all } h \in \mathfrak{h},\ \alpha \in \Sigma(\mathfrak{g}_{\bar{0}}). \tag{3.3.2}$$

Proof. It is obvious that the element $v_\lambda \in P_A(\bar{V}(\lambda))$ satisfies the relations (3.3.2). To check that these are all the relations, let W be the $\mathfrak{g} \otimes A$-module generated by a vector w with defining relations (3.3.2). Then we have a surjective homomorphism of $\mathfrak{g} \otimes A$-modules $\pi_1 \colon W \to P(\bar{V}(\lambda))$ which maps w to v_λ. Now, by relations (3.3.2), $w \in W$ generates a $\mathfrak{g}$-submodule of W isomorphic to $\bar{V}(\lambda)$. Thus, we have an epimorphism

$$\pi_2 : P(\bar{V}(\lambda)) \to W, \quad u_1 \otimes_{U(\mathfrak{g})} u_2 v_\lambda \mapsto u_1 u_2 w, \quad u_1 \in U(\mathfrak{g} \otimes A), u_2 \in U(\mathfrak{g}).$$

Since $\pi_1 = \pi_2^{-1}$, we have $W \cong P(\bar{V}(\lambda))$. □

For $\nu \in \Lambda^+$ and $V \in \mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$, we let V^ν be the unique maximal $\mathfrak{g} \otimes A$-module quotient of V such that the weights of V^ν lie in $\nu - Q^+$, where $Q^+ = \sum_{\alpha \in \Sigma} \mathbb{N}\alpha$ is the positive root lattice of $\mathfrak{g}$. In other words,

$$V^\nu = V / \sum_{\mu \notin \nu - Q^+} U(\mathfrak{g} \otimes A) V_\mu.$$

Note that a morphism $\varphi \colon V \to W$ of objects in $\mathcal{I}_A(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ induces a morphism $\varphi^\nu \colon V^\nu \to W^\nu$.

Definition 3.3.5 (The category $\mathcal{I}_A^\nu(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$)**.** Let $\mathcal{I}_A^\nu(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ be the full subcategory of $\mathcal{I}_A(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ whose objects are those $V \in \mathcal{I}_A(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ such that $V^\nu = V$.

Proposition 3.2.2 and Lemma 3.3.3 imply that $P_A(\bar{V}(\lambda)) \in \mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ for all $\lambda \in \Lambda^+$.

Definition 3.3.6 (Global Weyl module)**.** We define the *global Weyl module* associated to $\lambda \in \Lambda^+$ to be

$$W(\lambda) := P_A(\bar{V}(\lambda))^\lambda.$$

We let w_λ denote the image of v_λ in $W(\lambda)$.

Proposition 3.3.7. *For $\lambda \in \Lambda^+$, the global Weyl module $W(\lambda)$ is generated by w_λ, with defining relations*

$$(\mathfrak{n}^+ \otimes A) w_\lambda = 0, \quad h w_\lambda = \lambda(h) w_\lambda, \quad Y_\alpha^{\lambda(H_\alpha)+1} w_\lambda = 0, \quad \textit{for all } h \in \mathfrak{h},\ \alpha \in \Sigma(\mathfrak{g}_{\bar{0}}). \tag{3.3.3}$$

Proof. Since the weights of $W(\lambda)$ lie in $\lambda - Q^+$, it follows that $(\mathfrak{n}^+ \otimes A) w_\lambda = 0$. The remaining relations are clear since they are already satisfied by v_λ. To prove that these are the only relations, let W be the module generated by an element w with relations (3.3.3), so that we have an epimorphism $\pi_1 \colon W \twoheadrightarrow W(\lambda)$ sending w to w_λ. Since the relations (3.3.3) imply the relations (3.2.2), the vector $w \in W$ generates a $\mathfrak{g}$-submodule of W isomorphic to a quotient of $\bar{V}(\lambda)$. Thus we have a surjective homomorphism

$$\pi_2 \colon P_A(\bar{V}(\lambda)) \to W, \quad u_1 \otimes_{U(\mathfrak{g})} u_2 v_\lambda \mapsto u_1 u_2 w, \quad u_1 \in U(\mathfrak{g} \otimes A),\ u_2 \in U(\mathfrak{g}).$$

Since the $\mathfrak{g}$-weights of W are bounded above by λ, it follows that π_2 induces a map $W(\lambda) \to W$ inverse to π_1. □

In the non-super setting, Proposition 3.3.7 was proved in [CFK10, Prop. 4].

Proposition 3.3.8. *The global Weyl module $W(\lambda)$ is the unique object of $\mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$, up to isomorphism, that is generated by a highest weight vector of weight λ and admits a surjective homomorphism to any object of $\mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$ also generated by a highest weight vector of weight λ.*

Proof. Let $V \in \mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$ be generated by a highest weight vector v of weight λ. Then

$$(\mathfrak{n}^+ \otimes A)v = 0, \quad hv = \lambda(h)v, \quad \text{for all } h \in \mathfrak{h}.$$

Since the $\mathfrak{g}_{\bar{0}}$-module generated by v is finite-dimensional, we have that $Y_\alpha^{\lambda(H_\alpha)+1}v = 0$ for all $\alpha \in \Sigma(\mathfrak{g}_{\bar{0}})$. Thus, by Proposition 3.3.7, we have a surjective homomorphism $W(\lambda) \twoheadrightarrow V$ such that $w_\lambda \mapsto v$.

Suppose that W is another object of $\mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$ that is generated by a highest weight vector w of weight λ and admits a surjective homomorphism to any object of $\mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$ also generated by a highest weight vector of weight λ. In particular, we have a surjective homomorphism $\pi_1 \colon W \twoheadrightarrow W(\lambda)$. It follows from the PBW Theorem that $W(\lambda)_\lambda = U(\mathfrak{h}\otimes A_+)\otimes_{\mathbb{C}} w_\lambda$. The only elements of this weight space that generate $W(\lambda)$ are the $\mathbb{C}$-multiples of w_λ. Thus, possibly after rescaling, we have $\pi_1(w) = w_\lambda$. Now, as above, w satisfies the relations (3.3.3). Thus there exists a homomorphism $\pi_2 \colon W(\lambda) \to W$ sending w_λ to w. It follows that π_1 and π_2 are mutually inverse homomorphisms, and so $W \cong W(\lambda)$. □

Note that, when $A = \mathbb{C}$, the global Weyl module $W(\lambda)$ coincides with the generalized Kac module $\bar{V}(\lambda)$. In this case, Proposition 3.3.8 reduces to the universal property given in Lemma 3.2.3.

3.4 Local Weyl modules

Recall that $\mathfrak{g}$ is either a basic classical Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$, and that A is an associative commutative unital $\mathbb{C}$-algebra. The aim now is to describe, in terms of generators and relations, a universal object in the full subcategory of $\mathcal{I}(\mathfrak{g}\otimes A, \mathfrak{g}_{\bar{0}})$ whose objects are the finite-dimensional modules generated by a highest map-weight vector of a fixed highest map-weight (see Definition 3.4.2).

Definition 3.4.1 (Local Weyl module)**.** Let $\psi \in (\mathfrak{h}\otimes A)^*$ such that $\psi|_{\mathfrak{h}} \in \Lambda^+$. We define the *local Weyl module* $W(\psi)$ associated to ψ to be the $\mathfrak{g}\otimes A$-module generated by a vector w_ψ with defining relations

$$(\mathfrak{n}^+ \otimes A)w_\psi = 0, \quad xw_\psi = \psi(x)w_\psi, \quad Y_\alpha^{\lambda(H_\alpha)+1}w_\psi = 0, \quad \text{for all } x \in \mathfrak{h}\otimes A,\ \alpha \in \Sigma(\mathfrak{g}_{\bar{0}}). \tag{3.4.1}$$

Definition 3.4.2 (Highest map-weight module)**.** A $\mathfrak{g}\otimes A$-module generated by a vector w_ψ satisfying the first and second relations of (3.4.1) is called a *highest map-weight module* with *highest map-weight* ψ. The vector w_ψ is called a *highest map-weight vector* of *map-weight* ψ.

Recall that, for each $\alpha \in \Delta_{\bar{0}}^+$, we have an $\mathfrak{sl}(2)$-triple $\{X_\alpha, Y_\alpha, H_\alpha\}$.

Lemma 3.4.3. *Suppose $\psi \in (\mathfrak{h}\otimes A)^*$ such that $\lambda = \psi|_{\mathfrak{h}} \in \Lambda^+$. If $\alpha \in \Delta_{\bar{0}}^+$, then $Y_\alpha^{\lambda(H_\alpha)+1}w_\psi = 0$.*

Proof. The vector $Y_\alpha^{\lambda(H_\alpha)+1}w_\psi$ has weight $\lambda - (\lambda(H_\alpha)+1)\alpha$. On the other hand, it follows from Proposition 3.3.7 that $W(\psi)$ is a quotient of the global Weyl module $W(\lambda)$, and so it is a direct sum of irreducible finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules. This implies that the weights of $W(\lambda)$ are invariant under the action of the Weyl group of $\mathfrak{g}_{\bar{0}}$. But, if s_α denotes the reflection associated to the root α, then $s_\alpha(\lambda - (\lambda(H_\alpha)+1)\alpha) = \lambda + \alpha$ does not lie below λ. Therefore, $Y_\alpha^{\lambda(H_\alpha)+1}w_\psi = 0$. □

Let u be an indeterminate and, for $a \in A$, $\alpha \in \Delta_{\bar{0}}^+$, define the following power series with coefficients in $U(\mathfrak{h} \otimes A)$:

$$p(a,\alpha) = \exp\left(-\sum_{i=1}^{\infty} \frac{H_\alpha \otimes a^i}{i} u^i\right). \tag{3.4.2}$$

For $i \in \mathbb{N}$, let $p(a,\alpha)_i$ denote the coefficient of u^i in $p(a,\alpha)$. In particular, $p(a,\alpha)_0 = 1$.

Lemma 3.4.4. *Suppose $m \in \mathbb{N}$, $a \in A$, and $\alpha \in \Delta_{\bar{0}}^+$. Then*

$$(X_\alpha \otimes a)^m (Y_\alpha \otimes 1)^{m+1} - (-1)^m \sum_{i=0}^{m} (Y_\alpha \otimes a^{m-i}) p(a,\alpha)_i \in U(\mathfrak{g} \otimes A)(\mathfrak{n}^+ \otimes A). \tag{3.4.3}$$

Proof. This formula was proved in [Gar78, Lem. 7.5] for the algebra generated by the elements

$$\begin{aligned} t^j \otimes E, &\quad j \geq 0, \\ t^j \otimes F, &\quad j \geq 1, \\ t^j \otimes H, &\quad j \geq 1, \end{aligned}$$

where $t \in \mathbb{C}[t]$ and the set $\{E,\ F,\ H\}$ is an $\mathfrak{sl}(2)$-triple. Now, applying the Lie algebra homomorphism

$$\mathfrak{sl}(2) \otimes \mathbb{C}[t] \to \mathfrak{sl}(2) \otimes A, \quad x \otimes t^m \mapsto x \otimes a^m, \quad m \in \mathbb{N},\ x \in \mathfrak{sl}(2),$$

gives our result. □

Remark 3.4.5. Similar relations as the one given in Lemma 3.4.4 were proved in various settings. See for instance [BC14, Cha13].

For the rest of the chapter we assume that

$$A \text{ is finitely generated.}$$

Proposition 3.4.6. *Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\lambda = \psi|_{\mathfrak{h}} \in \Lambda^+$. If $\alpha \in \Delta_{\bar{0}}^+$, $a_1, a_2, \ldots, a_t \in A$, and $m_1, \ldots, m_t \in \mathbb{N}$, then*

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi \in \operatorname{span}_{\mathbb{C}} \{(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha),\ i = 1, \ldots, t\}. \tag{3.4.4}$$

In particular, $(Y_\alpha \otimes A)w_\psi$ is finite dimensional.

Proof. From the first and third relations in (3.4.1), together with (3.4.3), it follows that, for $a \in A$ and $m \geq \lambda(H_\alpha)$, we have

$$0 = (X_\alpha \otimes a)^m (Y_\alpha \otimes 1)^{m+1} w_\psi = \sum_{i=0}^{m} (-1)^m (Y_\alpha \otimes a^{m-i}) p(a,\alpha)_i w_\psi,$$

for any $a \in A$. Since $p(a,\alpha)_0 = 1$, we have

$$(Y_\alpha \otimes a^m) w_\psi \in \operatorname{span}_{\mathbb{C}}\{(Y_\alpha \otimes a^\ell) w_\psi \mid 0 \leq \ell < m\}.$$

This implies, by induction, that

$$(Y_\alpha \otimes a^m) w_\psi \in \operatorname{span}_{\mathbb{C}}\{(Y_\alpha \otimes a^\ell) w_\psi \mid 0 \leq \ell < \lambda(H_\alpha)\}, \quad \text{for all } m \in \mathbb{N},\ a \in A. \tag{3.4.5}$$

We will now prove (3.4.4) by induction on t. The case $t = 1$ follows immediately from (4.4.2). Assume that (3.4.4) holds for some $t \geq 1$. Let $m_1, \dots, m_{t+1} \in \mathbb{N}$ and choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}) w_\psi = \big(- \alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) + (Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})(h \otimes a_{t+1}^{m_{t+1}})\big) w_\psi,$$

and so

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}) w_\psi + \alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) w_\psi \in \operatorname{span}_{\mathbb{C}}\{(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}) w_\psi\}, \tag{3.4.6}$$

since $(h \otimes a_{t+1}^{m_{t+1}}) w_\psi \in \mathbb{C} w_\psi$. By the inductive hypothesis, we have

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) w_\psi \in \operatorname{span}_{\mathbb{C}} \left\{ (h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t}) w_\psi, (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t}) w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha) \right\}.$$

Then, by (4.4.3) (with $m_i = \ell_i$ for $i = 1, \dots, t$), we have

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) w_\psi \in \operatorname{span}_{\mathbb{C}} \left\{ (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t} a_{t+1}^{m_{t+1}}) w_\psi, (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t}) w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha) \right\}.$$

Since the above inclusion holds for all $m_1, \dots, m_{t+1} \in \mathbb{N}$, we can interchange the roles of m_1 and m_{t+1} to obtain

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) w_\psi \in \operatorname{span}_{\mathbb{C}} \left\{ (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t} a_{t+1}^{\ell_{t+1}}) w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha) \right\}.$$

This completes the proof of the inductive step. The final statement of the lemma follows from the fact that A is finitely generated. □

Let

$$\mathcal{L}(\mathfrak{h} \otimes A) = \{\psi \in (\mathfrak{h} \otimes A)^* \mid \psi(\mathfrak{h} \otimes I) = 0, \text{ for some finite-codimensional ideal } I \text{ of } A\}.$$

Proposition 3.4.7. *Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\lambda = \psi|_{\mathfrak{h}} \in \Lambda^+$. If $\psi \notin \mathcal{L}(\mathfrak{h} \otimes A)$, then $W(\psi) = 0$.*

Proof. Let $\alpha \in \Delta_{\bar{0}}^{+}$ and let I_α be the kernel of the linear map

$$A \to \mathrm{Hom}_{\mathbb{C}}\left(W(\psi)_\lambda \otimes \mathfrak{g}_{-\alpha}, (\mathfrak{g}_{-\alpha} \otimes A)w_\psi\right),$$
$$a \mapsto (v \otimes u \mapsto (u \otimes a)v), \quad a \in A,\ v \in W(\psi)_\lambda,\ u \in \mathfrak{g}_{-\alpha}.$$

Since $\mathfrak{g}_{-\alpha} = \mathbb{C}Y_\alpha$, Proposition 3.4.6 implies that $(\mathfrak{g}_{-\alpha} \otimes A)w_\psi$ is finite dimensional. Thus, I_α is a linear subspace of A of finite codimension. We claim that I_α is, in fact, an ideal of A. Indeed, since $\alpha \neq 0$, we can choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then, for all $g \in A$, $a \in I_\alpha$, $v \in W(\psi)_\lambda$, and $u \in \mathfrak{g}_{-\alpha}$, we have

$$0 = (h \otimes g)(u \otimes a)v = [h \otimes g, u \otimes a]v + (u \otimes a)(h \otimes g)v = -\alpha(h)(u \otimes ga)v + (u \otimes a)(h \otimes g)v.$$

Since $(h \otimes g)v \in W(\psi)_\lambda$ and $a \in I_\alpha$, the last term above is zero. Since we also have $\alpha(h) \neq 0$, this implies that $(u \otimes ga)v = 0$. As this holds for all $v \in W(\psi)_\lambda$ and $u \in \mathfrak{g}_{-\alpha}$, we have $ga \in I_\alpha$. Hence I_α is an ideal of A.

Let I be the intersection of all the I_α, $\alpha \in \Delta_{\bar{0}}^{+}$. Since $\mathfrak{g}$ has a finite number of positive roots, this intersection is finite, and thus I is also an ideal of A of finite-codimension. We have

$$(\mathfrak{n}_{\bar{0}}^{-} \otimes I)W(\psi)_\lambda = 0 \quad \text{and} \quad (\mathfrak{n}^{+} \otimes A)W(\psi)_\lambda = 0.$$

Then, since $\mathfrak{h} \otimes I \subseteq [\mathfrak{n}^{+} \otimes A, \mathfrak{n}_{\bar{0}}^{-} \otimes I]$, we have $(\mathfrak{h} \otimes I)W(\psi)_\lambda = 0$. In particular, $(\mathfrak{h} \otimes I)w_\psi = 0$.

Assume $\psi \notin \mathcal{L}(\mathfrak{h} \otimes A)$. Then there exists $a \in I$ such that $\psi(h \otimes a) \neq 0$ for some $h \in \mathfrak{h}$, which implies that $w_\psi = 0$, since

$$0 = (h \otimes a)w_\psi = \psi(h \otimes a)w_\psi.$$

Therefore $W(\psi) = 0$. □

Definition 3.4.8 (The ideal I_ψ)**.** For $\psi \in (\mathfrak{h} \otimes A)^*$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$, let I_ψ be the sum of all ideals $I \subseteq A$ such that $(\mathfrak{h} \otimes I)w_\psi = 0$.

Remark 3.4.9. It follows from the proof of Proposition 3.4.7 that I_ψ has finite codimension in A and that $(Y_\alpha \otimes I_\psi)w_\psi = 0$ for all $\alpha \in \Delta_{\bar{0}}^{+}$. Furthermore, by Lemma 1.2.2, parts (a) and (c), since I_ψ has finite codimension and A is finitely generated, we have that I_ψ^N has finite codimension, for all $N \in \mathbb{N}$.

For the rest of the chapter, we assume that

$$\Sigma \text{ is a system of simple roots for } \mathfrak{g} \text{ satisfying (3.1.1).}$$

Recall that, by Proposition 3.1.1, such a system always exists.

Lemma 3.4.10. *Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$. Then there exists $N_\psi \in \mathbb{N}$ such that*

$$\left(\mathfrak{n}^{-} \otimes I_\psi^{N_\psi}\right) w_\psi = 0.$$

Proof. Recall from Section 1.4.1 that the set $\{Y_\alpha \mid \alpha \in \Sigma\}$ generates $\mathfrak{n}^-$. We claim that

$$(Y_\alpha \otimes I_\psi)w_\psi = 0, \quad \text{for all } \alpha \in \Sigma. \tag{3.4.7}$$

By Remark 3.4.9, it suffices to consider the case $\alpha \in \Sigma_{\bar{1}}$. Fix such an α. By (3.1.1), there exists $\alpha' \in \Delta_{\bar{1}}$ such that $\beta := \alpha + \alpha' \in \Delta_{\bar{0}}^+$.

First suppose $\mathfrak{g}$ is not $A(1,1)$ or $\mathfrak{sl}(2,2)$. Then $\dim \mathfrak{g}_\nu = 1$ for any $\nu \in \Delta$ (see Remark 1.4.5). Thus, rescaling if necessary,

$$[X_{\alpha'}, Y_\beta] = Y_\alpha. \tag{3.4.8}$$

Then,

$$(Y_\alpha \otimes I_\psi)w_\psi = [X_{\alpha'} \otimes A, Y_\beta \otimes I_\psi]w_\psi \subseteq (X_{\alpha'} \otimes A)(Y_\beta \otimes I_\psi)w_\psi + (Y_\beta \otimes I_\psi)(X_{\alpha'} \otimes A)w_\psi = 0,$$

where the last equality follows from the fact that $(Y_\beta \otimes I_\psi)w_\psi = 0$ by Remark 3.4.9 and $(X_{\alpha'} \otimes A)w_\psi = 0$ by the first relation in (3.4.1). This proves (3.4.7).

To prove (3.4.7) for $\mathfrak{sl}(2,2)$ and $A(1,1)$, we consider $\mathfrak{g} = \mathfrak{gl}(2,2)$ and we let $\mathfrak{h}$ be the subalgebra of diagonal matrices of $\mathfrak{g}$. Denote by $\{\epsilon_i \mid i = 1, \dots, 4\}$ the basis of $\mathfrak{h}^*$ dual to $\{E_{i,i} \mid i = 1, \dots, 4\}$. In this case,

$$\Delta_{\bar{0}} = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_3 - \epsilon_4)\}, \quad \Delta_{\bar{1}} = \{\pm(\epsilon_1 - \epsilon_3), \pm(\epsilon_1 - \epsilon_4), \pm(\epsilon_2 - \epsilon_3), \pm(\epsilon_2 - \epsilon_4)\},$$

and $\mathfrak{g}_{\epsilon_r - \epsilon_s} = \mathbb{C}E_{r,s}$, for $1 \le r \ne s \le 4$. In particular, if we fix $\alpha \in \Sigma_{\bar{1}}$ and $\alpha' \in \Delta_{\bar{1}}^+$ such that $\beta := \alpha + \alpha' \in \Delta$, then there exist $k, \ell, p, q \in \{1,2,3,4\}$ with $k \ne \ell$ and $p \ne q$, such that $\mathfrak{g}_{\alpha'} = \mathbb{C}E_{k,\ell}$ and $\mathfrak{g}_{-\beta} = \mathbb{C}E_{p,q}$. Since $\beta \in \Delta_{\bar{0}}^+$ and $\alpha' \in \Delta_{\bar{1}}^+$, Remark 3.4.9 and the first relation in (3.4.1) give us that $([E_{k,\ell}, E_{p,q}] \otimes I_\psi)w_\psi = 0$. Regarding the $\mathfrak{sl}(2,2)$ case, we choose $Y_\alpha = [E_{k,\ell}, E_{p,q}]$. For the $A(1,1)$ case, we choose Y_α to be the image of $[E_{k,\ell}, E_{p,q}]$ in $A(1,1)$. Then $(Y_\alpha \otimes I_\psi)w_\psi = 0$. Since the choice of $\alpha \in \Sigma_{\bar{1}}$ was arbitrary, we conclude that $(Y_\alpha \otimes I_\psi)w_\psi = 0$ for all roots $\alpha \in \Sigma_{\bar{1}}$.

Now, for $\beta = \sum_{\alpha\in\Sigma}^n m_\alpha \alpha \in \Delta^+$, we define the *height* of β to be $\operatorname{ht}\beta := \sum_{\alpha\in\Sigma}^n m_\alpha$. We prove, by induction on the height of β, that $(Y_\beta \otimes I_\psi^{\operatorname{ht}\beta})w_\psi = 0$ for all $\beta \in \Delta^+$. Since $\mathfrak{g}$ is finite dimensional, the heights of elements of Δ^+ are bounded above, and thus the lemma will follow.

The base case of height one is precisely (3.4.7). Suppose $\beta \in \Delta^+$ with $\operatorname{ht}\beta > 1$. Then there exist $\beta', \beta'' \in \Delta^+$ with $\operatorname{ht}\beta', \operatorname{ht}\beta'' < \operatorname{ht}\beta$ such that $Y_\beta \in \mathbb{C}[Y_{\beta'}, Y_{\beta''}]$. Then

$$(Y_\beta \otimes I_\psi^{\operatorname{ht}\beta})w_\psi = [Y_{\beta'} \otimes I_\psi^{\operatorname{ht}\beta'}, Y_{\beta''} \otimes I_\psi^{\operatorname{ht}\beta''}]w_\psi = 0. \qquad \square$$

Corollary 3.4.11. *Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$, and let N_ψ be as in Lemma 4.4.3. Then*

$$\left(\mathfrak{g} \otimes I_\psi^{N_\psi}\right) w_\psi = 0.$$

Proof. It follows from the first relation in (3.4.1) that $(\mathfrak{n}^+ \otimes I_\psi^{N_\psi})w_\psi = 0$. Since $(\mathfrak{h} \otimes I_\psi)w_\psi = 0$ by the definition of I_ψ, we have $(\mathfrak{h} \otimes I_\psi^{N_\psi})w_\psi = 0$. Finally Lemma 4.4.3 implies that $(\mathfrak{n}^- \otimes I_\psi^{N_\psi})w_\psi = 0$. $\square$

Lemma 3.4.12. *For all $\lambda \in (\mathfrak{h} \otimes A)^*$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$, the set of $\mathfrak{g}$-weights (equivalently, $\mathfrak{g}_{\bar{0}}$-weights) of $W(\psi)$ is finite.*

Proof. Since the weights of $W(\lambda)$ are contained in $\lambda - Q^+$, finitely many of weights of $W(\lambda)$ are dominant integral. Since $W(\lambda)$ is a direct sum of $\mathfrak{g}_{\bar{0}}$-modules, its weights are invariant under the (finite) Weyl group of $\mathfrak{g}_{\bar{0}}$. The result follows. □

Recall that A is finitely generated and the system of simple roots Σ satisfies (3.1.1)

Theorem 3.4.13. *The local Weyl module $W(\psi)$ is finite dimensional for all $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} \in \Lambda^+$.*

Proof. By Definition 3.4.1, we have $W(\psi) = U(\mathfrak{n}^- \otimes A)w_\psi$. By Lemma 4.4.3, we have $(\mathfrak{n}^- \otimes I_\psi^{N_\psi})w_\psi = 0$. Thus $W(\psi) = U(\mathfrak{n}^- \otimes A/I_\psi^{N_\psi})w_\psi$. By Lemma 3.4.12, there exists $N \in \mathbb{N}$ such that

$$W(\psi) = U_n\left(\mathfrak{n}^- \otimes A/I_\psi^{N_\psi}\right) w_\psi, \quad \text{for all } n \geq N,$$

where $U(\mathfrak{a}) = \sum_{n=0}^{\infty} U_n(\mathfrak{a})$ is the usual filtration on the universal enveloping algebra of a Lie superalgebra $\mathfrak{a}$ induced from the natural grading on the tensor algebra. Since the Lie superalgebra $\mathfrak{n}^- \otimes A/I_\psi^{N_\psi}$ is finite dimensional (see Remark 3.4.9), $W(\psi)$ is also finite dimensional. □

In the non-super setting, Theorem 3.4.13 was proved in [CP01, Th. 1] for $A = \mathbb{C}[t, t^{-1}]$, and in [FL04, Th. 1] for A the algebra of functions on a complex affine variety.

Proposition 3.4.14. *Let $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ such that $\psi|_{\mathfrak{h}} = \lambda \in \Lambda^+$. Then the local Weyl module $W(\psi)$ is the unique (up to isomorphism) finite-dimensional object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ that is generated by a highest map-weight vector of map-weight ψ and admits a surjective homomorphism to any finite-dimensional object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ also generated by a highest map-weight vector of map-weight ψ.*

Proof. Let V be a finite-dimensional object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ that is generated by a highest map-weight vector v of map-weight ψ. It follows immediately from the definition of a highest map-weight $\mathfrak{g} \otimes A$-module that the two first relations in (3.4.1) are satisfied by v. Since the $\mathfrak{g}_{\bar{0}}$-module generated by v must be finite dimensional, we have also that $Y_\alpha^{\lambda(H_\alpha)+1} v = 0$, for all $\alpha \in \Sigma(\mathfrak{g}_{\bar{0}})$. Therefore, there exists a surjective homomorphism $W(\psi) \to V$ sending w_ψ to v.

To show that $W(\psi)$ is the unique representation with the given property, suppose that W is another module with this property. Then W is a quotient of $W(\psi)$ and vice-versa. Since both modules are finite dimensional, it follows that $W(\psi) \cong W$. □

Corollary 3.4.15. *Let $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ such that $\psi|_{\mathfrak{h}} = \lambda \in \Lambda^+$. Then the local Weyl module $W(\psi)$ is the maximal finite-dimensional quotient of the global Weyl module $W(\lambda)$ that is a highest map-weight module of highest map-weight ψ.*

By [Sav14, Th. 4.16], any irreducible finite-dimensional $\mathfrak{g} \otimes A$-module is a highest map-weight module, for some $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$. Then, by Proposition 3.4.14, there exists a surjective homomorphism from the local Weyl module $W(\psi)$ to such an irreducible module. In other words, all irreducible finite-dimensional $\mathfrak{g} \otimes A$-modules are quotients of local Weyl modules.

3.5 Tensor product decomposition

We conclude this chapter by showing that the local Weyl modules possess a tensor product property analogous to the one satisfied in the non-super setting (see, [CP01, Th. 2] and [FL04, Th. 2]). Recall that we are assuming that A is finitely generated and the system of simple roots Σ satisfies (3.1.1).

Theorem 3.5.1. *For $i = 1, 2$, let $\psi_i \in \mathcal{L}(\mathfrak{h} \otimes A)$ with $\lambda_i = \psi_i|_{\mathfrak{h}} \in \Lambda^+$, and suppose that I_{ψ_1} and I_{ψ_2} have disjoint support. Then*

$$W(\psi_1 + \psi_2) \cong W(\psi_1) \otimes W(\psi_2)$$

as $\mathfrak{g} \otimes A$-modules.

Proof. By Corollary 3.4.11, there exist $N_1, N_2 \in \mathbb{N}$ such that $(\mathfrak{g} \otimes I_{\psi_i}^{N_i})w_{\psi_i} = 0$ for $i = 1, 2$. Then the action of $\mathfrak{g} \otimes A$ on $W(\psi_1) \otimes W(\psi_2)$ factors through the composition

$$\mathfrak{g} \otimes A \overset{d}{\hookrightarrow} (\mathfrak{g} \otimes A) \oplus (\mathfrak{g} \otimes A) \overset{\pi}{\twoheadrightarrow} (\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2}), \tag{3.5.1}$$

where d is the diagonal embedding. Since $\mathrm{Supp}(I_{\psi_1}) \cap \mathrm{Supp}(I_{\psi_2}) = \varnothing$, Lemma 1.2.2(a) implies that $\mathrm{Supp}(I_{\psi_1}^{N_1}) \cap \mathrm{Supp}(I_{\psi_2}^{N_2}) = \varnothing$. Then, by Lemma 1.2.2(d), we have $A = I_{\psi_1}^{N_1} + I_{\psi_2}^{N_2}$ and $I_{\psi_1}^{N_1} \cap I_{\psi_2}^{N_2} = I_{\psi_1}^{N_1} I_{\psi_2}^{N_2}$. Thus, $A/I_{\psi_1}^{N_1} I_{\psi_2}^{N_2} \cong (A/I_{\psi_1}^{N_1}) \oplus (A/I_{\psi_2}^{N_2})$. We therefore have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{g} \otimes A & \overset{d}{\hookrightarrow} & (\mathfrak{g} \otimes A) \oplus (\mathfrak{g} \otimes A) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{g} \otimes A/I_{\psi_1}^{N_1} I_{\psi_2}^{N_2} & \overset{\cong}{\longrightarrow} & (\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2})
\end{array}$$

It follows that the composition (3.5.1) is surjective.

Since $W(\psi_1) \otimes W(\psi_2)$ is generated as a $(\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2})$-module by the vector $w_{\psi_1} \otimes w_{\psi_2}$, it follows from the above that it is also generated by this vector as a $\mathfrak{g} \otimes A$-module. Moreover, $\mathfrak{h} \otimes A$ acts on $w_{\psi_1} \otimes w_{\psi_2}$ via $\psi := \psi_1 + \psi_2$. Thus $W(\psi_1) \otimes W(\psi_2)$ is a finite-dimensional highest map-weight module of highest map-weight ψ. Therefore, by Proposition 3.4.14, it is a quotient of $W(\psi)$.

To simplify notation, let $I_1 = I_{\psi_1}$, $I_2 = I_{\psi_2}$ and $N = N_\psi$. Let $I = I_1 I_2 = I_1 \cap I_2$. Then $I \subseteq I_\psi$. Therefore, the action of $\mathfrak{b} \otimes A$ on $\mathbb{C}w_\psi$ descends to an action of $\mathfrak{b} \otimes A/I^N$ on $\mathbb{C}w_\psi$. Consider the induced module

$$M(\psi) := U(\mathfrak{g} \otimes A/I^N) \otimes_{U(\mathfrak{b} \otimes A/I^N)} \mathbb{C}w_\psi.$$

It follows from Corollary 3.4.11 that $W(\psi)$ is a quotient of $M(\psi)$. On the other hand, it is clear that the one-dimensional $\mathfrak{b} \otimes A$-modules $\mathbb{C}w_\psi$ and $\mathbb{C}w_{\psi_1} \otimes \mathbb{C}w_{\psi_2}$ are isomorphic. Hence,

$$\begin{aligned}
M(\psi) &= U(\mathfrak{g} \otimes A/I^N) \otimes_{U(\mathfrak{b}\otimes A/I^N)} \mathbb{C}w_\psi \\
&\cong U\left(\mathfrak{g} \otimes \left(A/I_1^N \oplus A/I_2^N\right)\right) \otimes_{U(\mathfrak{b}\otimes(A/I_1^N\oplus A/I_2^N))} (\mathbb{C}w_{\psi_1} \otimes \mathbb{C}w_{\psi_2}) \\
&\cong \left(U\left(\mathfrak{g} \otimes \left(A/I_1^N\right)\right) \otimes U\left(\mathfrak{g} \otimes \left(A/I_2^N\right)\right)\right) \otimes_{U(\mathfrak{b}\otimes(A/I_1^N))\otimes U(\mathfrak{b}\otimes(A/I_2^N))} (\mathbb{C}w_{\psi_1} \otimes \mathbb{C}w_{\psi_2}) \\
&\cong \left(U\left(\mathfrak{g} \otimes \left(A/I_1^N\right)\right) \otimes_{U(\mathfrak{b}\otimes(A/I_1^N))} \mathbb{C}w_{\psi_1}\right) \otimes \left(U\left(\mathfrak{g} \otimes \left(A/I_2^N\right)\right) \otimes_{U(\mathfrak{b}\otimes(A/I_2^N))} \mathbb{C}w_{\psi_2}\right) \\
&= M(\psi_1) \otimes M(\psi_2).
\end{aligned}$$

So $W(\psi)$ is a quotient of $M(\psi_1) \otimes M(\psi_2)$. Fix a surjection $\theta\colon M(\psi_1) \otimes M(\psi_2) \to W(\psi)$.

We claim that the image of $M(\psi_1)_\mu \otimes M(\psi_2)_\nu$ under θ is zero except for a finite number of weights μ and ν. By Lemma 3.4.12, the set D of weights occurring in $W(\psi)$ is finite. Thus, the sets

$$D_1 = (\lambda_1 - Q^+) \cap (-\lambda_2 + D + Q^+) \quad \text{and} \quad D_2 = (\lambda_2 - Q^+) \cap (-\lambda_1 + D + Q^+)$$

are also finite. Since, for $i = 1, 2$, the weights of $M(\psi_i)$ are contained in $\lambda_i - Q^+$, the image of $M(\psi_1)_\mu \otimes M(\psi_2)_\nu$ under θ is zero unless $\mu \in \lambda_1 - Q^+$, $\nu \in \lambda_2 - Q^+$ and $\mu + \nu \in D$. Thus it is nonzero only if $\mu \in D_1$ and $\nu \in D_2$, and hence the claim is proved.

For $i = 1, 2$, let $M(\psi_i)'$ be the submodule of $M(\psi_i)$ generated by the weight subspaces $M(\psi_i)_\mu$ with $\mu \notin D_i$, and let $\bar{M}(\psi_i) = M(\psi_i)/M(\psi_i)'$. Then $W(\psi)$ is a quotient of $\bar{M}(\psi_1) \otimes \bar{M}(\psi_2)$. Because I_i has finite codimension and there are only a finite number of weights occurring in the quotient $\bar{M}(\psi_i)$, this module is a finite-dimensional highest map-weight module of highest map-weight ψ_i. Then, by Proposition 3.4.14, it is a quotient of $W(\psi_i)$. Thus, $\bar{M}(\psi_1) \otimes \bar{M}(\psi_2)$ is a quotient of $W(\psi_1) \otimes W(\psi_2)$, which implies that $W(\psi)$ is a quotient of $W(\psi_1) \otimes W(\psi_2)$. Since the modules $W(\psi)$ and $W(\psi_1) \otimes W(\psi_2)$ are both finite dimensional, the fact that one is a quotient of the other implies the isomorphism in the statement of the theorem. □

3.6 Appendix

In this appendix we describe the root systems, the distinguished system of simple roots and the distinguished Cartan matrix of all basic Lie superalgebras. See [FSS00, Tables 3.54, 3.57–3.60], for details.

Recall that if $\mathfrak{g}$ is either a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$, then a Cartan subalgebra of $\mathfrak{g}$ is the same as a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$.

3.6.1 The non-exceptional cases

In all the cases in this section, the Cartan subalgebra of $\mathfrak{g}$ is a subspace of a space of diagonal matrices. In what follows, the roots will be expressed in terms of the functionals given by

$$\varepsilon_i(\operatorname{diag}(a_1, \dots, a_N)) = a_i,$$

for any diagonal matrix.

The basic Lie superalgebra $\mathfrak{g} = A(m-1, n-1)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$. Set $\delta_j := \varepsilon_{m+j}$, for $1 \leq j \leq n$. If $1 \leq i \neq j \leq m$ and $1 \leq k \neq \ell \leq n$, then

$$\Delta = \{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_\ell,\ \varepsilon_i - \delta_k,\ \delta_k - \varepsilon_i\},$$
$$\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_\ell\}, \quad \Delta_{\bar{1}} = \{\varepsilon_i - \delta_k,\ \delta_k - \varepsilon_i\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \delta_1 - \delta_2,\ \dots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \dots,\ \alpha_{n+m-1} = \varepsilon_{m-1} - \varepsilon_m,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & & \ddots & 0 \\
 & & & & & & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{array}\right).$$

The basic Lie superalgebra $\mathfrak{g} = A(n-1, n-1)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$. Set $\delta_j := \varepsilon_{n+j}$, for $1 \leq j \leq n$. If $1 \leq i \neq j \leq n$, then

$$\Delta = \{\varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j,\ \varepsilon_i - \delta_j,\ \delta_j - \varepsilon_i\},$$
$$\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j,\ \delta_i - \delta_j\}, \quad \Delta_{\bar{1}} = \{\varepsilon_i - \delta_j,\ \delta_j - \varepsilon_i\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{2n-1} = \varepsilon_{n-1} - \varepsilon_n,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & & \ddots & 0 \\
 & & & & & & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -1 & 2
\end{array}\right).$$

The basic Lie superalgebra $\mathfrak{g} = B(m, n)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$. Set $\delta_j := \varepsilon_{m+j}$, for $1 \leq j \leq n$. If $1 \leq i \neq j \leq m$ and $1 \leq k \neq \ell \leq n$, then

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k,\ \pm\varepsilon_i \pm \delta_k,\ \pm\delta_k\},$$
$$\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k\}, \quad \Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k,\ \pm\delta_k\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n+m-1} = \varepsilon_{m-1} - \varepsilon_m,\ \alpha_{m+n} = \varepsilon_m,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\left(\begin{array}{ccccc|c|ccccc}
2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\
0 & \ddots & & \ddots & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\
0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\
\hline
\vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\
\hline
\vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\
 & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & 0 & \ddots & \ddots & -1 & 0 \\
 & & & & & & & \ddots & -1 & 2 & -1 \\
0 & \cdots & & & \cdots & \cdots & 0 & \cdots & 0 & -2 & 2
\end{array}\right).$$

The basic Lie superalgebra $\mathfrak{g} = B(0, n)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2n)$. Set $\delta_j := \varepsilon_j$, for $1 \leq j \leq n$. If $1 \leq k \neq \ell \leq n$, then

$$\Delta = \{\pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k,\ \pm\delta_k\},$$
$$\Delta_{\bar{0}} = \{\pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k\}, \quad \Delta_{\bar{1}} = \{\pm\delta_k\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 & -1 & 0 \\
\vdots & & \ddots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -2 & 2
\end{pmatrix}.$$

The basic Lie superalgebra $\mathfrak{g} = C(n + 1)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(2) \oplus \mathfrak{sp}(2n)$. Set $\delta_j := \varepsilon_{2+j}$, for $1 \leq j \leq n$. If $1 \leq k \neq \ell \leq n$, then

$$\Delta = \{\pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k,\ \pm\varepsilon_1 \pm \delta_k\},$$
$$\Delta_{\bar{0}} = \{\pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k\}, \quad \Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \delta_k\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \varepsilon_1 - \delta_1,\ \alpha_2 = \delta_1 - \delta_2,\ \ldots,\ \alpha_n = \delta_{n-1} - \delta_n,\ \alpha_{n+1} = 2\delta_n,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\left(\begin{array}{c|ccccc} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{array}\right).$$

The basic Lie superalgebra $\mathfrak{g} = D(m,n)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$. Set $\delta_j := \varepsilon_{m+j}$, for $1 \le j \le n$. If $1 \le i \ne j \le m$ and $1 \le k \ne \ell \le n$, then

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k,\ \pm\varepsilon_i \pm \delta_k\},$$
$$\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k\}, \quad \Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \delta_1 - \delta_2,\ \ldots,\ \alpha_{n-1} = \delta_{n-1} - \delta_n,\ \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n+m-1} = \varepsilon_{m-1} - \varepsilon_m,\ \alpha_{m+n} = \varepsilon_{m-1} + \varepsilon_m,$$

where α_n is the only odd root in Π. The distinguished Cartan matrix is

$$\left(\begin{array}{ccccc|c|ccccc} 2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & & & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & & & & & & & \vdots \\ 0 & \ddots & & \ddots & 0 & & & & & & \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & & & & & \\ 0 & & 0 & -1 & 2 & -1 & \ddots & & & & \vdots \\ \hline \vdots & & & \ddots & -1 & 0 & 1 & \ddots & & & \vdots \\ \hline \vdots & & & & \ddots & -1 & 2 & -1 & 0 & & 0 \\ & & & & & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 & \ddots & \ddots & -1 & -1 \\ & & & & & & & \ddots & -1 & 2 & 0 \\ 0 & \cdots & & & \cdots & \cdots & 0 & \cdots & -1 & 0 & 2 \end{array}\right).$$

3.6.2 The exceptional cases

Recall that basic Lie superalgebras possesses a non-degenerated bilinear form. Let $(\cdot,\cdot)$ denote such a form.

The basic Lie superalgebra $\mathfrak{g} = F(4)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$. Consider the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (corresponding to $\mathfrak{so}(7)$) and δ (corresponding to $\mathfrak{sl}(2)$) such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta, \delta) = -3$, and $(\varepsilon_i, \delta) = 0$. In terms of these vectors, we have that

$$\Delta = \{\pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\},$$
$$\Delta_{\bar{0}} = \{\pm\delta,\ \pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i\}, \quad \Delta_{\bar{1}} = \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3),\ \alpha_2 = \varepsilon_3,\ \alpha_3 = \varepsilon_2 - \varepsilon_3,\ \alpha_4 = \varepsilon_1 - \varepsilon_2,$$

where α_1 is the only odd root in Π. The distinguished Cartan matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The basic Lie superalgebra $\mathfrak{g} = G(3)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus G_2$. Consider the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (corresponding to G_2) and δ (corresponding to $\mathfrak{sl}(2)$) such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, $(\varepsilon_i, \varepsilon_j) = 1 - 3\delta_{ij}$, $(\delta, \delta) = 2$, and $(\varepsilon_i, \delta) = 0$. In terms of these vectors, we have that

$$\Delta = \{\pm 2\delta,\ \pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j,\ \pm\delta,\ \pm\varepsilon_i \pm \delta\},$$
$$\Delta_{\bar{0}} = \{\pm 2\delta,\ \pm\varepsilon_i,\ \varepsilon_i - \varepsilon_j\}, \quad \Delta_{\bar{1}} = \{\pm\delta, \pm\varepsilon_i \pm \delta\}.$$

The distinguished system of simple roots Π is given as follows:

$$\alpha_1 = \delta + \varepsilon_3,\ \alpha_2 = \varepsilon_1,\ \alpha_3 = \varepsilon_2 - \varepsilon_1,$$

where α_1 is the only odd root in Π. The distinguished Cartan matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

The basic Lie superalgebra $\mathfrak{g} = D(2,1;\alpha)$.

In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Consider the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (each one corresponding to one of the copies of $\mathfrak{sl}(2)$) such that $(\varepsilon_1, \varepsilon_1) = -(1+\alpha)/2$, $(\varepsilon_2, \varepsilon_2) = 1/2$, $(\varepsilon_3, \varepsilon_3) = \alpha/2$, and $(\varepsilon_i, \varepsilon_j) = 0$ if $i \neq j$. In terms of these vectors, we have that

$$\Delta = \{\pm 2\varepsilon_i,\ \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\},$$
$$\Delta_{\bar{0}} = \{\pm 2\varepsilon_i\}, \quad \Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

The distinguished system of simple roots Π is given by

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3,\ \alpha_2 = 2\varepsilon_2,\ \alpha_3 = 2\varepsilon_3,$$

where α_1 is the only odd root in Π. The distinguished Cartan matrix is

$$\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Chapter 4

Super-Weyl functors

In this chapter we study Weyl modules from the point of view of Weyl functors, i.e., we introduce the super version of the functors defined in [CFK10]. We consider map superalgebras $\mathfrak{g} \otimes A$, where A is an associative commutative unital $\mathbb{C}$-algebra and $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. Via such functors, we recover the local Weyl modules. We also prove properties that are analogues of those of Weyl functors in the non-super setting.

4.1 A good system of simple roots II

In Section 4.4, we will be interested in systems of simple roots Σ satisfying the following property:

$$\begin{array}{c}\text{If } \theta \text{ is the lowest root of } \mathfrak{g} \text{ with respect to} \\ \text{the triangular decomposition induced by } \Sigma, \text{ then } \theta \text{ is even.}\end{array} \tag{4.1.1}$$

Recall that $\Pi = \{\gamma_1, \ldots, \gamma_n\}$ denotes a distinguished system of simple roots of $\mathfrak{g}$, where γ_s is the only odd root in Π. The next proposition shows that a system of simple roots satisfying (4.1.1) always exists.

Proposition 4.1.1. *Let* Π *be a distinguished system of simple roots for* $\mathfrak{g}$.

a. If $\mathfrak{g}$ *is a basic Lie superalgebra of type II, then* Π *satisfies condition* (4.1.1).

b. If $\mathfrak{g}$ *is* $\mathfrak{gl}(n,n)$, $n \geq 2$, $\mathfrak{sl}(n,n)$, $n \geq 2$, *or a basic Lie superalgebra other than* $B(0,n)$, *then* $r_{\gamma_s}(\Pi)$ *satisfies condition* (4.1.1).

In particular, $\mathfrak{g}$ *admits at least one system of simple roots satisfying* (4.1.1).

Proof. Part (a) follows from the fact that the $\mathbb{Z}$-grading of $\mathfrak{g}$ induced by Π gives us $\mathfrak{g}_{-2} \neq 0$ and $\mathfrak{g}_{-k} = 0$ for all $k > 2$. This implies that the lowest root vector of $\mathfrak{g}$ lies in $\mathfrak{g}_{-2}$, and so $\theta \in \Delta_{\bar{0}}^-$.

To prove part (b), notice that if $\mathfrak{g}$ is of type I, then the $\mathbb{Z}$-grading of $\mathfrak{g}$ induced by Π is of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In particular, a highest (resp. lowest) vector root of $\mathfrak{g}$ lies in $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$), and so a distinguished system Π does not satisfies (4.1.1). Let $X_\beta \in \mathfrak{g}_\beta$ be a highest weight of $\mathfrak{g}$ with respect to Π, it is clear that it is an element of $\mathfrak{g}_1$. On the other hand, since $\mathfrak{g}$ is simple and

$\mathfrak{g}_0 \neq 0$ we must have that $[Y_{\gamma_s}, X_\beta] = Z \neq 0$. Notice that $Z \in \mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$. We claim that Z is a highest weight vector with respect to $r_\alpha(\Pi) = \Pi'$. To prove this, we will see that $[X_{\gamma'}, Z] = 0$ for all $\gamma' \in \Pi'$. Recall from the proof of Proposition 3.1.1 that if $\mathfrak{g}$ is $\mathfrak{gl}(n,n)$ $(n \geq 2)$, $\mathfrak{sl}(n,n)$ $(n \geq 2)$, or a basic Lie superalgebra other than $B(0,n)$ or $D(2,1;\alpha)$, then

$$\gamma'_s = r_{\gamma_s}(\gamma_s) = -\gamma_s, \quad \gamma'_{s\pm1} = r_{\gamma_s}(\gamma_{s\pm1}) = \gamma_s + \gamma_{s\pm1} \text{ and } \gamma'_{s\pm j} = r_{\gamma_s}(\gamma_{s\pm j}) = \gamma_{s\pm j}, \text{ for all } j \geq 2.$$

Thus

$$\begin{aligned}[X_{\gamma'_s}, Z] = [Y_{\gamma_s}, [Y_{\gamma_s}, X_\beta]] = [[Y_{\gamma_s}, Y_{\gamma_s}], X_\beta] - [Y_{\gamma_s}, [Y_{\gamma_s}, X_\beta]] = 0,\\ [X_{\gamma'_{s\pm1}}, Z] = [X_{\gamma_s+\gamma_{s\pm1}}, [Y_{\gamma_s}, X_\beta]] = [[X_{\gamma_s+\gamma_{s\pm1}}, Y_{\gamma_s}], X_\beta] + [Y_{\gamma_s}, [X_{\gamma_s+\gamma_{s\pm1}}, X_\beta]] = 0,\\ [X_{\gamma'_{s\pm j}}, Z] = [X_{\gamma_{s\pm j}}, [Y_{\gamma_s}, X_\beta]] = [[X_{\gamma_{s\pm j}}, Y_{\gamma_s}], X_\beta] + [Y_{\gamma_s}, [X_{\gamma_{s\pm j}}, X_\beta]] = 0, \text{ for all } j \geq 2,\end{aligned}$$

where the first line follows from the fact that $[Y_{\gamma_s}, Y_{\gamma_s}] = 0$, the second line follows from the fact that X_β is a highest weigh vector with respect to Π, and the third line follows from the fact that $[X_{\gamma_{s\pm j}}, Y_{\gamma_s}] = 0$ along with the fact that X_β is a highest weight vector with respect to Π.

If $\mathfrak{g} = D(2,1;\alpha)$, then again by the proof of Proposition 3.1.1, we have that $\Pi' = \{-\gamma_1, \gamma_1 + \gamma_2, \gamma_1 + \gamma_3\}$. Thus

$$\begin{aligned}[X_{\gamma'_1}, Z] = [Y_{\gamma_1}, [Y_{\gamma_1}, X_\beta]] = [[Y_{\gamma_1}, Y_{\gamma_1}], X_\beta] - [Y_{\gamma_1}, [Y_{\gamma_1}, X_\beta]] = 0,\\ [X_{\gamma'_j}, Z] = [X_{\gamma_1+\gamma_j}, [Y_{\gamma_1}, X_\beta]] = [[X_{\gamma_1+\gamma_j}, Y_{\gamma_1}], X_\beta] + [Y_{\gamma_1}, [X_{\gamma_1+\gamma_j}, X_\beta]] = 0, \text{ for all } j = 1, 2,\end{aligned}$$

where the first line follows from the fact that $[Y_{\gamma_1}, Y_{\gamma_1}] = 0$, and the second line follows from the fact that X_β is a highest weigh vector with respect to Π.

Therefore we have proved that $Z \in \mathfrak{g}_{\bar{0}}$ is a highest vector with respect to the triangular decomposition induced by Π'. In particular, $\beta - \gamma_s$ (resp. $\beta - \gamma_1$, for $\mathfrak{g} = D(2,1;\alpha)$) is the root associated to Z. Thus $\theta = \gamma_s - \beta$ (resp. $\gamma_1 - \beta$, for $\mathfrak{g} = D(2,1;\alpha)$) is the lowest root desired. □

Remark 4.1.2. There are finite-dimensional simple Lie superalgebras that are not contragredient Lie superalgebras. In particular, we do not have systems of simple roots satisfying nice properties as those given in Section 1.4.1. On the other hand, any finite-dimensional simple Lie superalgebra has a triangular decomposition. So it make sense talk about highest (or lowest) root with respect to such a decomposition. Due this fact, Property 4.1.1 is a more applicable condition than Property 3.1.1.

4.2 Projective objects

Let $\mathfrak{g}$ be either a basic classical Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$, and let A be an associative commutative unital $\mathbb{C}$-algebra. Let $\lambda \in \Lambda^+$. Throughout this chapter we let $\mathcal{I}_A^\lambda$ (resp. $\mathcal{I}_A$) denote the category $\mathcal{I}_A^\lambda(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$ (resp. $\mathcal{I}_A(\mathfrak{g} \otimes A, \mathfrak{g}_{\bar{0}})$). See Section 3.3, for details.

For all $V \in \operatorname{Ob} \mathfrak{g}$-mod, consider the module $P_A(V)$ given by (3.3.1). Recall that one can regard V as a $\mathfrak{g}$-submodule of $P_A(V)$ via the natural identification $V \cong \mathbb{C} \otimes V \subseteq P_A(V)$.

Lemma 4.2.1. *Let V be a $\mathfrak{g}$-module whose restriction to $\mathfrak{g}_{\bar{0}}$ lies in $\mathcal{I}$. Then $P_A(V)$ is a projective object in $\mathcal{I}_A$. Further, there exists a surjective homomorphism $P_A(V) \to V$ given by $u \otimes v \mapsto uv$. In particular, the category $\mathcal{I}_A$ has enough projectives.*

Proof. Recall that by Proposition 3.3.3, the module $P_A(V)$ is an object in $\mathcal{I}_A$. The fact that $P_A(V)$ is projective in this category is a particular case of a standard result proved in [Hoc56, Lem. 2]. The map in the statement is clearly a surjective homomorphism of $\mathfrak{g} \otimes A$-modules. Finally, the fact that any $V \in \mathcal{I}_A$ is a quotient of $P_A(V)$, which is a projective object in $\mathcal{I}_A$, implies that $\mathcal{I}_A$ has enough projectives. □

Recall that $P_A(\bar{V}(\lambda)) \in \mathcal{I}_A$ for all $\lambda \in \Lambda^+$.

Corollary 4.2.2. *For any $\lambda \in \Lambda^+$ and $V \in \mathcal{I}_A^\lambda$, the module $P_A(V)^\lambda$ is a projective object in $\mathcal{I}_A^\lambda$. In particular, the category $\mathcal{I}_A^\lambda$ has enough projectives.*

Proof. Let $W \in \operatorname{Ob} \mathcal{I}_A^\lambda$. Notice that any homomorphism from $P_A(V)$ to W descends to a homomorphism from $P_A(V)^\lambda$ to W. On the other hand, $P_A(V)$ is a projective object in $\mathcal{I}_A$, and $\mathcal{I}_A^\lambda$ is a subcategory of $\mathcal{I}_A$. Thus $P_A(V)^\lambda$ is a projective object in $\mathcal{I}_A^\lambda$. □

4.3 The super-Weyl functor

Let A be an associative commutative unital $\mathbb{C}$-algebra and let $\mathfrak{g}$ be either a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. In this section we define the *super-Weyl functors*. These are the analogues, in the super setting, of the Weyl functors defined in [CFK10, p. 525].

For $\lambda \in \Lambda^+$, we set

$$\begin{aligned} \operatorname{Ann}_{\mathfrak{g}\otimes A}(w_\lambda) &= \{u \in U(\mathfrak{g} \otimes A) \mid uw_\lambda = 0\}, \\ \operatorname{Ann}_{\mathfrak{h}\otimes A}(w_\lambda) &= \operatorname{Ann}_{\mathfrak{g}\otimes A}(w_\lambda) \cap U(\mathfrak{h} \otimes A). \end{aligned}$$

It is easy to see that $\operatorname{Ann}_{\mathfrak{h}\otimes A}(w_\lambda)$ is an ideal of $U(\mathfrak{h} \otimes A)$.

Definition 4.3.1 (The algebra $\mathbf{A}_\lambda$)**.** We define the quotient algebra

$$\mathbf{A}_\lambda = U(\mathfrak{h} \otimes A)/\operatorname{Ann}_{\mathfrak{h}\otimes A}(w_\lambda). \tag{4.3.1}$$

Lemma 4.3.2. *The $\mathfrak{g} \otimes A$-module $W(\lambda)$ is a right $\mathfrak{h} \otimes A$-module, where*

$$(uw_\lambda)x = uxw_\lambda, \text{ for all } u \in U(\mathfrak{g} \otimes A) \text{ and } x \in U(\mathfrak{h} \otimes A).$$

Proof. We shall prove that

$$uw_\lambda = u'w_\lambda \Rightarrow uxw_\lambda = u'xw_\lambda, \text{ for all } u, u' \in U(\mathfrak{g} \otimes A).$$

Or equivalently,

$$(u - u')w_\lambda = 0 \Rightarrow (u - u')xw_\lambda = 0, \text{ for all } u, u' \in U(\mathfrak{g} \otimes A).$$

Note that it is enough to prove that xw_λ satisfies the relations (3.3.3), for all $x \in \mathbf{A}_\lambda$. Since $x \in \mathbf{A}_\lambda$, it follows that $xw_\lambda \in W(\lambda)_\lambda$ and hence the first and second relations in (3.3.3) are held. Since the $\mathfrak{g}_{\bar{0}}$-module generated by aw_λ is finite-dimensional, the second relation in (3.3.3) is also satisfied. □

By Lemma 4.3.2, $W(\lambda)$ is a $(\mathfrak{g}\otimes A, \mathbf{A}_\lambda)$-bimodule, where each weight space $W(\lambda)_\mu$ is a right $\mathbf{A}_\lambda$-submodule. In particular, the assignment $W(\lambda)_\lambda \to \mathbf{A}_\lambda$ such that $w_\lambda \mapsto 1_{\mathbf{A}_\lambda}$, defines a surjective homomorphism of $\mathbf{A}_\lambda$-modules whose kernel is $\mathrm{Ann}_{\mathfrak{h}\otimes A}(w_\lambda)$. Thus we have that

$$W(\lambda)_\lambda \cong \mathbf{A}_\lambda$$

as right $\mathbf{A}_\lambda$-modules.

For $\lambda \in \Lambda^+$, we let $\mathbf{A}_\lambda\text{-mod}$ denote the category of left $\mathbf{A}_\lambda$-modules. Let $M, M' \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and let $f \in \mathrm{Hom}_{\mathbf{A}_\lambda}(M, M')$. Since $\mathrm{id}\colon W(\lambda) \to W(\lambda)$ is even, it is clear that

$$\mathrm{id}\otimes f\colon W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M \to W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M'$$

is also an even homomorphism of $\mathfrak{g} \otimes A$-modules.

Definition 4.3.3 (The super-Weyl functor $\mathbf{W}_A^\lambda$)**.** Let $\lambda \in \Lambda^+$. The super-Weyl functor is defined to be

$$\mathbf{W}_A^\lambda : \mathbf{A}_\lambda\text{-mod} \to \mathcal{I}_A^\lambda, \quad \mathbf{W}_A^\lambda M = W(\lambda) \otimes_{\mathbf{A}_\lambda} M, \quad \mathbf{W}_A^\lambda f = \mathrm{id}\otimes f,$$

for all $M, M' \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and $f \in \mathrm{Hom}(M, M')$.

Note that

$$\mathbf{W}_A^\lambda \mathbf{A}_\lambda \cong W(\lambda)$$

as $U(\mathfrak{g} \otimes A)$-modules. Moreover

$$(\mathbf{W}_A^\lambda M)\mu \cong W(\lambda)_\mu \otimes_{\mathbf{A}_\lambda} M, \text{ as } \mathbf{A}_\lambda\text{-modules for all weights } \mu \in \mathfrak{h}^* \text{ and } M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}\,.$$

Lemma 4.3.4. *For all $\lambda \in \Lambda^+$ and $V \in \mathcal{I}_A^\lambda$, we have $(\mathrm{Ann}_{\mathfrak{h}\otimes A}(w_\lambda))V_\lambda = 0$.*

Proof. If $v \in V_\lambda$, then it is clear that $\mathfrak{n}^+ v = 0$ and $hv = \lambda(h)v$, for all $h \in \mathfrak{h}$. Since the $U(\mathfrak{g}_{\bar 0})v \subseteq V$ is a finite-dimensional $\mathfrak{g}_{\bar 0}$-submodule, we have also that $Y_\alpha^{\lambda(H_\alpha)+1} v = 0$, for all $\alpha \in \Sigma(\mathfrak{g}_{\bar 0})$. Thus the $\mathfrak{g}$-submodule $U(\mathfrak{g})v \subseteq V$ is a quotient of $\bar V(\lambda)$. If $\pi : \bar V(\lambda) \twoheadrightarrow U(\mathfrak{g})v$ is a projection, then after rescaling, if necessary, we may assume that $\pi(v_\lambda) = v'$. We therefore have an even linear map

$$\varphi : U(\mathfrak{g} \otimes A) \otimes_{U(\mathfrak{g})} \bar V(\lambda) \to V, \quad u \otimes_{U(\mathfrak{g})} v \mapsto u\pi(v), \quad u \in U(\mathfrak{g} \otimes A),\, v \in \bar V(\lambda),$$

where

$$U(\mathfrak{n}^+ \otimes A) \otimes v_\lambda \subseteq \ker \varphi.$$

Thus φ descends to a homomorphism $\tilde\varphi : W(\lambda) \to V$ sending w_λ to v'. In particular, if $u \in \mathrm{Ann}_{\mathfrak{h}\otimes A}(w_\lambda)$, then $uv = \tilde\varphi(uw_\lambda) = \varphi(0) = 0$. □

It follows from Lemma 4.3.4 that the left action of $U(\mathfrak{h}\otimes A)$ on $V \in \mathrm{Ob}\,\mathcal{I}_A^\lambda$ induces a left action of $\mathbf{A}_\lambda$ on V_λ. The resulting left $\mathbf{A}_\lambda$-module will be denoted by $\mathbf{R}_A^\lambda V$. Furthermore, it is clear that for $\pi \in \mathrm{Hom}_{\mathcal{I}_A^\lambda}(V, V')$, the restriction $\pi_\lambda : V_\lambda \to V'_\lambda$ is a morphism of $\mathbf{A}_\lambda$-module. Thus

$$V \mapsto \mathbf{R}_A^\lambda V, \quad \pi \mapsto \mathbf{R}_A^\lambda \pi = \pi_\lambda$$

defines a functor $\mathbf{R}_A^\lambda : \mathcal{I}_A^\lambda \to \mathbf{A}_\lambda\text{-mod}$. Notice that $\mathbf{R}_A^\lambda$ is exact since the restriction to a weight space is exact.

In the non-super setting, the next proposition was proved in [CFK10, Prop. 5] (in the untwisted case) and in [FMS15, Prop. 4.8] (in the twisted case).

Proposition 4.3.5. *The functors* $\mathbf{W}_A^\lambda$ *and* $\mathbf{R}_A^\lambda$ *satisfy the following properties:*

a. $\mathbf{R}_A^\lambda \mathbf{W}_A^\lambda \cong \mathrm{id}_{\mathbf{A}_\lambda\text{-mod}}$ *as functors.*

b. $\mathbf{W}_A^\lambda$ *is left adjoint to* $\mathbf{R}_A^\lambda$*.*

c. The functor $\mathbf{W}_A^\lambda$ *maps projective objects to projective objects.*

Proof. To prove part (a), consider $M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and note that

$$\mathbf{R}_A^\lambda \mathbf{W}_A^\lambda M = (\mathbf{W}_A^\lambda M)_\lambda = W(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda} M \cong \mathbf{A}_\lambda \otimes_{\mathbf{A}_\lambda} M \cong M.$$

To prove part (b), we need to find natural transformations

$$\epsilon : \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda \mapsto \mathrm{id}_{\mathcal{I}_A^\lambda}, \quad \eta : \mathrm{id}_{\mathbf{A}_\lambda\text{-mod}} \mapsto \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda,$$

such that for any $M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and $V \in \mathrm{Ob}\,\mathcal{I}_A^\lambda$, we have the following equality of morphisms

$$\mathrm{id}_{\mathbf{W}_A^\lambda M} = \epsilon_{\mathbf{W}_A^\lambda M} \circ \mathbf{W}_A^\lambda(\eta_M), \quad \mathrm{id}_{\mathbf{R}_A^\lambda V} = \mathbf{R}_A^\lambda(\epsilon_V) \circ \eta_{\mathbf{R}_A^\lambda V}. \tag{4.3.2}$$

Let us start defining η_M, for $M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$. Let

$$\eta_M : M \to \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda M, \quad m \mapsto w_\lambda \otimes m.$$

It is easy to see that for any $M, M' \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and $f \in \mathrm{Hom}_{\mathbf{A}_\lambda}(M, M')$, we have that

$$\eta_{M'} \circ f = \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda(f) \circ \eta_M.$$

Thus, the collection $\{\eta_M \in \mathrm{Hom}_{\mathbf{A}_\lambda}(M, \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda M) \mid M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}\}$ defines a natural transformation $\eta : \mathrm{id}_{\mathbf{A}_\lambda\text{-mod}} \to \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda$.

For ϵ we proceed in the following way: for each $V \in \mathrm{Ob}\,\mathcal{I}_A^\lambda$, one regards $W(\lambda) \otimes_{\mathbb{C}} \mathbf{R}_A^\lambda V$ as a left $\mathfrak{g} \otimes A$-module via the action of $\mathfrak{g} \otimes A$ on $W(\lambda)$. With respect to this action, we have a homomorphism of $\mathfrak{g} \otimes A$-modules denoted by $\epsilon_1 : W(\lambda) \otimes_{\mathbb{C}} \mathbf{R}_A^\lambda V \to V$, such that $uw_\lambda \otimes v \mapsto uv$ for $u \in U(\mathfrak{g} \otimes A)$ and $v \in V_\lambda$. Next, notice that for any $a \in \mathbf{A}_\lambda$, $u \in U(\mathfrak{g} \otimes A)$ and $v \in V_\lambda$

$$\epsilon_1(uw_\lambda a \otimes v) = \epsilon_1(uaw_\lambda \otimes v) = uav = \epsilon_1(uw_\lambda \otimes av).$$

Thus the map ϵ_1 factors through a homomorphism of $\mathfrak{g} \otimes A$-modules

$$\epsilon_V : W(\lambda) \otimes_{\mathbf{A}_\lambda} \mathbf{R}_A^\lambda V = \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \to V.$$

One can see that the collection $\{\epsilon_V \in \mathrm{Hom}_{\mathfrak{g}\otimes A}(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V, V) \mid V \in \mathrm{Ob}\,\mathcal{I}_A^\lambda\}$ defines a natural transformation $\epsilon : \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda \to \mathrm{id}_{\mathbf{A}_\lambda\text{-mod}}$.

Finally, for any $M \in \mathrm{Ob}\,\mathbf{A}_\lambda\text{-mod}$ and $m \in M$, we have

$$(\epsilon_{\mathbf{W}_A^\lambda M} \circ \mathbf{W}_A^\lambda(\eta_M))(uw_\lambda \otimes m) = \epsilon_{\mathbf{W}_A^\lambda M}(uw_\lambda \otimes w_\lambda \otimes m) = uw_\lambda \otimes m,$$

and for $V \in \mathrm{Ob}\,\mathcal{I}_A^\lambda$, $m \in \mathbf{R}_A^\lambda V$

$$(\mathbf{R}_A^\lambda(\epsilon_V) \circ \eta_{\mathbf{R}_A^\lambda V})(m) = \mathbf{R}_A^\lambda(\epsilon_V)(w_\lambda \otimes m) = m.$$

Then part (b) is proved.

Part (c) follows from the fact that $\mathbf{W}_A^\lambda$ is left adjoint to a right exact functor. □

Corollary 4.3.6. *For any $\lambda \in \Lambda^+$, the global Weyl module $W(\lambda)$ is a projective object in $\mathcal{I}_A^\lambda$.*

Proof. Since $\mathbf{A}_\lambda$ is a projective object in $\mathbf{A}_\lambda$-mod and $\mathbf{W}_A^\lambda \mathbf{A}_\lambda \cong W(\lambda)$, the result follows from Proposition 4.3.5(c). □

By Corollary 4.2.2, the category $\mathcal{I}_A^\lambda$ has enough projectives. Thus, for $M,\ N \in \operatorname{Ob}\mathcal{I}_A^\lambda$ and $n \in \mathbb{N}$, we can consider $\operatorname{Ext}^n_{\mathcal{I}_A^\lambda}(M,N)$ as defined in Section 1.1.

Theorem 4.3.7. *Let $V \in \mathcal{I}_A^\lambda$. Then $V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$ if and only if for any $U \in \mathcal{I}_A^\lambda$ such that $U_\lambda = 0$, we have*

$$\operatorname{Hom}_{\mathcal{I}_A^\lambda}(V,U) = 0, \quad \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(V,U) = 0. \tag{4.3.3}$$

Proof. Let $V \in \operatorname{Ob}\mathcal{I}_A^\lambda$ with $V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$. Let $U \in \operatorname{Ob}\mathcal{I}_A^\lambda$ with $U_\lambda = 0$ and let $\varphi \in \operatorname{Hom}_{\mathcal{I}_A^\lambda}(V,U)$. Since we are assuming $V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$, we have that $v_\lambda \otimes V_\lambda$ generates V as a $U(\mathfrak{g}\otimes A)$-module. On the other hand, it is clear that homomorphisms of modules preserve weights, and so $\varphi(v_\lambda \otimes V_\lambda) \subseteq U_\lambda = 0$. Thus $\varphi = 0$.

For the second statement, we first notice that the category $\mathbf{A}_\lambda$-mod has enough projectives. Therefore, there exists a surjective homomorphism $\pi : P \to \mathbf{R}_A^\lambda V$, where P is a projective object in $\mathbf{A}_\lambda$-mod. Since the functor $\mathbf{W}_A^\lambda$ is exact, we have an epimorphims of $\mathfrak{g}\otimes A$-modules

$$1 \otimes \pi : \mathbf{W}_A^\lambda P \to \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \cong V,$$

where, by Proposition 4.3.5(c), $\mathbf{W}_A^\lambda P$ is a projective module. Since $\operatorname{id}\otimes\pi$ is even, $K = \ker(1\otimes\pi)$ is a submodule of $\mathbf{W}_\lambda^\lambda P$, and so we have the short exact sequence

$$0 \to K \to \mathbf{W}_A^\lambda P \to V \to 0. \tag{4.3.4}$$

Notice now that $K = W(\lambda) \otimes_{\mathbf{A}_\lambda} \ker\pi$ is generated by $W(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda} \ker\pi$. Again, since homomorphisms between modules preserve weights, and $U_\lambda = 0$, we must have that $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(K,U) = 0$. Now, the long exact sequence obtained by applying the functor $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(-,U)$ to (4.3.4) is given by

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(V,U) \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda P,U) \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(K,U) \to \\ \to \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(V,U) \to \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda P,U) \to \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(K,U) \to \cdots, \end{aligned}$$

where $\operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda P,U) = 0$ (since $\mathbf{W}_A^\lambda P$ is projective) and $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(K,U) = 0$. But this shows that $\operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(V,U) = 0$, as we wanted.

Conversely, suppose $V \in \operatorname{Ob}\mathcal{I}_A^\lambda$ satisfies (4.3.3). Set $V' = U(\mathfrak{g}\otimes A)V_\lambda$. It is clear that $V/V' \in \operatorname{Ob}\mathcal{I}_A^\lambda$ and $(V/V')_\lambda = 0$. Thus, the first condition in (4.3.3) implies that $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(V, V/V') = 0$, and hence $V = V'$. Consider now the homomorphism of $\mathfrak{g}\otimes A$-modules $\epsilon_V : \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \to V$ defined in the proof of Proposition 4.3.5. Notice that $V' = V$ implies that such a map is surjective. Since ϵ_V is even, we have that $U = \ker \epsilon_V$ is a submodule of $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$. We claim that $U_\lambda = 0$. Indeed, an arbitrary element of $(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V)_\lambda$ is of the form $v' = w_\lambda a \otimes_{\mathbf{A}_\lambda} v$, where $a \in \mathbf{A}_\lambda$ and $v \in V_\lambda$. If such an element is in U, then $0 = \epsilon_V(w_\lambda a \otimes_{\mathbf{A}_\lambda} v) = av$. Thus $v' = w_\lambda \otimes_{\mathbf{A}_\lambda} av = 0$.

Consider the short exact sequence

$$0 \to U \to \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \to V \to 0.$$

The long exact sequence yielded by applying $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(-, U)$ is

$$0 \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(V, U) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V, U) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(U, U) \to \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(V, U) \to \cdots,$$

where $\mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(V, U) = 0$ (by hypotheses) and $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V, U) = 0$ (by the first part). Thus $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(U, U) = 0$, which implies that $U = 0$. Therefore ϵ_V defines an isomorphism between $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$ and V. □

Theorem 4.3.8. *The functor $\mathbf{W}_A^\lambda$ is exact if and only if*

$$\mathrm{Ext}^2_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V, U) = 0, \tag{4.3.5}$$

for all $V \in \mathrm{Ob}\, \mathbf{A}_\lambda$-mod, and for all $U \in \mathcal{I}_A^\lambda$, with $U_\lambda = 0$.

Proof. Let $V \in \mathrm{Ob}\, \mathbf{A}_\lambda$-mod and consider a short exact sequence of $\mathbf{A}_\lambda$-modules

$$0 \to V'' \to V \to V' \to 0.$$

If the condition (4.3.5) is satisfied, we can consider the induced short exact sequence

$$0 \to K \to \mathbf{W}_A^\lambda V \to \mathbf{W}_A^\lambda V' \to 0,$$

where K is the kernel of the map $\mathbf{W}_A^\lambda V'' \to \mathbf{W}_A^\lambda V$. Applying $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(-, U)$ to this sequence gives

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V', U) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V, U) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(K, U) \to \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V', U) \to \\ \to \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V, U) \to \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(K, U) \to \mathrm{Ext}^2_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V', U) \to \cdots. \end{aligned}$$

Thus Theorem 4.3.7 (applied to $\mathbf{W}_A^\lambda V$ and $\mathbf{W}_A^\lambda V'$) along with condition (4.3.5) imply that

$$\mathrm{Hom}_{\mathcal{I}_A^\lambda}(K, U) = 0, \quad \text{and} \quad \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(K, U) = 0, \quad \text{for all } U \in \mathrm{Ob}\, \mathcal{I}_A^\lambda \text{ such that } U_\lambda = 0.$$

Hence, again by Theorem 4.3.7, $K \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda K \cong \mathbf{W}_A^\lambda K_\lambda$. Let now M be the kernel of the map $\mathbf{W}_A^\lambda V'' \to K$. Using the fact that $\mathbf{R}_A^\lambda \mathbf{W}_A^\lambda \cong \mathrm{id}_{\mathbf{A}_\lambda\text{-mod}}$ and that the map $V'' \to V'$ is injective, we obtain that $M_\lambda = 0$. Applying $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(-, M)$ to the short exact sequence

$$0 \to M \to \mathbf{W}_A^\lambda V'' \to K \to 0$$

we get the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(K, M) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V'', M) \to \mathrm{Hom}_{\mathcal{I}_A^\lambda}(M, M) \to \mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(K, M) \to \cdots,$$

where $\mathrm{Ext}^1_{\mathcal{I}^\lambda_A}(K, M) = 0$ (since $M_\lambda = 0$) and $\mathrm{Hom}_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V'', M) = 0$ (by Theorem 4.3.7). Thus $\mathrm{Hom}_{\mathcal{I}^\lambda_A}(M, M) = 0$, and so is M.

Conversely, suppose $\mathbf{W}^\lambda_A$ is exact. Let $V \in \mathrm{Ob}\,\mathbf{A}_\lambda$-mod and let $P \to V$ be a surjective homomorphism of $\mathbf{A}_\lambda$-modules, where P is a projective object in $\mathbf{A}_\lambda$-mod. Let V' be the kernel of this map and consider the short exact sequence $0 \to V' \to P \to V \to 0$. Since $\mathbf{W}^\lambda_A$ is exact, the sequence

$$0 \to \mathbf{W}^\lambda_A V' \to \mathbf{W}^\lambda_A P \to \mathbf{W}^\lambda_A V \to 0$$

is exact as well. If $U \in \mathcal{I}^\lambda_A$ with $U_\lambda = 0$, then we can apply $\mathrm{Hom}_{\mathcal{I}^\lambda_A}(-, U)$ to the above sequence to get the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V, U) \to \mathrm{Hom}_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A P, U) \to \mathrm{Hom}_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V', U) \to \mathrm{Ext}^1_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V, U) \to$$
$$\to \mathrm{Ext}^1_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A P, U) \to \mathrm{Ext}^1_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V', U) \to \mathrm{Ext}^2_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V, U) \to \mathrm{Ext}^2_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A P, U) \cdots,$$

where $\mathrm{Ext}^1_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V', U) = 0$ by Theorem 4.3.7 (here we are using that $\mathbf{W}^\lambda_A \mathbf{R}^\lambda_A(\mathbf{W}^\lambda_A V') \cong \mathbf{W}^\lambda_A V'$), and $\mathrm{Ext}^2_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A P, U) = 0$, since $\mathbf{W}^\lambda_A P$ is projective in $\mathcal{I}^\lambda_A$. Thus $\mathrm{Ext}^2_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A V, U) = 0$. □

In the non-super setting, Theorems 4.3.7 and 4.3.8 were proved in [CFK10, Th. 1 and Cor. 3] and in [FMS15, Prop. 4.8], for untwisted and twisted cases, respectively.

4.4 The right $\mathbf{A}_\lambda$-module $W(\lambda)$

For the remainder of the chapter, we assume that

$$A \text{ is finitely generated.}$$

Lemma 4.4.1. *If $\lambda \in \Lambda^+$, $\alpha \in \Delta_0^+$, $a_1, a_2, \dots, a_t \in A$, and $m_1, \dots, m_t \in \mathbb{N}$, then*

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}) w_\lambda \in \mathrm{span}_{\mathbb{C}} \ \{(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t}) w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell_i < \lambda(H_\alpha),\ i = 1, \dots, t\}. \quad (4.4.1)$$

In particular, $(Y_\alpha \otimes A) w_\lambda$ is a finitely generated right $\mathbf{A}_\lambda$-module.

Proof. By Lemma 3.4.3, we have that $Y_\alpha^{\lambda(H_\alpha)+1} w_\lambda = 0$. Thus from the first and third relations in (3.3.7) along with (3.4.3), it follows that for $a \in A$ and $m \ge \lambda(H_\alpha)$

$$0 = (X_\alpha \otimes a)^m (Y_\alpha \otimes 1)^{m+1} w_\lambda = \sum_{i=0}^{m} (-1)^m (Y_\alpha \otimes a^{m-i}) p(a, \alpha)_i w_\lambda,$$

for any $a \in A$. Since $p(a, \alpha)_0 = 1$, this shows that

$$(Y_\alpha \otimes a^m) w_\lambda \in \mathrm{span}_{\mathbb{C}}\{(Y_\alpha \otimes a^\ell) w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell < m\}.$$

This implies, by induction, that

$$(Y_\alpha \otimes a^m) w_\lambda \in \mathrm{span}_{\mathbb{C}}\{(Y_\alpha \otimes a^\ell) w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell < \lambda(H_\alpha)\}, \quad \text{for all } m \in \mathbb{N},\, a \in A. \quad (4.4.2)$$

We will now prove (4.4.1) by induction on t. The case $t = 1$ follows immediately from (4.4.2). Assume that (4.4.1) holds for some $t \geq 1$. Let $m_1, \dots, m_{t+1} \in \mathbb{N}$ and choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda = (-\alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) + (Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})(h \otimes a_{t+1}^{m_{t+1}}))w_\lambda,$$

and so

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda + \alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\lambda \in \operatorname{span}_{\mathbb{C}}\{(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda \mathbf{A}_\lambda\}, \tag{4.4.3}$$

since $(h \otimes a_{t+1}^{m_{t+1}})w_\lambda \in w_\lambda \mathbf{A}_\lambda$. By the inductive hypothesis, we have that elements of the form $(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\lambda$ are in

$$\operatorname{span}_{\mathbb{C}}\left\{(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda, (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$

Then, by (4.4.3) (with $m_i = \ell_i$ for $i = 1, \dots, t$), we have

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\lambda \in \operatorname{span}_{\mathbb{C}}\left\{(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t} a_{t+1}^{m_{t+1}})w_\lambda, (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$

Since the above inclusion holds for all $m_1, \dots, m_{t+1} \in \mathbb{N}$, we can interchange the roles of m_1 and m_{t+1} to obtain

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\lambda \in \operatorname{span}_{\mathbb{C}}\left\{(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t} a_{t+1}^{\ell_{t+1}})w_\lambda \mathbf{A}_\lambda \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$

This completes the proof of the inductive step. The final statement of the lemma follows from the fact that A is finitely generated. □

Proposition 4.4.2. *For all $\lambda \in \Lambda^+$, the algebra $\mathbf{A}_\lambda$ is finitely generated.*

Proof. Suppose $a_1, a_2, \dots, a_t \in A$ is a set of generators of A and let $m_1, m_2, \dots, m_t \in \mathbb{N}$. One can see from the proof of Lemma 4.4.1 that $(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda$ can be written as a linear combination of elements of the form $(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})P(\alpha, \ell_1, \dots, \ell_t)w_\lambda$, where $0 \leq \ell_i \leq \lambda(H_\lambda)$, for all $i = 1, \dots, t$, and each $P(\alpha, \ell_1, \dots, \ell_t)$ is a linear combination of finite products of elements of $U(\mathfrak{h} \otimes A)$ of the form $(H_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})$.

On the other hand,

$$(H_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda = [X_\alpha \otimes 1, Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}]w_\lambda = (X_\alpha \otimes 1)(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda,$$

where the last is a linear combination of elements of the form

$$(X_\alpha \otimes 1)(Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})P(\alpha, \ell_1, \dots, \ell_t)w_\lambda = (H_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})P(\alpha, \ell_1, \dots, \ell_t)w_\lambda.$$

Hence, we obtain that

$$H_\alpha \otimes (a_1^{m_1}, \dots, a_t^{m_t})w_\lambda = P(\alpha, m_1, \dots, m_t)w_\lambda,$$

where $P(\alpha, m_1, \dots, m_t)$ lies in the subalgebra of $U(\mathfrak{h} \otimes A)$ generated by the elements of the form $H_\alpha \otimes (a_1^{\ell_1}, \dots, a_t^{\ell_t})$, with $1 \leq \ell_1, \dots, \ell_t \leq \lambda(h_\alpha)$. This implies that $\mathbf{A}_\lambda$ is a quotient of a finitely generated algebra, and so it is finitely generated as well. □

For the remainder of the section, we assume that

$$\Sigma \text{ is a system of simple roots for } \mathfrak{g} \text{ satisfying (4.1.1).}$$

Recall that, by Proposition 4.1.1, such a system always exists. Furthermore, since θ is even, Lemma 4.4.1 implies that for all $a_1, a_2, \ldots, a_t \in A$, and $m_1, \ldots, m_t \in \mathbb{N}$

$$(Y_\theta \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda \in \operatorname{span}_{\mathbb{C}} \{(Y_\theta \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell_i < \lambda(H_\theta),\ i = 1, \ldots, t\}.$$

Theorem 4.4.3. *For all $\lambda \in \Lambda^+$, the global Weyl module $W(\lambda)$ is a finitely generated right $\mathbf{A}_\lambda$-module*

Proof. Since $\theta \in \Delta_{\bar{0}}$ is a highest root and $\mathfrak{g}$ is simple, any element in $\mathfrak{g}$ lies in the span of $\{[X_{i_1}, [X_{i_2}, [\ldots [X_{i_k}, Y_\theta] \ldots]] \mid i_1, \ldots, i_k \in I,\ k \in \mathbb{N}\}$. Suppose $a_1, a_2, \ldots, a_t \in A$, and $m_1, \ldots, m_t \in \mathbb{N}$. Let us prove, by induction on $k \in \mathbb{N}$, that $([X_{i_1}, [X_{i_2}, [\ldots [X_{i_k}, Y_\theta] \ldots]] \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda \in \operatorname{span}_{\mathbb{C}}\{([X_{i_1}, [X_{i_2}, [\ldots [X_{i_k}, Y_\theta] \ldots]] \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell_i < \lambda(H_\theta),\ i = 1, \ldots, t\}$. Indeed, for any $i \in I$ we have

$$\begin{aligned}([X_i, Y_\theta] \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda &= [X_i \otimes 1, Y_\theta \otimes a_1^{m_1} \cdots a_t^{m_t}]w_\lambda \\ &= (X_i \otimes 1)(Y_\theta \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda \\ &\in \operatorname{span}_{\mathbb{C}}\{([X_i, Y_\theta] \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell_i < \lambda(H_\theta),\ i = 1, \ldots, t\},\end{aligned}$$

where the last line follows from Lemma 4.4.1 applied to $\theta \in \Delta_{\bar{0}}$. Considering $i_1, \ldots, i_k \in I$, $k \in \mathbb{N}$, we get

$$\begin{aligned}&([X_{i_1}, [X_{i_2}, [\cdots [X_{i_k}, Y_\theta] \cdots]]] \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda = [X_{i_1} \otimes 1, [X_{i_2} \otimes 1, [\cdots [X_{i_k} \otimes 1, Y_\theta \otimes a_1^{m_1} \cdots a_t^{m_t}] \cdots]]]w_\lambda \\ &= [X_{i_1} \otimes 1, [X_{i_2}, [\cdots [X_{i_k}, Y_\theta] \cdots]] \otimes a_1^{m_1} \cdots a_t^{m_t}]w_\lambda = (X_{i_1} \otimes 1)([X_{i_2}, [\cdots [X_{i_k}, Y_\theta] \cdots]] \otimes a_1^{m_1} \cdots a_t^{m_t})w_\lambda \\ &\in \{([X_{i_1}, [X_{i_2}, [\cdots [X_{i_k}, Y_\theta] \cdots]]] \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\lambda \mathbf{A}_\lambda \mid 0 \le \ell_i < \lambda(H_\theta),\ i = 1, \ldots, t\},\end{aligned}$$

where the last line follows by induction.

Since $\mathfrak{g}$ is finite dimensional, there exists $N \in \mathbb{N}$ such that $[X_{i_1}, [X_{i_2}, [\cdots [X_{i_k}, Y_\theta] = 0$ for all $k \ge N$. Thus the theorem follows. □

Corollary 4.4.4. *If $M \in \mathbf{A}_\lambda$-mod is finitely generated (resp. finite dimensional), then $\mathbf{W}_A^\lambda M$ is a finite generated (resp. finite dimensional) $\mathfrak{g} \otimes A$-module.* □

Remark 4.4.5. In the non-super setting, Proposition 4.4.2 and Theorem 4.4.3 were first proved for the untwisted case in [CFK10, Th. 2(i)]. The twisted version of these results was proved later in [NS13, Theorems 5.8 and 5.10]. We would like to point out that in both cases the analogues of Theorem 4.4.3 do not depend on the choice of the system of simple roots.

4.5 A tensor product property

Recall that $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. Let C be an associative, commutative unital $\mathbb{C}$-algebra and let $\Delta: U(\mathfrak{h}\otimes C) \to U(\mathfrak{h}\otimes C)\otimes U(\mathfrak{h}\otimes C)$ be the diagonal map. Let $\lambda, \mu \in \Lambda^+$, and consider the Weyl modules $W_C^{\lambda+\mu}(\lambda+\mu)$, $W_C^{\lambda}(\lambda)$ and $W_C^{\mu}(\mu)$. Since

$$\Delta\left(\operatorname{Ann}_{\mathfrak{h}\otimes C}(w_{\lambda+\mu})\right) \subseteq \operatorname{Ann}_{\mathfrak{h}\otimes C}(w_\lambda)\otimes U(\mathfrak{h}\otimes C) + U(\mathfrak{h}\otimes C)\otimes \operatorname{Ann}_{\mathfrak{h}\otimes C}(w_\mu),$$

the map Δ induces a homomorphism of algebras $\Delta_{\lambda,\mu}: \mathbf{C}_{\lambda+\mu} \to \mathbf{C}_\lambda \otimes \mathbf{C}_\mu$ (recall that the symbol $\mathbf{C}_\bullet$ is defined to be the quotient $U(\mathfrak{h}\otimes C)/\operatorname{Ann}_{\mathfrak{h}\otimes C}(w_\bullet)$). If $M \in \mathbf{C}_\lambda$-mod and $N \in \mathbf{C}_\mu$-mod, then the tensor product $M\otimes N$ can be view as a left $\mathbf{C}_{\lambda+\mu}$, where the action is induced by $\Delta_{\lambda,\mu}$. We denote this $\mathbf{C}_{\lambda+\mu}$-module by $\Delta^*_{\lambda,\mu}(M\otimes N)$.

Let $\pi: C \twoheadrightarrow A$ be a surjective homomorphism of algebras. The map $\operatorname{id}\otimes\pi: \mathfrak{g}\otimes C \to \mathfrak{g}\otimes A$ is clearly even, and so it is a homomorphism of superalgebras. This homomorphism induces an action of $\mathfrak{g}\otimes C$ on any $\mathfrak{g}\otimes A$-module V. We let $(\operatorname{id}\otimes\pi)^*(V)$ denote such a $\mathfrak{g}\otimes C$-module.

In the non-super setting, the next results in this section were proved in [CFK10, §4].

Proposition 4.5.1. *Assume that A and B are commutative, associative unital $\mathbb{C}$-algebras. Let $C = A\oplus B$, and let $\pi_A: C \twoheadrightarrow A$ and $\pi_B: C \twoheadrightarrow B$ be the two canonical surjective homomorphisms of algebras. If $\lambda,\mu\in\Lambda^+$, $M \in \operatorname{Ob}\mathbf{A}_\lambda$-mod and $N\in\operatorname{Ob}\mathbf{B}_\mu$-mod, then there exists a surjective homomorphism of $\mathfrak{g}\otimes C$-modules*

$$\mathbf{W}_C^{\lambda+\mu}\left(\Delta^*_{\lambda,\mu}(M\otimes N)\right) \twoheadrightarrow (\operatorname{id}\otimes\pi_A)^*(\mathbf{W}_A^\lambda M)\otimes(\operatorname{id}\otimes\pi_B)^*(\mathbf{W}_B^\mu N).$$

Proof. Since $\mathfrak{g}\otimes C \cong (\mathfrak{g}\otimes A)\oplus(\mathfrak{g}\otimes B)$, the $\mathfrak{g}\otimes C$-module $(\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu))$ is generated by the vector $w_\lambda\otimes w_\mu$. Such a vector satisfies (3.3.3). Therefore there exists a surjective homomorphism $\varphi: W_C(\lambda+\mu) \twoheadrightarrow (\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu))$ of $\mathfrak{g}\otimes C$-modules, such that $\varphi(w_{\lambda+\mu}) = w_\lambda\otimes w_\mu$. Since $(\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))$ is a right $\mathbf{C}_\lambda$-module and $(\operatorname{id}\otimes\pi_B)^*(W_B(\mu))$ is a right $\mathbf{C}_\mu$-module, we have that the tensor product $(\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu))$ is a right $\mathbf{C}_\lambda\otimes\mathbf{C}_\mu$-module, and so $\Delta^*_{\lambda,\mu}((\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu)))$ is a right $\mathbf{C}_{\lambda+\mu}$-module. It is clear that φ is also a homomorphism of right $\mathbf{C}_{\lambda+\mu}$-modules. Thus

$$\begin{gathered} W_C(\lambda+\mu)\otimes_{\mathbf{C}_{\lambda+\mu}}\Delta^*_{\lambda,\mu}(M\otimes N) \twoheadrightarrow \Delta^*_{\lambda,\mu}((\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu)))\otimes_{\mathbf{C}_{\lambda+\mu}}(M\otimes N) \\ w_{\lambda+\mu}\otimes_{\mathbf{C}_{\lambda+\mu}}(m\otimes n) \mapsto (w_\lambda\otimes w_\mu)\otimes_{\mathbf{C}_{\lambda+\mu}}(m\otimes n) \end{gathered}$$

defines a surjective homomorphism of $(\mathfrak{g}\otimes C, \mathbf{C}_{\lambda+\mu})$-bimodules. On the other hand, the map

$$\begin{gathered} \Delta^*_{\lambda,\mu}((\operatorname{id}\otimes\pi_A)^*(W_A(\lambda))\otimes(\operatorname{id}\otimes\pi_B)^*(W_B(\mu)))\otimes_{\mathbf{C}_{\lambda+\mu}}(M\otimes N) \to (\operatorname{id}\otimes\pi_A)^*(\mathbf{W}_A^\lambda M)\otimes(\operatorname{id}\otimes\pi_B)^*(\mathbf{W}_B^\mu N) \\ (w\otimes w')\otimes_{\mathbf{C}_{\lambda+\mu}}(m\otimes n) \mapsto (w\otimes_{\mathbf{C}_\lambda} m)\otimes(w'\otimes_{\mathbf{C}_\mu} n) \end{gathered}$$

is an isomorphism of $(\mathfrak{g}\otimes C, \mathbf{C}_{\lambda+\mu})$-bimodules. The composition of these two maps yields the desired map. ☐

The next theorem gives a refinement Theorem 4.3.7.

Lemma 4.5.2. *Let $\lambda, \mu \in \Lambda^+$ with $\lambda \not\leq \mu$ and $\mu \not\leq \lambda$. If $U \in \operatorname{Ob}\mathcal{I}_A^\mu$ is irreducible and $U_\mu \neq 0$, then:*

a. For all $M \in \operatorname{Ob}\mathbf{A}_\lambda$-mod, we have

$$\operatorname{Ext}^m_{\mathcal{I}_A}(\mathbf{W}_A^\lambda M, U) = 0, \quad \textit{for } m = 0, 1.$$

b. Let $V \in \operatorname{Ob}\mathcal{I}_A^\lambda$ with $\dim V_\lambda < \infty$. Then $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \cong V$, if and only if,

$$\operatorname{Ext}^m_{\mathcal{I}_A}(V, U) = 0, \quad \textit{for } m = 0, 1, \tag{4.5.1}$$

for all $U \in \operatorname{Ob}\mathcal{I}_A^\lambda$ with $\dim U < \infty$ and $U_\lambda = 0$.

Proof. For (a), consider an irreducible module $U \in \operatorname{Ob}\mathcal{I}_A^\lambda$ and observe that any nonzero morphism $\varphi\colon \mathbf{W}_A^\lambda M \to U$ must be surjective. However this is not possible because $(\mathbf{W}_A^\lambda M)_\mu = 0$, and homomorphisms preserve weight spaces. This proves that $0 = \operatorname{Hom}(\mathbf{W}_A^\lambda M, U) = \operatorname{Ext}^0_{\mathcal{I}_A}(\mathbf{W}_A^\lambda M, U)$. Next, suppose that

$$0 \to U \to V \to \mathbf{W}_A^\lambda M \to 0$$

is a short exact sequence of objects in $\mathcal{I}_A^\lambda$. Then $V_\lambda \neq 0$, the weights of V are contained in $(\mu - Q^+) \cup (\lambda - Q^+)$ and $(\mathfrak{n}^+ \otimes A)V_\lambda = 0$, since $\lambda \not\leq \mu$. If $V' = U(\mathfrak{g} \otimes A)V_\lambda$, then weights of V' lie in $\lambda - Q^+$. In order to prove that the sequence splits, it suffices to show that $V' \cap U = 0$. Indeed, since U is irreducible, if the intersection is not zero, then $U \cap V' = U$, which would imply that $\mu \in \mu - Q^+$, contradicting the fact that $\mu \not\leq \lambda$. Thus the sequence splits, and so the only element in $\operatorname{Ext}^1(\mathbf{W}_A^\lambda M, U)$ is 0 (due to the 1-1 correspondence between classes of extensions of $\mathbf{W}_A^\lambda M$ by U and elements of $\operatorname{Ext}^1(\mathbf{W}_A^\lambda M, U)$, see [Wei94]).

For part (b), notice that the forward implication follows from Theorem 4.3.7. Now, let us prove that $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V = V$ if (4.5.1) holds for all irreducible finite-dimensional module $U \in \operatorname{Ob}\mathcal{I}_A^\lambda$ such that $U_\lambda = 0$. Repeating the arguments used in the proof of Theorem 4.3.7, one prove that $V = U(\mathfrak{g} \otimes A)V_\lambda$. Then the map ϵ_V of the proof of Proposition 4.3.5 is surjective, and we have an exact sequence

$$0 \to K \to \mathbf{W}_A^\lambda V_\lambda \xrightarrow{\epsilon_V} V \to 0.$$

By Corollary 4.4.4, $\dim \mathbf{W}_A^\lambda V_\lambda < \infty$, since $\dim V_\lambda < \infty$. Thus $\dim K < \infty$, and $K_\lambda = 0$. Suppose $K \neq 0$. Then $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(K, U) \neq 0$ for some irreducible module $U \in \mathcal{I}_A^\lambda$ with $U_\lambda = 0$. Next, applying $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(-, U)$, we get the following long exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(V, U) \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V_\lambda, U) \to \operatorname{Hom}_{\mathcal{I}_A^\lambda}(K, U) \to \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(V, U) \to \cdots,$$

where $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(V, U) = \operatorname{Ext}^1_{\mathcal{I}_A^\lambda}(V, U) = 0$, since V satisfies (4.5.1), and $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(\mathbf{W}_A^\lambda V_\lambda, U) = 0$ by part (a). But this implies $\operatorname{Hom}_{\mathcal{I}_A^\lambda}(K, U) = 0$, which is a contradiction. Therefore $K = 0$, proving that $\mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V = \mathbf{W}_A^\lambda V_\lambda \overset{\epsilon_V}{\cong} V$. □

Theorem 4.5.3. *Suppose that A and B are finite-dimensional commutative, associative unital $\mathbb{C}$-algebras and let $\pi_A\colon A \oplus B \twoheadrightarrow A$ and $\pi_B\colon A \oplus B \twoheadrightarrow B$ be the canonical projections. Assume that the system of simple roots Σ satisfies* (4.1.1)*. If $M \in \operatorname{Ob} \mathbf{A}_\lambda$-mod, $N \in \operatorname{Ob} \mathbf{A}_\mu$, $\dim M < \infty$ and $\dim N < \infty$, then*

$$\mathbf{W}^{\lambda+\mu}_{A\oplus B}\left(\Delta^*_{\lambda,\mu}(M \otimes N)\right) \cong (\operatorname{id} \otimes \pi_A)^*(\mathbf{W}^\lambda_A M) \otimes (\operatorname{id} \otimes \pi_B)^*(\mathbf{W}^\mu_B N).$$

as $\mathfrak{g} \otimes (A \oplus B)$-modules.

Proof. To simplify notation in this proof, we denote the $\mathfrak{g} \otimes (A \oplus B)$-modules $\Delta^*_{\lambda,\mu}(M \otimes N)$, $(\operatorname{id} \otimes \pi_A)^*(\mathbf{W}^\lambda_A M)$ and $(\operatorname{id} \otimes \pi_B)^*(\mathbf{W}^\mu_B N)$ by $M \otimes N$, M and N, respectively.

By Proposition 4.5.1, we have an epimorphism of $\mathfrak{g} \otimes (A \oplus B)$-modules

$$\mathbf{W}^{\lambda+\mu}_{A\oplus B}(M \otimes N) \twoheadrightarrow \mathbf{W}^\lambda_A M \otimes \mathbf{W}^\mu_B N.$$

Using Lemma 4.5.2(a), to prove that this epimorphism is an isomorphism, it suffices to prove that

$$\operatorname{Ext}^m_{\mathcal{I}^{\lambda+\mu}_{A\oplus B}}(\mathbf{W}^\lambda_A M \otimes \mathbf{W}^\mu_B N, U) = 0, \quad m = 0, 1$$

for all irreducible $U \in \operatorname{Ob} \mathcal{I}^{\lambda+\mu}_{A\oplus B}$ with $U_{\lambda+\mu} = 0$. By [Sav14, Cor. 4.11] we have that $U \cong U_A \otimes U_B$, where U_A (resp. U_B) is an irreducible $\mathfrak{g} \otimes A$-module (resp. $\mathfrak{g} \otimes B$-module). Moreover, if ν_A and ν_B are the highest weights of U_A and U_B, respectively, then $\nu_A + \nu_B \in \lambda + \mu - Q^+$ (i.e. $\nu_A + \nu_B$ lies in the set of weights of U).

Notice that $\mathfrak{g} \otimes A$ and $\mathfrak{g} \otimes B$ are finite-dimensional Lie superalgebras, and that the modules $\mathbf{W}^\lambda_\lambda M$, U_A and U_B are finite-dimensional as well. Thus, by (1.4.1), if either

$$\operatorname{Ext}^m_{\mathcal{I}^\lambda_A}(\mathbf{W}^\lambda_A M, U_A) = 0 \text{ or } \operatorname{Ext}^m_{\mathcal{I}^\mu_B}(\mathbf{W}^\mu_B N, U_B) = 0 \text{ for } m = 0, 1, \tag{4.5.2}$$

then we have our result.

Assume first that either $U_A \in \operatorname{Ob} \mathcal{I}^\lambda_A$ or $U_B \in \operatorname{Ob} \mathcal{I}^\mu_B$. The fact that $U_{\lambda+\mu} = 0$, implies that either $(U_A)_{\nu_A} = 0$ or $(U_B)_{\nu_B} = 0$. Thus, since $\mathbf{W}^\lambda_A \mathbf{R}^\lambda_A(\mathbf{W}^\lambda_A M) \cong \mathbf{W}^\lambda_A M$ and $\mathbf{W}^\mu_B \mathbf{R}^\mu_B(\mathbf{W}^\mu_B N) \cong \mathbf{W}^\lambda_B N$, (4.5.2) follows from Lemma 4.5.2(b). Now assume $\nu_A \not\leq \lambda$ and $\nu_B \not\leq \mu$. Since $\nu_A + \nu_B < \lambda + \mu$, we have that either $\lambda \not\leq \nu_A$ or $\mu \not\leq \nu_B$ and therefore (4.5.2) follows from Lemma 4.5.2(a). ☐

4.6 Recovering the local Weyl modules

Assume that $\mathfrak{g}$ is either a basic classical Lie superalgebra or $\mathfrak{sl}(n, n)$, $n \geq 2$, and that A is a finitely-generated associative commutative unital $\mathbb{C}$-algebra. Let $\lambda \in \Lambda^+$. In this section, using the super-Weyl functors, we recover the local Weyl modules defined in Section 3.4.

Since $\mathbf{A}_\lambda$ is a finitely generated abelian algebra, any irreducible finite-dimensional $\mathbf{A}_\lambda$-module is one-dimensional. For $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_\mathfrak{h} \in \Lambda^+$, we let $H(\psi)$ denote the one-dimensional irreducible $\mathbf{A}_\lambda$-module, where $xv = \psi(x)v$ for all $x \in \mathbf{A}_\lambda$ and $v \in H(\psi)$. The next result shows that it is possible to recover the local Weyl module via the super-Weyl functor.

Theorem 4.6.1. *Let $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_{\mathfrak{h}} \in \Lambda^+$. Then*

$$\mathbf{W}_A^\lambda H(\psi) \cong W(\psi).$$

Proof. Let $V = \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W(\psi)$. Since $W(\psi) = U(\mathfrak{g} \otimes A)w_\psi$, the map $\epsilon_{W(\psi)} : \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W(\psi) \to W(\psi)$ is surjective (see Proposition 4.3.5). On the other hand, if $U = V/W(\psi)$, then $U_\lambda = 0$. Thus, by Theorem 4.3.7, $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(V, U) = 0$, which implies that

$$W(\psi) \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W(\psi) = \mathbf{W}_A^\lambda H(\psi). \qquad \square$$

The next corollary was proved in the non-super setting, for the twisted case, in [NS13, Lem. 7.5].

Corollary 4.6.2. *A $\mathfrak{g} \otimes A$-module V is isomorphic to the local Weyl module $W(\psi)$ if and only if it satisfies the following conditions:*

a. $V \in \mathrm{Ob}\, \mathcal{I}_A^\lambda$, *where* $\lambda = \psi|_{\mathfrak{h}}$;

b. $\mathbf{R}_A^\lambda V \cong H(\psi)$;

c. $\mathrm{Hom}_{\mathcal{I}_A^\lambda}(V, U) = 0$ *and* $\mathrm{Ext}^1_{\mathcal{I}_A^\lambda}(V, U) = 0$, *for all irreducible finite-dimensional* $U \in \mathrm{Ob}\, \mathcal{I}_A^\lambda$ *with* $U_\lambda = 0$.

Proof. If V satisfies all three conditions, then it follows from Theorem 4.3.7 that

$$V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V \cong \mathbf{W}_A^\lambda H(\psi) = W(\psi).$$

Conversely, notice that the local Weyl module $W(\lambda)$ satisfy the two first properties by definition. Furthermore, it follows from the proof of Theorem 4.6.1 that $W(\psi) \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda W(\psi)$. Therefore the last property follows again by Theorem 4.3.7. $\square$

Remark 4.6.3. As a consequence of Theorem 4.6.1 and Corollary 4.4.4, we obtain that the local Weyl module is finite-dimensional if the system of simple roots satisfies Property (4.1.1).

Chapter 5

Further directions

The definition of global and local Weyl modules for Lie superalgebras given in Chapter 3 opens a number of directions of possible further research. We conclude this thesis by listing some of these.

(a) One should be able to define Weyl modules when $\mathfrak{g}$ is not basic. For example, in Chapter 2, the finite-dimensional irreducible $\mathfrak{g} \otimes A$-modules have been classified in the case that $\mathfrak{g}$ is the queer Lie superalgebra. The nature of the classification (in terms of evaluation modules) seems to indicate that the theory of Weyl modules should be relatively similar to the case considered in this Thesis.

(b) *Twisted* versions of Weyl modules have been defined and investigated in the non-super setting (see [CFS08, FMS13, FKKS12, FMS15]). One should similarly be able to develop a twisted theory of Weyl modules for equivariant map Lie superalgebras.

(c) Kac modules related to different systems of simple roots were compared in [Ser11, Cou]. Since the Weyl modules defined in this Thesis generalizes Kac modules, it is natural to ask what kind of relations do we have between Weyl modules related to different systems. Also, most of properties concerning local Weyl modules were proved under a specific choice of a system of simple roots (see Section 3.4). So it would be interesting to verify if such properties depend on this choice.

(d) Recently, in [SVV, BHLW], local Weyl modules for current algebras have appeared as trace decategorifications of categories used to categorify quantum groups. It is natural to ask how the super analogues of Weyl modules defined in the current paper are related to the super analogues, defined in [KKT], of the afore-mentioned categories.

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