A NEW EXPANSION FOR JACOBI FUNCTION

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ABSTRACT - We obtain an expansion for the product of two hypergeometric functions with different parameters and different arguments. As a by product we obtain a new expansion for the Jacobi function of second kind. Particular cases are discussed.

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INTRODUCTION

In a recent paper [1] we presented a way to obtain many general expansions involving a product of two hypergeometric functions with different arguments. In another recent paper we obtained many new generating functions for the hypergeometric and confluent hypergeometric functions using a know expansion for the confluent hypergeometric function [2]. In boths papers we obtain general expansions which generalize many others results [3], (Brafman, Osscini, Toscano, Montaldi et al).

In this paper we use an expression for a product of two hypergeometric functions with different arguments and different parameters which is interpreted as generating function for the product of two hypergeometric functions with different arguments (or alternatively, an expansion for a hypergeometric function where the argument is a double product) to obtain an expansion for a product of two Jacobi polynomials. A product of two Jacobi polynomials appear, for example, when we solve the Schrödinger differential equation for the symmetrical top parametrized in terms of Euler angles [4], using a Green's function technique.

The paper is organized as follow: In section 1 we present a brief review to obtain the expansion for the product of two hypergeometric function with different arguments.

In the section 2 we discuss the new expansion for the Jacobi function of second kind and some other cases and discussions we present in the last section.

1. A REVIEW OF THE METHOD

In this section we present an algebraic derivation of an expansion for a product of two hypergeometric functions by means of an expansion for the confluent hypergeometric function. We consider the following expansion

$$
\begin{aligned}
{}_1F_1\left[\beta; 1+\alpha\,; \frac{vxz}{(1-z)(1-z+vz)}\right] &= (1-z)^{-\beta+\alpha+1}(1-z+vz)^{\beta}\exp\left(\frac{xz}{1-z}\right)\cdot \\
&\cdot \sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n,\beta;1+\alpha;v)\ {}_1F_1(-n;1+\alpha;x)
\end{aligned}
\tag{1}
$$

where $|z| < 1$ and ${}_2F_1(a,b;c;x)$ is a hypergeometric function and ${}_1F_1(a;c;x)$ is a confluent hypergeometric function.

Introducing the above expansion in the following integral representation for the hypergeometric function

$$
\Gamma(b)s^{-b}\ {}_2F_1(a,b;c;k/s) = \int_0^{\infty}\exp(-st)t^{b-1}\ {}_1F_1(a;c;kt)dt
\tag{2}
$$

which is valid for $|s| > |k|$ with $Re\, b > 0$ and making the integral (Laplace transform of the confluent hypergeometric function) for $s > z/(1-z)$ we obtain

$$
\begin{aligned}
&(1-z)^{1+\alpha-\beta-b}(1-z+vz)^{\beta}(1-z+uv)^{b}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n,\beta+1;1+\alpha;v)\cdot \\
&\cdot\ {}_2F_1(-n,b;1+\alpha;u) = \ {}_2F_1\left[\beta,b;1+\alpha\,; \frac{zuv}{(1-z+vz)(1-z+uz)}\right]
\end{aligned}
\tag{3}
$$

where $|u| < 1$, $|v| < 1$, $|z| < 1$ and we have defined the variable u by $1/u = s - z/(1-z)$.

The above expression is an expansion for the product of two hypergeometric functions with different arguments which can be interpreted as a generating function for the product of two hypergeometric functions with different arguments or an expansion for a hypergeometric function. We note that this expansion is a symmetrical expansion; i.e. the expansion does not change when we interchange $\beta \rightleftharpoons b$ and $v \rightleftharpoons u$. We also note that a formal relation among the parameters does not exist. This fact is important because it permits us to obtain a serie of interesting results.

2. JACOBI POLYNOMIALS

In this section we obtain a new expansion for the Jacobi polynomial with different arguments and different orders. Firstly, we introduce the following variables

$$\frac{v}{x} = \frac{1-z}{1-zx} \quad \text{and} \quad \frac{u}{y} = \frac{1-z}{1-zy} \tag{4}$$

in eq.(3) and we obtain

$$\begin{aligned}(1-z)^{\alpha+1}(1-xz)^{-\beta}(1-zy)^{-b}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\cdot \\ \cdot\ {}_2F_1\left(-n,\beta+1;\alpha+1;\frac{x-xz}{1-xz}\right)\cdot\ {}_2F_1\left(-n,b;\alpha+1;\frac{y-yz}{1-yz}\right) = \\ = {}_2F_1(\beta,b;\alpha+1;xyz)\,.\end{aligned} \tag{5}$$

This expression can be interpreted as an expansion for a hypergeometric function where the argument is a double product or a bilinear formula for the hypergeometric function with different arguments and different parameters.

In consequence of the freedon of the parameters and using the following relation for [5] the Jacobi polynomials, $P_m^{(\mu,\nu)}(x)$,

$$P_m^{(\mu;\nu)}(x) = \frac{\Gamma(m+\mu+1)}{m!\Gamma(\mu+1)}\ {}_2F_1\left(-m,\mu+\nu+m+1;\mu+1;\frac{1-x}{2}\right) \tag{6}$$

we obtain

$$\begin{aligned}(1-z)^{\alpha+1}(1-xz)^{-\beta}(1-zy)^{-b}\sum_{n=0}^{\infty}z^n\frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\cdot \\ \cdot P_n^{(\alpha;b-n-\alpha-1)}\left(\frac{1+yz-2y}{1-yz}\right)P_n^{(\alpha;\beta-n-\alpha)}\left(\frac{1+xz-2x}{1-xz}\right) = \\ = {}_2F_1(\beta,b;\alpha+1;xyz)\end{aligned} \tag{7}$$

where $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$.

In consequence of the freedon of the parameters we can obtain many other results, e.g. when we have $\nu = -\mu$ we obtain an expansion for Legendre function of second kind, and so on. We believe that this results are importants because when we discuss the Laplace or D'Alembert generalized differential equation [6], the Jacobi function appear almost naturally but this an object of the another paper.

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