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Graph Diagram Groups

Grupos de Diagramas de Grafos

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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"Ítaca te brindó tan hermoso viaje.
Sin ella no habrías emprendido el camino.
Pero no tiene ya nada que darte.
(Ítaca, Konstantino Kavafis)

Resumo

Neste presente trabalho nós definimos a família de grupos de diagrama de grafos a qual é a generalização dos grupos de diagrama estudados em [1]. Nós seguimos ideias similares para definir esta família mas demos nossa própria versão das noções de células, derivações, diagramas e concatenações. Nós também provamos que nossa família é grande em dois sentidos. Em primeiro lugar, mostrando que ela é fechada sob produto direto e em segundo lugar, mostrando que algumas famílias de grupos conhecidos podem ser vistas como grupos de diagrama de grafos. Por exemplo, nós estudamos a família dos grupos de rearranjos [2], a qual contém vários grupos tipo Thompson.

Palavras-chave: Grupos de Diagramas de Grafos, grupos de rearranjos, grupos tipo Thompson.

Abstract

In the present work we define the family of Graph Diagram Groups that is a generalization of Diagram Groups studied in [1]. We follow similar ideas to define our family but give our own version of the notions of cells, derivations, diagrams and concatenations. We also show that our family is big in two ways. Firstly, by proving that it is closed under direct product and secondly by showing that some known families of groups are contained in the family of graph diagram groups. For instance, we study the family of the Rearrangement Group of Fractals [2] that is a family containing many Thompson-like groups.

Keywords: Diagram groups. Rearrangement groups of fractals, Thompson-like groups.

Contents

Introduction

In this thesis we define and study Graph Diagram Groups, a family of groups generalizing Guba and Sapir's diagram groups. We will show that this family contains some important family of groups in geometric group theory. For instance, Thompson's groups F, T and V introduced by Richard Thompson in [3] are graph diagram groups. Elements of these groups can be defined by piecewise-linear homeomorphisms and, in particular. F acts on $[0, 1]$, T acts on S^1 and V acts on the Cantor set. The groups T and V are the first examples of infinite, finitely presented simple groups.

We wish to describe the groups in our family, so we are interested in finding common properties in some groups of this family. For instance, Max Dehn in [4] defined the following decision problems that are interesting in Geometric Group Theory.

- Word Problem: Given a group G generated by a finite generating set X, is there an algorithm that decides if, given a word w in the alphabet X, it is the identity element when viewed as an element of G?
- Conjugacy Problem: Given a group G with a given finite presentation $\langle X \mid R\rangle$, is there an algorithm that decides if, given two words w_1 and w_2 in the alphabet X, they are conjugate when viewed as elements of G?
- Isomorphism Problem: Given two finitely presented groups G_1 and G_2, is there an algorithm that decides whether or not they are isomorphic?

In general these problems are a useful tool to understand how a group works. It has been shown that F, T and V have solvable conjugacy problem, see [5]. On the other hand, the elements of the Thompson group F can be defined as rearrangements of $[0, 1]$ that is, as piecewise linear homeomorphism between two dyadic partitions of $[0, 1]$ that map a standard dyadic interval of $[0, 1]$ in the first partition into one in the second. Belk and Forrest in [2] extend this idea to a more complicated class of groups and provide a framework that covers Thompson's groups. From their point of view F is the group of rearrangements of the unit interval that preserves the self-similar structure defined by the standard dyadic intervals. They define a family of groups that act by homeomorphisms on a large family of self-similar topological spaces.

There have been many attempts to generalize the Thompson's groups F, T, and V into larger families. Among these we mention Higman's groups [6], the piecewise-linear groups of Bieri and Strebel [7] and Stein [8], the braided version studied by Brin

[9] and Dehornoy [10], the two-dimensional version of V described by Brin [11] and the rearrangement groups of fractals studied by Belk and Forrest [2].

In this doctoral thesis, we will study one more attempt to generalize Thompson's groups into a family of groups. We will define Graph Diagram Groups, a new family of groups inspired by the family of diagram groups. Diagram groups are a family of groups that was first suggested by Meakin and Sapir. The first theory about this family was developed by Kilibarda in her thesis [12] and in [13]. This family was further studied by Guba and Sapir in [1], [14], [15] and [16].

The importance of the study of such kind of families relies on the fact that they provided a framework that allow to prove properties for many groups in the family. For example, Guba and Sapir prove the solvability of the conjugacy problem for the elements in the family of diagram groups with the restriction that the semigroup presentation associated with the group has solvable word problem. In particular for Thompson's group F, they show new presentations for some groups that have definitions similar to F and give characterizations for the centralizer of F.

The study of diagram groups was also followed for other authors, for instance Farley [17] shows that if $\mathcal{P}$ is a finite semigroup presentations the diagram group associated to this presentation and a single word w is of type $\mathcal{F}_\infty$. Farley constructs a contractible, free complex for each diagram group $\mathcal{D}(\mathcal{P}, w)$ and shows that if $\mathcal{P}$ is a finite semigroup presentation this complex is a proper $CAT(0)$ cubical complex with respect to its natural metric and $\mathcal{D}(\mathcal{P}, w)$ act properly, freely, cellularly and by isometries.

James Belk and Bradley Forrest conceived of the idea of graph diagram groups as part of their work on rearrangement groups [2], but did not explore the idea fully. The goal of this thesis is to provide a solid foundation, finding examples and start the investigation on their structure.

The family of graph diagram groups has already shown to be of importance. For example, Belk and Forrest worked independently on generalizations of the Farley complex and showed that every graph diagram group over finite graph rewriting system acts properly by isometries on a $CAT(0)$ cubical complex. They also use the graph diagram group structure to show that the group of rearrangements for the airplane defined in Example 1.55 has type $\mathcal{F}_\infty$.

We divide this work as follows, in Chapter 1 we introduce some families related to the graph diagram groups such as Thompson groups, right angled Artin groups, rearrangement groups of fractals, (see [2]), and diagram groups. In Chapter 2 we define Graph diagrams and show that, under certain conditions, they are a group under concatenation.

Theorem. *The set of* $\mathcal{D}(\mathcal{R}, \Gamma)$ *of equivalence classes of* $Diag(\mathcal{R}, \Gamma)$ *forms a group under concatenation.*

On the other hand, in Chapter 3 and Chapter 4 we show that the family of graph diagram groups is large. In particular in Chapter 3 we show that all the groups in Chapter 1 can be seen as Graph diagram groups. More precisely, we have the following theorem,

Theorem. *If the rewriting system $\mathcal{R}'$ associated to a graph rewriting system $\mathcal{R}$ is reductive and symmetric, the rearrangement group $\mathcal{G}(\mathcal{R}')$ is isomorphic to $\mathcal{D}(\mathcal{R},\Gamma)$.*

Moreover, in Chapter 4 we see that the class of graph diagram groups is closed under direct product and a direct consequence of the work realized for the graph diagram groups is that the class of rearrangement group of fractals is also closed under direct products.

We stress that the main contributions of this work are:

1. Each graph diagram is equivalent to a unique reduced graph diagram (Theorem 2.50);
2. $\mathcal{D}(\mathcal{R},\Gamma)$ is a group (Theorem 2.57);
3. Right angled Artin groups are graph diagram groups (Theorem 3.4);
4. Rearrangement groups of fractals are graph diagram groups (Theorems 3.13, 3.14 and 3.22).

1 Some important groups

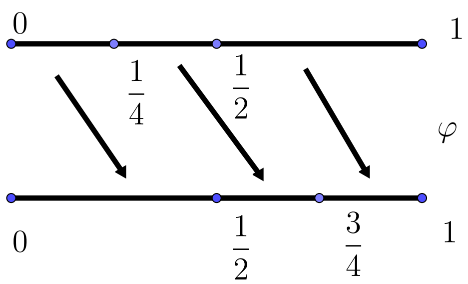
In this chapter we introduce all the groups that will be of interest throughout this doctoral thesis. We define Thompson groups F, T and V in Section 1.1, while in Section 1.2 we introduce another Thompson like group, the Basilica Thompson Group. In Section 1.3 we discuss some preliminary theory to understand the family of Diagram groups, defined in Section 1.4. Finally, we will define right angled Artin groups and the rearrangement group of fractals in Sections 1.5 1.6, respectively. We will concentrate on the principal properties of these groups, though we only prove some of them. Furthermore, none of the results presented in this chapter are new.

1.1 Thompson groups F, T and V

Definition 1.1. *A **dyadic subdivision** of the interval $[0,1]$ is recursively defined by taking some intervals of a dyadic subdivision and cutting them in half, starting from the basic subdivision given by $[0,1]$ itself. This means that the intervals of a dyadic subdivision are all of the form $\left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]$.*

Definition 1.2. *Given two dyadic subdivisions with the same number of subintervals, the corresponding **dyadic rearrangement of** $[0,1]$ is the orientation-preserving piecewise-linear homeomorphism obtained by mapping each interval of the first subdivision linearly onto the corresponding interval of the second subdivision.*

Example 1.3. *Consider the following rearrangement of $[0,1]$*



The domain of φ is a dyadic subdivision of $[0,1]$, since it can be constructed by first cutting the interval $[0,1]$ in half and then cutting the interval $[0,1/2]$ in half.

Moreover φ is a dyadic rearrangement that maps $[0,1/4]$ linearly onto $[0,1/2]$, maps $[1/4,1/2]$ linearly onto $[1/2,3/4]$, and maps $[1/2,1]$ linearly onto $[3/4,1]$.

Theorem 1.4. *The set of all dyadic rearrangements of the interval $[0,1]$ forms a group under composition; this is called Thompson's group F.*

Proof. See [18], p.2, Theorem 1.1.2 and Corollary 1.1.3. □

One of the advantages of F is that we can have multiple interpretation of its elements. This idea allows us to find more than one prove for some of its properties. There are other definitions for F. One of these is the following,

Definition 1.5. *We define Thompson's group F as the group (under composition) of those homeomorphisms of the interval $[0,1]$, which satisfy the following conditions:*

1. *they are piecewise linear and orientation-preserving,*
2. *there are only finitely many breakpoints on the interval $[0,1]$,*
3. *in the pieces where the maps are linear, the slope is always a power of 2, and*
4. *the breakpoints are dyadic.*

Thompson's group F is one of three groups discovered by Richard J. Thompson in the 1960's [3]. The others are T, which acts on the unit circle by suitable piecewise-linear homeomorphisms, and V which acts on the standard Cantor set by suitable piecewise-linear homeomorphisms. See [19] for a general introduction to these groups.

Thompson groups F, T, and V are considered interesting in geometric group theory because of their unique properties. In particular:

1. F, T, and V are finitely generated and finitely presented. Indeed, Brown and Geoghegan showed in [20] that F has type $\mathcal{F}_\infty$ and Brown then showed in [21] that all groups belonging to a large family of groups (containing the groups F, T and V) have type $\mathcal{F}_\infty$. A group has type $\mathcal{F}_\infty$ if it can be realized as the fundamental group of an aspherical CW-complex with finitely many cells in each dimension.
2. T and V are simple, and F has simple commutator subgroup.
3. T and V are the first known examples of infinite, finitely presented simple groups.
4. Each of F, T and V acts properly by isometries on a $CAT(0)$ cubical complex.
5. F, T and V have exponential word growth.

1.2 The Basilica Thompson Group

In [22] Belk and Forrest define the Basilica Thompson group T_B. The figure below shows the invariant lamination for the Basilica fractal. This lamination consists of a circle $[0,1]/\{0,1\}$ together with a single hyperbolic arc between each pair of points

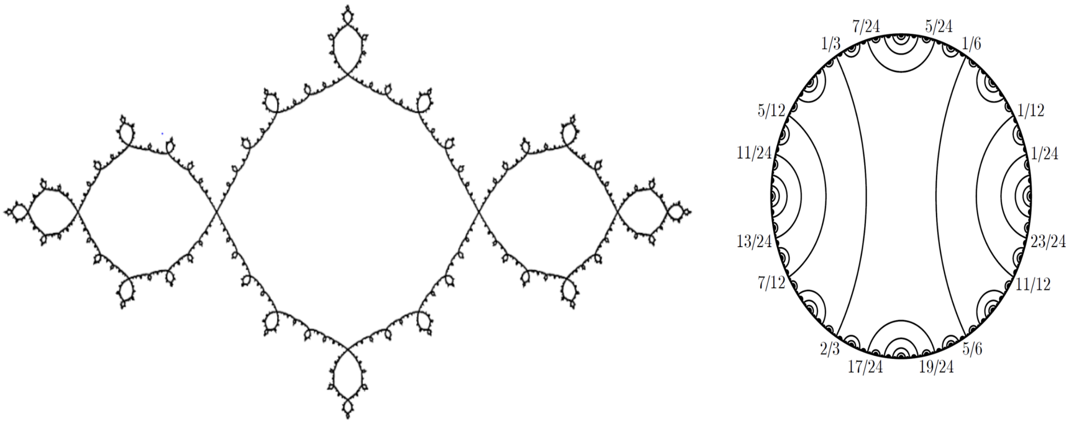


Figure 1 – The Basilica Julia set and the lamination of the Basilica

that should be identified. Any homeomorphism of the circle that preserves this lamination descends to a homeomorphism of the Basilica Julia set.

Then, topologically, the Basilica Julia set can be obtained from this lamination by identifying the end point of each arc. Similarly to Thompson's group F, we have multiple definitions for this group, for example,

Definition 1.6. *The* ***Basilica Thompson group*** T_B *consists of all piecewise-linear homeomorphisms* h *of the circle satisfying the following conditions:*

1. *All the slopes of* h *are powers of* 2.
2. *All of the breakpoints of* h *and* h^{-1} *are at angles of the form* $\dfrac{k\pi}{2^n 3}$ *, where* $k, n \in Z$.
3. h *preserves the invariant lamination for the Basilica.*

In [2] the same authors see that this group is a Rearrangement Group of Fractals. In fact, the rearrangement group T_B for the Basilica replacement system from Example 2.48 in the case $n = 1$ is the Basilica Thompson group.

Belk and Forrest proved the following facts about this group:

1. T_B is generated by the four elements in Figure 2.
2. Thompson's group T contains copies of T_B, and T_B contains T.
3. The commutator subgroup $[T_B : T_B]$ has index two in T_B and is simple. It is not isomorphic to T.
4. They also conjecture that T_B is not finitely presented, which was proven by Witzel and Zaremsky in [23].

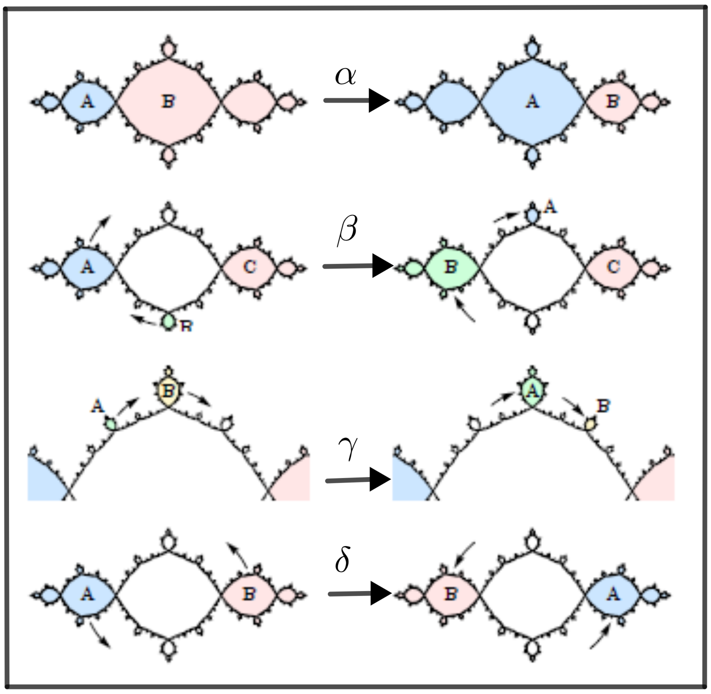


Figure 2 – The generators of the Basilica group

1.3 Rewriting Systems

The topics presented in this section will be of importance many times throughout this thesis. Mainly, we will use them to define equivalence classes of elements that belong to a graph diagram group, see Chapter 2.

Definition 1.7. *A **graph in the sense of Serre** is a 5− tuple* $< V, E, ^{-1}, \iota, \tau >$*. where* V *and* E *are disjoint sets of **vertices** and **edges** respectively, and* $^{-1}$ *is an **involution** on* E*, and* ι *and* τ *are functions from* E *to* V*. Moreover, we must have:*

1. $e^{-1} \neq e$ *for every* $e \in E$
2. $\iota(e^{-1}) = \tau(e), \tau(e^{-1}) = \iota(e)$.

Here $\iota(e)$ *is called the **initial vertex of*** e *and* $\tau(e)$ *is called the **terminal vertex of*** e.

Definition 1.8. *An **(undirected) graph** is a pair* $G = (V, E)$*, where* V *is a set whose elements are called **vertices**, and* E *is a set of 2-sets (sets with two distinct elements) of vertices, whose elements are called **edges**. The vertices* x *and* y *of an edge* $\{x, y\}$ *are called the **endpoints** of the edge. The edge* $\{x, y\}$ *is said to **join*** x *and* y *and to be **incident** to* x *and* y*. A **directed graph** is a pair* $G = (V, E)$ *where*

- V *a set of **vertices**;*
- E *is a set of ordered pairs* $E \subseteq \{(x, y) \mid (x, y) \in V^2 \text{ and } x \neq y\}$ *a set of edges called **directed edges**.*

We also define a ***multigraph*** *as a graph which is allowed to have multiple edges that have the same initial and terminal vertices.*

Remark 1.9. *Notice that the definition of a graph in the sense of Serre (Definition 1.7) is independent of (but related to) the notions of directed graph and undirected graph. In general, one can turn a directed graph into a graph in the sense of Serre and viceversa. It will be clear from the various contexts of this thesis which definition we will be using. More precisely, we use the definition of graphs in the sense of Serre when we discuss rewriting systems in Section 1.3, while we work with undirected/directed graphs in the context for graphs of graph diagrams that we will now introduce.*

Definition 1.10. *A* ***path*** *on a graph in the sense of Serre is either a vertex or a non-empty sequence of edges* $e_1 e_2, \dots e_n$ *where* $\tau(e_i) = \iota(e_{i+1})$ *for every* $i = 1, 2, \dots n-1$. *If* $p = e_1 e_2, \dots e_n$ *then we call the path* $p^{-1} = e_n^{-1} e_{n-1}^{-1}, \dots e_1^{-1}$ ***inverse path****. A path that is formed by only one vertex is called* ***empty path****.*

Definition 1.11. *An* ***orientation*** *on a graph* Γ *in the sense of Serre is a subset* E_0 *of the set* E *of edges such that* $E_0 \cup E_0^{-1} = E$ *and* $E_0 \cap E_0^{-1} = \varnothing$. *The edges of* E_0 *are called* ***positive*** *and the edges of* E_0^{-1} *are called* ***negative****. An* ***oriented graph*** *is a graph with an orientation. A path in an oriented graph is called* ***positive*** *if all its edges are positive.*

Definition 1.12. *A* ***rewriting system*** *is an oriented graph in the sense of Serre. The vertices of* Γ *are called* ***objects*** *and the positives edges are called* ***moves****.*

If $a = \iota(e), b = \tau(e)$ for some positive edge e then we write $a \to_\Gamma b$ and we denote the reflexive, transitive closure of the relation $\to_\Gamma$ by $\xrightarrow{*}_\Gamma$.

Definition 1.13. *A rewriting system is called* ***terminating*** *if every sequence* $a_1 \to a_2, \to \dots a_n \to \dots$ *terminates, that is, it is finite. A rewriting system* Γ *is called* ***confluent*** *if for every three objects* a, b, c *such that* $a \xrightarrow{*} b$ *and* $a \xrightarrow{*} c$ *there exists an object* d *such that* $b \xrightarrow{*} d$ *and* $c \xrightarrow{*} d$. *A rewriting system* Γ *is called* ***locally confluent*** *if for every three objects* a, b, c *such that* $a \to b$ *and* $a \to c$ *there exists an object* d *such that* $b \xrightarrow{*} d$ *and* $c \xrightarrow{*} d$.

Lemma 1.14 (Diamond Lemma)**.** *Every terminating locally confluent rewriting system is confluent.*

Proof. See [24] and also [25]. □

Remark 1.15. *Consider the equivalence relation generated by* $\to_\Gamma$, *that is the reflexive symmetric closure of* $\to_\Gamma$, *denoted by* $\xleftrightarrow{*}_\Gamma$. *Observe that each connected component* C *has a unique element* $\nu(C)$ *such that the moves of* Γ *cannot be applied to* $\nu(C)$. *In fact, suppose that* x *is an object that belong to* C. *Note that each sequence* $x \to x_1 \to x_2 \to \dots$ *must terminate since* Γ *is terminating and the last element in each sequence must be* $\nu(C)$ *since* Γ *is confluent.*

A special class of rewriting system is a the class of string rewriting system. To define this class we need to set the following basic concepts.

Definition 1.16. *An **alphabet** is an arbitrary nonempty set Σ, the elements in Σ are called **letters**. The **free semigroup** Σ^+ over Σ is the set of all nonempty strings of letters from Σ with operation given by concatenation. The **free monoid** Σ^* is obtained from Σ^+ by adding the empty string denoted by 1.*

Definition 1.17. *A **string rewriting system** is a rewriting system where objects (vertices) are elements of a free monoid Σ^* and the moves (edges) are the applications of relations from some fixed set of relations $\mathcal{R}$ where a relation is pair of words from Σ^*. Formally, an edge of the string rewriting system $\langle \Sigma \mid \mathcal{R} \rangle$ is a triple $(x, u \to v, y)$ where $x, y \in \Sigma^*, (u, v) \in \mathcal{R}$. The initial vertex of this edge is xuv and the terminal vertex is xvy.*

1.4 Diagram groups

Definition 1.18. *Given a $\mathcal{R}$ a rewriting system, a **derivation** is a sequence of words*

$$u_1 \overset{r_1}{\Longrightarrow} u_2 \overset{r_2}{\Longrightarrow} \ldots \overset{r_n}{\Longrightarrow} u_n.$$

*such that for every $i = 0, \ldots, n-1$ either $u_i \to u_{i+1}$ or $u_{i+1} \to u_i$ are in $\mathcal{R}$. Here u_1 is called the **initial word** of the derivation and u_n is called the **terminal word** of the derivation.*

Guba and Sapir associate to each derivation a geometrical object called diagram. Each diagram consists of a top path, a bottom path together with a set of cells that are between the top and the bottom of the diagram. We will motivate the definition of a diagram with the following examples.

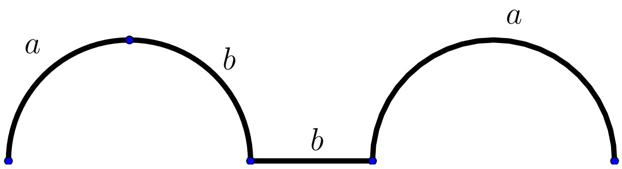
Example 1.19. *Let $\mathcal{P} = \langle a, b, c \mid b = c, ac = a, ca = a \rangle$ be a semigroup presentation. We denote the relations $b = c, ac = a,$ and $ca = a$ (read in this order) by the letters r_1, r_2 and r_3 respectively, to denote directed edges of a graph. Consider the following derivations under the semigroup presentation $\mathcal{P}$.*

$$abba \overset{r_1}{\Longrightarrow} acba \overset{r_2}{\Longrightarrow} aba \overset{r_3^{-1}}{\Longrightarrow} abca \overset{r_1^{-1}}{\Longrightarrow} abba$$

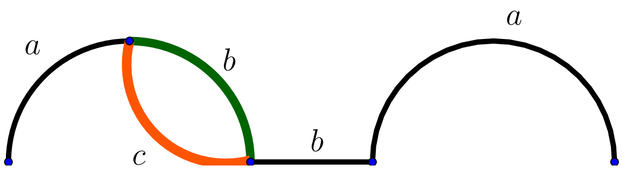
We associate the word

$$abba$$

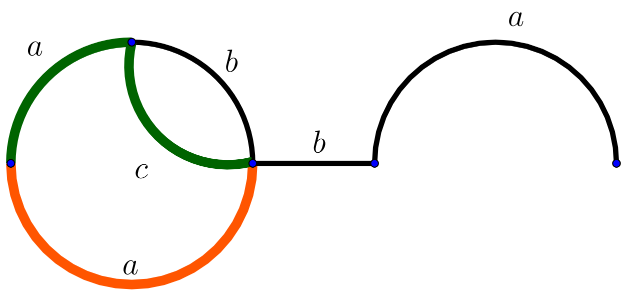
*to the **linear diagram** labeled with the letters of the word abba in each edge. We denote this graph by $\epsilon(abba)$ and we say that this diagram has **top** and **bottom** $\epsilon(abba)$. In general, given a diagram Δ obtained from a derivation ρ, with initial word u and terminal word w we define $top(\Delta) = \epsilon(u)$ and $bot(\Delta) = \epsilon(w)$.*



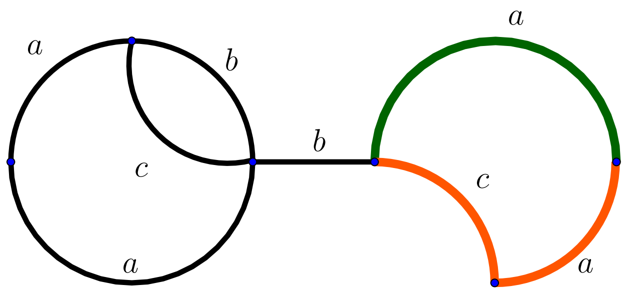
In the first move of the rewriting system, $abba \overset{r_1}{\Longrightarrow} acba$, *we add a* ***cell*** (b, c) *with top* b *and bottom* c *to* $\epsilon(abba)$, *then the derivation induces a diagram whose top is* $\epsilon(abba)$ *and the bottom* $\epsilon(acba)$.



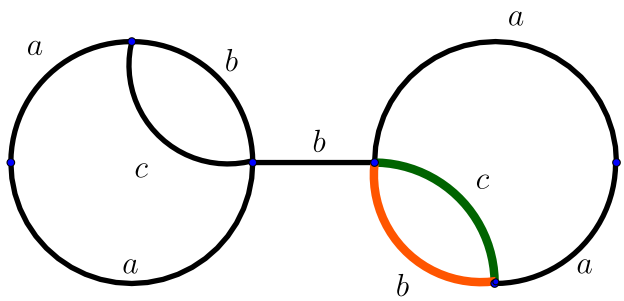
$$abba \overset{r_1}{\Longrightarrow} acba \overset{r_2}{\Longrightarrow} aba$$



$$abba \overset{r_1}{\Longrightarrow} acba \overset{r_2}{\Longrightarrow} aba \overset{r_3^{-1}}{\Longrightarrow} abca$$



$$abba \overset{r_1}{\Longrightarrow} acba \overset{r_2}{\Longrightarrow} aba \overset{r_3^{-1}}{\Longrightarrow} abca \overset{r_1^{-1}}{\Longrightarrow} abba$$



Finally, we obtain a diagram with the same top and bottom labeled with the word abba.

Example 1.20. *Consider the semigroup presentation* $\mathcal{P} = \langle x, y, z, s_1, s_2, l_1, l_2 \mid s_1 = s_2, l_1 = l_2 \rangle$*. Consider the following derivations corresponding to the figure below.*

$$xs_1yl_1z \overset{r_1}{\Longrightarrow} xs_2yl_1z \overset{r_2}{\Longrightarrow} xs_2yl_2z \text{ and } xs_1yl_1z \overset{r_1}{\Longrightarrow} xs_1yl_2z \overset{r_2}{\Longrightarrow} xs_2yl_2z$$

Observe that we can write these derivations as sequences of graphs on the next figure.

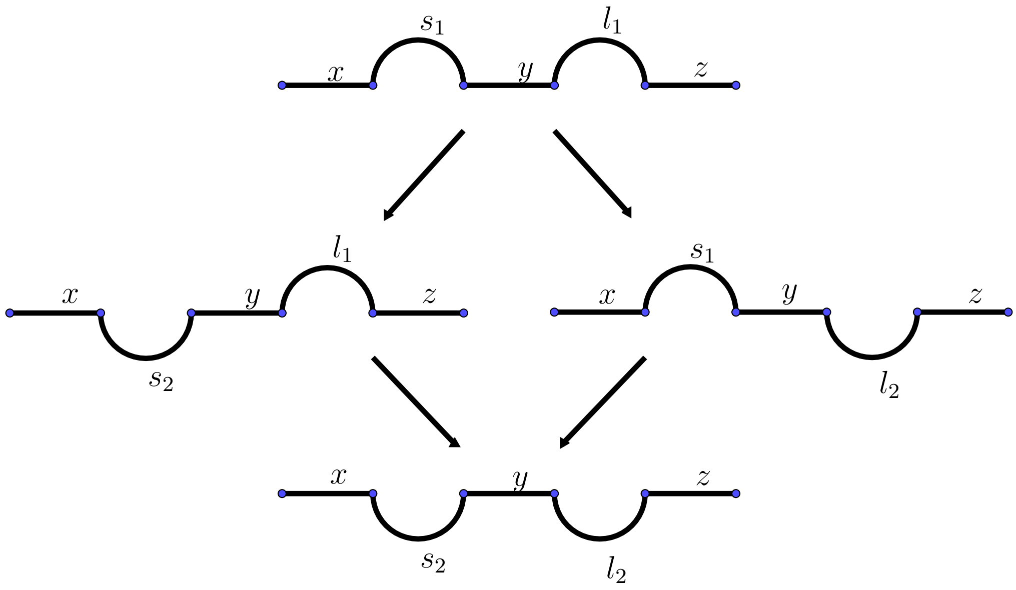


Figure 3

Also note that we can add the cells following the same procedure of the last example and produce the graph diagram in Figure 4.

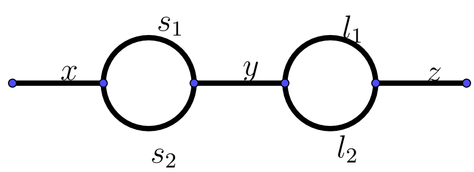


Figure 4

From these examples we can understand a method to create a **diagram**: Given a derivation we associate a labelled plane graph by taking $\epsilon(u)$ and attaching one by one cells corresponding to the relations used in this derivation. Notice that following this process we obtain a planar graph at every step, so a diagram is a planar graph.

Given a diagram Δ we can define its initial vertex as the leftmost vertex denoted $\iota(\Delta)$ and its terminal vertex as the rightmost vertex denoted $\tau(\Delta)$. This vertex together with the cells induce oriented paths with initial vertex $\iota(\Delta)$ and terminal vertex $\tau(\Delta)$. For example in Example 4 we have paths $\epsilon(xs_1yl_1z)$, $\epsilon(xs_2yl_1z)$, $\epsilon(xs_1yl_2z)$ and $\epsilon(xs_2yl_2z)$.

Definition 1.21. *A diagram whose top path is labelled by u and whose bottom path is labelled by v is called a (u,v)-**diagram**. The closure of a bounded complementary region of a diagram Δ is a **cell** of Δ. A diagram is **trivial** if it has no cells. An **atomic diagram** is one with at most one cell. Recall that an isotopy of a plane R^2 is a continuous function δ from $R^2 \times [0,1]$ to R^2 such that every function $\delta_t = \delta(*,t)$ from R^2 to R^2 is a homeomorphism. Let $\Gamma = (V,E)$ and $\Gamma' = (V',E')$ be plane graphs. We say that these graphs are **isotopic** if there exists an isotopy $\delta : R^2 \times [0,1] \to R^2$ of the plane into itself which takes V to V', E to E' and for every $t \in [0,1]$ we have that $(\delta(V,t), \delta(E,t))$ is a plane graph.*

*Therefore, two diagrams Δ_1 and Δ_2 are **isotopic**, denoted by $\Delta_1 \equiv \Delta_2$, if there is an isotopy of the plane carrying Δ_1 to Δ_2 which takes vertices to vertices and edges to edges, and matches orientations and labels on the edges.*

If Δ_1 is a (u,v)–diagram and Δ_2 is a (v,w)–diagram, then the concatenation $\Delta_1 \circ \Delta_2$ is obtained by identifying the bottom path of Δ_1 with the top path of Δ_2, using suitable representatives of the isotopy classes of Δ_1 and Δ_2. This operation is well-defined on isotopy classes and it is associative.

A pair of cells C_1 and C_2 in a diagram forms a **dipole** if the bottom path of C_1 is identical to the top path of C_2 and the label of the top path of C_1 is equal to the label of the bottom path of C_2 in the free semigroup Σ^+. To **reduce a dipole** one removes the portion of the diagram lying between the top path of C_1 and the bottom path of C_2, and identifies the top path of C_1 with the bottom path of C_2 (see Figure 5). Reducing a dipole is also called an **elementary reduction**. The inverse process is called **inserting a dipole**. A diagram without dipoles is called **reduced**.

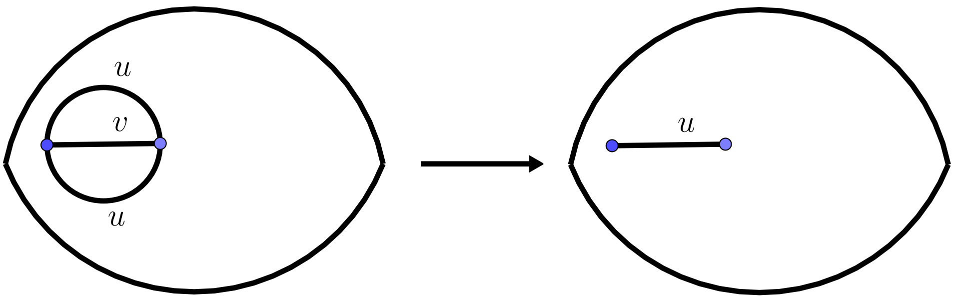


Figure 5 – A dipole reduction

We can define a rewriting system $\mathcal{R}(\mathcal{P})$. The objects of this rewriting system are diagrams over $\mathcal{P}$ and the moves are elementary reductions.

Lemma 1.22. *The diagram rewriting system $\mathcal{R}(\mathcal{P})$ is confluent and terminating.*

If a diagram Δ_1 can be obtained from Δ_2 by repeatedly inserting and removing dipoles (up to isotopy), then Δ_1 and Δ_2 are **equivalent** modulo dipoles, $\Delta_1 = \Delta_2$. Observe that concatenation is well defined under the class of diagrams modulo dipoles.

Notice that Lemma 1.22 and Remark 1.15 implies the following theorem that was proved by Kilibarda in [12].

Theorem 1.23. *Every equivalence class of diagrams over a semigroup presentation contains exactly one reduced diagram.*

Theorem 1.24. *Let $\mathcal{D}(\mathcal{P}, w)$ the set of $(w, w)-$diagrams modulo dipoles over the semigroup presentation $\mathcal{P}$. Then $\mathcal{D}(\mathcal{P}, w)$ forms a group under concatenation and it is called the* ***diagram group over $\mathcal{P}$ with the base word*** *w.*

Diagram groups are important for several reasons. For example Guba and Sapir in [1] show the following:

- The family of diagram groups contains Thompson's group F,
- Using the Squier complex for diagram groups, Guba and Sapir find new presentations for the Generalized Thompson Groups F_n.
- Diagram groups over semigroup presentations with solvable word problems have solvable conjugacy problems.
- Every diagram group is torsion free and has unique extraction roots, this is given two elements $\Delta_1, \Delta_2 \in \mathcal{D}(\mathcal{P}, w)$, $\Delta_1^2 = \Delta_2^2$ implies $\Delta_1 = \Delta_2$.
- The centralizer of every diagram group is isomorphic to the product of other diagram groups under the same presentation (but possibly different initial word) and a finite direct product of cyclic groups.
- Moreover, Crisp, Sageev and Sapir show in [26] that surface groups are contained in some diagram groups.

Moreover, Farley builds in [17] a contractible, free $\mathcal{D}(\mathcal{P}, w)$-complex $\widetilde{K}(\mathcal{P}, w)$ for each diagram group $\mathcal{D}(\mathcal{P}, w)$. Guba and Sapir define in [1] a 2− dimensional complex $\mathcal{K}(\mathcal{P})$, called Squier Complex. The 2−skeleton of $\widetilde{K}(\mathcal{P}, w)$ can be described as the universal cover of the connected component of $\mathcal{K}(\mathcal{P})$ containing the base point w. (Vertices in $\mathcal{K}(\mathcal{P})$ are words in the alphabet, and vice versa.) The complex $\widetilde{K}(\mathcal{P}, w)$ is a natural extension of $\widetilde{\mathcal{K}}(\mathcal{P})$ into higher dimensions.
The following results are due to Daniel Farley in [17].

Theorem 1.25. *If $\mathcal{P}$ is a finite semigroup presentation, then $\widetilde{K}(\mathcal{P}, w)$ is a proper $CAT(0)$ cubical complex with respect to its natural metric and $\mathcal{D}(\mathcal{P}, w)$ acts properly, freely, cellularly and by isometrices.*

Corollary 1.26. *If $\mathcal{P}$ is a finite semigroup presentation, then $\mathcal{D}(\mathcal{P}, w)$satisfies the Baum-Connes conjecture.*

We refer the reader to [27] and [28] for a proper introduction about the Baum-Connes conjecture.

Corollary 1.27. *If $\mathcal{P}$ is a finite semigroup presentation, then $\mathcal{D}(\mathcal{P}, w)$ is of type $\mathcal{F}_\infty$.*

1.5 Right Angled Artin Groups

In this section we briefly discuss the class of right angled Artin groups. We recommend the reader to see [29] for more details.

Definition 1.28. *An Artin group A is a group with the following presentation*

$$A = \langle s_1, s_2, \ldots, s_n \mid \underbrace{s_i s_j s_i \ldots}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots}_{m_{ji}} \rangle.$$

where $m_{ij} = m_{ji} \in N$, $m_{ij} \geq 2$ or $m_{ij} = \infty$ if there is no relation between s_i and s_j.

Example 1.29. *The Artin group*

$$A_1 = \langle S \mid \varnothing \rangle$$

is the free group with basis S. In this case $m_{st} = \infty$ for all $s, t \in S$. On the other hand, the Artin group

$$A_2 = \langle S \mid st = ts, \forall s, t \in S \rangle$$

is the free abelian group with basis S. In this case $m_{st} = m_{ts} = 2$ for all $s, t \in S$.

We will study a particular case of the Artin groups called **right angled Artin group** in which $m_{ij} \in \{2, \infty\}$ for all i, j so that, in this case, we have relations of the form $s_i s_j = s_j s_i$. The groups in Example 1.29 are right angled Artin groups. The most natural way to give the presentation for a right-angled Artin group is by means of the **defining graph** Γ, where Γ is the graph with vertices labeled by the generators $S = \{s_1, ..., s_n\}$ and where there is an edge between a pair of vertices s_i, s_j if and only if $m_{ij} = 2$.

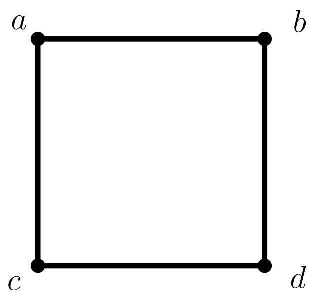


Figure 6 – $A_\Gamma = F_{\{a,d\}} \times F_{\{b,c\}}$

Example 1.30.
- *If Γ is a graph with no edges, then $A_\Gamma = F_n$, the free group of n generators,*

- *If Γ is a complete graph, then $A_\Gamma = \mathbb{Z}^n$,*
- *If Γ is a square (as in the figure 6), then A_Γ decomposes as a direct product of two free groups $A_\Gamma = F_{\{a,d\}} \times F_{\{b,c\}}$,*
- *If Γ is an n-agon, for $n \geq 5$, then A_Γ cannot be decomposed as either a direct product or a free product.*

1.6 Rearrangement Groups of Fractals

The definitions of the three Thompson groups depend heavily on the self-similar structure of the spaces on which these groups act. For example, each half of the unit interval is similar to the whole interval, as is each quarter. For this reason, it is natural to study Thompson-like groups associated with self-similar structures such as fractals. The results and definitions presented in this section are from Belk and Forrest in [2]. We will prove some of them to give some notion of the objects and techniques which are involved in the future chapters of this thesis.

Definition 1.31. *An edge **replacement system** is a pair $(G_0, e \to R)$, where G_0 is a finite, directed graph, e is a (non loop) oriented edge with distinguished initial and terminal vertices and R is a directed graph that contain the vertex v and w. We refer to the vertices v and w respectively as the initial and terminal vertices of R. The vertices v, w are called the **boundary vertices** of R. A **simple expansion** of an oriented graph G, denoted by $G \triangleleft \epsilon$ consists of replacing an edge ϵ of G by a copy of R by gluing the initial (terminal) vertex of ϵ with the initial (terminal) vertex of R. The graph obtained by doing a simple expansion in every edge of a graph G is called **full expansion of** G. We also denote G_{i+1} the full expansion of the graph G_i.*

Given a graph G obtained from applying cosecutive simple expansions to a graph G_0 we may refer to their edges by using finite sequences, $\epsilon_0\epsilon_1 \dots \epsilon_n$ where $\epsilon_0 \in G_0$ and $\epsilon_i \in R$ for $0 < i \leq n$. Analogously, we may refer to the vertices of the simple expansions as a sequence $\epsilon_0\epsilon_1 \dots \epsilon_n v$ where $\epsilon_0 \in G_0$ and $\epsilon_i \in R$ for $0 < i \leq n$ and v is a vertex in R. We refer to the sequences $\epsilon_0\epsilon_1 \dots \epsilon_n$ and $\epsilon_0\epsilon_1 \dots \epsilon_n v$ as **addresses** of an edge and a vertex, respectively. To illustrate this, observe the following example.

Example 1.32. *Consider the replacement system in Figure 7.*

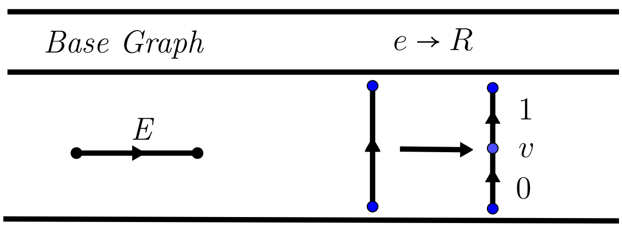


Figure 7

The graph E is a single directed edge and R is a graph with edges 0 and 1, interior vertex v and initial vertice $\iota(R) = x$ and terminal vertex $\tau(R) = y$. We can obtain the graphs $E \triangleleft E$, $E \triangleleft E \triangleleft 0$ and $E \triangleleft E \triangleleft 0$ by applying cosecutive simple expansions, see Figure 8 (for simplicity we omit arrows on the edges).

Figure 8

For instance the graph $E \triangleleft E \triangleleft 1$ has edges $E0, E10$ and $E11$ and vertices Ev and $E1v$, see Figure 8. Notice that all the edges are sequences $\epsilon_0\epsilon_1 \dots \epsilon_n$ with ϵ_0 in the base graph and $\epsilon_1 \dots \epsilon_n$ edges in the graph R of the replacement system.

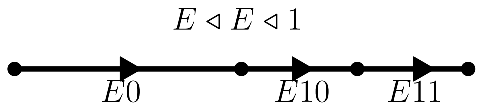


Figure 9

Example 1.33. *Consider the replacement system given by $(\Gamma, e \to R)$ as in Figure 10*

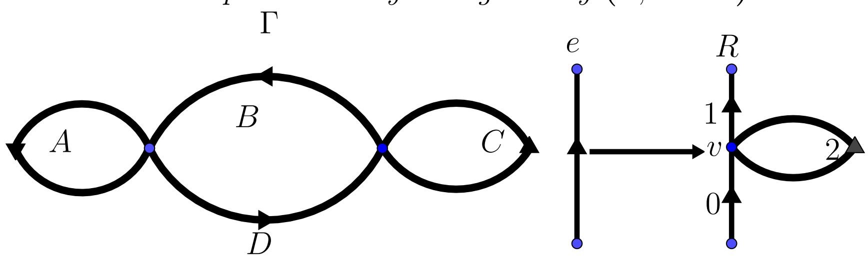


Figure 10

In Figure 11 we have the graph $\Gamma \triangleleft C$ with edges $A, B, D, C0, C1, C2$ and a new vertex Cv.

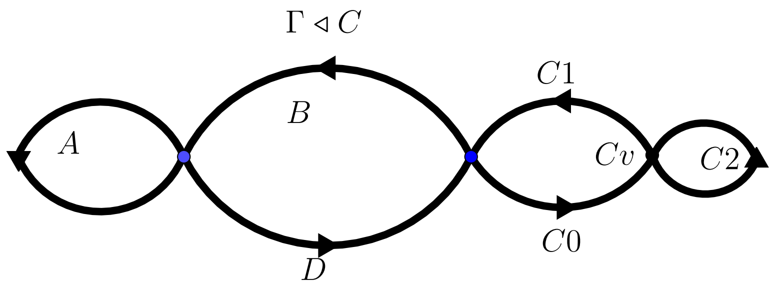


Figure 11

Let Ω be the set of symbols that consists of elements $\epsilon_0\epsilon_1\epsilon_2 \dots$ with $\epsilon_0 \in G_0$ and $\epsilon_i \in R$ for $i \geq 1$. The set Ω allows to refer to any edge in the limit space, but does not give information about the adjacency relations between its edges.The adjacency is given by the following relation,

Definition 1.34. *Let $\mathcal{R} = (G_0, e \to R)$ be a replacement system with full expansion sequence $\{G_n\}$ and symbol space Ω. Consider the relation $\sim$ defined as follows: two sequences*

$$\epsilon_0\epsilon_1\epsilon_2\dots \text{ and } \epsilon'_0\epsilon'_1\epsilon'_2\dots$$

are equivalent if for all n the edges of G_n with addresses

$$\epsilon_0\epsilon_1\epsilon_2\dots\epsilon_n \text{ and } \epsilon'_0\epsilon'_1\epsilon'_2\dots\epsilon'_n$$

*share at least one vertex. If $\sim$ is an equivalence relation, we call it the **gluing relation** on Ω. In this case, we define the **limit space** X for $\mathcal{R}$ as the quotient $\dfrac{\Omega}{\sim}$.*

Definition 1.35. *A replacement system $\mathcal{R} = (G_0, e \to R)$ is **expanding** if the following conditions are satisfied:*

1. *Neither G_0 nor R has any isolated vertices.*
2. *The initial and terminal vertices of R are not connected by an edge.*
3. *R has at least three vertices and two edges.*

*We thus call a replacement system satisfying these conditions an **expanding replacement system**.*

Recall that the idea is to generalize Thompson's groups to see them as rearrangement groups acting on self-similar structures. Observe that in Definition 1.34 we require the gluing relation to be an equivalence relation, however the relation $\sim$ does not always have this property. In the case that it is not an equivalence relation, we may be tempted to define the limit space using the transitive closure of the gluing relation. This approach often produces a space X that is not Hausdorff and therefore it cannot be guaranteed that the rearrangements of X are self-similar or conformal maps. On the other hand, when the replacement system is expanding we can guarantee that X is Hausdorff and therefore that is compact and metrizable. This is since Ω is compact and metrizable and X is a quotient of Ω, so it is enough to prove that X is Hausdorff (see [30], Proposition IX.17).

Theorem 1.36. *If $\mathcal{R}$ is an expanding replacement system, then the gluing relation $\sim$ is an equivalence relation, and the limit space $X = \dfrac{\Omega}{\sim}$ is a compact metrizable space.*

Hence, expanding replacement systems are important to ensure that X is compact and metrizable. Therefore, given an expanding replacement system we can endow X with a metric and we can prove that the canonical homeomorphism between any pair of cells in X that have the same number of boundary vertices is a similitude, that is, having the same shape but perhaps different size, so rearrangements are piecewise-similar homeomorphisms.

Convention 1.37. *In this thesis all replacement systems are assumed to be expanding.*

Definition 1.38. *Let $\pi : \Omega \to \frac{\Omega}{\sim}$ and $\Omega(e)$ the set of all the words in Ω with prefix e we call a* ***cell*** *of the limit space as $C(e) = \pi(\Omega(e))$.*

Definition 1.39. *Let $C(e)$ and $C(e')$ be cells in X, where both e and e' are loops or both are not loops. There exists a homeomorphism $\Phi : \Omega(e) \to \Omega(e')$ that $\Phi(e\zeta_1\zeta_2\dots) = e'\zeta_1\zeta_2\dots$ for any edges $\zeta_1\zeta_2\dots$ in R and Φ descends to a* ***canonical homeomorphism*** *$\phi : C(e) \to C(e')$.*

Definition 1.40.

1. *A* ***cellular partition*** *of X is a cover of X by finitely many cells whose interiors are disjoint.*
2. *A homeomorphism f, is a* ***rearrangement*** *if there exists a cellular partition P such that f is a canonical homeomorphism in each cell of P.*

Theorem 1.41. *The set of rearrangements of X is a group under the composition.*

Proof. The identity rearrangement is the rearrangement that sends any partition of the limit space to itself. If f is a rearrangement that sends a partition P_1 of X to another partition P_2, then f^{-1} is a homeomorphism that sends each cell of P_2 canonically to the corresponding cell of P_1. Lastly, if f and g^{-1} are rearrangements that restrict to a canonical homeomorphisms on each cell of P_1 and P_2 respectively. Consider the least common refinement Q of P_1 and P_2, that is the set of all cells of $P_1 \cup P_2$ that are not properly contained in other cells of $P_1 \cup P_2$. So, f and g^{-1} restricts to a canonical homeomorphism in each cell of Q and $f(g(g^{-1}(Q)))$ is well defined. □

Example 1.42. *In Section 1.2 we introduced the Basilica Thompson group T_B. In [2] Belk and Forrest prove that the Basilica is homeomorphic to the limit space given by Figure 12 and so the rearrangement group of that limit space is isomorphic to the Basilica Thompson group. In Figure 2 we have some examples of elements of the Basilica group. In general a* ***rearrangement*** *of the Basilica is any homeomorphism that maps conformally between corresponding pairs of cells in two cellular partitions. The group of all such rearrangements is the* ***Basilica Thompson group*** *T_B.*

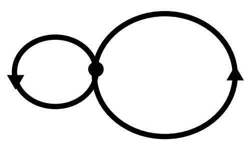
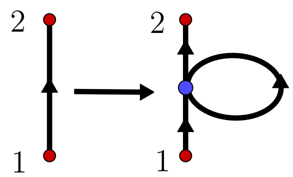
Limit Space	*Base Graph*	*Replacement Rule*
Basilica Julia Set		

Figure 12

The composition of elements in the rearrangement groups is analogous to that of the Thompson groups. For instance, consider $\alpha, \beta \in T_B$ as in Figure 13. Recall that these elements are generators of T_B the Basilica Thompson group, see Figure 2. Suppose that we wish to calculate $\beta \circ \alpha$.

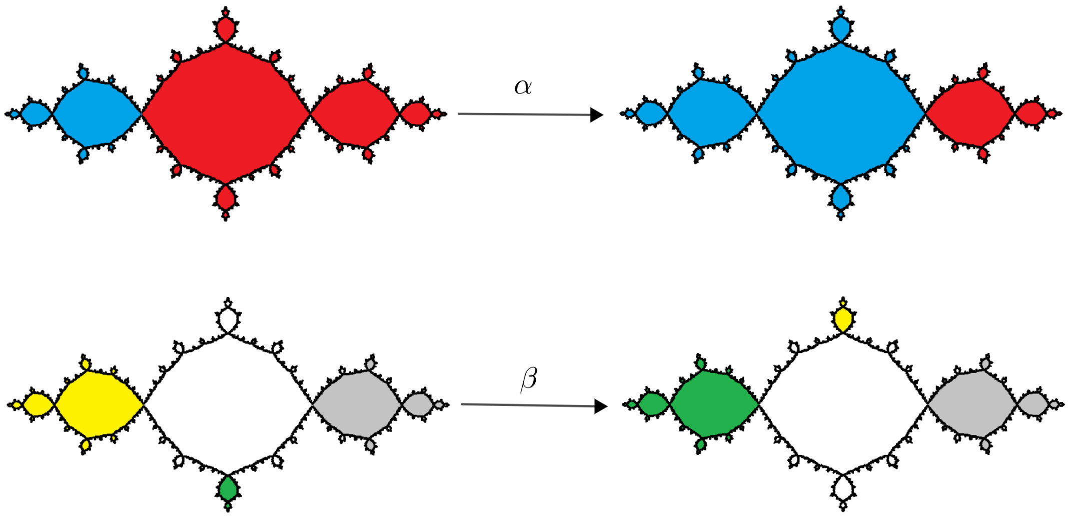


Figure 13

Let $\mathcal{P}_1$ the domain of β, $\mathcal{P}_2$ the domain of α^{-1} and $\mathcal{Q}$ the least common refinement of $\mathcal{P}_1 \cup \mathcal{P}_2$, this is the set of all the cells in $\mathcal{P}_1 \cup \mathcal{P}_2$ that are not properly contained in other cells of $\mathcal{P}_1 \cup \mathcal{P}_2$. In this case $\beta \circ \alpha$ is as in Figure 14.

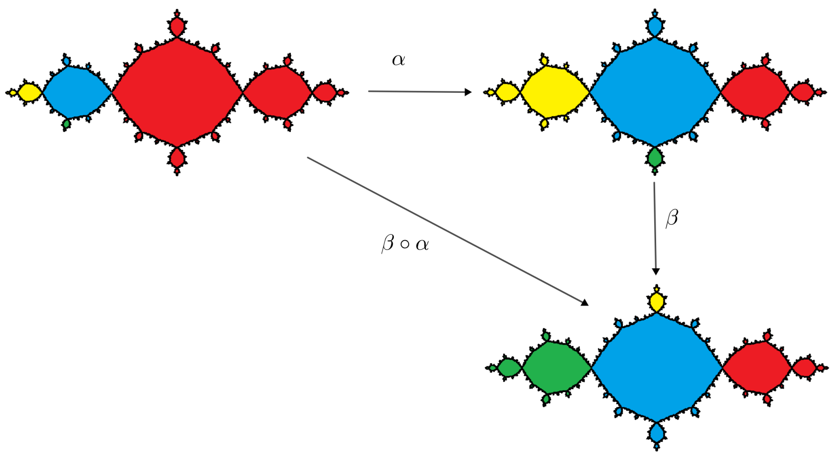


Figure 14

Example 1.43. *(The Family of Rabbits) This family correspond to the Julia sets for functions of the form $f(z) = z^2 + c$, where c lies in any interior component of the Mandelbrot set that is adjacent to the main cardioid (see Figure 15). There is a rabbit replacement system for each natural number. For example*

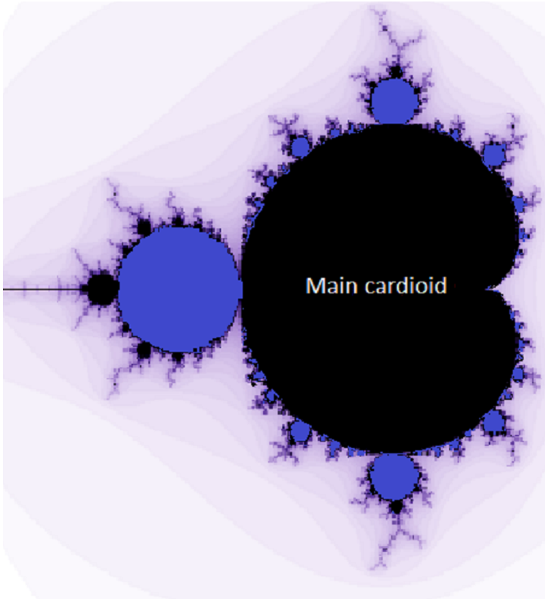


Figure 15 – The main cardioid and in blue the components adjacent to the main cardioid

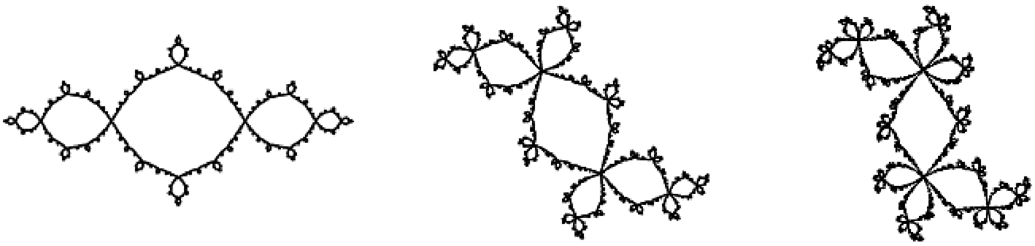
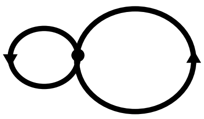
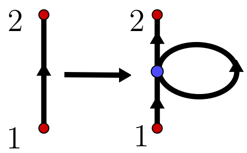
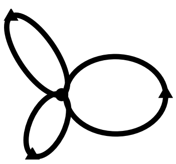
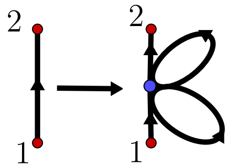


Figure 16 – The basilica, the Douady rabbit and a three eared rabbit

for $n = 1$ we have the basilica, $n = 2$ correspond to Douady rabbit, n=3 correspond to a three eared rabbit, and so forth (see Figure 16 and Figure 17).

Index	*Base Graph*	*Rule*
Basilica $n = 1$		
Douady Rabbit $n = 2$		
⋮	⋮	⋮

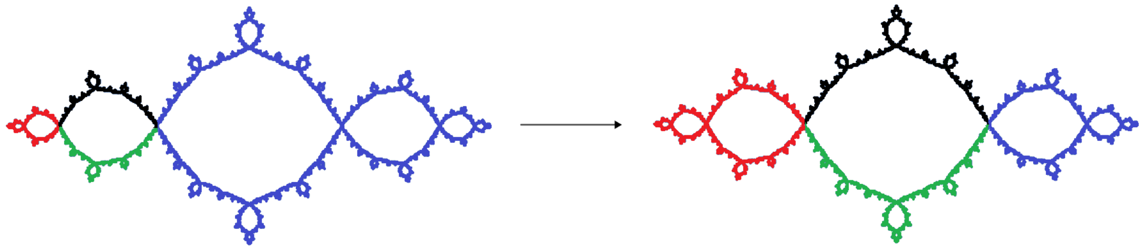


Figure 18 – An element of the Basilica Thompson's group

Figure 17

Definition 1.44. *A* ***graph pair diagram*** *for the rearrangement f is a triple (E, E', ϕ), where E and E' are expansions of G_0 and ϕ is an isomorphism between these graphs. The homeomorphism f maps the cell $C(e)$ canonically into the cell $C(\phi(e))$ for each $e \in E$.*

Remark 1.45. *Given a replacement system $(\mathcal{R}, \Gamma)$. There is a one to one correspondence between partitions of the limit space and the graphs that can be obtained by applying consecutive simple expansions beginning with the graph Γ. This relation implies that each element in the rearrangement group of fractals can be interpreted by using a graph pair diagram.*

Example 1.46. *Note that, by Remark 1.45, there is a one to one correspondence between partitions of the Basilica and the graphs obtained by expanding the base graph given in the replacement system defined in Figure 17. Also, note the relation between the rearrangement between the two partitions of the Basilica in Figure 18 and the graph pair diagram (G, H, φ) in Figure 19.*

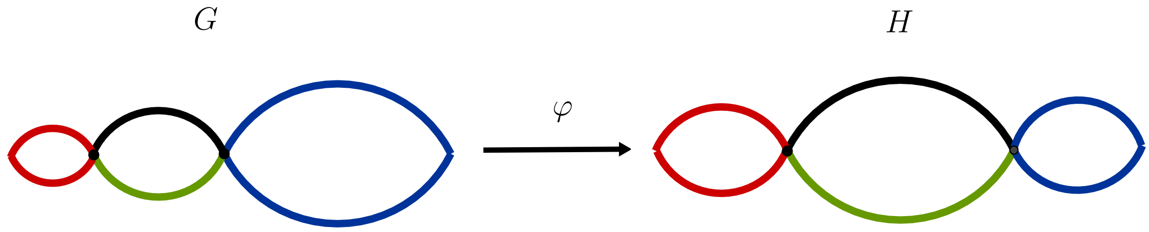


Figure 19

Example 1.47. *Recall that Thompson's groups F, T and V are groups of piecewise-linear, preserving orientation homeomorphisms respectively acting over $[0,1], S^1$ and a Cantor set.*

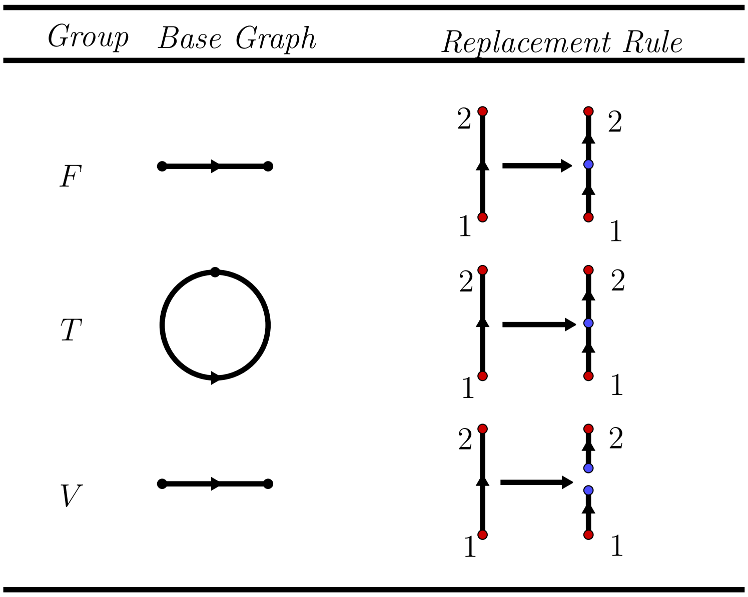


Figure 20

Theorem 1.48. *The rearrangement groups corresponding to the replacement systems shown in Figure 20 are isomorphic to Thompson's groups F, T and V.*

Proof. We will prove the result for the replacement system corresponding to T, the other cases being similar. Let the base graph E for the replacement be the unit circle, and let 0 and 1 be the left and right edges in the replacement graph respectively.

Each graph G_n in the full expansion sequence for this replacement system is a directed path of length 2^n. The gluing relation on the symbol space $E \times \{0,1\}^\infty$ is given by $e0\overline{1} \sim e1\overline{0}$ for any edge $e \in G_n$ and $E\overline{0} = E\overline{1}$. It follows that the limit space X is homeomorphic to S^1 with each point in the symbol space mapping to the point in S^1 whose binary expansion is the given binary sequence.

Then the gluing vertices correspond to dyadic fractions in S^1. The cells in X correspond to intervals of the form $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for $n \in N$ and $k \in \{1, \ldots, 2^n\}$, with 1 identified with 0 and a cellular partition of X is a subdivision of S^1 into standard dyadic intervals.

Hence a homeomorphism $h : S^1 \to S^1$ is a rearrangement if and only if there exist two partitions of S^1 into standard dyadic intervals such that h maps the first partition to the second linearly and preserving the orientation. It is immediate that the group of rearrangements is precisely Thompson's group T. □

Example 1.49. *Another important rearrangement group of fractals is the rearrangement group of the Vicsek fractal. In Figure 21 we have an element of this group. The replacement system to obtain this group is the following:*

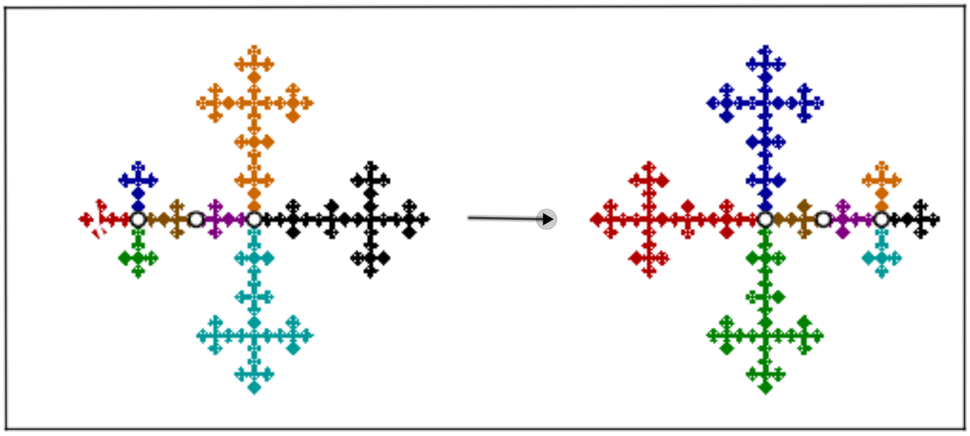


Figure 21 – A rearrangement of the Vicsek fractal

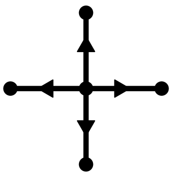
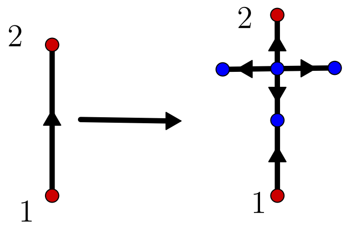
Limit Space	*Base Graph*	*Replacemnt Rule*
Vicsek Fractal		

Figure 22

Theorem 1.50. *Let $\mathcal{G}$ be the rearrangement group for the Vicsek fractal, and let H be a finite group. Then H is isomorphic to a subgroup of $\mathcal{G}$ if and only if H is solvable of order $2^j 3^k$ for some $j, k \geq 0$*

Theorem 1.50 implies that $\mathcal{G}$ is not isomorphic to any currently known Thompson-like group. Recall that the basilica is a fractal in the rabbit family in Figure 17. Analogously, the Vicsek fractal is part of a family the fractals called the Vicsek Family that can be obtained for the following replacement systems

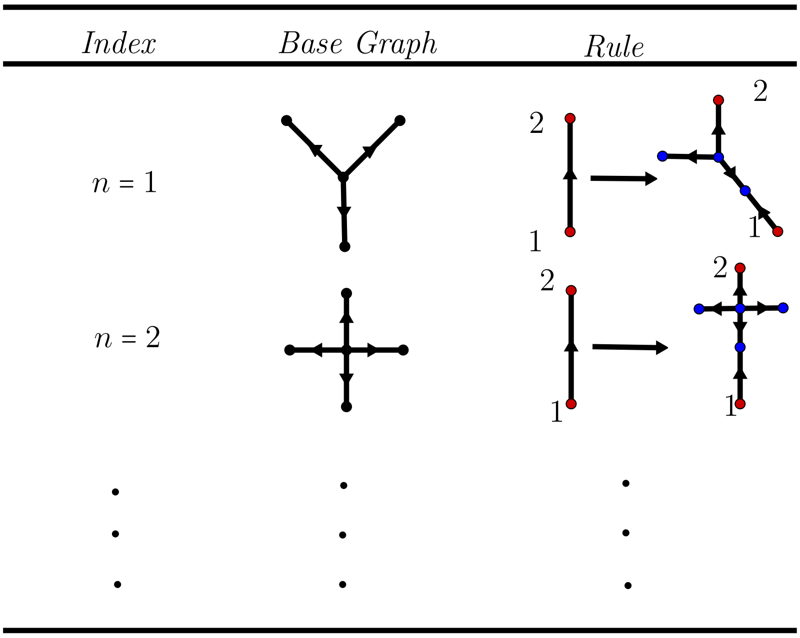


Figure 23 – The Vicsek family of replacement systems.

In particular, the case $n = 2$ corresponds to the Vicsek replacement system.

Theorem 1.51. *The rearrangement groups for the Vicsek family are all of type $\mathcal{F}_\infty$.*

1.6.1 Colored Replacement systems

Belk and Forrest also work with a wider class of replacement systems by allowing a different replacement graph for each color. This generalization allows to construct rearrangement groups for many different fractals. They provide the example of the airplane Julia set Figure 26 with replacement system in Figure 25. Additionally, rearrangement groups of this kind of replacement systems generalize a certain class of diagram group [2].

Definition 1.52. *A **colored replacement system** $\mathcal{R}$ consists of the following:*

1. *A finite set $\mathcal{C}$ of colors.*
2. *A directed base graph G_0, whose edges have been colored by the elements of $\mathcal{C}$.*
3. *For each $c \in \mathcal{C}$, a directed replacement graph R_c, whose edges have been colored by elements of $\mathcal{C}$.*

Similar to the case without colors we can define a limit space, replacement rules, canonical homeomorphism and graph pair diagram. In fact, given a colored edge e with the color c and a graph R_c with an initial vertex and a terminal vertex, we can apply a replacement rule $(e \to R_c)$ to a graph by erasing e and gluing the initial (terminal) vertex of e with the initial (terminal) vertex of R_c. For such a replacement system, the symbol space can be defined in an obvious way, and it endow of a topology as a closed subspace of the Cantor space $E(G_0) \times (\bigcup_{c \in C} E(R_c))^{\infty}$. Moreover, if the base graph G_0 and each of the replacement graphs R_c satisfy the requirements for an expanding replacement system, we have that the gluing relation $\sim$, in Definition 1.34 is an equivalence relation, and the limit space $X = \frac{\Omega}{\sim}$ is compact and metrizable.

For a colored replacement system $\mathcal{R}$, each cell $C(e)$ in the corresponding limit space has a color, and it only makes sense to talk about the canonical homeomorphism between cells of the same color. Then we can define a rearrangement similar to that of the non-colored case. Here each such rearrangement has a graph pair diagram of the form (E_1, E_2, φ), where E_1 and E_2 are colored expansions of the base graph G_0, and $\varphi : E_1 \to E_2$ is a color-preserving isomorphism.

Example 1.53. *(Diagram groups) The case of the* ***linear*** *colored replacement system, that is where the base graph G_0 is a directed path, and the replacement graph R_c for each color is a directed path of length two or greater from the initial vertex to the terminal vertex, always has a limit space homeomorphic to a closed interval. For example, consider the replacement system based on the rules in Figure 24. with two colors green and blue.*

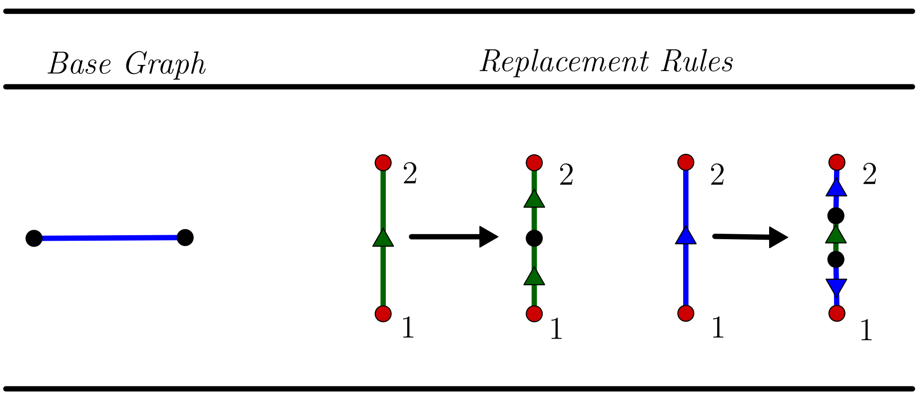


Figure 24 – A linear colored replacement system

Then the corresponding rearrangement group $\mathcal{R}$ acts on a closed interval. Algebraically, $\mathcal{R}$ is isomorphic to the restricted wreath product $F \wr F$.

Theorem 1.54. *The rearrangement group corresponding to a linear colored replacement system is always isomorphic to a diagram group, where the replacement rules determine the presentation of the corresponding semigroup.*

The graph diagram group given by the graph rewriting system $\mathcal{R}$ in Figure 24 coincides with the diagram group of the semigroup presentation

$$\mathcal{P} = \langle G, B \mid G = G^2, B = BGB \rangle.$$

Example 1.55. *Let* $C = \{green, blue\}$*, and consider the replacement system given in Figure 25.*

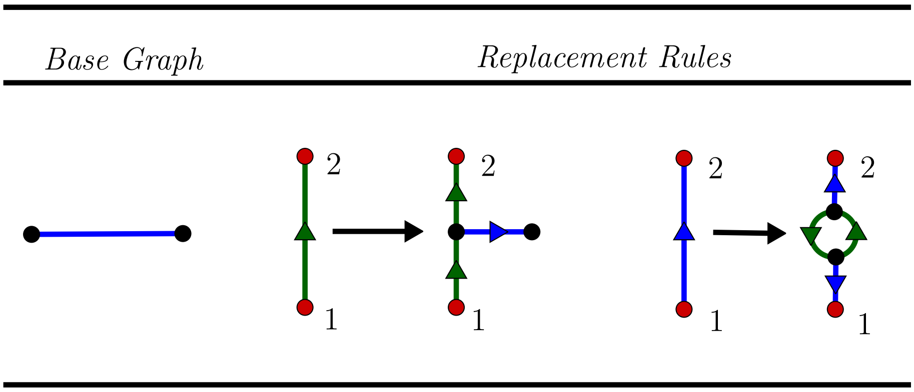


Figure 25 – The airplane replacement system

The limit space for this replacement system is a fractal homeomorphic to the Julia set for $z^2 - 1.755$*, which is known as the* ***airplane****.*

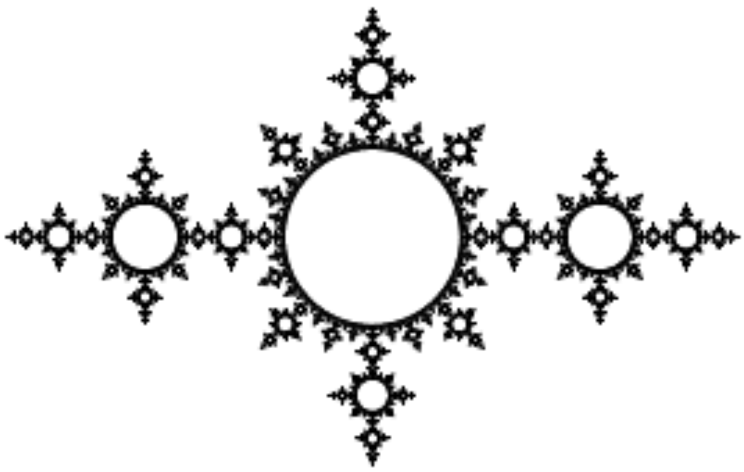


Figure 26 – The limit space for the airplane replacement system

Theorem 1.56. *Every rearrangement group acts properly by isometries on a* $CAT(0)$ *cubical complex.*

2 Graph rewriting systems

In the present chapter we establish the necessary concepts to introduce graph diagram groups. We will follow some of the ideas of Guba and Sapir, who work in diagrams groups in [1]. Recall that a diagram is a geometrical object whose definition relies on that of a rewriting system. Given a finite sequence of words (called *derivation*) obtained by successively applying rules in the string rewriting system Guba and Sapir associate a geometrical object (called *diagram*) by adding step by step cells that correspond to the rules used in the derivation. In a rough way a diagram is a set of cells together with two distinguished paths denoted top and bottom of the diagram. A product of two of these objects by concatenation is defined by identifying the top of the first diagram with the bottom of the second and giving some conditions to get a group structure. Among the required conditions we must consider two isotopic diagrams as the same and define the equivalence class of the diagram modulo some reductions called *dipole reductions*. One must then prove that every diagram yields only one reduced diagram. In general, diagram groups depend on a string rewriting system together with an initial word. In our case we will follow the same strategy to generalize diagram groups, but with some differences, for instance, we will use graph rewriting systems instead of a string rewriting system and therefore we will have more difficulties to endow these objects with a group structure. For example, in order to do this we will need to define a partial order on the cells of the graph diagram that allows us to define an analogue of the dipole reduction and the concatenation of two diagrams. Then we will consider two graph diagrams as equal if they are in the same equivalence class modulo a dipole reduction. Graph diagram groups depend on a graph rewriting system together with a fixed initial graph (that is the top of the diagram, which plays the role of the initial word for diagram groups) and an isomorphism between its top and its bottom. In this Chapter, we will introduce the necessary theory to define graph rewriting systems and graph diagram groups.

2.1 Context for Graphs

We start by choosing a **context for graphs**, which can be

1. The graphs can be both directed and undirected, see Definition 1.8.
2. The edges or labels might be labeled with the elements of some alphabet or with some set of colors.

Remark 2.1. *Independently of the context of the graphs, we consider the graphs to be multigraphs. From this Chapter onwards isomorphisms are graph isomorphisms that depend* *on the context for graphs in the sense that they preserve it. For instance, if the context for graphs is given by directed graphs, then an isomorphism is a graph isomorphism that preserves directions.*

2.2 Boundaries and Portions

Definition 2.2. *Consider a graph G with a subgraph X (denoted by ∂G), which we call the **boundary** of G. Any vertex of ∂G is called a **boundary vertex**, and any edge of ∂G is a **boundary edge**. The remaining edges and vertices are respectively called **interior edges** and **interior vertices**. In particular, there can be interior edges with a vertex in the boundary and a vertex in the interior of G. Note that an edge in the interior of G cannot be in its boundary ∂G too.*

Definition 2.3.
1. *Let G a graph. A **subgraph of G with boundary** is a pair $(H, \partial H)$ where H is a subgraph of G and ∂H is a boundary of H.*
2. *Given $(G, \partial G)$ and $(\widetilde{G}, \partial\widetilde{G})$ a **boundary preserving isomorphism** $\alpha : G \to \widetilde{G}$ is a graph isomorphism between G and $\widetilde{G}$ that satisfies $\alpha(\partial G) = \partial\widetilde{G}$.*

Definition 2.4. *Let $G = P \cup Q$ with P and Q subgraphs of G. We define a **portion** as a pair $(P, P \cap Q)$ where $P \cap Q$ is the boundary of P.*

Note that $(Q, \partial P)$ is also a portion where we endowed Q with the same boundary of P. We usually denote $(Q, \partial P)$ by $(P^c, \partial P)$ and call it the **complementary portion** of $(P, \partial P)$ in G. Thus, when we say that $(P, \partial P)$ is a portion, we are also saying that its complementary portion exists too.

Remark 2.5. *We use the notion of portion to avoid the notion of pushout complement used in [31], p.23, Definition 8. Given a graph G, it turns out that the pair of embeddings i_1 and i_2 associated with a subgraph P with boundary ∂P in the diagram below*

$$\begin{tikzcd}
& \partial P \arrow[dl, "i_1"'] \arrow[dr, dashed] & \\
P \arrow[dr, "i_2"'] & & P^c \arrow[dl, dashed] \\
& G &
\end{tikzcd}$$

has a pushout complement in the category of graphs and graph morphisms if and only if $(P, \partial P)$ is a portion, in which case the pushout complement is precisely the complement of the portion, that is $(P^c, \partial P)$.

Proposition 2.6 (Characterization of boundaries). *Let G be a graph and let P be a subgraph of G with boundary ∂P. Then $(P, \partial P)$ is a portion of G if and only if it satisfies* *the following condition: for every interior vertex v of P, every edge of G incident to v is an (interior) edge of P.*

Proof. We start proving that, if $(P, \partial P)$ is a portion of G and e is an edge of G that is incident to a vertex v in the interior of P, then e is in the interior of P. By definition of portion, there exists $(Q, \partial P)$ complementary portion of P in G. Since v is in the interior of P, e cannot be in ∂P and v cannot be in Q, which implies that e cannot be in the interior of Q. Therefore assuming that $G = P \cup Q$ and $\partial P = P \cap Q$ implies that e is in the in the interior of P.
On the other hand, consider the set

$$W = \{\text{vertices of } G \text{ that are not vertices in the interior of } P\}$$

and define the subgraph of G given by

$$Q = \{W, \{\text{edges incident to } W \text{ but not to the interior of } P \text{ in } G\}\}.$$

Note that ∂P is a subgraph of Q, so we can consider the graph with boundary $(Q, \partial P)$ and $\partial P \subseteq P \cap Q$. Actually, we have that $\partial P = P \cap Q$ since, by hypothesis, each edge with a vertex in the interior of P is an interior edge of P, so there are no edges with an endpoint in the interior of P and other endpoint in the interior of Q. Moreover, by construction of W (respectively, Q) all the vertices (respectively, edges) of G are in $P \cup Q$. Thus, $G = P \cup Q$. □

Definition 2.7. *A* ***partial isomorphism*** *between G and H is a boundary preserving isomorphism $\varphi : P \to Q$ where $(P, \partial P)$ is a portion of G and $(Q, \partial Q)$ is a portion of H.*

The partial isomorphism can be used to amalgamate graphs as follows,

Definition 2.8. *If $\varphi : G \to H$ is a partial isomorphism, the* ***amalgamation*** *$G \cup_\varphi H$ is the graph obtained from the disjoint union $G \cup H$ by identifying each vertex and edge in the domain of φ with its image in the range of φ.*

Definition 2.9. *A* ***replacement rule*** *is a triple $t = (R, S, \nu)$ where R, S are graphs with boundaries and non-empty interior, and $\nu : \partial R \to \partial S$ is an isomorphism.*

Given G graph, there is a **match** for t in G if there exists a boundary preserving isomorphism $\alpha : R \to P$ where $(P, \partial P)$ is a portion of G and we can **apply** the replacement rule by removing the interior of P and attaching a copy of the interior of S. The resulting graph is $\widetilde{G} = P^c \cup_{\nu \circ \alpha^{-1}} S$.
Note that the isomorphic portion between G and $\widetilde{G}$ is $(P^c, \partial P)$. Then we have a partial isomorphism $\varphi : G = P^c \cup P \to \widetilde{G} = P^c \cup_{\nu \circ \alpha^{-1}} S$ that identifies the graph P^c in the domain with its copy in the image. So the **domain** of the partial isomorphism is the complement of the portion $(P, \partial P)$. The **support** of a replacement is the complement of its domain,

i.e. the portion of G isomorphic to R. Similarly, the **cosupport** of a replacement is the complement of its range, i.e. the portion of $\widetilde{G}$ isomorphic to S.

Definition 2.10. *Two replacement rules* (R, S, ν) *and* (R', S', ν') *are isomorphic if there exist boundary-preserving isomorphism* $\rho : R \to R'$ *and* $\gamma : S \to S'$ *making the following diagram commute.*

$$\begin{tikzcd} \partial R \arrow[r, "\rho"] \arrow[d, "\nu"] & \partial R' \arrow[d, "\nu'"] \\ \partial S \arrow[r, "\gamma"] & \partial S' \end{tikzcd}$$

Definition 2.11. *Let t be a replacement rule, and let $\varphi : G \to \widetilde{G}$ be a partial isomorphism of graphs. We say that φ* ***is a replacement of type t*** *denoted*

$$G \overset{\varphi,t}{\Longrightarrow} \widetilde{G},$$

if there exists a pair of boundary preserving isomorphisms $\alpha : R \to dom(\varphi)^c$ and $\beta : S \to range(\varphi)^c$ making the following diagram commute.

$$\begin{tikzcd} \partial R \arrow[r, "\nu"] \arrow[d, "\alpha"] & \partial S \arrow[d, "\beta"] \\ \partial\, dom(\varphi) \arrow[r, "\varphi"] & \partial\, range(\varphi) \end{tikzcd}$$

That is, $G \overset{\varphi,t}{\Longrightarrow} \widetilde{G}$ is a replacement if $\widetilde{G} \cong P^c \cup S$

2.3 Independence

Following [31] we consider two types of independence for replacements.

Definition 2.12.

- *Two subgraphs H and K of G with boundary are said to* ***overlap*** *if H intersects the interior of K, or if K intersects the interior of H.*

- *Two replacements*

$$H \overset{\varphi,s}{\Longleftarrow} G \overset{\psi,r}{\Longrightarrow} H'$$

with initial graph G are ***parallel independent*** *if their supports do not overlap.*

- *Two consecutive replacements*

$$G \overset{\varphi,s}{\Longrightarrow} H \overset{\psi,r}{\Longrightarrow} K$$

are ***sequentially independent*** *if the cosupport of the first does not overlap with the support of the second.*

Proposition 2.13. *Given a parallel (respectively, sequential) independent replacements, it is possible to find suitable sequential (respectively, parallel) independent replacements that form the following diamond*

$$\begin{tikzcd} & G \arrow[dl, Rightarrow, "\varphi{,}s"'] \arrow[dr, Rightarrow, "\pi{,}r"] & \\ H \arrow[dr, Rightarrow, "\psi{,}r"'] & & H' \arrow[dl, Rightarrow, "\rho{,}s"] \\ & K & \end{tikzcd}$$

which commutes in the sense that $\psi \circ \varphi = \rho \circ \pi$.

Proof. Let $G \overset{\varphi,s}{\Longrightarrow} H \overset{\psi,r}{\Longrightarrow} K$ be such that the two replacements are sequentially independent, then

$$G = supp(\varphi)^c \cup supp(\varphi) \text{ and } H = supp(\varphi)^c \cup cosupp(\varphi) \tag{2.1}$$

and, by definition, $cosupp(\varphi)$ does not overlap $supp(\psi)$, thus

1. $cosupp(\varphi)$ does not intersect the interior of $supp(\psi)$.

2. $supp(\psi)$ does not intersect the interior of $cosupp(\varphi)$.

By item 1 and by (2.1) we have that $supp(\psi) \subset supp(\varphi)^c$, so we can apply the replacement r over G and obtain a graph $H' = supp(\pi) \cup supp(\pi)^c$ where $supp(\pi) = supp(\psi)$. Moreover $supp(\psi) \subset supp(\varphi)^c$ implies that $supp(\varphi) \subset supp(\psi)^c = supp(\pi)^c$ and we can apply the rule s over the graph H', then we obtain

$$G \overset{\pi,r}{\Longrightarrow} H' \overset{\rho,s}{\Longrightarrow} K$$

On the other hand, let $H \overset{\varphi,s}{\Longleftarrow} G \overset{\rho,r}{\Longrightarrow} H'$ be two parallel independent replacements, then

$$G = supp(\varphi)^c \cup supp(\varphi) = supp(\rho)^c \cup supp(\rho), \tag{2.2}$$

$$H = supp(\varphi)^c \cup cosupp(\varphi) \text{ and } H' = supp(\rho)^c \cup supp(\rho) \tag{2.3}$$

and, by definition, $supp(\varphi)$ does not overlap $supp(\rho)$, thus

1. $supp(\varphi)$ does not intersect the interior of $supp(\rho)$.

2. $supp(\rho)$ does not intersect the interior of $supp(\varphi)$.

By item 2 and by (2.2) we have that $supp(\rho) \subset supp(\varphi)^c$, so we can apply the replacement r over H and obtain a graph K. Analogously $supp(\varphi) \subset supp(\rho)^c$ and we can apply the replacement s over H' and obtain the graph K. Therefore, we have derivations

$$G \overset{\varphi,s}{\Longrightarrow} H \overset{\psi,r}{\Longrightarrow} K \text{ and } G \overset{\pi,r}{\Longrightarrow} H' \overset{\rho,s}{\Longrightarrow} K.$$

□

2.4 Graph rewriting systems

In Section 1.3 we introduced rewriting systems (see Definition 1.12) and we presented a type of rewriting systems called string rewriting system (see Definition 1.17). Specifically, string rewriting systems were fundamental to define the Guba and Sapir diagrams in Section 1.4. Analogously, in this Section we will introduce the graph rewriting systems that will be important to define a general class of diagrams called the graph diagrams.

Definition 2.14. *A* ***graph rewriting system*** $\mathcal{R}$ *consists of the following data:*

1. *A context for graphs.*
2. *A set of replacement rules in the given context.*

Convention 2.15. *We assume that no two replacement rules for the same graph rewriting system* $\mathcal{R}$ *are isomorphic. Thus, given a partial isomorphism* $\varphi : G \to \widetilde{G}$ *between graphs, there is at most one replacement rule* t *in* $\mathcal{R}$*. In this case we can denote the replacement of type* t, $G \overset{\varphi}{\Longrightarrow} \widetilde{G}$ *without specifying the replacement rule.*

Definition 2.16. *If* $\mathcal{R}$ *is a graph rewriting system, a* ***derivation over*** $\mathcal{R}$ *is a finite sequence*

$$G_0 \overset{\varphi_1}{\Longrightarrow} G_1 \overset{\varphi_2}{\Longrightarrow} G_2 \dots \overset{\varphi_n}{\Longrightarrow} G_n$$

of replacements from $\mathcal{R}$*. The graphs* G_0 *and* G_n *are called* ***initial*** *and* ***terminal*** *graph of the derivation respectively.*
Given two derivations $G_0 \overset{\varphi_1}{\Longrightarrow} G_1 \overset{\varphi_2}{\Longrightarrow} G_2 \dots \overset{\varphi_n}{\Longrightarrow} G_n$ *and* $H_0 \overset{\psi_1}{\Longrightarrow} H_1 \overset{\psi_2}{\Longrightarrow} H_2 \dots \overset{\psi_n}{\Longrightarrow} H_n$ *we say that they are* ***isomorphic*** *if there exist isomorphisms* $\tau_0, \tau_1, \dots, \tau_n$ *that satisfy* $\tau_i(supp(\varphi_{i+1})) = supp(\psi_{i+1})$ *for* $0 \le i < n$ *and make the following diagram commute*

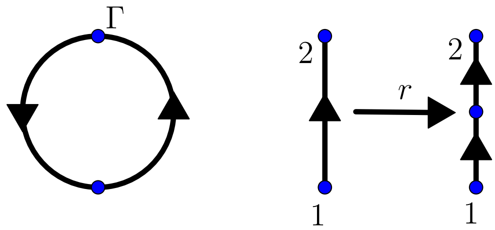
$$\begin{tikzcd}
G_0 \arrow[r, Rightarrow, "\varphi_1"] \arrow[d, "\tau_0"] & G_1 \arrow[r, Rightarrow, "\varphi_2"] \arrow[d, "\tau_1"] & \dots \arrow[r, Rightarrow, "\varphi_n"] & G_n \arrow[d, "\tau_n"] \\
H_0 \arrow[r, Rightarrow, "\psi_1"] & H_1 \arrow[r, Rightarrow, "\psi_2"] & \dots \arrow[r, Rightarrow, "\psi_n"] & H_n
\end{tikzcd}$$

Two replacement rules (R_1, S_1, ν_1) *and* (R_2, S_2, ν_2) *are called* ***inverse*** *if there exists boundary-preserving isomorphisms* $\alpha_1 : R_1 \to S_2$ *and* $\alpha_2 : S_1 \to R_2$ *making the following diagram commute*

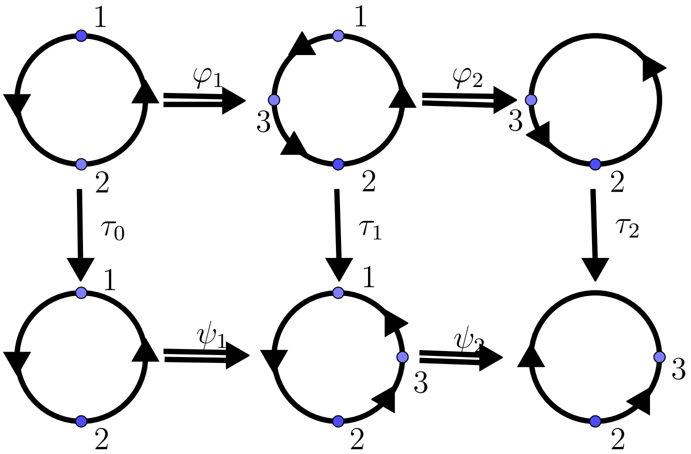
$$\begin{tikzcd}
\partial R_1 \arrow[r, "\nu_1"] \arrow[d, "\alpha_1"] & \partial S_1 \arrow[d, "\alpha_2"] \\
\partial S_2 & \partial R_2 \arrow[l, "\nu_2"]
\end{tikzcd}$$

Later on in Proposition 2.41 we will better see the meaning of this definition

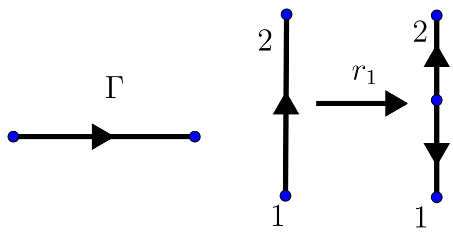
Example 2.17. *Consider the graph rewriting system with initial graph* Γ *and replacement rules* r *and* r^{-1}.



In this case the derivations below are isomorphic if we take τ_i *as the rotation by* π.

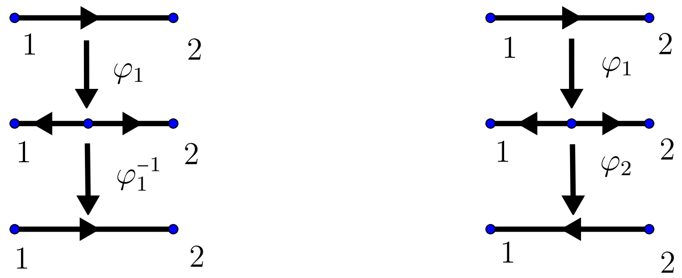


Example 2.18. *Consider the following rule with initial graph* Γ*, and directed edges in the context for graphs. Note that the numbers are giving us the information about how the boundary of the graphs, in this case the end vertices of the graphs, are identified by the boundary preserving isomorphism.*



The definition of replacement rule says that a replacement rule is a triple (R, S, ν) *where* $(R, \partial R)$ *and* $(S, \partial S)$ *are graphs with boundary, non-empty interiors and* $\nu(\partial R) = \partial S$*. In this case* ∂P *is given by the vertices labelled by* 1 *and* 2*. On the other hand the characterization of a boundary says that there are no edges between the interior of* P *and the interior of* P^c*, then* $P^c = \partial P$ *and* R *is isomorphic to the directed edge* Γ.

The following derivations are not isomorphic. In fact, the isomorphism in the derivation fixes the edges labels with 1 *and* 2 *and there is no isomorphism between the last graphs of the derivation that fixes the end vertices and changes the direction of the edges.*



2.5 Graph Diagrams

In this Section we fix a set of replacement rules $\mathcal{R}$. Every graph will be a graph over $\mathcal{R}$, this is a graph which we can apply rules from $\mathcal{R}$. We say that replacements from $\mathcal{R}$ are **valid replacements**.

2.5.1 Cells and Diagrams

Definition 2.19. *Let Δ be a graph. A* ***cell*** *in Δ is a subgraph C together with a pair (T, B) of complementary portions of C such that*

$$T \stackrel{i}{\Longrightarrow} B$$

is a valid replacement and where $i : \partial T \to \partial B$ denotes the identity isomorphism. Here we call the portion T as the ***top*** *of the cell and B as the* ***bottom*** *of it and we denote them $top(C)$ and $bot(C)$.*

Definition 2.20. *An* ***ordered graph diagram*** *is a graph Δ with a finite collection of cells in Δ such that there exists an ordering $C_1, C_2, \dots, C_n$ of the cells and a sequence $G_0, G_1, \dots, G_n$ of subgraphs of Δ satisfying the following conditions:*

1. $G_0 \cup G_1 \cup \dots \cup G_n = \Delta$
2. $G_i \cap G_k \subseteq G_j$ *for* $i < j < k$
3. *$top(C_i)$ is a portion of G_{i-1}, $bot(C_i)$ is a portion of G_i, and the complement of $top(C_i)$ in G_{i-1} is equal to the complement of $bot(C_i)$ in G_i.*

Definition 2.21. *Given a derivation*

$$\rho := G_0 \stackrel{\varphi_1}{\Longrightarrow} G_1 \stackrel{\varphi_2}{\Longrightarrow} G_2 \dots \stackrel{\varphi_n}{\Longrightarrow} G_n$$

a graph diagram associated with this derivation*, denoted,*

$$\Delta = Diag(G_0 \stackrel{\varphi_1}{\Longrightarrow} G_1 \stackrel{\varphi_2}{\Longrightarrow} G_2 \dots \stackrel{\varphi_n}{\Longrightarrow} G_n) = Diag(\rho)$$

is defined as follows:

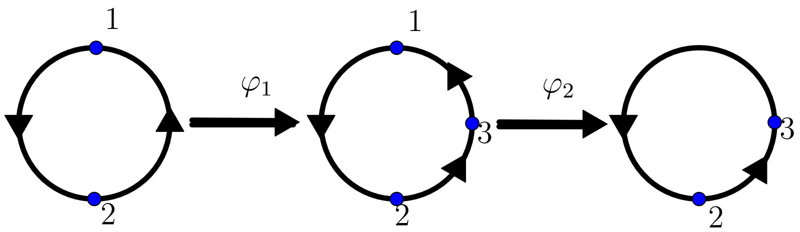
1. *A graph* $\Delta = G_0 \cup_{\varphi_1} G_1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} G_n$

2. *The cells* $\{C_1, C_2, \ldots C_n\}$ *are defined by*

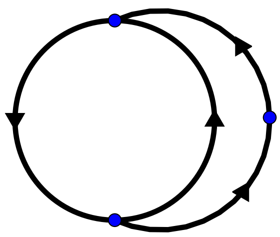
$$C_i = supp(\varphi_i) \cup cosupp(\varphi_i)$$

where $top(C_i) = supp(\varphi_i)$ *and* $bot(C_i) = cosupp(\varphi_i)$*. We call* $\{C_1, \ldots, C_n\}$ ***the cells induced*** *by the derivation.*

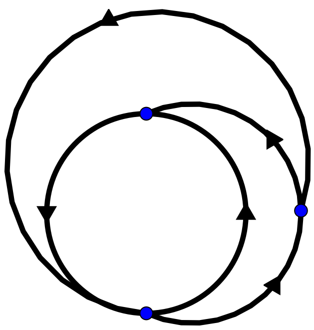
Example 2.22. *In Example 2.17 we consider two derivations. We will see how to produce the graph diagrams of one of these derivations.*
Consider



Here the partial isomorphism φ_1 *identifies the directed edges with initial vertex labeled with* 1 *and terminal vertex labeled with* 2 *and joins the disjoint union of the other edges as follows:*



Finally, φ_2 *identifies the labeled edges with initial vertex* 2 *and terminal vertex* 3 *and joins the disjoint union of the other edges to obtain the following graph diagram:*



This notion is similar to the construction of a Guba and Sapir diagram where for each derivation we add a cell to the diagram.
Remember that each time that we apply a replacement rule $((R, \partial R), (S, \partial S), \nu)$ we have a portion $(P, \partial P)$ of G isomorphic to $(R, \partial R)$, then we identify the border of S with ∂P and replace the interior of P by the interior of S to obtain a graph G', that is $G \overset{\varphi}{\Longrightarrow} G'$. The graph obtained by this process is $G \cup_\varphi G' = G \cup_\varphi cosupp(\varphi)$, which means that $cosupp(\varphi)$ does not overlap with G.

Definition 2.23. *If Δ is an ordered graph diagram, the* ***top*** *of Δ (denoted by $top(\Delta)$) is the subgraph consisting of all the edges and vertices that are not in the interior of the bottom of any cell. Analogously, the* ***bottom*** *of Δ (denoted by $bot(\Delta)$) is the subgraph consisting of all the edges and vertices that are not in the interior of the top of any cell.*

Definition 2.24. *An ordering on the set of cells $C_1, \dots, C_n$ of a diagram that satisfies the conditions of Definition 2.21 is called a* ***valid ordering****.*

Remark 2.25. *Observe that Definition 2.21 says that, given a derivation, we can obtain a valid order of its cells given by its partial isomorphisms and that also, given a valid ordering for the cells of a graph diagram, we can obtain a derivation for this graph diagram. This means that each ordered graph diagram arises from a derivation, in the sense that any valid ordering $C_1, C_2, \dots, C_n$ with subgraphs $G_0, G_1, \dots, G_n$ is a derivation corresponding to a derivation*

$$G_0 \overset{i_1}{\Longrightarrow} G_1 \overset{i_2}{\Longrightarrow} G_2 \dots \overset{i_n}{\Longrightarrow} G_n$$

where each i_k denotes the identity isomorphism on $G_{k-1} \cap G_k$. This means that the cells of a valid order for a graph diagram induce replacement rules $(supp(i_j), cosupp(i_j), i_j)$ that consist of the partial isomorphism of the derivation above.

Denote $top(C_i) \equiv R_i$, $bottom(C_i) \equiv S_i$, for every G_i we have

$$G_0 = P_1^c \cup R_1,$$

$$G_1 = P_1^c \cup S_1 = P_2^c \cup R_2,$$

$$G_i = P_i^c \cup S_i = P_{i+1}^c \cup R_{i+1}, \;\; 0 < i \leq n-1$$

where $P_i^c \subseteq G_{i-1}$ and $G_{i-1} \cap S_i \subseteq \partial P_i^c$.

Proposition 2.26. $\Delta = Diag(G_0 \overset{\varphi_1}{\Longrightarrow} G_1 \overset{\varphi_2}{\Longrightarrow} G_2 \dots \overset{\varphi_n}{\Longrightarrow} G_n)$ *is an ordered graph diagram.*

Proof. We must verify the items of Definition 2.20.

1. By definition, we have $\Delta = G_0 \cup_{\varphi_1} G_1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} G_n$

2. We will show that

$$G_i \cap G_k \subseteq G_j, \text{ for } 0 \leq i < j < k \leq n$$

   Let $\Delta_{(l,m)} = G_l \cup_{\varphi_{l+1}} G_{l+1} \cup_{\varphi_{l+2}} \cdots \cup_{\varphi_m} G_m$ for $l < m$.

   Consider $\Delta_{(i,j)} \cup_{\varphi_{j+1}} \Delta_{(j+1,k)}$ where, by definition, the amalgamation by φ_{j+1} of these diagrams consists of identifying the domain of φ_{j+1} with its range and make the disjoint union of the rest of the graph diagram. Thus an edge or vertex in $G_i \cap G_k$ must be in G_j. Indeed, it must be in $(supp(\varphi_{j+1}))^c$ and in $(cosupp(\varphi_{j+1}))^c$ otherwise they would be disjoint (again by the definition of amalgamation).

3. By construction $R_i \subseteq G_{i-1}$, $S_i \subseteq G_i$. Note that $G_i = P_i^c \cup S_i$ and $G_{i-1} = P_i^c \cup R_i$ implies that the complement of R_i in G_{i-1} coincides with the complement of S_i in G_i. $(R_i, \partial R_i)$ is a portion of G_i.

□

Definition 2.27. *Given Δ and Δ' graph diagrams, we say that these graph diagrams are **isomorphic** if there exists a graph isomorphism $\rho : \Delta \to \Delta'$ which maps the cells of Δ to the cells of Δ' in a way that preserves their tops and bottoms.*

Proposition 2.28. *Given two isomorphic derivations, their respective graph diagrams are isomorphic.*

Proof. It is enough to see it for derivations with $n = 1$. Suppose that the derivations are isomorphic,

$$\begin{array}{ccc} G_0 & \overset{\varphi_1}{\Longrightarrow} & G_1 \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_1} \\ H_0 & \overset{\psi_1}{\Longrightarrow} & H_1 \end{array}$$

By Definition 2.21 it is enough to observe that

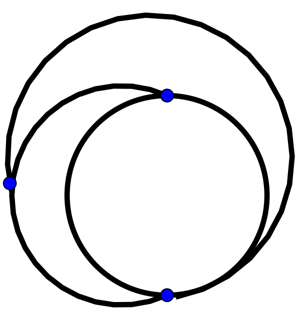
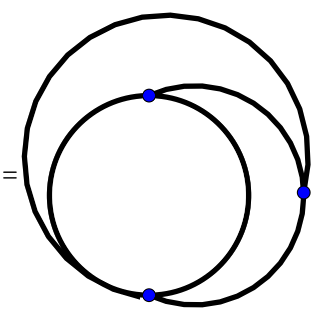
1. the first cell in the derivation C_1, is isomorphic to the first cell of the second derivation C_1', and this isomorphism preserves tops and bottoms of the cells and
2. $G_0 \cup_{\varphi_1} G_1$ is isomorphic to $H_0 \cup_{\psi_1} H_1$.

To show point 1, we observe that, by definition

$$\begin{aligned} \tau_0(top(C_1)) = \tau_0(supp(\varphi_1)) = supp(\psi_1) = top(C_1'), \text{ so that} \\ \psi_1(\tau_0(supp(\varphi_1))) = cosupp(\psi_1) \text{ and} \\ \tau_1(bottom(C_1)) = \tau_1(cosupp(\varphi_1)) = cosupp(\psi_1) = bottom(C_1') \end{aligned}$$

since the square diagram commutes. On the other hand, point 2 follows from point 1 and the isomorphism of the graphs in the derivations. □

Example 2.29. *In example 2.17 we have two isomorphic derivations whose amalgamation produce the graph diagrams below.*

$\Delta_\varphi =$  $\Delta'_\psi =$ 

Note that Δ_φ is isomorphic to Δ'_ψ.

2.5.2 The Partial Order

Given a graph Δ and a set of cells $\mathcal{C}$ in Δ, we can define a partial order over these cells. Let C and D cells in Δ, we say that C **directly precedes** D, denoted $C \lessdot D$ if the bottom of C overlaps with the top of D. More generally, we say that C **precedes** D, denoted $C < D$, if there exist cells $C_0, \dots, C_n$ in $\mathcal{C}$ so that $C = C_0 \lessdot C_1 \lessdot C_2 \cdots \lessdot C_n$.

Lemma 2.30. *Let Δ be an ordered graph diagram. Let $C_1, C_2 \dots C_n$ be a valid order of its cells,and let $G_0, G_1, \dots, G_n$ be the corresponding sequence of subgraphs. Then*

1. *$top(C_i)$ and $top(C_j)$ have disjoint interiors for $i < j$.*

2. *$bot(C_i)$ and $bot(C_j)$ have disjoint interiors for $i < j$.*

3. *$top(C_i)$ and $bot(C_j)$ are non-overlapping for $i < j$.*

4. *$C_i \cap G_{i-1} = top(C_i)$ and $C_i \cap G_i = bot(C_i)$.*

Proof.

1. We start by observing that $top(C_i) \cap top(C_j) \subseteq G_{i-1} \cap G_{j-1} \subseteq G_i$. On the other hand, by definition $top(C_i)$ is a portion of G_{i-1}, therefore there exists a complementary portion K that satisfies $G_{i-1} = K \cup top(C_i)$, $G_i = K \cup bot(C_i)$, $\partial top(C_i) = K \cap top(C_i)$, and $\partial bot(C_i) = K \cap bot(C_i)$.

   Then $top(C_i) \cap top(C_j) \subseteq top(C_i) \cap G_i = top(C_i) \cap (K \cup bot(C_i)) \subseteq \partial top(C_i)$, since by definition $bot(C_i)$ and $top(C_i)$ are complementary portions of C_i.

2. We start by observing that $bot(C_i) \cap bot(C_j) \subseteq G_i \cap G_j \subseteq G_{j-1}$. On the other hand, by definition $bot(C_j)$ is a portion of G_j, so there exists a complementary portion K in G_{j-1} such that $G_{j-1} = K \cup top(C_j)$, $G_j = K \cup bot(C_j)$, $\partial top(C_j) = K \cap top(C_j)$ and $\partial bot(C_j) = K \cap bot(C_j)$.

   So $bot(C_i) \cap bot(C_j) \subseteq bot(C_j) \cap G_{j-1} \subseteq bot(C_j) \cap (K \cup top(C_j)) = \partial bot(C_j)$.

3. $top(C_i) \cap bot(C_j) \subseteq G_{i-1} \cap G_j \subseteq G_i \cap G_{j-1}$. Then, as in the previous cases we have

   $top(C_i) \cap G_i = \partial top(C_i)$ and $bot(C_j) \cap G_{j-1} = \partial bot(C_j)$. Thus

   $top(C_i) \cap bot(C_j) \subseteq \partial top(C_i) \cap \partial bot(C_j)$. Therefore $top(C_i)$ and $bot(C_j)$ intersect only in their boundaries.

4. $top(C_i)$ is a portion of G_{i-1}, so there exists K a complementary portion in G_{i-1} and it satisfies $K \cap top(C_i) = \partial top(C_i)$, $K \cup top(C_i) = G_{i-1}$, $K \cup bot(C_i) = G_i$ and $K \cap bot(C_i) = \partial bot(C_i)$. Then, $K \cap C_i = \partial top(C_i) = \partial bot(C_i)$, so $C_i \cap G_{i-1} = C_i \cap (K \cup top(C_i)) = (C_i \cap K) \cup (C_i \cap top(C_i)) = top(C_i)$.

   On the other hand, $C_i \cap G_i = C_i \cap (K \cup bot(C_i)) = (C_i \cap K) \cup (C_i \cap bot(C_i)) = bot(C_i)$, then $C_i \cap G_i = bot(C_i)$.

□

The following result relates valid orderings introduced in Definition 2.24 together with the partial order introduced at the beginning of Subsection 2.5.2.

Lemma 2.31. *Let be Δ an ordered graph diagram and $C_1, \dots, C_n$ a valid ordering for Δ. Suppose that $C_k \nless C_{k+1}$ for some $k \in \{1, \dots, n-1\}$. Then the ordering $C_1, \dots, C_{k+1}, C_k, \dots, C_n$ is also valid for Δ.*

Proof. Let $G_0, \dots, G_n$ the graphs corresponding to $C_1, \dots, C_n$. Note that $bot(C_k)$ and $top(C_{k+1})$ are portions of G_k,, so there exist portions H_1 and H_2 such that

$$bot(C_k) \cup H_1 = top(C_{k+1}) \cup H_2 = G_k.$$

In particular, $H = H_1 \cap H_2$ satisfies

$$\begin{aligned}&top(C_{k+1}) \cup bot(C_k) \cup H = top(C_{k+1}) \cup bot(C_k) \cup (H_1 \cap H_2) = \\ &(top(C_{k+1}) \cup bot(C_k) \cup H_1)) \cap (bot(C_k) \cup H_2 \cup top(C_{k+1})) = G_k\end{aligned}$$

We claim that $top(C_k), bot(C_k), top(C_{k+1})$ and $bot(C_{k+1})$ are pairwise non-overlapping, with

$$top(C_k) \cap H = \partial top(C_k) = \partial bot(C_k) = bot(C_k) \cap H \text{ and}$$

$$top(C_{k+1}) \cap H = \partial top(C_{k+1}) = \partial bot(C_{k+1}) = bot(C_{k+1}) \cap H$$

Since $bot(C_k)$ and $top(C_{k+1})$ do not overlap, we know that $bot(C_k) \cap H = \partial bot(C_k)$ and $top(C_{k+1}) \cap H = \partial top(C_{k+1})$.
Note that $H \cup top(C_{k+1})$ is the complement of $bot(C_k)$ in G_k and by definition of graph diagram $H \cup top(C_{k+1})$ is also the complement of $top(C_k)$ in G_{k-1}. Thus,

$$(H \cup top(C_{k+1})) \cap top(C_k) = (H \cup top(C_{k+1})) \cap bot(C_k) = \partial top(C_k) = \partial bot(C_k) \subseteq H$$

Moreover, $\partial top(C_k) = \partial bot(C_k) \subseteq H$ implies $top(C_k) \cap H = \partial top(C_k) = \partial bot(C_k)$.
Finally, note that $C_k \nless C_{k+1}$, imply that $top(C_{k+1})$ and $bot(C_k)$ are non-overlapping and the pairs $top(C_k), bot(C_k)$ and $top(C_{k+1}), bot(C_{k+1})$ are non overlapping by the definition of a cell. On the other hand, by Lemma 2.30 we have that the pairs $top(C_k), top(C_{k+1}), bot(C_k), bot(C_{k+1})$ and $top(C_k), bot(C_{k+1})$ have disjoint interiors. More than that, as the interiors of all these are contained in H, it follows that these pairs are non-overlapping.
Observe that, given the graphs $G_{k-1} = H \cup top(C_k) \cup top(C_{k+1})$ and $G_{k+1} = H \cup bot(C_k) \cup bot(C_{k+1})$ we can consider the graph $\widetilde{G}_k = H \cup top(C_k) \cup bot(C_{k+1})$. We will prove that $G_0, \dots \widetilde{G}_k, \dots, G_n$ satisfy the required properties.

1. Recall that $G_k = H \cup bot(C_k) \cup top(C_{k+1})$, so $G_{k-1} \cup \widetilde{G}_k \cup G_{k+1} = G_{k-1} \cup G_k \cup G_{k+1}$, and

$$G_1 \cup \dots \cup \widetilde{G}_k \cup \dots G_n = G_1 \cup \dots \cup G_k \cup \dots G_n = \Delta.$$

2. If $i < k < j$, then $i \leq k - 1 < j$ and $i < k + 1 \leq j$, so

$$G_i \cap G_j \subseteq G_{k-1} \cap G_{k+1} = H \subseteq \widetilde{G}_k.$$

If $k < i < j$, then since $\widetilde{G}_k \subseteq G_{k-1} \cup G_{k+1}$, we have

$$\widetilde{G}_k \cap G_j \subseteq (G_{k-1} \cap G_j) \cup (G_{k+1} \cap G_j) \subseteq G_i \cup G_i = G_i.$$

Analogously, if $i < j < k$, then $\widetilde{G}_k \subseteq G_{k-1} \cup G_{k+1}$ implies

$$G_i \cap \widetilde{G}_k \subseteq (G_i \cap G_{k-1}) \cup (G_i \cap G_{k+1}) \subseteq G_j \cup G_j = G_j.$$

3. $top(C_{k+1})$ is a portion of G_{k-1} and $bot(C_{k+1})$ is a portion of $\widetilde{G}_k$, where the complement is $H \cup top(C_k)$ in each of these two cases. Similarly, $top(C_k)$ is a portion of $\widetilde{G}_k$ and $bot(C_k)$ is a portion of G_{k+1}, where the complement is $H \cup bot(C_{k+1})$ in each these two cases.

Therefore the ordering $C_1, \dots, C_{k+1}, C_k, \dots C_n$ is valid.

□

Definition 2.32. *A **strict partial order** $<$ is a binary relation on a set X that satisfies, for every $a, b, c \in X$*

- $a \nless a$
- *if $a < b$ and $b < c$, then $a < c$.*

Theorem 2.33. *Let Δ be an ordered graph diagram. Then, $\prec$ is a strict partial order on the cells of Δ. Furthermore, an ordering $C_1, \dots, C_n$ of the cells of Δ is valid if and only if*

$$C_i \prec C_j \Rightarrow i < j \tag{2.4}$$

for all $i, j \in 1, \dots, n$.

Proof. Suppose that $C_1, \dots, C_n$ is a valid order for Δ and $G_0, \dots, G_n$ are its corresponding graph diagrams. Observe that, by Lemma 2.30, the interior of $top(C_i)$ can only intersect $G_1, \dots, G_{i-1}$ while the interior of $bot(C_i)$ can only intersect $G_i, \dots, G_n$. Therefore the hypothesis (2.4) is satisfied. Moreover, since $\prec$ is a transitive subrelation of $<$ the linear ordering given by the numbering of the cells, that is $C_1 < C_2 < \dots < C_n$ then $\prec$ is a strict partial order.
On the other hand, let $C_1, \dots, C_n$ be an ordering of the cells of Δ that satisfies the hypothesis (2.4). Consider $C'_1, \dots, C'_n$ a valid order for Δ where $\{C_1, \dots, C_n\} = \{C'_1, \dots, C'_n\}$. Note that this order exists since Δ is an ordered graph diagram. We will show that $C_1, \dots, C_n$ is a valid order.

Note that $C_k \nprec C_1$ for $k \in \{2, \dots, n\}$, otherwise $C_k \prec C_1 \Rightarrow k < 1$ by hypothesis (2.4) and we would have a contradiction. Let $j \in \{1, \dots, n\}$ such that $C_1 = C'_j$ and observe that $C'_k \nprec C'_j = C_1$ for $k \in \{1, \dots, j-1\}$. Then, by Lemma 2.31, the ordering of the cells given by $C_1 = C'_j, C'_1, C'_2, \dots, C'_{j-1}, C'_{j+1}, \dots, C'_n$ is valid.
Analogously, note that $C_k \nprec C_2$ for $k \in \{3, \dots, n\}$, otherwise $C_k \prec C_2 \Rightarrow k < 2$ by hypothesis (2.4) and we would have a contradiction. Let $s \in \{1, \dots, n\}$ such that $C_2 = C'_s$ and observe that $C'_k \nprec C'_s = C_2$ for $k \notin \{j, s\}$. By Lemma 2.31, the ordering of the cells given by $C_1 = C'_j, C_2 = C'_s, C'_1, \dots, C'_{s-1}, C'_{s+1}, \dots, C'_{j-1}, C'_{j+1}, \dots, C'_n$ is valid.
Following this process we can reorganize the valid ordering $C'_1, \dots, C'_n$ using Lemma 2.31 until we get $C_1, \dots, C_n$ implying that this ordering is also valid.

□

Definition 2.34. *Consider the derivation*

$$G \overset{\varphi,s}{\Longrightarrow} H \overset{\psi,r}{\Longrightarrow} K$$

and suppose that r and s are sequentially independent, and let

$$\begin{tikzcd}
 & G \arrow[dl, Rightarrow, "\varphi{,}s"'] \arrow[dr, Rightarrow, "\pi{,}r"] & \\
H \arrow[dr, Rightarrow, "\psi{,}r"'] & & H' \arrow[dl, Rightarrow, "\rho{,}s"] \\
 & K &
\end{tikzcd}$$

be the corresponding replacement square. Then the derivation

$$G \overset{\pi,r}{\Longrightarrow} H' \overset{\rho,s}{\Longrightarrow} K$$

*is called a **transposition** of the original, we also say that we **move to the left** the replacement r in the derivation. We can **transpose** any two consecutive, sequentially independent replacements in a derivation.*
*A **permutation** of a derivation is obtained by any sequence of transpositions.*

Remark 2.35. *Consider the derivations ρ with graphs $G_0, G_1, \dots, G_i, G_{i+1}, \dots G_n$ and the derivation ρ' with graphs $G_0, G_1, \dots, G_{i-1}, G'_i, G_{i+1}, \dots G_n$ such that they are a transposition of one another,*

$$\begin{tikzcd}
G_0 \arrow[r, Rightarrow, "\varphi_1{,}r_1"] & G_1 \arrow[r, Rightarrow, "\varphi_2{,}r_2"] & \dots \arrow[r, Rightarrow, "\varphi_{i-1}{,}r_{i-1}"] & G_{i-1} \arrow[r, Rightarrow, "\varphi_i{,}r_i"] \arrow[dr, Rightarrow, "\varphi'_i{,}r_{i+1}"'] & G_i \arrow[r, Rightarrow, "\varphi_{i+1}r_{i+1}"] & G_{i+1} \arrow[r, Rightarrow, "\varphi_{i+2}{,}r_{i+2}"] & \dots \arrow[r, Rightarrow, "\varphi_n{,}r_n"] & G_n \\
 & & & & G'_i \arrow[ur, Rightarrow, "\varphi'_{i+1}{,}r_i"'] & & &
\end{tikzcd}$$

These derivations give us a valid ordering for its cells. In fact, we have a graph diagram Δ with graphs $G_0, G_1, \dots, G_{i-1}, G_i, G_{i+1}, \dots G_n$ and cells $C_1, \dots, C_{i-1}, C_i, C_{i+1}, \dots C_n$

where $C_k = supp(\varphi_k) \cup cosupp(\varphi_k)$ and a graph diagram Δ' with graphs $G_0, G_1, \ldots, G_{i-1}, G'_i, G_{i+1}, \ldots G_n$ and cells $C_1, \ldots, C_{i-1}, C_{i+1}, C_i, \ldots C_n$ with $C_{i+1} = supp(\varphi'_i) \cup cosupp(\varphi'_i)$ and $C_i = supp(\varphi'_{i+1}) \cup cosupp(\varphi'_{i+1})$. On the other hand, if we have this structure in the cells of the graph diagrams we have that the replacement rules r_i and r_{i+1} are sequentially independent and the two derivations induced by these orders a transposition of one another. In Lemma 2.38, we will prove that $\Delta = \Delta'$.

Example 2.36. *Consider the following graph rewriting system,*

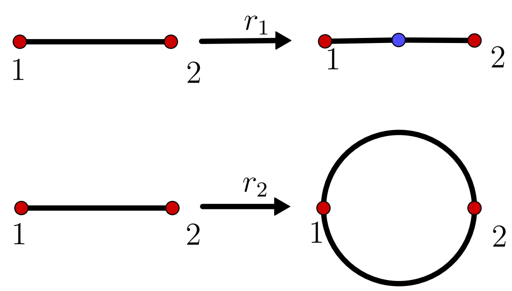


Figure 27 – A graph rewriting system

In Figure 28 the replacements r_1 and r_2 are sequentially independent, so we can move the replacement r_2 to the left.

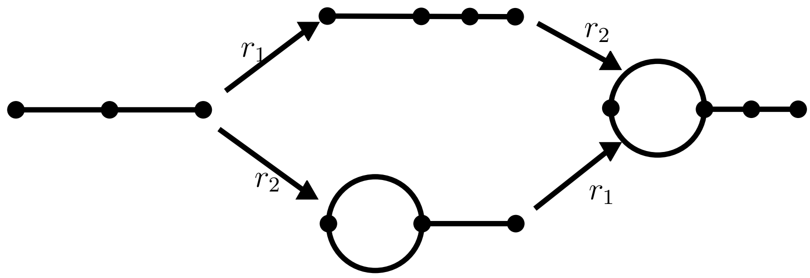


Figure 28

Example 2.37. *Let $\mathcal{P} = \langle a, b, c \mid b = c, ac = a, ca = a \rangle$ be a semigroup presentation. Denote the relations $b = c, ac = a,$and $ca = a$ by the replacement rules r_1, r_2 and r_3 respectively. Consider the following derivations under the presentation $\mathcal{P}$:*

$$abba \overset{r_1}{\Longrightarrow} acba \overset{r_2}{\Longrightarrow} aba \overset{r_3^{-1}}{\Longrightarrow} abca \overset{r_1^{-1}}{\Longrightarrow} abba$$

$$abba \overset{r_3^{-1}}{\Longrightarrow} abbca \overset{r_1^{-1}}{\Longrightarrow} abbba \overset{r_1}{\Longrightarrow} acbba \overset{r_2}{\Longrightarrow} abba$$

Note that the two derivations are a permutation of one another. In Example 1.19 we show how to obtain the diagram of the first derivation (we can do something analogous to get the same graph diagram). We can follow a similar process to realize that the graph diagram of both derivations corresponds to the diagram in Figure 29.

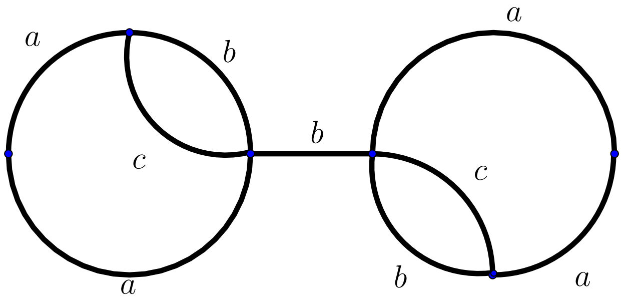


Figure 29

Lemma 2.38. *Suppose that (φ_i, r_i) and (φ_{i+1}, r_{i+1}) are sequentially independent replacement rules, then the derivation ρ with graphs $G_0, G_1, \ldots, G_{i-1}, G_i, G_{i+1}, \ldots, G_n$ and ρ' with graphs $G_0, G_1, \ldots, G_{i-1}, G'_i, G_{i+1}, \ldots, G_n$ have isomorphic graph diagrams.*

$$\begin{tikzcd}
G_0 \arrow[r, Rightarrow, "\varphi_1, r_1"] & G_1 \arrow[r, Rightarrow, "\varphi_2, r_2"] & \ldots \arrow[r, Rightarrow, "\varphi_{i-1}, r_{i-1}"] & G_{i-1} \arrow[r, Rightarrow, "\varphi_i, r_i"] \arrow[dr, Rightarrow, "\varphi'_i, r_{i+1}"'] & G_i \arrow[r, Rightarrow, "\varphi_{i+1} r_{i+1}"] & G_{i+1} \arrow[r, Rightarrow, "\varphi_{i+2}, r_{i+2}"] & \ldots \arrow[r, Rightarrow, "\varphi_n, r_n"] & G_n \\
& & & & G'_i \arrow[ur, Rightarrow, "\varphi'_{i+1}, r_i"'] & & &
\end{tikzcd}$$

Proof. Let $C_1, \ldots, C_{i-1}, C_i, C_{i+1}, \ldots C_n$ be a valid ordering for the cells of $\Delta = Diag(\rho)$, and $C_1, \ldots, C_{i-1}, C_{i+1}, C_i, \ldots C_n$ be a valid ordering of the cells of $\Delta' = Diag(\rho')$.
It is enough to show that $C_i \nless C_{i+1}$, since in this case Lemma 2.31 says that both orders are valid orders for the same graph diagram.
Suppose that $C_i < C_{i+1}$. By hypothesis (φ_i, r_i) and (φ_{i+1}, r_{i+1}) are sequentially independent replacement rules, so $bot(C_i)$ and $top(C_{i+1})$ are non overlapping which implies that $C_i \not\lessdot C_{i+1}$. Moreover $C_i < C_{i+1}$ and $C_i \not\lessdot C_{i+1}$ implies that there exists a sequence
$C_{i_1} = C_i \lessdot C_{i_2} \lessdot \cdots \lessdot C_{i_j} \lessdot C_{i_{j+1}} = C_{i+1}$, for suitable $i_k \in \{1, \ldots, n\}$, $C_{i+1} \neq C_{i_2}$ and $C_i \neq C_{i_j}$, which means that $i+1 \neq i_2$ and $i \neq i_j$.
Note that $i_2 \in \{i+2, \ldots, n\}$, since $C_i < C_{i_2} \Rightarrow i < i_2$ (by Theorem 2.33) and the fact that $i+1 \neq i_2$. Similarly we have $i_j \in \{1, \ldots, i-1\}$, since $C_{i_j} < C_{i+1} \Rightarrow i_j < i+1$ (by Theorem 2.33) and the fact that $i \neq i_j$.
Since $C_{i_2} < C_{i_j}$, Theorem 2.33 implies $i < i_2 < i_j < i$, a contradiction. Therefore $C_i \nless C_{i+1}$. □

Theorem 2.39. *Two derivations correspond to isomorphic graph diagrams if and only if the second derivation is isomorphic to a permutation of the first.*

Proof. $(\Rightarrow)$ Let

$$\rho := G_0 \xRightarrow{r_1, \varphi_1} G_1 \xRightarrow{r_2, \varphi_2} G_2 \ldots \xRightarrow{r_n, \varphi_n} G_n$$
$$\rho' := G'_0 \xRightarrow{r'_1, \varphi'_1} G'_1 \xRightarrow{r'_2, \varphi'_2} G'_2 \ldots \xRightarrow{r'_n, \varphi'_n} G'_n$$

such that

$$\Delta = Diag(\rho) \cong Diag(\rho') = \Delta'$$

Recall that $C_i = supp(\varphi_i) \cup cosupp(\varphi_i)$ for $i \in \{1, \dots, n\}$ and $C'_i = supp(\varphi'_i) \cup cosupp(\varphi'_i)$ for $i \in \{1, \dots, n\}$

Having isomorphic graph diagrams means that there is a graph isomorphism that maps the cells of Δ in the cells of Δ' preserving the tops and bottoms.

Moreover, the graph diagram isomorphism guarantees that the rules applied in the first derivation are isomorphic to the rules applied in the second. In particular, there exists j_1 such that $r_1 \cong r'_{j_1}$ with cells satisfying the same relations in the partial order. Recall that a valid order of the cells of a graph diagram has the same information of a derivation in the sense that given a derivation we can recognize a valid order for the graph diagram of the derivation and given an order in the cells we can obtain the associated permutation.

Furthermore, note that as r_1 is a rule applied over the graph G_0, we have that r'_{j_1} can be applied over the graph G'_0. In fact, rule r_1 corresponds to the cell C_1 in the valid order of Δ, while rule r'_{j_1} corresponds to the cell C'_{j_1} in the valid order of Δ' and $C_1 \cong C'_{j_1}$.

Thus $C_i \nprec C_1$ for $i \in \{2, \dots, n\}$, since $C_i \prec C_1$ would imply $i < 1$ by Theorem 2.33, a contradiction. The same relation must be satisfied for C_{j_1}, that is $C'_i \nprec C'_{j_1}$ for $i \neq j_1$. Then Lemma 2.31 implies that $C'_{j_1}, C'_1, \dots, C'_{j_1-1}, C'_{j_1+1}, \dots, C'_n$ is also a valid order for Δ' and these orders are obtained from derivations that are a permutation of one another. Indeed, recall that in Remark 2.35 we explain that each time that we use Lemma 2.31 we are transposing two replacements in a derivation.

Similarly, there exists j_2 such that $r_2 \cong r'_{j_2}$ and r'_{j_2} is sequentially independent of r'_k for $k \neq j_1$. In fact, in this case $C_2 \cong C'_{j_2}$ and we have that $C_k \nprec C_2$ for $k \neq 1$, since $C_k \prec C_2$ implies $1 \neq k < 2$ by Theorem 2.33, a contradiction. Then we have that $C'_k \nprec C'_{j_2}$ and Lemma 2.31 implies that $C'_{j_1}, C'_{j_2}, C'_1, \dots, C'_{j_1-1}, C'_{j_1+1}, \dots, C'_{j_2-1}, C'_{j_2+1}, \dots C'_n$ is a valid order for Δ' and the derivation associated with this valid order is a permutation of ρ'.

Following this process we can reorganize the cells of Δ' until get a new order $C'_{j_1}, C'_{j_2}, \dots, C'_{j_n}$ for this graph diagram that satisfies $C'_{j_k} \cong C_k$ for $k \in \{1, \dots, n\}$ and the permutation of this new order of Δ' is a permutation of ρ'.

($\Leftarrow$) It is enough to prove the statement in the case when the derivation ρ is isomorphic to a transposition of ρ'. Let $C_1, \dots, C_n$ be the valid order induced by ρ and let $C'_1, \dots, C'_n$ be the valid order induced by ρ'. Since ρ is a transposition of ρ' there exists $i \in \{1, \dots, n-1\}$ such that $C_i \cong C'_{i+1}$ and $C_{i+1} \cong C'_i$ and $C_k \cong C'_k$ for $k \notin \{i, i+1\}$ preserving tops and bottoms. Let ρ_1 be the derivation that is isomorphic to ρ' and is a transposition of ρ. Note that, by Lemma 2.38, the derivation ρ_1 is obtained from the valid order $C_1, C_2, \dots, C_{i+1}, C_i, \dots, C_n$ satisfying $Diag(\rho_1) = Diag(\rho)$. We claim that $Diag(\rho_1) \cong Diag(\rho')$.

In fact, by hypothesis, ρ_1 and ρ' have isomorphic derivations. Thus, by Lemma 2.28, we have

$$Diag(\rho') \cong Diag(\rho_1) = Diag(\rho).$$

□

2.5.3 Reductions of diagrams

In what follows, we will paint in red the boundary of the rules in the graph rewriting system, in black the interior edges, and in blue the interior vertices. We also will use black to represent the base graph of the graph rewriting system. We will use numbers to underline how to identify the boundaries.

Definition 2.40. *Let Δ be a graph diagram. A* ***dipole*** *in Δ is a pair (C, D) of cells satisfying the following conditions*

1. $bot(C) = top(D)$,
2. *No other cell E of Δ satisfies $C \prec E \prec D$, and*
3. *There exists an isomorphism $bot(D) \to top(C)$ that restricts to the identity on $\partial bot(D)$.*

If (C, D) is a dipole we can **reduce** it by removing the interior of $bot(C)$, and identifying $bot(D)$ to $top(C)$ via a boundary-fixing isomorphism φ.
Recall that two replacement rules (R_1, S_1, ν_1) and (R_2, S_2, ν_2) are inverse if there exist boundary-preserving isomorphisms $\alpha_1 : R_1 \to S_2$ and $\alpha_2 : S_1 \to R_2$ making the following diagram commute

$$\begin{tikzcd} \partial R_1 \arrow[r, "\nu_1"] \arrow[d, "\alpha_1"] & \partial S_1 \arrow[d, "\alpha_2"] \\ \partial S_2 & \partial R_2 \arrow[l, "\nu_2"] \end{tikzcd}$$

We will prove that the graph diagram $\Delta = R_1 \cup_{\nu_1} S_1 \cup_{\nu_2 \circ \alpha_2} S_2$ is a dipole.

Proposition 2.41. *Following the notation above, let $r_1 = (R_1, S_1, \nu_1)$ and $r_2 = (R_2, S_2, \nu_2)$ be inverse replacement rules. Consider the derivation*

$$\rho \coloneqq R_1 \overset{\varphi_1, r_1}{\Longrightarrow} S_1 \overset{\varphi_2, r_2}{\Longrightarrow} S_2,$$

then $Diag(\rho)$ can be reduced to R_1.

Proof. The valid order given by ρ has cells $C_1 = (R_1 \cup_{\nu_1} S_1)$ and $C_2 = (S_1 \cup_{\nu_2 \circ \alpha_2} S_2)$ and defining graphs $G_0 = R_1, G_1 = S_1$ and $G_2 = S_2$. Thus $bot(C_1) = S_1 = top(C_2)$ by the commutativity of the diagram after Definition 2.40 we have $\alpha_1(\partial R_1) = \nu_1 \circ \alpha_2 \circ \nu_2(\partial(R_1)) = \alpha_2 \circ \nu_2(\partial S_1) = \nu_2(\partial R_2) = \partial S_2$, implying that $\alpha_1^{-1}(\partial S_2) = \partial R_1$, that is, α_1 is a boundary fixing isomorphism and (C_1, C_2) is a dipole. We can reduce such dipole by eliminating G_1, C_1 and C_2 from the valid order and identifying G_0 with G_2. □

Proposition 2.42. *Let Δ be a graph diagram, and let $\mathcal{C}$ be its collection of cells. Let (C, D) be a dipole in Δ, and let $\widetilde{\Delta}$ be a graph obtained by reducing this dipole, with cells $\mathcal{C} - \{C, D\}$. Then $\widetilde{\Delta}$ is a graph diagram.*

Proof. By condition (2) in the dipole definition and the Theorem 2.33 there exists a valid ordering in $\mathcal{C}$ as follows:

$$C_1, \dots, C_k = C, C_{k+1} = D, \dots, C_n$$

for some $1 \leq k \leq n$. Let $G_0, G_1, \dots, G_n$ the corresponding graphs of Δ.
Let φ the isomorphism that identifies $top(C)$ to $bot(D)$. Given a cell C_j in the valid order, we can apply the reduction of the dipole by applying φ. Observe that $top(C)$ and $top(C_j)$ have disjoint interior by Lemma 2.30, so that for each edge in C_j and G_j we apply φ on the edges that are in $bot(C_{k+1})$. After applying this process to each cell and corresponding graph in Δ, we obtain a graph diagram $\widetilde{\Delta}$ with cells $\widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_{n-2}$ and corresponding graphs $\widetilde{G}_0, \widetilde{G}_1, \dots, \widetilde{G}_{n-2}$, where C_k, C_{k+1}, G_k and G_{k+1} were removed from this list. We will check that $\widetilde{\Delta}$ is a graph diagram.

1. The same edges were removed from Δ and $G_0, G_1, \dots, G_n$ to obtain $\widetilde{\Delta}$ and its corresponding graphs $\widetilde{G}_0, \widetilde{G}_1, \dots, \widetilde{G}_{n-2}$. The valid order on the cells of Δ induces a valid order on the cells of $\widetilde{\Delta} = \widetilde{G}_0 \cup \widetilde{G}_1 \cup \dots \cup \widetilde{G}_{n-2}$.

2. We will show that $\widetilde{G}_a \cap \widetilde{G}_c \subseteq \widetilde{G}_b$. We will analyze the different positions (with respect to the partial order of the cells) where the dipole might be.

   We will use constantly the fact that the dipole reduction does not affect edges and vertices in G_j for $j < k$.

   Suppose that $a < b < k \leq c$.

   Notice that $G_a \cap G_{c+2} \subseteq G_b$ implies that the relation it is true for the edges in $\widetilde{G}_a \cap \widetilde{G}_c$ that are not modified by the dipole. We will prove that the edges (vertices) modified by the dipole also satisfy the required relation. Observe that, once we reduce the dipole, the edges and vertices in $bot(C_{k+1})$ and $top(C_k)$ are identified. Moreover, since $G_{k+1} = bot(C_{k+1}) \cup bot(C_{k+1})^c$ where $bot(C_{k+1})^c = top(C_k)^c \subseteq G_{k-1}$ and $G_a \cap G_{c+2} \subseteq G_b$, then we have $\widetilde{G}_a \cap (\widetilde{G}_c \cap bot(C_{k+1})^c) \subseteq \widetilde{G}_b$. Furthermore, the image of $bot(C_{k+1})$ in $\widetilde{G}_c$ is contained in $\widetilde{G}_{k-1} = G_{k-1}$. Thus $G_a \cap G_{k-1} \subseteq G_b$ implies $\widetilde{G}_a \cap \widetilde{G}_{k-1} \subseteq \widetilde{G}_b$, so $\widetilde{G}_a \cap \widetilde{G}_c \subseteq \widetilde{G}_b$.

   Suppose that $a < k \leq b \leq c$.

   In a fashion similar to the previous case, note that $G_a \cap G_{c+2} \subseteq G_{b+2}$ implies that the relation it is true for the edges (vertices) in $\widetilde{G}_a \cap \widetilde{G}_c$ that are not modified by the dipole.

   Moreover $G_{k-1} = top(C_k) \cup top(C_k)^c$, together with $G_a \cap G_{c+2} \subseteq G_{b+2}$ and the fact that the dipole reduction does not affect G_j for $j < k$ implies that $(\widetilde{G}_a \cap top(C_k)^c) \cap \widetilde{G}_c \subseteq \widetilde{G}_b$. Note also that $top(C_k) \subseteq G_{k-1} = \widetilde{G}_{k-1}$ and $\widetilde{G}_{k-1} \cap \widetilde{G}_c \subseteq \widetilde{G}_b$. Hence, $\widetilde{G}_a \cap \widetilde{G}_c \subseteq \widetilde{G}_b$.

   Similarly, if $a < b < c \leq k$, then $G_a \cap G_c \subseteq G_b$ immediately implies that $\widetilde{G}_a \cap \widetilde{G}_c \subseteq \widetilde{G}_b$ and likewise if $k < a < b < c$, then $G_{a+2} \cap G_{c+2} \subseteq G_{b+2}$ immediately implies that $\widetilde{G}_a \cap \widetilde{G}_c \subseteq \widetilde{G}_b$.

3. The only edges affected by φ are those in $bot(C_{k+1})$, and such edges in G_j and C_j for $j \geq k+1$ were subjected to the same mapping. So, since Δ is a graph diagram, the top of each $\widetilde{C}_i$ is a portion of $\widetilde{G}_{i-1}$, the bottom of each $\widetilde{C}_i$ is a portion of $\widetilde{G}_i$, and the complement of $top(\widetilde{C}_i)$ in $\widetilde{G}_{i-1}$ is equal to the complement of $bot(\widetilde{C}_i)$ in $\widetilde{G}_i$.

   In particular, note that when the dipole is reduced G_{k-1} and G_{k+1} are identified, so $G_{k-1} = \widetilde{G}_{k-1}$ and $\widetilde{C}_k$ and $\widetilde{G}_k$ are obtained by applying the reduction to C_{k+2} and G_{k+2} respectively making the claim true for $\widetilde{C}_k$ and $\widetilde{G}_{k-1}$.

□

We say that $\widetilde{\Delta}$ in Proposition 2.42 is obtained from a **reduction** of Δ.

Definition 2.43. *A graph diagram is* ***reduced*** *if it contains no dipoles.*

Given a dipole (C, D) there may be more than one isomorphism between $bot(D)$ and $top(C)$, and therefore there may be more than one way to reduce a dipole. This implies that, in some occasions, two different graph diagrams can be obtained from reducing the same dipole. We wish to avoid such situation as we look for conditions helping us prove that each graph diagram only has one reduced element under the dipole reduction. This motivates the following definition.

Definition 2.44. *A replacement rule $t = (R, S, \nu)$ is said to be* ***reductive*** *if the only automorphism of R that fixes ∂R pointwise is the identity automorphism, and similarly for S. A graph rewriting system is* ***reductive*** *if each of its replacement rules is.*

Remark 2.45. 1. *Note that in a reductive graph rewriting system if φ_1, φ_2 are isomorphisms from the bottom of the graph diagram to the top of it, we have that $\varphi_1 \circ \varphi_2^{-1}$ is an automorphism of the bottom of the graph diagram, then $\varphi_1 \circ \varphi_2^{-1}$ is the identity automorphism and $\varphi_1 = \varphi_2$.*

2. *Given a dipole (C, D) in a graph diagram Δ we have boundary fixing isomorphisms $\varphi : top(C) \to bot(D)$, $\iota : bot(C) \to top(D)$ making the following diagram commute*

$$\begin{tikzcd} \partial top(C) \arrow[r, "\iota"] \arrow[d, "\varphi"] & \partial bot(C) \arrow[d, "\iota"] \\ \partial bot(D) & \partial top(D) \arrow[l, "\iota"] \end{tikzcd}$$

where ι is the identity map. Therefore the replacement rule used to obtain the cell C is the inverse that the replacement rule used to obtain the cell D and by Convention 2.15 we have that φ must be the identity map.

Definition 2.46. *A graph diagram Δ is* ***equivalent*** *to a graph diagram $\widetilde{\Delta}$ if there is a sequence*

$$\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \widetilde{\Delta}$$

where, for all i, the diagram Δ_i is obtained by a reduction of Δ_{i-1} or Δ_{i-1} is obtained by a reduction of Δ_i.

The rearrangement group of fractals provides several rules that induce graph rewriting systems, some of these rules are reductive and some of them are not.

Example 2.47. *In the next sections we will explain how we can obtain a graph rewriting system from a replacement system of a rearrangement group of fractals. For example, in Figure 30 we have graph rewriting systems for the Thompson groups F and V. Note that each rule in this system is reductive since the only automorphism of the graphs in the replacement rules that fixes pointwise the boundary is the identity. Recall that these automorphisms must preserve the context of the graphs, in this case directed graphs.*

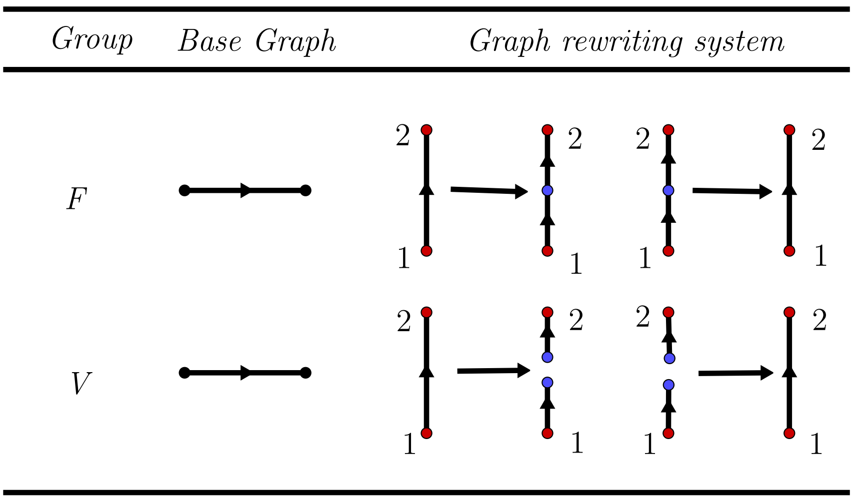


Figure 30 – The graph rewriting system for the groups F and V.

The Generalized Thompson's Groups $F_{n,k}, T_{n,k}$ and $V_{n,k}$ where n, k are positive integers are also graph diagram groups. The groups $F_{3,2}, T_{3,2}$ and $V_{3,2}$ correspond to the graph rewriting systems in Figure 31. When $n = 2$ and $k = 1$ we have the case of the Thompson's groups F, T and V respectively.

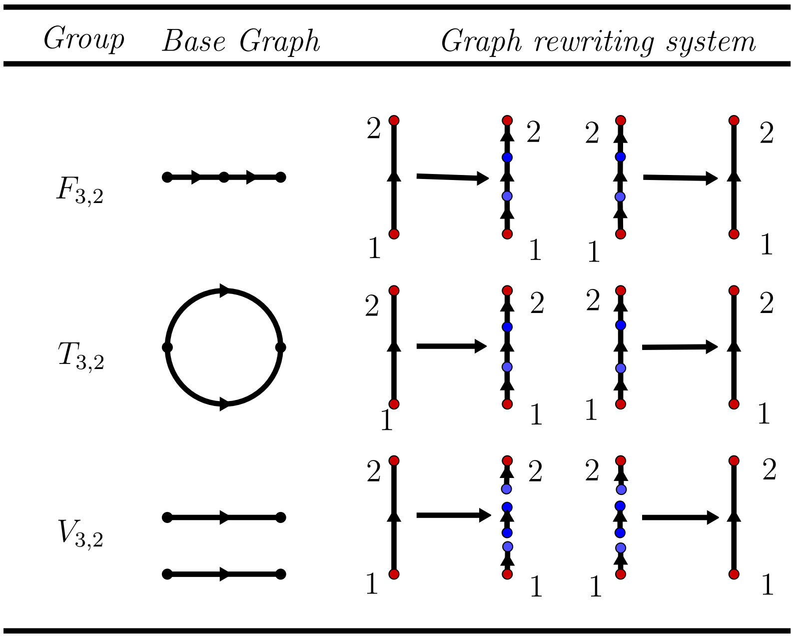


Figure 31 – The graph rewriting system for the groups $F_{3,2}, T_{3,2}$ and $V_{3,2}$.

Example 2.48. *We can turn some non-reductive rules into reductive ones. Observe that in Figure 17 and Figure 23 we have replacement rules for the rabbit and Vicsek families. In particular for $n = 2$ the limit spaces are the basilica and the Vicsek fractal. Note that in both figures the rule for $n = 1$ is reductive, but for $n \geq 2$ we have an automorphism that fix pointwise the boundaries but maps an edge (loop) to another. We can turn all these non-reductive rules (such as the one in Figure 32) into reductive ones by changing the context to allow edge labels. For example for $n = 2$ in the rabbit family we can add labels "A" and "B" and then replace the non-reductive rule into the three rules in Figure 32.*

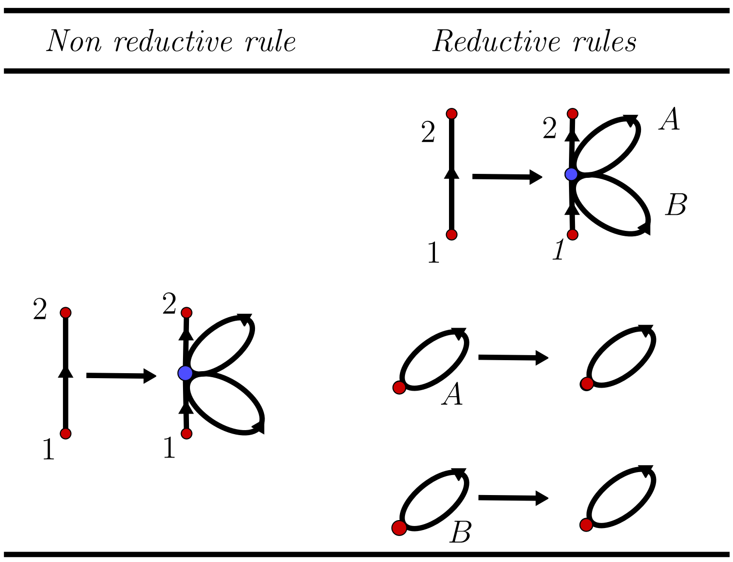


Figure 32 – From non-reductive rule to **auxiliary** reductive ones.

Remember that when we apply a reduction of graph diagrams, given a dipole (C, D) such that $\varphi(bot(D)) = top(C)$ we construct the graph diagram by applying φ to each cell C_j and subgraph G_j where $j > k+1$. That is, for each edge in C_j and G_j we apply φ if the edge is in interior of $bot(C_{k+1})$. We obtain cells $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_{n-2}$ and graphs $\widetilde{G}_0, \widetilde{G}_1, \ldots, \widetilde{G}_{n-2}$. Note that C_k, C_{k+1}, G_k and G_{k+1} have been removed from this list.
Observe that if C, D, E, F are pairwise distinct we can reduce two dipoles from a diagram without caring about the order in which we do it. Indeed, given two dipoles (C, D) and (E, F), we can reduce the dipoles using the isomorphism φ_1 and φ_2 respectively.

Lemma 2.49. *Let Δ be a graph diagram over a reductive graph rewriting system with dipoles $(C, D), (E, F)$ such that C, D, E, F are pairwise distinct. Then, once we reduce one of these dipoles we still can reduce the other.*

Proof. It is enough to see that we can reduce the dipole (C, D) and after the dipole (E, F). Observe that by definition of dipole we have that there exists a valid order with cells $C_1, C_2, \ldots, C_k = C, C_{k+1} = D, \ldots, C_l = E, C_{l+1} = F, \ldots, C_n$. If we reduce (C, D) we identify $bot(D)$ with $top(C)$ using φ_1. Note that as the graph rewriting system is reductive by Remark 2.45, then φ_1 is the identity map on $bot(D)$. Moreover, by Lemma 2.30, $bot(C) = top(D)$ does not intersect the interior of $top(E)$ and $bot(F)$, so there are no erased edges or vertices from the interior of $top(E)$ and $bot(F)$ when the dipole (C, D) is reduced and no edges or vertices of (E, F) have been affected. Therefore, we can still use φ_2 to reduce the dipole (E, F). □

Theorem 2.50. *Every graph diagram over a reductive graph rewriting system is equivalent to a unique reduced diagram.*

Proof. Under the graph reduction and the equivalence class of diagrams, we will prove that the rewriting system is confluent, so we can use Lemma 1.14. Note that with every reduction the number of cells is reduced by two, so the system is terminating. It suffices to show that the system is locally confluent. We consider first some cases where cells coincide in some way and after that we will consider the case when the cells in the dipoles are pairwise distinct.
Let (C, D) and (E, F) be dipoles in a graph diagram Δ.

- Suppose that $C = E$. Then the definition of dipole implies $bot(C) = top(D) = bot(E) = top(F)$ and by Lemma 2.30 we have that $D = F$.
- Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be the graph diagrams obtained by reducing from Δ the dipoles (C, D) and (E, F), respectively. Suppose $D = E$. Note that since the rewriting system is reductive, the only boundary preserving graph isomorphism from $bot(D)$ to $top(C)$ is the identity map, and so removing the dipole (C, D) identifies $top(F)$ with $top(C)$.

Analogously, the identity map is the only boundary preserving graph isomorphism from $top(E)$ to $bot(F)$, and so removing the dipole (E,F) identifies $bot(C)$ and $bot(F)$. Therefore, if we reduce both dipoles from Δ we obtain the following relations, $bot(E) \cong top(C) \cong top(F)$, and $top(E) = bot(F) = bot(C)$, Thus

$$C = top(C) \cup bot(C) \cong bot(E) \cup top(E),$$

$$F = top(F) \cup bot(F) \cong bot(E) \cup top(E)$$

This means that the replacement rules that induce C and F are the inverses of the replacement rules that induce E. Observe that the graph diagrams $\Delta^{(1)}$ and $\Delta^{(2)}$ have the same cells with the exception of the cell C_k that is C in $\Delta^{(1)}$ and F in $\Delta^{(2)}$, therefore $\Delta^{(1)}$ and $\Delta^{(2)}$ are isomorphic graph diagrams.

- Let (C,D) and (E,F) be dipoles in Δ with isomorphisms φ_1 and φ_2 respectively to identify their tops and bottoms and so that C,D,E,F are pairwise distinct. Consider the graph diagrams $\Delta, \Delta^{12}, \Delta^{21}$ where Δ^{ij} is the graph diagram obtained from reducing the dipoles applying $\varphi_i\varphi_j$ with $i+j=3$ and $1 \le i,j \le 2$.

  We will show that $\Delta^{12} \cong \Delta^{21}$.

  Δ is a graph diagram with dipoles (C,D) and (E,F). Then, by definition of dipole we have that there exists a valid order with cells $C_1, C_2, \dots, C_k = C, C_{k+1} = D, \dots, C_l = E, C_{l+1} = F, \dots, C_n$. We will denote C_i^{12} and C_i^{21} the cells of Δ^{12} and Δ^{21} respectively and G_i^{12} and G_i^{21} the defining graphs of Δ^{12} and Δ^{21}, respectively.

  We must show that the cells and defining graphs of both graph diagrams are isomorphic. In general, the defining graphs of a graph diagram are determined by the cells and the top of the diagram. This is because the other defining graphs are obtained successively from G_0 by taking complements using the third property in the definition of graph diagram. Thus, it is enough to show that the cells of both graphs are isomorphic (and preserving tops and bottoms) and $top(\Delta^{12}) = G_0^{12} \cong G_0^{21} = top(\Delta^{21})$.

  The second statement that we need to show is true. Indeed, observe that the top of a diagram is invariant under reductions since, by definition, it consists of the edges and vertices of Δ that are not in the interior of the bottom of any cell, in fact $top(\Delta) = G_0$, and when we eliminate the dipoles (C,D) and (E,F) we erase $bot(C)$ and $bot(E)$ from Δ, then no edges and vertices of G_0 are erased when we apply such reductions. Moreover, no edges and vertices of G_0 are affected by the dipole reduction since $\mathcal{R}$ is reductive and this implies that the map that identifies the bottom and the top of each dipole is the identity. Therefore, to reduce the dipoles do not move edges in $top(\Delta)$ and

$$top(\Delta^{12}) = G_0^{12} \cong G_0^{21} = top(\Delta^{21}) \cong top(\Delta).$$

On the other hand, to prove that the cells of both diagrams are isomorphic, we must note that independently of the order of the reductions the edges and vertices in the interior of $bot(C)$ and $bot(E)$ are erased from the new graph diagrams when both reductions are made. Observe that, by Lemma 2.49, we can do these reductions in any order we prefer.

We will prove that each cell of Δ that is not in the dipole remains the same once a dipole reduction is made.

Note that if $C_i \prec C$ in the valid order given in Δ, then the dipole reduction does not affect the cell C_i.

On the contrary, if $D \prec C_i$, again we have by Lemma 2.30 that, given a cell C_i of Δ and so $top(D)$ and $bot(C_i)$ are non-overlapping. This means that the intersection of the interior of $top(D)$ and $bot(C_i)$ is empty, so the reduction of the dipole does not eliminate any vertex or edge in $bot(C_i)$. In particular, it does not eliminate vertices and edges in $\partial bot(C_i) = \partial top(C_i)$. Similarly, $bot(C)$ and $bot(C_i)$ have disjoint interiors, thus the dipole reduction does not eliminate edges or vertices in C_i.

Now we can prove that when a dipole is reduced, the new cell $\varphi_j(C_i)$ is equal to C_i. Indeed, suppose that we reduced the dipole (C, D). Thus,

- If C_i doe not intersect the interior $bot(D), C_i = \varphi_1(C_i)$.
- If C_i intersects the interior of $bot(D)$, we call such intersection W. Observe that since the graph rewriting system is reductive, φ_1 is the only boundary preserving graph isomorphism from $bot(D)$ to $top(C)$ and it is the identity map. Then, $\varphi_1(W) \cong W$ and $\varphi_1(W) \cup (C_i \setminus W) \cong C_i$.

We have a analogous situation each time that we reduce a dipole, therefore we can conclude that $\varphi_1(\varphi_2(C_i)) \cong \varphi_2(\varphi_1(C_i)) \cong C_i$.

So we have that the graph diagrams Δ^{12} and Δ^{21} have isomorphic initial graphs and set of cells, therefore $\Delta^{12} \cong \Delta^{21}$.

□

2.5.4 Graph Diagram Groups

Definition 2.51. *Let Δ and $\widetilde{\Delta}$ be two ordered graph diagrams where $C_1, C_2, \dots C_n$ and $G_0, G_1, \dots G_n$ are the cells and the corresponding subgraphs of Δ and $\widetilde{C}_1, \widetilde{C}_2, \dots \widetilde{C}_n$ and $\widetilde{G}_0, \widetilde{G}_1, \dots, \widetilde{G}_m$ are the cells and the corresponding subgraphs of $\widetilde{\Delta}$. Suppose that $\varphi : G_n \to \widetilde{G}_0$ is a graph isomorphism. The* ***concatenation*** *$\widetilde{\Delta} \circ \Delta$ is a graph diagram given by $\Delta \cup_\varphi \widetilde{\Delta}$ with cell set $C_1, C_2, \dots C_n, \widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_m$ and corresponding subgraphs $G_0, G_1, \dots G_n = \widetilde{G}_0, \widetilde{G}_1, \dots, \widetilde{G}_m$.*

Proposition 2.52. *$\widetilde{\Delta} \circ \Delta$ is an ordered graph diagram.*

Proof. 1. We have $\Delta \cup_\varphi \widetilde{\Delta} = G_0 \cup G_1 \cup \cdots \cup (\varphi(G_n) = \widetilde{G}_0) \cup \widetilde{G}_1 \cup \cdots \cup \widetilde{G}_m$.

2. Observe that $G_i \cap G_k \subseteq G_j$ for $i < j < k$ and $\widetilde{G}_i \cap \widetilde{G}_k \subseteq \widetilde{G}_j$ for $i < j < k$ are satisfied.

   Remember that $\Delta \cup_\varphi \widetilde{\Delta}$ is the amalgamated union of Δ and $\widetilde{\Delta}$ that is obtained by doing the disjoint union of Δ and $\widetilde{\Delta}$ but identifying G_n and $\widetilde{G}_0$ using φ.

   We need to check,

   - $G_i \cap \widetilde{G}_k \subseteq G_j$ for $i < j$ and for any $0 \leq k \leq m$
     In fact, note that if $k = 0$

     $$G_i \cap \widetilde{G}_0 = G_i \cap G_n \subseteq G_j \text{ for } i < j < n$$

     If $k \neq 0$ then, by definition of $\Delta \cup_\varphi \widetilde{\Delta}$, then $\widetilde{G}_k$ in this new diagram has edges (and vertices) that are either disjoint from G_i or that belong to $G_n \cong \widetilde{G}_0$ and so they are not disjoint from G_i. In this last case,

     $$G_i \cap \widetilde{G}_k \subseteq G_i \cap G_n \subseteq G_j \text{ for } i < j < n.$$

   - $G_i \cap \widetilde{G}_k \subseteq \widetilde{G}_j$ for $j < k$.
     Indeed, if $i = n$,

     $$G_n \cap \widetilde{G}_k = \widetilde{G}_0 \cap \widetilde{G}_k \subseteq \widetilde{G}_j \text{ for } 0 < j < k.$$

     If $i \neq n$ by definition of $\Delta \cup_\varphi \widetilde{\Delta}$, the graph G_i in this new diagram has edges (vertices) that are either disjoint from $\widetilde{G}_k$, or edges (vertices) that belong to $G_n \cong \widetilde{G}_0$ and so they are not disjoint from $\widetilde{G}_k$. Then,

     $$G_i \cap \widetilde{G}_k \subseteq \widetilde{G}_0 \cap \widetilde{G}_k \subseteq \widetilde{G}_j \text{ for } i < j < m.$$

3. Since Δ and $\widetilde{\Delta}$ satisfy property 3 of Definition of 2.20, then clearly $\widetilde{\Delta} \circ \Delta$ satisfies it too, by construction.

□

Let be $\rho_1 := G_0 \overset{\varphi_1}{\Longrightarrow} G_1 \overset{\varphi_2}{\Longrightarrow} \ldots \overset{\varphi_n}{\Longrightarrow} G_n$ and $\rho_2 := \widetilde{G}_0 \overset{\widetilde{\varphi}_1}{\Longrightarrow} \widetilde{G}_1 \overset{\widetilde{\varphi}_2}{\Longrightarrow} \ldots \overset{\widetilde{\varphi}_m}{\Longrightarrow} \widetilde{G}_m$ with cells $C_1, C_2, \ldots C_n$ and $\widetilde{C}_1, \widetilde{C}_2, \ldots \widetilde{C}_n$ respectively where $C_i = supp(\varphi_i) \cup cosupp(\varphi_i)$ $\widetilde{C}_i = sup(\widetilde{\varphi}_i) \cup cosupp(\widetilde{\varphi}_i)$.
Observe that the graph diagram $\widetilde{\Delta} \circ \Delta$ coincides with the graph diagram of the concatenation of the derivations,

$$\rho := G_0 \overset{\varphi_1}{\Longrightarrow} G_1 \overset{\varphi_2}{\Longrightarrow} G_2 \ldots \overset{\varphi_n}{\Longrightarrow} G_n \overset{\widetilde{\varphi}_1}{\Longrightarrow} \widetilde{G}_1 \overset{\widetilde{\varphi}_2}{\Longrightarrow} \widetilde{G}_2 \ldots \overset{\widetilde{\varphi}_m}{\Longrightarrow} \widetilde{G}_m.$$

Indeed, note that the defining grahs of $Diag(\rho)$ are $G_0, G_1, \ldots G_n = \widetilde{G}_0, \widetilde{G}_1, \ldots, \widetilde{G}_m$. and the cells are $C_1, C_2, \ldots C_n, \widetilde{C}_1, \widetilde{C}_2, \ldots \widetilde{C}_n$.

Example 2.53. *In the example 2.18 and the upcoming example 3.10 we have a graph rewriting system for $F \rtimes \mathbb{Z}_2$ and derivations using this graph rewriting system. Consider the element in Figure 33,*

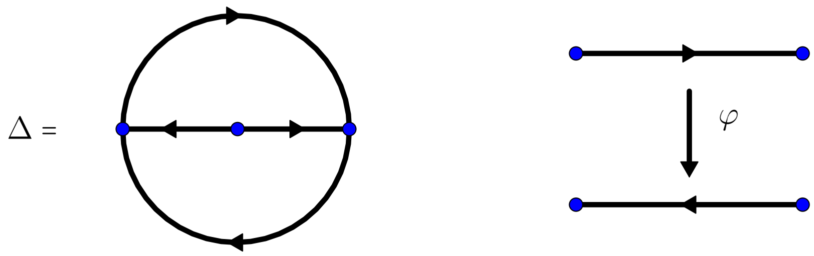


Figure 33 – On the left side a graph diagram Δ and in the right the isomorphism between its top and bottom.

We can calculate the square of this element Δ^2 by using the isomorphism φ, that is, φ tells us how to concatenate Δ with itself.

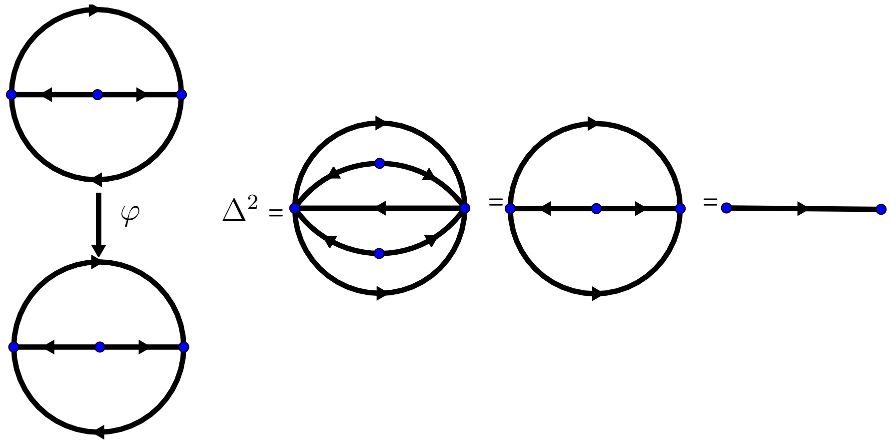
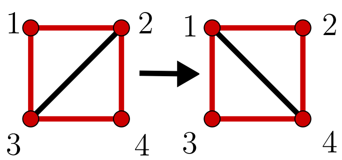


Figure 34 – Concatenation of an element with itself

In Figure 34 the map φ tells us how to identify the bottom of the first copy of Δ, that we call Δ_1 with the top of the second that we call Δ_2. This means that, when we perform this identification, all the edges of Δ_2 are affected, that is, when we rotate the top of Δ_2 to identify it with bot(Δ_1), we also rotate the whole graph diagram Δ_2 and obtain the graph Δ^2.

Remark 2.54. *For simplicity we restrict the replacement rules of our graph rewriting system to those that are not their own inverse. For example the following rule called **diagonal flip** is forbidden,*



Definition 2.55. *A graph rewriting system is called* ***symmetric*** *if for each replacement rule in it, we have the inverse rule in it.*

We wish to capture in the same equivalence class those graphs that produce isomorphic graph diagrams and the same top. In Theorem 2.28 we prove that the graph diagrams of isomorphic derivations are isomorphic, see also Examples 2.17 and Example 2.29. This motivates the following definition.

Definition 2.56. *Given a symmetric reductive graph rewriting system $\mathcal{R}$, consider the set $Diag(\mathcal{R},\Gamma)$ of pairs (Δ,φ) where Δ is a reduced graph diagram with top graph Γ and graph isomorphism $\varphi : bot(\Delta) \to \Gamma$. Two elements (Δ_1,φ_1) and (Δ_2,φ_2) of this set are* ***equivalent*** *if there exist isomorphic derivations for Δ_1 and Δ_2 that are the identity on Γ (this is the top of both diagrams is Γ and the map τ_0 in Definition 2.16 is the identity) and $\varphi_1 = \varphi_2 \circ \varphi$ where φ is the isomorphism from $bot(\Delta_1)$ to $bot(\Delta_2)$.*

Let (Δ,φ) with cells $C_1,\dots,C_n$. Suppose that C_i corresponds to an application of the replacement rule $(R,\widetilde{R},\nu)$ and let C_i^{-1} be the cell given by applying $(\widetilde{R},R,\nu^{-1})$ with $bot(C_i) = top(C_i^{-1})$ and $top(C) = bot(C_i^{-1})$. The inverse of (Δ,φ), $(\Delta^{-1},\varphi^{-1})$, is the diagram $(\Delta \cup_\varphi \Gamma, \varphi^{-1})$ with cells $C_n^{-1},\dots,C_1^{-1}$ where the Γ in the amalgamated union is disjoint from Δ.

Theorem 2.57. *The set of $\mathcal{D}(\mathcal{R},\Gamma)$ of equivalence classes of $D(\mathcal{R},\Gamma)$ forms a group under concatenation.*

Proof. We denote Δ_φ a representative of the equivalence class of (Δ,φ) in $\mathcal{D}(\mathcal{R},\Gamma)$. Suppose $\Delta_\varphi \in \mathcal{D}(\mathcal{R},\Gamma)$ with defining graphs $G_0 = \Gamma,\dots,G_n$ and cells $C_1,C_2,\dots,C_n$. Note that given $(\Delta_1,\varphi_1),(\Delta_2,\varphi_2) \in \mathcal{D}(\mathcal{R},\Gamma)$ we have that the composition in the group is given by $(\Delta_2,\varphi_2)\circ(\Delta_1,\varphi_1) = (\Delta_1 \cup_{\varphi_1} \Delta_2)_{\varphi_2\circ\tau_n^{-1}}$ where τ_n is given by an isomorphic derivation as in the diagram below and $\varphi_2 \circ \tau_n^{-1}(G'_n) = \Gamma$.

$$
\begin{tikzcd}
G_o = \Gamma \arrow[r, Rightarrow] \arrow[d, "\iota"] \arrow[rr, bend left, "\Delta_1"] & \dots \arrow[r, Rightarrow] & G_l =_{\varphi_1} \Gamma \arrow[r, Rightarrow] \arrow[d, "\tau_l=\varphi_1^{-1}"] \arrow[rrr, bend left, "\Delta_2"] & G_{l+1} \arrow[r, Rightarrow] \arrow[d, "\tau_{l+1}"] & \dots \arrow[r, Rightarrow] & G_n \arrow[r, "\varphi_2"] \arrow[d, "\tau_n"] & \Gamma \arrow[d, "\iota"] \\
G_0 = \Gamma \arrow[r, Rightarrow] \arrow[rrrrr, bend right, "(\Delta_1\cup_{\varphi_2}\Delta_2)_{\varphi_2\circ\tau_n^{-1}}"'] & \dots \arrow[r, Rightarrow] & G_l =_\iota G'_l \arrow[r, Rightarrow] & G'_{l+1} \arrow[r, Rightarrow] & \dots \arrow[r, Rightarrow] & G'_n \arrow[r, "\varphi_2\circ\tau_n^{-1}"] & \Gamma
\end{tikzcd}
$$

Note that given $\Delta_{\varphi_1}, \Delta'_{\varphi_2} \in \mathcal{D}(\mathcal{R},\Gamma)$ we have $\Delta_{\varphi_1} \circ \Delta'_{\varphi_2} = (\Delta' \cup_{\varphi_2} \Delta)_{\varphi_2\circ\varphi_1} \in \mathcal{D}(\mathcal{R},\Gamma)$, this is $\tau_n = \varphi_1^{-1}$ and therefore is satisfied that

$$\tau_n^{-1}(bot(\Delta_{\varphi_1} \circ \Delta'_{\varphi_2})) = \varphi_1(bot(\Delta_{\varphi_1} \circ \Delta'_{\varphi_2})) = \varphi_1(G'_n) = bot(\Delta_2). \tag{2.5}$$

- The equivalence class of the concatenated and reduced graph diagram does not depend on the representative of the class. Let (Δ_1, φ_1) and (Δ_1', φ_1') be respectively equivalent to (Δ_2, φ_2) and (Δ_2', φ_2') with $\varphi_1 = \varphi_2 \circ \varphi$ and $\varphi_1' = \varphi_2' \circ \varphi'$. Then $\Pi_1 = (\Delta_1', \varphi_1') \circ (\Delta_1, \varphi_1) = (\Delta_1 \cup_{\varphi_1} \Delta_1', \varphi_1' \circ \varphi_1)$ and $\Pi_2 = (\Delta_2', \varphi_2') \circ (\Delta_2, \varphi_2) = (\Delta_2 \cup_{\varphi_2} \Delta_2', \varphi_2' \circ \varphi_2)$ are equivalent. Indeed, by hypothesis there exist isomorphic derivations for the graph diagrams (Δ_1, φ_1) and (Δ_2, φ_2) and for the graph diagrams (Δ_1', φ_1') and (Δ_2', φ_2') then the composition of these derivations produce isomorphic derivations for $(\Delta_1 \cup_{\varphi_1} \Delta_1')$ and $(\Delta_2 \cup_{\varphi_2} \Delta_2')$, see diagram below.

$$\begin{tikzcd}
top(\Delta_1) = G_0 \arrow[r, Rightarrow] \arrow[d, "\tau_0=\iota"] & G_1 \arrow[r, Rightarrow] \arrow[d, "\tau_1"] & \dots G_k = bot(\Delta_1) = top(\Delta_1') \arrow[ll, bend right, "\Delta_1"'] \arrow[r, Rightarrow] \arrow[d, "\tau_k=\varphi"] & \dots \arrow[r, Rightarrow] & G_n = bot(\Delta_1') \arrow[ll, bend right, "\Delta_1'"'] \arrow[d, "\tau_n=\varphi'"] \\
top(\Delta_2) = H_0 \arrow[r, Rightarrow] & H_1 \arrow[r, Rightarrow] & \dots H_k = bot(\Delta_2) = top(\Delta_2') \arrow[ll, bend left, "\Delta_2"] \arrow[r, Rightarrow] & \dots \arrow[r, Rightarrow] & H_n = bot(\Delta_2') \arrow[ll, bend left, "\Delta_2'"]
\end{tikzcd}$$

On the other hand, by 2.5 we have that $\varphi_1' \circ \varphi_1 : bot(\Pi_1) \to \Gamma$, $\varphi_2' \circ \varphi_2 : bot(\Pi_2) \to \Gamma$ and $\varphi_1 = \varphi_2 \circ \varphi$ and $\varphi_1' = \varphi_2' \circ \varphi'$ implies that exist $\psi : bot(\Pi_1) \to bot(\Pi_2)$ such that $\varphi_1' \circ \varphi_1 = \varphi_2' \circ \varphi_2 \circ \psi$. In fact, it is enough to consider $\psi = \varphi_2^{-1} \circ \varphi' \circ \varphi_1$ and see that $\varphi_2^{-1}(\varphi'(\varphi_1(bot(\Pi_1))) = \varphi_2^{-1}(\varphi'(bot(\Delta_1')) = \varphi_2^{-1}(bot(\Delta_2')) = bot(\Pi_2)$.

- Let be Γ_ι a graph diagram with $\iota : \Gamma \to \Gamma$ the identity map and $top(\Gamma_\iota) = bot(\Gamma_\iota) = \Gamma$. Note that $\Delta_\varphi \circ \Gamma_\iota = (\Gamma \cup_\iota \Delta)_\varphi = (\Gamma \cup_\iota G_0 \cup \dots \cup G_n)_\varphi = \Delta_\varphi$ and $\Gamma_\iota \circ \Delta_\varphi = (\Delta \cup_\varphi \Gamma)_\iota = (G_0 \cup \dots \cup G_n \cup_\varphi \Gamma)_\iota = \Delta_\varphi$. Furthermore $\Delta_\varphi \circ \Gamma_\iota$ and $\Gamma_\iota \circ \Delta_\varphi$ have cells $C_1, C_2, \dots, C_n$ and defining graphs $G_0 = \Gamma, \dots, G_n$. Therefore Γ_ι is the identity in $\mathcal{D}(\mathcal{R}, \Gamma)$.

- Consider $\Delta_\varphi, \Delta^{-1}_{\varphi^{-1}} = (\Delta \cup_\varphi \Gamma)_{\varphi^{-1}}$, and $\Delta^{-1}_{\varphi^{-1}} \circ \Delta_\varphi = (\Delta \cup_\varphi \Delta^{-1})_{\varphi^{-1} \circ \varphi}$. The last diagram results from the amalgamation of the bottom of Δ (that is, the edges that are not in the top of any cell) with the top of Δ^{-1}(that is, the bottom of Δ). We can have a better view of $\Delta^{-1}_{\varphi^{-1}} \circ \Delta_\varphi = (\Delta \cup_\varphi \Delta^{-1})_{\varphi^{-1} \circ \varphi}$ from the following diagrams. In the first diagram we calculate $\Delta^{-1}_{\varphi^{-1}} = (\Delta \cup_\varphi \Gamma)_{\varphi^{-1}}$

$$\begin{tikzcd}
\Gamma =_{\varphi^{-1}} G_n \arrow[r, Rightarrow] \arrow[d, "\varphi"] \arrow[rr, bend left, "\Delta^{-1}"] & G_{n-1} \dots \arrow[r, Rightarrow] \arrow[d, "\tau_1"] & G_0 \arrow[r, "\iota"] \arrow[d, "\tau_n"] & \Gamma \arrow[d, "\iota"] \\
\Gamma = \varphi(G_n) \arrow[r, Rightarrow] \arrow[rr, bend right, "(\Delta \cup_\varphi \Gamma)_{\varphi^{-1}}"'] & G'_{n-1} \dots \arrow[r, Rightarrow] & G'_0 \arrow[r, "\iota"] & \Gamma
\end{tikzcd}$$

In the second diagram we use the same notation as in the previous one to calculate

$\Delta^{-1}_{\varphi^{-1}} \circ \Delta_{\varphi} = (\Delta \cup_{\varphi} \Delta^{-1})_{\varphi^{-1}\circ\varphi}$

$$\begin{tikzcd}
G_0 = \Gamma \arrow[r, Rightarrow] \arrow[d, "\iota"] \arrow[rr, bend left, "\Delta"] & \dots \arrow[r, Rightarrow] & G_n =_{\varphi} \varphi(G_n) \arrow[r, Rightarrow] \arrow[d, "\tau_0^{-1} = \varphi^{-1}"] \arrow[rrr, bend left, "\Delta^{-1}"] & G'_{n-1} \arrow[r, Rightarrow] \arrow[d, "\tau_1^{-1}"] & \dots \arrow[r, Rightarrow] & G'_0 \arrow[r, "\varphi^{-1}"] \arrow[d, "\tau_n^{-1} = \varphi^{-1}"] & \Gamma \arrow[d, "\iota"] \\
G_0 = \Gamma_1 \arrow[r, Rightarrow] \arrow[rrrrr, bend right, "(\Delta \cup_{\varphi} \Delta^{-1})_{\varphi^{-1}\circ\varphi}"'] & \dots \arrow[r, Rightarrow] & G_n \arrow[r, Rightarrow] & G_{n-1} \arrow[r, Rightarrow] & \dots \arrow[r, Rightarrow] & G_0 \arrow[r, "\iota"] & \Gamma
\end{tikzcd}$$

Note that when the graphs are amalgamated each cell will eventually end up in a dipole that can be reduced leaving only the top of Δ. In fact, recall that Δ has cells $C_1, C_2, \dots C_n$ and Δ^{-1} has cells $C_n^{-1}, \dots, C_1^{-1}$. Notice that when we identify $bot(\Delta)$ with $top(\Delta^{-1}) = bot(\Delta)$ we also identify $bot(C_n)$ with $top(C'_1) = top(C_n^{-1}) = bot(C_n)$ therefore the diagram $\Delta^{-1}_{\varphi^{-1}} \circ \Delta_{\varphi}$ has cells $\{C'_i\}$

$$C'_1 = C_1, C'_2 = C_2, \dots C'_n = C_n, C'_{n+1} = C_n^{-1}, \dots, C'_{2n+2} = C_1^{-1}$$

and defining graphs $\{G'_i\}$, defined as follows

$$G'_0 = G_0 = \Gamma, G'_1 = G_1, \dots G'_n = G_n, G'_{n+1} = G_n, G'_{n+2} = G_{n-1}, \dots G'_{2n+2} = G_0 = \Gamma$$

and a dipole (C_n, C_n^{-1}). We can reduce this dipole by erasing these cells and the graph G'_n from these lists and identifying the graph G'_{n+1} with the graph G'_{n-1}. Similarly to the previous case, $bot(C_{n-1})$ is identified with $top(C'_{n+1}) = top(C^{-1}_{n-1}) = bot(C_{n-1})$ and we have again a dipole (C_{n-1}, C^{-1}_{n-1}). We can repeat this argument until we identify the graph $G'_0 = \Gamma$ with the graph $G'_{2n+2} = \Gamma$.

- Let $\Delta_{\varphi_1}, \Omega_{\varphi_2}$ and $\Lambda_{\varphi_3} \in \mathcal{D}(\mathcal{R}, \Gamma)$. Consider $((\Delta_{\varphi_1} \circ \Omega_{\varphi_2}) \circ \Lambda_{\varphi_3})$ and $(\Delta_{\varphi_1} \circ (\Omega_{\varphi_2} \circ \Lambda_{\varphi_3})) \in \mathcal{D}(\mathcal{R}, \Gamma)$. Observe that, by definition

$$(\Delta_{\varphi_1} \circ \Omega_{\varphi_2}) \circ \Lambda_{\varphi_3} = (\Omega \cup_{\varphi_2} \Delta)_{\varphi_1\circ\varphi_2} \circ \Lambda_{\varphi_3} = (\Lambda \cup_{\varphi_3} \Omega \cup_{\varphi_2} \Delta)_{\varphi_1\circ\varphi_2\circ\varphi_3}$$

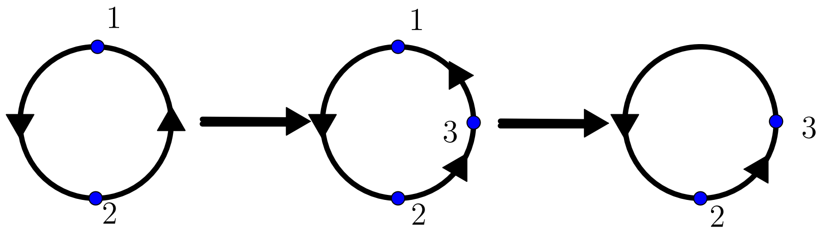
On the other hand,

$$\Delta_{\varphi_1} \circ (\Omega_{\varphi_2} \circ \Lambda_{\varphi_3}) = \Delta_{\varphi_1} \circ (\Lambda \cup_{\varphi_3} \Omega)_{\varphi_2\circ\varphi_3} = (\Lambda \cup_{\varphi_3} \Omega \cup_{\varphi_2} \Delta)_{\varphi_1\circ\varphi_2\circ\varphi_3}$$

By definition of composition we have that both diagrams have the same cells (respectively, defining graphs), that is the union of the set of cells (respectively, defining graphs) in each graph diagram is the same. Therefore $(\Delta_{\varphi_1} \circ \Omega_{\varphi_2}) \circ \Lambda_{\varphi_3}$ and $\Delta_{\varphi_1} \circ (\Omega_{\varphi_2} \circ \Lambda_{\varphi_3})$, are equal in $\mathcal{D}(\mathcal{R}, \Gamma)$.

□

Example 2.58. *Consider the following derivation*



Observe that there are two isomorphisms between the terminal graph and the initial graph of the derivation. These isomorphisms determine two different graph diagrams. First φ_1 corresponds to the isomorphism $\varphi_1(3) = 1, \varphi_1(2) = 2$, while the second corresponds to the isomorphism $\varphi_2(3) = 2, \varphi_2(2) = 1$.
In Example 3.10 we give a graph rewriting system for the Thompson group T. The graph diagrams in Figure 35 belong to this group. In fact, these elements have the same valid order, but different isomorphism between its bottom and top. In Figure 35 we use colors to indicate the isomorphism between the top (the circle inside the figure) and the bottom of the graph diagram (colored edges in the outside of the diagram).

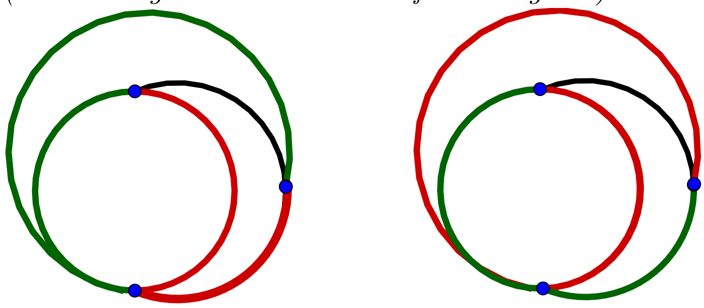


Figure 35

Remark 2.59. *We can define the product of two graphs diagrams under certain conditions. In fact, let Π_φ be a graph diagram with top$(\Pi_\varphi) = \Gamma_0$ and bot$(\Pi_\varphi) = \Gamma_1$, $\varphi : \Gamma_1 \to \Gamma_1$ an automorphism and Π'_φ be a graph diagram with top$(\Pi'_{\varphi'}) = \Gamma_1$ and bot$(\Pi'_{\varphi'}) = \Gamma_2$, $\varphi' : \Gamma_2 \to \Gamma_2$ and automorphism. We can find the concatenation $\Delta_\pi = \Pi'_{\varphi'} \circ \Pi_\varphi$ and to do this we wish to know how change the graph diagram $\Pi'_{\varphi'}$ when we identified bot(Π_φ) with top$(\Pi'_{\varphi'})$ in $\Pi'_{\varphi'} \circ \Pi_\varphi$. So, we will construct a derivation ρ with initial graph bot(Π_φ) and isomorphic to the one that is associated with the graph diagram $\Pi'_{\varphi'}$. Consider the diagram below where the first n coordinates of the first line correspond to a derivation and the map φ' correspond to the isomorphism between bot$(\Pi'_{\varphi'})$ and Γ_2. We can obtain the second line of this diagram by identifying bot(Π_φ) with top$(\Pi'_{\varphi'})$ and applying the same replacements that in the first line successively from bot(Π_φ) and defining $\psi = \varphi' \circ \tau_n^{-1}$. Notice that the derivation ρ in the diagram below, tell us how the defining graphs and cells of $\Pi'_{\varphi'}$ change when we identified top$(\Pi'_{\varphi'})$ and bot(Π_φ). Therefore the graph diagram of the derivation, Δ'_ψ is by construction the graph diagram obtained when we identify bot(Π_φ) and top$(\Pi'_{\varphi'})$.*

In the diagrams below the map ι correspond to the identity isomorphism of a graph.

$$\rho := \begin{tikzcd}
top(\Pi'_{\varphi'}) \arrow[r, Rightarrow, "\varphi_1"] \arrow[d, "\tau_0=\varphi"] \arrow[rrr, bend left, "\Pi'_{\varphi'}"] & G_1 \arrow[r, Rightarrow, "\varphi_2"] \arrow[d, "\tau_1"] & \dots \arrow[r, Rightarrow, "\varphi_n"] & bot(\Pi'_{\varphi'}) \arrow[r, "\varphi'"] \arrow[d, "\tau_n"] & \Gamma_2 \arrow[d, "\iota"] \\
bot(\Pi_{\varphi}) \arrow[r, Rightarrow, "\psi_1"] \arrow[rrr, bend right, "\Delta'_{\psi}"'] & H_1 \arrow[r, Rightarrow, "\psi_2"] & \dots \arrow[r, Rightarrow, "\psi_n"] & H_n \arrow[r, "\psi"] & \Gamma_2
\end{tikzcd}$$

So,

$$\Delta_{\psi} = \Pi'_{\varphi'} \circ \Pi_{\varphi} = (\Pi \cup_{\iota} \Delta')_{\psi} = (\Pi \cup_{\varphi} \Pi')_{\psi}. \tag{2.6}$$

Observe that this operation is associative. In fact, let be Δ_i an element of the groupoid such that $top(\Delta_i) = \Gamma_{i-1}, bot(\Delta_i) \cong_{\varphi_i} \Gamma_i$ for $i = 1, 2, 3$. Consider also the derivations,

$$\begin{aligned}
\rho_1 &:= \Gamma_0 \overset{\psi_1}{\Rightarrow} G_1 \overset{\psi_2}{\Rightarrow} G_2 \dots \overset{\psi_k}{\Rightarrow} G_k \to_{\varphi_1} \Gamma_1 \\
\rho_2 &:= \Gamma_1 \overset{\psi_{k+1}}{\Longrightarrow} G_{k+1} \overset{\psi_2}{\Rightarrow} G_{k+2} \dots \overset{\psi_l}{\Rightarrow} G_l \to_{\varphi_2} \Gamma_2 \\
\rho_3 &:= \Gamma_2 \overset{\psi_{l+1}}{\Longrightarrow} G_{l+1} \overset{\psi_2}{\Rightarrow} G_{l+2} \dots \overset{\psi_k}{\Rightarrow} G_n \to_{\varphi_3} \Gamma_3
\end{aligned}$$

such that $Diag(\rho_i) = \Delta_i$. We will prove that $\Delta_3 \circ (\Delta_2 \circ \Delta_1) = (\Delta_3 \circ \Delta_2) \circ \Delta_1$. Consider the derivation associated to $\Delta_3 \circ \Delta_2 \circ \Delta_1$,

$$\Gamma_0 \Rightarrow G_1 \Rightarrow \dots \Rightarrow G_k =_{\varphi_1} \Gamma_1 \Rightarrow G_{k+1} \Rightarrow \dots \Rightarrow G_l =_{\varphi_2} \Gamma_2 \Rightarrow G_{l+1} \Rightarrow \dots \Rightarrow G_n \to_{\varphi_3} \Gamma_3.$$

We will use 2.6 to obtain $(\Delta_3 \circ \Delta_2) \circ \Delta_1$. This is, we first do the amalgamated union of Δ_3 and Δ_2,$\Delta_2 \cup_{\varphi_2} \Delta_3$, by identifying $G_l =_{\varphi_2} \Gamma_2$ as $G_l =_{\iota} \varphi_2^{-1}(\Gamma_2) = G'_l$ and after we amalgamate this union to Δ_3. Observe that in the second derivation of the diagram below we obtain $\Delta_3 \circ \Delta_2 = (\Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \tau_n^{-1}}$.

$$\begin{tikzcd}
\Gamma_0 \arrow[r, Rightarrow] \arrow[d, "\iota"] \arrow[rr, bend left, "\Delta_1"] & \dots \arrow[r, Rightarrow] & G_k =_{\varphi_1} \Gamma_1 \arrow[r, Rightarrow] \arrow[d, "\iota"] \arrow[rr, bend left, "\Delta_2"] & \dots \arrow[r, Rightarrow] & G_l =_{\varphi_2} \Gamma_2 \arrow[r, Rightarrow] \arrow[d, "\tau_l=\varphi_2^{-1}"] \arrow[rrr, bend left, "\Delta_3"] & G_{l+1} \arrow[r, Rightarrow] \arrow[d, "\tau_{l+1}"] & \dots \arrow[r, Rightarrow] & G_n \arrow[r, "\varphi_3"] \arrow[d, "\tau_n"] & \Gamma_3 \arrow[d, "\iota"] \\
\Gamma_0 \arrow[r, Rightarrow] & \dots \arrow[r, Rightarrow] & G_k =_{\varphi_1} \Gamma_1 \arrow[r, Rightarrow] \arrow[rrrrr, bend right, "(\Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \tau_n^{-1}}"'] & \dots \arrow[r, Rightarrow] & G_l =_{\iota} G'_l \arrow[r, Rightarrow] & G'_{l+1} \arrow[r, Rightarrow] & \dots \arrow[r, Rightarrow] & G'_n \arrow[r, "\varphi_3 \circ \tau_n^{-1}"] & \Gamma_3
\end{tikzcd}$$

Moreover we can identify $G_k =_{\varphi_1} \Gamma_1$ as $G_k =_{\iota} \varphi_1^{-1}(\Gamma_1) := \widetilde{G}_k$.

$$\begin{tikzcd}
\Gamma_0 \arrow[r, Rightarrow] \arrow[d, "\iota"] & \dots G_k =_{\varphi_1} \Gamma_1 \arrow[r, Rightarrow] \arrow[d, "\tau_k=\varphi_1^{-1}"] \arrow[rrrr, bend left, "(\Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \tau_n^{-1}}"] & G_{k+1} \dots \arrow[r, Rightarrow] \arrow[d, "\tau_{k+1}"] & G'_l \arrow[r, Rightarrow] \arrow[d, "\tau'_l"] & G'_{l+1} \dots \arrow[r, Rightarrow] \arrow[d, "\tau'_{l+1}"] & G'_n \arrow[r, "\varphi_3 \circ \tau_n^{-1}"] \arrow[d, "\tau'_n"] & \Gamma_3 \arrow[d, "\iota"] \\
\Gamma_0 \arrow[r, Rightarrow] \arrow[rrrrr, bend right, "(\Delta_1 \cup_{\varphi_1} \Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \tau_n^{-1} \circ (\tau'_n)^{-1}}"'] & \dots G_k =_{\iota} \widetilde{G}_k \arrow[r, Rightarrow] & \widetilde{G}_{k+1} \dots \arrow[r, Rightarrow] & \widetilde{G}_l \arrow[r, Rightarrow] & \widetilde{G}_{l+1} \dots \arrow[r, Rightarrow] & \widetilde{G}_n \arrow[r, "\gamma"] & \Gamma_3
\end{tikzcd}$$

where $\gamma = \varphi_3 \circ \tau_n^{-1} \circ (\tau_n')^{-1}$. *We proceed in a similar way to calculate* $(\Delta_3 \circ \Delta_2) \circ \Delta_1$,

$$
\begin{tikzcd}
\Gamma_0 \arrow[r, Rightarrow] \arrow[r, bend left, "\Delta_1"] \arrow[d, "\iota"] & \ldots G_k =_{\varphi_1} \Gamma_1 \arrow[r, Rightarrow] \arrow[rr, bend left, "\Delta_2"] \arrow[d, "\tau_k = \varphi_1^{-1}"] & G_{k+1} \ldots \arrow[r, Rightarrow] \arrow[d, "\tau_{k+1}"] & G_l =_{\varphi_2} \Gamma_2 \arrow[r, Rightarrow] \arrow[rr, bend left, "\Delta_3"] \arrow[d, "\hat{\tau}_l"] & G_{l+1} \ldots \arrow[r, Rightarrow] \arrow[d, "\iota"] & G_n \arrow[r, "\varphi_3"] \arrow[d, "\iota"] & \Gamma_3 \arrow[d, "\iota"] \\
\Gamma_0 \arrow[r, Rightarrow] \arrow[rrr, bend right, "(\Delta_1 \cup_{\varphi_1} \Delta_2)_{\varphi_2 \circ \hat{\tau}_l^{-1}}"'] & \ldots G_k =_{\varphi_1} \widetilde{G}_k \arrow[r, Rightarrow] & \widetilde{G}_{k+1} \ldots \arrow[r, Rightarrow] & \widetilde{G}_l =_{\varphi_2 \circ \hat{\tau}_l^{-1}} \Gamma_2 \arrow[r, Rightarrow] & G_{l+1} \ldots \arrow[r, Rightarrow] & G_n \arrow[r, "\varphi_3"] & \Gamma_3
\end{tikzcd}
$$

Finally by the following diagram establish through equivalent derivations with identity isomorphism ι *between its initial graphs that* $\Delta_3 \circ (\Delta_1 \cup_{\varphi_1} \Delta_2)_{\varphi_2 \circ \tau_l^{-1}} = (\Delta_1 \cup_{\varphi_1} \Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \widetilde{\tau}_n^{-1}}$.

$$
\begin{tikzcd}
\Gamma_0 \arrow[r, Rightarrow] \arrow[rrr, bend left, "(\Delta_1 \cup_{\varphi_1} \Delta_2)_{\varphi_2 \circ \hat{\tau}_l^{-1}}"] \arrow[d, "\iota"] & \ldots G_k =_{\iota} \widetilde{G}_k \arrow[r, Rightarrow] \arrow[d, "\iota"] & \widetilde{G}_{k+1} \ldots \arrow[r, Rightarrow] \arrow[d, "\iota"] & \widetilde{G}_l =_{\varphi_2 \circ \hat{\tau}_l^{-1}} \Gamma_2 \arrow[r, Rightarrow] \arrow[rr, bend left, "\Delta_3"] \arrow[d, "\widetilde{\tau}_l = \hat{\tau}_l \circ \varphi_2^{-1}"] & G_{l+1} \ldots \arrow[r, Rightarrow] \arrow[d, "\widetilde{\tau}_{l+1}"] & G_n \arrow[r, "\varphi_3"] \arrow[d, "\widetilde{\tau}_n"] & \Gamma_3 \arrow[d, "\iota"] \\
\Gamma_0 \arrow[r, Rightarrow] \arrow[rrrrr, bend right, "(\Delta_1 \cup_{\varphi_1} \Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \widetilde{\tau}_n^{-1}}"'] & \ldots G_k =_{\iota} \widetilde{G}_k \arrow[r, Rightarrow] & \widetilde{G}_{k+1} \ldots \arrow[r, Rightarrow] & \widetilde{G}_l \arrow[r, Rightarrow] & \widetilde{G}_{l+1} \ldots \arrow[r, Rightarrow] & \widetilde{G}_n \arrow[r, "\varphi_3"] & \Gamma_3
\end{tikzcd}
$$

Notice that the graph $\widetilde{G}_{l+1}$ *is the same in the derivations for* $\Delta_3 \circ (\Delta_2 \circ \Delta_1)$ *and* $(\Delta_3 \circ \Delta_2) \circ \Delta_1$ *since the amalgamation of graphs is associative, so we can relate the maps* τ_i, τ_i' *with the map* $\widetilde{\tau}$. *This is,* $\widetilde{\tau}_{l+1}(G_{l+1}) = \widetilde{G}_{l+1} = \tau_{l+1}'(\tau_{l+1}(G_{l+1}))$ *implies that* $\tau_{l+1}' \circ \tau_{l+1} = \widetilde{\tau}_{l+1}$. *Moreover,* $\tau_{l+1}' \circ \tau_i = \widetilde{\tau}_i$ *for* $l < i \leq n$ *since we apply the same rules in each step. In particular,* $\tau_n' \circ \tau_n = \widetilde{\tau}_n$, *therefore* $\Delta_3 \circ (\Delta_2 \circ \Delta_1) = (\Delta_1 \cup_{\varphi_1} \Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \widetilde{\tau}_n^{-1}} = (\Delta_1 \cup_{\varphi_1} \Delta_2 \cup_{\varphi_2} \Delta_3)_{\varphi_3 \circ \tau_n^{-1} \circ (\tau_n')^{-1}} = (\Delta_3 \circ \Delta_2) \circ \Delta_1$.

3 Families of Graph Diagram Groups

In this Chapter we see some examples of graph diagram groups. Our principal results in this chapter are that right angled Artin groups, the Guba and Sapir diagram groups and rearrangement groups of fractals are all graph diagram groups.

3.1 Free Groups

Free groups are a fundamental example of diagram groups. The proof of the next theorem gives a blueprint for the proof that right angled Artin groups and diagram groups are graph diagram groups (which we will see later in this chapter).

Theorem 3.1. *Free groups are graph diagram groups.*

Proof. We will first prove that F_S with free generating set $S = \{a, b\}$ is a graph diagram group. Let the context for graphs be given by labeled graphs in the vocabulary $\Sigma = \{x, a_1, a_2, b_1, b_2\}$ and consider the following graph rewriting system $\mathcal{R}$ with base graph Γ given by a single segment labeled with the letter x.

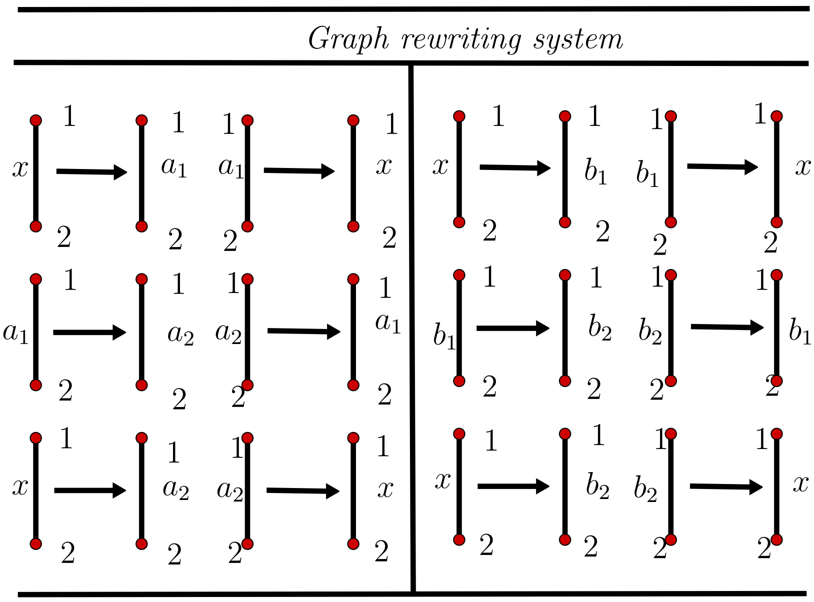


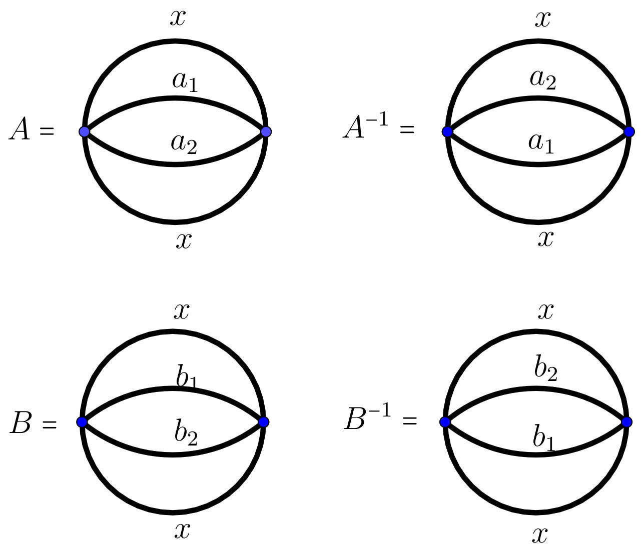
Figure 36

First note also that in Remark 2.54 we forbid rearrangement rules that are its own inverse, so $a_1 \neq a_2$ and $b_1 \neq b_2$. Also observe that $\mathcal{R}$ is reductive and symmetric, so by Theorem 2.57 $\mathcal{D}(\mathcal{R}, \Gamma)$ is a group. Moreover, the graph diagram group arising from $\mathcal{R}$ coincides with the diagram group (in the sense of Guba and Sapir) for the semigroup presentation given by

$$\mathcal{P} = \langle x, a_1, b_1, a_2, b_2 \mid x = a_1, a_1 = a_2, x = a_2, x = b_1, b_1 = b_2, b_2 = x \rangle$$

, see [1], Example 6.3.

Notice that an element in $\mathcal{D}(\mathcal{R},\Gamma)$ is a graph diagram with top and bottom Γ and cells induced by $\mathcal{R}$ and graph isomorphism the identity between the bottom and the top of each graph diagram.
Consider the following graph diagrams in $\mathcal{D}(\mathcal{R},\Gamma)$,



Consider the graph A and note that we are using the auxiliary edges labeled with a_1, a_2 to avoid a dipole reduction each time that appear consecutive copies of A in a graph diagram. In particular A^2 is reduced. On the other hand, we have that $AB \neq BA$.
We denote a cell as a pair $C = (a, b)$ where the $top(C)$ is an edge labeled with a and the $bot(C)$ is an edge labeled with the letter b.
Let $\alpha : F_S \to \mathcal{D}(\mathcal{R},\Gamma)$ be a group homomorphism naturally defined by $\alpha(a) = A, \alpha(b) = B, \alpha(a^{-1}) = A^{-1}$ and $\alpha(b^{-1}) = B^{-1}$ and extend it naturally over words in F_S.

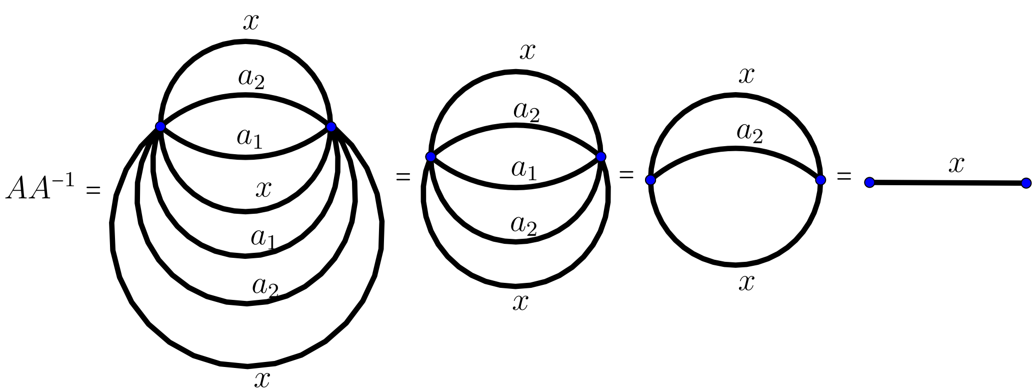


Figure 37

Observe that $\alpha(1) = \alpha(a)\alpha(a^{-1}) = \alpha(b)\alpha(b^{-1}) = \Gamma$ and $\alpha(ab) = \alpha(a)\alpha(b) = AB$.
On the other hand, α is an isomorphism.
First, note that $\ker(\alpha) = \Gamma$. Indeed, let $x_1 x_2 \ldots x_n \in F_S$ with $x_i \in \{a, b, a^{-1}, b^{-1}\}$ for $1 \leq i \leq n$. We define $\alpha(x_i) = D_{x_i} = A^{\pm 1}$ if $x_i = a^{\pm 1}$ and $\alpha(x_i) = D_{x_i} = B^{\pm 1}$ if $x_i = b^{\pm 1}$. We will proceed

by induction over n. For $n = 2$, then $\alpha(x_1x_2) = D_{x_1}D_{x_2} = \Gamma$, this is $x_2 = x_1^{-1}$. Now assume the induction hypothesis for $k < n$. Suppose that $\Delta = \alpha(x_1x_2 \dots x_n) = D_{x_1}D_{x_2} \dots D_{x_n} = \Gamma$ where $D_{x_i} \in \mathcal{D}(\mathcal{R}, \Gamma)$ for $1 \leq i \leq n$. This implies that we can reduce Δ until get the identity in $\mathcal{D}(\mathcal{R}, \Gamma)$. We claim that there exists a j such that $D_{x_j}D_{x_{j+1}} = \Gamma$. First, note that each D_{x_i} is reduced and this implies that $D_{x_1}D_{x_2} \dots D_{x_n}$ cannot have a dipole (C, D) such that $bot(C) = top(D) = y$ with $y \in \{a_1, a_2, b_1, b_2\}$. Indeed, suppose that $bot(C) = top(D) = a_1$ and observe that, by construction, the only possibilities for $top(C)$ and $bot(D)$ are a_2 or x with $bot(D) \neq top(C)$ and neither of these cases is a dipole. On the other hand, suppose that we have a dipole (C, D) such that $bot(C) = top(D) = x$ so the only possibilities for $top(C)$ and $bot(D)$ are a_1, a_2, b_1 and b_2. In each case we obtain a graph diagram D_{x_j} with $bot(C) = bot(D_{x_j}) = x$ and a graph diagram $D_{x_{j+1}}$ with $top(D) = top(D_{x_{j+1}})$ that satisfies $D_{x_j}D_{x_{j+1}} = \Gamma$. To clarify, suppose that $top(C) = a_1$, so as (C, D) is a dipole, then $bot(D) = a_1$. Since the elements of the groups are graph diagrams with top and bottom edge labeled with x, Δ must have cells E and F such that $bot(E) = top(C) = a_1$ and $top(F) = bot(D) = a_1$, so $top(E) = bot(F) = a_2$. The result follows from doing one more time the same argument to finally obtain A^{-1} with $bot(A^{-1}) = bot(C) = x$ and A with $top(A) = top(D) = x$ (see Figure 37).
Hence,

$$D_{x_1}D_{x_2} \dots D_{x_n} = D_{x_1}D_{x_2} \dots D_{x_j}D_{x_{j+1}} \dots D_{x_n} = D_{x_1}D_{x_2} \dots D_{x_{j-1}}D_{x_{j+2}} \dots D_{x_n} = \Gamma$$

Therefore,

$$\alpha(x_1x_2 \dots x_n) = \alpha(x_1x_2 \dots x_jx_{j+1} \dots x_n) = \alpha(x_1x_2 \dots x_{j-1}x_{j+2} \dots x_n) = \Gamma$$

Observe that the last term of the last equality has less than n factors, so we have that by inductive hypothesis $1 = x_1x_2 \dots x_{j-1}x_{j+2} \dots x_n = x_1x_2 \dots x_n$, since we know that $D_{x_j}D_{x_{j+1}} = \Gamma$, then $x_jx_{j+1} = 1$.
Let us now show that α is surjective. Consider a non-trivial reduced graph diagram $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma)$. It is enough to see that this diagram can be decomposed as a concatenation of copies of A, A^{-1}, B and B^{-1}. First, note that as Δ_φ is a diagram with top a segment labeled with the letter x, then its bottom must be also an edge labeled with the letter x, since φ in this case is the identity isomorphism. Note also that each time that a segment labeled with the letter x appears in Δ_φ, we can find a copy of either A, A^{-1}, B or B^{-1} after x in a reduced graph diagram.
In fact, suppose that we have a cell with top an edge labeled with x and bottom an edge labeled with b_1, denote this cell by $C_{i-1} = (x, b_1)$, then as Δ_φ is reduced the next cell with top b_1 must be $C_i = (b_1, b_2)$ and analogously the next cell must be $C_{i+1} = (b_2, x)$. Then after x we have a copy of B.
So, by the First Isomorphism Theorem, we have that F_S is isomorphic to $\mathcal{D}(\mathcal{R}, \Gamma)$.
Now we will argue why the result is true when $n > 2$. Let be $S = \{a, b, c\}$ and consider the graph rewriting system given by

the rules in Figure 36 together with the following additional rules

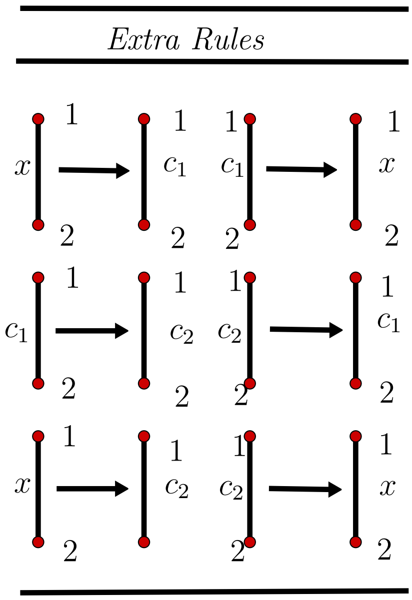


Figure 38

Notice that we can use the same technique used in the case when the free generating set has 2 elements to prove that F_S is isomorphic to $\mathcal{D}(\mathcal{R},\Gamma)$ when $S = \{a,b,c\}$. Moreover, observe that, for each element in the free generating set, we are using a modified copy of the rule in Figure 38. In fact, the rules in Figure 36 are two sets of rules similar to those rules in Figure 38. The general case where S is the free group generated by a free generating set of n elements follows from adding n copies of the rules in Figure 38 to the graph rewriting system and then use the same strategy followed when S had two generators.

□

3.2 Right Angled Artin Groups

Recall that given a right angled Artin group A we can express a presentation for this group using a **defining graph** X with vertices the generators of the group and edges indicating that two elements commute in A. Consider the graph $\widetilde{\Gamma}$ with the same set of vertices as g and set of edges determined by the following rule: for each pair of vertices $g_1, g_2 \in X$ that are not connected in X, we add an edge labeled by $a_{g_1g_2}$, but we do not add any more edges in $\widetilde{\Gamma}$. We call Γ the graph obtained from $\widetilde{\Gamma}$ by changing each isolated vertex g by a new edge $a_{g\overline{g}}$ with vertices labeled by g and $\overline{g}$. Observe that the new edges are disjoint from the other edges in Γ (see Example 3.3). Note also that if $\widetilde{\Gamma}$ has no isolated vertices $\Gamma = \widetilde{\Gamma}$ (see Example 3.2). Throughout this section Γ will be the base graph for our graph diagram groups.

Given g a generator of Γ we consider the set of edges $\{a_{gy_1}, a_{gy_2}, \dots, a_{gy_n}\}$ that are adjacent to g in Γ. We call a_g the graph consisting of all these edges. Observe that $\Gamma = \bigcup_{g \in V} a_g$ where V is the set of vertices of Γ. Note also that if g is an isolated vertex, we have that $a_g = \{a_{g\bar{g}}\}$. Given a graph a_g denote b_g the graph that results from changing to b's all the a's that appear in the graph a_g as in Figure 40. Analogously, given a graph b_g denote c_g the graph that results from changing to c's all the b's that appear in the graph b_g. We assume that

$$a_{gy_1} = a_{y_1 g}, \dots, a_{gy_n} = a_{y_n g} \text{ but all the other labels are different.} \tag{3.1}$$

This means for example that $b_{y_n g} \neq b_{gy_n}$ and $c_{y_n g} \neq b_{y_n g}$. Consider the graph rewriting system $\mathcal{R}$ with replacement rules given by $r_1 = (a_g \to b_g)$, $r_2 = (b_g \to c_g)$ and $r_3 = (c_g \to a_g)$ together with the inverse rules $b_g \to a_g$, $c_g \to b_g$ and $a_g \to c_g$ where the vertices are the boundaries and the isomorphism consists in identifying the vertices with the same label. Similarly to what we did for the free groups we will denote the cells as (t_g, m_g) with top t_g and bottom m_g where $t, m \in \{a, b, c\}$ and g is a generator of A. For example in Figure 41 we have a cell (a_x, b_x).
Let A_x be the diagram of the derivation

$$\rho := G_0 = \Gamma \xRightarrow{\varphi_1, r_1} G_1 \xRightarrow{\varphi_2, r_2} G_2 \xRightarrow{\varphi_3, r_3} G_3$$

that is, the graph diagram with initial graph Γ and cells $C_1 = (a_g, b_g), C_2 = (b_g, c_g)$ and $C_1 = (c_g, a_g)$. See Figure 42 and Figure 46.
In this case we also use some extra labeled edges as in the case of the free groups.

Example 3.2. *Consider the presentation* $A = \langle x, y, z, w \mid xy = yx, yz = zy, zw = wz \rangle$. *Notice that in this case* $\Gamma = \widetilde{\Gamma}$.

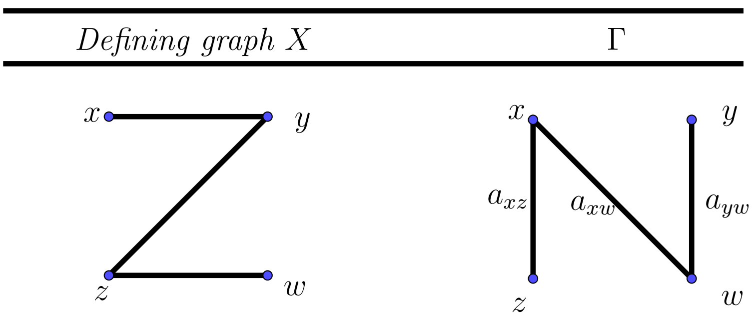


Figure 39

As in 3.1, we assume that $a_{xz} = a_{zx}, a_{xw} = a_{wx}$ *and* $a_{yw} = a_{wy}$ *but for example* $b_{xz} \neq b_{zx}, b_{xw} \neq b_{wx}$ *and* $b_{yw} \neq b_{wy}$ *and* $c_{xz} \neq c_{zx}, c_{xw} \neq c_{wx}$ *and* $c_{yw} \neq c_{wy}$. *We have these conditions to avoid forbidden relations as in Remark 2.54 and to avoid reductions when we do the product of* *some of A_g's with g a generator of A. In this case a_x and b_x are as follows:*

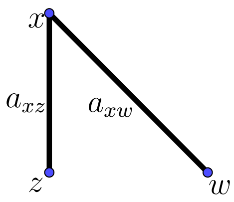
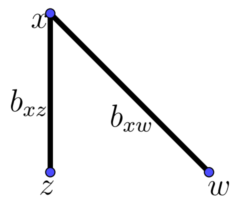
a_x	b_x
	

Figure 40

In this case the amalgamation of $a_x \cup_\varphi b_x$ is the graph,

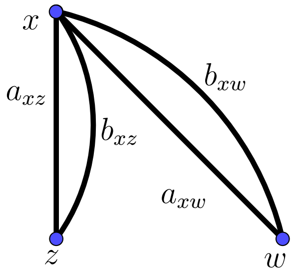


Figure 41

Therefore, given

$$\rho \coloneqq G_0 = \Gamma \xrightarrow{\varphi_1, r_1} G_1 \xrightarrow{\varphi_2, r_2} G_2 \xrightarrow{\varphi_3, r_3} G_3$$

and $A_x = Diag(\rho)$ is the following graph diagram,

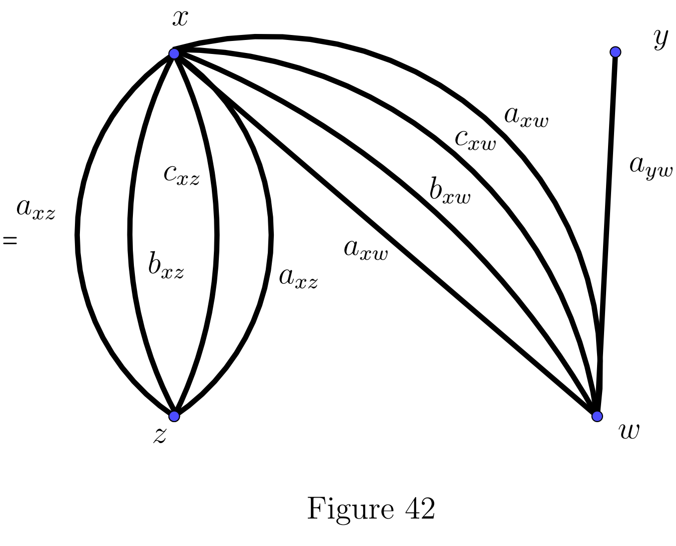
$A_x =$ 

Figure 42

the graph A_x has top and bottom the graph diagram Γ. On the other hand consider

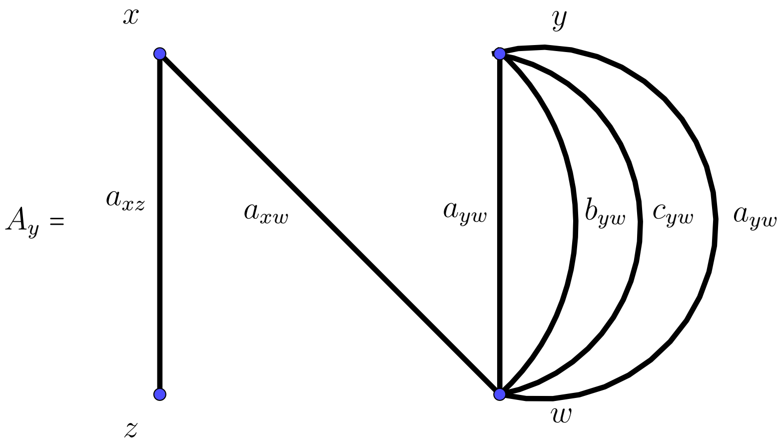


Figure 43

Notice that a_x and b_y do not have common edges, implying that $A_xA_y = A_yA_x$ (see Figure 44).

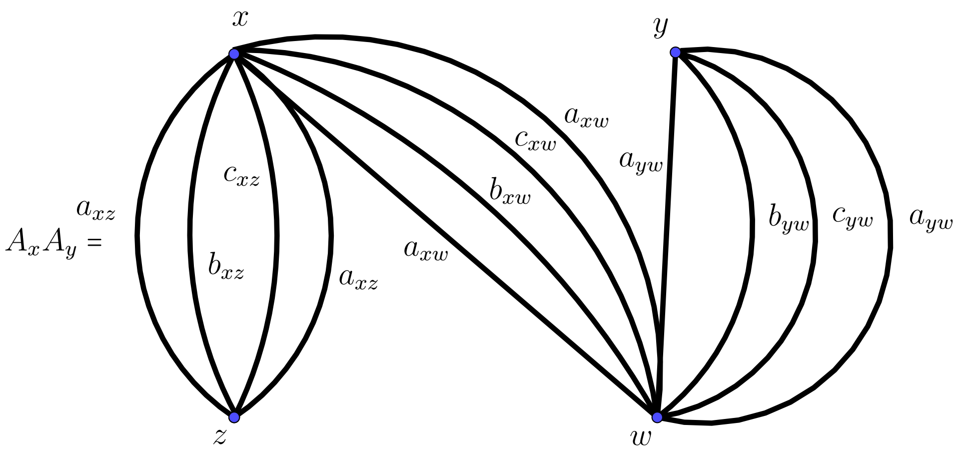


Figure 44

Example 3.3. *Let be $A = \langle x, y, z \mid xy = yx, xz = zx, yz = zy\rangle$. In this case we define Γ in the following way: for each element g that commutes with all the other elements in A we create an edge $a_{g\overline{g}}$ in Γ.*

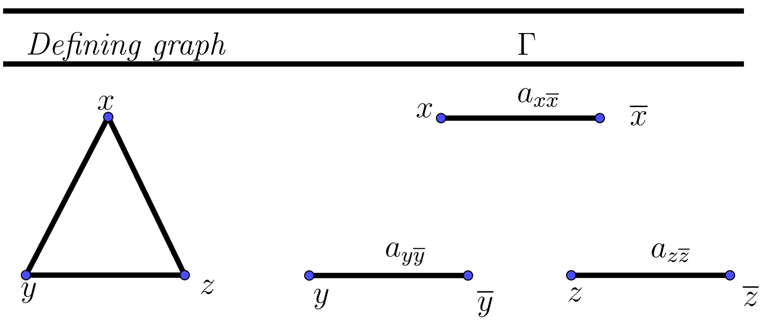


Figure 45

In this case we define a_x as a single edge labeled with $a_{x\bar{x}}$ and A_x is the graph diagram of the derivation ρ defined in a way analogous to Example 3.2.

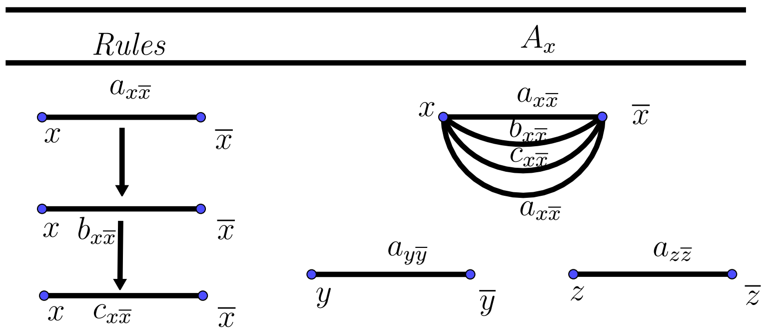


Figure 46

Note also that similarly to Theorem 3.1 the graph diagram formed by the initial graph Γ and cells $(a_x, b_x), (b_x, c_x)$ and (c_x, a_x) generate a free group and that A_x has top and bottom the diagram Γ. Observe that a similar analysis can be done for each A_g with g a generator of A.

Guba and Sapir find that many right angled Artin groups were already diagram groups [16]. In the next results we manage to prove that the whole family of right angled Artin groups are graph diagram groups.

Theorem 3.4. *Every right angled Artin group A can be seen as a graph diagram group.*

Proof. Let A be a right angled Artin group, Γ be the base graph and $\mathcal{R}$ be the graph rewriting system given by the rules $r_g = (a_g \to b_g)$, $s_g = (b_g \to c_g)$ and $t_g = (c_g \to a_g)$ together with the inverse rules $(b_g \to a_g)$, $(c_g \to b_g)$ and $(a_g \to c_g)$ for each $g \in A$. Note that, by construction, $\mathcal{R}$ is reductive and symmetric and so, by Theorem 2.57, we have that $\mathcal{D}(\mathcal{R}, \Gamma)$ is a group. Given S a set of generators of A we define $\widetilde{\alpha}$ that maps a generator g of A to the graph A_g and g^{-1} to the graph A_g^{-1}, so we set $A_{g^{-1}} := A_g^{-1}$ if $g \in S$. We can extend this to $\widetilde{\alpha} : F_S \to \mathcal{D}(\mathcal{R}, \Gamma)$ group homomorphism from the free group F_S and such that $\widetilde{\alpha}(g_1 g_2 \dots g_n) = A_{g_1} A_{g_2} \dots A_{g_n}$.
First we will prove the following statement.
Suppose that $g_1 \neq g_2$. Then g_1 commutes with g_2 if and only if A_{g_1} commute with A_{g_2}.
Assume $g_1 g_2 = g_2 g_1$ and recall that Γ is a graph that has edges between the non commuting elements of A (except for the edges with initial and final vertex with the same label). Then $g_1 g_2 = g_2 g_1$ if and only if a_{g_1} and a_{g_2} do not have edges in common in Γ. Note that the last statement is equivalent to say that the relations that produce A_{g_1} are sequentially independent of those that produce A_{g_2} which means that A_{g_1} commute with A_{g_2}.
Moreover, if $g_1 \neq g_2$ and $g_1 g_2 \neq g_2 g_1$, we have that A_{g_1} does not commute with A_{g_2}. Indeed, observe that $A_{g_1} A_{g_2}$ and $A_{g_2} A_{g_1}$ are reduced graph diagrams with different sets of cells.
By Von Dyck's Theorem, the map $\widetilde{\alpha}$ induces a group homomorphism $\alpha : A \to \mathcal{D}(\mathcal{R}, \Gamma)$.

We will see that $\ker(\alpha) = 1$. We proceed by induction over the number of generators in the factorization of $g \in A$. If $g = g_1 g_2$ and $\alpha(g_1 g_2) = A_{g_1} A_{g_2} = \Gamma$ we have that $A_{g_2} = A_{g_1^{-1}}$ which means that $g_2 = g_1^{-1}$.
Suppose that $\alpha(g_1 g_2 \dots g_n) = A_{g_1} A_{g_2} \dots A_{g_n} = \Gamma$ where $A_{g_j} \in \mathcal{D}(\mathcal{R}, \Gamma)$ for $j \in \{1, \dots, n\}$. Note that if $g_i^{-1} \neq g_j$ then $A_{g_i} A_{g_j}$ is reduced. In fact the cells of A_{g_i} and A_{g_j} can not produce dipoles by condition 3.1. Therefore a dipole (C, D) for $A_{g_1} A_{g_2} \dots A_{g_n}$ must have $bot(A_{g_j}) = bot(C) = top(D) = top(A_{g_k})$.
$\Delta_\varphi = A_{g_1} A_{g_2} \dots A_{g_n} = \Gamma$ implies that we can reduce Δ_φ until get the identity of $\mathcal{D}(\mathcal{R}, \Gamma)$. In particular at the beginning of such reduction we can reduce A_{g_j} with A_{g_k} for some $1 \leq j < k \leq n$, this is $A_{g_j} A_{g_k} = \Gamma$. We will prove first that, given cells C, D of a of Δ_φ such that $bot(C) = top(D)$, then we can find a valid order in which these cells appear at consecutive positions. In fact, note that the boundaries in each cell of the valid order are given by vertices since all the rules in the graph rewriting system has vertices as its boundaries satisfy that condition while the interior of all cells are given by the edges. In fact, note that all the vertices of a cell C are generators of A (or auxiliary vertices $\overline{g}$) and that, by construction, they are boundary points of $top(C)$ and $bot(C)$. On the other hand, by Definition 2.19, the interiors of $top(C)$ and $bot(C)$ cannot be empty and therefore must be given by the edges. Thus, in this case for a graph to overlap with another, they must share at least one edge. We claim that given a valid order $C_1, \dots C_j = C, \dots C_l, \dots, C_k = D, \dots, C_n$ for Δ_φ such that $bot(C) = top(D)$, then $bot(C_l)$ and $top(D)$ are non- overlapping. Note that by Lemma 2.5.2 $bot(C_l)$ and $bot(D)$ have disjoint interior, moreover $\partial bot(C) = \partial top(D)$ can not intersect the interior of $bot(C_l)$ since $\partial top(D)$ is formed by a set of vertices while the interior of $bot(C_l)$ is formed by edges. By the same argument $\partial bot(C_l)$ can not intersect the interior of $bot(D)$, that is, $bot(C_l)$ and $top(D)$ are not overlapping. Notice that by Lemma 2.38 we can permute D with all the cells C_l such that $j < l < k$, therefore we can find a valid order for Δ_φ such that C and D are consecutive cells.
Hence, given a valid order for $A_{g_1} A_{g_2} \dots A_{g_n}$ we can first obtain another valid order where the dipole (C, D) appears in consecutive cells and $bot(C) = bot(A_{g_j}) = top(A_{g_k}) = top(D)$. Then we can apply the same strategy again until we obtain a valid order where the cells of A_{g_j} and A_{g_k} appear in consecutive positions. This implies that $A_{g_j} A_{g_k} = \Gamma$. In fact suppose that the last cell of A_{g_j} is (b_g, a_g), then the first dipole of A_{g_k} is (a_g, b_g) and the only A_g's that have that cell in its top and bottom are A_g^{-1} and A_g respectively.
Then we can find a valid order for Δ_φ such that

$$\begin{aligned} A_{g_1} A_{g_2} \dots A_{g_n} &= A_{g_1} A_{g_2} \dots A_{g_j} A_{g_k} A_{g_{j+1}} \dots A_{g_{k-1}} A_{g_{k+1}} \dots A_{g_n} \\ &= A_{g_1} A_{g_2} \dots A_{g_{j-1}} A_{g_{j+1}} \dots A_{g_{k-1}} A_{g_{k+1}} \dots A_{g_n} = \Gamma \end{aligned}$$

So we have that,

$$\begin{aligned} \alpha(g_1 g_2 \dots g_n) &= \alpha(g_1 g_2 \dots g_j g_k g_{j+1} \dots g_{k-1} g_{k+1} \dots g_n) \\ &= \alpha(g_1 g_2 \dots g_{j-1} g_{j+1} \dots g_{k-1} g_{k+1} \dots g_n) = \Gamma \end{aligned}$$

The last equation has less than n factors, so by inductive hypothesis this implies that $g_1 g_2 \dots g_n = 1$.
We will prove that α is surjective. Consider a reduced element $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma)$ and note that

$$\Delta_\varphi = A_{g_1} A_{g_2} \dots A_{g_n} \text{ where } g_1, g_2 \dots, g_n \text{ are either generators of } A \text{ or their inverses.}$$

In fact, since Γ can be written as union of a_{g_i}'s where g_i are generators of A, it is enough to show that each time that we have a copy of a_g in $\mathcal{D}(\mathcal{R}, \Gamma)$, then either a_g is not the top of a cell, or there is a copy of A_g or A_g^{-1} in Δ_φ.
Note first that similarly to the argument that we did for the free groups, a graph diagram has bottom diagram isomorphic to Γ and this isomorphism must preserve the context for graphs (in this case labeled graphs).
Secondly, in a reduced graph diagram it holds that each time that there is a copy of a_g in Δ_φ that is the top of a non-trivial cell we must have b_g, c_g and a_g as the tops and bottoms of the subsequent cells. For example, if we have a cell $C = (a_g, c_g)$ in Δ_φ, then immediately after such cell we will have cells $D = (c_g, b_g)$, and $E = (b_g, a_g)$ that satisfying $C \prec D \prec E$ any without any other cell between them, because no other relations with support c_g and b_g can be applied. Therefore in this case we have a copy of A_g^{-1} after a_g.
Thus, by the First Isomorphim Theorem, we have that A is isomorphic to $\mathcal{D}(\mathcal{R}, \Gamma)$. □

3.3 Diagram Groups

We already worked with several diagram groups. For instance free groups, Thompson's group F and Example 3.2 are part of this family. These examples allow us to provide a graph rewriting system $\mathcal{R}$ and a base graph Γ for $\mathcal{D}(\mathcal{R}, \Gamma)$ from the classical rewriting system $\mathcal{G}$ arising from the semigroup presentation $\mathcal{P}$ and the initial word w of the diagram group $\mathcal{D}(\mathcal{P}, w)$. Indeed, given a word v let $l(v)$ be the linear graph with all the edges directed from left to right and labeled with the letters of v. Moreover, if $v = x_1 x_2 \dots x_n$ we say that the initial vertex of $l(v)$ is the leftmost vertex of $l(x_1)$ and the final vertex of $l(v)$ is the rightmost vertex of $l(x_n)$. Then, given a diagram group $D(\mathcal{P}, w)$, we define the initial graph Γ as $l(w)$ and for each rule $r = s$ in $\mathcal{G}$ we will add to $\mathcal{R}$ the rules $l(r) \to l(s)$ and $l(s) \to l(r)$ where the boundaries of these rules map the initial (final) point of $l(r)$ to the initial (final) point of $l(s)$. Therefore is straightforward the following result

Theorem 3.5. *Consider the semigroup presentation $\mathcal{P} = \langle \Sigma \mid \mathcal{R}' \rangle$ and the graph rewriting system $\mathcal{R}$ obtained from $\mathcal{R}'$ as in the paragraph above. Then $\mathcal{D}(\mathcal{P}, w) \cong \mathcal{D}(\mathcal{R}, \Gamma)$.*

3.4 The Rearrangement Group of Fractals

In the current section we will show that the family of the rearrangement group of fractals is contained in the family of the graph diagram groups. In order to do that we will follow the next steps:

1. We will show how each element in a rearrangement group of fractals that we denote by $\mathcal{G}(\mathcal{R}', G_0)$ can be used to obtain a graph diagram $\mathcal{D}(\mathcal{R}, G_0)$.

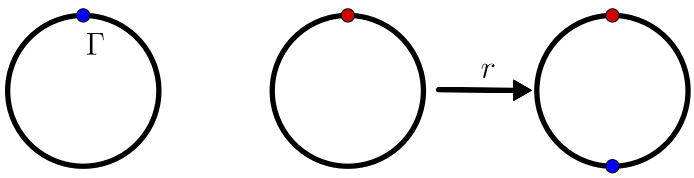
2. We show how a replacement system of the rearrangement group of fractals can be used to obtain a graph rewriting system such that the groups generated by these are isomorphic.

To explain the first point recall that given an element in a rearrangement group $\mathcal{G}(\mathcal{R}', G_0)$ we can describe this element using a graph pair diagram, see Definition 1.44 and Remark 1.45. Therefore, this element can be obtained by applying simple expansions to the initial graph G_0 until we get the domain and rank of the rearrangement. Notice that each application of the replacement rule to a graph G_i produces a new graph G_{i+1}. On the other hand, each replacement rule of the rearrangement group of fractals $(e \to R)$ (where u, v are the vertices of e and $\overline{u}, \overline{v}$ are distinguished vertices in R) also **induces** a replacement rule for the graph rewriting system given by $r = (e, R, \nu)$ and $r^{-1} = (R, e, \nu^{-1})$ where e, R are graphs with boundary $\{u, v\}$ and $\{\overline{u}, \overline{v}\}$ respectively, and ν identifies the vertex u with the vertex $\overline{u}$ and the vertex v with the vertex $\overline{v}$. A rule $r_i = (e_i, R_i, \nu_i)$ induces a cell $C_i = (e_i, R_i)$ and a rule $r_i^{-1} = (R_i, e_i, \nu_i^{-1})$ induces a cell $C_i^{-1} = (R_i, e_i)$. This motivates the following definition:

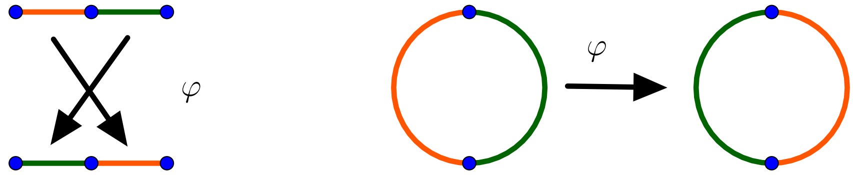
Definition 3.6. *Consider a graph pair diagram $(G_k, \widetilde{G}_k, \varphi)$ for the rearrangement f and diagrams $\Delta(G_k)$ and $\Delta(\widetilde{G}_k)$ given by $G_0, G_1, \dots G_k$ and $G_0 = \widetilde{G}_0, \widetilde{G}_1, \dots \widetilde{G}_k$ and $C_1, C_2, \dots C_k$ and $\widetilde{C}_1, \widetilde{C}_2, \dots \widetilde{C}_k$ as their respective cells. Here each G_{i+1} and $\widetilde{G}_{i+1}$ is obtained from G_i and $\widetilde{G}_i$, respectively by applying a replacement rule in $\mathcal{R}'$. We define a graph diagram $\Delta(f)$ for f as $\Delta^{-1}(\widetilde{G}_k) \circ_\varphi \Delta(G_k)$.*

Observe that, by definition of concatenation, we have that $\Delta(f)$ has defining graphs $G_0, G_1, \dots G_k = \widetilde{G}_k, \widetilde{G}_{k-1}, \dots G_0$ and cells $C_1, C_2, \dots C_k, \widetilde{C}_k^{-1}, \widetilde{C}_{k-1}^{-1}, \dots \widetilde{C}_1^{-1}$. Recall that in Remark 2.59 we explain how to make the product of $\Delta^{-1}(G_k)_\iota$ and $\Delta(G_k)_\varphi$.

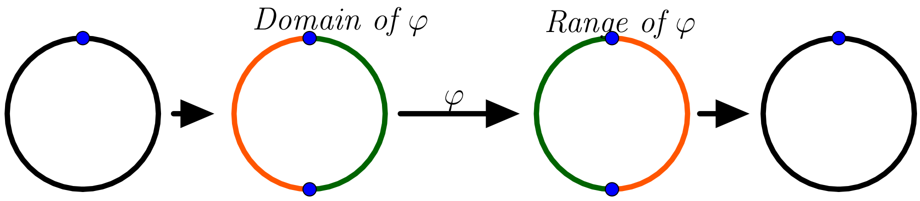
Example 3.7. *To illustrate this construction suppose that the replacement system for $\mathcal{G}(\mathcal{R}', G_0)$ is determined by the initial graph $\Gamma = G_0$ and the rule r as follows:*



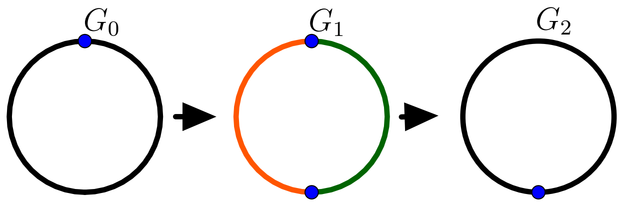
We want to see which diagram is produced by the following rearrangement φ.



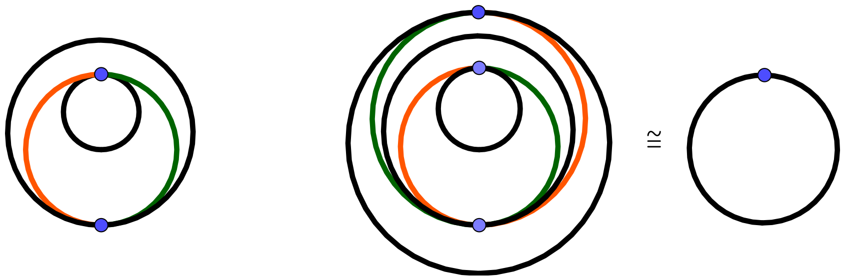
In order to create a derivation for this rearrangement, first we produce the domain and the range using the replacement rules induced by the cells as follows:



Then, we use φ^{-1} to identify the domain and the range of φ and we get the following derivation



In the next picture we obtain the graph diagram induced by this derivation and the one induced by its square.



Note that Δ^2, and φ^2 are the identity elements of their respective groups.

Given a replacement system $(\mathcal{R}', G_0)$ of the rearrangement group of fractals, we will produce a graph rewriting system $\mathcal{R}$. In order to do that, it is necessary to understand the differences between the rules in $\mathcal{R}'$ and rules in $\mathcal{R}$. For example, in the case of $\mathcal{R}'$ we can apply a replacement over any edge (including a loop) while in $\mathcal{R}$ we need a boundary preserving isomorphism that by definition is a graph isomorphism.

Remark 3.8. ***How to produce a graph rewriting system $\mathcal{R}$ from a replacement system $\mathcal{R}'$:*** *The replacement system of the rearrangement group* ***induces*** *a graph rewriting* *system with simple expansions playing the role of partial isomorphisms. Sometimes we need to add a finite number of extra rules to produce the same graphs with $\mathcal{R}$ and $\mathcal{R}'$. Indeed, in general, it is enough to consider rules that can be obtained from applying the replacement* $(e \to R)$ *to the edges of* R *and to the edges of the initial graph* Γ *as in Example 3.11. We also need to add the inverse of each of these rules to produce a symmetric graph rewriting system. In fact, this makes sense since all the edges in the limit space are obtained by successively applying replacement rules on the initial graph or in copies of the graph* R.

Definition 3.9. *Given a replacement system* $\mathcal{R}'$ *of the rearrangement group of fractals, we call* $\mathcal{R}$ *the graph rewriting system induced by* $\mathcal{R}'$ *described in Remark 3.8.*

We will clarify this in the following examples.

Example 3.10. *Consider the graph rewriting system given in [2] Belk and Forrest gave a replacement system for* $F \rtimes \mathbb{Z}_2$*, this replacement consist of the base graph* Γ *and the rule* r_1 *in Figure 47. Notice that the replacement system is reductive since the only automorphism of* e *and* S *that fix* ∂e *and* ∂S *pointwise is the identity automorphism and following Remark 3.8 we add the inverse replacement to obtain a graph rewriting system for* $F \rtimes \mathbb{Z}_2$.

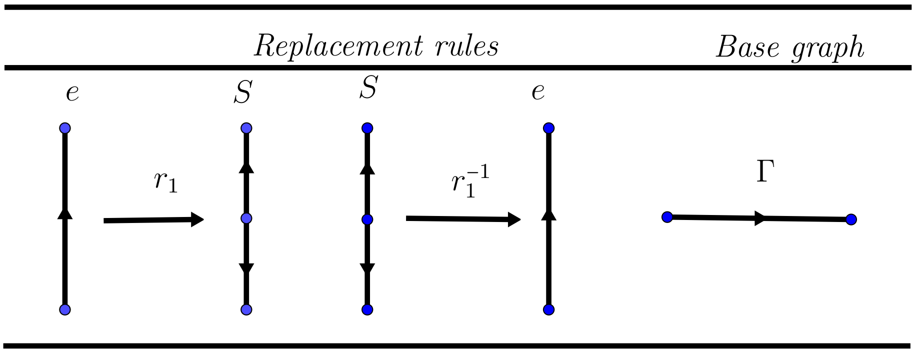


Figure 47 – A graph rewriting system for $F \rtimes Z_2$

Example 3.11. *Consider the rules* r_1, r_2 *in Figure 49 and the graphs* Γ *and* Γ_1 *in Figure 48. Observe that given a graph rewriting system* $\mathcal{R}' = \{r_1\}$ *we cannot apply the rule* r_1 *to the graph* Γ *and obtain* Γ_1*, since we need a boundary preserving isomorphism to apply this rule and there is not even a graph isomorphism. However seen as a rule from a rearrangement group we can apply the rule to* Γ *and obtain the graph* Γ_1.

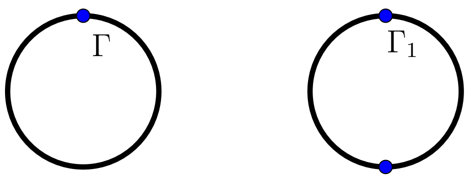


Figure 48

Moreover, observe that the replacement system given by $\mathcal{R}' = \{r_1\}$ *and the initial graph* Γ *is enough to produce the Thompson group* T *as a rearrangement group of fractals and* *we need $\mathcal{R} = \{r_1, r_2, r_1^{-1}, r_2^{-1}\}$ and the initial graph Γ to produce the same group as graph diagram group.*

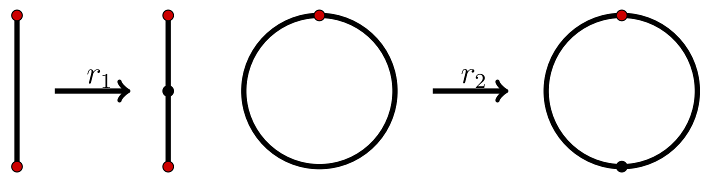


Figure 49

If the graph rewriting system induced by the replacement rule of a rearrangement group is reductive and symmetric, then we have that by Theorem 2.57 that we can consider the graph diagram group it generates.
Note that the associated derivation does not depend on the order of how we construct the graph pair diagram. In fact, suppose that we have two constructions of the graph pair diagram using the same number of simple expansions to get (G_k, G'_k, φ). In this case we obtain the graph pair diagram applying the same rules in a different order, this means that both derivations are transpositions of each other and, by Theorem 2.39. their diagrams are isomorphic.
The graph pair diagram obtained from a rearrangement is not unique. In fact, given (G_i, G'_i, φ) and e is an edge of G_i, then $(G_i \triangleleft e, G'_i \triangleleft \varphi(e), \varphi')$ is another graph pair diagram for the same rearrangement, where φ' coincide with φ in $G_i - \{e\}$ and maps $e\epsilon$ to $\varphi(e)\epsilon$ for every ϵ in R. However, the next proposition proved by Belk and Forrest in [2] guarantees that each graph diagram has a unique reduced element. We will show that two rearrangements with the same reduced graph pair diagram have equivalent graph diagram.

Proposition 3.12. *Every rearrangement has a unique reduced graph pair diagram.*

We will prove that rearrangement groups of fractals are contained in the family of the graph diagram groups. For clarity, we will see the following cases and in all of them $\mathcal{R}$ will be a symmetric graph rewriting system

1. first show the result in Theorem 3.13 in the case when the graph rewriting system $\mathcal{R}$ induced by $\mathcal{R}'$ is reductive,
2. then in Theorem 3.14 we explain that almost the same proof works for colored rewriting systems, and
3. finally we explain how to modify the proof in the case when $\mathcal{R}$ is not reductive.

Theorem 3.13. *If the graph rewriting system $\mathcal{R}$ induced by $\mathcal{R}'$ is reductive and symmetric, then the rearrangement group $\mathcal{G}(\mathcal{R}', \Gamma)$ is isomorphic to $\mathcal{D}(\mathcal{R}, \Gamma)$.*

Proof. First off, note that, by construction, we can obtain the same graphs from the replacement system $(\mathcal{R}', \Gamma)$ and with the graph rewriting system $(\mathcal{R}, \Gamma)$. Also note that by Theorem 2.57 $\mathcal{D}(\mathcal{R}, \Gamma)$ is a group since $\mathcal{R}$ is a reductive and symmetric graph rewriting system.

Consider a map $\alpha : \mathcal{G}(\mathcal{R}', \Gamma) \to \mathcal{D}(\mathcal{R}, \Gamma)$ that maps each rearrangement f to the graph diagram $\Delta(f)$. We will prove that this application is well defined, bijective and preserves the product of $\mathcal{G}(\mathcal{R}', \Gamma)$.

1. The map α does not depend on the valid orders used to obtain the reduced graph pair diagram. In fact, given two valid orders for $\Delta(f)$ associated to the same reduced graph pair diagram, we know that these produce isomorphic graph diagrams by Theorem 2.39.

2. Two equivalent elements in $\mathcal{G}(\mathcal{R}', \Gamma)$ produce equivalent graph diagrams. It is enough to prove that given $f, g \in \mathcal{G}(\mathcal{R}', \Gamma)$ such that, if the graph pair diagram of f is obtained from the graph pair diagram of g by applying one single reduction, then their diagrams are equivalent. Suppose that f has a graph pair diagram $(G_k, \widetilde{G}_k, \varphi)$ and g has graph pair diagram $(G_{k+1}, \widetilde{G}_{k+1}, \varphi')$ where G_{k+1} is obtained from G_k by applying a simple expansion $(e \to R)$ in an edge f of G_k and $\widetilde{G}_{k+1}$ is obtained from $\widetilde{G}_k$ by applying the same rule on the edge $\varphi(e)$ of $\widetilde{G}_k$.

   In this case $\Delta(f)$ has defining graphs and cells given as follow

   $$G_0 = \Gamma, G_1, \ldots, G_k = \widetilde{G}_k, \ldots, \widetilde{G}_0 = \Gamma \text{ and cells } C_1, \ldots, C_k, \widetilde{C}_k^{-1}, \ldots \widetilde{C}_1^{-1}$$

   and $\Delta(g)$ has defining graphs and cells given by

   $$G_0, G_1, \ldots, G_k, G_{k+1} = \widetilde{G}_{k+1}, \widetilde{G}_k, \ldots, \widetilde{G}_0 \text{ and } C_1, \ldots, C_k, C_{k+1}, \widetilde{C}_{k+1}^{-1}, \ldots, \widetilde{C}_1^{-1}$$

   Observe that, when G_{k+1} and $\widetilde{G}_{k+1}$ are identified, also $R = bot(C_{k+1})$ and $R = top(\widetilde{C}_{k+1}^{-1}) = bot(\widetilde{C}_{k+1}^{-1})$ are identified. Then (C_{k+1}, C_{k+1}^{-1}) is a dipole and when we reduce it, we eliminate G_k, G_{k+1} and C_k, C_{k+1} from the list of $\Delta(g)$ and we obtain the list of defining graphs and cells of $\Delta(f)$. Thus $\Delta(f)$ is equivalent to $\Delta(g)$.

3. Observe that α is a homomorphism that satisfies $ker(\alpha) = \Gamma$. In fact, let f and g be elements in $\mathcal{G}(\mathcal{R}', \Gamma)$ and suppose that $g \circ f$ is well defined. Then, we can write the graph pair diagrams for f and g as $(E_1, E_2, \varphi), (E_2, E_3, \varphi')$ respectively. By Definition 3.6, $\Delta(f) = \Delta(E_2)^{-1} \circ_\varphi \Delta(E_1)$ and $\Delta(g) = \Delta(E_3)^{-1} \circ_{\varphi'} \Delta(E_2)$. Then

   $$\begin{aligned}\Delta(g) \circ_\iota \Delta(f) &= \Delta(E_2)^{-1} \circ_\varphi \Delta(E_1) \cup_\iota \Delta(E_3)^{-1} \circ_{\varphi'} \Delta(E_2) \\ &= \Delta(E_1) \cup_\varphi (\Delta(E_2)^{-1} \cup_\iota \Delta(E_2)) \cup_{\varphi'} \Delta(E_3)^{-1} \\ &= \Delta(E_1) \cup_\varphi \Gamma \cup_{\varphi'} \Delta^{-1}(E_3) \\ &= \Delta(E_1) \cup_{\varphi' \circ \varphi} \Delta^{-1}(E_3) = \Delta(g \circ f)\end{aligned}$$

On the other hand, assume that $\alpha(f) = \Gamma$ with $(G_k, \widetilde{G}_k, \varphi)$ a graph pair diagram for f. So,

$$\alpha(f) = \Gamma \Rightarrow \Delta(\widetilde{G}_k) \circ_\varphi \Delta^{-1}(G_k) = \Gamma \Leftrightarrow G_k = \widetilde{G}_k,$$

thus f is the identity rearrangement.

4. On the other hand, α is surjective. Indeed, let $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma)$ and let $\mathcal{R} = \{\mathcal{R}^+ \cup \mathcal{R}^-\}$ where $\mathcal{R}^+$ is the set of replacement rules associated with simple expansions, $(e \to R)$ and $\mathcal{R}^-$ is the set of replacement rules associated with the inverses of simple expansions. We will use the replacements in $\mathcal{R}^+$ to produce a graph diagram $\Delta(G_k)$ and the replacements in $\mathcal{R}^-$ to produce a graph diagram $\Delta(\widetilde{G}_j)$ in such a way that $(G_k, \widetilde{G}_j, \tau)$ will be a graph pair diagram for f that satisfies $\alpha(f) = \Delta_\varphi$.

   Δ_φ arises from a valid order with cells $C_1, C_2, \ldots, C_n$, and defining graphs $G_0, G_1, \ldots, G_n$.

   We will show that we can find a valid order for Δ_φ such that we have first all of the cells (e, R) induced by $r \in \mathcal{R}^+$ and, after, all of the cells (R, e) obtained from rules $r \in \mathcal{R}^-$. It is enough to show that we can transpose two consecutive cells $C_j = (R_1, e_1)$ and $C_{j+1} = (e_2, R_2)$ in this order, where e_1, e_2 are isomorphic to e and R_1, R_2 are isomorphic to R. Indeed, let $C_j = (R_1, e_1)$ and $C_{j+1} = (e_2, R_2)$, by hypothesis Δ_φ is reduced, so $e_1 \neq e_2$ and the replacements that induce these cells are sequentially independent. In fact note that $bot(C_1) = e_1$ does not intersect the interior of $top(C_2) = e_2$ and $top(C_2) = e_2$ does not intersect the interior of $bot(C_1) = e_1$, so $top(C_2)$ and $bot(C_1)$ are non overlapping by Lemma 2.38 the order $C_1, C_2, \ldots, C_{j+1}, C_j, \ldots C_n$ is a valid order for Δ_φ.

   We proceed now to define $\Delta(G_k)$ and $\Delta(\widetilde{G}_j)$ and therefore the graph pair diagram $(G_k, \widetilde{G}_j, \tau)$. Notice that we wish that both graph diagrams have top given by the graph G_0. Firstly suppose that the C_i's are the cells induced by $r_i \in \mathcal{R}$ and that $r_1, \ldots, r_k \in \mathcal{R}^+$ and $r_{k+1}, \ldots, r_n \in \mathcal{R}^-$. Then we can produce a sequence of graphs G_i applying the rule r_i over the graph G_{i-1}. Moreover, note that $G_0 = \varphi(G_n)$ and that we can apply the rule r_n^{-1} to the graph $\varphi(G_n)$ and obtain a graph $\widetilde{G}_1$ in such a way that $G_{n-1} \cong \widetilde{G}_1$, then we apply the rule r_{n-1} to the graph $\widetilde{G}_1$ in such a way that we obtain a graph $\widetilde{G}_2$ that satisfies $G_{n-2} \cong \widetilde{G}_2$ and we follow this process until we get

$$\begin{tikzcd}
G_n \arrow[r, Rightarrow, "r_n^{-1}"] \arrow[d, "\tau_n"] & G_{n-1} \arrow[r, Rightarrow, "r_{n-1}^{-1}"] \arrow[d, "\tau_{n-1}"] & \ldots \arrow[r, Rightarrow, "r_{k+1}^{-1}"] & G_k \arrow[d, "\tau_k"] \\
\varphi(G_n) \arrow[r, Rightarrow, "r_n^{-1}"] & \widetilde{G}_1 \arrow[r, Rightarrow, "r_{n-1}^{-1}"] & \ldots \arrow[r, Rightarrow, "r_{k+1}^{-1}"] & \widetilde{G}_j
\end{tikzcd}$$

   where the τ_i's are isomorphisms. Hence, the two derivations are isomorphic and therefore they produce the same graph diagram.

   Note that the rearrangement f with graph pair diagram $(G_k, \widetilde{G}_j, \tau_k)$ belongs to $\mathcal{G}(\mathcal{R}, \Gamma)$ and satisfies $\alpha(f) = \Delta_\varphi$. In fact, the graphs G_k and $\widetilde{G}_j$ are obtained by applying replacements from $\mathcal{R}'$ and $\Delta(G_k)$ is given by the defining graphs $G_0, G_1, \ldots G_k$ and cells $C_1, C_2, \ldots C_k$ and $\Delta(\widetilde{G}_j)$ is given by the defining graphs $G_0 = \varphi(G_n) = \widetilde{G}_0, \widetilde{G}_1, \ldots \widetilde{G}_j$ and cells $C_n, C_{n-1}, \ldots, C_{k+1}$.

On the other hand, $\Delta^{-1}(\widetilde{G}_j) \circ_{\tau_k} \Delta(G_k) = \Delta(G_k) \cup_{\tau_k} \Delta^{-1}(\widetilde{G}_j) \cong \Delta_\varphi$.

□

Theorem 3.14. *Let $\mathcal{R}$ be the graph rewriting system induced by a colored graph rewriting system $(\mathcal{R}', \Gamma)$. Then $\mathcal{G}(\mathcal{R}', \Gamma)$ is isomorphic to $\mathcal{D}(\mathcal{R}, \Gamma)$.*

Proof. Observe that the proofs of items 1, 2 and 3 are analogous to the same items of the proof of Theorem 3.13.

We will prove item 4, that is, that α is surjective.

In this case we have an initial graph Γ with edges colored by a finite set of colors $\mathcal{C}$ and cells of the form (e_c, R_c) where e_c is colored by an element of $\mathcal{C}$ and R_c have edges colored with some colors of $\mathcal{C}$. Each replacement graph R_c has distinguished initial and terminal vertices. In fact, let $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma)$ and let $\mathcal{R} = \{\mathcal{R}^+ \cup \mathcal{R}^-\}$ where $\mathcal{R}^+$ is the set of replacement rules associated with simple expansions (e_c, R_c), and $\mathcal{R}^-$ is the set of replacement rules associated with the inverse of simple expansions (R_c, e_c), where $c \in \mathcal{C}$.

We notice that Δ_φ arises from a valid order with cells $C_1, C_2, \ldots, C_n$, and defining graphs $G_0, G_1, \ldots, G_n$ We will show that we can find a valid order for Δ_φ such that the cells appearing first are of the form (e_c, R_c) induced by $r \in \mathcal{R}^+$ and those appearing afterwards are those of the form (R_c, e_c) obtained from rules $r \in \mathcal{R}^-$. To prove this, it is enough to show that we can transpose two consecutive cells $C_j = (R_{c_j}, e_{c_j})$ and $C_{j+1} = (e_{c_{j+1}}, R_{c_{j+1}})$. Indeed, let $C_j = (R_{c_j}, e_{c_j})$ and $C_{j+1} = (e_{c_{j+1}}, R_{c_{j+1}})$. Now, if e_{c_j} has different color than $e_{c_{j+1}}$, then the result follows as in Theorem 3.13 while, on the other hand, $c_j \neq c_{j+1}$ implies that the replacements that induce the cells are sequentially independent since $bot(C_j)$ and $top(C_{j+1})$ have different colors in their edges, so they are non-overlapping and, by Lemma 2.38 the order $C_1, C_2, \ldots, C_{j+1}, C_j, \ldots C_n$ is a valid order for Δ_φ.

The proof now follows as in Theorem 3.13.

□

There are cases where the graph rewriting system $\mathcal{R}$ induced by the rewriting system $\mathcal{R}'$ is not reductive. In these cases we need to modify the map α defined in Theorem 3.14 a little bit.

Example 3.15. *We will work in the case $n = 2$ of the Basilica family.*

In Figure 32 we constructed a reductive replacement system for the element $n = 2$ of the basilica set family. We will add to that graph rewriting system the inverse rules and *consider the same initial graph seen in Figure 17.*

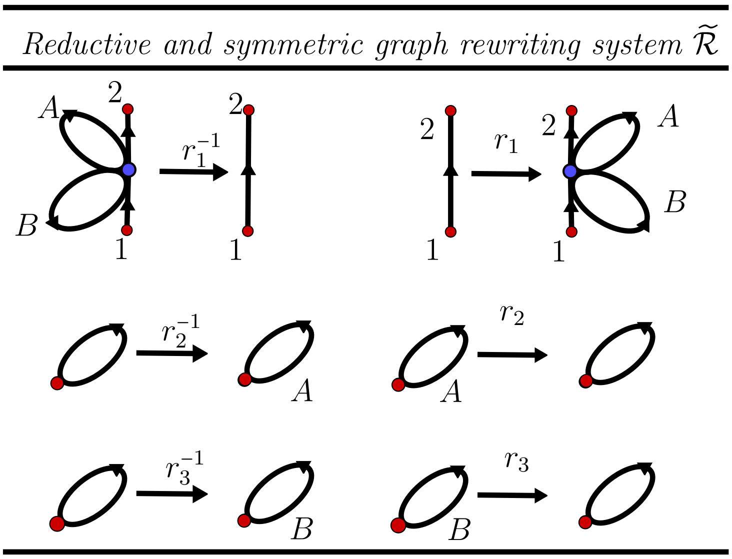


Figure 50

Recall that the initial graph and top of the diagram in the rabbit family for $n = 2$ is an unlabeled graph and that a graph diagram is a diagram that has a set of cells and defining graphs together with an isomorphism between its top and its bottom. This isomorphism must preserve the context for graphs and so the bottom of the graph diagram must be also an unlabeled graph.

Once we use the rule r_1 in Figure 50 we will eventually either use the rule r_1^{-1} (in this case we have a dipole in the graph diagram) or we must use the rules r_2 and r_3 to eliminate the labels in the graph since the bottom of the diagram has no labels. Apart of the cases mentioned above, there are no other ways of removing the labels in a derivation.

Observe that in a valid order for a graph diagram the rules r_1^{-1}, r_2^{-1} and r_3^{-1} are sequentially independent from the other rules and the same statement is also true for their inverse rules r_1, r_2 and r_3. So, by Lemma 2.38 we can transpose the cells in the valid order until put them in consecutive positions as in Figure 51.

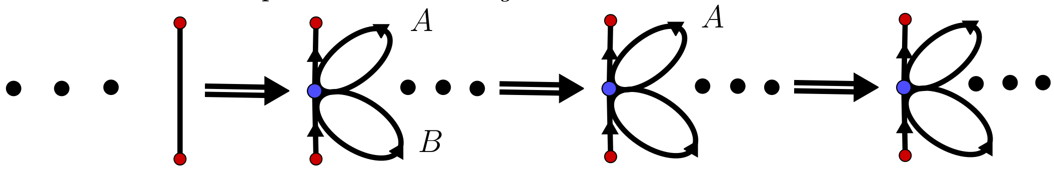


Figure 51

We will show in Lemma 3.21 that we can move to the left the replacement until obtains all the auxiliary rules one after the other as in the figure below,

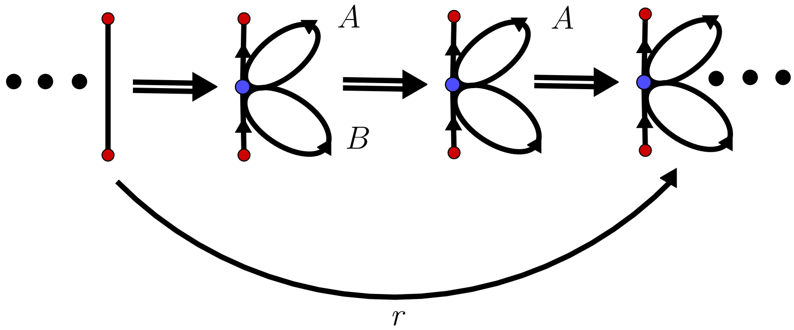


Figure 52 – Replacing the rule $r = (e, R, \nu)$ by a sequence of auxiliary reductive rules. *Therefore, having the auxiliary rules can be seen as having a unique rule r.*

In the following we will define a graph rewriting system $\widetilde{\mathcal{R}}$ obtained from $\mathcal{R}'$ where $\mathcal{G}(\mathcal{R}', \Gamma)$ is a rearrangement group of fractals. We will show the relation between these objects later on in Theorem 3.22.
We will focus our attention to the case when $(\mathcal{R}', \Gamma)$ is a colored replacement system (see Definition 1.52). In particular, in this case we can have more than one replacement rule where the color of the edge tells us which rules we can apply over that edge. So, in this case we can also have more than one non-reductive rule in $\mathcal{R}'$.

Remark 3.16. *Given a non-reductive rule, we wish to replace it by some reductive rules as in Example 3.15. Consider $r_i = (e, R, \nu)$, note that e in the rule r_i is already reductive since the only automorphism of e that fixes ∂e pointwise is the identity. However, (e, R) is not reductive if R admits automorphisms different from the identity. In these cases, to make r_i reductive, we must replace the rule (e, R) by some other rules. In order to do this, first we change R by the graph $\widetilde{R}$ that consists in assigning a different label to each edge (loop) that is identified with some other edge (loop) in an automorphism of R. After that, for each labeled edge (loop) e_A in $\widetilde{R}$ we add a rule (e_A, e, ν) where e is the edge (loop) obtained from erasing the label of e_A. Then these rules correspond to cells (e_A, e), where e_A is an edge (loop) with label A and e is an edge (loop) without label. Observe that in Example 3.10 we obtain the rules r_1, r_2 and r_3 by applying the former process to the rule r in Figure 50.*

Proposition 3.17. *Given a rearrangement group of fractals $\mathcal{G}(\mathcal{R}', \Gamma)$.Then the graph rewriting system $\widetilde{\mathcal{R}}$ obtained from $\mathcal{R}'$ as in Remark 3.16 is reductive.*

Proof. Note that the rules from $\widetilde{\mathcal{R}}$ are all reductive. In fact, we have rules of two forms.

- $(e, \widetilde{R}, \nu)$ where $\widetilde{R}$ is a graph with different labels in each edge that can be identified with a different edge through a boundary fixing automorphism of R. Thus, in $\widetilde{R}$ the only boundary fixing automorphism that preserves the labels is the identity automorphism. The same happens with e and therefore $(e, \widetilde{R}, \nu)$ is a reductive rule.

- (e_A, e, ν) where e is an edge (loop) and e_A is an edge (loop) labeled by the letter A. Thus, the only boundary fixing automorphism that preserves e_A and e is the identity automorphism, so that (e_A, e, ν) is reductive.

□

Remark 3.18. *In virtue of Proposition 3.17 given a rearrangement group of fractals with replacement system $\mathcal{R}'$ we can obtain a symmetric and reductive graph rewriting system $\widetilde{R}$ by applying the process described in Remark 3.16 to each rule in $\mathcal{R}'$ and adding the inverse rules of each rule that we obtain in this process.*

Definition 3.19. *Let r_i be a non-reductive rule in $\mathcal{R}$, then we call **auxiliary rules** the set of reductive rules $S_i = \{r_{i1}, r_{i2}, \dots, r_{in}\}$ in $\widetilde{\mathcal{R}}$ constructed as in Remark 3.18. In this case, we say that, if $r \in \mathcal{R}^+$, then $r_{i1}, r_{i2}, \dots, r_{in} \in \widetilde{\mathcal{R}}^+$ and, if $r \in \mathcal{R}^-$, then $r_{i1}, r_{i2}, \dots, r_{in} \in \widetilde{\mathcal{R}}^-$. We denote C_{ij} the cell associated with the auxiliary replacement r_{ij}.*

Remark 3.20. *Note that all the rules in this graph rewriting system have vertices as boundaries. Thus, given cells $C = (R, e)$ and $D = (f, S)$, where e and f are edges, we have that they only can overlap if bot(C) and top(D) have a common edge, this means in this case that $e = f$.*

Lemma 3.21. *Let $\Delta_\varphi \in \mathcal{D}(\widetilde{\mathcal{R}}, \Gamma)$ be a reduced graph diagram. Then there exists a valid order where appear first all the rules $r \in \widetilde{\mathcal{R}}^+$ and after all the rules $r \in \widetilde{\mathcal{R}}^-$.*

Proof. Note that, by construction, the cells induced by replacements in $\widetilde{\mathcal{R}}^+$ have two forms, say $(e, \widetilde{R}_0)$ and (e_A, e), and consequently the cells induced by rules in $\widetilde{\mathcal{R}}^-$ have forms $(\widetilde{R}_1, e)$ and (e, e_B) for some labels A and B. We will study the cases in which two consecutive cells C and D appear in a valid order and such that C is induced by a replacement in $\widetilde{\mathcal{R}}^-$ and D is induced by a replacement in $\widetilde{\mathcal{R}}^+$ and we will prove that in all these cases, we can transpose these cells and obtain a valid order where cell D comes before cell C. Before analyzing the cases we will establish some notation. Let e, f be edges labeled with a certain color such that the cells $(e, \widetilde{R}_1)$ and $(f, \widetilde{R}_2)$ are induced by replacement in $\widetilde{\mathcal{R}}$.

1. $C = (\widetilde{R}_1, f)$ and $D = (e, \widetilde{R}_0)$ Note that each replacement is applied to a different color. So, if $e = f$, then $\widetilde{R}_1 = \widetilde{R}_0$ and we would have a dipole in Δ_φ, but this graph diagram is reduced. On the other hand, if $e \neq f$ then, by Remark 3.20, *bot*(C) and *top*(D) are non-overlapping and, by Lemma 2.38, we can transpose C and D in a valid order.

2. $C = (\widetilde{R}_1, f)$ and $D = (e_A, e)$.

3. $C = (f, f_B)$ and $D = (e, \widetilde{R}_0)$.

4. $C = (f, f_B)$ and $D = (e_A, e)$.

   Observe that the cases 2. 3. and 4. can be analyzed all at the same time. Indeed, either $bot(C)$ or $top(D)$ are labeled vertices, therefore by Remark 3.20 these graphs can not be overlapping. Again, by Lemma 2.38, we can transpose C and D in each of these cases, obtaining a valid order for Δ_φ.

□

We have seen in Proposition 3.10 that, if the rearrangement system of $\mathcal{G}(\mathcal{R}', \Gamma)$ contains rules that are not reductive, we can replace them by some reductive rules that we called auxiliary rules and get a symmetric and reductive graph rewriting system $\widetilde{\mathcal{R}}$ as in Example 3.15. In this case, for each element in $\mathcal{G}(\mathcal{R}, \Gamma)$, we can do something similar to Definition 3.6, but using rules from $\widetilde{\mathcal{R}}$ instead of the rules from $\mathcal{R}'$. Hence, given a rearrangement f, we define $\widetilde{\Delta}(f)$ in the same way that we defined $\Delta(f)$. Observe that this graph can be obtained from $\Delta(f)$ by taking $\{G_i\}$ as its collection of graphs and that, if G_{k+1} were obtained from G_k using a non-reductive rule, we change it to a sequence of graphs obtained by using rules in $\widetilde{\mathcal{R}}$ as in the Figure 52.

Theorem 3.22. *Consider the rearrangement group given by $\mathcal{G}(\mathcal{R}', \Gamma)$. Let $\widetilde{\mathcal{R}}$ be a reductive and symmetric graph rewriting system obtained from $\mathcal{R}'$ by adding some auxiliary rules. Then $\mathcal{G}(\mathcal{R}', \Gamma)$ is isomorphic to $\mathcal{D}(\widetilde{\mathcal{R}}, \Gamma)$.*

Proof. Consider $\alpha : \mathcal{G}(\mathcal{R}', \Gamma) \to \mathcal{D}(\widetilde{\mathcal{R}}, \Gamma)$ that maps each rearrangement f to the graph diagram $\Delta(f)$.
Observe that the proofs of items 1, 2 and 3 are analogous to the same items of the proof of Theorem 3.13. We will prove that α is surjective.
Let $\Delta_\varphi \in \mathcal{G}(\widetilde{\mathcal{R}}, \Gamma)$ be a reduced graph diagram with a valid order given by cells $C_1, C_2, \ldots, C_n$, and defining graphs $G_0, G_1, \ldots, G_n$.
By Lemma 3.21 we can assume that the valid order that we have is given by taking first all of the cells induced by $r \in \widetilde{\mathcal{R}}^+$ and, after, all of the cells obtained from rules $r \in \widetilde{\mathcal{R}}^-$.
On the other hand, analogously to the proof of Theorem 3.13, we can find a graph pair diagram $(G_k, \widetilde{G}_j, \tau_k)$ for the rearrangement f. Observe first that the graphs G_k and $\widetilde{G}_j$ can be obtained by applying replacements from $\widetilde{\mathcal{R}}$. Indeed, by Lemma 3.21, once we apply an auxiliary rule $r_1 = (e \to R_1)$, we must apply replacements $r_2 = (R_1 \to R_2), r_3 = (R_2 \to R_3), \ldots, r_n = (R_{n-1} \to R)$ that are the reductive rules associated with a replacement rule $r = (e \to R) \in \mathcal{R}'$. Notice that $r_1, r_2, \ldots r_n \in \widetilde{\mathcal{R}}^+$. So G_k is the graph obtained from applying successively and beginning from G_0 all the rules in $\mathcal{R}^+$. A similar analysis happens with $\widetilde{G}_j$.
On the other hand $\alpha(f) = \Delta_\varphi$. In fact, the graph diagram $\Delta(G_k)$ is given by the defining graphs $G_0, G_1, \ldots G_k$ and cells $C_1, C_2, \ldots C_k$ and $\Delta(\widetilde{G}_j)$ is given by the defining graphs $G_0 = \varphi(G_n) = \widetilde{G}_0, \widetilde{G}_1, \ldots \widetilde{G}_j$ and cells $C_n, C_{n-1}, \ldots, C_{k+1}$.
On the other hand, $\Delta^{-1}(\widetilde{G}_j) \circ_{\tau_k} \Delta(G_k) = \Delta(G_k) \cup_{\tau_k} \Delta^{-1}(\widetilde{G}_j) \cong \Delta_\varphi$.

□

In conclusion, we can use Remark 3.8 to find a symmetric and reductive graph rewriting system $\widetilde{\mathcal{R}}$ from a replacement system of a rearrangement group of fractals. In particular, we use this strategy in examples 3.10, 3.11, 3.15 and in Figure 30 to get symmetric and reductive graph rewriting systems for the following groups

- Thompson groups F, T and V.
- The rearrangement group $F \rtimes \mathbb{Z}_2$.
- The Generalized Thompson groups $F_{n,k}, T_{n,k}$ and $V_{n,k}$.
- The Basilica family of rearrangement groups.

and we can use the same technique to find $\widetilde{\mathcal{R}}$ for

- The Vicsek family of rearrangement groups.
- The rearrangement group the colored replacement system (for example The Airplane).

Then, in virtue of the Theorems 3.13, 3.14 and 3.22, these groups can all be seen as Graph Diagram Groups.

4 Equivalent graph rewriting systems and groups

Definition 4.1. *We say that a graph* Γ_0 *is* ***equivalent*** *to a graph* Γ_1 *in a graph rewriting system* $\mathcal{R}$ *(and write* $\Gamma_0 \cong_{\mathcal{R}} \Gamma_1$*) if there exists a derivation*

$$G_0 = \Gamma_0 \xRightarrow{\varphi_1, r_1} G_1 \xRightarrow{\varphi_2, r_2} G_2 \ldots \xRightarrow{\varphi_n, r_n} G_n = \Gamma_1$$

where $r_1, r_2, \ldots, r_n \in \mathcal{R}$*.*

Theorem 4.2. *Let* $\mathcal{R}$ *be a reductive and symmetric graph rewriting system. Let* Γ_0 *and* Γ_1 *graphs over* $\mathcal{R}$*. If* $\Gamma_0 \cong_{\mathcal{R}} \Gamma_1$*, then* $\mathcal{D}(\mathcal{R}, \Gamma_0) \cong \mathcal{D}(\mathcal{R}, \Gamma_1)$*.*

Proof. Observe that $\Gamma_0 \cong_{\mathcal{R}} \Gamma_1$ implies that exist a graph diagram with top Γ_1, bottom Γ_0 and cells induced by relations in $\mathcal{R}$. We call this graph diagram Π_{ι_0} where $\iota_0 : \Gamma_0 \to \Gamma_0$ is the identity isomorphism in Γ_0. Analogously $\Pi_{\iota_1}^{-1}$ is the inverse graph diagram of Π_{ι_0} where $\iota_1 : \Gamma_1 \to \Gamma_1$ is the identity isomorphism in Γ_1. This is, $\Pi_{\iota_0} \circ \Pi_{\iota_1}^{-1} = \Gamma_{\iota_1}$ and $\Pi_{\iota_1}^{-1} \circ \Pi_{\iota_0} = \Gamma_{\iota_0}$ We define a map $\Theta : \mathcal{D}(\mathcal{R}, \Gamma_0) \to \mathcal{D}(\mathcal{R}, \Gamma_1)$ where $\Theta(\Delta_\varphi) = \Pi_{\iota_0}^{-1} \circ \Delta_\varphi \circ \Pi_{\iota_1}$. Notice that $top(\Theta(\Delta_\varphi)) = \Gamma_1$ and $bot(\Theta(\Delta_\varphi)) \cong \Gamma_1$. Θ is a homomorphism, in fact, given $\Delta_\varphi, \Delta'_{\varphi'} \in \mathcal{D}(\mathcal{R}, \Gamma_0)$ by the associative property explained in Remark 2.59, we have that

$$\begin{aligned}
\Theta(\Delta_\varphi) \circ \Theta(\Delta'_{\varphi'}) &= \Pi_{\iota_1}^{-1} \circ \Delta_\varphi \circ (\Pi_{\iota_0} \circ \Pi_{\iota_1}^{-1}) \circ \Delta'_{\varphi'} \circ \Pi_{\iota_0} \\
&= \Pi_{\iota_1}^{-1} \circ \Delta_\varphi \circ \Delta'_{\varphi'} \circ \Pi_{\iota_0} \\
&= \Theta(\Delta_\varphi \circ \Delta'_{\varphi'}).
\end{aligned}$$

Θ is injective, indeed,

$$\begin{aligned}
\Theta(\Delta_\varphi) = (\Gamma_1)_{\iota_1} &\Leftrightarrow \Pi_{\iota_1}^{-1} \circ \Delta_\varphi \circ \Pi_{\iota_0} = (\Gamma_1)_{\iota_1} \Leftrightarrow \\
\Delta_\varphi &= \Pi_{\iota_0} \circ (\Gamma_1)_{\iota_1} \circ \Pi_{\iota_1}^{-1} = \Pi^{-1} \cup_\iota (\Pi_{\iota_0} \circ (\Gamma_1)_{\iota_1}) \\
&= (\Pi^{-1} \cup_\iota (\Gamma_1)_\iota \cup_\iota \Pi)_{\iota_0} = (\Gamma_0)_{\iota_0}.
\end{aligned}$$

Moreover, Θ is surjective since if $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma_1)$, $\Pi_{\iota_0} \circ \Delta_\varphi \circ \Pi_{\iota_1}^{-1} \in \mathcal{D}(\mathcal{R}, \Gamma_0)$ and $\Theta(\Pi_{\iota_0} \circ \Delta_\varphi \circ \Pi_{\iota_1}^{-1}) = \Delta_\varphi$. □

Example 4.3. *Consider the graph rewriting system $\mathcal{R}'$ given by the rules r_1, r_2, r_3 in Figure 53 together with the inverse rules $r_1^{-1}, r_2^{-1}, r_3^{-1}$.*

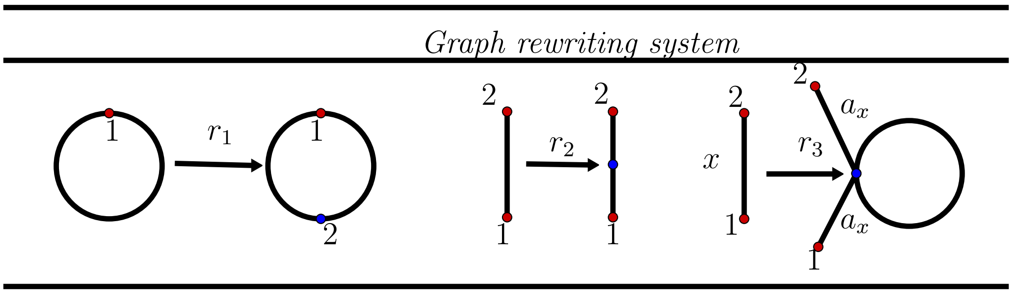


Figure 53

Note that $\mathcal{R} = \{r_1, r_2, r_1^{-1}, r_2^{-1}\}$ with base graph Γ given in Figure 54, is a graph rewriting system for Thompson's group T, see Example 3.11. We wish to compare $T \cong \mathcal{D}(\mathcal{R}, \Gamma)$ with $\mathcal{D}(\mathcal{R}', E)$ where E is an edge label with the letter x.

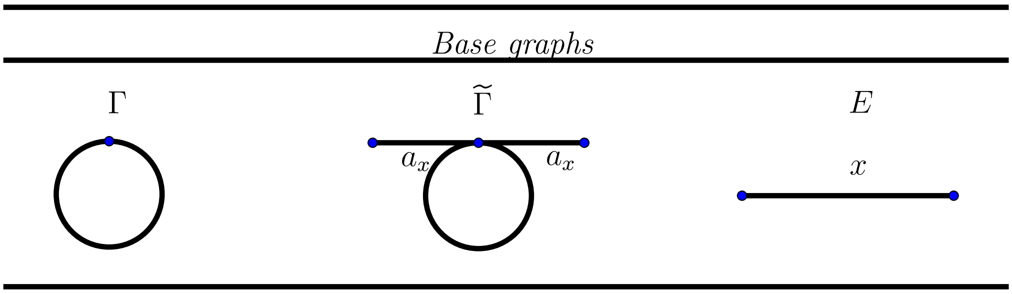


Figure 54

First, note that $\mathcal{D}(\mathcal{R}, \Gamma) \cong \mathcal{D}(\mathcal{R}, \widetilde{\Gamma})$. In fact, it is enough to define the $\alpha_1 : \mathcal{D}(\mathcal{R}, \Gamma) \to \mathcal{D}(\mathcal{R}, \widetilde{\Gamma})$ that map each diagram Δ_φ with top Γ to a diagram with two dangling edges labeled with the letter a_x in the only vertex of Γ. This is, α map the top of Δ_φ, says Γ to $\widetilde{\Gamma}$ and the other cells Δ_φ to themselves. Note that α_1 is an isomorphism. On the other hand, $E \equiv_{\mathcal{R}'} \widetilde{\Gamma}$, so by the last theorem $\mathcal{D}(\mathcal{R}', E) \cong \mathcal{D}(\mathcal{R}', \widetilde{\Gamma})$. We will prove that $\mathcal{D}(\mathcal{R}, \Gamma) \cong \mathcal{D}(\mathcal{R}', E)$ by proving that $\mathcal{D}(\mathcal{R}, \widetilde{\Gamma}) \cong \mathcal{D}(\mathcal{R}', \widetilde{\Gamma})$. In fact, let be $\alpha_2 : \mathcal{D}(\mathcal{R}, \widetilde{\Gamma}) \to \mathcal{D}(\mathcal{R}', \widetilde{\Gamma})$ such that $\alpha_2(\Delta_\varphi) = \Delta_\varphi$. Note that α_2 is a homomorphism injective. We will show that it is surjective. In fact, note that a reduced element in $\mathcal{D}(\mathcal{R}', \Gamma)$ can not have a cell induced by the relation r_3^{-1}, on the contrary, this cell would have a labeled edge x but bot(Δ_φ) have not edges labeled with the letter x, so this cell must be in a dipole, which is a contradiction since Δ_φ is reduced. In resume we prove that $\mathcal{D}(\mathcal{R}', E) \cong \mathcal{D}(\mathcal{R}', \widetilde{\Gamma}) \cong \mathcal{D}(\mathcal{R}, \widetilde{\Gamma}) \cong \mathcal{D}(\mathcal{R}, \Gamma)$.

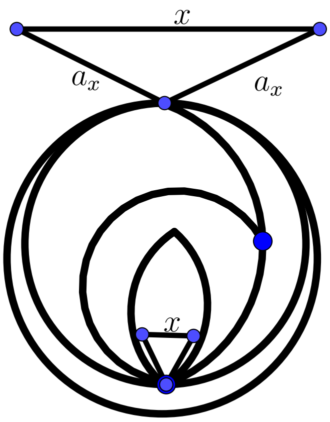


Figure 55 – An element of $\mathcal{D}(\mathcal{R}', E)$

4.1 Graph Diagram Groups and Group Theoretic Constructions

All the graph rewriting systems in this section are assumed to be symmetric and reductive.

Lemma 4.4. *Let $\mathcal{R}_{c_0}$ and $\mathcal{R}_{c_1}$ graph rewriting systems with disjoint sets of colors C_0 and C_1. Let Γ_0 and Γ_1 respectively be disjoint graphs over the sets of colors C_0 and C_1. Let $\mathcal{R} = \mathcal{R}_{c_0} \cup \mathcal{R}_{c_1}$ and Γ be the disjoint union $\Gamma_0 \biguplus \Gamma_1$. Then $\mathcal{D}(\mathcal{R}, \Gamma)$ is isomorphic to the direct product $\mathcal{D}(\mathcal{R}_{c_0}, \Gamma_0) \times \mathcal{D}(\mathcal{R}_{c_1}, \Gamma_1)$.*

Proof. Define $\alpha : \mathcal{D}(\mathcal{R}_{c_0}, \Gamma_0) \times \mathcal{D}(\mathcal{R}_{c_1}, \Gamma_1) \to \mathcal{D}(\mathcal{R}, \Gamma)$ that maps $(\Delta_{\varphi_0}, \Pi_{\varphi_1})$ to the graph diagram $\Delta_{\varphi_0} \biguplus \Pi_{\varphi_1}$ defined as follows: if Δ_{φ_0} is the graph diagram given by the valid order $G_0, G_1, \ldots, G_n$ and cells $C_1, C_2, \ldots, C_n$ and Π_{φ_1} given by the valid order $\widetilde{G}_0, \widetilde{G}_1, \ldots, \widetilde{G}_m$ and cells $\widetilde{C}_1, \ldots, \widetilde{C}_m$ then we define $\alpha(\Delta_{\varphi_0}, \Pi_{\varphi_1})$ as the graph diagram with defining graphs

$$G_0 \cup \Gamma_1, G_1 \cup \Gamma_1, \ldots, G_n \cup \Gamma_1, G_n \cup \widetilde{G}_1, \ldots, G_n \cup \widetilde{G}_m$$

and cells

$$C_1, C_2, \ldots, C_n, \widetilde{C}_1, \ldots, \widetilde{C}_m$$

where $\varphi(bot(\Delta_{\varphi_0}) \biguplus bot(\Pi_{\varphi_1})) = \varphi_0(bot(\Delta_{\varphi_0})) \biguplus \varphi_1(bot(\Pi_{\varphi_1})) = \Gamma_0 \biguplus \Gamma_1 = \Gamma$.

- Suppose that $(\Delta_{\varphi_0}, \Pi_{\varphi_1})$ is equivalent to $(\Delta'_{\varphi'_0}, \Pi'_{\varphi'_1})$. Then Δ_{φ_0} is equivalent to $\Delta'_{\varphi'_0}$ and Π_{φ_1} is equivalent to $\Pi'_{\varphi'_1}$, so this implies that the derivations given by the following orderings are also equivalent

  $$G_0 \cup \Gamma_1, G_1 \cup \Gamma_1, \ldots, G_n \cup \Gamma_1, G_n \cup \widetilde{G}_1, \ldots, G_n \cup \widetilde{G}_m$$

  and

  $$G_0 \cup \Gamma_1 = G'_0 \cup \Gamma_1, G'_1 \cup \Gamma_1, \ldots, G'_n \cup \Gamma_1, G'_n \cup \widetilde{G}'_1, \ldots, G'_n \cup \widetilde{G}'_m$$

  Moreover, these equivalences also imply that exist ψ_0 and ψ_1 such that $\varphi_0 = \varphi'_0 \circ \psi_0$ and $\varphi_1 = \varphi'_1 \circ \psi_1$ which, in turn, imply the existence of ψ satisfying $\varphi = \varphi' \circ \psi$.

- α is a homomorphism. In fact, the product works in the same way on both sides and this homomorphism is injective since, if $\Delta_{\varphi_0} \dot{\bigcup} \Pi_{\varphi_1}$ are equivalent, then Δ_{φ_0} and Π_{φ_1} are also equivalent.

- α is surjective. Let $\Delta_\varphi \in \mathcal{D}(\mathcal{R}, \Gamma)$, then we have a valid order for this graph diagram. In particular, we have that the cells inducing replacement rules of color C_0 are sequentially independent of those of color c_1 since they belong to a different connected component in Δ_φ. Hence, we can consider a valid order for Γ where the first cells (graphs) are induced by the replacement rules of color C_0 and the other cells (graphs) correspond to the replacement rules of color C_1. Thus, the defining graphs are

$$G_0 \cup \Gamma_1, G_1 \cup \Gamma_1, \ldots, G_n \cup \Gamma_1, G_n \cup \widetilde{G}_1, \ldots, G_n \cup \widetilde{G}_m$$

  Then the graph diagrams Δ_{φ_0} and Π_{φ_1} with ordering $G_0, G_1, \ldots, G_n$ and cells $C_1, C_2, \ldots, C_n$ and $\widetilde{G}_0, \widetilde{G}_1, \ldots, \widetilde{G}_m$ and cells $\widetilde{C}_1, \ldots, \widetilde{C}_m$ respectively satisfy that $\alpha(\Delta_{\varphi|_{\Gamma_0}}, \Pi_{\varphi|_{\Gamma_1}}) = \Delta_\varphi$ by construction of α.

□

Theorem 4.5. *The class of the graph diagram groups over a graph rewriting system is closed under taking finite direct products.*

Proof. It is a consequence of Lemma 4.4. □

Corollary 4.6. *Given rearrangement groups of fractals $\mathcal{G}(\mathcal{R}'_i, \Gamma_i)$ such that each $\mathcal{R}'_i$ for $1 \leq i \leq n$ induces a symmetric and reductive graph rewriting system $\mathcal{R}_i$, then the finite direct product of these rearrangement groups of fractals is a rearrangement group of fractal.*

Proof. We will prove the case when we have two replacement systems since the other cases are analogous.Given $(\mathcal{R}'_0, \Gamma_0)$ and $(\mathcal{R}'_1, \Gamma_1)$ we define the replacement systems $(\mathcal{R}'_{c_0}, \Gamma_{c_0})$ and $(\mathcal{R}'_{c_1}, \Gamma_{c_1})$ by respectively coloring the edges in their rules and their initial graphs with disjoint color sets c_0 and c_1, that is, we consider the rewriting systems $(\mathcal{R}'_0, \Gamma_0)$ and $(\mathcal{R}'_1, \Gamma_1)$, but using disjoint sets of colors for each of them. In particular, if $(\mathcal{R}'_0, \Gamma_0)$ and $(\mathcal{R}'_1, \Gamma_1)$ are uncolored rewriting systems, then we use only two colors c_0 and c_1, where we use c_i on every replacement rule in $(\mathcal{R}'_i, \Gamma_i)$.Note that, by construction, $\mathcal{G}(\mathcal{R}'_0, \Gamma_0) \cong \mathcal{G}(\mathcal{R}'_{c_0}, \Gamma_{c_0})$ and $\mathcal{G}(\mathcal{R}'_1, \Gamma_1) \cong \mathcal{G}(\mathcal{R}'_{c_1}, \Gamma_{c_1})$. Let $\mathcal{R}' = \mathcal{R}'_{c_0} \cup \mathcal{R}'_{c_1}$ then, by Theorems 3.14 and 4.5, we have that

$$\mathcal{G}(\mathcal{R}', \Gamma) \cong \mathcal{D}(\mathcal{R}, \Gamma) \cong \mathcal{D}(\mathcal{R}_{c_0}, \Gamma_0) \times \mathcal{D}(\mathcal{R}_{c_1}, \Gamma_{c_1}) \cong \mathcal{G}(\mathcal{R}'_{c_0}, \Gamma_{c_0}) \times \mathcal{G}(\mathcal{R}'_{c_1}, \Gamma_{c_1})$$

where $\mathcal{R}, \mathcal{R}_{c_0}$ and $\mathcal{R}_{c_1}$ are, respectively, the reductive and symmetric graph rewriting systems induced by $\mathcal{R}', \mathcal{R}'_{c_0}$ and $\mathcal{R}'_{c_1}$ and $\Gamma = \Gamma_{c_0} \dot{\bigcup} \Gamma_{c_1}$.

□

Final Remarks

In this doctoral thesis, we define a family of groups called Graph Diagram Groups. This family was inspired by other families containing Thompson-like groups such as diagram groups [1] and rearrangement group of fractals [2].Let us recall the topics presented in the chapters.

In Chapter 2 we follow a structure similar to the work of Guba and Sapir [1] to define this family. For instance, we use a graph rewriting system instead of a rewriting system or a semigroup presentation to capture a larger variety of groups in our family. To do this, we use the notion of portion of a graph that allows us to define cells and apply replacement rules in a wider sense than in rearrangement groups of fractals [2]. This abstract concept of cell requires a more precise concept of dipole that depends on a strict partial order on the cells of the graph diagram (see Lemma 2.30 and Definition 2.40). Furthermore, we also need to restrict the dipole reduction to be reductive (Definition 2.44) to prove that each graph diagram is equivalent to a unique reduced element. Finally, we define some conditions to give group structure to $\mathcal{D}(\mathcal{R}, \Gamma)$.

In Chapter 3 we show that the families of right angled Artin groups, diagram groups and rearrangement groups of fractals are contained in the family of graph diagram groups. For each such family, we give concrete graph rewriting systems $\mathcal{R}$ and find isomorphisms between $\mathcal{D}(\mathcal{R}, \Gamma)$ and groups in these families. An important characteristic between the graph rewriting systems that we use to prove these isomorphisms is that all the rules in the graph rewriting system that we define have vertices as boundary. Therefore, it is easier to prove that two cells are non-overlapping and so when two valid orders represent the same graph diagrams.

Finally, in Chapter 4, we prove that our family is closed under direct products by adapting ideas from [1] and, as a corollary, we obtain the same result for the family of rearrangement group of fractals in its colored version (see Subsection 1.6.1).

The work in the family of graph diagram groups is still in progress, and we have many possible options to continue it. For example the fundamental group of the groupoid associated with the groups in this family is a useful tool to find presentations for some groups using standards techniques from algebraic topology. Therefore, we will continue looking for more elements in the family and we will try to give alternative presentations for these elements taking this approach. In particular Belk and Forrest prove independently that every graph diagram group over finite graph rewriting systems acts properly by isometries on a $CAT(0)$ cubical complex and, therefore, we can study first groups that act properly on a $CAT(0)$ cubical complex.

Another direction consists in using the group structure to look for solutions of the conjugacy problem. Note that the techniques used by Guba and Sapir for diagram groups do not seem to lead us to a solution of the conjugacy problem in most of the groups in our family. This happens since in their case all the groups are torsion-free while the new interesting examples of graph diagram groups are groups with torsion. However, in [5] James Belk and Francesco Matucci give a solution of the conjugacy problem for F, T and V using a geometrical object called strand diagram that can be induced by the dipole reductions of the Thompson group F when this is seen as a diagram group. Following the same approach we are trying to solve conjugacy in the Basilica group T_B, that is we define an abstract strand diagram that preserves the information about the cells, edges and vertices in the graph diagram and have partial results towards a solution of the conjugacy problem.

Bibliography

[1] GUBA, V.; SAPIR, M. Diagram groups. *American Mathematical Soc.*, v. 620, n. 130, p. 1–117, 1997.

[2] BELK, J.; FORREST, B. Rearrangement groups of fractals. *Transactions of the American Mathematical Society,* to appear, 2019.

[3] THOMPSON, R. Handwritten notes. Unpublished. 1965.

[4] DEHN, M. Über unendliche diskontinuierliche gruppen. *Mathematische Annalen*, Springer, v. 71, n. 1, p. 116–144, 1911.

[5] BELK, J.; MATUCCI, F. Conjugacy and dynamics in thompson's groups. *Geometriae Dedicata*, Springer, v. 169, n. 1, p. 239–261, 2014.

[6] HIGMAN, G. *Finitely presented infinite simple groups.* [S.l.]: Dept. of Pure Mathematics, Dept. of Mathematics, IAS, Australian National Univ., 1974. v. 8.

[7] BIERI, R.; STREBEL, R. On groups of pl-homeomorphisms of the real line. Preprint. 2014.

[8] STEIN, M. Groups of piecewise linear homeomorphisms. *Transactions of the American Mathematical Society*, v. 332, n. 2, p. 477–514, 1992.

[9] BRIN, M. The algebra of strand splitting. i. a braided version of thompson's group v. *J. Group Theory*, v. 10, n. 6, p. 757–788, 2007.

[10] DEHORNOY, P. The group of parenthesized braids. *Advances in Mathematics*, Elsevier, v. 205, n. 2, p. 354–409, 2006.

[11] BRIN, M. Higher dimensional thompson groups. *Geometriae Dedicata*, Springer, v. 108, n. 1, p. 163–192, 2004.

[12] KILIBARDA, V. On the algebra of semigroup diagrams. *Ph.D. Thesis, Univ. Nebraska*, 1994.

[13] KILIBARDA, V. On the algebra of semigroup diagrams. *International Journal of Algebra and Computation*, World Scientific, v. 7, n. 03, p. 313–338, 1997.

[14] GUBA, V.; SAPIR, M. On subgroups of r.thompson's group and other diagram groups. *Sbornik Mathematics*, IOP Publishing, v. 190, n. 8, p. 1077, 1999.

[15] ARZHANTSEVA, G.; GUBA, V.; SAPIR, M. Metrics on diagram groups and uniform embeddings in a hilbert space. *Comment. Math. Helv.*, Elsevier, v. 81, n. 4, p. 911–929, 2006.

[16] GUBA, V.; SAPIR, M. Diagram groups are totally orderable. *Journal of Pure and Applied Algebra*, Elsevier, v. 205, n. 1, p. 48–73, 2006.

[17] FARLEY, D. Finiteness and cat (0) properties of diagram groups. *Topology*, Elsevier, v. 42, n. 5, p. 1065–1082, 2003.

[18] BELK, J. Thompson's group f. *ProQuest LLC, Ann Arbor, MI, Ph.D. Thesis, Cornell University.*, 2004.

[19] CANNON, J.; FLOYD, W.; PARRY, W. Introductory notes on richard thompson's groups. *Enseignement Mathématique*, SWETS & ZEITLINGER, v. 42, p. 215–256, 1996.

[20] BROWN, K.; GEOGHEGAN, R. An infinite-dimensional torsion-free fp_∞ group. *Inventiones mathematicae*, Springer, v. 77, n. 2, p. 367–381, 1984.

[21] BROWN, K. Finiteness properties of groups. *Journal of Pure and Applied Algebra*, North-Holland, v. 44, n. 1, p. 45–75, 1987.

[22] BELK, J.; FORREST, B. A thompson group for the basilica. *Groups Geom. Dyn.*, Elsevier, v. 9, n. 4, p. 975–1000, 2015.

[23] WITZEL, S.; ZAREMSKY, M. The basilica thompson group is not finitely presented. *Groups Geom. Dyn.,* to appear, 2019.

[24] NEWMAN, M.; HERMAN, A. On theories with a combinatorial definition of equivalence. *Annals of mathematics*, JSTOR, p. 223–243, 1942.

[25] DERSHOWITZ, N.; JOUANNAUD, J. Rewrite systems. In: *Formal models and semantics*. [S.l.]: Elsevier, 1990. p. 243–320.

[26] CRISP, J.; SAGEEV, M.; SAPIR, M. Surface subgroups of right-angled artin groups. *International Journal of Algebra and Computation*, World Scientific, v. 18, n. 03, p. 443–491, 2008.

[27] VALETTE, A. *Introduction to the Baum-Connes conjecture*. [S.l.]: Birkhäuser, 2012.

[28] BAUM, P.; CONNES, A.; HIGSON, N. Classifying space for proper actions and k-theory of group cˆ*-algebras. *Contemporary Mathematics*, American Mathematical Society, v. 167, p. 241–241, 1994.

[29] CHARNEY, R. An introduction to right-angled artin groups. *Geometriae Dedicata*, Springer, v. 125, n. 1, p. 141–158, 2007.

[30] BOURBAKI, N. *General Topology*. [S.l.]: Chapters 5-10, Springer-Verlag Berlin Heidelberg, 1998.

[31] CORRADINI, A.; MONTANARI, U.; ROSSI, F.; EHRIG, H.; HECKEL, R.; LOWE, M. Algebraic approaches to graph transformation–part i: Basic concepts and double pushout approach. In: *Handbook Of Graph Grammars And Computing By Graph Transformation: Volume 1: Foundations*. [S.l.]: World Scientific, 1997. p. 163–245.