UNIVERSIDADE ESTADUAL DE CAMPINAS
Faculdade de Engenharia Mecânica

REGIANE AKEMI HIRATA

Limit cycles output feedback stabilization of discrete-time switched affine systems

Estabilização de ciclos limite via realimentação de saída de sistemas afins com comutação a tempo discreto

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Orientadora: Profa. Dra. Grace Silva Deaecto

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UNIVERSIDADE ESTADUAL DE CAMPINAS
FACULDADE DE ENGENHARIA MECÂNICA

DISSERTAÇÃO DE MESTRADO

Estabilização de ciclos limite via realimentação de saída de sistemas afins com comutação a tempo discreto

Autor: Regiane Akemi Hirata

Orientador: Profa. Dra. Grace Silva Deaecto

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RESUMO

Este trabalho trata do projeto de controle de uma função de comutação dependente da saída para sistemas afins com comutação a tempo discreto, de forma a assegurar estabilidade assintótica global de um ciclo limite adequado. Antes de abordar este objetivo principal, a classe de sistemas afins com comutação é apresentada e suas características intrínsecas discutidas. Uma delas está relacionada à limitação da frequência de comutação, o que justifica o estudo da estabilidade assintótica de ciclos limite. Neste contexto, uma família de ciclos limite é determinada de forma a satisfazer critérios de desempenho relacionados à resposta do sistema em regime permanente. Posteriormente, baseado em uma função de Lyapunov convexa e variante no tempo, um conjunto de sub-problemas convexos, expressos em termos de desigualdades matriciais lineares, é fornecido para a determinação de uma função de comutação dependente da saída. Esta função deve estar associada ao ciclo limite, pertencente à família determinada, que minimiza um limitante superior dos índices de desempenho $\mathcal{H}_2$ ou $\mathcal{H}_\infty$. Alguns exemplos acadêmicos e o controle de um conversor CC-CC de três células são usados para validação e comparação.

Palavras–chave: Sistemas afins com comutação; Estabilidade de ciclos limite; Realimentação estática de saída; Domínio do tempo discreto.

ABSTRACT

This work deals with control design of a static output-dependent switching function for discrete-time switched affine systems to ensure global asymptotic stability of a suitable limit cycle. Before tackling this main goal, the class of switched affine systems is presented and its intrinsic characteristics are discussed. One of them is related to the switching frequency limitation that justifies the study of asymptotic stability of limit cycles. In this context, a family of limit cycles is determined that satisfies performance criteria of interest related to the system steady-state response. Afterwards, based on a time-varying convex Lyapunov function, a set of convex sub-problems, expressed in terms of linear matrix inequalities, is provided to determine an output-dependent switching function. This function must be associated to the limit cycle, belonging to the determined family, that minimizes an upper bound to the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance indexes. Some academical examples and the control of a DC-DC three-cells power converter are used for validation and comparison.

Keywords: Switched affine systems; Limit cycle asymptotic stability; Static output feedback; Discrete-time domain.

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS AND ACRONYMS

DC	Direct Current
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant
SISO	Single-Input Single-Output

LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{R}^n$	Set of real vectors of order n
$\mathbb{R}^{n\times m}$	Set of real matrices of dimension $n \times m$
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_-$	Set of natural numbers including $\{-1\}$, that is, $\mathbb{N}_- = \mathbb{N} \cup \{-1\}$
$\mathbb{K}$	Set of the first N positive natural numbers (i.e. $\mathbb{K} = \{1, \cdots, N\}$)
$\mathcal{H}_2$	$\mathcal{H}_2$ norm
$\mathcal{H}_\infty$	$\mathcal{H}_\infty$ norm
$\mathcal{J}_2$	$\mathcal{H}_2$ performance index
$\mathcal{J}_\infty$	$\mathcal{H}_\infty$ performance index
$\mathscr{L}$	Laplace operator
s	Laplace transform variable
$\mathfrak{z}$	$\mathcal{Z}$ transform variable
$X > (<)0$	Positive (negative) definite matrix
$\mathrm{diag}(X, Y)$	Diagonal matrix with submatrices X and Y
$\bullet$	Symmetric matrix block
$(\cdot)^\sim$	Transpose conjugate of a signal $(\cdot)$
$\mathrm{Tr}(X)$	Trace of matrix a X
Λ	Unit simplex, i.e. $\Lambda = \left\{\lambda_i \geq 0,\ \sum_{i=1}^{N} \lambda_i = 1\right\}$
X_λ	Convex combination of matrices X_i, i.e. $X_\lambda = \sum_{i=1}^{N} \lambda_i X_i,\ \lambda \in \Lambda$

$\mu_{max}(X)$	Maximum singular value of matrix X
$\gamma_i(X)$	i-th eigenvalue of matrix X
$c = a \bmod b$	Remainder of the Euclidean division between a and b
$k(n)$	n mod κ, where $\kappa > 0$
$\lvert z \rvert$	Absolute value of z
$\\|z\\|_2$	L_2-norm of a trajectory $z[n]$, that is, $\\|z\\|_2^2 = \sum_{n\in\mathbb{N}} z[n]'z[n]$
L_2	Set composed of all trajectories $z[n]$ such that $\\|z\\|_2 < \infty$
$\\|z\\|$	Euclidean norm of vector z, that is, $\\|z\\|^2 = z'z$
$\\|z\\|_\infty$	∞-norm of vector z, that is, $\\|z\\|_\infty = \max_i \lvert z_i \rvert$

CONTENTS

1 INTRODUCTION

Hybrid systems are characterized by the interaction between continuous dynamics and discrete events. They consist in a research field that has attracted the attention of the scientific community in the last decades, mainly for the great applicability in several engineering areas and for the theoretical challenges brought by their intrinsic properties. A subclass of great importance is composed of the switched systems. They are defined by a finite number of subsystems and by a rule (or function) responsible for selecting a subsystem at each instant of time. This rule can be modeled as a perturbation or as a control signal to be determined in order to ensure stability and performance to the closed-loop system. In this thesis our focus is to consider the switching rule as a control signal. The books (LIBERZON, 2003), (SUN; GE, 2011) and the article (SHORTEN *et al.*, 2007) are basic references about the theme.

Inside the class of switched systems, the switched affine systems are characterized by having affine terms in its dynamic equation. When all affine terms are null, the system becomes linear and the unique equilibrium point, common to all subsystems, is the origin. For this simpler case, the literature presents several results for both, continuous and discrete-time domains. These results include stability analysis (GEROMEL; COLANERI, 2006b), (LIN; ANTSAKLIS, 2009), state feedback control design (GEROMEL; DEAECTO, 2009), (OGURA *et al.*, 2016), output feedback control design (DEAECTO; DAIHA, 2020), (DEAECTO *et al.*, 2011a), (GEROMEL *et al.*, 2008) and robust control (DEAECTO *et al.*, 2011b).

However, if at least one affine term is non-null, the system presents a set of attainable equilibrium points, forming a region of great interest in the state-space. For this case, the control goal is to design a switching function in order to ensure global asymptotic stability of a desired equilibrium point as well as a suitable performance to the overall system. This goal is only accomplished for continuous-time systems, when the switching frequency is arbitrarily high. This occurs due to the fact that the equilibrium point of interest generally does not coincide with those of the subsystems and, consequently, an extremely high switching frequency is mandatory to keep the state trajectories fixed in this point during the steady-state, which indicates that the system always evolves on a stable sliding mode. References (DEAECTO *et al.*, 2010), (BOLZERN; SPINELLI, 2004) and (TROFINO *et al.*, 2009) are some examples where the global asymptotic stability is ensured.

A hindrance of this control technique is that in real systems a high switching frequency (or chattering) is undesirable, since it may cause equipment wear or it is not implementable due to physical limitations. This is always the case of discrete-time systems obtained through a suitable discretization procedure, which imposes an upper bound to the switching frequency. As an alternative, the literature provides two different strategies to deal with constraints on the switching frequency. The first one deals with practical stability. In this case, the system trajectories are guided to an invariant set of attraction, as small as possible, that contains the desired equilibrium point, see (HETEL; FRIDMAN, 2013), (SANCHEZ *et al.*, 2019) and (EGIDIO; DEAECTO, 2019). The inconvenience of this approach is that nothing can be concluded about the steady-state response, because there is no information about the state trajectories once they are inside the invariant set, see (BENMILOUD *et al.*, 2019). Moreover, neither $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes can be taken into account, since they are defined, exclusively, for asymptotically stable systems.

To circumvent this problem, references (BENMILOUD *et al.*, 2019), (PATINO *et al.*, 2010), (EGIDIO *et al.*, 2020) and (SERIEYE *et al.*, 2023) have treated stabilization of limit cycles. This approach allows the designer to determine a suitable limit cycle that satisfies criteria associated to the steady-state response. For continuous-time systems, reference (BENMILOUD *et al.*, 2019) treated local stabilization using a hybrid Poincaré map approach and (PATINO *et al.*, 2010) proposed a methodology based on predictive control that uses sensitivity functions and Newton algorithm. For discrete-time systems, (SERIEYE *et al.*, 2023) treated state feedback global asymptotic stabilization of limit cycles, but without considering any performance index. The reference (EGIDIO *et al.*, 2020) has dealt with the same problem and provided a state-dependent switching function that minimizes upper bounds of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes. The continuous-time counterpart of this last result, but without considering performance indexes, is available in (EGIDIO *et al.*, 2020).

In this context, and also motivated by practical situations in which not all states are available for measurement, the present work provides sufficient conditions, based on a convex time-varying Lyapunov function, expressed in terms of linear matrix inequalities (LMIs) to the design of an output-dependent switching function in order to ensure global asymptotic stability of a desired limit cycle, without considering any additional dynamic structure. Moreover, the results are associated to the existence of an asymptotically stable periodic switching function, whose $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance index is worse or at most equal the one obtained with the proposed closed-loop switching function. Since, this result is a generalization of the reference (EGIDIO *et al.*, 2020), that treats state feedback control, exclusively, we have compared this technique with the more recent one proposed in (SERIEYE *et al.*, 2023) that does not consider any peformance index. For this reason, we have included a guaranteed cost in the results of (SERIEYE *et al.*, 2023) for the sake of comparison. Throughout this thesis the authors will find academical examples to illustrate the main theoretical features and a practical application in the area of power electronics.

1.1 Publication List

- G. S. Deaecto, R. A. Hirata, M. C. M. Teixeira, “Static output feedback global asymptotic stability of limit cycles for discrete-time switched affine systems”, *Preprints of the IFAC World Congress*, Yokohama-JP, pp. 4534–4539, 2023.
- R. A. Hirata, G. S. Deaecto, M. C. M. Teixeira, "Estabilização via realimentação estática de saída de ciclos limites para sistemas afins com comutação a tempo discreto", *Anais do Simpósito Brasileiro de Automação Inteligente-SBAI*, Manaus-AM, submetido.

1.2 Thesis Structure

This thesis is structured in four chapters as presented in the sequel:

- **Chapter 1 - Introduction** :
  In the introduction, it is provided the state of the art of switched affine systems. More specifically, it is presented the motivation to study this subclass of systems, as well as the challenges to be faced when stability and performance is taken into account. The peculiarities of these systems in the discrete-time domain or when the switching frequency is limited are also presented and discussed.
- **Chapter 2 - Fundamental Concepts** :
  In the second chapter, some well-known concepts related to the study of dynamical systems are provided. They go from discretization of a linear time invariant (LTI) continuous-time systems to some definitions regarding stability of dynamic systems, which allow us to present the Lyapunov’s stability criterion, and finalizes with the definition of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms with their calculus through LMIs.

- **Chapter 3 - Switched Affine Systems** :

  In this chapter, some intrinsic properties of the switched affine systems are highlighted and discussed. At first, sufficient conditions, borrowed from the literature, for the state feedback control design of continuous-time switched affine systems are presented. Afterwards, an academic example is provided to illustrate the main features of this class of systems and the challenges to overcome when the switching frequency is limited.

- **Chapter 4 - Stabilization of Limit Cycles** :

  This chapter is dedicated to the main results of this thesis. Initially, the limit cycle generation is provided. Then, some results from (EGIDIO *et al.*, 2020) are recalled, which are the basis for generalization to obtain the output feedback control design. To validate these results with respect to recent ones from the literature, a guaranteed cost has been included in the results of (SERIEYE *et al.*, 2023) in order to compare both strategies. At last, sufficient conditions to the design of an output-dependent switching function ensuring global asymptotic stability of the desired limit cycle and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performance indexes are presented, and illustrated by means of a practical application example.

- **Chapter 5 - Conclusion** :

  An overview on the main topics of this thesis is provided, highlighting the main contributions. Then, a discussion with respect future works is presented.

The numerical simulations were performed in Matlab - R2017a using LMI Solver routines in an Apple computer with operating system Mac OS X version 10.15.7.

2 FUNDAMENTAL CONCEPTS

In this chapter, some fundamental concepts are presented forming the basis for the study of dynamical systems. Initially, a linear time-invariant system and its discretized model are presented. Afterward, some important stability concepts including the well-known Lyapunov theorem are provided for the system in the discrete-time domain. This chapter ends with the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms and their calculus through the solution of convex optimization problems described in terms of linear matrix inequalities. All the results of this chapter are well-established in the literature of control theory and can be found in several books of the area, as for instance, (KHALIL, 2002), (CHEN, 2013), (GEROMEL; KOROGUI, 2011) and (BOYD *et al.*, 1994).

2.1 Linear Time-Invariant Systems

Consider a Linear Time-Invariant (LTI) system in the continuous-time domain described in the state space form as

$$\dot{x}(t) = A_o x(t) + H_o w(t),\ x(0) = x_0,\ t \geq 0 \tag{2.1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, x_0 is the initial condition and $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input. Its general solution is given by

$$x(t) = e^{A_o t} x_0 + \int_0^t e^{A_o(t-\tau)} H_o w(\tau)\, d\tau \tag{2.2}$$

which represents the dynamical behavior of several real-world systems. When some control law is included, it generally occurs by means of digital controllers due to the flexibility to make changes in the control algorithm and the recent low-cost production of digital computers. In this case, the behavior of the overall system can be obtained by the equivalent discretized system, which is one of the motivations of studying discrete-time systems, besides the ones that are intrinsically defined in this time-domain. This thesis is focused on discrete-time systems, which represents the behavior of several real-world models, mainly when some digital actuation is taken into account.

To obtain the equivalent discrete-time system of (2.1) let us suppose that the external input is piecewise constant $w(t) = w(t_n),\ \forall t \in [t_n, t_{n+1})$ with $t_{n+1} - t_n = T > 0$ and $n \in \mathbb{N}$. The time instants $t_n = nT$ and $t_{n+1} = (n+1)T$ are sampling instants and T is the sampling period. The general solution of (2.1) for $t \in [t_n, t_{n+1})$ is given by

$$\begin{aligned} x(t) &= e^{A_o(t-t_n)}x(t_n) + \int_{t_n}^{t} e^{A_o(t-\tau)}H_o\, d\tau \;\; w(t_n) \\ &= e^{A_o(t-t_n)}x(t_n) + \int_{0}^{t-t_n} e^{A_o\varphi}H_o\, d\varphi \;\; w(t_n) \end{aligned} \tag{2.3}$$

where it was used the change of variable $\varphi = t - \tau$ and the fact that $w(t_n)$ is constant in this time interval. Hence, defining $x(t_n) = x(nT) = x[n]$, we obtain

$$x(t_{n+1}) = e^{A_oT}x(t_n) + \int_0^T e^{A_o\varphi}H_o\, d\varphi \;\; w(t_n) \tag{2.4}$$

which leads to the equivalent discrete-time system

$$x[n+1] = Ax[n] + Hw[n],\; x[0] = x_0 \tag{2.5}$$

with $A = e^{A_oT}$ and $H = \int_0^T e^{A_o\varphi}H_o\, d\varphi$. The general solution of this system is given by

$$x[n] = A^n x[0] + \sum_{i=0}^{n-1} A^{(n-1-i)}Hw[i] \tag{2.6}$$

Notice that the discretized matrices can be obtained in a more direct way as follows

$$e^{\mathcal{A}T} = \begin{bmatrix} A & H \\ 0 & I \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A_o & H_o \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

as provided in (SOUZA *et al.*, 2014). This can seen using the Laplace transformation

$$\begin{aligned} e^{\mathcal{A}t} &= \mathscr{L}^{-1}\left\{(sI - \mathcal{A})^{-1}\right\} \\ &= \mathscr{L}^{-1}\left\{\begin{bmatrix} sI - A_o & -H_o \\ 0 & sI \end{bmatrix}^{-1}\right\} \\ &= \mathscr{L}^{-1}\left\{\begin{bmatrix} (sI - A_o)^{-1} & \frac{1}{s}(sI - A_o)^{-1}H_o \\ 0 & \frac{1}{s}I \end{bmatrix}\right\} \\ &= \begin{bmatrix} e^{A_ot} & \int_0^t e^{A_o\varphi}\, d\varphi H_o \\ 0 & I \end{bmatrix} \end{aligned} \tag{2.8}$$

which evaluated for $t = T$ provides (2.7).

2.2 Basic Concepts on Dynamical Systems

In this section, a series of definitions is presented that will be extensively used in this thesis. Since these definitions are general for dynamical systems, let us consider a generic system as follows

$$x[n+1] = f(x[n]) \tag{2.9}$$

where $f : \mathcal{D} \to \mathbb{R}^{n_x}$ with $\mathcal{D} \subset \mathbb{R}^{n_x}$.

Definition 2.2.1. *A point x_e is said to be an equilibrium point of the system* (2.9) *if it satisfies*

$$f(x_e) = x_e, \ \forall n \in \mathbb{N}$$

In other words if $x[n_0] = x_e$ then $x[n] = x_e$ for all $n \geq n_0$. Notice that the origin is the unique equilibrium point of the linear system $x[n+1] = Ax[n]$ whenever matrix $(I - A)$ is non-singular.

The next definitions consider that the origin is the unique equilibrium point of the system (2.9). This is a very common approach, since it is always possible to shift the equilibrium point $x_e \neq 0$ to the origin using a simple change of variable such as $\xi[n] = x[n] - x_e$, which implies that $\xi[n] \to 0$ whenever $x[n] \to x_e$. A illustration of each definition is given in Figure 2.1.

Definition 2.2.2. *(SLOTINE; LI, 1991) The equilibrium point $x_e = 0$ is said to be stable, if for any $R > 0$, there exists $r > 0$ such that*

$$\|x[0]\| < r \implies \|x(n)\| < R, \ \forall n \geq 0$$

Otherwise, the equilibrium point $x_e = 0$ is unstable.

Assuming that the origin is a stable equilibrium point, then if the initial condition of the system is close enough of the origin, the trajectories are guaranteed to stay inside the ball of radius $R > 0$ for any instant $n \geq 0$.

Definition 2.2.3. *The equilibrium point $x_e = 0$ is said to be asymptotically stable if both statements are true:*

- *It is stable*
- $\|x[0]\| < r \implies \|x\| \to 0$ *as* $n \to \infty$

Hence, whenever the initial condition is within the radius r, then the state trajectories reach the origin as $n \to \infty$.

Definition 2.2.4. *The equilibrium point $x_e = 0$ is said to be globally asymptotically stable if*

$$\|x\| \to 0 \ \ as \ \ n \to \infty$$

is valid for all $x_0 \in \mathbb{R}^{n_x}$.

It means that independently of the initial condition the state trajectories go always to the origin.

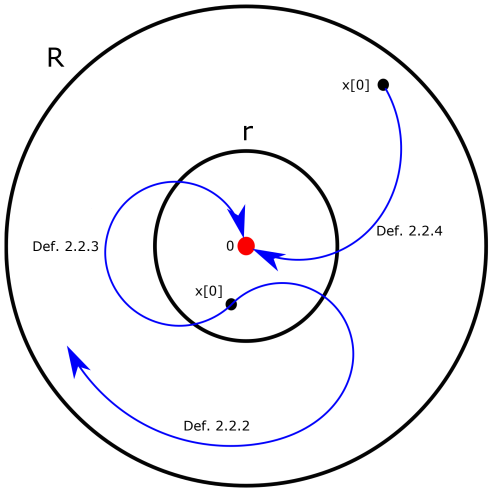


Figure 2.1 – Stability concepts

2.2.1 Stability of Dynamical Systems

The stability of a system is defined considering an equilibrium point. A general approach to verify stability is through the Lyapunov's direct method. This method is based on the physical observation that if the total energy of a mechanical (or electrical) system is continuously dissipated, then the system trajectories must tend to an equilibrium point (SLOTINE; LI, 1991). With this idea, the scientist Aleksandr M. Lyapunov showed, in 1892, that a certain scalar function can be used to analyze the stability of a generic dynamical system (KHALIL, 2002). Based on this function, the next theorem from (KHALIL, 2002), adapted for the discrete-time domain, provides the Lyapunov's stability theorem.

Theorem 2.1. *(KHALIL, 2002) Let $x_e = 0$ be an equilibrium point for* (2.9) *and $\mathcal{D} \subset \mathbb{R}^{n_x}$ be a domain containing $x_e = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a scalar function such that*

- $V(0) = 0$ *and* $V(x) > 0, \ \forall x \in \mathcal{D}$, $x \neq 0$

- $\Delta V(x) \leq 0 \ \ \forall x \in \mathcal{D}$

Then $x_e = 0$ *is stable. Moreover, if*

- $\Delta V(x) < 0 \ \ \forall x \in \mathcal{D}$, $x \neq 0$

Then $x_e = 0$ *is asymptotically stable.*

The function $V(x)$ that satisfies the conditions of this theorem is named Lyapunov function. Notice that, whenever the domain $\mathcal{D}$ is defined as the full state space $\mathcal{D} \equiv \mathbb{R}^{n_x}$ the stability of the equilibrium point x_e is global. The next theorem provides this result.

Theorem 2.2. *Let* $x_e = 0$ *be an equilibrium point for* (2.9). *Let* $V : \mathbb{R}^{n_x} \to \mathbb{R}$ *such that*

- $V(0) = 0$ *and* $V(x) > 0, \ \forall x \neq 0$

- $\|x\| \to \infty \implies V(x) \to \infty$

- $\Delta V(x) < 0, \ \forall x \neq 0$

Then $x_e = 0$ *is globally asymptotically stable.*

More details about these results are presented in (SLOTINE; LI, 1991) e (KHALIL, 2002). The next subsection applies these concepts to study the stability of the LTI system (2.5).

2.2.2 Stability of LTI systems

Let us consider the dynamical system (2.5) with $w[n] = 0 \ \forall n \in \mathbb{N}$ given by

$$x[n+1] = Ax[n], \ x[0] = x_0 \tag{2.10}$$

The origin of this system is globally asymptotically stable whenever all eigenvalues of A lie within the unit circle centered at the origin. In this case, the system is said to be Schur stable, see (FRANKLIN *et al.*, 2002).

For this class of systems, the quadratic function given by

$$V(x) = x[n]'Px[n] \tag{2.11}$$

defined $\forall n \in \mathbb{N}$ and $P > 0$ is a good candidate to Lyapunov function. Indeed, notice that from Theorem 2.2, the first and second itens are clearly satisfied. In order to analyze the third item, we have

$$\Delta V(x) = x[n+1]'Px[n+1] - x[n]'Px[n] \tag{2.12}$$

$$= (Ax[n])'P(Ax[n]) - x[n]'Px[n] \tag{2.13}$$

$$= x[n]'(A'PA - P)x[n] \tag{2.14}$$

To guarantee that $\Delta V(x) < 0, \ \forall x \neq 0$, it is imposed that

$$\Delta V(x) = x[n]'(A'PA - P)x[n] = -x[n]'Qx[n] < 0 \tag{2.15}$$

with Q being any given positive definite matrix. Hence, the third item of Theorem 2.2 is satisfied indicating that $x_e = 0$ is globally asymptotically stable. Actually, the solution to the so called Lyapunov equation

$$A'PA - P + Q = 0, \quad P > 0 \tag{2.16}$$

is not only a sufficient condition but also necessary for the stability of the LTI system (2.10), as formalized in the next theorem.

Theorem 2.3. *The LTI system* (2.10) *is globally asymptotically stable if and only if for a given* $Q > 0$*, there exists* $P > 0$ *that satisfies the Lyapunov equation*

$$A'PA - P = -Q$$

Moreover, $P > 0$ *is the unique solution of this equation.*

Proof: The proof is available in (CHEN, 2013), but will be repeated here for convenience. To prove the necessity, let us assume that matrix A is Schur stable and, consequently, its eigenvalues satisfy $|\gamma_i| < 1 \ \forall i \in \{1, \cdots, n_x\}$. We need to show that under this assumption, matrix $P > 0$ is the unique solution of the Lyapunov equation

$$A'PA - P = -Q \tag{2.17}$$

Consider

$$P = \sum_{j=0}^{\infty} (A')^j Q (A)^j \tag{2.18}$$

for a given $Q > 0$, which implies that the candidate P is also positive definite. Since matrix A is Schur-stable, then the summation (2.18) converges and is well defined. Replacing (2.18) in (2.17), we obtain

$$
\begin{aligned}
A'PA - P &= A'\left(\sum_{j=0}^{\infty}(A')^jQ(A)^j\right)A - \sum_{j=0}^{\infty}(A')^jQ(A)^j \\
&= \sum_{j=0}^{\infty}(A')^{j+1}Q(A)^{j+1} - \sum_{j=0}^{\infty}(A')^jQ(A)^j \\
&= -Q
\end{aligned}
\tag{2.19}
$$

which indicates that (2.18) satisfies indeed the Lyapunov equation. To show that (2.18) is the unique solution of this equation, let us suppose that there is another solution $\tilde{P} > 0$ satisfying $A'\tilde{P}A - \tilde{P} = -Q$. Subtracting this equation from (2.17), we have

$$
A'(P - \tilde{P})A - P - \tilde{P} = 0 \tag{2.20}
$$

Multiplying this equality to the right by A^j and the the left by the transpose, and summing the result for all $j = 0$ up to infinity, we have

$$
\sum_{j=0}^{\infty}(A')^{j+1}(P - \tilde{P})(A)^{j+1} - \sum_{j=0}^{\infty}(A')^j(P - \tilde{P})(A)^j = -(A')^0(P - \tilde{P})A^0 = 0 \tag{2.21}
$$

which indicates that $P = \tilde{P}$, concluding the proof of necessity. To show the sufficiency, we need to suppose that $P > 0$ is the unique solution of the Lyapunov equation (2.17) and to show that A is Schur stable. Consider that γ_i as the i-th eigenvalue of A associated to eigenvector v_i, such as $Av_i = \gamma_i v_i$ then

$$
\begin{aligned}
-v_i^{\sim}Qv_i &= v_i^{\sim}A'PAv_i - v_i^{\sim}Pv_i \\
&= v_i^{\sim}\bar{\gamma}_iP\gamma_iv_i - v_i^{\sim}Pv_i \\
&= (|\gamma_i|^2 - 1)v_i^{\sim}Pv_i
\end{aligned}
\tag{2.22}
$$

Since $v_i^{\sim}Qv_i > 0$ and $v_i^{\sim}Pv_i > 0$, then $1 - |\gamma_i|^2 > 0 \implies |\gamma_i|^2 < 1$ for all $i \in \{1, \cdots, n_x\}$, concluding the proof. □

The next subsection defines the two most important performance indexes to analyze dynamical systems. They will be extensively adopted in this thesis.

2.3 Performance Indexes

In the previous section, important concepts regarding stability were presented. However, as it is well known in the literature, to ensure stability is the first step in a control design problem, the next one is to guarantee some performance towards an objective. In the single input and output (SISO) control theory, considering the hypothesis of dominant poles, some performance indexes, well-defined, are the overshooting and stabilization time for the transient state. Yet, this approach cannot be extended when robustness is taken into account. Considering a more general scenario, two very important performance criteria are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, which are defined for the more general system

$$x[n+1] = Ax[n] + Hw[n],\ x[0] = 0 \tag{2.23}$$

$$z[n] = Ex[n] + Gw[n] \tag{2.24}$$

for all $n \in \mathbb{N}$, where $z[n] \in \mathbb{R}^{n_z}$ is the controlled output. The transfer function of this system is given by

$$F(\mathfrak{z}) = E(\mathfrak{z}I - A)^{-1}H + G \tag{2.25}$$

and the associated impulse response $h[n]$ is as follows

$$h[n] = \begin{cases} G & , \quad n = 0 \\ EA^{n-1}H & , \quad n \geq 1 \end{cases} \tag{2.26}$$

The next subsections treat both norms separately.

2.3.1 $\mathcal{H}_2$ norm

This norm is defined for asymptotically stable systems, with transfer function $F(\mathfrak{z})$ analytic in all complex plane except in the interior of the unit circle centered at the origin, as being

$$\|F(\mathfrak{z})\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\big(F(e^{j\omega})^{\sim}F(e^{j\omega})\big) \tag{2.27}$$

where the symbol $^{\sim}$ indicates the transpose conjugate of the signal. Alternatively, using the Parseval theorem, we can express this norm in the time-domain as follows

$$\|F(\mathfrak{z})\|_2^2 = \sum_{n=0}^{\infty} \mathrm{Tr}(h[n]'h[n])) \tag{2.28}$$

Replacing the impulse response $h[n]$ of (2.26) in (2.28), we obtain

$$\begin{aligned}\|F(\mathfrak{z})\|_2^2 &= \sum_{n=1}^{\infty} \mathrm{Tr}\left((EA^{n-1}H)'(EA^{n-1}H)\right) + \mathrm{Tr}(G'G)\\ &= \mathrm{Tr}\left(H'\Big(\sum_{m=0}^{\infty}(A^m)'E'EA^m\Big)H\right) + \mathrm{Tr}(G'G) \qquad (2.29)\end{aligned}$$

where we have used the change of variable $m = n - 1$ and considered that the trace is a linear function in order to put the summation inside $\mathrm{Tr}(\cdot)$. Notice that

$$P_o = \sum_{n=0}^{\infty}(A^n)'E'EA^n$$

is the solution of the Lyapunov equation

$$A'P_oA - P_o + E'E = 0$$

and is named observability gramian. Hence, the $\mathcal{H}_2$ norm can be calculated by means of the observability gramian as follows

$$\|F(\mathfrak{z})\|_2^2 = \{\mathrm{Tr}(H'P_oH + G'G) \ : \ A'P_oA - P_o + E'E = 0, \ P_o > 0\} \qquad (2.30)$$

Alternatively, this norm can be determined by the solution of the following convex optimization problem expressed in terms of LMIs

$$\|F(\mathfrak{z})\|_2^2 = \inf_{P>0} \mathrm{Tr}(H'PH + G'G) \qquad (2.31)$$

subject to

$$A'PA - P + E'E < 0 \qquad (2.32)$$

Indeed, notice that (2.32) can be rewritten as the Lyapunov equation

$$A'PA - P + E'E = -S \qquad (2.33)$$

for some $S > 0$. Hence, we have that

$$\begin{aligned}P &= \sum_{n=0}^{\infty}(A^n)'(E'E + S)A^n\\ &> P_o \qquad (2.34)\end{aligned}$$

and the minimum operator in (2.31) makes the solution of this problem to reach the value obtained in (2.30).

2.3.2 $\mathcal{H}_\infty$ Norm

As before, this norm is determined for systems with transfer function $F(\mathfrak{z})$, analytic in all complex plane, except in the interior of the unit circle centered at the origin. It is defined as

$$\|F(\mathfrak{z})\|_\infty = \max_{\omega\in[-\pi,\pi]} \mu_{max}(F(e^{j\omega})) \tag{2.35}$$

where $\mu_{max}(F(e^{j\omega}))$ is the maximum singular value of the transfer matrix $F(e^{j\omega})$, that is

$$\mu_{max}(F(e^{j\omega})) = \max_{i\in\{1,\cdots,n_x\}} \sqrt{\gamma_i(F(e^{j\omega})^\sim F(e^{j\omega}))} \tag{2.36}$$

Notice that for the scalar case, this definition assume the simplest form

$$\|F(\mathfrak{z})\|_\infty = \max_{\omega\in[-\pi,\pi]} |F(e^{j\omega})| \tag{2.37}$$

which is the peak of the magnitude Bode plot.

We can determine this norm in the time domain by using the fact that $\hat{z} = F(\mathfrak{z})\hat{w}$. Moreover, with $\|F(\mathfrak{z})\|_\infty^2 < \rho$ we obtain $F(e^{j\omega})^\sim F(e^{j\omega}) < \rho$, which multiplied to the right by $\hat{w}$ and to the left by the transpose conjugate provides

$$\hat{z}^\sim \hat{z} - \rho\hat{w}^\sim\hat{w} < 0 \tag{2.38}$$

which implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\hat{z}^\sim \hat{z} - \rho\hat{w}^\sim\hat{w})\, d\omega < 0 \tag{2.39}$$

Applying the Parseval theorem, we obtain the equivalent of (2.39) in the time-domain as being

$$\sum_{n=0}^{\infty}(z[n]'z[n] - \rho w[n]'w[n]) < 0 \tag{2.40}$$

We can observe that this inequality is valid for all trajectory $w \neq 0$ such that

$$\|w\|_2^2 = \sum_{n=0}^{\infty} w[n]'w[n] < \infty \tag{2.41}$$

Let us recall that the set of all trajectories whose this summation is finite belongs to the set L_2. Hence, it is clear that for a given $\rho > 0$ the inequality $\|F(\mathfrak{z})\|_\infty^2 < \rho$ is true if and only if

$$\sup_{w\neq 0\in L_2} \|z\|_2^2 - \rho\|w\|_2^2 < 0 \tag{2.42}$$

is satisfied. The solution of this problem allows us to determine upper bounds for the $\mathcal{H}_\infty$ norm, being the smallest one close to its value.

Notice that, imposing the inequality

$$\Delta V(x[n]) < -z[n]'z[n] + \rho w[n]'w[n] \tag{2.43}$$

with the quadratic Lyapunov function defined in (2.11) and summing for all $n = 0$ up to infinity, we obtain

$$\lim_{n\to\infty} V(x[n]) - V(x[0]) < -\|z\|_2^2 + \rho\|w\|_2^2 \tag{2.44}$$

The left hand side of this inequality is null since $\lim_{n\to\infty} V(x[n]) = 0$, as a consequence of the system stability, and $V(x[0]) = 0$ because $x[0] = 0$, which leads to (2.40). Hence, we can obtain the $\mathcal{H}_\infty$ norm of the system by imposing directly (2.43). Indeed, taking into account that

$$\Delta V(x) = \begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} A'PA - P & \bullet \\ H'PA & H'PH \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{2.45}$$

and

$$z[n]'z[n] - \rho w[n]'w[n] = \begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} E'E & \bullet \\ G'E & G'G - \rho I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{2.46}$$

then, imposing (2.43) is equivalent to satisfy

$$\begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} A'PA - P + E'E & \bullet \\ H'PA + G'E & H'PH + G'G - \rho I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0 \tag{2.47}$$

which allows us to formulate the calculus of the $\mathcal{H}_\infty$ norm as the solution of the following convex optimization problem

$$\|F(\mathfrak{z})\|_\infty^2 = \inf_{P>0,\ \rho>0} \rho \tag{2.48}$$

subject to

$$\begin{bmatrix} A'PA - P + E'E & \bullet \\ H'PA + G'E & H'PH + G'G - \rho I \end{bmatrix} < 0 \tag{2.49}$$

Differently from the $\mathcal{H}_2$ norm that depends on the system response to impulsive external inputs, the $\mathcal{H}_\infty$ norm admits a different interpretation that is associated to system robustness. This norm is one of the most important concepts that make possible the study of systems subject to uncertainties as, for instance, those associated to a delay in the state variables, (GEROMEL; KOROGUI, 2011).

2.4 Final Considerations

In this chapter, the main concepts concerning the study of dynamical systems have been presented with focus in the Lyapunov's stability criterion and in the calculus of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, including their description as convex optimization problems expressed in terms of LMIs. The main references were the books (KHALIL, 2002), (CHEN, 2013) and (SLOTINE; LI, 1991) for stability study and (GEROMEL; KOROGUI, 2011) and (COLANERI *et al.*, 1997) for norms calculations.

3 SWITCHED AFFINE SYSTEMS

This chapter is dedicated to present the class of switched affine systems that are the focus of our attention in this thesis. For continuous-time systems, we present sufficient conditions, borrowed from the literature, to the control design of a switching rule able to ensure global asymptotic stability of a desired equilibrium point and a guaranteed cost of performance. Through an academical example, the intrinsic characteristics of these systems are explored and the main challenges discussed. One of them that motivates the study of the next chapter is that the asymptotic stability is generally impossible to be ensured when the switching frequency is limited, which indicates difficulties to be faced in studying these systems in the discrete-time domain. The basic references on switched affine systems are the books (LIBERZON, 2003), (SUN; GE, 2011) and the paper (DEAECTO *et al.*, 2010).

3.1 Stability and Guaranteed Cost

Switched systems represent a subclass of hybrid systems and are characterized by presenting a finite number of subsystems and a rule (or function) that is responsible for choosing one of them at each instant of time. In this thesis, the subclass of interest is composed by the switched affine systems whose state space realization is given as follows

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)} \tag{3.1}$$

$$z(t) = E_{\sigma(t)}x(t) \tag{3.2}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output and $\sigma(t) : \mathbb{R}_+ \to \mathbb{K} := \{1, \cdots, N\}$ is the switching rule that selects at each instant of time one of the N available subsystems. An interesting point about affine systems is that when $b_i = 0,\ \forall i \in \mathbb{K}$, the system becomes linear and the origin is the unique equilibrium point. For this class, the control problem is simpler and the literature presents several results dealing with analysis and control design, see (GEROMEL; COLANERI, 2006a), (DEAECTO *et al.*, 2011a).

Let us define the unit simplex as being

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N : \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \right\} \tag{3.3}$$

and the convex combination of matrices $\{X_1, \cdots X_N\}$ as

$$A_\lambda = \sum_{i=1}^{N} \lambda_i A_i, \ \lambda \in \Lambda \tag{3.4}$$

A very simple but important result in the context of switched linear systems is presented in the next lemma, see (FERON, 1996).

Lemma 3.1. *For the system* (3.1)-(3.2) *with* $b_i = 0, \ \forall i \in \mathbb{K}$, *assume that there exist* $P > 0$ *and* $\lambda \in \Lambda$ *satisfying the inequality*

$$A_\lambda' P + P A_\lambda + Q_\lambda < 0 \tag{3.5}$$

with $Q_i = E_i' E_i, \ \forall i \in \mathbb{K}$. *Then, the state-dependent switching function* $\sigma(t) = u(x(t))$ *with*

$$u(x) = \arg\min_{i \in \mathbb{K}} x'(2P_i A_i + Q_i)x \tag{3.6}$$

is globally asymptotically stabilizing and satisfies the guaranteed cost

$$\|z\|_2^2 < x_0' P x_0 \tag{3.7}$$

Proof: For an arbitrary trajectory of the linear system $\dot{x} = A_\sigma x$ and adopting a quadratic Lyapunov function $V(x) = x'Px, \ P > 0$, we have

$$\begin{aligned} \dot{V}(x) &= \dot{x}'Px + x'P\dot{x} \\ &= x'(A_\sigma' P + P A_\sigma + E_\sigma' E_\sigma)x - z'z \\ &= \min_{i \in \mathbb{K}} x'(A_i' P + P A_i + E_i' E_i)x - z'z \\ &= \min_{\lambda \in \Lambda} x'(A_\lambda' P + P A_\lambda + Q_\lambda)x - z'z \\ &\leq x'(A_\lambda' P + P A_\lambda + Q_\lambda)x - z'z \\ &< -z'z \end{aligned} \tag{3.8}$$

where the third equality is due to the switching function (3.6), the fourth equality and the first inequality is a consequence of the minimum operator and the last inequality comes from the validity of (3.5). Moreover, integrating both sides of (3.8) from 0 up to ∞ we obtain the guaranteed cost (3.7). The proof is concluded. □

Notice that the condition (3.5) is not an LMI due to the presence of the variables λ and P. A manner of solving this Lemma is by doing a search with respect to λ and solving the resulting LMI in order to obtain the smallest guaranteed cost.

While the linear system has a unique equilibrium point, the switched affine system has several equilibrium points, forming a region of great interest in the state space defined by

$$X_e = \left\{ x_e \in \mathbb{R}^{n_x} \ : x_e = -A_\lambda^{-1} b_\lambda, \ \lambda \in \Lambda \right\} \tag{3.9}$$

Generally, the equilibrium point of interest is not common to the subsystems, which requires an arbitrarily high switching frequency in order to make the state trajectories reach this point, ensuring asymptotic stability. In this case, the system always evolves on a sliding mode, which causes a behavior that is significantly different from the behavior of each individual subsystem (LIBERZON, 2003).

Adopting the change of variable $\xi(t) = x(t) - x_e$ we can rewrite the system (3.1)-(3.2) as

$$\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + \ell_{\sigma(t)} \tag{3.10}$$

$$z_e(t) = E_{\sigma(t)}\xi(t) \tag{3.11}$$

where $\ell_i = A_i x_e + b_i$ and $z_e = z - E_\sigma x_e$. Similarly to the linear case, our main goal is to design a state dependent switching function $\sigma(t) = u(\xi(t))$ in order the ensure the global asymptotic stability of the equilibrium point $x_e \in X_e$ in (3.1)-(3.2), which is equivalent to ensure the same for the origin $\xi = 0$ in (3.10)-(3.11). Moreover, a guaranteed cost must be taken into account. The next theorem borrowed from (DEAECTO *et al.*, 2010) provides this result.

Theorem 3.1. *For the switched affine system* (3.10)-(3.11)*, let the equilibrium point $x_e \in X_e$ and its associated vector $\lambda \in \Lambda$ be given. If there exists $P > 0$ solution to the LMI*

$$A_\lambda' P + A_\lambda P + Q_\lambda \leq 0 \tag{3.12}$$

with $Q_i = E_i' E_i$. Then, the state-dependent switching function $\sigma(t) = u(\xi(t))$ with

$$u(\xi) = \arg\min_{i \in \mathbb{K}} \xi'(2PA_i + Q_i)\xi + 2\xi' P \ell_i \tag{3.13}$$

makes the equilibrium point $x_e \in X_e$ globally asymptotically stable and ensures that the guaranteed cost

$$\|z_e\|_2^2 < (x_0 - x_e)' P (x_0 - x_e) \tag{3.14}$$

holds.

Proof: The proof is presented in (DEAECTO *et al.*, 2010) but will be repeated here for convenience. Considering the switching strategy (3.13) and adopting the quadratic Lyapunov function

$V(\xi) = \xi' P \xi$, its time derivative along an arbitrary trajectory of the switched system (3.10)-(3.11) leads

$$\begin{aligned}
\dot{V}(\xi) &= \dot{\xi}' P \xi + \xi' P \dot{\xi} \\
&= \xi'(A_\sigma' P + P A_\sigma + E_\sigma' E_\sigma)\xi + 2\xi' P \ell_\sigma - z_e' z_e \\
&= \min_{i \in \mathbb{K}} \xi'(A_i' P + P A_i + E_i' E_i)\xi + 2\xi' P \ell_i - z_e' z_e \\
&= \min_{\lambda \in \Lambda} \xi'(A_\lambda' P + P A_\lambda + Q_\lambda)\xi + 2\xi' P \ell_\lambda - z_e' z_e \\
&\leq \xi'(A_\lambda' P + P A_\lambda + Q_\lambda)\xi + 2\xi' P \ell_\lambda - z_e' z_e \\
&< -z_e' z_e
\end{aligned} \tag{3.15}$$

where the third equality comes from the switching rule (3.13), the fourth equality and the first inequality are a consequence of the $\min$ operator, and the last inequality comes from the condition (3.12) and the fact that $\ell_\lambda = 0$ since $x_e \in X_e$. The guaranteed cost is obtained by integrating both sides of (3.15). The proof is concluded. □

Although this theorem presents some similarities with respect to Lemma 3.1, there are some differences inherited from the intrinsic nature of the affine system. Indeed, the conditions (3.5) and (3.12) are the same and do not require any stability property of the individual subsystems. A necessary and sufficient condition for feasibility is that A_λ be Hurwitz stable[1]. On the other hand, while in Lemma 3.1 the vector $\lambda \in \Lambda$ is searched to optimize the guaranteed performance, in Theorem 3.1 it is associated to the equilibrium point $x_e \in X_e$. Several other aspects of this intriguing class of systems will be explored in the next example.

3.2 Illustrative Example

Consider the continuous-time switched affine system (3.1)-(3.2) with the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ b_1 = \begin{bmatrix} -3 \\ 20 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ E_1 = E_2 = I \tag{3.16}$$

The equilibrium points of the first and second subsystems are, respectively

$$x_{e1} = -A_1^{-1} b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ \ x_{e2} = -A_2^{-1} b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{3.17}$$

[1] It has all the eigenvalues in the open left hand side of the complex plane

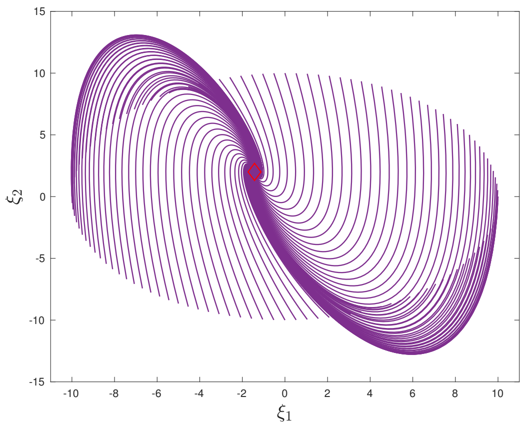
Moreover, x_{e1} is a stable focus and the eigenvalues of A_1 are $-2 \pm 2i$ and x_{e2} is a saddle and the eigenvalues of A_2 are ± 1.

From all the attainable equilibrium points given in (3.9) the desired one and its associated vector λ are as follows

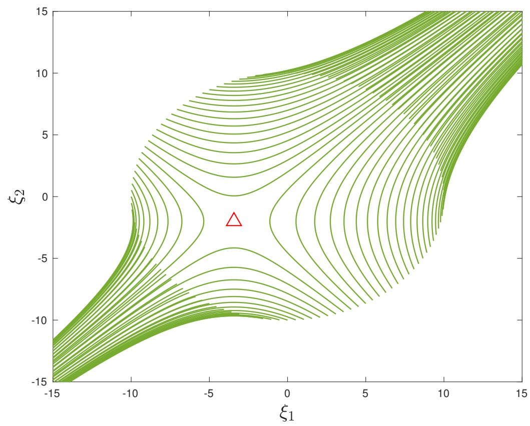
$$x_e = \begin{bmatrix} 2.4286 \\ 1 \end{bmatrix}, \lambda = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{3.18}$$

which does not coincide with the equilibrium points of the subsystems. Differently from the linear case, where a trivial switching function exists when all the subsystems are stable, the actuation of the switching rule in switched affine systems is essential to govern the state trajectories to the equilibrium point x_e independently of the stability of the subsystems.

Figure 3.1 shows the phase portrait of each isolated subsystem regarding the state variable $\xi = x - x_e$ and considering initial conditions taken in the circumference of radius 10 centered at the origin. In this these plots the equilibrium points of the first $\xi_1 = x_{e1} - x_e$ and the second $\xi_2 = x_{e2} - x_e$ subsystems are indicated, respectively, by the symbols $\diamond$ and $\triangle$.



(a) Subsystem 1.



(b) Subsystem 2.

Figure 3.1 – Phase portrait of the isolated subsystems

Solving the optimization problem

$$\inf_{P>0} \ \mathrm{Tr}(P) \tag{3.19}$$

subject to (3.12), we have obtained

$$P = \begin{bmatrix} 1.4107 & 0.1429 \\ 0.1429 & 0.3214 \end{bmatrix} \tag{3.20}$$

which is important for the switching rule implementation (3.13). As in (DEAECTO *et al.*, 2010), this objective function corresponds to that of (3.14) with the vector $x_0 - x_e$ assumed to be uniformly distributed over the unit sphere. The advantage of (3.19) is to determine $P > 0$ only once independently of the initial conditions.

Figure 3.2 presents the phase portrait of the controlled system resulting from the actuation of switching function (3.13). The equilibrium point $\xi = 0$ is represented by the symbol $\circ$ and the red trajectory was highlighted to indicate that it corresponds to the plots of Figure 3.3, which show the dynamical behavior of the system with respect to time.

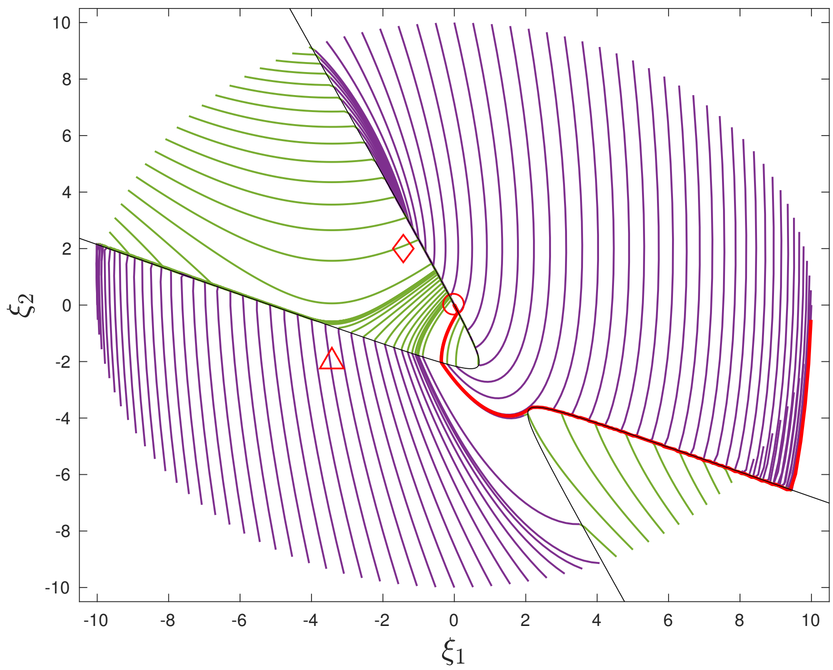
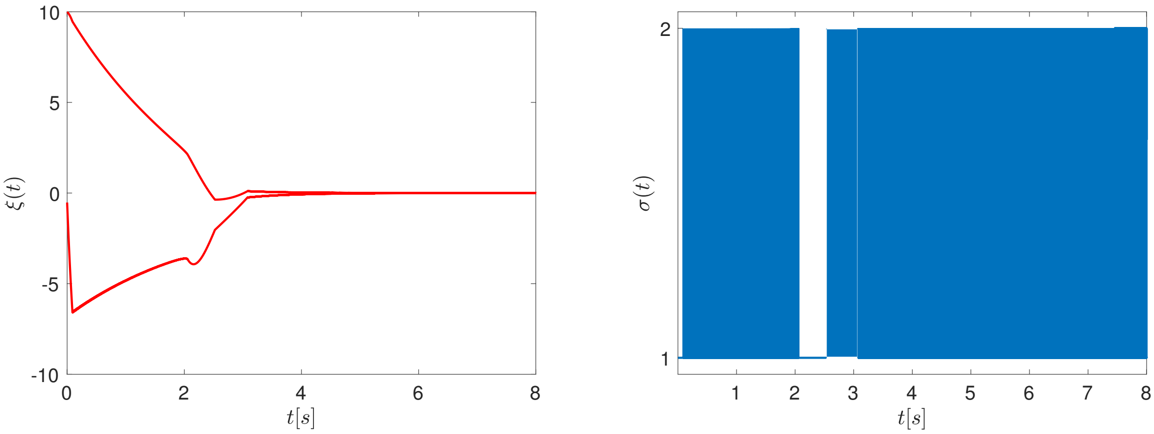


Figure 3.2 – Phase portrait of the switched system

It can be seen in Figure 3.2 that all state trajectories converge to the origin as expected. Following the color pattern, it is easy to see when the switching rule changes the subsystem. This always occurs at the switching surface represented by the hyperboloid in Figure 3.2, which is the locus of the equation

$$\xi'\Big(2P(A_1 - A_2) + (Q_1 - Q_2)\Big)\xi + 2\xi' P(\ell_1 - \ell_2) = 0 \tag{3.21}$$

By analyzing the red curve in the phase portrait and its evolution with respect to time in Figure 3.3, it is clear that in two time intervals $t \in [0.09, 2.04]$ [s] and $t \geq 3.079$ [s] the switching frequency is infinitely fast and the state trajectories slide on the switching surface, assuming a dynamical behavior completely different from that of the isolated subsystems. This



(a) System trajectories

(b) Switching function

Figure 3.3 – State evolution through time

behavior is known as sliding mode and the region of the surface where it occurs is named sliding surface. Although this phenomenon is essential to ensure asymptotic stability in switched affine systems, it is often undesirable in mathematical models of real systems, due to the very fast switching, that may cause excessive equipment wear, (LIBERZON, 2003). Moreover, in several situations this high switching frequency cannot be implemented due to physical limitations. For this reason, it is important to study cases where the switching frequency has an upper bound, which is equivalent to impose the following constraint on the switching rule

$$\sigma(t) = \sigma(t_n),\ t \in [t_n, t_{n+1}) \tag{3.22}$$

where t_n and t_{n+1} are two subsequent switching instants and $t_{n+1} - t_n = T > 0$. Under this constraint, asymptotic stability is impossible to be ensured. The reason is simple. If the rule is kept constant in $\sigma(t) = i$ during T seconds, the trajectory will assume the dynamical behavior of the i-th subsystem, which is associated with the equilibrium point x_{ei}, thus diverging from x_e. This limitation of the switching frequency naturally occurs in discrete-time systems due to the sampling period. This will be the focus of the next chapter, which is dedicated to study stabilization of discrete-time switched affine systems.

The literature proposes two different approaches to treat switching frequency limitation. The first is to study practical stability, where the state trajectories are governed by the switching rule to an invariant set of attraction containing the equilibrium point of interest, see (HETEL; FRIDMAN, 2013), (SANCHEZ *et al.*, 2019), (DEAECTO; GEROMEL, 2017), (EGIDIO; DEAECTO, 2019). The idea is to minimize the volume of this set in order to make the steady-state response as near as possible of x_e. A negative point is that nothing can be concluded about the trajectories, once they are inside the set of attraction. Moreover, the well-known $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes cannot be taken into account, since they are defined only for asymptotically stable systems.

A recent methodology is to ensure asymptotic stability of a suitable limit-cycle. References (BENMILOUD *et al.*, 2019) and (EGIDIO *et al.*, 2020) are some examples where this approach is adopted. Stabilization of limit cycles is the main theme of this thesis and will be treated with details in the next chapter. Its main advantage is to ensure a suitable performance for the steady-state and the transient responses by determining a suitable limit-cycle and imposing $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes.

3.3 Final Considerations

In this chapter, we have presented the switched affine systems that are our focus of study in this thesis and discussed some of their intrinsic characteristics. Sufficient conditions borrowed from the literature have been provided to ensure global asymptotic stability of a desired equilibrium point. An academical example was used to emphasize the main properties of this class of system and show some difficulties that serve as motivation to the next chapter.

4 STABILIZATION OF LIMIT-CYCLES

This chapter is dedicated to present the main results of this thesis, which are also available in (DEAECTO *et al.*, 2023) and (HIRATA *et al.*, 2023). They consist in the control design of a static output-dependent switching function for discrete-time switched affine systems to ensure global asymptotic stability of a suitable limit cycle and $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance indexes. The conditions are based on a time-varying convex Lyapunov function and expressed in terms of LMIs. These results are a generalization of reference (EGIDIO *et al.*, 2020) that treats state feedback control, exclusively. For the sake of comparison, we have included a guaranteed cost in the conditions of (SERIEYE *et al.*, 2023) to show that the technique used as basis to our results is not more conservative than recent ones from the literature. An academical example and a practical application concerning the voltage regulation of a DC-DC multicellular converter are used for validation and comparison.

4.1 Problem statement

Consider a discrete-time switched affine system

$$x[n+1] = A_{\sigma[n]}x[n] + b_{\sigma[n]} + H_{\sigma[n]}w[n] \tag{4.1}$$

$$y[n] = C_{\sigma[n]}x[n] \tag{4.2}$$

$$z[n] = E_{\sigma[n]}x[n] + G_{\sigma[n]}w[n] \tag{4.3}$$

defined for all $n \in \mathbb{N}_- = \mathbb{N} \cup \{-1\}$, where $x[n] \in \mathbb{R}^{n_x}$ is the state and $w[n] \in \mathbb{R}^{n_w}$ is the exogenous input, $z[n] \in \mathbb{R}^{n_z}$ is the controlled output and $y[n] \in \mathbb{R}^{n_y}$ is the measured output. The switching function $\sigma[n] : \mathbb{N}_- \to \mathbb{K}$ selects at each instant of time one of the N subsystems and is the unique control variable $\sigma[n] = u(y[n]) \in \mathbb{K}$ to be determined in order to ensure stability and performance for the overall system. At this moment, consider that the modulo operator is defined as $d = a \mod b$ where d is the remainder of the Euclidean division between the integers a and b. Moreover, for a positive $\kappa \in \mathbb{N}$, we define the function $k(n) = n \mod \kappa$.

The limit cycle is a periodic solution with period $\kappa > 0$ of the system

$$x_e[n+1] = A_{\sigma[n]}x_e[n] + b_{\sigma[n]} \tag{4.4}$$

associated to a periodic switching sequence $\sigma[n] = c[k(n)]$ with $c = (c[0], \dots, c[\kappa - 1]) \in \mathbb{K}^\kappa$ and is denoted by

$$\mathcal{X}_e(c) = \{x_e[k(n)] : (4.4), n \in \mathbb{N}_-\} \tag{4.5}$$

The fundamental period $(x_e[0], \dots, x_e[\kappa-1])$ is determined from one of the N^κ possible switching periodic sequences $\sigma[n] = c[k(n)]$ chosen by the designer. This point will be further explored afterwards, but is also discussed in details in (EGIDIO *et al.*, 2020).

Defining the auxiliary variable $\xi[n] = x[n] - x_e[n]$, we obtain the equivalent system

$$\xi[n+1] = A_{\sigma[n]}\xi[n] + \ell_{\sigma[n]}[n] + H_{\sigma[n]}w[n] \tag{4.6}$$

$$y_e[n] = C_{\sigma[n]}\xi[n] \tag{4.7}$$

$$z_e[n] = E_{\sigma[n]}\xi[n] + G_{\sigma[n]}w[n] \tag{4.8}$$

with $\ell_i[n] = A_i x_e[n] - x_e[n+1] + b_i,\ i \in \mathbb{K}$, $z_e[n] = z[n] - E_{\sigma[n]}x_e[n]$ and $y_e[n] = y[n] - C_{\sigma[n]}x_e[n]$ for all $n \in \mathbb{N}_-$. Notice that $\xi[n] \to 0$ whenever $x[n] \to x_e[k(n)]$. Hence, studying stabilization of the equilibrium point $\xi = 0$ in (4.6)-(4.8) is equivalent to ensure stabilization of the limit cycle $x_e[k(n)]$ for some $c[k(n)]$ in (4.1)-(4.3).

Our main goal is to generalize the results of the recent reference (EGIDIO *et al.*, 2020), that treats exclusively state feedback control, to cope with static output feedback control of the system (4.6)-(4.8). More specifically, the goal is to design an output-dependent switching function $\sigma[n] = u(y[n])$ with $u(y) : \mathbb{R}^{n_y} \to \mathbb{K}$ in order to ensure global asymptotic stability of the origin $\xi = 0$, which implies the global asymptotic stability of the limit-cycle of interest $\mathcal{X}_e(c)$ in (4.1)-(4.3). Notice that the switching rule $\sigma[n] = u(y[n])$ must depend directly on the measured output y, without considering any additional dynamic structure. Moreover, we are interested in ensuring guaranteed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes. As known in the literature, these indexes are defined to asymptotically stable systems and cannot be calculated in case of practical stability.

More specifically, the $\mathcal{H}_2$ performance index considers the system (4.6)-(4.8) with $\xi[-1] = 0$ and subject to $w[n] = \delta[n+1]e_r$, with e_r being the r-th column of the identity matrix and $\delta[n]$ the discrete-time impulse. It is defined by

$$J_2(\sigma) = \sum_{r=1}^{n_w} \|z_{er}\|_2^2 + e_r' G_{\sigma[-1]}' G_{\sigma[-1]} e_r \tag{4.9}$$

where z_{er} is the controlled output correspondent to the impulse applied in the r-th channel of the external input. On the other hand, the $\mathcal{H}_\infty$ performance index takes into account the system

(4.6)-(4.8) evolving from $\xi[0] = 0$ and subject to external input $w \in \mathcal{L}_2$. It is defined as

$$J_\infty(\sigma) = \sup_{w \in L_2 \backslash \{0\}} \frac{\|z_e\|_2^2}{\|w\|_2^2} \tag{4.10}$$

See references (GEROMEL *et al.*, 2008), (DEAECTO *et al.*, 2013) and (EGIDIO *et al.*, 2020) for details about these indexes.

4.1.1 Limit Cycle Generation

Let us define the set $\mathfrak{C}(\kappa) = \mathbb{K}^\kappa$ with N^κ elements $c = (c[0], \ldots, c[\kappa-1]) \in \mathfrak{C}(\kappa)$ each one associated to a limit cycle candidate $\mathcal{X}_e(c)$. The first κ points $x_e[n],\ n \in \{0, \ldots, \kappa-1\}$ are obtained as a solution of the equation

$$\mathbf{A}(c)\mathbf{x}_e = -\mathbf{b}(c) \tag{4.11}$$

where $\mathbf{x}_e = [x_e[0]'\ x_e[1]'\ x_e[\kappa-1]']'$ and

$$\mathbf{A}(c) = \begin{bmatrix} A_{c[0]} & -I & 0 & \cdots & 0 \\ 0 & A_{c[1]} & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I & 0 & 0 & \cdots & A_{c[\kappa-1]} \end{bmatrix}, \mathbf{b}(c) = \begin{bmatrix} b_{c[0]} \\ b_{c[1]} \\ \vdots \\ b_{c[\kappa-1]} \end{bmatrix} \tag{4.12}$$

derived from (4.4) with $\sigma[n] = c[k(n)]$ and taking into account the boundary condition $x_e[0] = x_e[\kappa]$. We can conclude from the solution of the linear equation (4.11) that $\det(\mathbf{A}(c)) \neq 0$ implies that the periodic sequence $c = (c[0], \ldots, c[\kappa-1]) \in \mathfrak{C}(\kappa)$ generates a unique limit cycle $\mathcal{X}_e(c)$ to the system.

From all the possible candidates $\mathfrak{X} = \{\mathcal{X}_e(c),\ c \in \mathfrak{C}(\kappa)\}$ let us consider a subset of great interest $\mathfrak{X}_s \subset \mathfrak{X}$ together with its associated set $\mathfrak{C}_s(\kappa) \subset \mathfrak{C}(\kappa)$ that satisfies some criterion defined by the designer, as for instance

$$\mathfrak{X}_s = \left\{ \mathcal{X}_e \in \mathfrak{X} : \max_{n \in \{0, \cdots, \kappa-1\}} \|\Gamma(x_e[n] - x_*)\|_\infty < 1 \right\} \tag{4.13}$$

where x_* is a reference point chosen by the designer. This criterion can be used to bound the maximum ripple, see (EGIDIO *et al.*, 2020) for details. The matrix Γ is given and allows to optimize the steady-state behavior of only one or a combination of state components. Notice that the period κ is chosen by the designer in order to obtain a non-empty set $\mathfrak{X}_s$. In case of several possibilities, this choice can be made by selecting the value associated with the smaller $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost.

4.2 Main Results

The main results consist in the static output feedback control design ensuring an $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performance. However, due to the importance for this thesis, we will recall some results from (EGIDIO *et al.*, 2020) that treats state feedback control exclusively.

4.2.1 Stability and Guaranteed Cost

Let us consider the simpler switched affine system

$$\xi[n+1] = A_{\sigma[n]}\xi[n] + \ell_{\sigma[n]}[n] \tag{4.14}$$

$$y_e[n] = C_{\sigma[n]}\xi[n] \tag{4.15}$$

$$z_e[n] = E_{\sigma[n]}\xi[n] \tag{4.16}$$

defined for all $n \in \mathbb{N}$ and evolving from an arbitrary initial condition $\xi[0] = x[0] - x_e[0]$.

4.2.1.1 State Feedback Control

At this first moment, let us consider that the state is available $y_e[n] = \xi[n]$ and recall the state feedback control design proposed in (EGIDIO *et al.*, 2020), which is based on the time-varying convex Lyapunov function

$$V(\xi[n], n) = \xi[n]' P[n] \xi[n] \tag{4.17}$$

where matrices $P[n] = P[k(n)]$ are periodic with period $\kappa > 0$. The next theorem provides stabilization conditions and a guaranteed cost, available in (EGIDIO *et al.*, 2020), to the control design of a state-dependent switching function $\sigma[n]$ that uses the functional matrix

$$\mathcal{M}_i[n] = \begin{bmatrix} A_i' P[n+1] A_i - P[n] & \bullet \\ \ell_i[n]' P[n+1] A_i & \ell_i[n]' P[n+1] \ell_i[n] \end{bmatrix} \tag{4.18}$$

defined for all $i \in \mathbb{K}$.

Theorem 4.1. *Consider system* (4.14)-(4.16) *with* $C_i = I$, $\forall i \in \mathbb{K}$, *the subset of limit cycles* $\mathfrak{X}_s$ *with the associated periodic sequences* $\mathfrak{C}_s(\kappa)$ *and the period* $\kappa > 0$ *be given. If there exist symmetric matrices* $P[n] > 0$ *satisfying the optimization problem*

$$\min_{\mathcal{X}_e(c) \in \mathfrak{X}_s} \inf_{P[n] > 0} (x[0] - x_e[0])' P[0] (x[0] - x_e[0]) \tag{4.19}$$

subject to the linear matrix inequalities

$$A'_{c[n]}P[n+1]A_{c[n]} - P[n] + E'_{c[n]}E_{c[n]} < 0 \tag{4.20}$$

for all $n \in \{0, \ldots, \kappa - 1\}$, $c \in \mathfrak{C}_s(\kappa)$, *with the boundary condition* $P[0] = P[\kappa]$, *then the state-dependent switching function* $\sigma[n] = u(\xi[n])$ *given by*

$$u(\xi) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} \xi \\ 1 \end{bmatrix}' \mathcal{M}_i[k(n)] \begin{bmatrix} \xi \\ 1 \end{bmatrix} \tag{4.21}$$

ensures that the limit cycle $\mathcal{X}_e^* = \mathcal{X}_e(c_*)$ *solution of* (4.19) *is globally asymptotically stable and*

$$\|z_e\|_2^2 < (x[0] - x_e[0])'P[0](x[0] - x_e[0]) \tag{4.22}$$

is a guaranteed cost of performance.

Proof: Available in (EGIDIO *et al.*, 2020). □

Now, let us present a recent result available in (SERIEYE *et al.*, 2023) that treats only limit cycle stabilization without taking into account any guaranteed cost. Unfortunately, that reference does not make clear the contribution of the proposed state-dependent switching function with respect to the one provided in (EGIDIO *et al.*, 2020), available three years before. For this reason, we have included a guaranteed cost in the conditions of (SERIEYE *et al.*, 2023) for the sake of comparison of both proposals. In that reference, the authors have adopted the min-type Lyapunov function

$$\nu(x[n]) = \min_{i\in\{0,\cdots,\kappa-1\}} (x[n] - x_e[i])'P[i](x[n] - x_e[i]) \tag{4.23}$$

which is non-convex and time-invariant. Defining $\mathcal{L}_i[n]$ as being

$$\mathcal{L}_i[n] = A'_iP[n+1]A_i - P[n] + E'_iE_i \tag{4.24}$$

the next theorem proposes a guaranteed cost for the conditions of (SERIEYE *et al.*, 2023).

Theorem 4.2. *Consider system* (4.1)-(4.3) *with* $w = 0$ *and* $C_i = I$, $\forall i \in \mathbb{K}$, *the subset of limit cycles* $\mathfrak{X}_s$ *with the associated periodic sequences* $\mathfrak{C}_s(\kappa)$ *and the period* $\kappa > 0$ *be given. If there exist symmetric matrices* $P[n] > 0$ *satisfying the optimization problem*

$$\min_{\mathcal{X}_e(c)\in\mathfrak{X}_s} \min_{i\in\{0,\cdots,\kappa-1\}} \inf_{P[i]>0} (x[0] - x_e[i])'P[i](x[0] - x_e[i]) \tag{4.25}$$

subject to the linear matrix inequalities (4.20) *for all* $n \in \{0, \dots, \kappa - 1\}$*,* $c \in \mathfrak{C}_s(\kappa)$*, with the boundary condition* $P[0] = P[\kappa]$*, then the state-dependent switching function* $\sigma[n] = u(x[n])$ *given by*

$$u(x) = \left\{ c[\theta],\ \theta = \arg \min_{i \in \{0, \cdots, \kappa-1\}} (x - x_e[i])' P[i] (x - x_e[i]) \right\} \tag{4.26}$$

ensures that the limit cycle $\mathcal{X}_e^* = \mathcal{X}_e(c_*)$ *solution of* (4.25) *is globally asymptotically stable and*

$$\|z_e\|_2^2 < \min_{i \in \{0, \kappa-1\}} (x[0] - x_e[i])' P[i] (x[0] - x_e[i]) \tag{4.27}$$

is a guaranteed cost of performance.

Proof: Consider an arbitrary trajectory of (4.1)-(4.3) and define $\Delta\nu(x[n]) = \nu(x[n+1]) - \nu(x[n])$, $\mathbb{K}_\theta = \{0, \cdots, \kappa - 1\}$. The Lyapunov function (4.23) and the associated switching function (4.26) provide

$$\begin{aligned}
\Delta\nu(x) &= \min_{i \in \mathbb{K}_\theta} (x[n+1] - x_e[i])' P[i] (x[n+1] - x_e[i]) - (x[n] - x_e[\theta])' P[\theta] (x[n] - x_e[\theta]) \\
&\le (x[n] - x_e[\theta])' \mathcal{L}_{c[\theta]}[\theta] (x[n] - x_e[\theta]) - z_e[n]' z_e[n] \\
&< -z_e[n]' z_e[n]
\end{aligned} \tag{4.28}$$

Defining $\zeta[n, \theta] = x[n] - x_e[\theta]$, the first inequality comes from the fact that

$$\begin{aligned}
\nu(x[n+1]) &= \min_{i \in \mathbb{K}_\theta} (x[n+1] - x_e[i])' P[i] (x[n+1] - x_e[i]) \\
&\le \zeta[n+1, \theta+1]' P[\theta+1] \zeta[n+1, \theta+1] \\
&= \zeta[n, \theta] A_{c[\theta]} P[\theta+1] A_{c[\theta]} \zeta[n, \theta]
\end{aligned} \tag{4.29}$$

where the inequality in (4.29) is a consequence of the minimum operator and the equality is a consequence from the fact that $x[n+1] = A_{c[\theta]} x[n] + b_{c[\theta]}$, due to the switching function (4.26), and $x_e[\theta + 1] = A_{c[\theta]} x_e[\theta] + b_{c[\theta]}$. The last inequality from (4.28) comes from the validity of (4.20). Now summing both sides of (4.28) for all $n \in \mathbb{N}$ we have that $\|z_e\|_2^2 < \nu(x[0])$ which concludes the proof of the theorem. □

Although both theorems present different switching functions (4.21) and (4.26), respectively, they are based on the same sufficient conditions (4.20). Concerning the guaranteed cost, notice that if a periodic sequence $c_* \in \mathfrak{C}_s$ is associated to the limit-cycle $\mathcal{X}_e(c_*)$ that respects the criterion (4.13), then all the shifted sequences of c_* also belong to the set $\mathfrak{C}_s(\kappa)$, because the condition (4.13) is associated to distance among points $x_e[n]$ and x_*, without any requirement on the temporal sequence. Hence, in this case, the guaranteed costs (4.19) and (4.20) are also identical. Concerning the switching functions (4.21) and (4.26), they can have advantages or disadvantages depending on the application, since they are not comparable in terms of conservatism. Moreover, notice that at each instant of time, (4.21) can choose any available subsystem $i \in \mathbb{K}$, while (4.26) can only select the subsystems that compose the associated periodic sequence $c = (c[0], \cdots, c[\kappa-1]) \in \mathfrak{C}_s$. In addition, it seems not trivial to generalize the state-feedback results to cope with output feedback control design adopting switching rule of (SERIEYE *et al.*, 2023), but it could be done with the switching rule of (EGIDIO *et al.*, 2020), as it will be clear in the sequel.

4.2.1.2 Output Feedback Control

At this moment, let us suppose that the state is not available for feedback and generalize the conditions of Theorem 4.1 to the control design of $\sigma[n] = u(y[n])$. The next theorem provides this result.

Theorem 4.3. *Consider system* (4.14)-(4.16) *with* $C_i = I,\ \forall i \in \mathbb{K}$, *the subset of limit cycles* $\mathfrak{X}_s$ *with the associated periodic sequences* $\mathfrak{C}_s(\kappa)$ *and the period* $\kappa > 0$ *be given. If there exist symmetric matrices* $P[n] > 0$, $R_i[n]$ *and* $U[n]$ *forming the solution set* Ψ *of the optimization problem*

$$\min_{\mathcal{X}_e(c)\in\mathfrak{X}_s} \inf_{\Psi} (x[0]-x_e[0])'P[0](x[0]-x_e[0]) \tag{4.30}$$

subject to the linear matrix inequalities

$$\mathcal{L}_i[n] < U[n] + C_i'R_i[n]C_i \tag{4.31}$$

$$U[n] + C_{c[n]}'R_{c[n]}[n]C_{c[n]} < 0 \tag{4.32}$$

for all $i \in \mathbb{K}$, $n \in \{0,\ldots,\kappa-1\}$, $c \in \mathfrak{C}_s(\kappa)$, *with the boundary condition* $P[0] = P[\kappa]$, *then the output-dependent switching function* $\sigma[n] = u(y[n])$ *given by*

$$u(y) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} y_e[n] \\ \ell_i[k(n)] \end{bmatrix}' \begin{bmatrix} R_i[k(n)] & \bullet \\ S_i[k(n)] & W_i[k(n)] \end{bmatrix} \begin{bmatrix} y_e[n] \\ \ell_i[k(n)] \end{bmatrix} \tag{4.33}$$

with matrices

$$S_i[n] = P[n+1]A_iC_i'(C_iC_i')^{-1} \tag{4.34}$$

$$W_i[n] = P[n+1] - \mathcal{J}_i[n]\mathcal{S}_i[n]^{-1}\mathcal{J}_i[n]' + \varepsilon I \tag{4.35}$$

with $\varepsilon > 0$ *arbitrarily small,* $\mathcal{S}_i[n] = \mathcal{L}_i[n] - U[n] - C_i' R_i[n] C_i < 0$ *and* $\mathcal{J}_i[n] = P[n+1]A_i - S_i[n]C_i$*, ensures that the limit cycle* $\mathcal{X}_e^* = \mathcal{X}_e(c_*)$ *solution of* (4.30) *is globally asymptotically stable and*

$$\|z_e\|_2^2 < (x[0] - x_e[0])' P[0] (x[0] - x_e[0]) \tag{4.36}$$

is a guaranteed cost of performance.

Proof: Consider system (4.14)-(4.16), denote $\xi[n] = \xi$, $\sigma[n] = \sigma$ and $\ell_i[n] = \ell_i$ and define the difference operator $\Delta V = V(\xi[n+1], n+1) - V(\xi[n], n)$ of the Lyapunov function (4.17), which for an arbitrary trajectory provides

$$\begin{aligned}
\Delta V &= \begin{bmatrix} \xi \\ \ell_\sigma \end{bmatrix}' \begin{bmatrix} \mathcal{L}_\sigma[n] & \bullet \\ P[n+1]A_\sigma & P[n+1] \end{bmatrix} \begin{bmatrix} \xi \\ \ell_\sigma \end{bmatrix} - z_e' z_e \\
&< \begin{bmatrix} \xi \\ \ell_\sigma \end{bmatrix}' \begin{bmatrix} U[n] + C_\sigma' R_\sigma[n] C_\sigma & \bullet \\ S_\sigma[n] C_\sigma & W_\sigma[n] \end{bmatrix} \begin{bmatrix} \xi \\ \ell_\sigma \end{bmatrix} - z_e' z_e \\
&= \min_{i \in \mathbb{K}} \begin{bmatrix} \xi \\ \ell_i \end{bmatrix}' \begin{bmatrix} U[n] + C_i' R_i[n] C_i & \bullet \\ S_i[n] C_i & W_i[n] \end{bmatrix} \begin{bmatrix} \xi \\ \ell_i \end{bmatrix} - z_e' z_e \\
&\le \xi' (U[n] + C_{c[n]}' R_{c[n]}[n] C_{c[n]}) \xi - z_e' z_e \\
&< -z_e' z_e
\end{aligned} \tag{4.37}$$

In (4.37), the first inequality is valid whenever

$$\begin{bmatrix} \mathcal{L}_i[n] & \bullet \\ P[n+1]A_i & P[n+1] \end{bmatrix} < \begin{bmatrix} U[n] + C_i' R_i[n] C_i & \bullet \\ S_i[n] C_i & W_i[n] \end{bmatrix} \tag{4.38}$$

is verified for all $i \in \mathbb{K}$ and $n \in \{0, \cdots, \kappa - 1\}$. Choosing $S_i[n]$ as in (4.34), which is the solution that minimizes the quadratic error norm $\|\mathcal{J}_i[n]\|$, we have that $\mathcal{J}_i[n]$ and $\mathcal{S}_i[n]$ are completely known from the solution of (4.31) and (4.32). Hence, taking into account that (4.38) can be rewritten as

$$\begin{bmatrix} \mathcal{S}_i[n] & \bullet \\ \mathcal{J}_i[n] & P[n+1] - W_i[n] \end{bmatrix} < 0 \tag{4.39}$$

and performing the Schur Complement with respect to $\mathcal{S}_i[n] < 0$ we conclude that $W_i[n]$ chosen as in (4.35) ensures that inequalities (4.38) are indeed verified for all $i \in \mathbb{K}$. The second equality in (4.37) comes from the switching function (4.33) and the second and third inequalities are consequences of the fact that $\ell_{c[n]}[n] = 0$ and that (4.32) holds, respectively. From the periodic continuation $P[n] = P[k(n)]$ we have that $\Delta V < -z_e' z_e$ for all $n \in \mathbb{N}$, which summing from $n = 0$ up to infinity, provides $\|z_e\|_2^2 < v(\xi[0], 0)$, concluding thus the proof. $\square$

This theorem provides sufficient conditions expressed as the solution of a finite set of convex subproblems described in terms of LMIs for the control design of a static output feedback switching function that ensures a upper bound for $\|z_e\|_2^2$. Differently from the state feedback case available in (EGIDIO *et al.*, 2020), the actual cost obtained from the periodic switching function $\sigma[n] = c[k(n)]$ does not coincide with the right hand side of (4.36). This occurs due to the structure imposed in the design conditions to make the switching function dependent only on the measured output $y \in \mathbb{R}^{n_y}$. Notice that the same strategy here adopted cannot be applied to the switching function (4.26) of Theorem 4.2 because the limit cycle $x_e[0]$ depends on the rule through θ and impose an structure on $P[i]$ generally makes the solution extremely conservative.

A remark of great importance about this theorem concerns the matrices $R_i[n]$ when $i \neq c[n]$. Notice in (4.31)-(4.32) that these matrices can be anyone great enough to satisfy (4.31) and that $R_i[n] = \alpha I$ with $\alpha \to \infty$ is always a feasible solution. However, with this choice the switching function is always the periodic one $\sigma[n] = c[k(n)]$ and the advantages of the closed-loop control obtained with the min-type switching function are lost. Hence, a suitable choice is crucial to enhance the actual performance. A good alternative is to choose $R_i[n],\ i \neq c[n]$ as near as possible the bound of feasibility of (4.32) as presented in the next corollary borrowed from (DAIHA; DEAECTO, 2021) that treats the control of switched linear systems.

Corollary 4.1. *Assume there exists a solution for the optimization problem of Theorem 4.1, take matrices $P[n]$, $U[n]$ and define $\Gamma_{1i} = C_i'(C_iC_i')^{-1}$, $\Gamma_{2i} = \mathcal{N}(C_i)$ and*

$$\mathcal{Q}_i[n] = A_i'P[n+1]A_i - P[n] + E_i'E_i - U[n] \tag{4.40}$$

for all $i \in \mathbb{K}$ and $n \in \{0, \cdots, \kappa - 1\}$. Then, for $\varepsilon > 0$ arbitrarily small, the output-dependent switching rule (4.33) is an asymptotically stabilizing switching function for matrices $R_i[n]$ determined as follows:

- *For $i \neq c[n]$*

$$R_i[n] = Q_{1i} - Q_{2i}'Q_{3i}^{-1}Q_{2i} + \varepsilon I \tag{4.41}$$

with $Q_{1i} = \Gamma'_{1i}\mathcal{Q}_i[n]\Gamma_{1i}$, $Q_{2i} = \Gamma'_{2i}\mathcal{Q}_i[n]\Gamma_{1i}$, $Q_{3i} = \Gamma'_{2i}\mathcal{Q}_i[n]\Gamma_{2i} < 0$ *when* $\dim(\mathcal{N}(C_i)) \neq 0$ *or*

$$R_i[n] = C_i'^{-1}\mathcal{Q}_i[n]C_i^{-1} + \varepsilon I \tag{4.42}$$

otherwise.

- *For* $i = c[n]$, *adopt the solution of Theorem 4.3.*

Proof: Available in (DAIHA; DEAECTO, 2021). □

As it will be illustrated in the examples, this choice can reduce considerably the actual cost compared to one resulting from the periodic sequence $\sigma[n] = c[k(n)]$.

4.2.1.3 Academical Example

In this section, let us consider the continuous-time switched affine system defined in (ALBEA; SEURET, 2021) by the matrices

$$A_{o1} = \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix}, \; A_{o2} = \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix}, \; A_{o3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$b_{o1} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \; b_{o2} = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} \; e \; b_{o3} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

A discretized model was obtained by making

$$A_i = e^{A_{oi}T}, \; b_i = \int_0^T e^{A_{oi}\tau}\, d\tau \; b_{oi} \tag{4.43}$$

with $T = 0.1$ seconds. We have considered an initial condition $x[0] = [10 \;\; -10]'$ and solved the conditions of Theorem 4.1 and 4.2 that suppose that the state is available for feedback with $C_i = I, \; i \in \mathbb{K}$ and the conditions of Theorem 4.3 associated with Corollary 4.1 with

$$C_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; i \in \mathbb{K} \tag{4.44}$$

which indicates that the second state is not available for measurement.

Our main goal is to ensure global asymptotic stability of a limit-cycle satisfying a maximum ripple of 0.5 with respect to the reference point $x_* = [2 \;\; 0]'$. This is obtained when the constraint (4.13) is satisfied with $\Gamma = 2I$. For a period of $\kappa = 6$, we have obtained 106 limit cycle candidates that compose the set $\mathfrak{X}_s$.

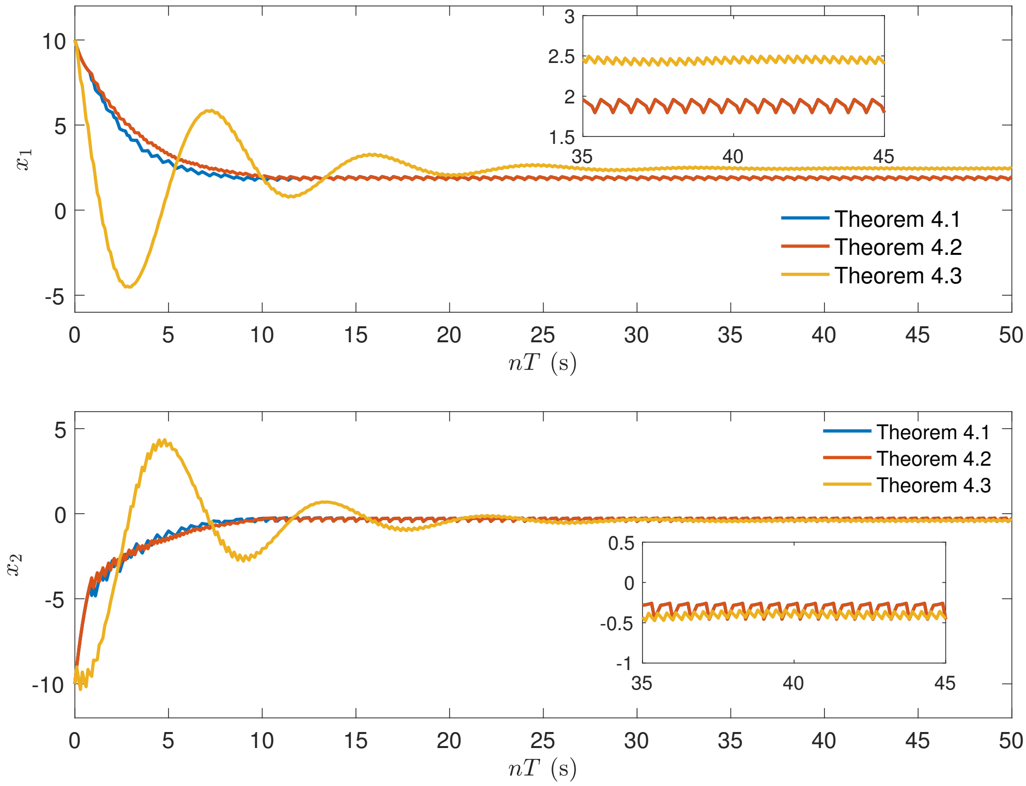


Figure 4.1 – State trajectories for the switching function of Theorem 4.1 proposed in (EGIDIO *et al.*, 2020), Theorem 4.2 based on (SERIEYE *et al.*, 2023) and for Theorem 4.3.

As expected, the solutions of Theorems 4.1 and 4.2 have provided the same guaranteed cost $\|z_e\|_2^2 < 2290.07$, but different actual costs $\|z_e\|_2^2 = 1518.74$ and $\|z_e\|_2^2 = 1617.78$, respectively. As already discussed before, it is not possible to conclude which switching rule is better, since for another initial condition $x[0]$ we could obtain a different conclusion. These solutions are associated to the periodic sequence $c = (1 \;\; 1 \;\; 3 \;\; 3 \;\; 3 \;\; 2)$ correspondent to the limit cycle $\mathcal{X}_e^*(c)$ given by

$$\begin{bmatrix} 1.7964 & 1.8759 & 1.9598 & 1.9318 & 1.9044 & 1.8780 \\ -0.4559 & -0.3650 & -0.2827 & -0.2772 & -0.2690 & -0.2581 \end{bmatrix}$$

For the same set $\mathfrak{X}_s$, we have solved the conditions of Theorem 4.3 together with Corollary 4.1 obtaining the costs $\|z_e\|_2^2 = 3565.60 < 154391.64$ associated to the periodic sequence $c = (1 \;\; 3 \;\; 3 \;\; 1 \;\; 3 \;\; 3)$ and to the limit cycle $\mathcal{X}_e^*(c)$ given by

$$\begin{bmatrix} 2.4076 & 2.4877 & 2.4499 & 2.4076 & 2.4877 & 2.4499 \\ -0.4440 & -0.3542 & -0.4011 & -0.4440 & -0.3542 & -0.4011 \end{bmatrix}$$

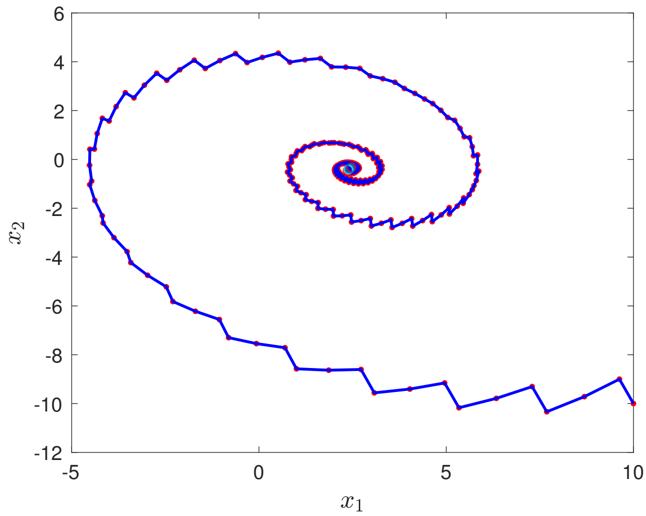


Figure 4.2 – Phase portrait considering the conditions of Theorem 4.3.

Figure 4.1 provides the state trajectories for the three studied switching strategies. For all of them the asymptotic stability of the desired limit cycle was successfully ensured. Figure 4.2 presents the phase portrait of the state trajectories for the conditions of Theorem 4.3. This example showed the validity and efficiency of the proposed theory. Moreover, it was important to make clear that the conditions we have used as basis of generalization provided in (EGIDIO *et al.*, 2020) is not more conservative than the more recent strategy proposed in (SERIEYE *et al.*, 2023).

4.2.2 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design

At this point, let us generalize the conditions of Theorem 4.3 to cope with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design. To cope with the $\mathcal{H}_2$ control, it is important to notice that the system (4.6)-(4.8) evolving from $\xi[-1] = 0$ and perturbed by an impulsive external input $w[n] = \delta[n+1]e_r$, with e_r being the r-th column of the identity matrix, can be equivalently rewritten as

$$\xi[n+1] = A_\sigma \xi[n] + \ell_\sigma[n], \ \xi[0] = \ell_m[-1] + H_m e_r \tag{4.45}$$

$$y_e[n] = C_\sigma \xi[n] \tag{4.46}$$

$$z_{er}[n] = E_\sigma \xi[n] \tag{4.47}$$

defined for all $n \in \mathbb{N}$ with $\sigma[-1] = m$, where z_{er} is the controlled output associated to an impulse applied in the r-th channel. The next corollary presents the conditions for the $\mathcal{H}_2$ control.

Corollary 4.2. *Consider system* (4.45)-(4.46)*. The conditions of Theorem 4.3 remains valid if the objective function* (4.30) *is replaced by*

$$\min_{\mathcal{X}_e(c)\in\mathfrak{X}_s} \inf_{\Psi} \mathrm{Tr}\left((L_m + H_m)' P[0] (L_m + H_m)\right) \tag{4.48}$$

with $L_m = [\ell_m[-1] \ \cdots \ \ell_m[-1]] \in \mathbb{R}^{n_x \times n_w}$ *with* $\sigma[-1] = m$*. In this case, the switching function* $\sigma[n] = u(y[n])$ *with* $u[y]$ *given in* (4.33) *ensures that the limit cycle* $\mathcal{X}_e^* = \mathcal{X}_e(c_*)$ *solution of* (4.48) *is globally asymptotically stable and the upper bound*

$$J_2(\sigma) < \mathrm{Tr}\left((L_m + H_m)' P[0] (L_m + H_m) + G_m' G_m\right) \tag{4.49}$$

is valid.

Proof: From the validity of Theorem 4.3 we have that the global asymptotic stability is ensured and that $\Delta V < -z_{er}' z_{e_r}, \forall n \in \mathbb{N}$. Summing both sides of this inequality from $n = 0$ up to infinite, we obtain $\|z_{er}\|_2^2 < v(\xi[0], 0)$ with $\xi[0] = \ell_m[-1] + H_m e_r$. From the $\mathcal{H}_2$ performance index defined in (4.9), we have

$$\begin{aligned} J_2(\sigma) &= \sum_{r=1}^{n_w} \|z_{er}\|_2^2 + e_r' G_m' G_m e_r \\ &< \sum_{r=1}^{n_w} \xi[0]' P[0] \xi[0] + e_r' G_m' G_m e_r \\ &= \mathrm{Tr}\left((L_m + H_m)' P[0] (L_m + H_m) + G_m' G_m\right) \end{aligned} \tag{4.50}$$

concluding thus the proof. □

Notice that $\sigma[-1] = m$ can be chosen by the designer or can be optimized to obtain the smaller $\mathcal{H}_2$ guaranteed cost.

Turning our attention to the $\mathcal{H}_\infty$ control design let us consider the system (4.6)-(4.8) evolving from $\xi[0] = 0$ and subject to an external input $w \in L_2$. The next theorem provides the main result.

Corollary 4.3. *Consider system* (4.6)-(4.8) *evolving from* $\xi[0] = 0$ *and with* $w \in L_2$*. Let the positive scalar* $\kappa \in \mathbb{N}$*, and the subset of limit cycles* $\mathfrak{X}_s$ *with the associated periodic sequences* $\mathfrak{C}_s(\kappa)$ *be given. If there exist symmetric matrices* $P[n] > 0$*,* $R_i[n]$*,* $U[n]$ *and a scalar* $\rho > 0$ *forming the solution set* Ψ *of the optimization problem*

$$\min_{\mathcal{X}_e(c)\in\mathfrak{X}_s} \inf_{\Psi} \rho \tag{4.51}$$

subject to the linear matrix inequalities

$$\begin{bmatrix} P[n]+U[n]+C_i'R_i[n]C_i & \bullet & \bullet & \bullet \\ 0 & \rho I & \bullet & \bullet \\ P[n+1]A_i & P[n+1]H_i & P[n+1] & \bullet \\ E_i & G_i & 0 & I \end{bmatrix} > 0 \tag{4.52}$$

$$U[n] + C_{c[n]}' R_{c[n]}[n] C_{c[n]} < 0 \tag{4.53}$$

for all $i \in \mathbb{K}$, $n \in \{0, \ldots, \kappa - 1\}$, $c \in \mathfrak{C}_s(\kappa)$, *with the boundary condition* $P[0] = P[\kappa]$, *then the output-dependent switching function* $\sigma[n] = u(y[n])$ *given by* (4.33) *with matrices*

$$S_i[n] = (P[n+1]A_i - \mathcal{N}_i[n]'\Xi_i[n]^{-1}\mathcal{M}_i[n])C_i'(C_iC_i')^{-1} \tag{4.54}$$

$$W_i[n] = P[n+1] - \mathcal{N}_i[n]'\Xi_i[n]^{-1}\mathcal{N}_i[n] - \mathcal{J}_i[n]\mathcal{S}_i[n]^{-1}\mathcal{J}_i[n]' + \varepsilon I \tag{4.55}$$

with $\varepsilon > 0$ *arbitrarily small, where*

$$\Xi_i[n] = H_i'P[n+1]H_i + G_i'G_i - \rho I \tag{4.56}$$

$$\mathcal{M}_i[n] = H_i'P[n+1]A_i + G_i'E_i \tag{4.57}$$

$$\mathcal{N}_i[n] = H_i'P[n+1] \tag{4.58}$$

$$\mathcal{S}_i[n] = \mathcal{L}_i[n] - \mathcal{M}_i[n]'\Xi_i[n]^{-1}\mathcal{M}_i[n] - U[n] - C_i'R_i[n]C_i \tag{4.59}$$

$$\mathcal{J}_i[n] = P[n+1]A_i - \mathcal{N}_i[n]'\Xi_i[n]^{-1}\mathcal{M}_i[n] - S_i[n]C_i \tag{4.60}$$

and $\mathcal{L}_i[n]$ *defined in* (4.24)*, ensures that the limit cycle* $\mathcal{X}_e^* = \mathcal{X}_e(c_*)$ *solution of* (4.51) *is globally asymptotically stable and the upper bound* $J_\infty(\sigma) < \rho$ *is valid.*

Proof: Consider system (4.6)-(4.8), denote $\xi[n] = \xi$, $\sigma[n] = \sigma$, $\ell_i[n] = \ell_i$ and $w[n] = w$, and define

$$\tilde{\mathcal{F}}_i[n] = \begin{bmatrix} \mathcal{L}_i[n] & \bullet & \bullet \\ \ell_i'P[n+1]A_i & \ell_i'P[n+1]\ell_i & \bullet \\ \mathcal{M}_i[n] & \mathcal{N}_i[n]\ell_i & \Xi_i[n] \end{bmatrix}$$

with $\mathcal{L}_i[n]$ given in (4.24), as well as the augmented state variable $\tilde{\xi} = [\xi' \;\; 1 \;\; w']'$. Adopting the Lyapunov function (4.17), we have, within the time interval $n \in \{0, \cdots, \kappa - 1\}$, the following developments

$$
\begin{aligned}
\Delta V(\xi,n) &= \tilde{\xi}'\tilde{\mathcal{F}}_\sigma[n]\tilde{\xi} - z_e'z_e + \rho w'w \\
&\leq \begin{bmatrix}\xi\\ \ell_\sigma\end{bmatrix}' \mathcal{F}_\sigma[n] \begin{bmatrix}\xi\\ \ell_\sigma\end{bmatrix} - z_e'z_e + \rho w'w \\
&< \begin{bmatrix}\xi\\ \ell_\sigma\end{bmatrix}' \begin{bmatrix} U[n]+C_\sigma' R_\sigma[n]C_\sigma & \bullet \\ S_\sigma[n]C_\sigma & W_\sigma[n]\end{bmatrix} \begin{bmatrix}\xi\\ \ell_\sigma\end{bmatrix} - z_e'z_e + \rho w'w \\
&= \min_{i\in\mathbb{K}} \begin{bmatrix}\xi\\ \ell_i\end{bmatrix}' \begin{bmatrix} U[n]+C_i' R_i[n]C_i & \bullet \\ S_i[n]C_i & W_i[n]\end{bmatrix} \begin{bmatrix}\xi\\ \ell_i\end{bmatrix} - z_e'z_e + \rho w'w \\
&\leq \xi'(U[n] + C_{c[n]}' R_{c[n]}[n] C_{c[n]})\xi - z_e'z_e + \rho w'w \\
&< -z_e'z_e + \rho w'w
\end{aligned} \tag{4.61}
$$

where

$$
\mathcal{F}_i[n] = \begin{bmatrix} \mathcal{L}_i[n] & \bullet \\ P[n+1]A_i & P[n+1] \end{bmatrix} - \begin{bmatrix} \mathcal{M}_i[n]' \\ \mathcal{N}_i[n]' \end{bmatrix} \Xi_i[n]^{-1} \begin{bmatrix} \mathcal{M}_i[n]' \\ \mathcal{N}_i[n]' \end{bmatrix}'
$$

The first inequality is due to the fact that the function $f_i(\xi, w) = \tilde{\xi}'\tilde{\mathcal{F}}_i[n]\tilde{\xi}$ is concave with respect to w because (4.52) ensures that $\Xi_i[n] < 0$ for all $i \in \mathbb{K}$. This can be verified by performing the Schur Complement with respect to the two last rows and columns and observing the second main diagonal. Hence, it is possible to determine $\sup_{w\in L_2} f_i(\xi, w)$ which occurs for

$$
w_*[n] = -\Xi_\sigma[n]^{-1}\left(\mathcal{M}_\sigma[n]\xi[n] + \mathcal{N}_\sigma[n]\ell_\sigma[n]\right) \tag{4.62}
$$

Then, replacing (4.62) into $f_\sigma(\xi, w)$ and rearranging, we obtain the expression in the right hand side of the first inequality of (4.61). The second inequality holds whenever

$$
\mathcal{F}_i[n] < \begin{bmatrix} U[n] + C_i' R_i[n] C_i & \bullet \\ S_i[n]C_i & W_i[n] \end{bmatrix} \tag{4.63}
$$

for all $i \in \mathbb{K}$. Notice that $S_i[n]$ given in (4.59) is the solution that minimizes the quadratic error norm $\|\mathcal{J}_i[n]\|$ and $\mathcal{J}_i[n]$ appears in the block (2,1) of (4.63). Moreover, performing the Schur Complement, it is possible to conclude that (4.52) ensures that $\mathcal{S}_i[n] < 0$, which appears in the first diagonal block of (4.63). With $\mathcal{J}_i[n]$ and $\mathcal{S}_i[n]$ completely known and taking into account that (4.63) can be rewritten as

$$
\begin{bmatrix} \mathcal{S}_i[n] & \bullet \\ \mathcal{J}_i[n] & P[n+1] - \mathcal{N}_i[n]'\Xi_i[n]^{-1}\mathcal{N}_i[n] - W_i[n] \end{bmatrix} < 0 \tag{4.64}
$$

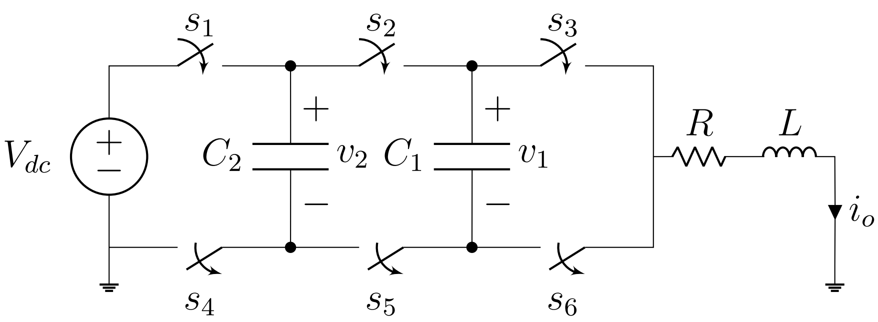


Figure 4.3 – Schema of a three-cell converter.

performing the Schur Complement with respect to $\mathcal{S}_i[n] < 0$, we conclude that $W_i[n]$ chosen as in (4.55) ensures that inequalities (4.63) are indeed verified for all $i \in \mathbb{K}$. The second equality comes from the switching function $\sigma[n] = u(y[n])$ defined in (4.33) and the third and fourth inequalities are consequences of the fact that $\ell_{c[n]} = 0$ since $\mathcal{X}_e(c) \in \mathfrak{X}_s$ and that the inequalities (4.53) are feasible, respectively. The periodic continuation $P[n] = P[k(n)]$ assures that $\Delta V(\xi, n) < -z_e'z_e + \rho w'w$ for all $n \in \mathbb{N}$. Summing both sides of this inequality from $n = 0$ up to infinity, and recalling that $V(\xi, 0) = 0$ since $\xi[0] = 0$ and $\lim_{n\to\infty} V(\xi, n) = 0$ as a consequence of the asymptotic stability of the origin $\xi = 0$, we obtain

$$\|z_e\|_2^2 - \rho\|w\|_2^2 < 0 \tag{4.65}$$

which ensures the validity of the $\mathcal{H}_\infty$ guaranteed cost. □

The same remark presented after Theorem 4.3 is valid here. Matrices $R_i[n]$ in (4.52) are restricted only for $i = c[n]$ by means of (4.53). When $i \neq c[n]$ they can be anyone great enough to satisfy (4.52). Hence, an alternative is to adopt Corollary 4.1 replacing $\mathcal{Q}_i[n]$ given in (4.40) by

$$\mathcal{Q}_i[n] = \mathcal{L}_i[n] - \mathcal{M}_i[n]'\Xi_i[n]^{-1}\mathcal{M}_i[n] - U[n] \tag{4.66}$$

to determine $R_i[n]$ for $i \neq c[n]$ as close as possible the bound of feasibility of (4.52). The next practical application illustrates the main features of the proposed theory.

4.3 Practical Application

This example was borrowed from (BENMILOUD *et al.*, 2019) and adopted also in (EGIDIO *et al.*, 2020). It consists of a three cells converter as depicted in Figure 4.3 where the control signal is $u_k = 1$ ($u_k = 0$) indicating that the upper switch s_k is closed (open) and the correspondent lower switch is open (closed). Table 4.1 shows the $N = 8$ operation modes resulting from the combinations of $\{u_1, u_2, u_3\}$.

Defining the state variable as $x(t) = [v_1(t) \;\; v_2(t) \;\; i_o(t)]'$ the continuous-time model is given by

$$\dot{x}(t) = A_{o\sigma(t)}x(t) + b_{o\sigma(t)} + H_{o\sigma(t)}w(t) \tag{4.67}$$

$$z_c(t) = E_{o\sigma(t)}x(t) + G_{o\sigma(t)}w(t) \tag{4.68}$$

with matrices

$$A_{oi} = \begin{bmatrix} 0 & 0 & \frac{u_2-u_1}{C_1} \\ 0 & 0 & \frac{u_3-u_2}{C_2} \\ \frac{u_1-u_2}{L} & \frac{u_2-u_3}{L} & \frac{-R}{L} \end{bmatrix}, \; b_{oi} = \begin{bmatrix} 0 \\ 0 \\ \frac{V_{dc}u_3}{L} \end{bmatrix} \tag{4.69}$$

$E_{oi} = \mathrm{diag}([1 \;\; 1 \;\; 1])$ and $G_{oi} = [0 \;\; 0 \;\; 0]'$ for all $i \in \mathbb{K}$. The system parameters are the same as in the mentioned references: $V_{dc} = 60$ [V], $C_1 = C_2 = 40$ $[\mu F]$, $L = 5$ [mH], and $R = 20$ $[\Omega]$. As in (EGIDIO *et al.*, 2020), it is considered that $\sigma(t) = \sigma(t_n), \; \forall t \in [t_n, t_{n+1})$ with $t_{n+1} - t_n = T$ with $T = 0.1$ ms being the sampling period. A discrete-time model as in (4.1)-(4.3) was obtained using the following discretization procedure

$$\begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix} = e^{\mathcal{A}_iT}, \begin{bmatrix} E_i' \\ F_i' \end{bmatrix}\begin{bmatrix} E_i' \\ F_i' \end{bmatrix}' = \int_0^T e^{\mathcal{A}_i't}\mathcal{C}_i'\mathcal{C}_ie^{\mathcal{A}_it}dt \tag{4.70}$$

with

$$\mathcal{A}_i = \begin{bmatrix} A_{oi} & B_{oi} \\ 0 & 0 \end{bmatrix}, \; \mathcal{C}_i = [E_{oi} \; F_{oi}]$$

of (SOUZA *et al.*, 2014). The pairs (B_i, F_i) and (B_{oi}, F_{oi}) are defined according to the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance index. In our context, it is supposed that only the voltages v_1 and v_2 are measured, leading to

$$C_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.71}$$

for all $i \in \mathbb{K}$ in the output (4.2). As in (EGIDIO *et al.*, 2020), the candidate limit cycles must satisfy (4.13) with $\Gamma = \mathrm{diag}(0.5, 0.5, 0)$ and $x_* = [20 \;\; 40 \;\; I_{ref}]'$ [A] with $I_{ref} \in [0, 3]$ which leads to 30 candidates.

$\mathcal{H}_2$ control: In this case $H_i = -x_e[0]$ and $\sigma[-1] = c[\kappa - 1]$ in order to make $x[0] = 0$ representing the system start-up. The discretization is performed with $B_{oi} = [b_{oi} \;\; 0]$ and $F_{oi} =$

Table 4.1 – Control signal u_i for each mode i

i	1	2	3	4	5	6	7	8
u_1	0	1	0	1	0	1	0	1
u_2	0	0	1	1	0	0	1	1
u_3	0	0	0	0	1	1	1	1

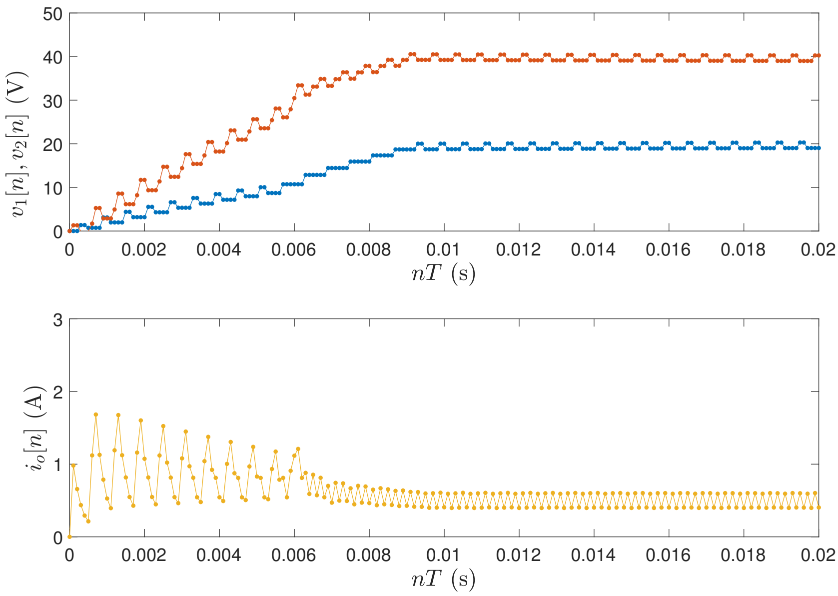


Figure 4.4 – State trajectories for $\mathcal{H}_2$ control design.

$[0\ \ G_{oi}]$ providing, besides A_i and E_i, the matrices $B_i = [b_i\ \ 0]$ and $F_i = [d_i\ \ G_i]$ in (4.70). Notice that the discrete-time controlled output of this form is $z_a[n] = E_\sigma x[n] + d_\sigma + G_\sigma w[n]$ and we redefine $z[n] = z_a[n] - d_\sigma$. Solving the optimization problem of Corollary 4.2 we have obtained a guaranteed cost of $J_2(\sigma) < 75.6657$ associated to the sequence $c = (5, 1, 3, 1, 2, 1)$, the same of (EGIDIO *et al.*, 2020). Implementing the switching function (4.33) with matrices $R_i[n]$ determined from Corollary 4.1 with $\varepsilon = 10^{-4}$, which is the same adopted in (4.35), we have obtained a non-periodic switching rule which has provided an actual cost of $J_2(\sigma) = 5.8967$ and the state trajectories of Figure 4.4. Figure 4.5 presents the corresponding phase portrait. If the periodic switching function $\sigma[n] = c[k(n)]$ is adopted, the state trajectories reach the limit cycle in a time interval 10 times greater and the associated actual cost is $J_2(\sigma) = 37.7900$. Moreover, the state trajectories of ($v_1[n]$,$v_2[n]$,$i_o[n]$) have a peak value of (26.5189 [V], 42.7869 [V], 1.0875 [A]) indicating overshoot in the two voltages, which is not observed in Figure 4.4.

$\mathcal{H}_\infty$ **control:** As in (EGIDIO *et al.*, 2020), the continuous-time matrices $H_{ci} = [0\,0\,u_3/L]'$ are used to model $w(t)$ as a deviation of the input voltage around V_{dc}. We have discretized the

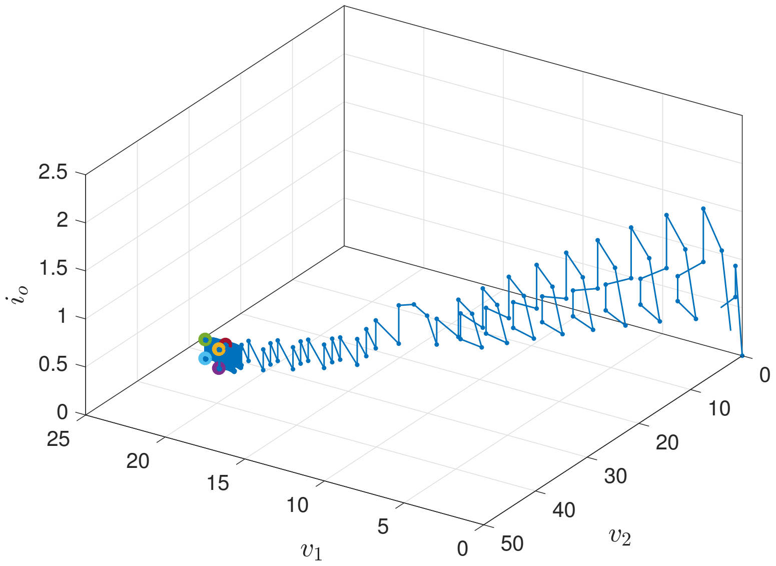


Figure 4.5 – State trajectories converging to the limit cycle $\mathcal{X}_e^*$.

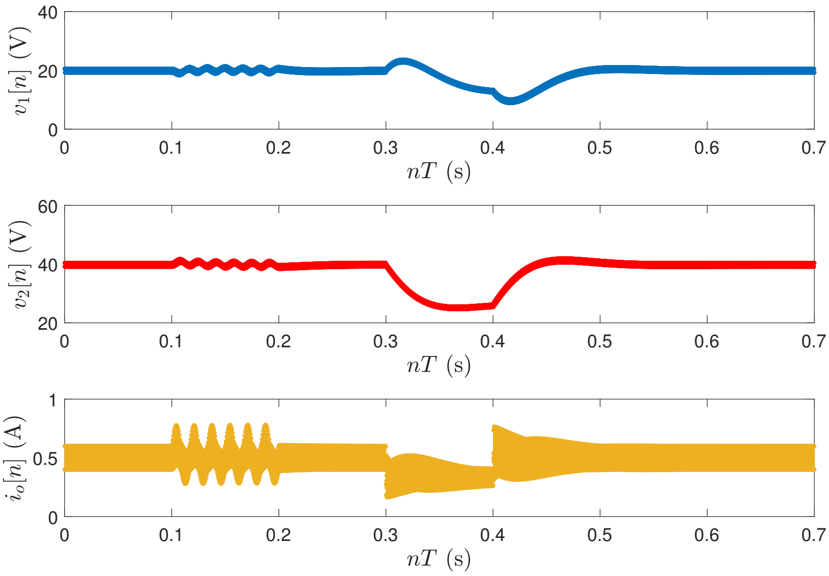


Figure 4.6 – State trajectories for $\mathcal{H}_\infty$ control design.

system using (4.70) with $B_{oi} = [b_{oi} \ \ H_{oi}]$ and $F_{oi} = [0 \ \ G_{oi}]$ providing, besides A_i and E_i, the matrices $B_i = [b_i \ \ H_i]$ and $F_i = [d_i \ \ G_i]$. Solving the conditions of Corollary 4.3 we have obtained a guaranteed cost of $J_\infty < 0.0012$ associated to the sequence $c = (5, 1, 2, 1, 3, 1)$ which is different from the one obtained in (EGIDIO *et al.*, 2020). Implementing the switching function (4.33) with $R_i[n]$ determined from Corollary 4.1 taking into account (4.41) with $\varepsilon = 10^{-4}$ adopted also in (4.55) and adopting

$$w[n] = \begin{cases} 10\sin(120\pi T n), & n \in [0.1/T, 0.2/T) \\ -20, & n \in [0.3/T, 0.4/T) \\ 0, & \text{otherwise} \end{cases} \tag{4.72}$$

we have obtained the state trajectories of Figure 4.6 for the system evolving from $x[0] = x_e[0]$. Notice that the asymptotic stability is preserved always after the perturbation. Moreover, the presented figures illustrated the validity and the efficiency of the proposed theory.

4.4 Final Considerations

Throughout this chapter, sufficient LMI conditions derived from a time-varying convex Lyapunov function were obtained to the design of a static output-dependent switching function for discrete-time switched affine systems. As preliminary result, we have included a guaranteed cost in a recent work from the literature (SERIEYE *et al.*, 2023) and compared with the results of (EGIDIO *et al.*, 2020) showing that the conditions we adopted as basis for generalization is not more conservative. The obtained output feedback conditions ensures global asymptotic stability of a suitable limit-cycle and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes. Hence, the designer is able to control suitably the transient and steady-state responses of the system even when the state is not available. A practical application example illustrated the main features of the proposed methodology.

5 CONCLUSION

The study of static output feedback control design of discrete-time switched affine systems was motivated by the interesting properties of this class of systems and by the relevance of considering more realistic practical situations. Among them, the physical limitations of real-world systems, which impose constraints on the switching frequency, and the fact that, generally, not all states are available for measurement.

In this context, we have treated asymptotic stability of suitable limit cycles, as a manner to deal with switching frequency limitation, which makes impossible to ensure asymptotic stability of an equilibrium point. The desired limit cycle is determined by the designer and must satisfy some criterion associated with the steady-state response, for example, the maximum allowed ripple. Afterwards, based on a convex time-varying Lyapunov function, we have obtained sufficient conditions expressed in terms of LMIs, to the design of an output-dependent switching function that ensures global asymptotic stability of the desired limit cycle and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performance indexes. This represents an advantage with respect to other approaches from the literature that take into account practical stability, because allows to ensure an adequate performance for the steady-state and transient responses. In the practical stability approach nothing can be stated about the steady-state behavior and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes cannot be considered, since they are defined exclusively for asymptotic stable systems. A practical application example concerning a three-cell DC-DC converter was used to illustrate these results.

Also in this work, we have included a guaranteed performance cost in the recent reference (SERIEYE *et al.*, 2023) that treated only stability to compare their results with the ones proposed in (EGIDIO *et al.*, 2020), that have been used as basis for generalization to obtain our output feedback sufficient conditions. Both techniques provided the same guaranteed cost, and are not comparable in terms of actual performance. This result was illustrated by an academic example.

As perspectives for future works, a natural step is the generalization of these results to treat asymptotic stability of limit cycles and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performance indexes for switched affine systems in the continuous-time domain. Also, in this context, an important topic to be dealt with is to treat robust control when the model is affected by parametric uncertainties.

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