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CONFORMAL STRUCTURES AND TWISTORS IN THE PARAVECTOR MODEL OF SPACETIME

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Some properties of the Clifford algebras $\mathcal{Cl}_{3,0}, \mathcal{Cl}_{1,3}, \mathcal{Cl}_{4,1} \simeq \mathbb{C} \otimes \mathcal{Cl}_{1,3}$ and $\mathcal{Cl}_{2,4}$ are presented, and three isomorphisms between the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ and $\mathcal{Cl}_{4,1}$ are exhibited, in order to construct conformal maps and twistors, using the paravector model of spacetime. The isomorphism between the twistor space inner product isometry group $SU(2,2)$ and the group $\text{Spin}_+(2,4)$ is also investigated, in the light of a suitable isomorphism between $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ and $\mathcal{Cl}_{4,1}$. After reviewing the conformal spacetime structure, conformal maps are described in Minkowski spacetime as the twisted adjoint representation of $\text{Spin}_+(2,4)$, acting on paravectors. Twistors are then presented via the paravector model of Clifford algebras and related to conformal maps in the Clifford algebra over the Lorentzian $\mathbb{R}^{4,1}$ spacetime. We construct twistors in Minkowski spacetime as algebraic spinors associated with the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ using one lower spacetime dimension than standard Clifford algebra formulations, since for this purpose, the Clifford algebra over $\mathbb{R}^{4,1}$ is also used to describe conformal maps, instead of $\mathbb{R}^{2,4}$. Our formalism sheds some new light on the use of the paravector model and generalizations.

Keywords: Twistors; conformal maps; Clifford algebras.

Mathematics Subject Classification 2000: 15A66, 81R05

0. Introduction

Twistor theory is originally based on spinors, from the construction of a space, the *twistor space*, in such a way that the spacetime structure emerges as a secondary

concept. According to this formalism, twistors are considered as more primitive entities than spacetime points. Twistors are used to describe some physical concepts, for example, momentum, angular momentum, helicity and massless fields [10, 11]. The difficulties in constructing a theory for quantum gravity, based on the continuous spacetime structure, suggests the discretization process of such structure [12]. One of the motivations to investigate twistor theory are the *spin networks*, related to a discrete description of spacetime [13–15].

Twistor formalism has been used to describe a lot of physical theories, and an increasing progress of wide-ranging applications of this formalism via Clifford algebras has been done [16–20] in the last two decades. Another branch of applications is the union between twistors, supersymmetric theories and strings (see, e.g. [21–32, 50] and many others). As a particular case, the classical Penrose twistor formalism [13–15, 34, 35] describes a spin $3/2$ particle, the gravitino, which is the graviton superpartner. Twistor formalism is also used in the investigation on the relativistic dynamics of elementary particles [33, 36] and about confined states [37].

The main aim of the present paper is to describe spinors and twistors as algebraic objects, from the Clifford algebra standview. In this approach, a twistor is an algebraic spinor [38], an element of a lateral minimal ideal of a Clifford algebra. This characterization is done using the representation of the conformal group and the structure of the Periodicity Theorem of Clifford algebras [1, 9, 39, 40]. Equivalently, a twistor on Minkowski spacetime is an element that carries the representation of the group $\text{Spin}_+(2, 4)$, the double covering^a of $\text{SO}_+(2, 4)$. This group $\text{SO}_+(2, 4)$ describes proper orthochronous rotations in $\mathbb{R}^{2,4}$, and is the invariance group of the bilinear invariants [41] in the Dirac relativistic quantum mechanics theory [7]. This group is also the double covering of $\text{SConf}_+(1, 3)$, the group of the proper special conformal transformations, that is the biggest one that preserves the structure of Maxwell equations, leaving invariant the light-cone in Minkowski spacetime.

Introducing the group $\text{SU}(2, 2)$ — of the inner product isometries in twistor spaces in the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3} \simeq \mathcal{Cl}_{4,1}$ — we describe twistors as a special class of algebraic spinors in $\mathcal{Cl}_{4,1}$. This paper is organized as follows. In Sec. 1, we give a brief introduction to Clifford algebras and fix the notation to be used in the rest of the paper. In Sec. 2, the Pauli algebra $\mathcal{Cl}_{3,0}$ is investigated jointly with the representation of the Pauli matrices [1] and quaternions. In Sec. 3, we point out some remarks on the spacetime algebra $\mathcal{Cl}_{1,3}$ and its quaternionic $\mathcal{M}(2, \mathbb{H})$ matrix representation, the 2×2 matrices with quaternionic entries. In Sec. 4, the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ is investigated. In Sec. 5, the algebra $\mathcal{Cl}_{2,4}$ is briefly investigated, and in Sec. 6, three explicit isomorphisms between $\mathcal{Cl}_{4,1}$ and $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ are obtained. In order to prove the correspondence of our twistor approach to the Penrose classical formalism, the isomorphism $\text{SU}(2, 2) \simeq \text{Spin}_+(2, 4)$ is constructed. Also, the isomorphism $\text{SU}(2, 2) \simeq \text{Spin}_+(2, 4)$ is constructed. The Clifford algebra morphisms of $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ are related to the ones of $\mathcal{Cl}_{4,1}$ and two new antiautomorphisms are introduced. In Sec. 7,

^aThe double covering of $\text{SO}_+(2, 4)$ is the group $\text{Spin}_+(2, 4)$. The notation $\text{Spin}_+(2, 4)$ is motivated by the use of paravectors, elements of $\mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{Cl}_{4,1}$.

the Periodicity Theorem of Clifford algebras is presented, from which Möbius maps in the plane are investigated. We also introduce the conformal compactification of $\mathbb{R}^{p,q}$ and then, the conformal group is defined. In Sec. 8, the conformal transformations in Minkowski spacetime are presented as the twisted adjoint representation of the group $SU(2,2) \simeq \text{Spin}_+(2,4)$ on paravectors of $\mathcal{Cl}_{4,1}$. Also, the Lie algebra of associated groups and the one of the conformal group, are presented. In Sec. 9, twistors, the incidence relation between twistors and the Robinson congruence, via multivectors and the paravector model of $\mathbb{C} \otimes \mathcal{Cl}_{1,3} \simeq \mathcal{Cl}_{4,1}$, are introduced. We show explicitly how our results can be led to the well-established ones of Keller [42], and consequently, to the classical formulation introduced by Penrose [34,35]. In Appendix the Weyl and standard representations of the Dirac matrices are obtained, as in [1].

1. Preliminaries

Let V be a finite n -dimensional real vector space. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^n \Lambda^k(V)$ of multivectors over V . $\Lambda^k(V)$ denotes the space of the antisymmetric k -tensors, isomorphic to the k -forms. Given $\psi_k \in \Lambda^k(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi}_k = (-1)^{[k/2]} \psi_k$ ($[k]$ denotes the integer part of k). $\hat{\psi}_k$ denotes the *main automorphism or graded involution*, given by $\hat{\psi}_k = (-1)^k \psi_k$. The *conjugation* is defined as the reversion followed by the main automorphism. If V is endowed with a non-degenerate, symmetric, bilinear map $g : V \times V \rightarrow \mathbb{R}$, it is possible to extend g to $\Lambda(V)$. Given $\psi = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ and $\phi = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_l$, $\mathbf{u}_i, \mathbf{v}_j \in V$, one defines $g(\psi, \phi) = \det(g(\mathbf{u}_i, \mathbf{v}_j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. Finally, the projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its p -vector part is given by $\langle \psi \rangle_p = \psi_p$. The Clifford product between $\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \lrcorner \psi$. Given $\psi, \phi, \xi \in \Lambda(V)$, the left contraction is defined implicitly by $g(\psi \lrcorner \phi, \xi) = g(\tilde{\psi}, \phi \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $\mathbf{v} \lrcorner a = 0$. Given $\mathbf{v} \in V$, the Leibniz rule $\mathbf{v} \lrcorner (\psi \wedge \phi) = (v \lrcorner \psi) \wedge \phi + \hat{\psi} \wedge (v \lrcorner \phi)$ holds. The right contraction is analogously defined $g(\psi \lrcorner \phi, \xi) = g(\phi, \psi \wedge \tilde{\xi})$ and its associated Leibniz rule $\mathbf{v} \lrcorner (\psi \wedge \phi) = (v \lrcorner \psi) \wedge \phi + \hat{\psi} \wedge (\mathbf{v} \lrcorner \phi)$ also holds. Both contractions are related by $\mathbf{v} \lrcorner \psi = -\hat{\psi} \lrcorner \mathbf{v}$. The Grassmann algebra $(\Lambda(V), g)$ endowed with the Clifford product is denoted by $\mathcal{Cl}(V, g)$ or $\mathcal{Cl}_{p,q}$, the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p+q = n$. In what follows, \mathbb{R}, \mathbb{C} and \mathbb{H} denote respectively the real, complex and quaternionic (scalar) fields, and the Clifford geometric product will be denoted by juxtaposition. The vector space $\Lambda_k(V)$ denotes the space of the k -vectors.

2. The Pauli Algebra $\mathcal{Cl}_{3,0}$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of \mathbb{R}^3 . The Clifford algebra $\mathcal{Cl}_{3,0}$, also called the *Pauli algebra*, is generated by $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, such that $\frac{1}{2}(\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i) = 2g(\mathbf{e}_i, \mathbf{e}_j) = 2\delta_{ij}$. An arbitrary element of $\mathcal{Cl}_{3,0}$ can be written as

$$\psi = a + a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3 + a^{12} \mathbf{e}_{12} + a^{13} \mathbf{e}_{13} + a^{23} \mathbf{e}_{23} + p \mathbf{e}_{123}, \quad a, a^i, a^{ij}, p \in \mathbb{R}. \quad (1)$$

The graded involution performs the decomposition $\mathcal{Cl}_{3,0} = \mathcal{Cl}_{3,0}^+ \oplus \mathcal{Cl}_{3,0}^-$, where $\mathcal{Cl}_{3,0}^\pm = \{\psi \in \mathcal{Cl}_{3,0} \mid \hat{\psi} = \pm \psi\}$. Here $\mathcal{Cl}_{3,0}^+$ denotes the even subalgebra of $\mathcal{Cl}_{3,0}$ and its elements are written as $\varphi_+ = a + a^{ij}\mathbf{e}_{ij}$.

2.1. Representation of $\mathcal{Cl}_{3,0}$: Pauli matrices

Now, a representation $\rho : \mathcal{Cl}_{3,0} \rightarrow \mathcal{M}(2, \mathbb{C})$ is obtained by the mapping $\rho : \mathbf{e}_i \mapsto \rho(\mathbf{e}_i) = \sigma_i$ given by

$$\rho(\mathbf{e}_1) = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\mathbf{e}_2) = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho(\mathbf{e}_3) = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2)$$

that are the Pauli matrices. In this representation, a multivector $\psi \in \mathcal{Cl}_{3,0}$ corresponds to the matrix $\Psi = \rho(\psi)$. If ψ is given by Eq. (1), then Ψ is given by

$$\Psi = \begin{pmatrix} (a + a^3) + i(a^{12} + p) & (a^1 + a^{13}) + i(a^{23} - a^2) \\ (a^1 - a^{13}) + i(a^{23} + a^2) & (a - a^3) + i(p - a^{12}) \end{pmatrix} := \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}.$$

Reversion, graded involution and conjugation of $\psi \in \mathcal{Cl}_{3,0}$, corresponds in $\mathcal{M}(2, \mathbb{C})$ to

$$\tilde{\Psi} = \begin{pmatrix} z_1^* & z_2^* \\ z_3^* & z_4^* \end{pmatrix}, \quad \hat{\Psi} = \begin{pmatrix} z_4^* & -z_2^* \\ -z_3^* & z_1^* \end{pmatrix}, \quad \bar{\Psi} = \begin{pmatrix} z_4 & -z_3 \\ -z_2 & z_1 \end{pmatrix}$$

and an element of $\mathcal{Cl}_{3,0}^+$ is represented by

$$\rho(\varphi_+) = \Phi_+ = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}, \quad w_1, w_2 \in \mathbb{C}. \quad (3)$$

2.2. Quaternions

The quaternion ring \mathbb{H} has elements of the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = q_0 + \mathbf{q}$, where $q_\mu \in \mathbb{R}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the \mathbb{H} -units. They satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \quad (4)$$

$q_0 = \text{Re}(q)$ denotes the real part of q and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ denotes its pure quaternionic part. Since $\varphi_+ = a + a^{12}\mathbf{e}_{12} + a^{13}\mathbf{e}_{13} + a^{23}\mathbf{e}_{23} \in \mathcal{Cl}_{3,0}^+$ then $\mathbb{H} \simeq \mathcal{Cl}_{0,2} \simeq \mathcal{Cl}_{3,0}^+$. Introducing the notation $\mathbf{i} = \mathbf{e}_2\mathbf{e}_3$, $\mathbf{j} = \mathbf{e}_3\mathbf{e}_1$, $\mathbf{k} = \mathbf{e}_1\mathbf{e}_2$, the isomorphism $\zeta : \mathbb{H} \rightarrow \mathcal{Cl}_{3,0}^+$, is explicitly constructed by $\zeta(\mathbf{i}) = \mathbf{i}$, $\zeta(\mathbf{j}) = \mathbf{j}$, $\zeta(\mathbf{k}) = \mathbf{k}$, and it is immediate that the bivectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfy Eqs. (4). Denoting $\mathcal{J} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, the element $\psi \in \mathcal{Cl}_{3,0}$ can be expressed as

$$\psi = (a + \mathcal{J}p) + (a^{12} - \mathcal{J}a^3)\mathbf{e}_{12} + (a^{23} - \mathcal{J}a^1)\mathbf{e}_{23} + (-a^{13} - \mathcal{J}a^2)\mathbf{e}_{31}, \quad (5)$$

which permits us to verify that $\mathbb{C} \otimes \mathbb{H} \simeq \mathcal{Cl}_{3,0}$ and therefore, $\mathbb{C} \otimes \mathbb{H} \simeq \mathcal{M}(2, \mathbb{C})$.

3. The Spacetime Algebra $\mathcal{Cl}_{1,3}$

Let $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ be an orthonormal basis in $\mathbb{R}^{1,3}$, satisfying $\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu}$, where $\eta_{ii} = -1$, $\eta_{00} = 1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, $(\mu, \nu = 0, 1, 2, 3)$. $\gamma_\mu \cdot \gamma_\nu$ denotes the scalar product between γ_μ and γ_ν . An element $\Upsilon \in \mathcal{Cl}_{1,3}$ is written as

$$\begin{aligned} \Upsilon = & c + c^0 \gamma_0 + c^1 \gamma_1 + c^2 \gamma_2 + c^3 \gamma_3 + c^{01} \gamma_{01} + c^{02} \gamma_{02} + c^{03} \gamma_{03} + c^{12} \gamma_{12} + c^{13} \gamma_{13} \\ & + c^{23} \gamma_{23} + c^{012} \gamma_{012} + c^{013} \gamma_{013} + c^{023} \gamma_{023} + c^{123} \gamma_{123} + c^{0123} \gamma_{0123}. \end{aligned} \quad (6)$$

The pseudoscalar $\gamma_5 := \gamma_{0123}$ satisfies $(\gamma_5)^2 = -1$ and $\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu$. In order to construct an isomorphism $\mathcal{Cl}_{1,3} \simeq \mathcal{M}(2, \mathbb{H})$, the primitive idempotent $f = \frac{1}{2}(1 + \gamma_0)$ is used. A left minimal ideal of $\mathcal{Cl}_{1,3}$ is denoted by $I_{1,3} := \mathcal{Cl}_{1,3} f$, which has arbitrary elements expressed as

$$\Xi = (a^1 + a^2 \gamma_{23} + a^3 \gamma_{31} + a^4 \gamma_{12})f + (a^5 + a^6 \gamma_{23} + a^7 \gamma_{31} + a^8 \gamma_{12})\gamma_5 f,$$

where

$$a^1 = c + c^0, \quad a^2 = c^{23} + c^{023}, \quad a^3 = -c^{13} - c^{013}, \quad a^4 = c^{12} + c^{012}, \quad (7)$$

$$a^5 = -c^{123} + c^{0123}, \quad a^6 = c^1 - c^{01}, \quad a^7 = c^2 - c^{02}, \quad a^8 = c^3 - c^{03}. \quad (8)$$

Denoting $\mathbf{i}_\circ = \gamma_{23}$, $\mathbf{j}_\circ = \gamma_{31}$, $\mathbf{k}_\circ = \gamma_{12}$, it is seen that the elements of the set $\{\mathbf{i}_\circ, \mathbf{j}_\circ, \mathbf{k}_\circ\}$ anticommute, satisfy the relations $\mathbf{i}_\circ \mathbf{j}_\circ = \mathbf{k}_\circ$, $\mathbf{j}_\circ \mathbf{k}_\circ = \mathbf{i}_\circ$, $\mathbf{k}_\circ \mathbf{i}_\circ = \mathbf{j}_\circ$, $\mathbf{i}_\circ \mathbf{j}_\circ \mathbf{k}_\circ = -1$, and

$$\Xi = (a^1 + a^2 \mathbf{i}_\circ + a^3 \mathbf{j}_\circ + a^4 \mathbf{k}_\circ)f + (a^5 + a^6 \mathbf{i}_\circ + a^7 \mathbf{j}_\circ + a^8 \mathbf{k}_\circ)\gamma_5 f \in \mathcal{Cl}_{1,3} f = I_{1,3}.$$

The set $\{1, \gamma_5\}f$ is a basis of the ideal $I_{1,3}$. From the rules in [1], we can write

$$\gamma_\mu = f \gamma_\mu f + f \gamma_\mu \gamma_5 f - f \gamma_5 \gamma_\mu f - f \gamma_5 \gamma_\mu \gamma_5 f, \quad (9)$$

and the following representation for γ_μ is obtained:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix}, \quad (10)$$

implying that $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\gamma_5 f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. and using the equations above, $\Upsilon \in \mathcal{Cl}_{1,3}$ is written as

$$\begin{aligned} \Upsilon = & \begin{pmatrix} (c + c^0) + (c^{23} + c^{023})\mathbf{i} & (-c^{123} - c^{0123}) + (c^1 + c^{01})\mathbf{i} \\ + (-c^{13} - c^{013})\mathbf{j} + (c^{12} + c^{012})\mathbf{k} & + (c^2 + c^{02})\mathbf{j} + (c^3 + c^{03})\mathbf{k} \\ (-c^{123} + c^{0123}) + (c^1 - c^{01})\mathbf{i} & (c - c^0) + (c^{23} - c^{023})\mathbf{i} \\ + (c^2 - c^{02})\mathbf{j} + (c^3 - c^{03})\mathbf{k} & + (-c^{13} + c^{013})\mathbf{j} + (c^{12} - c^{012})\mathbf{k} \end{pmatrix} \\ = & \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \in \mathcal{M}(2, \mathbb{H}). \end{aligned} \quad (11)$$

The reversion of Υ is given by $\tilde{\Upsilon} = \begin{pmatrix} \bar{q}_1 & -\bar{q}_3 \\ -\bar{q}_2 & \bar{q}_4 \end{pmatrix}$, where \bar{q} denotes the \mathbb{H} -conjugation of q .

As particular cases of the isomorphism $\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{q,p+1}^+$ [1], when $p = 3$ and $q = 0$, we have $\mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_{3,0}$, given by the application $\rho : \mathcal{C}\ell_{1,3}^+ \rightarrow \mathcal{C}\ell_{3,0}$ defined as $\rho(\gamma_i) = \mathbf{e}_i = \gamma_i \gamma_0$. Given a vector $\mathbf{x} = x^\mu \gamma_\mu \in \mathbb{R}^{1,3}$, from the isomorphism above we see that

$$\mathbf{x} \gamma_0 = x^\mu \gamma_\mu \gamma_0 = x^0 + x^i \gamma_i \gamma_0 = x^0 + x^i \mathbf{e}_i \in \mathbb{R} \oplus \mathbb{R}^3.$$

A vector in $\mathbb{R}^{1,3}$ is said to be isomorph to a *paravector* [4–6] of \mathbb{R}^3 , defined as an element of $\mathbb{R} \oplus \mathbb{R}^3 \hookrightarrow \mathcal{C}\ell_{3,0}$.

It can also be seen that the norm $s\bar{s}$ of $s \in \mathcal{C}\ell_{1,3}^+$ is equivalent to the norm $\sigma\bar{\sigma}$ of $\sigma \in \mathcal{C}\ell_{3,0}$, where $\sigma = \rho(s)$. In this sense, the group $\mathbb{S}\text{pin}_+(1, 3) = \{s \in \mathcal{C}\ell_{3,0} \mid s\bar{s} = 1\}$ is defined, as in [2].

4. The Dirac–Clifford Algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$

The standard Clifford algebra, usually found in relativistic quantum mechanics textbooks [3, 7], is *not* the real spacetime algebra $\mathcal{C}\ell_{1,3} \simeq \mathcal{M}(2, \mathbb{H})$, but its complexification $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{M}(4, \mathbb{C})$, the so-called Dirac algebra. In this section, the Weyl representation and the standard representation of the Dirac algebra are explicitly constructed. We follow and reproduce the steps described in [1], where a method to find a representation of $\mathcal{C}\ell_{3,1} \simeq \mathcal{M}(4, \mathbb{R})$ is explained. The set $\{e_0, e_1, e_2, e_3\} \in \mathbb{R}^{1,3}$ denotes an orthonormal basis and $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ ($\gamma_\mu := \gamma(e_\mu)$) denotes hereon the (matrix) representation of $\{e_0, e_1, e_2, e_3\}$. Since $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}(\mathbb{R}) \simeq \mathcal{C}\ell_{1,3}(\mathbb{C}) \simeq \mathcal{M}(4, \mathbb{C})$, we must obtain four primitive idempotents P_1, P_2, P_3 and P_4 such that $1 = P_1 + P_2 + P_3 + P_4$. It is enough [1] to obtain two idempotents e_{I_1}, e_{I_2} of $\mathcal{C}\ell_{1,3}(\mathbb{C})$ that commute. This is done in detail in Appendix, where we follow the idea presented in [1] to obtain the Weyl and the Standard representations.

5. The Clifford Algebra $\mathcal{C}\ell_{2,4}$

Consider the Clifford algebra $\mathcal{C}\ell_{2,4}$. Let $\{\varepsilon_{\tilde{A}}\}_{\tilde{A}=0}^5$ be a basis of $\mathbb{R}^{2,4}$, with $\varepsilon_0^2 = \varepsilon_5^2 = 1$ and $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = -1$. Let $\mathbb{R}^{4,1}$ be the vector space with a basis $\{E_A\}_{A=0}^4$, where $E_0^2 = -1$ and $E_1^2 = E_2^2 = E_3^2 = E_4^2 = 1$. The groups

$$\text{Pin}_+(2, 4) = \{R \in \mathcal{C}\ell_{2,4} \mid R\tilde{R} = 1\}, \quad \text{Spin}_+(2, 4) = \{R \in \mathcal{C}\ell_{2,4}^+ \mid R\tilde{R} = 1\} \quad (12)$$

are defined, together with the group

$$\mathbb{S}\text{pin}_+(2, 4) = \{D \in \mathcal{C}\ell_{4,1} \mid D\bar{D} = 1\} \quad (13)$$

The inclusion

$$\mathbb{S}\text{pin}_+(2, 4) \hookrightarrow \mathcal{C}\ell_{2,4}^+ \simeq \mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}.$$

follows from the definition. These groups are useful in the twistor definition to be presented in Sec. 9.

6. The Isomorphism $\mathcal{Cl}_{4,1} \simeq \mathbb{C} \otimes \mathcal{Cl}_{1,3}$

In this section, the conformal maps in Minkowski spacetime are described using the Dirac algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$. For this purpose, we explicitly exhibit three important isomorphisms between $\mathcal{Cl}_{4,1}$ and $\mathbb{C} \otimes \mathcal{Cl}_{1,3}$ in the following subsections. The relation between these algebras is investigated deeply, since twistors are defined as algebraic spinors in $\mathbb{R}^{4,1}$ or, equivalently, as classical spinors in $\mathbb{R}^{2,4}$, defined as elements of the representation space of the group $\text{Spin}_+(2, 4)$ (defined by Eq. (13)) in $\mathcal{Cl}_{4,1} \simeq \mathbb{C} \otimes \mathcal{Cl}_{1,3}$.

6.1. The $\gamma_\nu = E_\nu E_4$ identification

An isomorphism $\mathbb{C} \otimes \mathcal{Cl}_{1,3} \rightarrow \mathcal{Cl}_{4,1}$ is defined by

$$\begin{aligned} \gamma_\nu &\mapsto E_\nu E_4, \quad (\nu = 1, 2, 3, 4), \\ i &= \gamma_0 \gamma_1 \gamma_2 \gamma_3 \mapsto E_{01234}. \end{aligned} \quad (14)$$

It can be shown that

$$E_0 = -i\gamma_{123}, \quad E_1 = -i\gamma_{023}, \quad E_2 = i\gamma_{013}, \quad E_3 = -i\gamma_{012}, \quad E_4 = -i\gamma_{0123}. \quad (15)$$

An arbitrary element of $\mathcal{Cl}_{4,1}$ is written as:

$$\begin{aligned} Z &= H + H^A E_A + H^{AB} E_{AB} + H^{ABC} E_{ABC} + H^{ABCD} E_{ABCD} + H^{01234} E_{01234} \\ &= B + B^\mu \gamma_\mu + B^{\mu\nu} \gamma_{\mu\nu} + B^{\mu\nu\sigma} \gamma_{\mu\nu\sigma} + B^{0123} \gamma_{0123}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} B &= H + iH^{01234} & B^0 &= H^{04} - iH^{123}, & B^1 &= H^{14} - iH^{023}, \\ B^2 &= H^{24} + iH^{013}, & B^3 &= H^{34} - iH^{012}, & B^{12} &= -H^{12} + iH^{034}, \\ B^{13} &= -H^{13} - iH^{024}, & B^{23} &= -H^{23} + iH^{014}, & B^{01} &= -H^{01} - iH^{234}, \\ B^{02} &= -H^{02} + iH^{134}, & B^{03} &= -H^{03} - iH^{124}, & B^{012} &= -H^{0124} - iH^3, \\ B^{013} &= -H^{0134} + iH^2, & B^{023} &= -H^{0234} - iH^1, & B^{123} &= -H^{1234} - iH^0, \\ B^{0123} &= H^{0123} - iH^4. \end{aligned}$$

With these identifications, the (anti-)automorphisms of $\mathcal{Cl}_{4,1}$ are related to the ones of $\mathcal{Cl}_{1,3}$ by

$$\begin{aligned} \widehat{\mathcal{Cl}_{4,1}} &\simeq \mathbb{C}^* \otimes \mathcal{Cl}_{1,3}, \\ \widetilde{\mathcal{Cl}_{4,1}} &\simeq \mathbb{C} \otimes \overline{\mathcal{Cl}_{1,3}}, \\ \overline{\mathcal{Cl}_{4,1}} &\simeq \mathbb{C}^* \otimes \overline{\mathcal{Cl}_{1,3}}, \end{aligned}$$

Another two automorphisms of $\mathcal{Cl}_{4,1}$ can be defined:

$$\mathcal{Cl}_{4,1}^\bullet := E_4 \widehat{\mathcal{Cl}_{4,1}} E_4 \simeq \mathbb{C} \otimes \widetilde{\mathcal{Cl}_{1,3}}, \quad (17)$$

$$\mathcal{Cl}_{4,1}^\Delta := E_4 \mathcal{Cl}_{4,1} E_4 \simeq \mathbb{C} \otimes \widehat{\mathcal{Cl}_{1,3}}. \quad (18)$$

Using the standard representation for γ_μ , one obtains

$$\varrho(Z) \equiv \mathbf{Z} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix},$$

where

$$\begin{aligned} z_{11} &= (H + H^{04} + H^{034} - H^3) + i(H^{01234} - H^{123} + H^{12} + H^{0124}), \\ z_{12} &= (-H^{13} - H^{0134} + H^{014} - H^1) + i(-H^{024} + H^2 + H^{23} + H^{0234}), \\ z_{13} &= (H^{03} - H^{34} + H^4 + H^0) + i(H^{124} + H^{012} + H^{0123} - H^{1234}), \\ z_{14} &= (H^{01} - H^{14} + H^{134} - H^{013}) + i(H^{234} + H^{023} - H^{02} + H^{24}), \\ z_{21} &= (H^{13} + H^{0134} + H^{014} - H^1) + i(H^{024} - H^2 + H^{23} + H^{0234}), \\ z_{22} &= (H + H^{04} - H^{034} + H^3) + i(H^{01234} - H^{123} - H^{12} - H^{0124}), \\ z_{24} &= (-H^{03} + H^{34} + H^4 + H^0) + i(-H^{124} - H^{012} + H^{0123} - H^{1234}), \\ z_{31} &= (H^{03} + H^{34} + H^4 - H^0) + i(H^{124} - H^{012} + H^{0123} + H^{1234}), \\ z_{32} &= (H^{01} + H^{14} - H^{134} + H^{013}) + i(H^{234} - H^{023} - H^{02} - H^{24}), \\ z_{33} &= (H - H^{04} + H^{034} + H^3) + i(H^{01234} + H^{123} + H^{12} - H^{0124}), \\ z_{34} &= (-H^{13} + H^{0134} + H^{014} + H^1) + i(-H^{024} - H^2 + H^{23} - H^{0234}), \\ z_{41} &= (H^{01} + H^{14} + H^{134} - H^{013}) + i(H^{234} - H^{023} + H^{02} + H^{24}), \\ z_{42} &= (-H^{03} - H^{34} + H^4 - H^0) + i(-H^{124} + H^{012} + H^{0123} + H^{1234}), \\ z_{43} &= (H^{13} - H^{0134} + H^{014} + H^1) + i(H^{024} + H^2 + H^{23} - H^{0234}), \\ z_{44} &= (H - H^{04} - H^{034} - H^3) + i(H^{01234} + H^{123} - H^{12} + H^{0124}). \end{aligned}$$

From these expressions, we relate the (anti-)automorphisms of $\mathcal{C}\ell_{4,1}$ to the matrix operations below:

1. Conjugation:

$$\varrho(\bar{Z}) \equiv \bar{\mathbf{Z}} = \begin{pmatrix} z_{33}^* & z_{43}^* & -z_{13}^* & -z_{23}^* \\ z_{34}^* & z_{44}^* & -z_{14}^* & -z_{24}^* \\ -z_{31}^* & -z_{41}^* & z_{11}^* & z_{21}^* \\ -z_{32}^* & -z_{42}^* & z_{12}^* & z_{22}^* \end{pmatrix} = \begin{pmatrix} \phi_4^\dagger & -\phi_2^\dagger \\ -\phi_3^\dagger & \phi_1^\dagger \end{pmatrix},$$

where \dagger denotes Hermitean conjugation.

The relation $\bar{Z}Z = 1$ in $\mathcal{C}\ell_{4,1}$ is translated into $\mathcal{M}(4, \mathbb{C})$ by $\bar{\mathbf{Z}}\mathbf{Z} = 1$, i.e.

$$\begin{aligned} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1^\dagger & \phi_3^\dagger \\ \phi_2^\dagger & \phi_4^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \phi_1^\dagger & \phi_3^\dagger \\ \phi_2^\dagger & \phi_4^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

which means that $\mathbf{Z} \in \text{Sp}(2, \mathbb{C})$ [8]. From this relation, written in the form $\mathbf{Z}^\dagger J \mathbf{Z} = J$, it follows that $(\det \mathbf{Z})^2 = 1$, since $\det J = 1$, and

$$\det \mathbf{Z} = \pm 1. \quad (19)$$

These two possibilities cannot be accomplished in the symplectic case as in the orthogonal case. It is well known [6, 9] that, if $\mathbf{Z} \in \text{Sp}(n, \mathbb{K})$, then $\det \mathbf{Z} = 1$. Then Eq. (19) does not admit the solution $\det \mathbf{Z} = -1$ and only the relation

$$\det \mathbf{Z} = 1 \quad (20)$$

is valid.

2. Reversion:

$$\tilde{\mathbf{Z}} = \begin{pmatrix} z_{44} & -z_{34} & z_{24} & -z_{14} \\ -z_{43} & z_{33} & -z_{23} & z_{13} \\ z_{42} & -z_{32} & z_{22} & -z_{12} \\ -z_{41} & z_{31} & -z_{21} & z_{11} \end{pmatrix} = \begin{pmatrix} \text{adj}(\phi_4) & \text{adj}(\phi_2) \\ \text{adj}(\phi_3) & \text{adj}(\phi_1) \end{pmatrix},$$

where $\text{adj}(\phi) = (\det \phi) \phi^{-1}$, $\forall \phi \in \mathcal{M}(2, \mathbb{C})$.

3. Graded involution:

$$\hat{\mathbf{Z}} = \begin{pmatrix} z_{22}^* & -z_{21}^* & -z_{24}^* & z_{23}^* \\ -z_{12}^* & z_{11}^* & z_{14}^* & -z_{13}^* \\ -z_{42}^* & z_{41}^* & z_{44}^* & -z_{43}^* \\ z_{32}^* & -z_{31}^* & -z_{34}^* & z_{33}^* \end{pmatrix} = \begin{pmatrix} \text{cof}(\phi_1)^* & -\text{cof}(\phi_2)^* \\ -\text{cof}(\phi_3)^* & \text{cof}(\phi_4)^* \end{pmatrix},$$

where

$$\text{cof} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

4. The antiautomorphism Z^\bullet is defined as $Z^\bullet := E_4 \tilde{Z} E_4$:

$$\mathbf{Z}^\bullet = \begin{pmatrix} z_{22} & -z_{12} & z_{42} & -z_{32} \\ -z_{21} & z_{11} & -z_{41} & z_{31} \\ z_{24} & -z_{41} & z_{44} & -z_{34} \\ -z_{23} & z_{13} & -z_{43} & z_{33} \end{pmatrix} = \begin{pmatrix} \text{adj}(\phi_1) & \text{adj}(\phi_3) \\ \text{adj}(\phi_2) & \text{adj}(\phi_4) \end{pmatrix}.$$

5. We define $Z^\Delta := E_4 Z E_4 = \widetilde{E_4 \tilde{Z} E_4} = \widetilde{\mathbf{Z}^\bullet}$:

$$\mathbf{Z}^\Delta = \begin{pmatrix} z_{33} & z_{34} & z_{31} & z_{32} \\ z_{43} & z_{44} & z_{41} & z_{42} \\ z_{13} & z_{14} & z_{11} & z_{12} \\ z_{23} & z_{24} & z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} \phi_4 & \phi_3 \\ \phi_2 & \phi_1 \end{pmatrix}.$$

Although we explicitated the standard representation, and in other representations, the coefficients $z_{\mu\nu}$ of \mathbf{Z} are different, all the relations described by (1)–(5) above are valid in an arbitrary representation.

6.2. The identification $\gamma_\mu = iE_\mu$

Another isomorphism $\mathcal{C}\ell_{4,1} \rightarrow \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ is defined, denoting $i = \gamma_{0123} = E_{01234}$, by

$$\begin{aligned} E_0 &\mapsto -i\gamma_0, \\ E_1 &\mapsto -i\gamma_1, \\ E_2 &\mapsto -i\gamma_2, \\ E_3 &\mapsto -i\gamma_3, \\ E_4 &\mapsto -i\gamma_{0123}. \end{aligned} \tag{21}$$

Now the coefficients of Eq. (16) are related by

$$\begin{aligned} B &= H + iH^{01234}, & B^0 &= -H^{1234} - iH^0, & B^1 &= -H^{0234} - iH^1, \\ B^2 &= H^{0134} + iH^2, & B^3 &= -H^{0124} - iH^3, & B^{12} &= -H^{12} + iH^{034}, \\ B^{13} &= -H^{13} - iH^{024}, & B^{23} &= -H^{23} + iH^{014}, & B^{01} &= -H^{01} - iH^{234}, \\ B^{02} &= -H^{02} + iH^{134}, & B^{03} &= -H^{03} - iH^{124}, & B^{012} &= -H^{34} + iH^{012}, \\ B^{013} &= H^{24} + iH^{013}, & B^{023} &= -H^{14} + iH^{023}, & B^{123} &= -H^{04} + iH^{123}, \\ B^{0123} &= H^{0123} - iH^4. \end{aligned}$$

We conclude that

$$\begin{aligned} \widehat{\mathcal{C}\ell_{4,1}} &\simeq \mathbb{C}^* \otimes \mathcal{C}\ell_{1,3}, \\ \widetilde{\mathcal{C}\ell_{4,1}} &\simeq \mathbb{C} \otimes \widetilde{\mathcal{C}\ell_{1,3}}, \\ \overline{\mathcal{C}\ell_{4,1}} &\simeq \mathbb{C}^* \otimes \widetilde{\mathcal{C}\ell_{1,3}}, \\ \mathcal{C}\ell_{4,1}^\bullet &:= E_4 \widehat{\mathcal{C}\ell_{4,1}} E_4 \simeq \mathbb{C} \otimes \overline{\mathcal{C}\ell_{1,3}}, \\ \mathcal{C}\ell_{4,1}^\Delta &:= E_4 \mathcal{C}\ell_{4,1} E_4 \simeq \mathbb{C} \otimes \widehat{\mathcal{C}\ell_{1,3}}. \end{aligned}$$

The relations above are different from the ones exhibited in the last section, but can be led to them if we substitute the graded involution by the reversion in $\mathcal{C}\ell_{1,3}$ and vice-versa. Using, without loss of generality, again the standard representation of γ_μ , we obtain the matrix representation of $Z \in \mathcal{C}\ell_{4,1}$:

$$\varrho(Z) \equiv \mathbf{Z} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix},$$

where

$$\begin{aligned}
 z_{11} &= (H - H^{1234} + H^{034} + H^{012}) + i(H^{01234} - H^0 + H^{12} + H^{34}), \\
 z_{12} &= (-H^{13} + H^{24} + H^{014} + H^{023}) + i(-H^{024} + H^{013} + H^{23} + H^{14}), \\
 z_{13} &= (H^{03} - H^{0124} + H^4 - H^{123}) + i(H^{124} + H^3 + H^{0123} - H^{04}), \\
 z_{14} &= (H^{01} + H^{0234} - H^{134} + H^2) + i(H^{234} + H^1 - H^{02} + H^{0134}), \\
 z_{21} &= (H^{13} - H^{24} + H^{014} + H^{023}) + i(H^{024} - H^{013} + H^{23} + H^{14}), \\
 z_{22} &= (H - H^{1234} - H^{034} - H^{012}) + i(H^{01234} - H^0 - H^{12} - H^{34}), \\
 z_{23} &= (H^{01} + H^{0234} + H^{134} - H^2) + i(H^{234} + H^1 + H^{02} - H^{0134}), \\
 z_{24} &= (-H^{03} - H^{0124} + H^4 + H^{123}) + i(-H^{124} - H^3 + H^{0123} + H^{04}), \\
 z_{31} &= (H^{03} - H^{0124} + H^4 + H^{123}) + i(H^{124} - H^3 + H^{0123} + H^{04}), \\
 z_{32} &= (H^{01} - H^{0234} - H^{134} - H^2) + i(H^{234} - H^1 - H^{02} - H^{0134}), \\
 z_{33} &= (H + H^{1234} + H^{034} - H^{012}) + i(H^{01234} + H^0 + H^{12} - H^{34}), \\
 z_{34} &= (-H^{13} - H^{24} + H^{014} - H^{023}) + i(-H^{024} - H^{013} + H^{23} - H^{14}), \\
 z_{41} &= (H^{01} - H^{0234} + H^{134} + H^2) + i(H^{234} - H^1 + H^{02} + H^{0134}), \\
 z_{42} &= (-H^{03} + H^{0124} + H^4 + H^{123}) + i(-H^{124} + H^3 + H^{0123} + H^{04}), \\
 z_{43} &= (H^{13} + H^{24} + H^{014} - H^{023}) + i(H^{024} + H^{013} + H^{23} - H^{14}), \\
 z_{44} &= (H + H^{1234} - H^{034} + H^{012}) + i(H^{01234} + H^0 - H^{12} + H^{34}).
 \end{aligned} \tag{22}$$

Using the expression above, the (anti-)automorphisms in $\mathcal{C}\ell_{4,1}$ are translated into the ones of $\mathcal{M}(4, \mathbb{C})$:

1. Conjugation:

$$\varrho(\bar{Z}) \equiv \bar{\mathbf{Z}} = \begin{pmatrix} z_{11}^* & z_{21}^* & -z_{31}^* & -z_{41}^* \\ z_{12}^* & z_{22}^* & -z_{32}^* & -z_{42}^* \\ -z_{13}^* & -z_{23}^* & z_{33}^* & z_{43}^* \\ -z_{14}^* & -z_{24}^* & z_{34}^* & z_{44}^* \end{pmatrix} = \begin{pmatrix} \phi_1^\dagger & -\phi_3^\dagger \\ -\phi_2^\dagger & \phi_4^\dagger \end{pmatrix}.$$

The relation $\bar{Z}Z = 1$ in $\mathcal{C}\ell_{4,1}$ is equivalent, in $\mathcal{M}(4, \mathbb{C})$, to $\bar{\mathbf{Z}}\mathbf{Z} = 1$, i.e.

$$\begin{aligned}
 &\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1^\dagger & \phi_3^\dagger \\ \phi_2^\dagger & \phi_4^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &\Rightarrow \begin{pmatrix} \phi_1^\dagger & \phi_3^\dagger \\ \phi_2^\dagger & \phi_4^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
 \end{aligned}$$

which means that $\mathbf{Z} \in \text{U}(2, 2)$. Therefore, it follows the result

$$\text{Spin}_+(2, 4) \hookrightarrow \text{U}(2, 2) \tag{23}$$

But from Eq. (20), we have that $\det \mathbf{Z} = 1$. As an unitary transformation does not change the determinant, $\det \mathbf{Z} = 1$ and $\mathbf{Z} \in \text{SU}(2, 2)$, i.e.

$$\text{Spin}_+(2, 4) \hookrightarrow \text{SU}(2, 2) \tag{24}$$

An alternative way to see that $\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{SU}(2,2)$ is from relation $\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{U}(2,2)$ is given by Eq. (23). We have two possibilities:

1. $\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{SU}(2,2)$, or
2. $\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{U}(2,2)$, with determinant of the representation $\rho : \mathfrak{Spin}_+(2,4) \rightarrow \mathrm{End}(\mathbb{R}^{2,4})$ unitary and negative.

Since $\mathfrak{Spin}_+(2,4)$ is the connected (with the identity of $\mathfrak{Spin}(2,4)$) component, then

$$\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{SU}(2,2). \quad (25)$$

It is well known that the Lie algebra of $\mathfrak{Spin}_+(2,4)$ is generated by the bivectors. Now, we remember that the dimension of $\Lambda^2(\mathbb{R}^{p,q}) \hookrightarrow \mathcal{C}\ell_{p,q}$ is $n(n-1)/2$, where $n = p + q$ is the dimension of $\mathbb{R}^{p,q}$. Therefore, the group $\mathfrak{Spin}_+(2,4)$ has dimension 15. Since $\dim \mathrm{SU}(2,2) = 15$, because $\dim \mathrm{SU}(n,n) = (2n)^2 - 1$, from the inclusion $\mathfrak{Spin}_+(2,4) \hookrightarrow \mathrm{SU}(2,2)$ and that $\dim \mathfrak{Spin}_+(2,4) = \dim \mathrm{SU}(2,2) = 15$, it follows that

$$\mathfrak{Spin}_+(2,4) \simeq \mathrm{SU}(2,2) \quad (26)$$

Then the twistor space inner product isometry group $\mathrm{SU}(2,2)$ is expressed as

$$\mathrm{SU}(2,2) = \{Z \in \mathcal{C}\ell_{4,1} \mid Z\bar{Z} = 1\} \quad (27)$$

where \mathbf{Z} is the matrix representation of elements in $\mathfrak{Spin}_+(2,4) \simeq \mathrm{SU}(2,2)$.

2. Reversion:

$$\tilde{\mathbf{Z}} = \begin{pmatrix} z_{44} & -z_{34} & z_{24} & -z_{14} \\ -z_{43} & z_{33} & -z_{23} & z_{13} \\ z_{42} & -z_{32} & z_{22} & -z_{12} \\ -z_{41} & z_{31} & -z_{21} & z_{11} \end{pmatrix} = \begin{pmatrix} \mathrm{adj}(\phi_4) & \mathrm{adj}(\phi_2) \\ \mathrm{adj}(\phi_3) & \mathrm{adj}(\phi_1) \end{pmatrix},$$

where $\mathrm{adj}(\psi) = (\det \psi)\psi^{-1}$, $\forall \psi \in \mathcal{M}(2, \mathbb{C})$.

3. Graded involution:

$$\hat{\mathbf{Z}} = \begin{pmatrix} z_{22}^* & -z_{21}^* & -z_{24}^* & z_{23}^* \\ -z_{12}^* & z_{11}^* & z_{14}^* & -z_{13}^* \\ -z_{42}^* & z_{41}^* & z_{44}^* & -z_{43}^* \\ z_{32}^* & -z_{31}^* & -z_{34}^* & z_{33}^* \end{pmatrix} = \begin{pmatrix} \mathrm{cof}(\phi_1)^* & -\mathrm{cof}(\phi_2)^* \\ -\mathrm{cof}(\phi_3)^* & \mathrm{cof}(\phi_4)^* \end{pmatrix},$$

where

$$\mathrm{cof} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

4. $Z^\bullet := E_4 \tilde{Z} E_4$:

$$\mathbf{Z}^\bullet = \begin{pmatrix} z_{22} & -z_{12} & z_{42} & -z_{32} \\ -z_{21} & z_{11} & -z_{41} & z_{31} \\ z_{24} & -z_{41} & z_{44} & -z_{34} \\ -z_{23} & z_{13} & -z_{43} & z_{33} \end{pmatrix} = \begin{pmatrix} \mathrm{adj}(\phi_1) & \mathrm{adj}(\phi_3) \\ \mathrm{adj}(\phi_2) & \mathrm{adj}(\phi_4) \end{pmatrix}.$$

$$5. \quad Z^\Delta := E_4 Z E_4 = \widetilde{E_4 \widetilde{Z} E_4} = \widetilde{Z}^\bullet:$$

$$\mathbf{Z}^\Delta = \begin{pmatrix} z_{33} & z_{34} & z_{31} & z_{32} \\ z_{43} & z_{44} & z_{41} & z_{42} \\ z_{13} & z_{14} & z_{11} & z_{12} \\ z_{23} & z_{24} & z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} \phi_4 & \phi_3 \\ \phi_2 & \phi_1 \end{pmatrix}.$$

6.3. A useful identification to twistors

In this case, the isomorphism $\mathcal{C}\ell_{4,1} \rightarrow \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ is given by:

$$\begin{aligned} E_0 &\mapsto i\gamma_0, \\ E_1 &\mapsto \gamma_{10}, \\ E_2 &\mapsto \gamma_{20}, \\ E_3 &\mapsto \gamma_{30}, \\ E_4 &\mapsto \gamma_5\gamma_0 = -\gamma_{123}. \end{aligned} \tag{28}$$

In this case, the coefficients of Eq. (16) are given by

$$\begin{aligned} B &= H + iH^{01234}, & B^0 &= H^{1234} - iH^0, & B^1 &= H^{234} + iH^{01}, \\ B^2 &= H^{134} + iH^{02}, & B^3 &= -H^{124} + iH^{03}, & B^{12} &= -H^{12} + iH^{034}, \\ B^{13} &= -H^{13} - iH^{024}, & B^{23} &= -H^{23} + iH^{014}, & B^{01} &= H^1 + iH^{014}, \\ B^{02} &= -H^2 - iH^{0134}, & B^{03} &= -H^3 + iH^{0124}, & B^{012} &= -H^{34} + iH^{012}, \\ B^{013} &= H_{24} + iH^{013}, & B^{023} &= -H^{14} + iH^{023}, & B^{123} &= -H^4 - iH^{0123}, \\ B^{0123} &= H^{123} + iH^{04}. \end{aligned}$$

Twistor space inner product isometry group $\text{SU}(2,2)$ is written in the Clifford algebra $\mathcal{C}\ell_{4,1}$ from the isomorphism $\text{SU}(2,2) \simeq \mathbb{S}\text{pin}_+(2,4)$ shown via an appropriate isomorphism between $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{4,1}$.

7. Periodicity Theorem, Möbius Maps and the Conformal Group

The Periodicity Theorem of Clifford algebras has great importance and shall be used in the rest of the paper:

Periodicity Theorem. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra of the quadratic space $\mathbb{R}^{p,q}$. The following isomorphisms are verified:*

$$\begin{aligned} \mathcal{C}\ell_{p+1,q+1} &\simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q}, \\ \mathcal{C}\ell_{q+2,p} &\simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q}, \\ \mathcal{C}\ell_{q,p+2} &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{p,q}, \end{aligned} \tag{29}$$

where $p > 0$ or $q > 0$.

The isomorphism given by Eq. (29), the so-called *Periodicity Theorem*

$$\mathcal{C}\ell_{p+1,q+1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q} \quad (30)$$

is of primordial importance in what follows, since *twistors* are characterized via a representation of the conformal group.

Now, the extended periodicity theorem is presented, for details see, e.g. [1, 39, 43]. The reversion is denoted by α_1 , while the conjugation, by α_{-1} , in order to simplify the notation. The two antiautomorphisms are included in the notation $\alpha_\epsilon (\epsilon = \pm 1)$.

Periodicity Theorem (II). *The Periodicity Theorem (29) is also expressed, in terms of the associated anti-automorphisms, by*

$$(\mathcal{C}\ell_{p+1,q+1}, \alpha_\epsilon) \simeq (\mathcal{C}\ell_{p,q}, \alpha_{-\epsilon}) \otimes (\mathcal{C}\ell_{1,1}, \alpha_\epsilon).$$

Proof. The bases $\{\mathbf{e}_i\}$, $\{\mathbf{f}_j\}$ span the algebras $\mathcal{C}\ell_{p,q}$ and $\mathcal{C}\ell_{1,1}$, respectively. Let $\{\mathbf{e}_i \otimes \mathbf{f}_1 \mathbf{f}_2, 1 \otimes \mathbf{f}_j\}$ be a basis for $\mathcal{C}\ell_{p+1,q+1} \simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{1,1}$. The following relations are easily verified:

$$\begin{aligned} (\alpha_{-\epsilon} \otimes \alpha_\epsilon)(\mathbf{e}_i \otimes \mathbf{f}_1 \mathbf{f}_2) &= \alpha_{-\epsilon}(\mathbf{e}_i) \otimes \alpha_\epsilon(\mathbf{f}_1 \mathbf{f}_2) \\ &= \epsilon(\mathbf{e}_i \otimes \mathbf{f}_1 \mathbf{f}_2) \end{aligned} \quad (31)$$

$$\begin{aligned} (\alpha_{-\epsilon} \otimes \alpha_\epsilon)(1 \otimes \mathbf{f}_j) &= \alpha_{-\epsilon}(1) \otimes \alpha_\epsilon(\mathbf{f}_j) \\ &= \epsilon(1 \otimes \mathbf{f}_j) \end{aligned} \quad (32)$$

Therefore, the generators are multiplied by ϵ . □

7.1. Möbius maps in the plane

It is well known that rotations in the Riemann sphere \mathbb{CP}^1 are associated with rotations in the Argand–Gauss plane [35], which is (as a vector space) isomorphic to \mathbb{R}^2 . The algebra $\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$ is suitable to describe rotations in \mathbb{R}^2 .

From the periodicity theorem of Clifford algebras given by (29), it can be seen that Lorentz transformations in spacetime, generated by the vector representation of the group $\mathcal{Spin}_+(1,3) \hookrightarrow \mathcal{C}\ell_{3,0}$, are directly related to the Möbius maps in the plane, since we have the following correspondence:

$$\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{0,1}. \quad (33)$$

In this case, conformal transformations are described using $\mathcal{C}\ell_{1,1}$. From Eq. (33), it is possible to represent a paravector [4, 5] $\mathbf{a} \in \mathcal{C}\ell_{3,0}$ in $\mathcal{M}(2, \mathbb{C})$:

$$\mathbf{a} = \begin{pmatrix} z & \lambda \\ \mu & \bar{z} \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^3, \quad (34)$$

where $z \in \mathbb{C}$, $\mu, \lambda \in \mathbb{R}$.

Consider now an element of the group

$$\mathbb{S}pin_+(1, 3) := \{\phi \in \mathcal{C}\ell_{3,0} \mid \phi\bar{\phi} = 1\}. \quad (35)$$

From the Periodicity Theorem (II), it follows that

$$\widetilde{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} = \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix}. \quad (36)$$

The rotation of a paravector $\mathbf{a} \in \mathbb{R} \oplus \mathbb{R}^3$ can be performed by the twisted adjoint representation

$$\mathbf{a} \mapsto \mathbf{a}' = \eta \mathbf{a} \tilde{\eta}, \quad \eta \in \mathbb{S}pin_+(1, 3). \quad (37)$$

In terms of the matrix representation, we can write Eq. (37), using Eq. (34), as:

$$\begin{pmatrix} z & \lambda \\ \mu & \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z & \lambda \\ \mu & \bar{z} \end{pmatrix} \widetilde{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}, \quad (38)$$

and using Eq. (36), it follows that

$$\begin{pmatrix} z & \lambda \\ \mu & \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z & \lambda \\ \mu & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} \quad (39)$$

Taking $\mu = 1$ and $\lambda = z\bar{z}$, we see that the paravector \mathbf{a} is mapped on

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z & z\bar{z} \\ 1 & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} = \omega \begin{pmatrix} z' & z'\bar{z}' \\ 1 & \bar{z}' \end{pmatrix}, \quad (40)$$

where $z' := \frac{az+c}{bz+d}$ and $\omega := |bz+d|^2 \in \mathbb{R}$.

The map given by Eq. (40) is the spin-matrix $\mathbf{A} \in \mathrm{SL}(2, \mathbb{C})$, described in [35].

7.2. Conformal compactification

The results in this section are achieved in [9, 44]. Given the quadratic space $\mathbb{R}^{p,q}$, consider the injective map given by

$$\begin{aligned} \varkappa : \mathbb{R}^{p,q} &\rightarrow \mathbb{R}^{p+1,q+1} \\ x &\mapsto \varkappa(x) = (x, x \cdot x, 1) = (x, \lambda, \mu) \end{aligned} \quad (41)$$

The image of $\mathbb{R}^{p,q}$ is a subset of the quadric $Q \hookrightarrow \mathbb{R}^{p+1,q+1}$, described by the equation:

$$x \cdot x - \lambda\mu = 0, \quad (42)$$

the so-called *Klein absolute*. The map \varkappa induces an injective map from Q in the projective space $\mathbb{RP}^{p+1,q+1}$. Besides, Q is compact and defined as the conformal compactification $\widehat{\mathbb{R}^{p,q}}$ of $\mathbb{R}^{p,q}$.

$Q \approx \widehat{\mathbb{R}^{p,q}}$ is homeomorphic to $(S^p \times S^q)/\mathbb{Z}_2$ [44]. In the particular case where $p = 0$ and $q = n$, the quadric is homeomorphic to the n -sphere S^n , the compactification of \mathbb{R}^n via the addition of a point at infinity.

There also exists an injective map

$$\begin{aligned} s : \mathbb{R} \oplus \mathbb{R}^3 &\rightarrow \mathbb{R} \oplus \mathbb{R}^{4,1} \\ v &\mapsto s(v) = \begin{pmatrix} v & v\bar{v} \\ 1 & \bar{v} \end{pmatrix} \end{aligned} \quad (43)$$

The following theorem is introduced by Porteous [9, 44]:

Theorem.

- (i) the map $\kappa : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p+1,q+1}$; $x \mapsto (x, x \cdot x, 1)$, is an isometry.
- (ii) the map $\pi : Q \rightarrow \mathbb{R}^{p,q}$; $(x, \lambda, \mu) \mapsto x/\mu$ is defined where $\lambda \neq 0$ is conformal.
- (iii) if $U : \mathbb{R}^{p+1,q+1} \rightarrow \mathbb{R}^{p+1,q+1}$ is an orthogonal map, the the map $\Omega = \pi \circ U \circ \kappa : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$ is conformal.

The application Ω maps conformal spheres onto conformal spheres, which can be quasi-spheres or hiperplanes. A quasi-sphere is a submanifold of $\mathbb{R}^{p,q}$, defined by the equation

$$a x \cdot x + b \cdot x + c = 0, \quad a, c \in \mathbb{R}, \quad b \in \mathbb{R}^{p,q}. \quad (44)$$

A quasi-sphere is a sphere when a quadratic form g in $\mathbb{R}^{p,q}$ is positive defined and $a \neq 0$. A quasi-sphere is a plane when $a = 0$. From the assertion (iii) of the theorem above, we see that U and $-U$ induces the same conformal transformation in $\mathbb{R}^{p,q}$. The conformal group is defined as

$$\text{Conf}(p, q) \simeq \text{O}(p+1, q+1)/\mathbb{Z}_2. \quad (45)$$

$\text{O}(p+1, q+1)$ has four components and, in the Minkowski spacetime case, where $p = 1, q = 3$, the group $\text{Conf}(1,3)$ has four components. The component of $\text{Conf}(1,3)$ connected to the identity, denoted by $\text{Conf}_+(1,3)$ is known as the Möbius group of $\mathbb{R}^{1,3}$. Besides, $\text{SConf}_+(1,3)$ denotes the component connected to the identity, time-preserving and future-pointing.

8. Paravectors of $\mathcal{Cl}_{4,1}$ in $\mathcal{Cl}_{3,0}$ via the Periodicity Theorem

Consider the basis $\{\varepsilon_{\check{A}}\}_{\check{A}=0}^5$ of $\mathbb{R}^{2,4}$ that obviously satisfies the relations

$$\varepsilon_0^2 = \varepsilon_5^2 = 1, \quad \varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = -1, \quad \varepsilon_{\check{A}} \cdot \varepsilon_{\check{B}} = 0 \quad (\check{A} \neq \check{B}). \quad (46)$$

Consider also $\mathbb{R}^{4,1}$, with basis $\{E_A\}_{A=0}^4$, where

$$E_0^2 = -1, \quad E_1^2 = E_2^2 = E_3^2 = E_4^2 = 1, \quad E_A \cdot E_B = 0 \quad (A \neq B). \quad (47)$$

The basis $\{E_A\}$ is obtained from the basis $\{\varepsilon_{\check{A}}\}$, if we define the isomorphism

$$\begin{aligned} \xi : \mathcal{Cl}_{4,1} &\rightarrow \Lambda_2(\mathbb{R}^{2,4}) \\ E_A &\mapsto \xi(E_A) = \varepsilon_A \varepsilon_5. \end{aligned} \quad (48)$$

The basis $\{E_A\}$ defined by Eq. (48) obviously satisfies Eqs. (47).

Given a vector $\alpha = \alpha^{\check{A}} \varepsilon_{\check{A}} \in \mathbb{R}^{2,4}$, we obtain a paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{Cl}_{4,1}$ if the element ε_5 is left multiplied by \mathfrak{b} :

$$\mathfrak{b} = \alpha \varepsilon_5 = \alpha^A E_A + \alpha^5. \quad (49)$$

From the Periodicity Theorem, it follows the isomorphism $\mathcal{Cl}_{4,1} \simeq \mathcal{Cl}_{1,1} \otimes \mathcal{Cl}_{3,0}$ and so it is possible to express an element of $\mathcal{Cl}_{4,1}$ as a 2×2 matrix with entries in $\mathcal{Cl}_{3,0}$.

A homomorphism $\vartheta : \mathcal{Cl}_{4,1} \rightarrow \mathcal{Cl}_{3,0}$ is defined as:

$$E_i \mapsto \vartheta(E_i) = E_i E_0 E_4 \equiv \mathbf{e}_i. \quad (50)$$

It can be seen that $\mathbf{e}_i^2 = 1$, $E_i = \mathbf{e}_i E_4 E_0$ and that $E_4 = E_+ + E_-$, $E_0 = E_+ - E_-$, where $E_{\pm} := \frac{1}{2}(E_4 \pm E_0)$. Then,

$$\mathfrak{b} = \alpha^5 + (\alpha^0 + \alpha^4)E_+ + (\alpha^4 - \alpha^0)E_- + \alpha^i \mathbf{e}_i E_4 E_0. \quad (51)$$

If we choose E_4 and E_0 to be represented by $E_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, consequently we have

$$E_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_4 E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (52)$$

and then the paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{Cl}_{4,1}$ in Eq. (51) is represented by

$$\mathfrak{b} = \begin{pmatrix} \alpha^5 + \alpha^i \mathbf{e}_i & \alpha^4 - \alpha^0 \\ \alpha^0 + \alpha^4 & \alpha^5 - \alpha^i \mathbf{e}_i \end{pmatrix} \quad (53)$$

The vector $\alpha \in \mathbb{R}^{2,4}$ is in the Klein absolute, i.e. $\alpha^2 = 0$. Besides, this condition implies that $\alpha^2 = 0 \Leftrightarrow \mathfrak{b}\bar{\mathfrak{b}} = 0$, since $\alpha^2 = \alpha\alpha = \alpha 1\alpha = \alpha \varepsilon_5^2 \alpha = \alpha \varepsilon_5 \varepsilon_5 \alpha = \mathfrak{b}\bar{\mathfrak{b}}$. We denote

$$\lambda = \alpha^4 - \alpha^0, \quad \mu = \alpha^4 + \alpha^0. \quad (54)$$

Using the matrix representation of $\mathfrak{b}\bar{\mathfrak{b}}$, the entry $(\mathfrak{b}\bar{\mathfrak{b}})_{11}$ of the matrix is given by

$$(\mathfrak{b}\bar{\mathfrak{b}})_{11} = x\bar{x} - \lambda\mu = 0, \quad (55)$$

where

$$x := (\alpha^5 + \alpha^i \mathbf{e}_i) \in \mathbb{R} \oplus \mathbb{R}^3 \hookrightarrow \mathcal{Cl}_{3,0}. \quad (56)$$

If we fix $\mu = 1$, consequently, $\lambda = x\bar{x} \in \mathbb{R}$. This choice does correspond to a projective description. Then the paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{Cl}_{4,1}$ can be represented as

$$\mathfrak{b} = \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} = \begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix}. \quad (57)$$

From Eq. (55), we obtain $(\alpha^5 + \alpha^i \mathbf{e}_i)(\alpha^5 - \alpha^i \mathbf{e}_i) = (\alpha^4 - \alpha^0)(\alpha^4 + \alpha^0)$ which implies that

$$(\alpha^5)^2 - (\alpha^i \mathbf{e}_i)(\alpha^j \mathbf{e}_j) = (\alpha^4)^2 - (\alpha^0)^2, \quad (58)$$

and conclude that

$$(\alpha^5)^2 + (\alpha^0)^2 - (\alpha^1)^2 - (\alpha^2)^2 - (\alpha^3)^2 - (\alpha^4)^2 = 0 \quad (59)$$

which is the Klein absolute (Eq. (42)).

8.1. Möbius transformations in Minkowski spacetime

The matrix $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is in the group $\mathbb{S}pin_+(2, 4)$ if and only if its entries $a, b, c, d \in \mathcal{C}\ell_{3,0}$ satisfy the conditions [43]

$$\begin{aligned} \text{(i)} \quad & a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbb{R}, \\ \text{(ii)} \quad & a\bar{b}, c\bar{d} \in \mathbb{R} \oplus \mathbb{R}^3, \\ \text{(iii)} \quad & av\bar{c} + c\bar{v}a, cv\bar{d} + d\bar{v}c \in \mathbb{R}, \quad \forall v \in \mathbb{R} \oplus \mathbb{R}^3, \\ \text{(iv)} \quad & av\bar{d} + c\bar{v}b \in \mathbb{R} \oplus \mathbb{R}^3, \quad \forall v \in \mathbb{R} \oplus \mathbb{R}^3, \\ \text{(v)} \quad & a\tilde{c} = c\tilde{a}, b\tilde{d} = d\tilde{b}, \\ \text{(vi)} \quad & a\tilde{d} - c\tilde{b} = 1. \end{aligned} \quad (60)$$

Conditions (i), (ii), (iii), (iv) are equivalent to the condition $\hat{\sigma}(g)(\mathfrak{b}) := g\mathfrak{b}\tilde{g} \in \mathbb{R} \oplus \mathbb{R}^{4,1}$, $\forall \mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1}$, where $\hat{\sigma} : \mathbb{S}pin_+(2, 4) \rightarrow SO_+(2, 4)$ is the twisted adjoint representation. Indeed,

$$\begin{aligned} g\mathfrak{b}\tilde{g} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} \\ &= \begin{pmatrix} ax\bar{d} + \lambda a\bar{b} + \mu c\bar{d} + c\bar{x}\bar{b} & ax\bar{c} + \lambda a\bar{a} + \mu c\bar{c} + bx\bar{a} \\ bx\bar{d} + \lambda d\bar{b} + \mu d\bar{d} + d\bar{x}\bar{b} & bx\bar{c} + \lambda b\bar{a} + \mu d\bar{b} + d\bar{x}\bar{a} \end{pmatrix} \\ &= \begin{pmatrix} w & \lambda' \\ \mu' & \bar{w} \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \end{aligned} \quad (61)$$

where the last equality (considering $w \in \mathbb{R} \oplus \mathbb{R}^3$ and $\lambda', \mu' \in \mathbb{R}$) comes from the requirement that $g \in \mathbb{S}pin_+(2, 4)$, i.e. $g\mathfrak{b}\tilde{g} \in \mathbb{R} \oplus \mathbb{R}^{4,1}$. If these conditions are required, (i), (ii), (iii) and (iv) follow.

Conditions (v), (vi) express $g\bar{g} = 1$, since for all $g \in \mathbb{S}pin_+(2, 4)$, we have:

$$g\bar{g} = 1 \Leftrightarrow \begin{pmatrix} a\tilde{d} - c\tilde{b} & a\tilde{c} - c\tilde{a} \\ b\tilde{d} - d\tilde{b} & d\tilde{a} - b\tilde{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (62)$$

8.2. Conformal transformations

We have thus seen that a paravector $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{C}\ell_{4,1}$ is represented as

$$\begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix} = \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix}, \quad (63)$$

where $x \in \mathbb{R} \oplus \mathbb{R}^3$ is a paravector of $\mathcal{C}\ell_{3,0}$.

Consider an element of the group

$$\mathbb{S}pin_+(2, 4) := \{g \in \mathcal{C}\ell_{4,1} \mid g\bar{g} = 1\}. \quad (64)$$

It is possible to represent it as an element $g \in \mathcal{C}\ell_{4,1} \simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{3,0}$, i.e. $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $a, b, c, d \in \mathcal{C}\ell_{3,0}$. The rotation of $\mathfrak{b} \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{C}\ell_{4,1}$ is performed by the use of the twisted adjoint representation $\hat{\sigma} : \mathbb{S}pin_+(2, 4) \rightarrow SO_+(2, 4)$, defined as

$$\begin{aligned} \hat{\sigma}(g)(\mathfrak{b}) &= g\mathfrak{b}\hat{g}^{-1} \\ &= g\mathfrak{b}\tilde{g}, \quad g \in \mathbb{S}pin_+(2, 4). \end{aligned} \quad (65)$$

Using the matrix representation, the action of $\mathbb{S}pin_+(2, 4)$ is given by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} \widetilde{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} \quad (66)$$

Fixing $\mu = 1$, the paravector \mathfrak{b} is mapped on

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} = \Delta \begin{pmatrix} x' & x'x' \\ 1 & \bar{x}' \end{pmatrix}, \quad (67)$$

where

$$x' := (ax + c)(bx + d)^{-1}, \quad \Delta := (bx + d)(\overline{bx + d}) \in \mathbb{R}. \quad (68)$$

The transformation (68) is conformal [45, 46].

From the isomorphisms

$$\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{M}(4, \mathbb{C}), \quad (69)$$

elements of $\mathbb{S}pin_+(2, 4)$ are elements of the Dirac algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. From Eq. (56), we denote $x \in \mathbb{R} \oplus \mathbb{R}^3$ a paravector. The conformal maps are expressed by the action of $\mathbb{S}pin_+(2, 4)$, by the following matrices [9, 43, 45, 46]:

Conformal Map	Explicit Map	Matrix of $\mathbb{S}pin_+(2, 4)$
Translation	$x \mapsto x + h, \quad h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$
Dilation	$x \mapsto \rho x, \quad \rho \in \mathbb{R}$	$\begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix}$
Rotation	$x \mapsto \mathfrak{g}x\hat{\mathfrak{g}}^{-1}, \quad \mathfrak{g} \in \mathbb{S}pin_+(1, 3)$	$\begin{pmatrix} \mathfrak{g} & 0 \\ 0 & \hat{\mathfrak{g}} \end{pmatrix}$
Inversion	$x \mapsto -\bar{x}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Transvection	$x \mapsto x + x(hx + 1)^{-1}, \quad h \in \mathbb{R} \oplus \mathbb{R}^3$	$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$

This index-free geometric formulation allows us to trivially generalize the conformal maps of $\mathbb{R}^{1,3}$ to the ones of $\mathbb{R}^{p,q}$, if the Periodicity Theorem of Clifford algebras is used.

The group $\text{SConf}_+(1,3)$ is fourfold covered by $\text{SU}(2,2)$, and the identity element $id_{\text{SConf}_+(1,3)}$ of the group $\text{SConf}_+(1,3)$ corresponds to the following elements of $\text{SU}(2,2) \simeq \mathbb{S}\text{pin}_+(2,4)$:

$$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \quad \begin{pmatrix} -1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \begin{pmatrix} i_2 & 0 \\ 0 & i_2 \end{pmatrix}, \quad \begin{pmatrix} -i_2 & 0 \\ 0 & -i_2 \end{pmatrix}. \quad (70)$$

The element 1_2 denotes $id_{2 \times 2}$ and i_2 denotes the matrix $\text{diag}(i, i)$.

In this way, elements of $\mathbb{S}\text{pin}_+(2,4)$ give rise to the orthochronous Möbius transformations. The isomorphisms

$$\text{Conf}(1,3) \simeq \text{O}(2,4)/\mathbb{Z}_2 \simeq \text{Pin}(2,4)/\{\pm 1, \pm i\}, \quad (71)$$

are constructed in [9] and consequently,

$$\text{SConf}_+(1,3) \simeq \text{SO}_+(2,4)/\mathbb{Z}_2 \simeq \mathbb{S}\text{pin}_+(2,4)/\{\pm 1, \pm i\}. \quad (72)$$

The homomorphisms

$$\mathbb{S}\text{pin}_+(2,4) \xrightarrow{2-1} \text{SO}_+(2,4) \xrightarrow{2-1} \text{SConf}_+(1,3) \quad (73)$$

are explicitly constructed in [17, 47].

8.3. The Lie algebra of the associated groups

Consider $\mathcal{C}\ell_{p,q}^*$ the group of invertible elements. The function

$$\begin{aligned} \exp : \mathcal{C}\ell_{p,q} &\rightarrow \mathcal{C}\ell_{p,q}^* \\ a &\mapsto \exp a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \end{aligned} \quad (74)$$

is defined. The vector space $V := \mathcal{C}\ell_{p,q}$ endowed with the Lie bracket is identified with the Lie algebra $\mathcal{C}\ell_{p,q}^*$. As an example, consider the Clifford-Lipschitz $\Gamma_{p,q}$ group, a Lie subgroup of the Lie group $\mathcal{C}\ell_{p,q}^*$ and its Lie algebra is a vector subspace of $\mathcal{C}\ell_{p,q}$. Suppose that X is an element of $\Gamma_{p,q}$. Then $\exp(tX)$ is an element of $\Gamma_{p,q}$, i.e.

$$f(t) = \text{Ad } \exp(tX)(\mathbf{v}) \equiv \exp(tX) \mathbf{v} \exp(-tX) \in \mathbb{R}^{p,q}, \quad \forall \mathbf{v} \in \mathbb{R}^{p,q}. \quad (75)$$

Defining $(\text{ad}(X))(\mathbf{v}) = [X, \mathbf{v}] = X\mathbf{v} - \mathbf{v}X$, and using the well-known result $\text{Ad}(\exp(tX)) = \exp(\text{ad}(tX))$, we have that $f(t) \in \mathbb{R}^{p,q}$ if, and only if

$$\text{ad}(X)(\mathbf{v}) = [X, \mathbf{v}] = X\mathbf{v} - \mathbf{v}X \in \mathbb{R}^{p,q}. \quad (76)$$

It can be proved that $X \in \Gamma_{p,q}$ is written as $X \in \text{Cen}(\mathcal{C}\ell_{p,q}) \oplus \Lambda_2(\mathbb{R}^{p,q})$. In this way, $\exp(tX) \in \Gamma_{p,q}$.

If $R \in \text{Spin}(p, q)$, then $\hat{R} = R$, and for $R = \exp(tX)$, X must be written as $X = a + B$, where $a \in \mathbb{R}$, $B \in \Lambda_2(\mathbb{R}^{p,q})$. Besides, the condition $R\tilde{R} = 1$ implies that $1 = \exp(t\tilde{X})\exp(tX) = \exp(2ta)$, i.e. $a = 0$. Then

$$\text{Spin}_+(p, q) \ni R = \exp(tB), \quad B \in \Lambda_2(\mathbb{R}^{p,q}) \quad (77)$$

The Lie algebra of $\text{Spin}_+(p, q)$, denoted by $\mathfrak{spin}_+(p, q)$ is generated by the space of 2-vectors endowed with the commutator. Indeed, if B and C are bivectors, then

$$BC = \langle BC \rangle_0 + \langle BC \rangle_2 + \langle BC \rangle_4, \quad (78)$$

and since $\tilde{B} = -B$ e $\tilde{C} = -C$, it follows that $\widetilde{BC} = \tilde{C}\tilde{B} = CB$, and so $\widetilde{BC} = CB = \langle BC \rangle_0 - \langle BC \rangle_2 + \langle BC \rangle_4$, from which we obtain $BC - CB = [B, C] = 2\langle BC \rangle_2$, i.e. $(\Lambda_2(\mathbb{R}^{p,q}), [\ , \]) = \mathfrak{spin}_+(p, q)$

8.4. The Lie algebra of the conformal group

The Lie algebra of $\mathfrak{Spin}_+(2, 4)$ is generated by $\Lambda^2(\mathbb{R}^{2,4})$, which has dimension 15. Since $\dim \text{Conf}(1, 3) = 15$, the relation between these groups is investigated now. In Sec. 6.7, we have just seen that

$$E_0 = -i\gamma_0, \quad E_1 = -i\gamma_1, \quad E_2 = -i\gamma_2, \quad E_3 = -i\gamma_3, \quad E_4 = -i\gamma_{0123}, \quad (79)$$

and in Sec. 8, that

$$E_A = \varepsilon_A \varepsilon_5, \quad (80)$$

where $\{\varepsilon_A\}_{A=0}^5$ is basis of $\mathbb{R}^{2,4}$, $\{E_A\}_{A=0}^4$ is basis of $\mathbb{R}^{4,1}$ and $\{\gamma_\mu\}_{\mu=0}^3$ is basis of $\mathbb{R}^{1,3}$. The generators of $\text{Conf}(1, 3)$, as elements of $\Lambda_2(\mathbb{R}^{2,4})$, are defined as:

$$\begin{aligned} P_\mu &= \frac{i}{2}(\varepsilon_\mu \varepsilon_5 + \varepsilon_\mu \varepsilon_4), \\ K_\mu &= -\frac{i}{2}(\varepsilon_\mu \varepsilon_5 - \varepsilon_\mu \varepsilon_4), \\ D &= -\frac{1}{2}\varepsilon_4 \varepsilon_5, \\ M_{\mu\nu} &= \frac{i}{2}\varepsilon_\nu \varepsilon_\mu. \end{aligned} \quad (81)$$

From relations (79) and (80), the generators of $\text{Conf}(1, 3)$ are expressed from the $\{\gamma_\mu\} \in \mathcal{C}\ell_{1,3}$ as

$$\begin{aligned} P_\mu &= \frac{1}{2}(\gamma_\mu + i\gamma_\mu \gamma_5), \\ K_\mu &= -\frac{1}{2}(\gamma_\mu - i\gamma_\mu \gamma_5), \\ D &= \frac{1}{2}i\gamma_5, \\ M_{\mu\nu} &= \frac{1}{2}(\gamma_\nu \wedge \gamma_\mu). \end{aligned} \quad (82)$$

They satisfy the following relations:

$$\begin{aligned}
 [P_\mu, P_\nu] &= 0, \quad [K_\mu, K_\nu] = 0, \quad [M_{\mu\nu}, D] = 0, \\
 [M_{\mu\nu}, P_\lambda] &= -(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu), \\
 [M_{\mu\nu}, K_\lambda] &= -(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu), \\
 [M_{\mu\nu}, M_{\sigma\rho}] &= g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}, \\
 [P_\mu, K_\nu] &= 2(g_{\mu\nu}D - M_{\mu\nu}), \\
 [P_\mu, D] &= P_\mu, \\
 [K_\mu, D] &= -K_\mu.
 \end{aligned} \tag{83}$$

The commutation relations above are invariant under substitution $P_\mu \mapsto -K_\mu$, $K_\mu \mapsto -P_\mu$ and $D \mapsto -D$.

9. Twistors as Geometric Multivectorial Elements

In this section, we present and discuss the Keller approach, and introduce our definition, showing how our twistor formulation can be led to the Keller approach and consequently, to the Penrose classical twistor theory. The twistor defined as a minimal lateral ideal is also given in [20, 48]. Robinson congruences and the incidence relation, that determines a spacetime point as a secondary concept obtained from the intersection between two twistors, are also investigated.

9.1. The Keller approach

The twistor approach by Keller [42] uses the projectors $\mathbb{P}_{\mathcal{X}} := \frac{1}{2}(1 \pm i\gamma_5)$ ($\mathcal{X} = \mathcal{R}, \mathcal{L}$) and the element $T_{\mathbf{x}} = 1 + \gamma_5 \mathbf{x}$, where $\mathbf{x} = x^\mu \gamma_\mu \in \mathbb{R}^{1,3}$. Now we introduce some results obtained by Keller [42]:

Definition. The *reference twistor* $\eta_{\mathbf{x}}$, associated with the vector $\mathbf{x} \in \mathbb{R}^{1,3}$ and a Weyl covariant dotted spinor (written as the left-handed projection of a Dirac spinor ω) $\Pi = \mathbb{P}_{\mathcal{L}}\omega = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$ is given by

$$\eta_{\mathbf{x}} = T_{\mathbf{x}}\mathbb{P}_{\mathcal{L}}\omega = (1 + \gamma_5 \mathbf{x})\mathbb{P}_{\mathcal{L}}\omega = (1 + \gamma_5 \mathbf{x})\Pi \tag{84}$$

In order to show the equivalence of this definition with the Penrose classical twistor formalism, the Weyl representation is used:

$$\eta_{\mathbf{x}} = (1 + \gamma_5 \mathbf{x})\Pi = \left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -i_2 & 0 \\ 0 & i_2 \end{pmatrix} \begin{pmatrix} 0 & \vec{x} \\ \vec{x}^c & 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \tag{85}$$

Each entry of the matrices above denote 2×2 matrices, and the vector

$$\vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \tag{86}$$

is related to the point $\mathbf{x} \in \mathbb{R}^{1,3}$ and \bar{x}^c is the \mathbb{H} -conjugation of $\vec{x} \in \mathbb{R}^{1,3}$ given by Eq. (86). Therefore,

$$\eta_{\mathbf{x}} = \begin{pmatrix} -i\vec{x}\xi \\ \xi \end{pmatrix}. \quad (87)$$

That is, the index-free version of Penrose classical twistor [34]. The sign in the first component is different, since it used the Weyl representation.

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma(e_k) = \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}. \quad (88)$$

In order to get the correct sign, Keller uses a representation similar to the Weyl one, but with the vectors in \mathbb{R}^3 reflected ($\vec{x} \mapsto -\vec{x}$) through the origin:

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma(e_k) = \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \quad (89)$$

Then it is possible to get the Penrose twistor

$$\eta_{\mathbf{x}} = \begin{pmatrix} i\vec{x}\xi \\ \xi \end{pmatrix}. \quad (90)$$

Twistors are completely described by the multivectorial structure of the Dirac algebra $\mathbb{C} \otimes \mathcal{Cl}_{1,3} \simeq \mathcal{Cl}_{4,1} \simeq \mathcal{M}(4, \mathbb{C})$.

A classical spinor is an element that carries the irreducible representation of $\text{Spin}_+(p, q)$. Since this group is the set of even elements ϕ of the Clifford–Lipschitz group such that $\phi\bar{\phi} = 1$, the irreducible representation comes from the irreducible representation of the even subalgebra $\mathcal{Cl}_{p,q}^+$.

9.2. An alternative approach to twistors

We now define twistors as a special class of algebraic spinors in $\mathcal{Cl}_{4,1}$. The isomorphism $\mathcal{Cl}_{4,1} \simeq \mathbb{C} \otimes \mathcal{Cl}_{1,3}$ presented in Sec. 6.3, given by Eqs. (28)

$$E_0 = i\gamma_0, \quad E_1 = \gamma_{10}, \quad E_2 = \gamma_{20}, \quad E_3 = \gamma_{30}, \quad E_4 = \gamma_5\gamma_0 = -\gamma_{123}, \quad (91)$$

explicitly gives rise to the relations $E_0^2 = -1$ and $E_1 = E_2 = E_3 = E_4 = 1$. A paravector $x \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{Cl}_{4,1}$ is written as

$$\begin{aligned} x &= x^0 + x^A E_A \\ &= x^0 + \alpha^0 E_0 + x^1 E_1 + x^2 E_2 + x^3 E_3 + \alpha^4 E_4. \end{aligned} \quad (92)$$

We also define an element $\chi := xE_4 \in \Lambda^0(\mathbb{R}^{4,1}) \oplus \Lambda^1(\mathbb{R}^{4,1}) \oplus \Lambda^2(\mathbb{R}^{4,1})$ as

$$\chi = xE_4 = x^0 E_4 + \alpha^0 E_0 E_4 + x^1 E_1 E_4 + x^2 E_2 E_4 + x^3 E_3 E_4 + \alpha^4.$$

It can be seen that

$$\chi \frac{1}{2}(1 + i\gamma_5) = T_{\mathbf{x}} \frac{1}{2}(1 + i\gamma_5) = T_{\mathbf{x}} \mathbb{P}_{\mathcal{L}}. \quad (93)$$

We define the twistor as the algebraic spinor

$$\chi \mathbb{P}_{\mathcal{L}} U f \in (\mathbb{C} \otimes \mathcal{Cl}_{1,3}) f$$

where f is a primitive idempotent of $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}$ and $U \in \mathcal{C}\ell_{4,1}$ is arbitrary. Therefore, Uf is a Dirac spinor and $\mathbb{P}_{\mathcal{L}}Uf = \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \Pi \in \frac{1}{2}(1 + i\gamma_5)(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})$ is a covariant dotted Weyl spinor. The twistor is written as

$$\begin{aligned} \chi\Pi &= xE_4\Pi \\ &= (x^0E_4 + \alpha^0E_0E_4 + x^1E_1E_4 + x^2E_2E_4 + x^3E_3E_4 + \alpha^4)\Pi. \end{aligned} \tag{94}$$

From the relation $E_4\Pi = \gamma_5\gamma_0\Pi = -\gamma_0\gamma_5\Pi = -i\gamma_0\Pi$, it follows that

$$\begin{aligned} \chi\Pi &= (x^0E_4 + \alpha^0E_0E_4 + x^1E_1E_4 + x^2E_2E_4 + x^3E_3E_4 + \alpha^4)\Pi \\ &= x^0(E_4\Pi) + x^kE_k(E_4\Pi) + \alpha^0E_0(E_4\Pi) + \alpha^4\Pi \\ &= -ix^0\gamma_0\Pi - ix^k\gamma_k\Pi + \alpha^0\Pi + \alpha^4\Pi \\ &= (1 + \gamma_5\mathbf{x})\Pi \\ &= \begin{pmatrix} i\vec{x}\xi \\ \xi \end{pmatrix}. \end{aligned} \tag{95}$$

Then our definition is shown to be equivalent to the Keller one, and therefore, to the Penrose classical twistor, by Eq. (90).

The incidence relation, that determines a point in spacetime from the intersection between two twistors [13], is given by

$$\begin{aligned} J_{\bar{\chi}\chi} &:= \overline{x\bar{E}_4\bar{U}}x\bar{E}_4U \\ &= -\bar{U}E_4\bar{x}x\bar{E}_4U \\ &= 0, \end{aligned} \tag{96}$$

since the paravector $x \in \mathbb{R} \oplus \mathbb{R}^{4,1} \hookrightarrow \mathcal{C}\ell_{4,1}$ is in the Klein absolute, and consequently, $x\bar{x} = 0$.

Finally, the Robinson congruence is defined in our formalism from the product

$$\begin{aligned} J_{\bar{\chi}\chi'} &:= \overline{x\bar{E}_4\bar{U}}x'E_4U \\ &= -\bar{U}E_4\bar{x}x'E_4U. \end{aligned} \tag{97}$$

The above product is null if $x = x'$ and the Robinson congruence is defined when we fix x and let x' vary.

10. Concluding Remarks

The paravector model permits us to express vectors of $\mathbb{R}^{p,q} \hookrightarrow \mathcal{C}\ell_{p,q}$ as paravectors, elements of $\mathbb{R} \oplus \mathbb{R}^{q,p-1} \hookrightarrow \mathcal{C}\ell_{q,p-1}$. The conformal transformations (translations, inversions, rotations, transvections and dilations) are expressed via the adjoint representation of $\mathfrak{spin}_+(2,4)$ acting on paravectors of $\mathcal{C}\ell_{4,1}$. While the original formulation of the conformal transformations is described as rotations in $\mathbb{R}^{2,4}$, the paravector model allows us to describe them using the Clifford algebra $\mathcal{C}\ell_{4,1}$, isomorphic to the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. Then the redundant dimension is eliminated. Also, the Lie algebra related to the conformal group is described via the Dirac–Clifford algebra.

Twistors are defined in the index-free Clifford formalism as particular algebraic spinors (with an explicit dependence of a given spacetime point) of $\mathbb{R}^{4,1}$, i.e. twistors are elements of a left minimal ideal of the Dirac–Clifford algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. Equivalently, twistors are classical spinors of $\mathbb{R}^{2,4}$. Our formalism led to the well known formulations, e.g. Keller [42] and Penrose [13–15, 34, 35]. The first advantage of an index-free formalism is the explicit geometric nature of the theory, besides the easier comprehension of an abstract index-destituted theory. Besides, using the Periodicity Theorem of Clifford algebras, the present formalism can be generalized, in order to describe conformal maps and to generalize and extend the concept of twistors in any $(2n)$ -dimensional quadratic space. The relation between this formalism and exceptional Lie algebras, and the use of the pure spinor formalism is investigated in [49].

A. Appendix

A.1. Standard representation

Take the elements $e_{I_1} = e_0$ and $e_{I_2} = ie_1e_2$ are taken [1, 2]. Then

$$P_1 = \frac{1}{2}(1 + e_0)\frac{1}{2}(1 + ie_1e_2), \quad P_2 = \frac{1}{2}(1 + e_0)\frac{1}{2}(1 - ie_1e_2), \quad (98)$$

$$P_3 = \frac{1}{2}(1 - e_0)\frac{1}{2}(1 + ie_1e_2), \quad P_4 = \frac{1}{2}(1 - e_0)\frac{1}{2}(1 - ie_1e_2). \quad (99)$$

These four primitive idempotents are similar. Indeed, $e_{13}P_1(e_{13})^{-1} = P_2$, $e_{30}P_1(e_{30})^{-1} = P_3$, $e_{10}P_1(e_{10})^{-1} = P_4$. Then it is easily seen that $e_{13}P_1 \subset P_2\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$, $e_{30}P_1 \subset P_3\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$ and $e_{10}P_1 \subset P_4\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$. It follows that

$$\mathcal{E}_{11} = P_1, \quad \mathcal{E}_{21} = -e_{13}P_1, \quad \mathcal{E}_{31} = e_{30}P_1, \quad \mathcal{E}_{41} = e_{10}P_1. \quad (100)$$

Denoting $\{\mathcal{E}_{ij}\}_{i,j=1}^4$, a basis for $\mathcal{M}(4, \mathbb{C})$, with the conditions $\mathcal{E}_{1j} \subset P_1\mathcal{C}\ell_{1,3}(\mathbb{C})P_j$ and $\mathcal{E}_{1j}\mathcal{E}_{j1} = P_1$, it can be verified that

$$\mathcal{E}_{11} = P_1, \quad \mathcal{E}_{12} = e_{13}P_2, \quad \mathcal{E}_{13} = e_{30}P_3, \quad \mathcal{E}_{14} = e_{10}P_4. \quad (101)$$

The other \mathcal{E}_{ij} are in the following table:

\mathcal{E}_{ij}				
	P_1	$-e_{13}P_2$	$e_{30}P_3$	$e_{10}P_4$
$e_{13}P_1$	P_2	$e_{10}P_3$	$-e_{30}P_4$	
$e_{30}P_1$	$e_{10}P_2$	P_3	$e_{13}P_4$	
$e_{10}P_1$	$e_{03}P_2$	$-e_{13}P_3$	P_4	

Using the relations (98), the representations of e_μ , denoted by γ_μ , are constructed:

- $e_0 = P_1 + P_2 - P_3 - P_4 = \mathcal{E}_{11} + \mathcal{E}_{22} - \mathcal{E}_{33} - \mathcal{E}_{44}$. Then

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \text{where } I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (102)$$

- $e_{10} = e_{10}P_1 + e_{10}P_2 + e_{10}P_3 + e_{10}P_4 = \mathcal{E}_{41} + \mathcal{E}_{32} + \mathcal{E}_{23} + \mathcal{E}_{14}$. Therefore

$$\gamma(e_{10}) = \gamma_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and so it follows that}$$

$$\gamma(e_1) = \gamma_1 = \gamma_{10}\gamma_0 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \quad (103)$$

- $e_{30} = e_{30}P_1 + e_{30}P_2 + e_{30}P_3 + e_{30}P_4 = \mathcal{E}_{31} - \mathcal{E}_{42} + \mathcal{E}_{13} - \mathcal{E}_{24}$. Therefore,

$$\gamma(e_{30}) = \gamma_{30} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and then}$$

$$\gamma_3 = -\gamma_0\gamma_{03} = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \quad (104)$$

- $P_1 + P_3 - P_2 - P_4 = ie_1e_2$. It implies that

$$\begin{aligned} e_2 &= ie_0(e_{01}P_1 + e_{01}P_3 - e_{01}P_2 - e_{01}P_4) \\ &= ie_0(\mathcal{E}_{14} + \mathcal{E}_{32} - \mathcal{E}_{23} - \mathcal{E}_{41}). \end{aligned} \quad (105)$$

It follows that

$$\gamma_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \quad (106)$$

The standard representation of the Dirac matrices is then given by

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma(e_k) = \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (107)$$

A.2. Weyl representation

In this case, we have $e_{I_1} = e_5 := e_{0123}$ and $e_{I_2} = ie_1e_2$.

$$P_1 = \frac{1}{2}(1 + e_5)\frac{1}{2}(1 + ie_1e_2), \quad P_2 = \frac{1}{2}(1 + e_5)\frac{1}{2}(1 - ie_1e_2), \quad (108)$$

$$P_3 = \frac{1}{2}(1 - e_5)\frac{1}{2}(1 + ie_1e_2), \quad P_4 = \frac{1}{2}(1 - e_5)\frac{1}{2}(1 - ie_1e_2). \quad (109)$$

These idempotents are similar, as it can be easily verified:

$$e_0P_1e_0^{-1} = P_3, \quad e_1P_1e_1^{-1} = P_4, \quad e_{01}P_1(e_{01})^{-1} = P_2. \quad (110)$$

More generally, it can be asserted that in a simple algebra, all primitive idempotents are similar. Then it follows that $e_0P_1 \subset P_3\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$, $e_1P_1 \subset P_4\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$, $e_{01}P_1 \subset P_2\mathcal{C}\ell_{1,3}(\mathbb{C})P_1$ and

$$\mathcal{E}_{11} = P_1, \quad \mathcal{E}_{21} = e_{01}P_1, \quad \mathcal{E}_{31} = e_0P_1, \quad \mathcal{E}_{41} = e_1P_1. \quad (111)$$

With the conditions $\mathcal{E}_{1j} \subset P_1\mathcal{C}\ell_{1,3}(\mathbb{C})P_j$ and $\mathcal{E}_{1j}\mathcal{E}_{j1} = P_1$, it is immediate that

$$\mathcal{E}_{11} = P_1, \quad \mathcal{E}_{12} = e_{01}P_2, \quad \mathcal{E}_{13} = e_0P_3, \quad \mathcal{E}_{14} = -e_1P_4. \quad (112)$$

The other entries \mathcal{E}_{ij} are exhibited in the following table:

\mathcal{E}_{ij}				
	P_1	$e_{01}P_2$	e_0P_3	$-e_1P_4$
$e_{01}P_1$	P_2	$-e_1P_3$	e_0P_4	
e_0P_1	e_1P_2	P_3	$-e_{01}P_4$	
e_1P_1	e_0P_2	$e_{10}P_3$	P_4	

The representation of e_μ , denoted by γ_μ , are obtained:

- $e_0 = e_0P_1 + e_0P_2 + e_0P_3 + e_0P_4 = \mathcal{E}_{31} + \mathcal{E}_{42} + \mathcal{E}_{13} + \mathcal{E}_{24}$. Then

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (113)$$

- $e_1 = e_1P_1 + e_1P_2 + e_1P_3 + e_1P_4 = \mathcal{E}_{41} + \mathcal{E}_{32} - \mathcal{E}_{23} - \mathcal{E}_{14}$. Therefore, we have

$$\gamma(e_1) = \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \quad (114)$$

- $ie_5 = P_1 + P_2 - P_3 - P_4 \Rightarrow e_5 = -i(P_1 + P_2 - P_3 - P_4) = -i(\mathcal{E}_{11} + \mathcal{E}_{22} - \mathcal{E}_{33} - \mathcal{E}_{44})$.

It then follows that

$$\gamma(e_5) = \gamma_5 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \begin{pmatrix} -i_2 & 0 \\ 0 & i_2 \end{pmatrix}. \quad (115)$$

- $ie_1e_2 = P_1 + P_3 - P_2 - P_4$. Now it implies that

$$\begin{aligned} e_2 &= i(e_1P_1 + e_1P_3 - e_1P_2 - e_1P_4) \\ &= i(\mathcal{E}_{14} - \mathcal{E}_{32} - \mathcal{E}_{23} + \mathcal{E}_{41}). \end{aligned} \quad (116)$$

It is immediate to see that

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \quad (117)$$

From the notation $e_5 = e_{0123}$, it is seen that $e_3 = -e_{012}e_5$. Besides, we can show that

$$\gamma(e_3) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}.$$

Then the Weyl representation of the Dirac matrices is given by

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma(e_k) = \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (118)$$

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