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# Unified description of seagull cancellations and infrared finiteness of gluon propagators

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We present a generalized theoretical framework for dealing with the important issue of dynamical mass generation in Yang-Mills theories, and, in particular, with the infrared finiteness of the gluon propagators, observed in a multitude of recent lattice simulations. Our analysis is manifestly gauge invariant, in the sense that it preserves the transversality of the gluon self-energy, and gauge independent, given that the conclusions do not depend on the choice of the gauge-fixing parameter within the linear covariant gauges. The central construction relies crucially on the subtle interplay between the Abelian Ward identities satisfied by the nonperturbative vertices and a special integral identity that enforces a vast number of “seagull cancellations” among the one- and two-loop dressed diagrams of the gluon Schwinger-Dyson equation. The key result of these considerations is that the gluon propagator remains rigorously massless, provided that the vertices do not contain (dynamical) massless poles. When such poles are incorporated into the vertices, under the pivotal requirement of respecting the gauge symmetry of the theory, the terms comprising the Ward identities conspire in such a way as to still enforce the total annihilation of all quadratic divergences, inducing, at the same time, residual contributions that account for the saturation of gluon propagators in the deep infrared.

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## I. INTRODUCTION

The infrared finiteness of the gluon propagator, namely the fact that its scalar form factor,  $\Delta(q^2)$ , saturates at a finite (nonvanishing) value in the low-energy region, has been established in the Landau gauge by means of large-volume lattice simulations both for SU(2) [1–4] and SU(3) [5–8]. In addition, recent lattice simulations in the linear covariant ( $R_\xi$ ) gauges [9] reveal that this particular property is not special to the Landau gauge ( $\xi = 0$ ), given that it persists for values of  $\xi$  ranging in the interval  $[0, 0.5]$ . Moreover, the inclusion of a small number of dynamical quarks (“unquenching”) produces a relative suppression to the gluon propagator, but preserves the feature of saturation clearly intact [10]. Naturally, these results have attracted particular attention, since they offer a valuable opportunity to explore the nonperturbative dynamics of Yang-Mills theories, and even though some of the underlying ideas have a rather long history [11–25], various field-theoretic mechanisms that may account for this characteristic behavior have been considered in the more recent literature [26–46].

A particular set of physical concepts and formal techniques for dealing with this important issue has been developed in a series of articles [26,30,31], based on the Schwinger-Dyson equations (SDEs) [47] derived within the powerful framework obtained from the fusion of the pinch technique (PT) [12,48–52] and background field

method (BFM) [53,54]. As has been amply emphasized in the literature cited above, some of the most prominent features of the PT-BFM framework are the Abelian Slavnov-Taylor identities (STIs) satisfied by the full vertices, a fact that permits, among other things, the systematic organization of the one- and two-loop dressed contributions into manifestly gauge-invariant (transverse) subsets.

The main purpose of the present work is to demonstrate how an elaborate interplay between the Ward identities (WIs) and a fundamental nonperturbative cancellation operating at the level of the gluon self-energy leads invariably to the necessity of introducing massless poles in the vertices of the theory, in order to accommodate the aforementioned lattice findings. In the four items that follow we summarize the logical sequence of the pivotal concepts appearing in this article:

- (1) To begin with, a sharp distinction between the term STI and WI must be drawn, which is easily established in QED, in terms of the photon-electron vertex,  $\Gamma_\mu(q, p, p+q)$ , and the electron propagator,  $S(p)$ . In this textbook context, what we denominate Abelian STI is the standard Takahashi identity,  $q^\mu \Gamma_\mu(q, p, p+q) = S^{-1}(p+q) - S^{-1}(p)$ , whereas the term Ward identity refers to the relation  $\Gamma_\mu(0, p, p) = \partial S^{-1}(p)/\partial p^\mu$ , which may be obtained from the Taylor expansion of the Takahashi identity around  $q = 0$ .

- (2) In the SDE of the gluon propagator there appear three fully dressed vertices, namely the three-gluon, the gluon-ghost, and the four-gluon vertex. Since we work in the PT-BFM framework, these vertices contain one background gluon, namely the one entering into the gluon SDE, carrying the momentum  $q$  of the gluon propagator; all remaining legs are “quantum,” and are irrigated by the virtual momenta circulating in the loops where these vertices are inserted. Thus, when contracted by  $q^\mu$  from the side of their background leg, they satisfy Abelian STIs, in contradistinction to what happens when the contracted leg is quantum, in which case non-Abelian STIs are triggered (i.e., STIs whose tree-level form is modified by contributions from the ghost propagator and ghost kernels). Then, if the form factors comprising the vertices do not contain poles of the type  $1/q^2$ , then in the limit  $q \rightarrow 0$  one arrives at expressions that are qualitatively similar to the WI of the photon-electron vertex reported above.
- (3) These WIs, in turn, expose a set of crucial cancellations that take place between the diagrams belonging to each of the subsets shown in Figs. 1–3, leaving a residual contribution that assumes exactly the form of the “seagull identity,” namely a special type of integral that vanishes in any regularization procedure preserving translational invariance, such as the dimensional regularization. As a consequence, the net effect of the combined action between the WIs and the seagull identity is the total annihilation of any type of term, finite or divergent, that could possibly contribute to  $\Delta^{-1}(0)$ .
- (4) Therefore, in order for the gluon SDE to yield  $\Delta^{-1}(0) = c$ , where  $c$  is finite and positive, one of the conditions imposed in the previous steps must be relaxed. Given that the STIs must remain intact, since they are a direct reflection of the underlying Becchi-Ruet-Stora-Tyutin (BRST) symmetry of the theory,<sup>1</sup> one must allow for the possibility that some of the vertex form factors contain the aforementioned type of poles.<sup>2</sup> In the present work we do not concern ourselves with the question of how such poles may be dynamically produced [57–63]. Instead, we assume their formation, and explain how the WIs are rearranged in their presence, producing the aforementioned cancellations, but leaving residual terms that give rise to the desired effect.

<sup>1</sup>We return to this point shortly.

<sup>2</sup>This general notion dates back to Schwinger’s seminal observation [55,56], according to which a gauge boson may acquire a mass, even if the gauge symmetry forbids a mass term at the level of the fundamental Lagrangian, provided that its vacuum polarization develops a pole at zero momentum transfer.

Given that certain variations on the aforementioned concepts have appeared in some of the cited literature, it is useful to briefly highlight the main novel aspects of the present approach.

- (i) A new form of the seagull identity, Eq. (2.7), is derived, which makes the “two-loop dressed” analysis far more transparent; indeed, in earlier works only the “one-loop dressed” diagrams had been addressed [64], while the cancellations operating at two loops had been neither identified nor implemented. Here, instead, *the entire set* of dressed diagrams comprising the gluon SDE is treated in a unified way.
- (ii) While in previous works we have relied on the Abelian STIs satisfied by the PT-BFM vertices [52], the main tools employed here are the corresponding WIs, given in Eqs. (3.11), (3.12) and (3.16). Even though these WIs are contained in the STIs, in the sense described in item 1 above, their use allows for a more elegant and concise demonstration of some of the most central results.
- (iii) Whereas in the past all related demonstrations were carried out in the Landau gauge, the constructions presented here are valid *for any value* of the gauge-fixing parameter, within the context of the linear covariant gauges.
- (iv) From the conceptual point of view, the theoretical approach elaborated here marks a gradual departure from the strict notion of a momentum-dependent gluon mass, first introduced in [12] and subsequently employed in a large number of studies. Specifically, in recent articles [63,65,66], the infra-red finite gluon propagator was parametrized as  $\Delta^{-1}(q^2) = q^2 J(q^2) + m^2(q^2)$ , where  $J(q^2)$  plays the role of the “kinetic term” or “wave-function” contribution, and  $m^2(q^2)$  that of the “effective gluon mass,” satisfying the crucial condition  $m^2(0) > 0$ . However, in contradistinction to the case of the quark propagator, where the Dirac structure of the wave and mass functions makes their separation completely unambiguous,  $J(q^2)$  and  $m^2(q^2)$  are strictly distinguishable only at  $q = 0$ , while, away from the origin, an arbitrary amount of  $J(q^2)$  may be allotted to  $m^2(q^2)$ . Instead, throughout our analysis we have refrained from using any such parametrization, treating  $\Delta(q^2)$  as a single function, and focusing exclusively on its behavior at the origin.
- (v) The above point affects substantially our understanding of the way in which the massless poles enforce the preservation of the corresponding STIs. Roughly speaking, previously the pole part of the three-gluon vertex [left-hand side (lhs) of the STI] would furnish, after its contraction with the appropriate momentum, the  $m^2(q^2)$ -components of the gluon propagators appearing on the right-hand side

(rhs) of the STI [63,65,66]. Instead, in the current interpretation, the contraction of the pole parts furnishes a piece that accounts for the infrared finiteness of the corresponding propagators at the origin, but has no *a priori* restriction on its form for general  $q^2$ . This novel point of view leads to a considerable reassessment of previous standpoints [63], as is clearly reflected in the present treatment of the ghost-gluon vertex, which may also possess poles without clashing with the nonperturbative masslessness of the ghost.

- (vi) Last but not least, we identify a procedure based on the particular form factor content of the WIs in the presence of poles, which, at least in principle, may corroborate or falsify the proposed mass-generating mechanism.

Let us now return to point 4 of the previous discussion, and make some additional clarifications. One of the main advantages of the PT-BFM formalism that we employ is that the transversality of the gluon self-energy may be systematically enforced at the level of the corresponding SDE, because the fully dressed vertices entering in its diagrammatic representation satisfy special (Abelian) STIs [see Eq. (3.5)]. In fact, the PT-BFM is the only framework where the stronger version of the blockwise transversality is realized [see Eq. (3.6)]. The gluon transversality remains formally exact even when a dynamical gluon mass is generated, because there is no modification whatsoever at the level of the original Yang-Mills Lagrangian; the infrared finiteness of the gluon propagator stems as a special solution of the same SDEs that were derived at the beginning. Therefore, the transition from a massless to a “massive” gluon propagator introduces no explicit BRST breaking at any step of the way. A subtle objection may be raised, however: one could argue that artifacts may appear that are not directly related to the actual mass-generating mechanism that we put forth here, but rather originate from the fact that our SDEs are derived within a Faddeev-Popov-type gauge-fixing scheme (namely the BFM), which offers no *a priori* control on the issue of the Gribov copies. This possibility is difficult to confirm or discard within the specific context of the SDEs themselves, because the information on any possible Gribov overcounting is dissipated when the minimization principle is implemented during their derivation. A reasonable answer may be conjectured (but not rigorously demonstrated) at least about the gluon transversality, by studying what happens to it in the context of a formalism such as the Gribov-Zwanziger (GZ) quantization procedure [67,68], which keeps the Gribov problem under control.

Within this latter formalism, the impact of the Gribov copies on the nonperturbative physics is reduced because the gauge condition is implemented by restricting the functional integral over gauge-field configurations that reside within the so-called “first Gribov region”

[34,68,69]. This restriction is imposed by adding to the usual (Landau) gauge-fixed Yang-Mills action a nonlocal term, known as the “horizon function.” This special term is accompanied by a massive parameter  $\gamma$ , known as the Gribov parameter, whose value is dynamically determined by a self-consistent condition (horizon condition<sup>3</sup>). It turns out that the restriction thusly imposed on the space of field configurations gives rise to a soft breaking of the BRST symmetry [74–76]. The softness of the breaking follows from the fact that the added term has dimension 2 in the fields; this, in turn, ensures the renormalizability of the theory through suitable STIs.<sup>4</sup>

Returning to the question of the transversality of the gluon self-energy, recently it was proved that the GZ action exhibits a modified nonperturbative BRST symmetry [77]. In this framework, the usual BRST transformation corresponds to a symmetry at the perturbative level, which must be adjusted when one approaches the nonperturbative regime. This extended symmetry has two main effects: (a) it controls the gauge-parameter dependence and makes the construction consistent with the gauge invariance, and (b) it protects the longitudinal component of the gluon propagator, which is not affected by quantum effects, exactly as happens in the standard Faddeev-Popov quantization scheme; this, in turn, is tantamount to the nonperturbative transversality of the gluon self-energy. We therefore conclude that, even though no rigorous analogy between two distinct formalisms (SDEs and GZ) can be drawn at this stage, it seems reasonable to infer that no Gribov-related artifacts affect the main principles of the present work.

The article is organized as follows: Sec. II is dedicated to the derivation of a new, compact version of the seagull identity. In Sec. III we derive in the context of the PT-BFM framework the WIs satisfied by the fundamental vertices of the theory. Then, in Sec. IV we demonstrate that, under the assumption that these vertices do not contain massless poles, the mixed quantum-background gluon self-energy vanishes at the origin. In Sec. V we carry out the renormalization, and show that the conclusions of the previous section persist at the level of the renormalized gluon self-energy. The main result of this article is presented in Sec. VI, where it is shown that the inclusion of poles in the vertices leads to a subtle distortion of the seagull cancellations, giving rise to an infrared finite gluon propagator. In Sec. VII we derive some additional results, and carry out a numerical analysis within a simplified

<sup>3</sup>In an alternative approach [70,71], a special average procedure over the Gribov copies is performed; the resulting local field theory is equivalent to the massive Curci-Ferrari model [72], and the bare gluon mass is related to the averaging weight that lifts the degeneracy between Gribov copies [73].

<sup>4</sup>An exact nonperturbative version of the nilpotent BRST operator for the Gribov-Zwanziger action in the linear covariant gauges has been recently introduced in [77,78].

setting, thus offering a concrete realization of the formal results obtained previously. Finally, in Sec. VIII we present our discussion and conclusions, and in a short appendix we report for completeness the relevant Feynman rules. Let us end this section by clarifying that our analysis is carried out using Feynman rules derived in the Minkowski space, and the transition to the Euclidean space, where the formulas of dimensional regularization are valid, is implemented in the last step. For practical purposes, this transition needs to be made explicit only in arriving at Eq. (4.29), and, more importantly, in Sec. VII, where the numerical analysis is performed.

## II. GENERALIZED SEAGULL IDENTITY

In this section we derive a more general and compact version of the seagull identity than the one presented in [64], which makes the implementation of the resulting seagull cancellations at the level of the gluon SDE far more transparent and efficient, allowing for a unified treatment of one-loop and two-loop dressed diagrams. In fact, a major advantage of this new formulation, to be exploited in the next section, is that it disentangles the action of this identity from the presence of explicit seagull diagrams, such as  $(a_2)$  and  $(a_4)$ , permitting the treatment of contributions that are effectively “seagull-like” (in the sense that they are quadratically divergent), but are concealed inside diagrams of more complicated topology, such as  $(a_5)$  and  $(a_6)$ .

To proceed with the derivation, let us introduce the integral measure of dimensional regularization,

$$\int_k \equiv \frac{\mu^\epsilon}{(2\pi)^d} \int d^d k, \quad (2.1)$$

where  $d = 4 - \epsilon$  and  $\mu$  is the 't Hooft mass, and consider the class of vector functions

$$\mathcal{F}_\mu(k) = f(k^2)k_\mu, \quad (2.2)$$

where, for the time being,  $f(k^2)$  is some arbitrary scalar function. Since  $\mathcal{F}_\mu$  is an odd function of  $k$ , one has immediately that in dimensional regularization

$$\int_k \mathcal{F}_\mu(k) = 0. \quad (2.3)$$

Next, impose on  $f(k^2)$  the condition originally introduced by Wilson [79], namely that, as  $k^2 \rightarrow \infty$ , it vanishes rapidly enough so that the integral (in spherical coordinates, with  $y = k^2$ )

$$\int_k f(k^2) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} \int_0^\infty dy y^{\frac{d}{2}-1} f(y) \quad (2.4)$$

converges for all positive values of  $d$  below a certain value of  $d^*$ . Then, the integral is well defined for any  $d$  within

$(0, d^*)$ , and can be analytically continued outside this interval.<sup>5</sup>

Observe now that within dimensional regularization (or any other scheme that preserves translational invariance) one may shift the argument of the function (2.2) by an arbitrary momentum  $q$  without compromising the result (2.3). Then, carrying out a Taylor expansion around  $q = 0$ , and using the result

$$\begin{aligned} \mathcal{F}_\mu(q+k) &= \mathcal{F}_\mu(k) + q^\nu \left\{ \frac{\partial}{\partial q^\nu} \mathcal{F}_\mu(q+k) \right\}_{q=0} + \mathcal{O}(q^2) \\ &= \mathcal{F}_\mu(k) + q^\nu \frac{\partial \mathcal{F}_\mu(k)}{\partial k^\nu} + \mathcal{O}(q^2), \end{aligned} \quad (2.5)$$

we obtain

$$q^\nu \int_k \frac{\partial \mathcal{F}_\mu(k)}{\partial k^\nu} = 0, \quad (2.6)$$

since, in agreement with Eq. (2.3), if we integrate both sides of the above Taylor expansion, the result must vanish order by order.

Given that the integral has two free Lorentz indices and no momentum scale, it can only be proportional to the metric tensor  $g_{\mu\nu}$ ; in addition, since  $q$  is arbitrary, one concludes that Eq. (2.6) is realized through the seagull identity

$$\int_k \frac{\partial \mathcal{F}_\mu(k)}{\partial k^\mu} = 0. \quad (2.7)$$

Using finally

$$\frac{\partial \mathcal{F}_\mu(k)}{\partial k^\mu} = 2k^2 \frac{\partial f(k^2)}{\partial k^2} + df(k^2), \quad (2.8)$$

we recover the original version of this identity, namely [64]

$$\int_k k^2 \frac{\partial f(k^2)}{\partial k^2} + \frac{d}{2} \int_k f(k^2) = 0. \quad (2.9)$$

In order to elaborate further on some of the concepts introduced in this section, we consider an explicit example, namely the case when  $f(k^2)$  is a massive tree-level propagator

$$f(k^2) = \frac{1}{k^2 - m^2}. \quad (2.10)$$

The following points are then worth mentioning:

<sup>5</sup>Additional requirements, such as analyticity of the function  $f(k^2)$  at  $k^2 = 0$ , may be needed in order to perform the analytic continuation of the integrals. Such cases, however, are not relevant to the present work; for a general discussion, see [80].

- (1) It is easy to verify using Eq. (2.4) that for both integrals appearing in Eq. (2.9) we have that  $d^* = 2$ , so that they both converge in the interval (0,2).
- (2) The validity of Eq. (2.9) may be verified explicitly in the case of Eq. (2.10), by applying the standard integration rules of dimensional regularization, namely

$$\begin{aligned} \int_k \frac{k^2}{(k^2 - m^2)^2} &= -i(4\pi)^{-\frac{d}{2}} \left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) (m^2)^{\frac{d}{2}-1}, \\ \int_k \frac{1}{k^2 - m^2} &= -i(4\pi)^{-\frac{d}{2}} \Gamma\left(1 - \frac{d}{2}\right) (m^2)^{\frac{d}{2}-1}. \end{aligned} \quad (2.11)$$

- (3) It is clear that if the function  $f(k^2)$  were such that the two integrals appearing in Eq. (2.9) converged for  $d = 4$  [for example,  $f(k^2) = (k^2 - m^2)^{-3}$ ], then its validity could be demonstrated through simple integration by parts, namely (suppressing the angular contribution)

$$\begin{aligned} \int_0^\infty dy y^{\frac{d}{2}} \frac{\partial f(y)}{\partial y} &= y^{\frac{d}{2}} f(y) \Big|_0^\infty \\ &\quad - \frac{d}{2} \int_0^\infty dy y^{\frac{d}{2}-1} f(y), \end{aligned} \quad (2.12)$$

and dropping the surface term. For the  $f(k^2)$  of Eq. (2.10) one may still interpret Eq. (2.9) as a result of an integration by parts, where the surface term  $\frac{y^{\frac{d}{2}}}{y+m^2} \Big|_0^\infty$  can be dropped if  $d < d^* = 2$ . The fact that one obtains the same value for  $d^*$  as above suggests an underlying self-consistency of the notions and techniques employed.

We end this section by emphasizing that the demonstration of the seagull identity presented above relies crucially on translational invariance,<sup>6</sup> which ultimately permits the shift of the integration variable. Therefore, regularization procedures that do not possess this property (such as the use of a hard cutoff) are bound to invalidate Eq. (2.7). This fact, in turn, may appear to clash with the numerical treatment that these equations may undergo, given that one generally introduces such cutoffs in the integration routines employed. However, this potential inconsistency can be avoided by recognizing that one may first implement the seagull cancellations formally, encode their implications manifestly into the relevant equations, and introduce the cutoffs necessary for their numerical evaluation only at the last step.

<sup>6</sup>We thank M. Lavelle for calling our attention to this important point.

### III. ABELIAN WARD IDENTITIES OF THE PT-BFM VERTICES

In this section we derive the basic WIs satisfied by the fully dressed vertices of the theory. Given that these vertices appear in the gluon SDE derived within the PT-BFM framework, their WIs are of central importance for the considerations that follow.

#### A. General framework

As has been explained in detail in the related literature, the PT-BFM formalism provides a manifestly BRST preserving truncation for the gluon propagator SDE, which ultimately governs its nonperturbative dynamics [30,31]. Within this framework it is particularly expeditious to employ directly the standard BFM procedure, and write the gauge field  $A_\mu^a$  as the sum of a background ( $B_\mu^a$ ) and a quantum ( $Q_\mu^a$ ) component, i.e.,  $A_\mu^a = B_\mu^a + Q_\mu^a$ . This splitting introduces a considerable proliferation of Green's function, composed by combinations of  $B$  and  $Q$  fields. Thus, for example, in the two-point sector, one has (i) the conventional gluon propagator, with two  $Q$ -type gluons ( $Q^2$ ), denoted by  $\Delta_{\mu\nu}^{ab}(q)$ , (ii) the mixed background-quantum propagator, with one  $Q$ - and one  $B$ -type gluon ( $QB$  or  $BQ$ ), denoted by  $\tilde{\Delta}_{\mu\nu}^{ab}(q)$ , and (iii) the background propagator, with two  $B$ -type gluons, denoted by  $\hat{\Delta}_{\mu\nu}^{ab}(q)$ .

In what follows we identify the quantum gauge-fixing parameter  $\xi_Q$  of the BFM, which appears inside quantum loops, with the corresponding parameter  $\xi$  introduced in the renormalizable  $R_\xi$  gauges, i.e.,  $\xi_Q = \xi$ . Thus, the  $Q^2$  gluon propagator  $\Delta_{\mu\nu}^{ab}(q) = \delta^{ab} \Delta_{\mu\nu}(q)$  is given by

$$\begin{aligned} \Delta_{\mu\nu}(q) &= -i \left[ \Delta(q^2) P_{\mu\nu}(q) + \xi \frac{q_\mu q_\nu}{q^4} \right]; \\ P_{\mu\nu}(q) &= g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \end{aligned} \quad (3.1)$$

with inverse

$$\begin{aligned} \Delta_\mu^\nu(q) \Delta_{\nu\rho}^{-1}(q) &= g_{\mu\rho}; \\ \Delta_{\nu\rho}^{-1}(q) &= i [\Delta^{-1}(q^2) P_{\nu\rho}(q) + \xi^{-1} q_\nu q_\rho]. \end{aligned} \quad (3.2)$$

The corresponding self-energy  $\Pi_{\mu\nu}(q)$  is transverse,  $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q) \Pi(q^2)$ , with

$$\Delta^{-1}(q^2) = q^2 + i \Pi(q^2). \quad (3.3)$$

Completely analogous expressions hold for the  $QB$  propagator  $\tilde{\Delta}_{\mu\nu}^{ab}(q)$ , with  $\Delta(q^2) \rightarrow \tilde{\Delta}(q^2)$  and  $\Pi(q^2) \rightarrow \tilde{\Pi}(q^2)$ . It is important to mention that, unlike what happens in QED, all scalar quantities defined above depend also on  $\xi$ , but this dependence is suppressed throughout.

Let us finally emphasize that  $\Delta(q^2)$ ,  $\tilde{\Delta}(q^2)$ , and  $\hat{\Delta}(q^2)$  are related by a set of formal, all-order “background-quantum identities” [81–83], namely

$$\begin{aligned}\Delta(q^2) &= [1 + G(q^2)]\tilde{\Delta}(q^2); \\ \tilde{\Delta}(q^2) &= [1 + G(q^2)]\hat{\Delta}(q^2),\end{aligned}\quad (3.4)$$

with  $G(q^2)$  being the  $g_{\mu\nu}$  component of a special function describing the ghost-gluon dynamics, defined in [31]. These identities are a consequence of the anti-BRST invariance [83], and may be generalized to any  $n$ -point function.

Consider now the SDE that controls the self-energy  $\tilde{\Pi}_{\mu\nu}(q)$  of the mixed  $QB$  propagator. The six diagrams comprising this self-energy are shown in Figs. 1–3. The fully dressed vertices appearing in the corresponding diagrams are denoted by  $\tilde{\Gamma}_{\mu\alpha\beta}$  ( $BQ^2$ ),  $\tilde{\Gamma}_\alpha$  ( $B\bar{c}c$ ), and  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}$  ( $BQ^3$ ); their definition and tree-level expressions are given in Appendix.

When contracted with the momentum carried by the  $B$  gluon, these vertices are known to satisfy Abelian STIs; specifically (all momenta entering),

$$\begin{aligned}q^\mu \tilde{\Gamma}_{\mu\alpha\beta}(q, r, p) &= i\Delta_{\alpha\beta}^{-1}(r) - i\Delta_{\alpha\beta}^{-1}(p), \\ q^\mu \tilde{\Gamma}_\mu(q, r, p) &= iD^{-1}(r^2) - iD^{-1}(p^2), \\ q^\mu \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}(q, r, p, t) &= f^{mse} f^{ern} \Gamma_{\alpha\beta\gamma}(r, p, q + t) \\ &\quad + f^{mne} f^{esr} \Gamma_{\beta\gamma\alpha}(p, t, q + r) \\ &\quad + f^{mre} f^{ens} \Gamma_{\gamma\alpha\beta}(t, r, q + p),\end{aligned}\quad (3.5)$$

where  $D(q^2)$  denotes the fully dressed ghost propagator, and the vertices appearing on the rhs of the last equation represent the conventional ( $Q^3$ ) full three-gluon vertices (see Appendix again).

By virtue of the special Abelian STIs of Eq. (3.5), it is relatively straightforward to prove the transversality of each set of diagrams in Figs. 1–3 [26], namely

$$q^\nu \tilde{\Pi}_{\mu\nu}^{(i)}(q) = 0, i = 1, 2, 3. \quad (3.6)$$

## B. Ward identities

In the ensuing analysis we are interested in the behavior of  $\Delta(0)$ , or, given the identity (3.4),  $\tilde{\Delta}(0)$ . Thus, the relevant Abelian STIs we need to consider are those obtained by determining the limit of the corresponding identities in Eq. (3.5) as the momentum  $q$  of the background gluon is taken to vanish.

The main ingredient that one needs in order to accomplish this task is the Taylor expansion of a function  $f(q, r, p)$  around  $q = 0$  (and  $p = -r$ ), given by

$$\begin{aligned}f(q, r, -r) &= f(0, r, -r) + q^\mu \left\{ \frac{\partial}{\partial q^\mu} f(q, r, p) \right\}_{q=0} \\ &\quad + \mathcal{O}(q^2),\end{aligned}\quad (3.7)$$

where any possible Lorentz and color structure of the function has been suppressed; evidently, after taking the derivative of  $f(q, r, -r - q)$  with respect to  $q^\mu$  and setting  $q = 0$ , the term in curly brackets on the rhs of Eq. (3.7) becomes a function of  $r$  only [of the general form  $r_\mu H(r^2)$ ].

In order to fix the ideas, let us first turn to the well-known Abelian model describing the interaction of a photon with a complex scalar field (scalar QED), and consider the Abelian STI (or Takahashi identity) satisfied by the full photon-scalar vertex  $\Gamma_\mu(q, r, p)$  [with  $\Gamma_\mu^{(0)}(q, r, p) = (p - r)_\mu$ ],

$$q^\mu \Gamma_\mu(q, r, p) = i\mathcal{D}^{-1}(r^2) - i\mathcal{D}^{-1}(p^2), \quad (3.8)$$

where  $\mathcal{D}(p^2)$  is the fully dressed propagator of the scalar field [and  $\mathcal{D}^{(0)}(p^2) = i/p^2$ ]. In order to determine the corresponding WI we expand both sides of Eq. (3.8) around  $q = 0$ , to obtain

$$\begin{aligned}q^\mu \Gamma_\mu(q, r, p) &= q^\mu \Gamma_\mu(0, r, -r) + \mathcal{O}(q^2) \\ &= -iq^\mu \left\{ \frac{\partial}{\partial q^\mu} \mathcal{D}^{-1}((q + r)^2) \right\}_{q=0} + \mathcal{O}(q^2).\end{aligned}\quad (3.9)$$

Then, equating the coefficients of the terms linear in  $q^\mu$ , one obtains the relation

$$\begin{aligned}\Gamma_\mu(0, r, -r) &= -i \left\{ \frac{\partial}{\partial q^\mu} \mathcal{D}^{-1}((q + r)^2) \right\}_{q=0} \\ &= -i \frac{\partial}{\partial r^\mu} \mathcal{D}^{-1}(r^2),\end{aligned}\quad (3.10)$$

which is the exact analogue of the familiar textbook WI valid for the photon-electron vertex of spinor QED.

Applying the procedure described above to the first two STIs of Eq. (3.5), we obtain the corresponding WIs of the  $BQ^2$  and  $B\bar{c}c$ , given by

$$\begin{aligned}\tilde{\Gamma}_{\mu\alpha\beta}(0, -p, p) &= i \frac{\partial}{\partial p^\mu} \Delta_{\alpha\beta}^{-1}(p); \\ \tilde{\Gamma}_{\mu\alpha\beta}(0, r, -r) &= -i \frac{\partial}{\partial r^\mu} \Delta_{\alpha\beta}^{-1}(r),\end{aligned}\quad (3.11)$$

and, similarly,

$$\begin{aligned}\tilde{\Gamma}_\mu(0, -p, p) &= i \frac{\partial}{\partial p^\mu} D^{-1}(p^2); \\ \tilde{\Gamma}_\mu(0, r, -r) &= -i \frac{\partial}{\partial r^\mu} D^{-1}(r^2).\end{aligned}\quad (3.12)$$

In the case of the  $BQ^3$  vertex the derivation of the corresponding WI is slightly more involved. Specifically, consider the rhs of the last identity of Eq. (3.5); then the vertex  $\Gamma_{\mu\nu\rho}$  stays the same since it does not depend on  $q$ , whereas we have

$$\begin{aligned}\Gamma_{\beta\gamma\alpha}(p, t, q + r) &= \Gamma_{\beta\gamma\alpha}(p, -r - p, r) \\ &+ q^\mu \left\{ \frac{\partial}{\partial q^\mu} \Gamma_{\beta\gamma\alpha}(p, t, q + r) \right\}_{q=0} + \mathcal{O}(q^2),\end{aligned}\quad (3.13)$$

and similarly for the  $\Gamma_{\rho\mu\nu}$  term.

Then, since by Bose symmetry we have

$$\begin{aligned}\Gamma_{\alpha\beta\gamma}(r, p, -r - p) &= \Gamma_{\beta\gamma\alpha}(p, -r - p, r) \\ &= \Gamma_{\gamma\alpha\beta}(-r - p, r, p),\end{aligned}\quad (3.14)$$

the zeroth order terms vanish, by virtue of the Jacobi identity,

$$(f^{mse} f^{ern} + f^{mne} f^{esr} + f^{mre} f^{ens}) \Gamma_{\alpha\beta\gamma}(r, p, -r - p) = 0. \quad (3.15)$$

Finally, after elementary manipulations, the terms linear in  $q$  yield

$$\begin{aligned}\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnr s}(0, r, p, -r - p) &= \left( f^{mne} f^{esr} \frac{\partial}{\partial r^\mu} + f^{mre} f^{ens} \frac{\partial}{\partial p^\mu} \right) \\ &\times \Gamma_{\alpha\beta\gamma}(r, p, -r - p), \\ \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnr s}(0, -r, -p, r + p) &= - \left( f^{mne} f^{esr} \frac{\partial}{\partial r^\mu} + f^{mre} f^{ens} \frac{\partial}{\partial p^\mu} \right) \\ &\times \Gamma_{\alpha\beta\gamma}(-r, -p, r + p).\end{aligned}\quad (3.16)$$

Equations (3.11), (3.12) and (3.16) constitute the central results of this section. To the best of our knowledge these special WIs appear for the first time in the literature.

### C. Ward identities and vertex form factors

Let us finally consider how the above WIs reflect themselves at the level of the form factors that appear in the tensorial decomposition of the corresponding vertices.

The simplest case is that of the (background) gluon-ghost vertex, which, in terms of the two momenta  $q$  and  $r$ , has the general form

$$\tilde{\Gamma}^\mu(q, r, p) = \tilde{\mathcal{A}}_1(q^2, r^2, p^2) q^\mu + \tilde{\mathcal{A}}_2(q^2, r^2, p^2) r^\mu. \quad (3.17)$$

In compliance with the assumptions made when deriving the WIs of Eqs. (3.11)–(3.13), we postulate for the moment that the form factors  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_2$  do not contain poles (kinematic or dynamical) in  $q^2$ . From Eq. (3.17), and using

the above assumption, we obtain immediately that, when  $q = 0$ ,

$$\tilde{\Gamma}^\mu(0, r, -r) = \tilde{\mathcal{A}}_2(r^2) r^\mu \quad (3.18)$$

and, therefore, from Eq. (3.12) it follows directly that

$$\tilde{\mathcal{A}}_2(r^2) = -2i \frac{\partial}{\partial r^2} D^{-1}(r^2). \quad (3.19)$$

An elementary check of this result is to consider the tree-level vertex at  $q = 0$ , namely (see Appendix)

$$\tilde{\Gamma}_\mu^{(0)}(0, r, -r) = -2r_\mu, \quad (3.20)$$

which is indeed what one obtains from Eq. (3.19) after setting  $D^{-1}(r^2) = -ir^2$ .

Next, the tensorial decomposition for  $\tilde{\Gamma}^{\mu\alpha\beta}(q, r, p)$  that is suitable for our purposes reads

$$\tilde{\Gamma}^{\mu\alpha\beta}(q, r, p) = \sum_{i=1}^{14} \tilde{\mathcal{A}}_i(q^2, r^2, p^2) b_i^{\mu\alpha\beta}, \quad (3.21)$$

where the basis  $b_{\mu\alpha\beta}^i$  is chosen to be

$$\begin{aligned}b_1^{\mu\alpha\beta} &= q^\mu g^{\alpha\beta}; & b_2^{\mu\alpha\beta} &= q^\mu q^\alpha q^\beta; & b_3^{\mu\alpha\beta} &= q^\mu q^\alpha r^\beta; \\ b_4^{\mu\alpha\beta} &= q^\mu r^\alpha q^\beta; & b_5^{\mu\alpha\beta} &= q^\mu r^\alpha r^\beta; & b_6^{\mu\alpha\beta} &= r^\mu g^{\alpha\beta}; \\ b_7^{\mu\alpha\beta} &= r^\mu q^\alpha q^\beta; & b_8^{\mu\alpha\beta} &= r^\mu q^\alpha r^\beta; & b_9^{\mu\alpha\beta} &= r^\mu r^\alpha q^\beta; \\ b_{10}^{\mu\alpha\beta} &= r^\mu r^\alpha r^\beta; & b_{11}^{\mu\alpha\beta} &= q^\alpha g^{\beta\mu}; & b_{12}^{\mu\alpha\beta} &= q^\beta g^{\alpha\mu}; \\ b_{13}^{\mu\alpha\beta} &= r^\alpha g^{\beta\mu}; & b_{14}^{\mu\alpha\beta} &= r^\beta g^{\alpha\mu}.\end{aligned}\quad (3.22)$$

As in the previous case, the form factors  $\tilde{\mathcal{A}}_i$  are assumed without poles in  $q^2$ . Then, in the limit  $q \rightarrow 0$ , only the components  $b_6$ ,  $b_{10}$ ,  $b_{13}$ , and  $b_{14}$  survive, so that

$$\begin{aligned}\tilde{\Gamma}^{\mu\alpha\beta}(0, r, -r) &= \tilde{\mathcal{A}}_6(r^2) r^\mu g^{\alpha\beta} + \tilde{\mathcal{A}}_{10}(r^2) r^\mu r^\alpha r^\beta \\ &+ \tilde{\mathcal{A}}_{13}(r^2) r^\alpha g^{\beta\mu} + \tilde{\mathcal{A}}_{14}(r^2) r^\beta g^{\alpha\mu}.\end{aligned}\quad (3.23)$$

If we now use the general expression for the gluon inverse propagator given in Eq. (3.2) to evaluate the rhs of the WI in Eq. (3.11), and match the resulting tensorial structures with those of Eq. (3.23), we obtain

$$\begin{aligned}\tilde{\mathcal{A}}_6(r^2) &= 2 \frac{\partial}{\partial r^2} \Delta^{-1}(r^2); \\ \tilde{\mathcal{A}}_{10}(r^2) &= -2 \frac{\partial}{\partial r^2} \left( \frac{\Delta^{-1}(r^2)}{r^2} \right), \\ \tilde{\mathcal{A}}_{13}(r^2) &= \tilde{\mathcal{A}}_{14}(r^2) = \xi^{-1} - \frac{\Delta^{-1}(r^2)}{r^2}.\end{aligned}\quad (3.24)$$

It is easy to verify the validity of the above results at tree level, since the  $BQ^2$  vertex at  $q = 0$  reads (see Appendix)

$$\begin{aligned}\tilde{\Gamma}_{\mu\alpha\beta}^{(0)}(0, r, -r) &= 2r_\mu g_{\alpha\beta} + r_\alpha g_{\beta\mu}(\xi^{-1} - 1) \\ &\quad + r_\beta g_{\alpha\mu}(\xi^{-1} - 1),\end{aligned}\quad (3.25)$$

implying immediately that

$$\begin{aligned}\tilde{A}_6^{(0)}(r^2) &= 2; \quad \tilde{A}_{10}^{(0)}(r^2) = 0; \\ \tilde{A}_{13}^{(0)}(r^2) &= \tilde{A}_{14}^{(0)}(r^2) = \xi^{-1} - 1,\end{aligned}\quad (3.26)$$

which is exactly what Eq. (3.24) yields upon setting  $\Delta^{-1}(r^2) = r^2$ .

The analogous construction for the case of the four-gluon vertex  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}(q, r, p, t)$  is particularly cumbersome, given the vast proliferation of tensorial structures appearing in its Lorentz decomposition [84–86], and is not carried out. As we see in what follows, although results such as Eqs. (3.19) and (3.24) must be used in order to trigger Eq. (2.9) at the one-loop dressed level, the new form of the seagull identity makes no reference to them, and, most importantly, obviates the need to dwell on the tensorial decomposition of  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}(q, r, p, t)$ .

#### IV. VANISHING OF $\tilde{\Pi}_{\mu\nu}(0)$ IN THE ABSENCE OF POLES

Within dimensional regularization, formulas such as

$$\int_k \frac{\ln^n(k^2/\mu^2)}{k^2} = 0, n = 0, 1, 2, \dots \quad (4.1)$$

enforce the masslessness of the gluon to all orders in perturbation theory. The most obvious source of such contributions is the so-called seagull diagrams, such as  $(a_2)$  in Fig. 1. However, similar contributions are concealed inside diagrams that do not have the topological form associated with seagull graphs, such as  $(a_1)$  in Fig. 1, and most notably the two-loop diagrams of Fig. 3.

The situation becomes significantly more complicated nonperturbatively, since there is no mathematical justification in assuming, for example, that the equivalent expression of Eq. (4.1) for  $n = 0$ , namely  $\int_k \Delta(k^2)$ , vanishes. But if these contributions are not allowed to vanish individually, they are actually quadratically divergent, namely they behave as a  $\Lambda^2$  in the hard cutoff treatment, or as  $m^2(1/\epsilon)$  in dimensional regularization. The disposal of such divergences, in turn, requires the inclusion in the original Lagrangian of a counterterm of the form  $m^2 A_\mu^2$ , which is *strictly* forbidden by the gauge (viz. BRST) invariance.

In this section we use the Abelian WIs derived above in order to cast the formal expressions that determine the  $g_{\mu\nu}$  component of the gluon self-energy at the origin into a very particular form. Specifically, we demonstrate that various terms that could potentially lead to seagull-like contributions cancel against each other, both at one- and at two-loop dressed level, and that the remainder vanishes because it

triggers precisely the seagull identity of Eq. (2.7). The upshot of all this is that  $\tilde{\Pi}_{\mu\nu}(0)$  is not only finite, and, therefore, no modifications to the original Lagrangian are required in the sense described above, but it is, in fact, exactly 0.

In order to forestall possible confusion, we emphasize that all cancellations among different diagrams identified in this section persist unaltered in the case when the main assumption of the absence of poles is relaxed, to be presented in Sec. VI. As a result, all quadratically divergent terms cancel against each other as before, and the only crucial difference that converts  $\tilde{\Pi}_{\mu\nu}(0)$  from vanishing to finite appears in the last step of implementing Eq. (2.7).

#### A. General considerations

The exact (blockwise) transversality of  $\tilde{\Pi}_{\mu\nu}(q)$  guarantees that the form factors of  $g_{\mu\nu}$  and  $q_\mu q_\nu/q^2$  are equal and opposite in sign; therefore, at least in principle, one may obtain  $\tilde{\Pi}_{\mu\nu}(0)$  by studying the behavior of either one of these two form factors as  $q^2 \rightarrow 0$ . However, the mathematical steps required for reaching the final answer are completely different for both cases; in particular, the manipulation of the  $g_{\mu\nu}$  cofactor is highly nontrivial, requiring full use of the ingredients developed in the previous sections, whereas the treatment of the  $q_\mu q_\nu/q^2$  counterpart is fairly straightforward.

A typical example of this inequivalence is encountered in the treatment of the expression [66]

$$I_{\mu\nu}(q) = \int_k k_\mu k_\nu f(k, q), \quad (4.2)$$

where  $f(k, q)$  is an arbitrary function that remains finite in the limit  $q \rightarrow 0$ . Clearly,

$$I_{\mu\nu}(q) = g_{\mu\nu} A(q^2) + \frac{q_\mu q_\nu}{q^2} B(q^2), \quad (4.3)$$

and the form factors  $A(q^2)$  and  $B(q^2)$  are given by

$$\begin{aligned}A(q^2) &= \frac{1}{d-1} \int_k \left[ k^2 - \frac{(k \cdot q)^2}{q^2} \right] f(k, q), \\ B(q^2) &= -\frac{1}{d-1} \int_k \left[ k^2 - d \frac{(k \cdot q)^2}{q^2} \right] f(k, q).\end{aligned}\quad (4.4)$$

Then, setting  $(q \cdot k)^2 = q^2 k^2 \cos^2 \theta$ , and using that, for any function  $f(k^2)$

$$\int_k \cos^2 \theta f(k^2) = \frac{1}{d} \int_k f(k^2), \quad (4.5)$$

we obtain from Eq. (4.4) that, as  $q \rightarrow 0$ ,

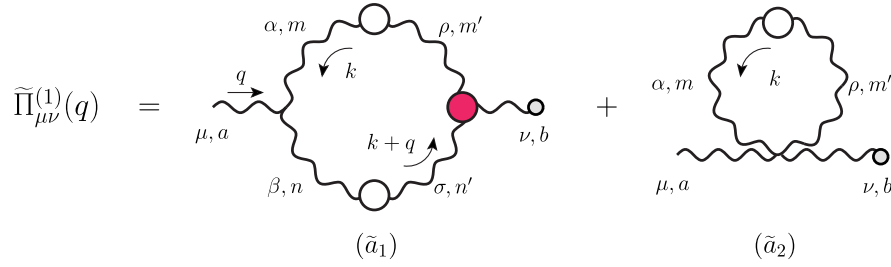


FIG. 1. One-loop dressed gluon diagrams contributing to the SDE of the  $QB$  gluon self-energy. White circles indicate fully dressed propagators and the red circle indicates a fully dressed three-gluon vertex  $\tilde{\Gamma}$ .

$$A(0) = \frac{1}{d} \int_k k^2 f(k^2); \quad B(0) = 0. \quad (4.6)$$

Evidently, the function  $f(k^2)$  may be such that the integral defining  $A(0)$  diverges, while, for the same function,  $B(0)$  vanishes; for example, if  $f(k^2) = \Delta(k^2)$  or  $f(k^2) = D(k^2)$  one obtains quadratically divergent  $g_{\mu\nu}$  components.

It is also clear from the above analysis that  $A(0)$  and  $B(0)$  may be determined through the simpler operation of setting  $q = 0$  directly in the rhs of Eq. (4.2),

$$I_{\mu\nu}(0) = \int_k k_\mu k_\nu f(k^2), \quad (4.7)$$

which can be only proportional to  $g_{\mu\nu}$ . Thus, in the absence of a contribution proportional to  $q_\mu q_\nu / q^2$ , one recovers immediately that  $B(0) = 0$ ; the value of  $A(0)$  may be obtained from Eq. (4.7) by simply taking the trace, and coincides with that given in Eq. (4.6).

In view of the observations made above, the procedure that we follow is to consider the fully dressed diagrams,  $(a_j)_{\mu\nu}(q)$ , of each subset in Figs. 1–3, and set in them directly  $q = 0$ . Since the resulting tensorial structures may be only saturated by  $g_{\mu\nu}$ , the  $QB$  self-energy at  $q = 0$  simply reads (color indices are suppressed whenever possible)

$$\begin{aligned} \tilde{\Pi}_{\mu\nu}(0) &= \tilde{\Pi}(0) g_{\mu\nu}; \\ \tilde{\Pi}(0) &= d^{-1} \tilde{\Pi}_\mu^\mu(0) = \sum_{i=1}^3 \tilde{\Pi}^{(i)}(0), \end{aligned} \quad (4.8)$$

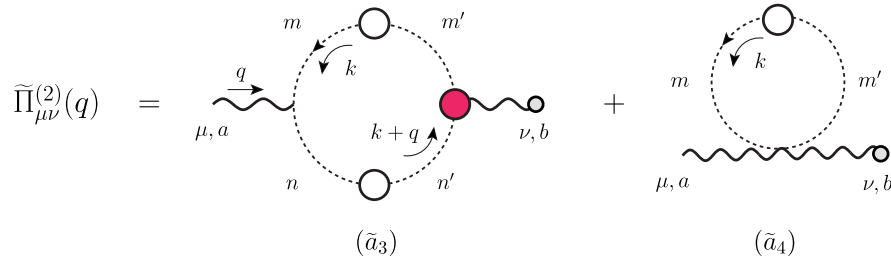


FIG. 2. One-loop dressed ghost diagrams.

where  $\tilde{\Pi}^{(i)}(0)$  are obtained by taking the Lorentz trace of the corresponding diagrams, evaluated at  $q = 0$ . We denote any such trace by  $a_j(q) \equiv (a_j)_\mu^\mu(q)$ , and, in particular,  $a_j(0) \equiv (a_j)_\mu^\mu(0)$ .

### B. One-loop dressed gluon diagrams

We start with  $\tilde{\Pi}_{\mu\nu}^{(1)}(q)$ , which is given by the sum of the two diagrams shown in Fig. 1; evidently

$$d\tilde{\Pi}^{(1)}(0) = \tilde{a}_1(0) + \tilde{a}_2(0), \quad (4.9)$$

with

$$\tilde{a}_1(0) = \frac{1}{2} g^2 C_A \int_k \Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) \Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k) \tilde{\Gamma}_{\sigma\rho}^\mu(0, -k, k), \quad (4.10)$$

$$\tilde{a}_2(0) = -ig^2 C_A (d-1) \int_k \Delta_\alpha^\alpha(k), \quad (4.11)$$

where  $C_A$  represents the Casimir eigenvalue of the adjoint representation [ $N$  for  $SU(N)$ ], and (see again Appendix)

$$\Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) = 2k_\mu g_{\alpha\beta} - k_\beta g_{\alpha\mu} - k_\alpha g_{\beta\mu}. \quad (4.12)$$

Using then the WI Eq. (3.11), we derive the relation

$$\Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k) \tilde{\Gamma}_{\sigma\rho}^\mu(0, -k, k) = -i \frac{\partial}{\partial k_\mu} \Delta^{\alpha\beta}(k), \quad (4.13)$$

so that, after integrating by parts, we may cast Eq. (4.10) in the form

$$\tilde{a}_1(0) = -\frac{i}{2}g^2C_A \left\{ \int_k \frac{\partial}{\partial k^\mu} [\Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) \Delta^{\alpha\beta}(k)] - \int_k \Delta^{\alpha\beta}(k) \frac{\partial}{\partial k^\mu} \Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) \right\}. \quad (4.14)$$

Due to the result

$$\frac{\partial}{\partial k^\mu} \Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) = 2(d-1)g_{\alpha\beta}, \quad (4.15)$$

we then see that the second term of Eq. (4.14) cancels exactly against the  $\tilde{a}_2(0)$  of Eq. (4.11), and we are left with the result

$$d\tilde{\Pi}^{(1)}(0) = -g^2C_A(d-1) \int_k \frac{\partial}{\partial k_\mu} \mathcal{F}_\mu^{(1)}(k); \quad \mathcal{F}_\mu^{(1)}(k) = k_\mu \Delta(k^2). \quad (4.16)$$

### C. One-loop dressed ghost diagrams

Turning to  $\tilde{\Pi}_{\mu\nu}^{(2)}(q)$ , the ghost diagrams of Fig. 2 at  $q = 0$  give

$$\tilde{a}_3(0) = g^2C_A \int_k k_\mu D^2(k^2) \tilde{\Gamma}^\mu(0, -k, k), \quad (4.17)$$

$$\tilde{a}_4(0) = -ig^2C_A d \int_k D(k^2). \quad (4.18)$$

Then, from the identity (3.12) we obtain

$$D^2(k^2) \tilde{\Gamma}^\mu(0, -k, k) = -i \frac{\partial D(k^2)}{\partial k^\mu}, \quad (4.19)$$

and therefore, after integrating by parts,  $a_3(0)$  yields

$$\tilde{a}_3(0) = -ig^2C_A \left\{ \int_k \frac{\partial}{\partial k^\mu} [\Gamma^\mu(0, -k, k) D(k^2)] - d \int_k D(k^2) \right\}. \quad (4.20)$$

Clearly, the second term in Eq. (4.20) cancels exactly against  $a_4(0)$ , and we are left with

$$d\tilde{\Pi}^{(2)}(0) = -ig^2C_A \int_k \frac{\partial}{\partial k_\mu} \mathcal{F}_\mu^{(2)}(k); \quad \mathcal{F}_\mu^{(2)}(k) = k_\mu D(k^2). \quad (4.21)$$

### D. Two-loop dressed diagrams

The two-loop dressed gluon self-energy,  $\tilde{\Pi}_{\mu\nu}^{(3)}(q)$ , is given by the sum of the two diagrams in Fig. 3, which at  $q = 0$  read

$$\begin{aligned} \tilde{a}_5^{ab}(0) &= -\frac{1}{6}g^4\Gamma_{\mu\alpha\beta\gamma}^{(0)amnr} \int_k \int_\ell \Delta^{\alpha\rho}(k+\ell) \Delta^{\beta\sigma}(\ell) \\ &\quad \times \Delta^{\gamma\tau}(k) \tilde{\Gamma}_{\mu\tau\sigma\rho}^{\text{brnm}}(0, -k, -\ell, k+\ell), \\ \tilde{a}_6(0) &= -i\mathcal{N}_{\mu\alpha\beta\gamma} \int_k Y_\delta^{\alpha\beta}(k) \Delta^{\gamma\tau}(k) \Delta^{\delta\lambda}(k) \tilde{\Gamma}_{\tau\lambda}^\mu(0, k, -k), \end{aligned} \quad (4.22)$$

where we have defined

$$\mathcal{N}_{\mu\alpha\beta\gamma} = \frac{3}{4}g^4C_A^2(g_{\mu\alpha}g_{\beta\gamma} - g_{\mu\beta}g_{\alpha\gamma}), \quad (4.23)$$

and

$$Y_\delta^{\alpha\beta}(k) = \int_\ell \Delta^{\alpha\rho}(k+\ell) \Delta^{\beta\sigma}(k+\ell) \Gamma_{\sigma\rho\delta}(\ell, -k-\ell, k), \quad (4.24)$$

which is proportional to the subdiagram nested inside ( $a_6$ ). One may show that, due to the Bose symmetry of the vertex  $\Gamma_{\sigma\rho\delta}$ ,  $Y_\delta^{\alpha\beta}(k)$  assumes the form [66]

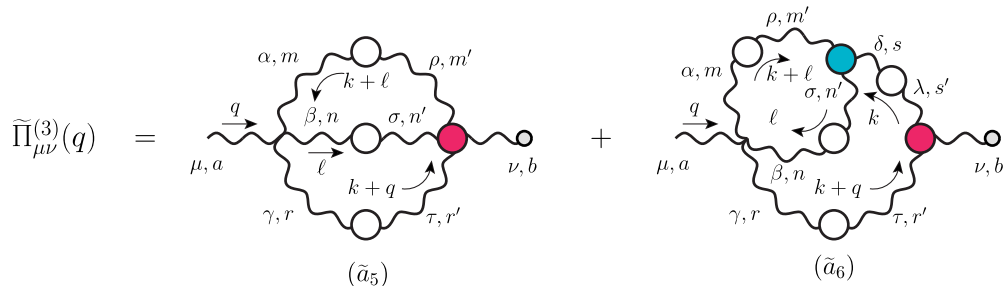


FIG. 3. Two-loop gluon dressed diagrams.

$$Y_{\delta}^{\alpha\beta}(k) = (k^{\alpha}g_{\delta}^{\beta} - k^{\beta}g_{\delta}^{\alpha})Y(k^2);$$

$$Y(k^2) = \frac{1}{d-1} \frac{1}{k^2} k_{\alpha} g_{\beta}^{\delta} Y_{\delta}^{\alpha\beta}(k). \quad (4.25)$$

Then, using Eq. (4.13), we immediately obtain

$$\tilde{a}_6(0) = -\mathcal{N}_{\mu\alpha\beta\gamma} \int_k Y_{\delta}^{\alpha\beta}(k) \frac{\partial \Delta^{\gamma\delta}(k)}{\partial k^{\mu}}. \quad (4.26)$$

Even though  $a_6(0)$  eventually cancels in its entirety against an analogous contribution from  $a_5(0)$ , it is instructive to evaluate its form a bit further. Specifically, choosing for simplicity the Landau gauge ( $\xi = 0$ ) and using Eq. (4.25), we find

$$\tilde{a}_6(0) = -\frac{3}{2}(d-1)g^4 C_A^2 \int_k [\Delta(k^2) + 2k^2 \Delta'(k^2)] Y(k^2). \quad (4.27)$$

At this point one may set  $\Delta(k^2) \rightarrow 1/k^2$  into Eq. (4.27), and use the lowest order perturbative expression for  $Y(k^2)$  [66],

$$Y(k^2) = \frac{-5i}{64\pi^2} \log\left(\frac{-k^2}{\mu^2}\right), \quad (4.28)$$

to obtain (in Euclidean space)

$$\tilde{a}_6^{\text{pert}}(0) = c \int_k \frac{\ln(k^2/\mu^2)}{k^2}, \quad (4.29)$$

with  $c$  being an irrelevant numerical constant; an exactly analogous expression is obtained for a general value of the gauge-fixing parameter. Evidently, Eq. (4.29) corresponds to the  $n = 1$  case of Eq. (4.1), and therefore vanishes. This, however, is no longer true nonperturbatively, and  $a_6(0)$  yields, up to logarithms, a quadratically divergent contribution.

Turning to the contribution coming from the other two-loop diagram, after employing the crucial WI of Eq. (3.16), we find that

$$\begin{aligned} \tilde{a}_5^{ab}(0) = & -\frac{1}{6} g^4 \Gamma_{\mu\alpha\beta\gamma}^{(0)amnr} \int_k \int_{\ell} \Delta^{\gamma\tau}(k) \Delta^{\beta\sigma}(\ell) \Delta^{\alpha\rho}(k+\ell) \\ & \times \left( f^{bre} f^{emn} \frac{\partial}{\partial k^{\mu}} + f^{bne} f^{erm} \frac{\partial}{\partial \ell^{\mu}} \right) \\ & \times \Gamma_{\tau\sigma\rho}(k, \ell, k+\ell). \end{aligned} \quad (4.30)$$

Next, we make use of the identities

$$\begin{aligned} f^{brx} f^{xmn} \Gamma_{\mu\alpha\beta\gamma}^{(0)amnr} &= \frac{3}{2} C_A^2 \delta^{ab} (g_{\mu\alpha} g_{\beta\gamma} - g_{\mu\beta} g_{\alpha\gamma}), \\ f^{bnx} f^{xrm} \Gamma_{\mu\alpha\beta\gamma}^{(0)amnr} &= \frac{3}{2} C_A^2 \delta^{ab} (g_{\mu\gamma} g_{\alpha\beta} - g_{\mu\alpha} g_{\beta\gamma}), \end{aligned} \quad (4.31)$$

integrate by parts and carry out the appropriate shifts in the integration momenta to obtain

$$\begin{aligned} \tilde{a}_5(0) = & -\mathcal{N}_{\alpha\beta\gamma}^{\mu} \left\{ \frac{2}{3} \int_k \frac{\partial}{\partial k^{\mu}} [\Delta^{\gamma\delta}(k) Y_{\delta}^{\alpha\beta}(k)] \right. \\ & \left. - \int_k Y_{\delta}^{\alpha\beta}(k) \frac{\partial \Delta^{\gamma\delta}(k)}{\partial k^{\mu}} \right\}, \end{aligned} \quad (4.32)$$

where the color factor  $\delta^{ab}$  has now been omitted. Evidently, the second term on the rhs cancels exactly the entire contribution Eq. (4.26), as anticipated, and we finally obtain

$$\begin{aligned} d\tilde{\Pi}^{(3)}(0) &= i(d-1)g^4 C_A^2 \int_k \frac{\partial}{\partial k^{\mu}} \mathcal{F}_{\mu}^{(3)}(k); \\ \mathcal{F}_{\mu}^{(3)}(k) &= k_{\mu} \Delta(k^2) Y(k^2). \end{aligned} \quad (4.33)$$

In summary, the above demonstration establishes that, under the pivotal assumption of the absence of  $q^2$ -type poles in the form factors of the fundamental vertices,

$$\tilde{\Pi}^{(i)}(0) = 0; \quad i = 1, 2, 3 \Rightarrow \tilde{\Pi}(0) = \sum_{i=1}^3 \tilde{\Pi}^{(i)}(0) = 0, \quad (4.34)$$

which, since  $\tilde{\Delta}^{-1}(q^2) = q^2 + i\tilde{\Pi}(q^2)$ , leads to the conclusion that

$$\tilde{\Delta}^{-1}(0) = 0. \quad (4.35)$$

In order to extract from Eq. (4.35) the behavior of the conventional (quantum) quantity  $\Delta^{-1}(q^2)$  at the origin, one additional step is necessary. Specifically, we must employ the crucial identity Eq. (3.4), which, at  $q = 0$  yields

$$\Delta^{-1}(0) = \frac{\tilde{\Delta}^{-1}(0)}{1 + G(0)}. \quad (4.36)$$

If we now introduce the additional assumption that  $1 + G(0)$  is finite for every  $\xi$  (see the discussion below), then we reach the final conclusion that, in the absence of poles,

$$\Delta^{-1}(0) = 0. \quad (4.37)$$

## E. Additional remarks

- (1) It is important to recognize that the proofs elaborated in the previous subsections are valid for any value of the gauge-fixing parameter  $\xi$  within the class of the linear covariant gauges ( $R_{\xi}$ ); indeed, at no moment has it been necessary to choose a specific value for  $\xi$ .

- (2) Related to this point, in a general  $R_\xi$  gauge there exists a relation between  $G$  and the ghost dressing function  $F$  which reads [83]

$$F^{-1}(q^2) = 1 + G(q^2) + L(q^2) + \xi K(q^2), \quad (4.38)$$

where  $L$  is the  $q_\mu q_\nu / q^2$  component of the function mentioned right after Eq. (3.4), and  $K$  originates from the coupling of the antighost to a certain anti-BRST source, necessary for formulating the theory in the background field method (or, equivalently, rendering the conventional theory anti-BRST invariant) [83]. When  $\xi = 0$ , we know that  $F(0) \neq 0$  while  $L(0) = 0$  [87]; therefore, from Eq. (4.38) one obtains

$$F^{-1}(0) = 1 + G(0), \quad (4.39)$$

which ensures that the quantity  $1 + G(0)$  is finite in the Landau gauge. For  $\xi \neq 0$  the situation is more complicated. Recent analytical studies [88,89] indicate that in this case the ghost dressing function goes to 0, which, in turn, suggests that the lhs of Eq. (4.38) diverges. At the same time, however, they also show that the gluon propagator continues to saturate in the IR, a result that has been confirmed by lattice simulations [9]. If the behavior of the ghost dressing function is to be confirmed, these results point towards some highly nontrivial dynamics in the ghost sector, which ensure that the function  $K$  (and possibly  $L$ ) appearing in Eq. (4.38) cancels the divergence of the ghost dressing function, leaving a finite  $1 + G(0)$  when  $\xi \neq 0$ .

- (3) Note that in the demonstrations presented no particular form for the propagators  $\Delta$  (or  $D$ ) appearing in the one- and two-loop dressed diagrams has been used. In fact, even if one were to *assume* that the gluon propagator circulating in them is of a massive type,<sup>7</sup> the seagull identity still annihilates the individual contributions  $\tilde{\Pi}^{(1,2,3)}(0)$ , yielding  $\Delta^{-1}(0) = 0$ , in contradiction with the original assumption. The main lesson drawn from this observation is that the theory, when properly treated, resists the generation of a mass, due to the operation of a very subtle cancellation mechanism. Of course, the realization of this mechanism hinges crucially on the absence of poles in the vertices of the theory; it is precisely the relaxation of this assumption that eventually permits the emergence of infrared finite gluon propagators, as explained in the continuation of this article.

<sup>7</sup>For instance, one could consider a propagator of the Cornwall type [12],  $\Delta(q^2) = 1/[q^2 + m^2(q^2)]$ , or of the Stingl form [90],  $\Delta(q^2) = c(1 + aq^2)/[(q^2 + m^2)^2 + b^2]$ , with  $a, b, c$  suitable parameters.

- (4) We emphasize that no specific ansatz for any of the fully dressed vertices appearing in the above derivations has been employed. This is to be contrasted with the original demonstration of the seagull cancellation presented in [64]: there, a gauge-technique-inspired ansatz was used for  $\tilde{\Gamma}_{\mu\alpha\beta}(q, r, p)$ , satisfying (by construction) the first Abelian STI of Eq. (3.5), which, when inserted into graph ( $a_1$ ), activated the original form of the seagull identity, Eq. (2.9). This is particularly relevant in the case of the four-gluon vertex, where, due to the complexity of the corresponding STI, the construction of such an Ansatz has never been presented in the literature.
- (5) It is instructive to demonstrate the vanishing of  $\tilde{\Pi}^{(1)}(0)$  and  $\tilde{\Pi}^{(2)}(0)$  using the form of the relevant form factors, Eqs. (3.19) and (3.24), to activate Eq. (2.9). In the case of  $\tilde{\Pi}^{(2)}(0)$ , we substitute directly Eqs. (3.18) and (3.19) into  $a_3(0)$  ( $r \rightarrow k$ ),

$$\begin{aligned} \tilde{a}_3(0) &= 2ig^2 C_A \int_k k^2 D^2(k^2) \frac{\partial}{\partial k^2} D^{-1}(k^2) \\ &= -2ig^2 C_A \int_k k^2 \frac{\partial}{\partial k^2} D(k^2), \end{aligned} \quad (4.40)$$

which, when added to  $a_4(0)$ , triggers Eq. (2.9) with  $f(k^2) = D(k^2)$ . The case of  $\tilde{\Pi}^{(1)}(0)$  proceeds in a similar fashion, but is operationally slightly more involved. Substituting Eqs. (3.23) and (3.24) into  $a_1(0)$ , it is straightforward to establish that the contribution from  $\tilde{A}_{10}$  vanishes, whereas  $\tilde{A}_6$ ,  $\tilde{A}_{13}$ , and  $\tilde{A}_{14}$  combine to yield

$$\begin{aligned} \tilde{a}_1(0) &= -g^2 C_A (d-1) \left[ 2 \int_k k^2 \frac{\partial}{\partial k^2} \Delta(k^2) \right. \\ &\quad \left. + \int_k \Delta(k^2) \right]. \end{aligned} \quad (4.41)$$

Then, using that

$$\tilde{a}_2(0) = -g^2 C_A (d-1)^2 \int_k \Delta(k^2), \quad (4.42)$$

we find that

$$\begin{aligned} d\tilde{\Pi}^{(1)}(0) &= -2g^2 C_A (d-1) \left[ \int_k k^2 \frac{\partial}{\partial k^2} \Delta(k^2) \right. \\ &\quad \left. + \frac{d}{2} \int_k \Delta(k^2) \right], \end{aligned} \quad (4.43)$$

which triggers Eq. (2.9), with  $f(k^2) = \Delta(k^2)$ .

Note, however, that the application of the above procedure at the two-loop dressed level, in order to demonstrate the vanishing of  $\tilde{\Pi}^{(3)}(0)$ , is completely

impractical, given that, as mentioned at the end of Sec. III C, expressions analogous to Eqs. (3.19) and (3.24) for the relevant form factors of  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}$  are rather difficult to derive. Instead, the use of the compact version of the seagull identity, Eq. (2.7), requires only the global form of the corresponding WI that  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}$  satisfies, making no reference whatsoever to its tensorial decomposition. This fact, in turn, exemplifies the advantages of the present formulation, and allows one to explore important aspects of the two-loop dressed structure, which otherwise would have been unattainable.

## V. RENORMALIZATION

In the previous section, the behavior of the gluon propagator at the origin has been derived using bare (unrenormalized) quantities. It is therefore important to establish that the main conclusion, namely the vanishing of  $\Delta^{-1}(0)$  in the absence of massless poles, persists after renormalization, i.e., that  $\Delta_R^{-1}(0) = 0$ .

Let us start by stating the renormalization conditions in the quantum sector of the theory,

$$\Delta_R = Z_Q^{-1} \Delta(q^2); \quad D_R = Z_c^{-1} D(q^2); \quad g_R = Z_g^{-1} g, \quad (5.1)$$

$$\begin{aligned} \Gamma_R^\mu &= Z_1 \Gamma^\mu; & \Gamma_R^{\mu\alpha\beta} &= Z_3 \Gamma^{\mu\alpha\beta}; \\ \Gamma_{R\mu\alpha\beta\nu}^{mnrs} &= Z_4 \Gamma_{\mu\alpha\beta\nu}^{mnrs}. \end{aligned} \quad (5.2)$$

Note in particular that  $Z_Q^{-1}$  is reserved for the renormalization of the quantum gauge field  $Q$ , whereas the corresponding constant renormalizing the background field  $B$ , to be introduced below, is denoted by  $Z_B^{-1}$ .

In the quantum sector, the constraints relating the above constants are exactly the same as in the conventional covariant gauges [91]; specifically, the standard STIs of the theory enforce the validity of

$$Z_g = Z_1 Z_Q^{-1/2} Z_c^{-1} = Z_3 Z_Q^{-3/2} = Z_4^{1/2} Z_Q^{-1}. \quad (5.3)$$

Consider now the background and mixed quantum-background sectors. The relevant two-point functions are renormalized as

$$\hat{\Delta}_R = Z_B^{-1} \hat{\Delta}; \quad \tilde{\Delta}_R = \mathcal{Z}^{-1} \tilde{\Delta}; \quad G_R = Z_G G, \quad (5.4)$$

which, due to the residual background symmetry [53,54], or relations such as Eqs. (3.4) and (4.38), satisfy

$$\begin{aligned} Z_g &= Z_B^{-1/2}; & \mathcal{Z} &= Z_Q^{1/2} Z_B^{1/2}; \\ Z_G &= Z_Q^{-1/2} Z_B^{1/2}; & Z_G &= Z_c Z_1^{-1} = \mathcal{Z} Z_Q^{-1}. \end{aligned} \quad (5.5)$$

Similarly, the renormalization constants of the three vertices involving one background gluon (in the  $q$  channel) are defined as

$$\begin{aligned} \tilde{\Gamma}_R^\mu &= \tilde{Z}_1 \tilde{\Gamma}^\mu; & \tilde{\Gamma}_R^{\mu\alpha\beta} &= \tilde{Z}_3 \tilde{\Gamma}^{\mu\alpha\beta}; \\ \tilde{\Gamma}_{R\mu\alpha\beta\nu}^{mnrs} &= \tilde{Z}_4 \tilde{\Gamma}_{\mu\alpha\beta\nu}^{mnrs}, \end{aligned} \quad (5.6)$$

and the corresponding Abelian STIs impose the crucial conditions,

$$\tilde{Z}_1 = Z_c; \quad \tilde{Z}_3 = Z_Q; \quad \tilde{Z}_4 = Z_3, \quad (5.7)$$

which must be preserved by the renormalization procedure, and in particular by the renormalization scheme chosen. We implicitly assume that all pertinent renormalization conditions are imposed at a renormalization point that lies in a region where the form of the relevant Green's functions, and especially of the gluon propagator, is under control, i.e., where perturbative considerations are still applicable.

Given the relations above, let us see how the SDE for  $\tilde{\Delta}$  is renormalized. One has

$$\begin{aligned} d\tilde{\Pi}^{(1)}(q^2) &= \tilde{a}_1(q^2) + \tilde{a}_2(q^2) \\ &= g^2 C_A \left[ \frac{1}{2} \int_k \Gamma_{\mu\alpha\beta}^{(0)} \Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k+q) \tilde{\Gamma}_{\sigma\rho}^\mu \right. \\ &\quad \left. + (d-1) \int_k \Delta_\alpha^\alpha(k) \right] \\ &= Z_g^2 Z_Q g_R^2 C_A \left[ \frac{1}{2} \int_k \Gamma_{\mu\alpha\beta}^{(0)} \Delta_R^{\alpha\rho}(k) \Delta_R^{\beta\sigma}(k+q) \tilde{\Gamma}_{R\sigma\rho}^\mu \right. \\ &\quad \left. + (d-1) \int_k \Delta_{R\alpha}^\alpha(k) \right] \\ &= Z_g^2 Z_Q [a_1^R(q^2) + a_2^R(q^2)] \\ &= Z_g^2 Z_Q d\tilde{\Pi}_R^{(1)}(q^2), \end{aligned} \quad (5.8)$$

and, similarly,

$$\begin{aligned} d\tilde{\Pi}^{(2)}(q^2) &= a_3(q^2) + a_4(q^2) = Z_g^2 Z_c [a_3^R(q^2) + a_4^R(q^2)] \\ &= Z_g^2 Z_c d\tilde{\Pi}_R^{(2)}(q^2), \\ d\tilde{\Pi}^{(3)}(q^2) &= a_5(q^2) + a_6(q^2) = Z_g^4 Z_3^{-1} Z_Q^3 [a_5^R(q^2) \\ &\quad + a_6^R(q^2)] = Z_g^4 Z_3^{-1} Z_Q^3 d\tilde{\Pi}_R^{(3)}(q^2), \end{aligned} \quad (5.9)$$

where we have introduced the combinations

$$\begin{aligned} Z_3 &= \mathcal{Z} Z_g^2 Z_Q; & Z_1 &= \mathcal{Z} Z_g^2 Z_c; \\ Z_4 &= \mathcal{Z} Z_g^4 Z_3^{-1} Z_Q^3. \end{aligned} \quad (5.10)$$

Combining all the above equations, we find the relation

$$\begin{aligned} \tilde{\Delta}_R^{-1}(q^2) &= \mathcal{Z} q^2 + i[Z_3 \tilde{\Pi}_R^{(1)}(q^2) + Z_1 \tilde{\Pi}_R^{(2)}(q^2) \\ &\quad + Z_4 \tilde{\Pi}_R^{(3)}(q^2)], \end{aligned} \quad (5.11)$$

which, under the assumption of a finite  $1 + G_R(0)$ , implies that  $\Delta_R^{-1}(0) = 0$ .

Let us go one step further and impose the momentum subtraction (MOM) renormalization condition  $\tilde{\Delta}_R^{-1}(\mu^2) = \mu^2$ , so that

$$\mathcal{Z} = 1 - \frac{i}{\mu^2} [Z_3 \tilde{\Pi}_R^{(1)}(\mu^2) + Z_1 \tilde{\Pi}_R^{(2)}(\mu^2) + Z_4 \tilde{\Pi}_R^{(3)}(\mu^2)]. \quad (5.12)$$

At this point one has

$$\begin{aligned} \tilde{\Delta}_R^{-1}(q^2) &= q^2 + i[Z_3 \tilde{\Pi}_R^{(1)}(q^2) + Z_1 \tilde{\Pi}_R^{(2)}(q^2) + Z_4 \tilde{\Pi}_R^{(3)}(q^2)] \\ &\quad - \frac{q^2}{\mu^2} i[Z_3 \tilde{\Pi}_R^{(1)}(\mu^2) + Z_1 \tilde{\Pi}_R^{(2)}(\mu^2) + Z_4 \tilde{\Pi}_R^{(3)}(\mu^2)], \end{aligned} \quad (5.13)$$

so that, once again,  $\Delta_R^{-1}(0) = 0$ .

## VI. EVADING THE SEAGULL CANCELLATIONS: VERTICES WITH POLES

In order to obtain an infrared finite gluon propagator self-consistently, one needs to introduce poles in the vertices, and, in particular, in the channel that is associated with the momentum flowing into the gluon SDE, where these vertices are eventually inserted. To be precise, and following the conventions of Figs. 1–3, (some of) the vertices  $\tilde{\Gamma}_{\mu\alpha\beta}$ ,  $\tilde{\Gamma}_\mu$ , and  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}$  must contain pole terms of the form  $q^\mu/q^2$ . The purpose of this section is to study how the inclusion of such terms circumvents the seagull identity, thus allowing for the possibility of  $\Delta^{-1}(0) \neq 0$ .

Specifically, let us assume that some of the form factors now contain two distinct parts, which are indicated by a superscript “p” (for pole parts) or “np” (for no-pole parts). In that sense, whereas before none of the form factors contained poles, now we can say that, using the same notation introduced in Eqs. (3.17) and (3.21),

$$\begin{aligned} \tilde{A}_1 &= \tilde{A}_1^{\text{np}} + \tilde{A}_1^{\text{p}}; & \tilde{A}_2 &= \tilde{A}_2^{\text{np}}, \\ \tilde{A}_i &= \tilde{A}_i^{\text{np}} + \tilde{A}_i^{\text{p}}, & i &= 1, \dots, 5; \\ \tilde{A}_i &= \tilde{A}_i^{\text{np}}, & i &= 6, \dots, 14. \end{aligned} \quad (6.1)$$

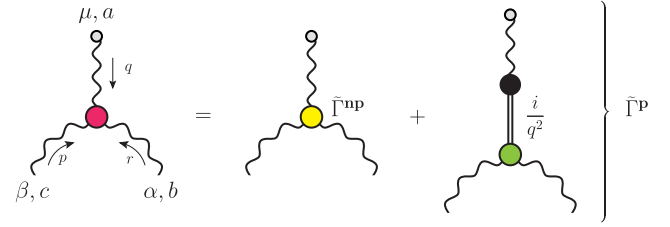


FIG. 4. The no-pole and pole parts of the  $BQ^2$  gluon vertex. Analogous decompositions hold for the  $Bc\bar{c}$  and the  $BQ^3$  vertices.

The fact that only the longitudinally coupled form factors are allowed to contain pole parts is dictated by the physical requirement that they should act like “dynamical Nambu-Goldstone bosons,” and decouple from physical observables [57–63].

Thus, one may cast the nonperturbative vertices in the form (see Fig. 4)

$$\begin{aligned} \tilde{\Gamma}_{\mu\alpha\beta}(q, r, p) &= \tilde{\Gamma}_{\mu\alpha\beta}^{\text{np}}(q, r, p) + \tilde{\Gamma}_{\mu\alpha\beta}^{\text{p}}(q, r, p), \\ \tilde{\Gamma}_\mu(q, r, p) &= \tilde{\Gamma}_\mu^{\text{np}}(q, r, p) + \tilde{\Gamma}_\mu^{\text{p}}(q, r, p), \\ \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{mnrs}(q, r, p, t) &= \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{np}, mnrs}(q, r, p, t) + \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{p}, mnrs}(q, r, p, t), \end{aligned} \quad (6.2)$$

where, due to the condition of longitudinality, one can write in full generality

$$\begin{aligned} \tilde{\Gamma}_{\mu\alpha\beta}^{\text{p}}(q, r, p) &= \frac{q_\mu}{q^2} \tilde{C}_{\alpha\beta}(q, r, p), \\ \tilde{\Gamma}_\mu^{\text{p}}(q, r, p) &= \frac{q_\mu}{q^2} \tilde{C}(q, r, p), \\ \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{p}, mnrs}(q, r, p, t) &= \frac{q_\mu}{q^2} \tilde{C}_{\alpha\beta\gamma}^{mnrs}(q, r, p, t). \end{aligned} \quad (6.3)$$

The precise way in which the form factors of Eq. (6.1) comprise  $\tilde{\Gamma}^{\text{np}}$  and  $\tilde{\Gamma}^{\text{p}}$ , and in particular the functions  $\tilde{C}$ , may be easily worked out. For example, in the case of  $\tilde{\Gamma}_\mu$ , which is the vertex with the simplest tensorial structure, we simply obtain

$$\begin{aligned} \tilde{\Gamma}_\mu^{\text{np}}(q, r, p) &= \tilde{A}_1^{\text{np}} q_\mu + \tilde{A}_2^{\text{np}} r_\mu, \\ \tilde{\Gamma}_\mu^{\text{p}}(q, r, p) &= \tilde{A}_1^{\text{p}} q_\mu \Rightarrow \tilde{A}_1^{\text{p}} \equiv \frac{\tilde{C}(q, r, p)}{q^2}. \end{aligned} \quad (6.4)$$

Of course, in order to keep the BRST invariance intact, we demand that all STIs maintain their exact form in the presence of these poles; therefore, Eq. (3.5) now reads

$$\begin{aligned}
q^\mu \tilde{\Gamma}_{\mu\alpha\beta}^{\text{np}}(q, r, p) + \tilde{C}_{\alpha\beta}(q, r, p) &= i\Delta_{\alpha\beta}^{-1}(r) - i\Delta_{\alpha\beta}^{-1}(p), \\
q^\mu \tilde{\Gamma}_\mu^{\text{np}}(q, r, p) + \tilde{C}(q, r, p) &= iD^{-1}(r^2) - iD^{-1}(p^2), \\
q^\mu \tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{np},mnrs}(q, r, p, t) + \tilde{C}_{\alpha\beta\gamma}^{mnrs}(q, r, p, t) &= f^{mne} f^{ern} \Gamma_{\alpha\beta\gamma}(r, p, q+t) + f^{mne} f^{esr} \Gamma_{\beta\gamma\alpha}(p, t, q+r) \\
&\quad + f^{mre} f^{ens} \Gamma_{\gamma\alpha\beta}(t, r, q+p).
\end{aligned} \tag{6.5}$$

Note that if  $\tilde{\Gamma}_{\mu\alpha\beta}(q, r, p)$  contains poles in  $q^2$ , by virtue of the corresponding background-quantum identity [31] so does the conventional ( $Q^3$ ) three-gluon vertex  $\Gamma_{\alpha\beta\gamma}(q, r, p)$ . However, the rhs of the third STI in Eq. (6.5) contains no such poles, because  $q$  never appears in the arguments of the  $\Gamma$ 's alone, but rather in the combinations  $q+t$ ,  $q+r$ , or  $q+p$ .

At this point it should be clear that  $\tilde{\Gamma}^{\text{np}}$  represents precisely the part of the total vertex  $\tilde{\Gamma}$  that enters in the calculation of  $\tilde{\Pi}(0)g_{\mu\nu}$ , and consequently participates in the seagull cancellation. On the other hand, the term with the massless pole in  $q^2$  contributes to the term  $\tilde{\Pi}(0)q_\mu q_\nu/q^2$ , which is not involved in the seagull cancellation. Of course, since the original STIs of Eq. (3.5), now replaced by the equivalent set given in Eq. (6.5), remain intact, the blockwise transversality property of Eq. (3.6) is automatically incorporated in the final answer.

Therefore, the total contribution of each block to the  $g_{\mu\nu}$  part (after the seagull cancellation) is exactly equal (and opposite in sign) to that proportional to  $q_\mu q_\nu/q^2$ .

In order to appreciate the above points in detail, let us study the  $q=0$  limit of the gluon self-energy as done previously in Sec. III. To this end we need to derive the equivalent of Eqs. (3.11)–(3.13). This can be done by taking directly a Taylor expansion around  $q=0$  of both sides of Eqs. (6.5), as they are both regular in this limit. Evidently, the zeroth order term vanishes in all three cases,

$$\begin{aligned}
\tilde{C}_{\alpha\beta}(0, r, -r) &= 0; & \tilde{C}(0, r, -r) &= 0; \\
\tilde{C}_{\alpha\beta\gamma}^{mnrs}(0, r, p, -p-r) &= 0,
\end{aligned} \tag{6.6}$$

whereas the first order terms yield for the  $BQ^2$  vertex

$$\begin{aligned}
\tilde{\Gamma}_{\mu\alpha\beta}^{\text{np}}(0, r, -r) &= -i \frac{\partial}{\partial r^\mu} \Delta_{\alpha\beta}^{-1}(r) - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha\beta}(q, r, -r-q) \right\}_{q=0}; \\
\tilde{\Gamma}_{\mu\alpha\beta}^{\text{np}}(0, -p, p) &= i \frac{\partial}{\partial p^\mu} \Delta_{\alpha\beta}^{-1}(p) - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha\beta}(q, -p-q, p) \right\}_{q=0},
\end{aligned} \tag{6.7}$$

for the  $B\bar{c}c$  vertex

$$\begin{aligned}
\tilde{\Gamma}_\mu^{\text{np}}(0, r, -r) &= -i \frac{\partial}{\partial r^\mu} D^{-1}(r^2) - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}(q, r, -r-q) \right\}_{q=0}; \\
\tilde{\Gamma}_\mu^{\text{np}}(0, -p, p) &= i \frac{\partial}{\partial p^\mu} D^{-1}(p^2) - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}(q, -p-q, p) \right\}_{q=0},
\end{aligned} \tag{6.8}$$

and, finally, for the  $BQ^3$  vertex

$$\begin{aligned}
\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{np},mnrs}(0, r, p, -r-p) &= \left( f^{mne} f^{esr} \frac{\partial}{\partial r^\mu} + f^{mre} f^{ens} \frac{\partial}{\partial p^\mu} \right) \Gamma_{\alpha\beta\gamma}(r, p, -r-p) \\
&\quad - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha\beta\gamma}^{mnrs}(q, r, p, -q-r-p) \right\}_{q=0}, \\
\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{np},mnrs}(0, -r, -p, r+p) &= - \left( f^{mne} f^{esr} \frac{\partial}{\partial r^\mu} + f^{mre} f^{ens} \frac{\partial}{\partial p^\mu} \right) \Gamma_{\alpha\beta\gamma}(-r, -p, r+p) \\
&\quad + \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha\beta\gamma}^{mnrs}(-q, -r, -p, q+r+p) \right\}_{q=0}.
\end{aligned} \tag{6.9}$$

These identities, in turn, provide symmetry constraints for the  $\tilde{C}$  functions. In particular, in the  $BQ^2$  and  $B\bar{c}c$  cases, the  $\tilde{C}$  is invariant upon inversion of all momenta, whereas it is antisymmetric when inverting the last two momenta (and the corresponding indices in the  $BQ^2$  case). For the  $BQ^3$  vertex instead,  $\tilde{C}$  behaves with respect to its arguments as a conventional three-gluon vertex.

Let us now repeat the calculation of Sec. IV following the exact same logic; specifically, the vertex contributions that survive the limit  $q = 0$  in the various self-energy

diagrams must be replaced by the lhs of the corresponding WIs. However, the crucial difference now is that the WIs to employ are those given in Eqs. (6.7)–(6.9), which, in addition to the terms already present in Eqs. (3.11)–(3.13), contain the derivatives of the functions  $\tilde{C}$ . As a result, whereas the first type of terms triggers again the seagull identities and vanishes exactly as before, the contributions originating from  $\tilde{C}$  escape the total annihilation. In particular, it is fairly straightforward to show that

$$d\tilde{\Pi}^{(1)}(0) = -\frac{1}{2}g^2C_A \int_k \Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) \Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k) \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\sigma\rho}(q, -k - q, k) \right\}_{q=0}, \quad (6.10)$$

$$d\tilde{\Pi}^{(2)}(0) = -g^2C_A \int_k \Gamma_\mu^{(0)}(0, k, -k) D^2(k^2) \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}(q, -k - q, k) \right\}_{q=0}, \quad (6.11)$$

and

$$\begin{aligned} d\tilde{\Pi}^{(3)}(0)\delta^{ab} &= \frac{1}{6}g^4\Gamma_{\mu\alpha\beta\gamma}^{(0)amnr} \int_k \int_\ell \Delta^{\alpha\rho}(k + \ell) \Delta^{\beta\sigma}(\ell) \Delta^{\gamma\tau}(k) \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\tau\sigma\rho}^{brnm}(q, -k - q, -\ell, k + \ell) \right\}_{q=0} \\ &+ iN_{\mu\alpha\beta\gamma}\delta^{ab} \int_k Y_\delta^{\alpha\beta}(k) \Delta^{\gamma\tau}(k) \Delta^{\lambda\delta}(k) \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\tau\lambda}(q, -k - q, k) \right\}_{q=0}. \end{aligned} \quad (6.12)$$

Thus, the presence of  $1/q^2$  poles allows for a nonvanishing value of the self-energy at  $q = 0$ , therefore providing the possibility for  $\tilde{\Delta}^{-1}(0) \neq 0$ .

Let us end this section by proving that one recovers the same answer for  $\tilde{\Pi}(0)$  by considering directly the part of the self-energy proportional to  $q_\mu q_\nu / q^2$ , where, of course, no seagull identity is operating. To work out a concrete example in detail, consider the one-loop dressed gluon diagrams; in this case, the desired contribution is obtained by simply replacing inside graph ( $\tilde{a}_1$ ) the full vertex by its pole part. One obtains

$$\begin{aligned} \frac{q_\mu q_\nu}{q^2} \tilde{\Pi}^{(1)}(q^2) &= \frac{1}{2}g^2C_A \frac{q_\nu}{q^2} \int_k \Gamma_{\mu\alpha\beta}^{(0)}(q, k, -k - q) \Delta^{\alpha\rho}(k) \\ &\times \Delta^{\beta\sigma}(k + q) \tilde{C}_{\sigma\rho}(q, -k - q, k) + \dots, \end{aligned} \quad (6.13)$$

where the dots indicate terms that vanish as  $q \rightarrow 0$ ; then one immediately has

$$\begin{aligned} \tilde{\Pi}^{(1)}(q^2) &= \frac{1}{2}g^2C_A \frac{q^\mu}{q^2} \int_k \Gamma_{\mu\alpha\beta}^{(0)}(q, k, -k - q) \Delta^{\alpha\rho}(k) \\ &\times \Delta^{\beta\sigma}(k + q) \tilde{C}_{\sigma\rho}(q, -k - q, k) + \dots \end{aligned} \quad (6.14)$$

We can now expand  $\tilde{C}$  around  $q = 0$  and use the fact that  $\tilde{C}_{\sigma\rho}(0, -k, k) = 0$  [see Eq. (6.6)] to obtain

$$\tilde{\Pi}^{(1)}(q^2) = \frac{1}{2}g^2C_A \frac{q^\mu q^\lambda}{q^2} X_{\mu\lambda}, \quad (6.15)$$

with

$$\begin{aligned} X_{\mu\lambda} &= \int_k \Gamma_{\mu\alpha\beta}^{(0)}(0, k, -k) \Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k) \\ &\times \left\{ \frac{\partial}{\partial q^\lambda} \tilde{C}_{\sigma\rho}(q, -k - q, k) \right\}_{q=0}. \end{aligned} \quad (6.16)$$

Then, since  $X_{\mu\lambda}$  has no dependence on  $q$ , it can only be proportional to the metric tensor  $g_{\mu\lambda}$ , namely one has

$$\begin{aligned} dX_{\mu\lambda} &= g_{\mu\lambda} \int_k \Gamma_{\nu\alpha\beta}^{(0)}(0, k, -k) \Delta^{\alpha\rho}(k) \Delta^{\beta\sigma}(k) \\ &\times \left\{ \frac{\partial}{\partial q_\nu} \tilde{C}_{\sigma\rho}(q, -k - q, k) \right\}_{q=0}, \end{aligned} \quad (6.17)$$

which, upon substitution into Eq. (6.15), gives the announced equality with Eq. (6.10). An exactly analogous procedure may be followed for the remaining contributions, thus establishing explicitly the transversality of the gluon self-energy even in the presence of massless poles.

## VII. FURTHER CONSIDERATIONS AND NUMERICAL ANALYSIS

The WIs given in Eqs. (6.7)–(6.9) furnish certain interesting relations among the form factors of the vertices that satisfy them. In order to derive them, let us consider a general scalar function  $f(q, r, p)$ , which is antisymmetric under  $r \leftrightarrow p$ , and expand it around  $q = 0$  (and  $p = -r$ ). Since in that case  $f(0, r, -r) = 0$ , we have that

$$f(q, r, p) = 2(q \cdot r)f'(r, -r) + \mathcal{O}(q^2), \quad (7.1)$$

where the prime denotes differentiation with respect to  $(r + q)^2$  and subsequently taking the limit  $q \rightarrow 0$ , i.e.,

$$f'(r, -r) \equiv \lim_{q \rightarrow 0} \frac{\partial}{\partial(r + q)^2} f(q, r, -r - q). \quad (7.2)$$

Evidently, due to Lorentz invariance,  $f'(r, -r) = f'(r^2)$ .

Then, if we expand Eq. (6.8) around  $q = 0$ , the analogue of Eq. (3.19) may be obtained after using that

$$\left\{ \frac{\partial}{\partial q^\mu} \tilde{C}(q, r, p) \right\}_{q=0} = 2r_\mu \tilde{C}'(r^2), \quad (7.3)$$

and so

$$\tilde{A}_2^{\text{np}}(r^2) = -2 \left[ i \frac{\partial}{\partial r^2} D^{-1}(r^2) + \tilde{C}'(r^2) \right]. \quad (7.4)$$

The above argument may be extended directly to the case of the three-gluon vertex; to simplify the situation, we assume that out of the five possible tensorial structures of  $\tilde{C}_{\sigma\rho}$  only the one proportional to  $g_{\alpha\beta}$  develops a pole in  $q^2$ ; at the level of  $\tilde{\Gamma}_{\mu\alpha\beta}^{\text{p}}$  this is equivalent to the statement that only  $A_1^{\text{p}} \equiv \tilde{C}_1/q^2 \neq 0$ . Then,

$$\left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha\beta}(q, r, p) \right\}_{q=0} = 2r_\mu g_{\alpha\beta} \tilde{C}'_1(r^2), \quad (7.5)$$

and the first identity in Eq. (3.24) assumes the form

$$\tilde{A}_6^{\text{np}}(r^2) = 2 \left[ \frac{\partial}{\partial r^2} \Delta^{-1}(r^2) - \tilde{C}'_1(r^2) \right]. \quad (7.6)$$

In order to gain a basic quantitative understanding of some of the relations derived here, we next carry out a numerical analysis, under a number of simplifying assumptions. In particular, we assume that the strength of the poles coming from  $\tilde{\Gamma}_\mu^{\text{p}}(q, r, p)$  and  $\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{\text{p},mnrs}(q, r, p, t)$  is suppressed with respect to that of  $\tilde{\Gamma}_{\mu\alpha\beta}^{\text{p}}(q, r, p)$ , and may therefore be neglected. In addition, we keep as before only the component  $\tilde{C}_1$ , so that Eq. (6.10) yields

$$d\tilde{\Pi}^{(1)}(0) = 2g^2 C_A \int_k \left[ (d-1)k^2 \Delta^2(k^2) \right] \tilde{C}'_1(k^2) \quad (7.7)$$

Considering for simplicity the Landau gauge,  $\xi = 0$ , one finds (Euclidean space)

$$\tilde{\Delta}^{-1}(0) = -2g^2 C_A \frac{d-1}{d} \int_k k^2 \Delta^2(k^2) \tilde{C}'_1(k^2), \quad (7.8)$$

or using Eqs. (4.36) and (4.39), introducing spherical coordinates, and setting  $k^2 = y$ ,  $d = 4$ , and  $\alpha_s = g^2/4\pi$ ,

$$\Delta^{-1}(0) = -\frac{3C_A\alpha_s}{8\pi} F(0) \int_0^\infty dy y^2 \Delta^2(y) \tilde{C}'_1(y). \quad (7.9)$$

Evidently, given that  $\Delta^{-1}(0)$  and  $F(0)$  are positive quantities, the function  $\tilde{C}'_1(y)$  must be such that, when inserted in Eq. (7.9), it will compensate for the overall minus sign. One way to accomplish this is by assuming that  $\tilde{C}'_1(y)$  is negative throughout the entire range of momenta. Alternatively, one may envisage a type of function that changes its sign, having positive and negative supports that are appropriately distributed with respect to the function  $y^2 \Delta^2(y)$ , eventually furnishing a more negative than positive contribution to the integral of Eq. (7.9). To be sure, this particular issue may be definitively resolved only after a careful study of the corresponding vertex SDE and the Bethe-Salpeter equation derived from it (see the related discussion and references in Sec. VIII).

Since, apart from the qualitative considerations given above, the precise form of the function  $\tilde{C}'_1(y)$  is undetermined at this level, in order to proceed with our analysis we consider three possible models describing its functional form. The general shapes chosen are inspired by the solutions obtained from the Bethe-Salpeter equations governing the formation of massless poles [62]. In particular, we use the following models,

$$\tilde{C}'_1(y) = \begin{cases} 1/(ay^2 + by + c), & \text{model A} \\ ay \exp(-y/b), & \text{model B} \\ 1/(ay + b\sqrt{y} + c), & \text{model C,} \end{cases} \quad (7.10)$$

with  $a$ ,  $b$ , and  $c$  being suitable parameters.

To begin with let us assume that the gluon propagator has a simple massive form,  $\Delta(y) = 1/(y + m^2)$ , so that Eq. (7.9) becomes

$$\Delta^{-1}(0) = m^2 = -\frac{3C_A\alpha_s}{8\pi} F(0) \int_0^\infty dy \frac{y^2}{(y + m^2)^2} \tilde{C}'_1(y). \quad (7.11)$$

To proceed further, we use as input in Eq. (7.11) the values for the saturation points of the gluon propagator and the ghost dressing function found in the lattice simulations of

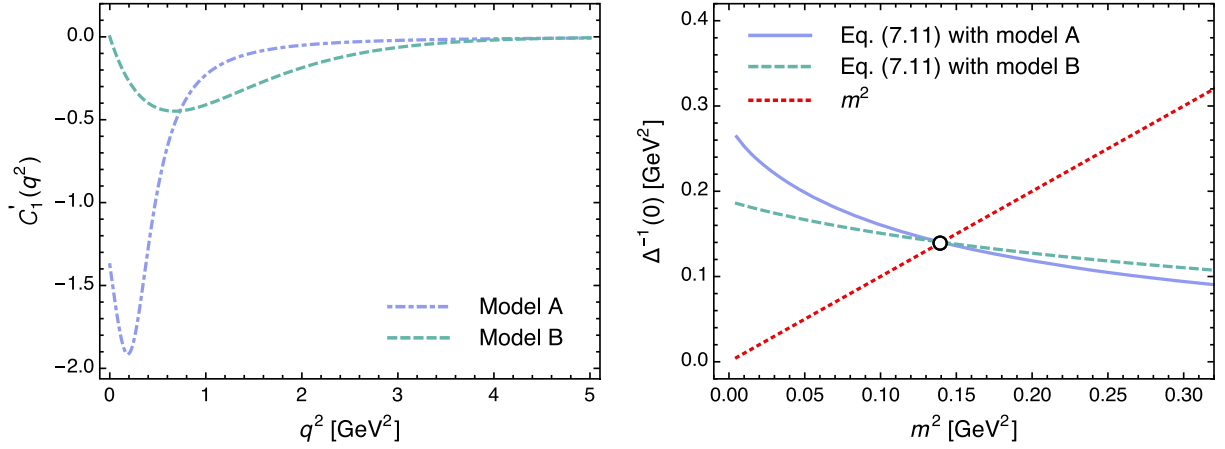


FIG. 5. Left panel: The functions  $\tilde{C}'_1(y)$  obtained from model A with  $a = -5.82$  GeV<sup>-4</sup>,  $b = 2.18$  GeV<sup>-2</sup>,  $c = -0.73$ , and from model B with  $a = -1.81$  GeV<sup>-2</sup>,  $b = 0.675$  GeV<sup>2</sup>. Right panel: The solution of Eq. (7.11) when  $m^2$  coincides with the SU(3) lattice saturation value  $\Delta^{-1}(0) = 0.14$  GeV<sup>2</sup>.

[5,6]; specifically, when the MOM subtraction point is chosen at  $\mu = 4.3$  GeV, one has that  $\Delta^{-1}(0) = m^2 = 0.15$  GeV<sup>2</sup> and  $F(0) = 2.91$ . The idea then is to try to determine the parameters of  $\tilde{C}'_1(y)$  in Eq. (7.10) so that Eq. (7.11) is satisfied. Of course, the solution of this problem is not unique, as there are many possible combinations of  $a$ ,  $b$ , and  $c$  leading to the same result. In addition, notice that, for the case of a simple massive propagator, model C cannot be considered, because the corresponding integral diverges logarithmically at its upper limit. In Fig. 5 we show how the desired solution is obtained using the particular set of values for the parameters of models A and B given in the caption.

Let us next repeat the above analysis employing a more realistic gluon propagator, namely the one obtained in the SU(3) quenched lattice simulations of Ref. [6]. On the left panel of Fig. 6 we show a physically motivated fit for the gluon propagator (green continuous line), together with the lattice data at the renormalization scale  $\mu = 4.3$  GeV (triangles). Notice that the fit displays the inflection point that must appear due to the presence of divergent ghost loops [92]; we comment on this issue shortly.

We clearly see that the lattice gluon propagator is significantly more enhanced in the region below 2 GeV<sup>2</sup> when compared to the naive constant mass propagator used in our previous discussion, with  $m^2 = 0.15$  GeV<sup>2</sup> (dashed curve). It is therefore interesting to study how the values of

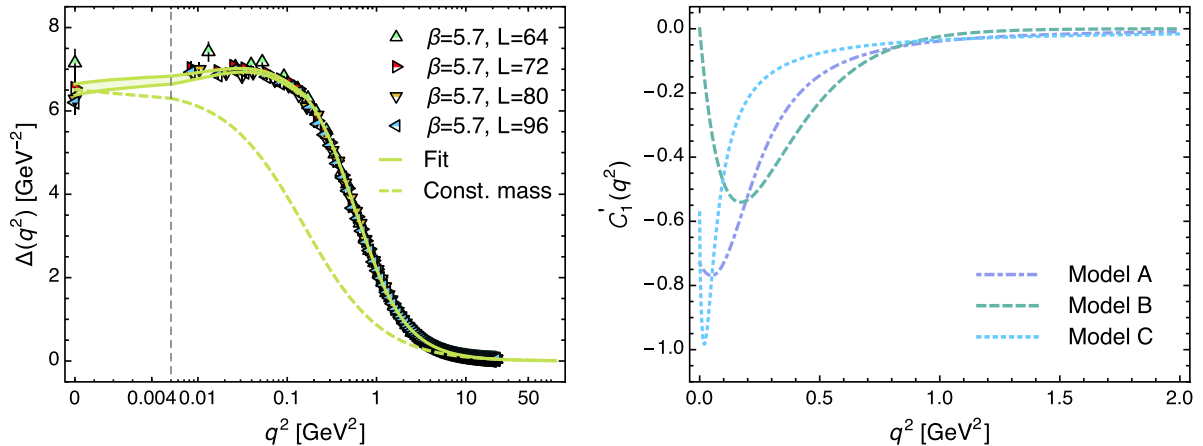


FIG. 6. Left panel: The quenched SU(3) lattice data for the gluon propagator,  $\Delta(q^2)$ , renormalized at  $\mu = 4.3$  GeV (triangles) and its corresponding fit (continuous line); for comparison, we also plot a simple constant mass propagator (dashed line). Notice that the scale on the  $x$  axis becomes linear past the dashed vertical line. Right panel: The functions  $\tilde{C}'_1(y)$  correspond to model A with  $a = -26.78$  GeV<sup>-4</sup>,  $b = 2.73$  GeV<sup>-2</sup>,  $c = -1.37$ , model B with  $a = -8.64$  GeV<sup>-2</sup>,  $b = 0.17$  GeV<sup>2</sup>, and model C with  $a = -37.2$  GeV<sup>-2</sup>,  $b = -10.44$  GeV<sup>-1</sup>,  $c = -1.74$ ; all three curves yield a solution to Eq. (7.9) when the lattice gluon propagator is used as input.

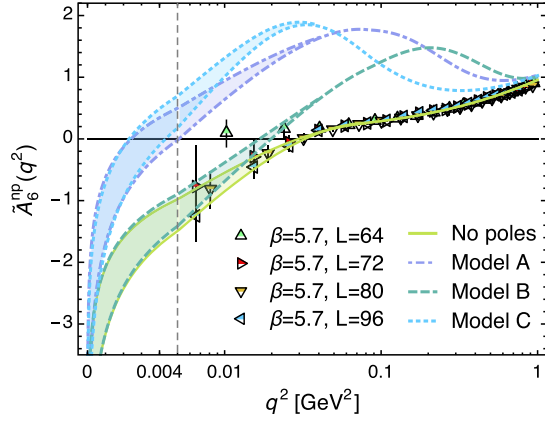


FIG. 7. The form factor  $\tilde{A}_6^{\text{np}}$  evaluated when using the quenched SU(3) lattice propagator as input. In the absence of massless poles, this quantity is proportional to the derivative of the inverse propagator (here evaluated directly from the lattice data); see Eq. (3.24); notice the zero crossing and logarithmic divergence in the deep infrared (again past the vertical dashed line the scale becomes linear). The band indicates the spreading between the  $L = 72$  and  $L = 96$  lattice data. When massless poles are generated, the presence of the function  $\tilde{C}'_1$  modifies this dependence according to Eq. (7.6), and a positive maximum appears in the region  $q^2 < 1 \text{ GeV}^2$ , the height and exact location of which depend on the details of the model.

the model parameters  $a$ ,  $b$ , and  $c$  must be modified in order to obtain from Eq. (7.9) the same solution as before.

A typical case is shown on the right panel of Fig. 6, where we plot the  $\tilde{C}'(y)$  obtained for each model considered, for the parameters quoted in the caption. Evidently, the  $\tilde{C}'_1(y)$  used in this case are suppressed compared to those of the simple massive case, because, precisely due to the aforementioned enhancement of the gluon propagator, less strength is required from them in order for the rhs of Eq. (7.9) to reach the fixed value of  $0.15 \text{ GeV}^2$ .

We end this section with a brief qualitative description of how the study of the form factor  $\tilde{A}_6^{\text{np}}(q^2)$ , and in particular of the relation Eq. (7.6), may corroborate or invalidate the necessity of longitudinally coupled massless poles. This, in turn, may be particularly interesting, especially in view of the comments following Eq. (6.1).

To begin with, let us recall that, in contradistinction to the massive behavior observed for the gluon propagator, the Landau gauge lattice simulations reveal that the ghost remains massless: the ghost propagator behaves like  $1/p^2$  in the IR, and is multiplied by a dressing function that saturates at a finite nonvanishing value [1–6]. Combining these two results together, one arrives at the (model-independent) conclusions that [92] (i) the gluon propagator must display a mild maximum in the deep infrared (a feature that can be seen in the left panel of Fig. 6), and (ii) the derivative of its inverse will display a logarithmic divergence (and a corresponding zero crossing). In particular, one has that

$$\frac{\partial}{\partial q^2} \Delta^{-1}(q^2) \underset{q \rightarrow 0}{\sim} \log q^2, \quad (7.12)$$

and indeed, by evaluating the derivative of the lattice data, one can see the appearance of the zero crossing and the logarithmic divergence predicted by Eq. (7.12) (see data points in Fig. 7).

Thus, if the observed finiteness of the gluon propagator were not due to the presence of massless poles, Eq. (3.24) implies that the three-gluon vertex form factor  $\tilde{A}_6^{\text{np}} \equiv \tilde{A}_6$  would display the same infrared behavior as that of Eq. (7.12). On the other hand, the presence of massless poles alters the shape of this form factor, as the function  $\tilde{C}'_1$  now enters in its determination [see Eq. (7.6)]. Given the finiteness of  $\tilde{C}'_1$ , this does not alter the deep infrared behavior of the form factor; however, it forces it to have a characteristic positive maximum, which could be detected against the dominant term, as shown in Fig. 7. In fact, the best strategy might be to determine simultaneously  $\Delta$  and  $\tilde{A}_6$  and form the difference  $\tilde{A}_6 - 2\partial\Delta^{-1}$ , which yields directly the function  $C'_1$  (multiplied by a factor of 2).

## VIII. DISCUSSION AND CONCLUSIONS

We have presented a unified framework for the self-consistent treatment of the infrared finiteness of the gluon propagator at the level of the SDEs of the theory, formulated within the PT-BFM framework. Particular attention has been dedicated to the extensive cancellations induced by the WIs of the theory, and the necessity of introducing massless poles in order to achieve the desired effect in a self-consistent way. Our analysis has been carried out for a general value of the gauge-fixing parameter, reverting to the Landau gauge only in order to simplify the numerical analysis presented in the last section.

Particularly relevant is the observation that the presence of these poles is bound to affect not only the two-point but also the three-point sector of the theory. Therefore, determining particular form factors of, e.g., the (background) three-gluon vertex by means of lattice techniques has the potential to confirm or discard massless poles as the infrared mechanism underlying the dynamical generation of a gluon mass. The main difficulty in actually realizing this idea stems from the fact that the form factors satisfying Eqs. (7.4) and (7.6) are those of the  $BQ^2$  and  $Bc\bar{c}$  vertices, rather than those comprising the conventional vertices  $Q^3$  and  $Qc\bar{c}$  [93–95]. The two sets of vertices are related through rather complicated background-quantum identities [31], which need to be studied in order to determine if any simplifications occur in the relevant  $q \rightarrow 0$  limit. At the same time, it might be also helpful to explore this issue from the lattice point of view [96,97], by implementing the BFM along the preliminary proposals put forth in [98,99].

It is clear from the analysis presented that the three-gluon vertex, and in particular its PT-BFM version,  $BQ^2$ , plays a

particularly important role in the entire construction that leads to an infrared finite gluon propagator. One of the most notable features is the presence of massless poles in a special subset of its form factors, which are therefore expected to diverge in the deep infrared. It is important to mention that several SDE studies of the gluon and ghost propagator, as well as of higher order Green's functions, have considered the possibility that the conventional three-gluon vertex,  $Q^3$ , displays a divergent behavior. Some of the most representative works in this direction include the following: (i) Focusing only on the form factors of the tensors appearing in the tree-level vertex [first equation in (A2)], and using a power-counting scheme, the authors of [100] found that the infrared limit of the “symmetric configuration” ( $q^2 = r^2 = p^2$ ) shows a divergent power-law behavior of the type  $(p^2)^{-3\kappa}$ , where  $\kappa = 0.59$  is the typical parameter of the so-called “scaling solutions.” (ii) A similar analysis was performed in [101], for one soft and two hard external momenta, finding a softer divergence, of the type  $(p^2)^{1-2\kappa}$ . (iii) A study with a more complete tensorial structure was carried out in [102], where the *transverse* parts of the three-gluon vertex turned out to be very mildly divergent, and with no appreciable impact on the gluon and ghost SDEs. We emphasize that, even though the main qualitative features between the aforementioned results and those of the present work appear to be similar, a direct quantitative comparison between them is not possible at present, mainly due to the fact that the fundamental properties of the  $Q^3$  and  $BQ^2$  vertices are very different. In fact, as mentioned in the previous paragraph, the two vertices are formally related by an exact, but rather complicated identity [31], the ingredients of which are, to a large extent, unexplored.

The actual generation of the poles as massless bound-state excitations may be studied in the generalized context of the Bethe-Salpeter equations, as proposed in the early works of [57,58,60,61], and as was further explored in [62,63]. Note, however, that in the analysis presented in [62,63] the possibility that the ghost-gluon vertex may contain such poles was not contemplated, and  $\tilde{\Gamma}_\mu^p$  was assumed to vanish identically. It is therefore especially interesting to explore whether or not the formation of a nontrivial  $\tilde{\Gamma}_\mu^p$  is dynamically favored. In particular, whereas

in previous considerations only the Bethe-Salpeter equation for the massless pole of the three-gluon vertex was studied, under the light of the analysis presented the corresponding dynamical equation for the ghost vertex must be derived and solved. In fact, the complete treatment of this problem requires the solution of a coupled system rather than a single integral equation, given that the pole part of the ghost vertex gets mixed with that of the three-gluon vertex, and vice versa.

The general formalism developed in this work sets up the stage for a detailed quantitative study of the precise field-theoretic mechanism that accounts for the infrared saturation of the gluon propagator observed in recent lattice simulations performed away from the Landau gauge [9]. In particular, the relevant set of Bethe-Salpeter equations must be derived for general  $\xi$ , and then appropriately coupled to Eqs. (6.10)–(6.12), which determine the value of  $\Delta^{-1}(0)$ . This analysis may be particularly revealing, given the observed tendency of the gluon saturation point to decrease as  $\xi$  increases, at least within the interval  $[0, 0.5]$ . Reproducing this characteristic behavior constitutes a considerable challenge, whose successful completion would considerably validate the proposed approach and general philosophy. We hope to be able to pursue this issue in the near future.

## ACKNOWLEDGMENTS

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## APPENDIX: FEYNMAN RULES

The following vertex definitions have been employed (all momenta entering):

$$\begin{aligned}
 i\Gamma_{Q_\mu^a Q_\alpha^m Q_\beta^n}(q, r, p) &= gf^{amn}\Gamma_{\mu\alpha\beta}(q, r, p); i\Gamma_{B_\mu^a Q_\alpha^m Q_\beta^n}(q, r, p) = gf^{amn}\tilde{\Gamma}_{\mu\alpha\beta}(q, r, p), \\
 i\Gamma_{c^n Q_\mu^a \bar{c}^m}(p, q, r) &= gf^{amn}\Gamma_\mu(q, r, p); i\Gamma_{c^n B_\mu^a \bar{c}^m}(p, q, r) = gf^{amn}\tilde{\Gamma}_\mu(q, r, p); \\
 \Gamma_{Q_\mu^a Q_\alpha^m Q_\beta^n Q_\gamma^r}(q, r, p, t) &= -ig^2\Gamma_{\mu\alpha\beta\gamma}^{amnr}(q, r, p, t); \Gamma_{B_\mu^a Q_\alpha^m Q_\beta^n Q_\gamma^r}(q, r, p, t) = -ig^2\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{amnr}(q, r, p, t).
 \end{aligned} \tag{A1}$$

At tree level one has

$$\begin{aligned}
\Gamma_{\mu\alpha\beta}^{(0)}(q, r, p) &= g_{\alpha\beta}(r-p)_\mu + g_{\mu\beta}(p-q)_\alpha + g_{\mu\alpha}(q-r)_\beta, \\
\tilde{\Gamma}_{\mu\alpha\beta}^{(0)}(q, r, p) &= g_{\alpha\beta}(r-p)_\mu + g_{\mu\beta}(p-q+\xi^{-1}r)_\alpha + g_{\mu\alpha}(q-r-\xi^{-1}p)_\beta, \\
\Gamma_\mu^{(0)}(q, r, p) &= -r_\mu, \\
\tilde{\Gamma}_\mu^{(0)}(q, r, p) &= (p-r)_\mu, \\
\Gamma_{\mu\alpha\beta\gamma}^{(0)amnr}(q, r, p, t) &= f^{are} f^{enm}(g_{\mu\beta}g_{\alpha\gamma} - g_{\mu\alpha}g_{\beta\gamma}) + f^{ame} f^{ern}(g_{\mu\gamma}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\gamma}) + f^{ane} f^{erm}(g_{\mu\gamma}g_{\alpha\beta} - g_{\mu\alpha}g_{\beta\gamma}), \\
\tilde{\Gamma}_{\mu\alpha\beta\gamma}^{(0)amnr}(q, r, p, t) &= \Gamma_{\mu\alpha\beta\gamma}^{(0)amnr}(q, r, p, t).
\end{aligned} \tag{A2}$$

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- [1] A. Cucchieri and T. Mendes, *Proc. Sci.*, LAT2007 (2007) 297 [[arXiv:0710.0412](#)].
- [2] A. Cucchieri and T. Mendes, *Phys. Rev. Lett.* **100**, 241601 (2008).
- [3] A. Cucchieri and T. Mendes, *Phys. Rev. D* **81**, 016005 (2010).
- [4] A. Cucchieri and T. Mendes, *Proc. Sci.*, QCD-TNT09 (2009) 026 [[arXiv:1001.2584](#)].
- [5] I. Bogolubsky, E. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, *Phys. Lett. B* **676**, 69 (2009).
- [6] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, *Proc. Sci.*, LAT2007 (2007) 290 [[arXiv:0710.1968](#)].
- [7] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, A. Sternbeck, L. von Smekal, A. G. Williams, and J. Zhang, *Phys. Rev. D* **76**, 094505 (2007).
- [8] O. Oliveira and P. Silva, *Proc. Sci.*, LAT2009 (2009) 226 [[arXiv:0910.2897](#)].
- [9] P. Bicudo, D. Binosi, N. Cardoso, O. Oliveira, and P. J. Silva, *Phys. Rev. D* **92**, 114514 (2015).
- [10] A. Ayala, A. Bashir, D. Binosi, M. Cristoforetti, and J. Rodriguez-Quintero, *Phys. Rev. D* **86**, 074512 (2012).
- [11] J. M. Cornwall, *Nucl. Phys.* **B157**, 392 (1979).
- [12] J. M. Cornwall, *Phys. Rev. D* **26**, 1453 (1982).
- [13] C. W. Bernard, *Phys. Lett.* **108B**, 431 (1982).
- [14] C. W. Bernard, *Nucl. Phys.* **B219**, 341 (1983).
- [15] J. F. Donoghue, *Phys. Rev. D* **29**, 2559 (1984).
- [16] M. Lavelle, *Phys. Rev. D* **44**, R26 (1991).
- [17] F. Halzen, G. I. Krein, and A. A. Natale, *Phys. Rev. D* **47**, 295 (1993).
- [18] O. Philipsen, *Nucl. Phys.* **B628**, 167 (2002).
- [19] A. P. Szczepaniak and E. S. Swanson, *Phys. Rev. D* **65**, 025012 (2001).
- [20] A. C. Aguilar, A. Mihara, and A. A. Natale, *Phys. Rev. D* **65**, 054011 (2002).
- [21] A. C. Aguilar, A. A. Natale, and P. S. Rodrigues da Silva, *Phys. Rev. Lett.* **90**, 152001 (2003).
- [22] P. Maris and C. D. Roberts, *Int. J. Mod. Phys. E* **12**, 297 (2003).
- [23] A. P. Szczepaniak, *Phys. Rev. D* **69**, 074031 (2004).
- [24] X.-d. Li and C. M. Shakin, *Phys. Rev. D* **71**, 074007 (2005).
- [25] A. C. Aguilar and A. A. Natale, *J. High Energy Phys.* **08** (2004) 057.
- [26] A. C. Aguilar and J. Papavassiliou, *J. High Energy Phys.* **12** (2006) 012.
- [27] K.-I. Kondo, *Phys. Rev. D* **74**, 125003 (2006).
- [28] J. Braun, H. Gies, and J. M. Pawłowski, *Phys. Lett. B* **684**, 262 (2010).
- [29] D. Eppe, H. Reinhardt, W. Schleifenbaum, and A. Szczepaniak, *Phys. Rev. D* **77**, 085007 (2008).
- [30] D. Binosi and J. Papavassiliou, *Phys. Rev. D* **77**, 061702 (2008).
- [31] D. Binosi and J. Papavassiliou, *J. High Energy Phys.* **11** (2008) 063.
- [32] A. C. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **78**, 025010 (2008).
- [33] P. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pène, and J. Rodríguez-Quintero, *J. High Energy Phys.* **06** (2008) 099.
- [34] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, *Phys. Rev. D* **78**, 065047 (2008).
- [35] C. S. Fischer, A. Maas, and J. M. Pawłowski, *Ann. Phys. (Amsterdam)* **324**, 2408 (2009).
- [36] A. P. Szczepaniak and H. H. Matevosyan, *Phys. Rev. D* **81**, 094007 (2010).
- [37] P. Watson and H. Reinhardt, *Phys. Rev. D* **82**, 125010 (2010).
- [38] J. Rodríguez-Quintero, *J. High Energy Phys.* **01** (2011) 105.
- [39] D. R. Campagnari and H. Reinhardt, *Phys. Rev. D* **82**, 105021 (2010).
- [40] M. Tissier and N. Wschebor, *Phys. Rev. D* **82**, 101701 (2010).
- [41] M. Pennington and D. Wilson, *Phys. Rev. D* **84**, 119901 (2011).
- [42] P. Watson and H. Reinhardt, *Phys. Rev. D* **85**, 025014 (2012).
- [43] K.-I. Kondo, *Phys. Rev. D* **84**, 061702 (2011).
- [44] I. C. Cloet and C. D. Roberts, *Prog. Part. Nucl. Phys.* **77**, 1 (2014).

- [45] F. Siringo, *Phys. Rev. D* **90**, 094021 (2014).
- [46] D. Binosi, L. Chang, J. Papavassiliou, and C. D. Roberts, *Phys. Lett. B* **742**, 183 (2015).
- [47] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [48] J. M. Cornwall and J. Papavassiliou, *Phys. Rev. D* **40**, 3474 (1989).
- [49] A. Pilaftsis, *Nucl. Phys.* **B487**, 467 (1997).
- [50] D. Binosi and J. Papavassiliou, *Phys. Rev. D* **66**, 111901 (R) (2002).
- [51] D. Binosi and J. Papavassiliou, *J. Phys. G* **30**, 203 (2004).
- [52] D. Binosi and J. Papavassiliou, *Phys. Rep.* **479**, 1 (2009).
- [53] L. F. Abbott, *Nucl. Phys.* **B185**, 189 (1981).
- [54] L. F. Abbott, *Acta Phys. Pol. B* **13**, 33 (1982).
- [55] J. S. Schwinger, *Phys. Rev.* **125**, 397 (1962).
- [56] J. S. Schwinger, *Phys. Rev.* **128**, 2425 (1962).
- [57] R. Jackiw and K. Johnson, *Phys. Rev. D* **8**, 2386 (1973).
- [58] R. Jackiw, in *Erice 1973, Proceedings, Laws Of Hadronic Matter, New York, 1975* (MIT Cambridge Report No. COO-3069-190, 1973), pp. 225–251.
- [59] J. M. Cornwall and R. E. Norton, *Phys. Rev. D* **8**, 3338 (1973).
- [60] E. Eichten and F. Feinberg, *Phys. Rev. D* **10**, 3254 (1974).
- [61] E. C. Poggio, E. Tomboulis, and S. H. H. Tye, *Phys. Rev. D* **11**, 2839 (1975).
- [62] A. Aguilar, D. Ibanez, V. Mathieu, and J. Papavassiliou, *Phys. Rev. D* **85**, 014018 (2012).
- [63] D. Ibañez and J. Papavassiliou, *Phys. Rev. D* **87**, 034008 (2013).
- [64] A. C. Aguilar and J. Papavassiliou, *Phys. Rev. D* **81**, 034003 (2010).
- [65] A. C. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **84**, 085026 (2011).
- [66] D. Binosi, D. Ibañez, and J. Papavassiliou, *Phys. Rev. D* **86**, 085033 (2012).
- [67] V. N. Gribov, *Nucl. Phys.* **B139**, 1 (1978).
- [68] D. Zwanziger, *Nucl. Phys.* **B323**, 513 (1989).
- [69] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde, *Phys. Rev. D* **77**, 071501 (2008).
- [70] J. Serreau and M. Tissier, *Phys. Lett. B* **712**, 97 (2012).
- [71] J. Serreau, M. Tissier, and A. Tresmontant, *Phys. Rev. D* **92**, 105003 (2015).
- [72] G. Curci and R. Ferrari, *Nuovo Cimento A* **32**, 151 (1976).
- [73] J. Serreau, *Proc. Sci.*, CPOD2014 (2015) 036 [arXiv: 1504.00038].
- [74] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde, *Phys. Rev. D* **79**, 121701 (2009).
- [75] D. Dudal and S. P. Sorella, *Phys. Rev. D* **86**, 045005 (2012).
- [76] A. Cucchieri, D. Dudal, T. Mendes, and N. Vandersickel, *Phys. Rev. D* **90**, 051501 (2014).
- [77] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimaraes, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P. Sorella, *Phys. Rev. D* **92**, 045039 (2015).
- [78] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimaraes, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P. Sorella, arXiv:1605.02610 [Phys. Rev. D (to be published)].
- [79] K. G. Wilson, *Phys. Rev. D* **7**, 2911 (1973).
- [80] J. C. Collins, *Renormalization*, Cambridge Monographs on Mathematical Physics Vol. 26 (Cambridge University Press, Cambridge, 1986).
- [81] P. A. Grassi, T. Hurth, and M. Steinhauser, *Ann. Phys. (N.Y.)* **288**, 197 (2001).
- [82] D. Binosi and J. Papavassiliou, *Phys. Rev. D* **66**, 025024 (2002).
- [83] D. Binosi and A. Quadri, *Phys. Rev. D* **88**, 085036 (2013).
- [84] D. Binosi, D. Ibañez, and J. Papavassiliou, *J. High Energy Phys.* **09** (2014) 059.
- [85] A. K. Cyrol, M. Q. Huber, and L. von Smekal, *Eur. Phys. J. C* **75**, 102 (2015).
- [86] G. Eichmann, C. S. Fischer, and W. Heupel, *Phys. Rev. D* **92**, 056006 (2015).
- [87] A. Aguilar, D. Binosi, and J. Papavassiliou, *J. High Energy Phys.* **11** (2009) 066.
- [88] A. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **91**, 085014 (2015).
- [89] M. Q. Huber, *Phys. Rev. D* **91**, 085018 (2015).
- [90] M. Stingl, *Phys. Rev. D* **34**, 3863 (1986); **36**, 651(E) (1987).
- [91] P. Pascual and R. Tarrach, *Lect. Notes Phys.* **194**, 1 (1984).
- [92] A. C. Aguilar, D. Binosi, D. Ibañez, and J. Papavassiliou, *Phys. Rev. D* **89**, 085008 (2014).
- [93] M. Q. Huber and L. von Smekal, *J. High Energy Phys.* **04** (2013) 149.
- [94] A. Blum, M. Q. Huber, M. Mitter, and L. von Smekal, *Phys. Rev. D* **89**, 061703 (2014).
- [95] G. Eichmann, R. Williams, R. Alkofer, and M. Vujanovic, *Phys. Rev. D* **89**, 105014 (2014).
- [96] A. Cucchieri, A. Maas, and T. Mendes, *Phys. Rev. D* **74**, 014503 (2006).
- [97] A. Cucchieri, A. Maas, and T. Mendes, *Phys. Rev. D* **77**, 094510 (2008).
- [98] D. Binosi and A. Quadri, *Phys. Rev. D* **85**, 121702 (2012).
- [99] A. Cucchieri and T. Mendes, *Phys. Rev. D* **86**, 071503 (2012).
- [100] R. Alkofer, C. S. Fischer, and F. J. Llanes-Estrada, *Phys. Lett. B* **611**, 279 (2005).
- [101] R. Alkofer, M. Q. Huber, and K. Schwenzer, *Phys. Rev. D* **81**, 105010 (2010).
- [102] R. Alkofer, M. Q. Huber, and K. Schwenzer, *Eur. Phys. J. C* **62**, 761 (2009).