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Decision Problems in Homeomorphism Groups

Problemas de Decisão em Grupos de Homeomorfismo

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(In memorian)

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Resumo

Na presente tese de doutorado, estudamos problemas de decisão no grupo $H := H(\mathbb{R})$ de Monod, o grupo de homeomorfismos projetivos por pedaços, que preservam orientação, da reta real projetiva $\mathbb{R}\mathrm{P}^1$, que estabilizam o infinito. Este grupo foi introduzido em [22] como contra-exemplo para a conjectura de von Neumann-Day. Generalizando as técnicas desenvolvidas em [9, 18, 21], desenvolvemos um invariante de conjugação para H. Além disso, descrevemos outro invariante de conjugação para o grupo H. Por fim, como aplicações desses dois invariantes, calculamos os subgrupos centralizadores de elementos de H.

Palavras-Chave: Problemas de Decisão. Problema da Conjugação. Grupo de Homeomorfismos. Algoritmo da Escada. Invariante de Mather. Centralizadores.

Abstract

In the present work we study decision problems in Monod's group $H := H(\mathbb{R})$, the group of piecewise projective orientation-preserving homeomorphisms of the projective real line $\mathbb{R}\mathrm{P}^1$ which stabilize infinity. This group was introduced in [22] as counterexample of the von Neumann-Day conjecture. By generalizing techniques from [9, 18, 21], we developed a conjugacy invariant for the group H. Moreover, we describe another conjugacy invariant for this group. As applications of both invariants, we compute centralizer subgroups of elements from H.

Keywords: Decision Problems. Conjugacy Problem. Homeomorphisms Group. Stair Algorithm. Mather Invariant. Centralizers.

Contents

Introduction

Our main goal in this doctoral thesis is the study of decision problems in group theory. In particular, we are interested in study of conjugacy on Monod's group $H := H(\mathbb{R})$, recently introduced by Nicolas Monod [22]. In 1910, Max Dehn published a work where he considered the problem concerning when two knots are the same, building on the work of Henri Poincaré about the fundamental group expressed with a given presentation. He realized that the knot theory problems were particular cases of more general questions about finitely presented groups. He published a paper in 1911 about these problems in group theory explicitly [12]. If a group G is defined by a finite set of generators X and a finite number of relations R, Dehn's problems ask if it is possible to solve, via an algorithm, the following questions:

(a) Given a word w in the X alphabet, can we decide if it is the identity element when viewed as an element of G? (*Word Problem - WP*);

(b) Given two words w_1, w_2 in the X alphabet, can we decide if they are conjugated as elements of G? (*Conjugacy Problem - CP*);

(c) If G' is another group with generators X' and relations R', can we decide if the groups G and G' are isomorphic? (*Isomorphism Problem - IP*).

Notice that word problem is a particular case of conjugacy problem. These decision problems play an important unifying role in group theory, giving an understanding of the complexity of these objects.

We say that a decision problem is **soluble** if there is an algorithm which decides when the problem has a positive answer. In particular, we say that a group G has **soluble conjugacy problem**, if there exists an algorithm which, given $y, z \in G$, determines whether there is, or not, an element $g \in G$ such that $g^{-1}yg = z$. We stress that the conjugacy problem does not depend on the choice of a finite presentation for the group.

We will be interested in groups of homeomorphisms of one-dimensional spaces. Some known results concerning conjugacy problem:

1. Thompson's group V has soluble conjugacy problem, by Graham Higman [15] and Olga Salazar-Díaz [26]. The group V is the group of right-continuous bijections of S^1 that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers

and such thaton each maximal interval on which the function is differentiable, the map is linear with slope 2^n;

2. Diagram groups have soluble conjugacy problem under certain conditions on the semigroup presentations, by Victor Guba and Mark Sapir [13]. In particular, Thompson's group F, the group of piecewise linear orientation-preserving homeomorphisms on $[0,1]$ with slopes 2^n and finitely many dyadic rational breakpoints;

3. $\mathrm{PL}_+(\mathbb{R})$, the group of piecewise linear orientation-preserving homeomorphisms of the real line with finitely many breakpoints, by Matthew G. Brin and Craig C. Squier [6]. They provided a criterion for describing conjugacy classes, since the group $\mathrm{PL}_+(\mathbb{R})$ is not finitely presented;

4. A unified solution for the conjugacy problem for Thompson groups F, T (a "circular" version of F) and V was given by James Belk and Francesco Matucci [3] by means of certain diagrams called *strand diagrams*;

5. A solution for the simultaneous conjugacy problem for many subgroups of $\mathrm{PL}_+(\mathbb{R})$, such as $\mathrm{PL}_2([0,1])$, was given by Martin Kassabov and Francesco Matucci [18]. For a fixed $k \in \mathbb{Z}_{>0}$, this problem asks if, given two k-tuples $(y_1, y_2, \ldots, y_k)$ and $(z_1, z_2, \ldots, z_k)$ of elements of a group G, we can decide if there exists an element $g \in G$ in the group so that $g^{-1}y_ig = z_i$ for every $i \in \{1, 2, \ldots, k\}$. In their work [18], the authors developed an algorithm (the *Stair Algorithm*) which constructs conjugators, if they exist;

6. A solution for the conjugacy problem and the power conjugacy problem in Higman-Thompson groups $V_{n,r}$, where $n \geqslant 2$ and $r \geqslant 1$, was given by Nathan Barker, Andrew J. Duncan and David M. Robertson [2]. The group $V_{n,r}$ is the automorphism group of the free Cantor algebra $C_n[r]$ of type n on r generators;

7. A solution for the *twisted conjugacy problem* for Thompson's group F, was given by Joseph Burillo, Francesco Matucci and Enric Ventura [9]. For a given $\varphi \in \mathrm{Aut}\,(G)$, the φ-twisted conjugacy problem asks if, given two elements y and z in a group G, we can decide if there is an element $g \in G$ such that $g^{-1}y\varphi\,(g) = z$. We say that G has soluble twisted conjugacy problem if the φ-twisted conjugacy problem is soluble for every $\varphi \in \mathrm{Aut}\,(G)$. In [9], the authors adapt a version of the Stair Algorithm even if the functions are periodic and have thus no initial slope. This allows one to find extensions of F with soluble/unsoluble conjugacy problem by finding necessary conditions that a "twisted" conjugator must satisfy.

We intend to adapt techniques from [9, 18, 21] and study conjugacy in Monod's groups $H(A)$ for A a subring of $\mathbb{R}$. These groups are interesting because they provide a family of counterexamples of the following conjecture:

Conjecture 0.1 (von Neumann-Day)**.** A group G is non-amenable if, and only if, G contains a subgroup which is a free group on two generators.

Thompson's group F (which we will introduce in Chapter 2 in more detail) has been proposed for a long time as a finitely presented potential counterexample for this conjecture, however the amenability of this group is an open question which is still challenging mathematicians nowadays. Conjecture 0.1 was disproved in 1980, in the work of Alexander Ol'shanskii, when he proved that the Tarski monster groups that he introduced do not have free subgroups of rank two and are non-amenable [23]. Many other counterexamples were constructed later. However, none of these counterexamples are "easy" to understand as Monod's groups.

The elements of these groups are piecewise projective orientation-preserving homeomorphisms of $\mathbb{R}\mathrm{P}^1$, with finitely many breakpoints, which stabilize infinity. If A is a subring of $\mathbb{R}$, $f \in H(A)$ if there are finitely many points $t_0, t_1, \ldots, t_{n+1} \in \mathbb{R}\mathrm{P}^1$ so that on each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n$,

$$f : t \mapsto \frac{a_i t + b_i}{c_i t + d_i},$$

where $a_i d_i - c_i b_i = 1$, for suitable $a_i, b_i, c_i, d_i \in A$, and $f(\infty) = \infty$. It is worth pointing out that we are making a slight abuse of notation by allowing $t = \infty$. The breakpoints lie in the set $\mathcal{P}_A$ of hyperbolic points. The product of two elements is given by composition of maps. We will present the definitions and some properties of H in Chapter 4.

As presented in Chapter 4, Monod's group $H = H(\mathbb{R})$ contains a copy of Thompson's group F and shares some properties with it, however H is not finitely presented. In this thesis, we produce a conjugacy invariant for H, as done by Matthew G. Brin and Craig C. Squier for the group $\mathrm{PL}_+(\mathbb{R})$ in [6], which is not finitely presented.

We divide this work as follows. In Chapter 2, we introduce Thompson's group F and present some of its properties. In Chapter 3, we introduce the Stair Algorithm for $\mathrm{PL}_2([0,1])$. In Chapter 4, we define Monod groups and present some properties, some of them shared with Thompson's group F, such as k-transitivity and the fact that H is a full group, this last fact is a original result of us, to the best of our knowledge. In Chapter 5, we adapt the Stair Algorithm, developed by Martin Kassabov and Francesco Matucci, in order to provide a criterion to describe conjugacy classes in Monod's group H and get the following theorem, where $\mathrm{Aff}(\mathbb{R})$ denotes the affine group $\mathrm{Aff}(\mathbb{R}) := (\mathbb{R}_{>0}, \cdot) \ltimes (\mathbb{R}, +)$,

Theorem A (Stair Algorithm)**.** *Let $y, z \in H^<$ and let $(-\infty, L]^2$ be the initial box. Let us consider $\left(a^2, ab\right) \in \mathrm{Aff}(\mathbb{R})$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that the unique conjugator g between y and z with initial germ $g_{-\infty} = \left(a^2, ab\right)$, is given by*

$$g(t) = y^{-N} g_0 z^N(t), \ \textit{for} \ \ t \in \left(-\infty, z^{-N}(L)\right],$$

and linear otherwise, where $g_0 \in H$ is an arbitrary map which is linear on the initial box and such that $g_{0_{-\infty}} = \left(a^2, ab\right)$.

Moreover, in this Chapter, we define the Mather invariants of elements from H. The Mather Invariant was developed by John N. Mather in [20] as a conjugacy invariant of one-bump maps f from $\mathrm{Diff}_+(I)$ so that $f'(0) > 1$ and $f'(1) < 1$ and proved that two elements from $\mathrm{Diff}_+(I)$ satisfying these conditions are conjugated if, and only if, they have the same initial and final slopes and the same Mather invariant. We show the relation between the Stair Algorithm and Mather Invariants of elements from H with the following result.

Theorem B. *Let $y, z \in H^>$ be such that $y(t) = z(t) = t + b_0$ for $t \in (-\infty, L]$ and $y(t) = z(t) = t + b_1$ for $t \in [R, +\infty)$ and let $\overline{y}^\infty, \overline{z}^\infty \colon C_0 \longrightarrow C_1$ be the corresponding Mather invariants. Then, y and z are conjugate in H if and only if $\overline{y}^\infty$ and $\overline{z}^\infty$ differ by rotations of the domain and range circles.*

As applications of the preceding two results, in Chapter 6 we compute the centralizer subgroups of elements from H. We get the following result.

Theorem C. *Given $z \in H$, then*

$$C_H(z) \cong (\mathbb{Z}, +)^n \times (\mathbb{R}, +)^m \times H^k,$$

for suitable $k, m, n \in \mathbb{Z}_{\geqslant 0}$.

We stress that the main contribution of this work is that we extended the techniques from [9, 18, 20, 21] to a group of piecewise projective homeomorphisms of the projective real line. The previous results and the other ones in Chapters 5 and 6 are new. They are being prepared for publication.

1. Group Theory

In this chapter, we present some preliminaries definitions and results concerning group theory. The reader can consult one of the references [16, 19, 25] for more details.

1.1 Group Presentations

Definition 1.1. If X is a subset of a group F, then F is a **free group with basis** X, and we write $F(X)$, if, for every group G and for each map $f\colon X \to G$, there exists a unique group homomorphism $\varphi\colon F(X) \to G$ such that $\varphi(x) = f(x)$ for every $x \in X$. In other words, φ makes the following diagram commute

$$\begin{tikzcd} F(X) \arrow[rr, dashed, "\varphi"] & & G \\ & X \arrow[ul, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

Remark 1.2. This property of making the preceding diagram commute is called the **universal property of free groups**.

We assume $X \neq \varnothing$ and choose a set X^{-1} that is in bijection to X and such that $X \cap X^{-1} = \varnothing$. We denote the image of $x \in X$ under this bijection by x^{-1}.

Definition 1.3. The set $\Sigma = X \cup X^{-1}$ is called **alphabet** and its elements are called as **letters**. A **word** of length n, where $n \in \mathbb{Z}_{>0}$, is a function

$$\begin{aligned} w\colon\ \{1, 2, \ldots, n\} &\longrightarrow \Sigma \\ i &\longmapsto w(i) := x_i^{\varepsilon_i}, \end{aligned}$$

where $x_i \in \Sigma$ and $\varepsilon_i = \pm 1$ for every $i \in \{1, 2, \ldots, n\}$, and write

$$w = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}.$$

The length n we denote by $|w|$. We allow the **empty word**, denoted by 1, and set it to have length zero, by convention. A **subword** of a word $w = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is either the empty word or a word of the form $x_r^{\varepsilon_r} \ldots x_s^{\varepsilon_s}$, where $1 \leqslant r \leqslant s \leqslant n$. The **inverse word** of $w = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is $w^{-1} = x_n^{-\varepsilon_n} \ldots x_1^{-\varepsilon_1}$. A word w is said to be **reduced** if either $w = 1$ or w has no subwords either of the form $x^{-1}x$ or xx^{-1}, for $x \in X$.

Any two words can be **multiplied** by juxtaposing them. For example, if $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ and $u = y_1^{\gamma_1} \dots y_n^{\gamma_n}$, then

$$wu = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} y_1^{\gamma_1} \dots y_n^{\gamma_n}.$$

We interpret $w1 = 1w = w$.

Definition 1.4. Let u and v be words on the alphabet Σ, possibly empty, and $w = uv$. An **insertion** is changing w to $uxx^{-1}v$ or $ux^{-1}xv$. A **deletion** is changing $uxx^{-1}v$ or $ux^{-1}xv$ to w. Both of them are said to be **elementary operations**. We write $w \to w'$ if w' is obtained from w by a finite number of elementary operations. Two words u and w on X are **equivalent**, and we write $u \sim w$, if we can transform one into another using finitely many elementary operations. The **equivalence class** of a word w is denoted $[w]$.

Proposition 1.5. *Let $F(X)$ be the set of equivalence classes of words with respect to the equivalence relation $\sim$. The multiplication on $F(X)$ is given by $[u][w] := [uw]$ is well defined and gives to $F(X)$ a group structure with identity element $[1]$ and inverse $[w]^{-1} = [w^{-1}]$.*

Proposition 1.6. *Let $i\colon X \hookrightarrow F(X)$ be the map defined by $i(x) = [x]$. For every group G and for each map $f\colon X \to G$, there exists a unique group homomorphism $\varphi\colon F(X) \to G$ so that $\varphi(i(x)) = f(x)$.*

Free groups are important because we have the following result.

Proposition 1.7. *Every group G is a quotient of a free group.*

If G is a group generated by a subset X and if we consider the free group $F(X)$, the universal property of free groups gives us a natural surjection

$$\begin{aligned} \pi\colon F(X) &\longrightarrow G \\ [x] &\longmapsto x \end{aligned}$$

We call the elements from $\ker(\pi)$ by **relations of the presentation** π. More generally, if S is a normal subgroup of $F(X)$ and R is subset of S so that S is the minimal normal subgroup of $F(X)$ containing R, we say that G is given by the set of generators X and the set of relations R and we write

$$G = \langle X \mid R \rangle.$$

This describes G as the quotient of $F(X)$ by the minimal normal subgroup containing R. If $R = \{[r_1], [r_2], \dots, [r_l]\}$ is finite, we also write

$$G = \langle X \mid r_1 = 1, r_2 = 1, \dots, r_l = 1 \rangle.$$

Definition 1.8. We call a group G by **finitely generated** (respectively, **finitely presented**) if it has a presentation $G = \langle X \mid R \rangle$ such that X is finite (respectively, X and R are finite).

If $G = \langle X \mid R \rangle$ and $S \subseteq F(X)$ is a set of words in X, then the presentation

$$\langle X \mid R \cup S \rangle \cong \frac{G}{S^G},$$

where S^G denotes the smallest normal subgroup in G containing S and we call it by **normal closure of S in G**. Similarly, the same group can be described as

$$\langle X \mid R \cup S \rangle \cong \frac{F(X)}{(R \cup S)^{F(X)}},$$

where $(R \cup S)^{F(X)}$ is the normal closure of $R \cup S$ in $F(X)$.

In general, it is not easy to prove that a group has a particular presentation. In order to do this, we need some tools. However, there is no procedure to prove that a group has a given presentation effectively. For each case, we use particular results or representation of a group. Given a random presentation, it is really hard to determine if the group is infinite, finite or even trivial. Next, we state, without proof, the tool which we used in Proposition 2.10.

Theorem 1.9 (von Dyck's Theorem)**.** *Let us consider a group $G = \langle X \mid R \rangle$. Given another group H, let $f\colon X \longrightarrow H$ be a map. Let us suppose that for each relation $r = [x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_l^{\varepsilon_l}] \in R$, we get*

$$f(x_1)^{\varepsilon_1} f(x_2)^{\varepsilon_2} \dots f(x_l)^{\varepsilon_l} = 1_H.$$

There exists a unique group homomorphism $\varphi\colon G \longrightarrow H$ such that $\varphi(i(x)) = f(x)$ for each $x \in X$.

1.2 On subgroups of $(\mathbb{R}, +)$

Proposition 1.10. *The discrete subgroups of $(\mathbb{R}, +)$ are the subgroups $(a\mathbb{Z}, +)$, where $a \geqslant 0$. Moreover, any nondiscrete subgroups of $(\mathbb{R}, +)$ is dense with respect to the Euclidean topology.*

Proof. See [24], p. 22, Propositions 1 and 2. $\square$

2. Thompson's Group F

In this chapter, we will define a group of piecewise linear homeomorphisms of the interval $[0, 1]$, called **Thompson's group** F, and also present some of its properties. This group is one of the groups introduced by Richard Thompson in some unpublished handwritten notes in 1965 in connection to his work in logic. It was later rediscovered by topologists, who were researching the structure of topological spaces with homotopy idempotents. From there on, F has become an important object of study in geometric group theory. None of the results in this chapter are new.

In the following, and in the rest of this doctoral thesis, we will denote the unit interval $[0, 1]$ by I. Moreover, we will adopt the convention that homeomorphisms will act on the left and will be composed from the right to left

$$(f \circ g)(t) = f(g(t)).$$

2.1 The Thompson's Group F

Since Thompson's group F is a group of piecewise linear homeomorphisms of I, we start by defining the standard dyadic interval and dyadic partition of I.

Definition 2.1. A **standard dyadic interval** is an interval of the form

$$\left[\frac{p}{2^q}, \frac{p+1}{2^q}\right],$$

where $p, q \in \mathbb{Z}_{\geqslant 0}$. We say that a partition $\mathcal{P}$ of I into intervals is a **dyadic partition** if all of its intervals are standard dyadic intervals.

Example 2.2. The following figure illustrates two examples of dyadic partitions of I. In the first partition, we take $p \in \{0, 1\}$ and $q = 1$. In the second partition, we take $p \in \{0, 1, 2, 6, 7\}$ and $q \in \{2, 3\}$.

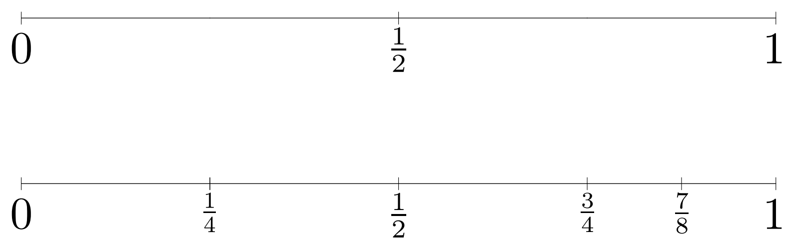

Figure 1 – Two dyadic partitions of I

Given two dyadic partitions $\mathcal{P}, \mathcal{R}$ of I with the same number of intervals, we can associate each interval of $\mathcal{P}$ to an interval of $\mathcal{R}$. Moreover, we can define a piecewise linear homeomorphism $f\colon I \to I$ by sending the n^{th} interval of $\mathcal{P}$ linearly onto the n^{th} interval of $\mathcal{R}$. This is called **dyadic rearrangement** of I.

Example 2.3. The following two maps are dyadic rearrangements of I:

$$x_0(t) := \begin{cases} \dfrac{t}{2}, & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ t - \dfrac{1}{4}, & \text{if } t \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right] \\ 2t - 1, & \text{if } t \in \left[\dfrac{3}{4}, 1\right] \end{cases} \quad \text{and} \quad x_1(t) := \begin{cases} t, & \text{if } t \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{t}{2} + \dfrac{1}{4}, & \text{if } t \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right] \\ t - \dfrac{1}{8}, & \text{if } t \in \left[\dfrac{3}{4}, \dfrac{7}{8}\right] \\ 2t - 1, & \text{if } t \in \left[\dfrac{7}{8}, 1\right]. \end{cases}$$

Their graphs are

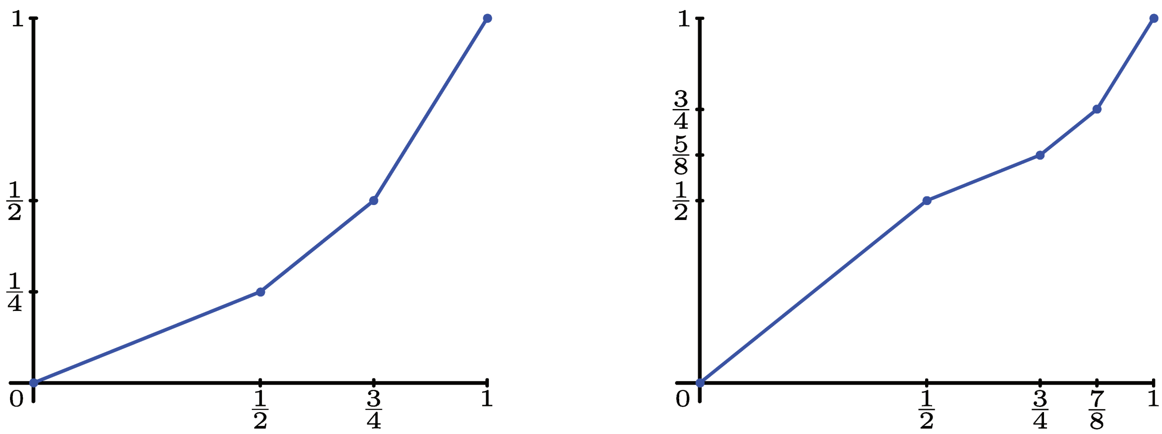

Figure 2 – Graphs of x_0 and x_1, respectively.

Notice that x_0 and x_1 are differentiable, except at finitely many dyadic rational numbers, and on each standard dyadic interval each slope is a power of two. In fact, all piecewise linear homeomorphism $f\colon I \to I$, which are dyadic rearrangements, satisfy these two properties. Conversely, if we have a piecewise linear homeomorphism $f\colon I \to I$ which satisfies these two properties, then f is a dyadic rearrangement. The set F of all dyadic rearrangements of I forms a group under composition. Thus F is a subgroup of the group of all homeomorphisms from I to I. This group is what is known as **Thompson's group** F. A formal definition is the following:

Definition 2.4. **Thompson's group** F is the group of all piecewise linear homeomorphisms of I to itself and with finitely many breakpoints where

1. All the slopes are power of two;

2. All the breakpoints have dyadic rational coordinates.

The group operation is function composition.

The maps in the Example 2.3 are elements of Thompson's group F and it can be shown that they generate F. In particular, the elements x_0 and x_1 can be used to define a family of elements $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ in F by

$$x_n := x_0^{-(n-1)}x_1x_0^{n-1},\ n \in \mathbb{Z}_{>0}$$

which will later be seen giving an infinite presentation of F.

Proposition 2.5. *The Thompson's group F is infinite and a torsion-free group.*

Lemma 2.6. *Given $f \in F$, there exists a dyadic partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I such that f is linear on each interval from the partition and $0 = f(t_0) < f(t_1) < \ldots < f(t_n) = 1$ also is a dyadic partition of I.*

Proof. Let $\mathcal{P}$ be a partition of I such that its points are dyadic rational numbers where f is linear on each interval of $\mathcal{P}$. Let $[a, b]$ be an interval of $\mathcal{P}$ and let us suppose that the slope of f on $[a, b]$ is 2^{-k}, with $k \in \mathbb{Z}$. Let $m \in \mathbb{Z}_{\geqslant 0}$ be such that $m + k \geqslant 0$, $2^m a, 2^m b, 2^{m+k}f(a), 2^{m+k}f(b) \in \mathbb{Z}$. Then,

$$a < a + \frac{1}{2^m} < a + \frac{2}{2^m} < a + \frac{3}{2^m} < \ldots < b$$

is a dyadic partition of $[a, b]$ and

$$f(a) < f(a) + \frac{1}{2^{m+k}} < f(a) + \frac{2}{2^{m+k}} < f(a) + \frac{3}{2^{m+k}} < \ldots < f(b)$$

is a dyadic partition of $[f(a), f(b)]$. □

The reader interested in more properties of Thompson's group F should consult the references [4, 7, 10].

2.2 Generators of F

In this section, we present the generators and some presentations for Thompson's group F. The reader can find more details, including the proofs of the statements, in [4, 7, 10].

First of all, let us recall that the functions from the Example 2.3 generate a family of elements in F of the form $x_n = x_0^{-(n-1)}x_1x_0^{n-1}$, where $n \in \mathbb{Z}_{>0}$. Then, we get elements in F of the type

$$x_0^{b_0}x_1^{b_1}x_2^{b_2}\ldots x_n^{b_n},$$

where $b_j \geqslant 0$, for all $j = 0, 1, \ldots, n$. These elements are called **positive elements**.

Proposition 2.7. *Each element from F can be expressed as pq^{-1}, where p and q are positive.*

We state the next result without proof.

Theorem 2.8. *The Thompson's group F is generated by x_0 and x_1.*

We will see in Theorem 2.11 that F has a finite presentation with the generating set being $\{x_0, x_1\}$.

2.3 Normal Forms

In this section we define what a normal form is for an element from Thompson's group F and the concept of reductions on it.

Definition 2.9. The **normal form** for an element from F is a word given by the form

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n} x_n^{-a_n} \dots x_1^{-a_1} x_0^{-a_0},$$

where the following two conditions are satisfied

(NF1) Exactly one of the exponents a_n and b_n are different from zero;

(NF2) If we have both x_i and x_i^{-1}, then we have either x_{i+1} or x_{i+1}^{-1}.

A word is said to be in the **seminormal form** if is given by the preceding form and it just satisfies (NF1).

We can write any element from F in its seminormal form by using the following **reductions**:

$$\begin{array}{lccc}
(\mathcal{R}1) & x_n^{-1} x_k & \rightarrow & x_k x_{n+1}^{-1}; \\
(\mathcal{R}2) & x_k^{-1} x_n & \rightarrow & x_{n+1} x_k^{-1}; \\
(\mathcal{R}3) & x_n x_k & \rightarrow & x_k x_{n+1}; \\
(\mathcal{R}4) & x_k^{-1} x_n^{-1} & \rightarrow & x_{n+1}^{-1} x_k^{-1}; \\
(\mathcal{R}5) & x_i^{-1} x_i & \rightarrow & 1;
\end{array}$$

for every $k < n$. We will denote them by $\mathcal{R}$. It is worth pointing out that no matter the order we apply the reductions $\mathcal{R}$, we always get the same result. The normal form of an element from F always exists and it is unique.

2.4 Presentations for F

For now, let us consider

$$F_1 = \langle y_0, y_1, \ldots \mid y_k^{-1} y_n y_k = y_{n+1},\ \text{for } 0 \leqslant k < n \rangle$$

and the following map

$$\begin{aligned} \Theta\colon F_1 &\longrightarrow F \\ y_n &\longmapsto x_n, \end{aligned}$$

where $x_n = x_0^{-(n-1)} x_1 x_0^{n-1}$, for $n \in \mathbb{Z}_{>0}$.

The following result is proved via von Dyck's Theorem 1.9.

Proposition 2.10. *The preceding map is a group isomorphism.*

Theorem 2.11. *The group F has the following finite presentation*

$$F_2 = \langle x_0, x_1 \mid x_1^{-1} x_2 x_1 = x_3, x_1^{-1} x_3 x_1 = x_4 \rangle,$$

where $x_{n+1} = x_0^{-1} x_n x_0$, for each $n \in \mathbb{Z}_{>0}$.

Remark 2.12. The defining relations from this finite presentation are long and complicated. Moreover, it is difficult to see anything from them. We presented them here in order to show that the group admits a presentation with only two generators and two defining relations, besides the infinite presentation.

2.5 Another Point of View of F

We can realize Thompson's group F in different ways. For example, as $\mathrm{PL}_2\left([a,b]\right)$, where $[a,b]$ is any dyadic interval, and as the group of piecewise linear homeomorphims from the real line. The reader can find more details, including the proofs of the statements, in [4, 7, 10].

Proposition 2.13. *Let $[a,b]$ be any dyadic interval and let $\mathrm{PL}_2\left([a,b]\right)$ be the subgroup of F consisting of all elements with support in I. Then $F \cong \mathrm{PL}_2\left([a,b]\right)$.*

Now, we give an idea of a representation of F as a group of piecewise linear maps of $\mathbb{R}$. This representation is basically the same as the one in $[a,b]$. Essentially, the idea is to conjugate F with a suitable piecewise linear map. Let us construct a map from $\mathbb{R}$ to the unit interval I by defining what its breakpoints are first

$$\phi(k) = \begin{cases} 1 - \dfrac{1}{2^{k+1}}, & \text{if } k \in \mathbb{Z}_{\geqslant 0} \\ 2^{k-1}, & \text{if } k \in \mathbb{Z}_{<0} \end{cases}$$

and then extending it linearly between any two of them. Now, we consider, for every element $f \in F$, its conjugate $\bar{f} := \phi^{-1} f \phi \colon \mathbb{R} \longrightarrow \mathbb{R}$. We stress that ϕ maps dyadic numbers in dyadic numbers, by definition. Thus, breakpoints of $\bar{f}$ have dyadic coordinates. Moreover, $\bar{f}$ has finitely many breakpoints, since at neighborhood of zero in the interval, we see that the map $y = 2^k$ maps an interval of the type $\left[\frac{1}{2^m}, \frac{1}{2^{m-1}}\right]$ to $\left[\frac{1}{2^{m-k}}, \frac{1}{2^{m-k-1}}\right]$. From this, we have that $\bar{f}(t) = t + k$ for t negative sufficiently large. Similarly, $\bar{f}(t) = t + m$ for t positive sufficiently large and a suitable $m \in \mathbb{Z}$.

3. The Conjugacy Problem in Thompson's Group F

In this chapter, we will present the Stair Algorithm developed in [18] by Martin Kassabov and Francesco Matucci. We will consider Thompson's group F as the group $\mathrm{PL}_2(I)$. We will see how to find a conjugator $g \in \mathrm{PL}_2(I)$ for two suitable functions $y, z \in \mathrm{PL}_2(I)$, if it exists. The idea is to assume that such a conjugator g exists and obtain conditions that g must satisfy. The results presented here are not new, but we will show some of them for the sole purpose of comparison with the results shown in Chapter 5. The interested reader can consult [18, 21].

3.1 The Setup

First of all, we start by denoting the set of all fixed points of some function $f \in \mathrm{PL}_2(I)$ by

$$\mathrm{Fix}\,(f) \coloneqq \{t \in I \mid f\,(t) = t\}\,.$$

We remark that the set $\mathrm{Fix}(f)$ is a disjoint union of a finite number of closed intervals and isolated points, since $f \in \mathrm{PL}_2(I)$ has finitely many breakpoints. Now, if we consider two maps $y, z \in \mathrm{PL}_2(I)$ such that y is conjugate to z by some function g, then given $t \in \mathrm{Fix}(z)$, we get

$$yg(t) = gz(t),$$

that is

$$y\,(g(t)) = g(t).$$

Then, $g\,(\mathrm{Fix}(z)) = \mathrm{Fix}(y)$, which implies that $\partial\mathrm{Fix}(y) = g\,(\partial\mathrm{Fix}(z))$. Thus, if y and z are conjugate, g makes the fixed points of z coincide with those of y. Since the boundary of the set of fixed points of either y or z is finite, the first step in order to study conjugacy is to check whether $\#\partial\mathrm{Fix}\,(y) = \#\partial\mathrm{Fix}\,(z)$. In case this is satisfied, we are able to build a map $h \in \mathrm{PL}_2(I)$ such that

$$h^{-1}\,(\partial\mathrm{Fix}\,(y)) = \partial\mathrm{Fix}\,(z)\,.$$

Then we reduce the study to checking if $h^{-1}yh$ and z, which now share the same boundary of their set of fixed points, are conjugate. This will be true if, for any two consecutive points $t_i, t_{i+1} \in \partial\mathrm{Fix}(z)$, we find a conjugator $g_i \in \mathrm{PL}_2\,([t_i, t_{i+1}])$ for the restrictions of $h^{-1}yh$ and z on $[t_i, t_{i+1}]$. We notice that such restrictions of $h^{-1}yh$ and z are either identity maps or their graphs are either entirely above or below the diagonal on $[t_i, t_{i+1}]$. Because of this observation, we will only consider the kind of function defined in the following definition.

Definition 3.1. Given $f \in \mathrm{PL}_2(I)$, this is called **one-bump function** if f lies above or below the identity map $\mathrm{id}(t) = t$, for all $t \in I$. In other words, f is a one-bump function either if $f(t) > t$ or if $f(t) < t$, for all $t \in I$.

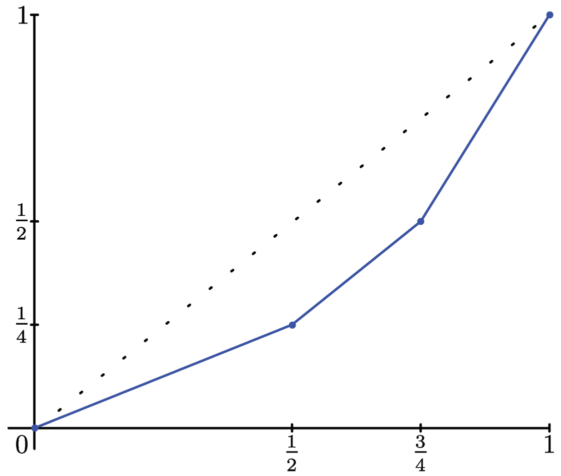

Figure 3 – The map $x_0 \in \mathrm{PL}_2(I)$ is an one-bump function

For convenience, we will denote by $\mathrm{PL}_2^>(I)$ the subset of $\mathrm{PL}_2(I)$ of all one-bump functions above the diagonal. Similarly, we denote by $\mathrm{PL}_2^<(I)$ the subset of all the ones below the diagonal.

Definition 3.2. Given $z \in \mathrm{PL}_2(I)$, we define **initial slope** and **final slope**, respectively, to be the numbers $z'(0)$ and $z'(1)$.

We notice that if two one-bump functions y and z are conjugate to each other, their initial and final slopes are equal. Moreover, the graphs of y and z coincide inside suitable boxes in neighborhoods of the points $(0, 0)$ and $(1, 1)$ in $I \times I$.

Lemma 3.3. *Given $y, z \in \mathrm{PL}_2(I)$ such that there is a function $g \in \mathrm{PL}_2(I)$ satisfying the equation $g^{-1}yg = z$, then there exists $L, R \in (0, 1)$ such that $z(t) = y(t)$ for all $t \in [0, L] \cup [R, 1]$.*

Proof. Let $g \in \mathrm{PL}_2(I)$ such that $g^{-1}yg = z$. Since $y, g, g^{-1} \in \mathrm{PL}_2(I)$, they fix 0. Then, consider $\varepsilon > 0$ sufficiently small such that $g(t) = at$, for all $t \in [0, \varepsilon]$, $y(t) = bt$, for each $t \in [0, g(\varepsilon)]$, and $g^{-1}(t) = a^{-1}t$, for all $t \in [0, yg(\varepsilon)]$. Then, let us define

$$L \coloneqq \min\{\varepsilon, g(\varepsilon), yg(\varepsilon)\}.$$

Then, given $t \in [0, L]$, we have

$$z(t) = g^{-1}yg(t) = g^{-1}y(at) = g^{-1}(bat) = a^{-1}bat = bt$$

That is, $z(t) = y(t)$ for all $t \in [0, L]$. In the same fashion, we prove the existence of some $\beta \in (0,1)$ such that y and z coincide on $[R, 1]$. $\square$

This fact is important because it is saying that if the graphs of two elements of $\mathrm{PL}_2(I)$ do not coincide on certain boxes around the points $(0,0)$ and $(1,1)$, then they are not conjugate. Thus, the previous lemma gives an obstruction to conjugacy.

3.2 Linearity Boxes Lemma

As mentioned before, the idea is to suppose that there exists a conjugator between y and z in $\mathrm{PL}_2(I)$ and get necessary conditions that this conjugator must satisfy. One of them is that, if there is a conjugator, it must be a linear map inside some boxes around $(0,0)$ and $(1,1)$.

Lemma 3.4 (Linearity Boxes Lemma)**.** *Let $y, z \in \mathrm{PL}_2^{>}(I)$ such that there is a map $g \in \mathrm{PL}_2(I)$ satisfying $g^{-1}yg = z$. If $y(t) = z(t) = bt$ for $t \in [0, L]$ for some $b > 1$ and $L > 0$, then g is linear inside the box $[0, L] \times [0, L]$.*

Proof. First of all, we notice that there exists $r \in (0,1)$ such that g is linear on $[0, r]$, since $g \in \mathrm{PL}_2(I)$ and $g(0) = 0$. Thus, let us define

$$\tilde{L} \coloneqq \sup \{r \in I \mid g \text{ is linear in } [0, r]\}.$$

If we prove that

$$\tilde{L} \geqslant \min\left\{L, g^{-1}(L)\right\},$$

then we are done. Assume the contrary, that is, $\tilde{L} < g^{-1}(L)$ and $\tilde{L} < L$, and let us suppose that $g(t) = at$ for all $t \in \left[0, \tilde{L}\right]$, for some $a > 0$. Given an arbitrary number $\sigma \in [0, 1)$, since $\tilde{L} < L$, we get $\sigma\tilde{L} < L$. By linearity of y, we have

$$g(y(\sigma\tilde{L})) = g(b\sigma\tilde{L}). \tag{3.1}$$

But, since $\tilde{L} < g^{-1}(L)$, we also get $\sigma\tilde{L} < g^{-1}(L)$. Since $g \in \mathrm{PL}_2(I)$, we obtain $g(\sigma\tilde{L}) < L$. From this, z is linear around $g(\sigma\tilde{L})$, so

$$z(g(\sigma\tilde{L})) = z(a\sigma\tilde{L}) = b(a\sigma\tilde{L}). \tag{3.2}$$

Since $gy = zg$, by (3.1) and (3.2) we get

$$g(b\sigma\tilde{L}) = a(b\sigma\tilde{L}), \ \forall\, \sigma \in [0, 1).$$

Considering $\sigma \in (b^{-1}, 1)$, we have $\tilde{L} < b\sigma\tilde{L}$ and g is linear up to $b\sigma\tilde{L}$, which is a contradiction with the definition of $\tilde{L}$. Thus,

$$\tilde{L} \geqslant \min\left\{L, g^{-1}(L)\right\}$$

and the result follows. $\square$

It is worth pointing out that the box $[0, L] \times [0, L]$ depends only on y and z. Moreover, the result also holds for $y, z \in \mathrm{PL}_2^<(I)$ with the same initial slope. To prove it, we just apply the previous Lemma to the inverse maps y^{-1} and z^{-1}. Thus, we can work either with $\mathrm{PL}_2^<(I)$ or $\mathrm{PL}_2^>(I)$. A similar result holds inside the box $[R, 1] \times [R, 1]$.

While the Linearity Boxes Lemma is stated for $y, z \in \mathrm{PL}_2^>(I)$, the next few

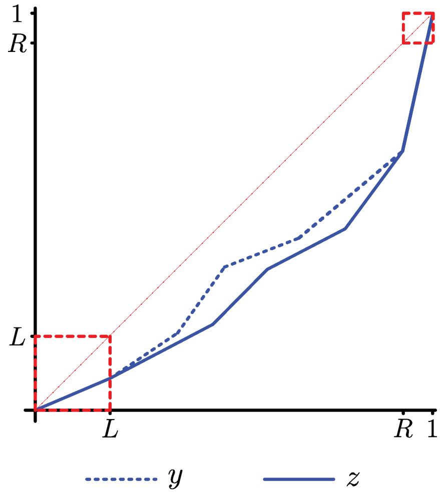

Figure 4 – Linearity boxes.

results, of which we only give the statements, are stated for $y, z \in \mathrm{PL}_2^<(I)$. This stems from the original treatment and proofs. Analogous statements can be given for $y, z \in \mathrm{PL}_2^>(I)$.

Lemma 3.5 (Identification Lemma - [21])**.** *Let $y, z \in \mathrm{PL}_2^<(I)$ and let $L \in (0, 1)$ be such that $y(t) = z(t)$ for $t \in [0, L]$. There exists an element $g \in \mathrm{PL}_2(I)$ so that $z(t) = g^{-1}yg(t)$, for $t \in [0, z^{-1}(L)]$ and $g(t) = t$ in $[0, L]$. Moreover, g is uniquely defined up to the point $z^{-1}(L)$.*

Remark 3.6. We can apply the preceding result several times in order to define a partial conjugator g_N on $\left[0, z^{-N}(L)\right]$.

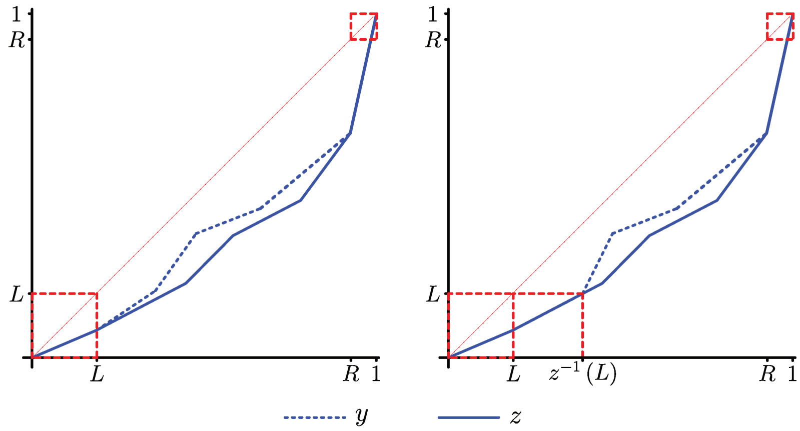

Figure 5 – How Identification Lemma works.

Lemma 3.7 (Uniqueness Lemma - [21]). *Let $y, z \in \mathrm{PL}_2^<(I)$. For any $q \in \mathbb{R}_{>0}$, there exists at most one $g \in \mathrm{PL}_2(I)$ such that $g^{-1}yg = z$ and $g'(0) = q$.*

Lemma 3.8 (Conjugator for Powers - [21]). *Let $y, z \in \mathrm{PL}_2^<(I)$, $g \in \mathrm{PL}_2(I)$ and $n \in \mathbb{Z}_{>0}$. Then, $g^{-1}yg = z$ if, and only if, $g^{-1}y^n g = z^n$.*

3.3 Stair Algorithm

Finally, we state the Stair Algorithm theorem.

Theorem 3.9 (Stair Algorithm - [21]). *Let $y, z \in \mathrm{PL}_2^<(I)$, let $[0, L]^2$ be the initial linearity box and let $0 < q < 1$ be a real number. There is an $N \in \mathbb{Z}_{>0}$ so that the unique candidate conjugator whose initial slope q is given by*

$$g(t) = y^{-N} g_0 z^N(t), \quad \forall \ \ t \in [0, z^{-N}(L)]$$

and linear otherwise, where g_0 is any map in $\mathrm{PL}_2(I)$ which is linear in the initial box and such that $g_0'(0) = q$.

In summary, we consider $y, z \in \mathrm{PL}_2^<(I)$ so that they coincide in $[0, L]$ and $[R, 1]$, for suitable $L, R \in I$. Then we choose some initial slope $q \in \mathbb{R}$ in order to construct a potential conjugator g between y^N and z^N, for a suitable $N \in \mathbb{Z}_{>0}$. By the Linearity Boxes Lemma, the graph of g must be linear on $[0, L]^2$ and $[R, 1]^2$. We then pass a "door", represented by the initial linearity box, by applying the Identification Lemma several times and build a candidate conjugator step-by-step, going up the stairs until we reach the other "door", represented by the final linearity box. By the Uniqueness Lemma, if there exists a conjugator, it must coincide with the function constructed by the Stair Algorithm and we check if it is a conjugator. If indeed it is a conjugator between y^N and z^N, by the Conjugator for Powers Lemma, it must conjugate y to z.

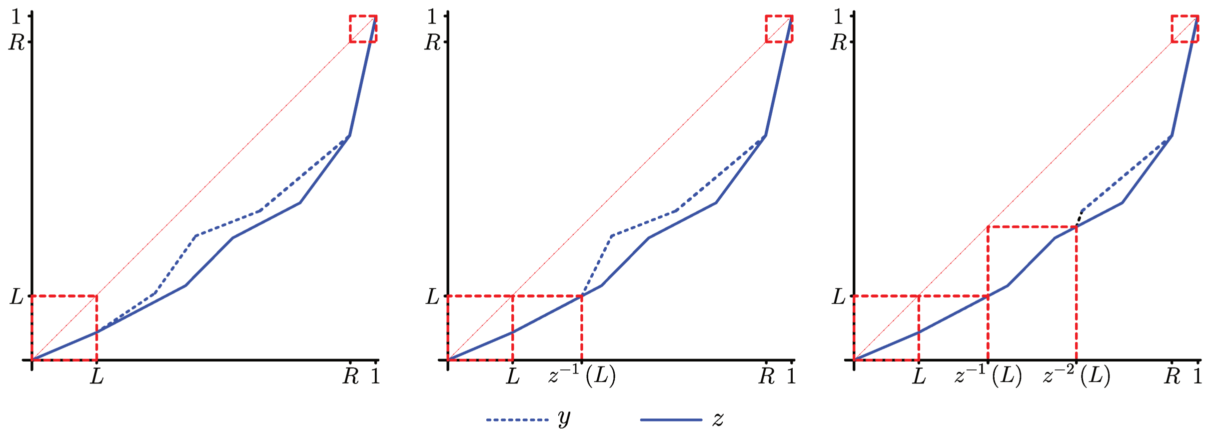

Figure 6 – Stair Algorithm.

Remark 3.10. We notice that by "unique candidate conjugator", we mean a map g so that, if there exists a conjugator between y and z with initial slope q, then it must be equal to g. Hence, we can check if $g^{-1}yg = z$ and we are done.

4. Piecewise Projective Homeomorphisms Groups

In this chapter, we will discuss the groups of piecewise projective orientation-preserving homeomorphism of $\mathbb{R}\mathrm{P}^1$ which stabilize infinity and discuss some of their properties. These groups are called *Monod groups* and were introduced by Nicolas Monod in 2013 as counterexamples to the von Neumman-Day conjecture [22]. Until then, they were the simplest counterexample for this conjecture, but still not finitely presented. Most results in this chapter are not new, while the one concerning the proof of fullness of the group is original, to the best of our knowledge.

4.1 The Group $\mathrm{PSL}_2(A)$

Let us consider the set of all lines in $\mathbb{R}^2$ passing through the origin. This topological space is called *projective real line* $\mathbb{R}\mathrm{P}^1$. It is important to note that every such line meets the sphere S^1 centered in the origin exactly twice, that is, the line through the point $(x, y) \in S^1$ also meets S^1 in its antipodal point $(-x, -y)$. We will consider the quotient topology induced by $\mathbb{R}^2 \setminus \{(0,0)\}$, making $\mathbb{R}\mathrm{P}^1$ a topological circle. Given a subring A of $\mathbb{R}$, we define the group $\mathrm{PSL}_2(A)$ as the group of Möbius transformations of $\mathbb{R}\mathrm{P}^1$ in itself such that their coefficients lie in A. In other words,

$$\mathrm{PSL}_2(A) := \left\{ t \longmapsto f(t) = \frac{at+b}{ct+d} \mid a, b, c, d \in A \text{ and } ad - bc = 1 \right\}.$$

We point out that an element $f(t) = \dfrac{at+b}{ct+d}$ from this group is related to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ from $\mathrm{SL}_2(A)$, which are the 2×2 matrices with determinant one and, moreover, composition of Möbius transformations corresponds to the product of the matrices associated to such transformations. There are three kind of elements in $\mathrm{PSL}_2(A)$ distinguished by the value of its **trace**

$$\mathrm{tr}(f) := a + d.$$

Definition 4.1. If $|\mathrm{tr}(f)| < 2$, we say that f is **elliptic**. If $|\mathrm{tr}(f)| = 2$, we say that f is **parabolic**. If $|\mathrm{tr}(f)| > 2$, we say that f is **hyperbolic**.

The fixed points of elements from $\mathrm{PSL}_2(A)$ can be found by solving the equation

$$\frac{at_0+b}{ct_0+d} = t_0. \tag{4.1}$$

A parabolic element from $\mathrm{PSL}_2(A)$ has just one fixed point in $\mathbb{RP}^1$ and a hyperbolic element has exactly two fixed points in $\mathbb{RP}^1$. These fixed points are classified as following.

Definition 4.2. Let $f \in \mathrm{PSL}_2(A)$ and let $t_0 \in \mathbb{RP}^1$ be a fixed point of f. We say that t_0 is **parabolic point** if so is f. We say that t_0 is **hyperbolic point** if f is hyperbolic.

An element $f \in \mathrm{PSL}_2(A)$ fixes ∞ if, and only if, $c = 0$ and, then, it is of the form $a^2t + ab$. If A is a subring so that the only units are ± 1, then $a^2 = 1$. As a consequence, f is parabolic. Otherwise, f is hyperbolic and the other fixed point is $\dfrac{ab}{1-a^2}$.

4.2 Definition of Monod Groups

Let us consider the action of the group $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{RP}^1$. We define the set $\mathrm{PPSL}_2(\mathbb{R})$ of all piecewise projective homeomorphisms of $\mathbb{RP}^1$ which will be a group under composition. We say that $f \in \mathrm{PPSL}_2(\mathbb{R})$ if there are finitely many points $t_0, t_1, \ldots, t_{n+1} \in \mathbb{RP}^1$ so that on each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n$,

$$f : t \mapsto \frac{a_i t + b_i}{c_i t + d_i},$$

where $a_i d_i - c_i b_i = 1$, for suitable $a_i, b_i, c_i, d_i \in \mathbb{R}$, and we have $t_0 = t_{n+1}$.

Remark 4.3. It is worth pointing out that we are making a small abuse of notation by considering the maps defined above for $t = \infty$ by saying that $f(\pm\infty) := \lim\limits_{t \to \pm\infty} \dfrac{a_i t + b_i}{c_i t + d_i}$.

We emphasize that it is easily seen that all elements of this group are orientation-preserving, since that the first derivative of any piece is

$$\frac{1}{(c_i t + d_i)^2}.$$

Since the elements from these groups are piecewise homeomorphisms, they may have "breakpoints". Let us define the concept of breakpoint in this context.

Definition 4.4. We say that a point $t_0 \in \mathbb{R}$ is a **breakpoint** of $f \in \mathrm{PPSL}_2(\mathbb{R})$ if there exists an $\varepsilon > 0$ such that there do not exist $a, b, c, d \in \mathbb{R}$, where $ad - cb = 1$ and $f(t) = \dfrac{at+b}{ct+d}$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Given a subring A of $\mathbb{R}$, we denote by $\mathcal{P}_A$ the set of all points in $\mathbb{RP}^1$ which are fixed by hyperbolic elements from $\mathrm{PSL}_2(A)$. It is worth reminding the reader that when we use the word "subring" in this thesis we always mean that the multiplicative identity of A is the same of the field $\mathbb{R}$. An interesting fact about the set $\mathcal{P}_A$ is that if we take subrings A of $\mathbb{R}$ so that the only units are ± 1, there is no hyperbolic element in $\mathrm{PSL}_2(A)$ fixing ∞.

Lemma 4.5 ([22]). *Let $A \subseteq \mathbb{R}$ be a subring. If A has some unit $a \neq \pm 1$, then $\infty \in \mathcal{P}_A$. Otherwise, $\infty \notin \mathcal{P}_A$.*

Proof. Let A be a subring of $\mathbb{R}$ such that there is an unit $a \neq \pm 1$. Let us consider the map $f : t \longmapsto a^2 t$ on $\mathbb{RP}^1$. Notice that $f \in \mathrm{PPSL}_2(\mathbb{R})$, with coefficient $a \in A$ and, rewriting the map as $f : t \mapsto \dfrac{at + 0}{0 \cdot t + a^{-1}}$ we see that its corresponding matrix in $\mathrm{SL}_2(A)$ is

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}.$$

The trace of this matrix is greater than two Then, the element f is hyperbolic. Moreover, f fixes $t = \infty$ and hence $\infty \in \mathcal{P}_A$. Now, let us consider the case where the only units of A are ± 1. We observe that ∞ is fixed by a Möbius transformation

$$\frac{at + b}{ct + d}$$

if and only if $c = 0$. Let us now suppose that there exists a hyperbolic element f so that the corresponding matrix is

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

where $a, d \neq \pm 1$, thus it follows that $ad = 1$ and, then, a and d are units. By assumption, this means that $a, d = \pm 1$ and so the absolute value of the trace $|a + d|$ can never be greater than 2 so f cannot be hyperbolic and we have a contradiction. This implies that $\infty \notin \mathcal{P}_A$. $\square$

Another important property of $\mathcal{P}_A$, for any subring A of $\mathbb{R}$, is the following.

Lemma 4.6 ([22]). *The set $\mathcal{P}_A$ is invariant under any $Z \in \mathrm{PSL}_2(A)$. In other words, $Z(\mathcal{P}_A) \subset \mathcal{P}_A$.*

Proof. Let us consider $L \in \mathcal{P}_A$. Thus, there exists a hyperbolic element $Y \in \mathrm{PSL}_2(A)$ so that $Y(L) = L$. Now, considering an arbitrary element Z in $\mathrm{PSL}_2(A)$, we have

$$Y(L) = \left(Z^{-1} Z\right)(L).$$

Then,

$$ZY(L) = Z(L).$$

Moreover, we get

$$\left(ZYZ^{-1}\right)(Z(L)) = Z(L).$$

It follows that $Z(L) \in \mathbb{R}$ is a fixed point of $ZYZ^{-1} \in \mathrm{PSL}_2(A)$. We observe that ZYZ^{-1} is a hyperbolic element from $\mathrm{PSL}_2(A)$. By an abuse of notation, we still denote the associated matrix by ZYZ^{-1} and we know that ZYZ^{-1} and Y have the same trace. Therefore, $Z(L) \in \mathcal{P}_A$. $\square$

Now, let us consider the subgroup $G(A)$ in $\mathrm{PPSL}_2(\mathbb{R})$ such that the coefficients $a_i, b_i, c_i, d_i \in A$ and that the set of breakpoints is a subset of $\mathcal{P}_A$. Thus, let us denote by $H(A)$ the subset of $G(A)$ of all elements which fix $t = \infty$. This subset is a group under composition.

Remark 4.7. The map f created in the proof of the Lemma 4.5 lies in $\mathrm{PPSL}_2(\mathbb{R})$ and, since it fixes $t = \infty$, it belongs to $H(A)$, where A is some subring of $\mathbb{R}$ which has an unit $a \neq \pm 1$.

Definition 4.8. The group $H(A)$, where A is a subring of $\mathbb{R}$, is called **Monod's group with coefficients in** A.

Of course $H(A)$ is a subgroup of $H(\mathbb{R})$ for any subring A of $\mathbb{R}$. As mentioned at the beginning of this chapter, one of the reasons making these groups interesting is that they do not contain any non-abelian free subgroup (since $H(\mathbb{R})$ does not contain any) and, for any non-trivial subring $A \subset \mathbb{R}$, so that $A \neq \mathbb{Z}$, the group $H(A)$ is non-amenable [22]. As a consequence, Monod groups provide a family of counterexamples for the von Neumman-Day Conjecture 0.1.

Remark 4.9. From now on, if $A = \mathbb{R}$, we will denote $H(\mathbb{R})$ and $\mathcal{P}_{\mathbb{R}}$ just by H and $\mathcal{P}$, respectively.

The Lemma 4.6 implies that if we consider any value in $L \in \mathcal{P}_A$ and apply an arbitrary element $z \in H(A)$, then $z(L) \in \mathcal{P}_A$. This is important, because in order to build the candidate to conjugator between two elements, we need that all its interval endpoints lie in $\mathcal{P}_A$. In addition, since in the case of $A = \mathbb{R}$ we get $\mathcal{P}_A = \mathbb{R}\mathrm{P}^1$, we can prove that it is possible to associate any two k-tuples of values from $\mathbb{R}$ via some element from H. This property means that H acts order k-transitively on $\mathbb{R}\mathrm{P}^1$ and it is analogous to the one in Lemma 2.6.

Lemma 4.10. *Let $t_1 < t_2 < \ldots < t_k$ and $s_1 < s_2 < \ldots < s_k$ be elements from $\mathbb{R}\mathrm{P}^1 \backslash \{\infty\}$ Then, there exists $f \in H$ such that $f(t_i) = s_i$, for all $i = 1, 2, \ldots, k$.*

Proof. For all $i \in \{1, 2, \ldots, k-1\}$, let us consider the intervals $[t_i, t_{i+1}]$ and $[s_i, s_{i+1}]$. Then, since $\mathrm{PSL}_2(\mathbb{R})$ is double-transitive on $\mathbb{R}\mathrm{P}^1$ (see p.218, Theorem 5.2.1 (ii), in [17]) there exists an element $f_i \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$f_i(t_i) = s_i \text{ and } f_i(t_{i+1}) = s_{i+1}.$$

Thus, it is enough to glue together these maps with functions $f_0, f_k \in \mathrm{PSL}_2(\mathbb{R})$ defined on $[\infty, t_1]$ and $[t_k, \infty]$, respectively, as

$$f_0(t) = \frac{a_0 t + b_0}{d_0} \text{ and } f_{k+1}(t) = \frac{a_k t + b_k}{d_k},$$

where $a_0 d_0 = a_k d_k = 1$ and $a_0, b_0, d_0, a_k, b_k, d_k$ are chosen in such way that $f_0(t_1) = s_1$ and $f_k(t_k) = s_k$. Thus, we construct the following element from H

$$f(t) := \begin{cases} f_0(t), & \text{if } t \in [\infty, t_1] \\ f_i(t), & \text{if } t \in [t_i, t_{i+1}] \\ f_k(t), & \text{if } t \in [t_k, \infty] \end{cases}$$

where $i \in \{1, 2, \ldots, k-1\}$, so that $f(t_i) = s_i$, for all $i \in \{1, 2, \ldots, k\}$. □

Another interesting property of the set $\mathcal{P}_A$ is the following.

Lemma 4.11 ([22])**.** *Let us consider a subring A of $\mathbb{R}$. Then, if A is countable, so is $\mathcal{P}_A$.*

4.3 Properties of Monod Groups

These groups have other interesting properties, which we will introduce in this section. Some of these properties are similar to those holding for Thompson's group F. Some of the following results are known.

Definition 4.12. Let G be a group of homeomorphisms of some topological space X.

(a) A homeomorphism h of X **locally agrees** with G if for every point $p \in X$, there exists a neighborhood U of p and an element $g \in G$ such that

$$h\big|_U = g\big|_U.$$

We denote the set of all homeomorphisms of X which locally agree with G by $[G]$;

(b) The group G is a **full** if every homeomorphism of X that locally agrees with G belongs to G. In other words, G is a full group if $G = [G]$.

Theorem 4.13. *Monod's group $H(A)$ is a full group for any subring A of $\mathbb{R}$.*

Proof. Given a subring A of $\mathbb{R}$, let h be an element of $[H(A)]$. By definition, for every $p \in \mathbb{R}\mathrm{P}^1$, there exists a neighborhood V_p of p and an element $g_p \in H(A)$ such that

$$h\big|_{V_p} = g_p\big|_{V_p}.$$

Consider the infinite family of neighborhoods $\{V_p\}_{p \in \mathbb{R}\mathrm{P}^1}$. Since $\mathbb{R}\mathrm{P}^1$ is a compact set, there exists a finite family $\{V_{p_j}\}_{j=1}^n \subseteq \{V_p\}_{p \in \mathbb{R}\mathrm{P}^1}$ such that

$$\mathbb{R}\mathrm{P}^1 \subseteq \bigcup_{j=1}^n V_{p_j}.$$

Then, we get that

$$h\big|_{V_{p_j}} = g_{p_j}\big|_{V_{p_j}},$$

for all $j \in \{1, \dots, n\}$. Note that we may have that $V_{p_j} \cap V_{p_{j+1}} \neq \emptyset$, where $j \in \{1, \dots, n-1\}$. In this case,

$$g_{p_j}\big|_{V_{p_j} \cap V_{p_{j+1}}} = h\big|_{V_{p_j} \cap V_{p_{j+1}}} = g_{p_j}\big|_{V_{p_j} \cap V_{p_{j+1}}}.$$

Since $g_{p_j} \in H(A)$, for all $j \in \{1, \dots, n\}$, we have that g_{p_j} has finitely many breakpoints in V_{p_j} for each $j \in \{1, \dots, n\}$. Therefore, h has finitely many breakpoints in V_{p_j}, for all $j \in \{1, \dots, n\}$. Now, since $\mathbb{R}\mathrm{P}^1 = \bigcup_{j=1}^{n} V_{p_j}$, we have that the number of breakpoints of h in $\mathbb{R}\mathrm{P}^1$ is finite. Moreover, close to ∞, h must be of the form

$$h(t) = \frac{at + b}{d},$$

since it coincides with some element from $H(A)$ around ∞. Then, $h \in H(A)$ and, more generally, $H(A) \supseteq [H(A)]$. Therefore, $[H(A)] = H(A)$, that is, $H(A)$ is a full group, for any subring A of $\mathbb{R}$. □

Remark 4.14. Another way to phrase the previous result is that if some homeomorphism of the $\mathbb{R}\mathrm{P}^1$ locally coincides with some element from $H(A)$, then it must to lie in $H(A)$.

We make some observations for elements of $H(A)$. We start by explicitly stating that Definition 4.8 says that an $f \in H(A)$ if there are finitely many points $t_1, t_2, \dots, t_n \in \mathcal{P}_A$ such that on each interval $[t_i, t_{i+1}]$

$$f\colon t \mapsto \frac{a_i t + b_i}{c_i t + d_i},$$

where $a_i d_i - c_i b_i = 1$, for suitable $a_i, b_i, c_i, d_i \in A$, and

$$f\colon t \mapsto (a_0 t + b_0)/d_0 \quad \text{and} \quad f\colon t \mapsto (a_n t + b_n)/d_n$$

on $(-\infty, t_1]$ and $[t_n, +\infty)$, respectively, where $a_0 d_0 = 1 = a_n d_n$, for $a_0, a_n, b_0, b_n \in A$. Then we can say that elements in $H(A)$ have **affine germs** at $\pm\infty$. In other words, when $t \in (-\infty, t_1]$ we rewrite f in this interval as $f(t) = a_0^2 t + a_0 b_0$, for all $t \in (-\infty, t_1]$, since $a_0 d_0 = 1$. Similarly, we can rewrite f as $f(t) = a_n^2 t + a_n b_n$, for all $t \in [t_n, +\infty)$, since $a_n d_n = 1$. Then, we have the following fact

Lemma 4.15. *Let $f \in H$. There is $K \in \mathbb{R}_{>0}$, such that*

1. *$f(t) = a_0^2 t + a_0 b_0$, for all $t < -K$;*
2. *$f(t) = a_n^2 t + a_n b_n$, for all $t > K$.*

Proof. In the preceding observations, considering $K = \max\{|t_1|, |t_n|\}$. □

Remark 4.16. Notice that, for all elements in $H(A)$, the germs at infinity satisfy that the slopes a_0^2 and a_n^2 are units of the ring A. Thus, if the only units of A are ± 1, the first and last parts of maps in $H(A)$ are translations. For instance, if $A = \mathbb{Z}$, the unique possibility is $a_0^2 = a_n^2 = 1$.

Lemma 4.17 ([22])**.** *Monod's group H is a torsion-free group. Thus, for any subring A of $\mathbb{R}$, $H(A)$ is also a torsion-free group.*

Next, we present interesting examples of elements from $H(A)$ which appeared first in the work by Burillo, Lodha and Reeves in [8].

Example 4.18. Let A be a subring of $\mathbb{R}$. Given any $r \in \mathbb{R}_{>0} \cap A$, consider the translation $t + r$ and take $\varepsilon \in (0, 1)$ so that

$$\frac{\varepsilon}{1-\varepsilon} = \varepsilon + r.$$

We get

$$\varepsilon = \frac{-r \pm \sqrt{r^2 + 4r}}{2}.$$

Finally, define

$$f_r(t) := \begin{cases} t, & \text{if } t \in (-\infty, 0] \\ \dfrac{t}{1-t}, & \text{if } t \in [0, \varepsilon] \\ t + r, & \text{if } t \in [\varepsilon, +\infty) \end{cases}$$

Notice that $\varepsilon \in \mathcal{P}_A$. In fact, The corresponding matrices of $t \mapsto \dfrac{t}{1-t}$ and $t \mapsto t + r$ are

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix},$$

respectively. If we now consider the matrix multiplication

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

we get the matrix

$$\begin{bmatrix} 1 & r \\ 1 & r+1 \end{bmatrix},$$

whose trace is clearly bigger than 2, so that the element $g \in \mathrm{PSL}_2(A)$ associated to this matrix is

$$g(t) = \frac{t + r}{t + r + 1}$$

and it is hyperbolic. Moreover,

$$g(t) = t \iff \frac{t + r}{t + r + 1} = t \iff t = \frac{-r \pm \sqrt{r^2 + 4r}}{2} \iff t = \varepsilon.$$

Thus, $\varepsilon \in \mathcal{P}_A$ and $f_r \in H(A)$. Now, considering an arbitrary $n \in \mathbb{Z}$ and conjugating f_r by the translation $t+n$, we get the following element in $H(A)$,

$$f_{r,n}(t) := \begin{cases} t, & \text{if } \ t \in (-\infty, n] \\ \dfrac{t-n}{1-(t-n)} + n, & \text{if } \ t \in [n, n+\varepsilon] \\ t+r, & \text{if } \ t \in [n+\varepsilon, +\infty). \end{cases}$$

In the same manner we obtain maps which lie in $H(A)$ by considering $r \in \mathbb{R}_{<0} \cap A$. In this case, we have $\varepsilon \in [-r, -r+1]$ so that

$$\frac{\varepsilon}{1+\varepsilon} = t + \varepsilon.$$

Similar arguments apply to this case in order to prove that $\varepsilon \in \mathcal{P}_A$ and that the maps

$$f_r(t) := \begin{cases} t, & \text{if } \ t \in (-\infty, 0] \\ \dfrac{t}{1+t}, & \text{if } \ t \in [0, \varepsilon] \\ t+r, & \text{if } \ t \in [\varepsilon, +\infty) \end{cases} \quad \text{and} \quad f_{r,n}(t) := \begin{cases} t, & \text{if } \ t \in (-\infty, n] \\ \dfrac{t-n}{1+(t-n)} + n, & \text{if } \ t \in [n, n+\varepsilon] \\ t+r, & \text{if } \ t \in [n+\varepsilon, +\infty). \end{cases}$$

lie in $H(A)$.

The last example gives us families of elements from $H(A)$, for any subring A of $\mathbb{R}$, which allows us to provide functions in $H(A)$ that have a bump between the identity around $-\infty$ and a translation $t+r$ around $+\infty$. Moreover, it allowed the authors in [8] to observe the next result, for which we need the following definition.

Definition 4.19. The **support** of an element $f \in H$ is the subset of points in $\mathbb{R}\mathrm{P}^1$, where f is different from the identity map, *i.e.*,

$$\operatorname{supp}(f) = \left\{ t \in \mathbb{R}\mathrm{P}^1 \mid f(t) \neq t \right\}.$$

Moreover, if $\operatorname{supp}(f)$ is a compact set, then we say that f has a **compact support**.

Remark 4.20. We notice that

$$\operatorname{supp}\left(g^{-1} y g\right) = g^{-1} \left(\operatorname{supp}\left(y\right)\right).$$

Lemma 4.21 ([8])**.** *For every $r \in A$ and $p \in \mathbb{R}$, there exists an element $f \in H(A)$ such that*

1. *For some $x \in \mathbb{R}$ so that $x < p$, f is supported on $[x, +\infty)$;*

2. *The restriction of f to $(p, +\infty)$ is equal to a translation by r.*

Similarly, there exists an element f from $H(A)$ so that

1. *For some* $y \in \mathbb{R}$ *so that* $y > p$*,* f *is supported on* $(-\infty, y]$*;*

2. *The restriction of* f *to* $(-\infty, p)$ *is equal to a translation by* r*.*

The important point to note with the preceding example is that, since these elements are translations around $+\infty$, we can get new examples of elements in $H(A)$ with compact support, in this case, maps which are trivial outside of some compact interval with endpoints in $\mathcal{P}_A$. We do this by gluing elements from the families of the previous example, which agree on some interval, to provide the identity on $[p, +\infty)$, for some $p \in \mathcal{P}_A$.

Definition 4.22. We call an interval $[a, b]$ a **gluing interval** if there exist $f, g \in H(A)$ so that the restrictions of these elements on $[a, b]$ agree. If there exist two elements which agree on a suitable interval, we say that these elements have a gluing interval.

Lemma 4.23 (Gluing, [8])**.** *Let us consider an ordered pair* (f, g) *where* $f, g \in H(A)$ *with gluing interval* $[a, b]$*. There exists an element* $h \in H(A)$ *so that*

1. *it agrees with* f *on* $(-\infty, a]$*;*

2. *it agrees with* g *on* $[b, +\infty)$*;*

3. *it agrees with* f *and* g *on* $[a, b]$*.*

The order of the ordered pair is important when we glue the elements. The result maps are different when we glue the pair (f, g) or the pair (g, f), as we can see in Figure 7. In [8], the authors show that some properties are shared between $H(A)$ and Thompson's group F when the only units of A are ± 1. We state the following result without proof.

Proposition 4.24 ([8])**.** *Let* $A \subseteq \mathbb{R}$ *be a subring. We have*

1. *If the only units of* A *are* ± 1*, then*
   (a) $H(A)'$ *is simple;*
   (b) *Every proper quotient of* $H(A)$ *is abelian;*
   (c) *All finite index subgroups of* $H(A)$ *are normal in* $H(A)$*.*

2. *If* A *has units other than* ± 1
   (a) $H(A)' \neq H(A)''$*;*
   (b) $H(A)''$ *is simple;*
   (c) *Every proper quotient of* $H(A)$ *is metabelian.*

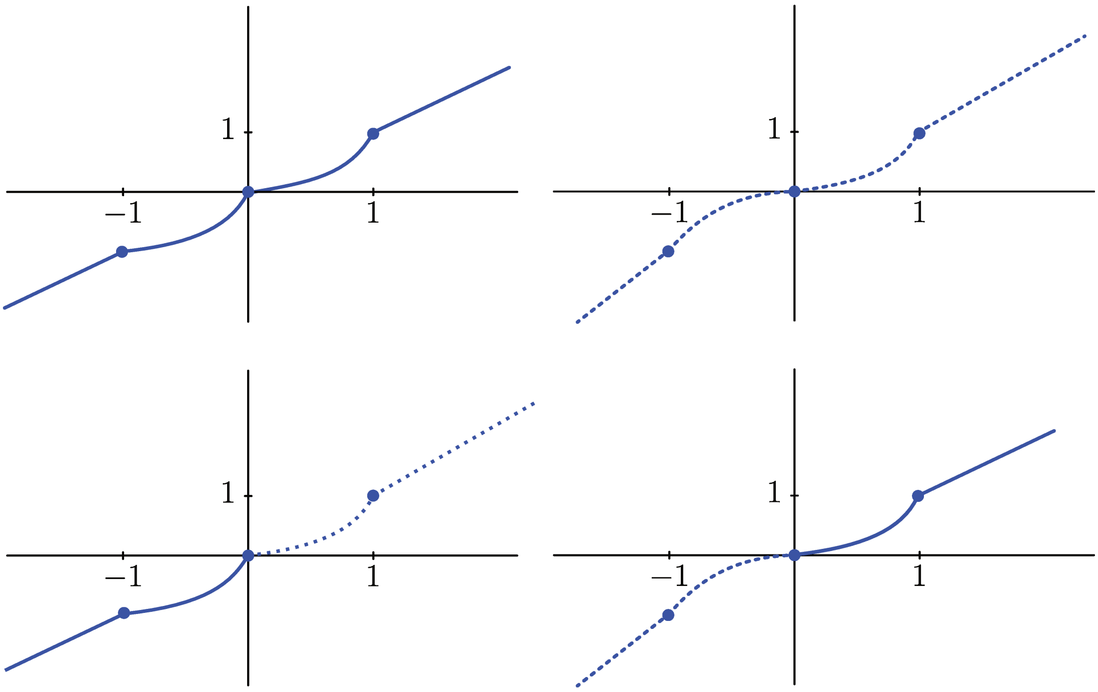

Figure 7 – At the top we see the solid line graph f and the dotted line one g. At the bottom we see the gluing of the pair (f, g) followed by that of the pair (g, f).

The groups $H(A)$ are not finitely presented (see [8] for more details).

We finish this section recommending the interested reader to check out references [8, 22] for more properties of Monod groups $H(A)$, for a given subring A of $\mathbb{R}$.

4.4 The Piecewise Projective Version of Thompson's Group F

In this section, we will relate elements of Thompson's group F to piecewise projective maps. The tool that we need is the Minkowski Question Mark function which we define recursively. In order to define this function, we start by defining it on the endpoints of the interval I:

$$?(0) = 0 \qquad \text{and} \qquad ?(1) = 1.$$

Since $0 = \dfrac{0}{1}$ and $1 = \dfrac{1}{1}$, we take the Farey mediant

$$0 \oplus 1 = \frac{0+1}{1+1} = \frac{1}{2}$$

and define

$$?\left(\frac{1}{2}\right) = ?(0 \oplus 1) = \frac{1}{2}(?(0) + ?(1)) = \frac{1}{2}.$$

Now we take the Farey mediants

$$0 \oplus \frac{1}{2} = \frac{1}{3} \qquad \text{and} \qquad \frac{1}{2} \oplus 1 = \frac{2}{3}$$

and define

$$?\left(\frac{1}{3}\right) =?\left(0 \oplus \frac{1}{2}\right) = \frac{1}{2}\left(?(0)+?\left(\frac{1}{2}\right)\right) = \frac{1}{4}$$

and

$$?\left(\frac{2}{3}\right) =?\left(\frac{1}{2} \oplus 1\right) = \frac{1}{2}\left(?\left(\frac{1}{2}\right) \oplus ?(1)\right) = \frac{3}{4}.$$

Proceeding this way, given a pair $\frac{p}{q}$ and $\frac{r}{s}$ of reduced fractions such that $|ps - qr| = 1$, we define $?\left(\frac{p}{q}\right)$ and $?\left(\frac{r}{s}\right)$ and then

$$?\left(\frac{p}{q} \oplus \frac{r}{s}\right) = \frac{1}{2}\left(?\left(\frac{p}{q}\right) +?\left(\frac{r}{s}\right)\right).$$

At the n^{th} stage, the function is defined for $2^n + 1$ values of x and the images corresponding to these values of x are of the form $\frac{m}{2^n}$, where $m = 0, 1, \ldots, 2^n$. We notice that this function is defined for all rational numbers in I and it maps them to dyadic rational numbers.

Definition 4.25. The function recursively defined above is called **Minkowski Question Mark function**.

We can extend the function, by continuity, to the whole interval I. Since it satisfies $?(t + 1) =?(t) + 1$, we can extend the function from the box I^2 to any other one of the type $[n, n + 1]^2$ for each $n \in \mathbb{Z}$ and, using this method, we can extend it to the real line. We have that $?(x)$ is a bijection between $\mathbb{Q}$ and the set of dyadic real numbers $\mathbb{Z}\left[\frac{1}{2}\right]$. Moreover, this function is a homeomorphism of $\mathbb{R}$ and its inverse is called the **Conway box function**. The reason of our interest for the function $?(x)$ is the following:

Proposition 4.26. *Let $q_1, q_2 \in \mathbb{Q}$ and let $d_i =?(q_i) \in \mathbb{Z}\left[\frac{1}{2}\right]$ for $i = 1, 2$. On the interval (q_1, q_2) let us consider a projective map $f(t) = \frac{at + b}{ct + d}$. Then the map $\tilde{f}(t) =?f?^{-1}(t)$, defined for all $t \in [d_1, d_2]$, is a linear map.*

Proof. Let $t_1, t_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$ and let us consider $x_i =?^{-1}(t_i) \in \mathbb{Q}$, for $i = 1, 2$. Let us write $x_i = \frac{p_i}{q_i}$, for $i = 1, 2$. It is enough to prove that

$$\tilde{f}\left(\frac{t_1 + t_2}{2}\right) = \frac{\tilde{f}(t_1) + \tilde{f}(t_2)}{2}.$$

We get that

$$\begin{aligned}\tilde{f}\left(\frac{t_1+t_2}{2}\right) &= ?f?^{-1}\left(\frac{t_1+t_2}{2}\right)\\ &= ?f?^{-1}\left(\frac{?(x_1)+?(x_2)}{2}\right)\\ &= ?f\left(\frac{p_1+p_2}{q_1+q_2}\right)\\ &= ?\left(\frac{ap_1+ap_2+bq_1+bq_2}{cp_1+cp_2+dq_1+dq_2}\right).\end{aligned}$$

By definition of $?(x)$

$$\begin{aligned}\tilde{f}\left(\frac{t_1+t_2}{2}\right) &= \frac{1}{2}\left(?\left(\frac{ap_1+bq_1}{cp_1+dq_1}\right)+?\left(\frac{ap_2+bq_2}{cp_2+dq_2}\right)\right)\\ &= \frac{1}{2}\left(?f(x_1)+?f(x_2)\right)\\ &= \frac{?f?^{-1}(t_1)+?f?^{-1}(t_2)}{2}\\ &= \frac{\tilde{f}(t_1)+\tilde{f}(t_2)}{2}.\end{aligned}$$

Thus $\tilde{f}$ is linear. □

This result means that we can transform projective maps defined on some interval to linear maps by conjugation by Minkowski Question Mark. The advantage is that we can apply this result to piecewise projective maps and transform them into piecewise linear maps and vice-versa. As a consequence, we can get a copy of Thompson's group F in $\mathrm{PPSL}_2(\mathbb{R})$, with rational breakpoints and integer coefficients. William Thurston was the first one to prove the existence of this projective version of F, but he did not publish it and it first appeared in the paper by Cannon, Floyd and Parry [10]. We only give an idea of the proof of the following result.

Theorem 4.27. *The group of piecewise projective homeomorphisms maps of the real line with rational breakpoints is isomorphic to F.*

Idea of the proof. First of all, let us consider f a piecewise projective map with rational breakpoints such that its coefficients lie in $\mathbb{Z}$. We will write these breakpoints as $t_1, t_2, \ldots, t_n$ and so that

$$-\infty = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = \infty.$$

Let us consider f_i the projective map given by restricting f to $[t_i, t_{i+1}]$, with $i = 0, 1, \ldots, n$. Since each f_i is projective with integer coefficients, we have that it preserves rational points.

If we conjugate each f_i by the Minkowski Question Mark Function, Proposition 4.26 returns a linear map $\tilde{f}_i \coloneqq ?f_i?^{-1}$ defined on $[?(t_i), ?(t_{i+1})]$ for each $i = 0, 1, \ldots, n$. Since $?(x)$ maps rational numbers to dyadic rational numbers, we get that the endpoints of each $[?(t_i), ?(t_{i+1})]$ are dyadic rational numbers. Moreover, if f_i preserves rational numbers on it domain, then $\tilde{f}_i$ preserves dyadic rational numbers on it domain. Notice that, since $f_i(t_{i+1}) = f_{i+1}(t_{i+1})$ for all $i = 0, 1, \ldots, n$, we have that $\tilde{f}_i(?(t_{i+1})) = \tilde{f}_{i+1}(?(t_{i+1}))$. By gluing all the linear maps $\tilde{f}_i$ with their successors, we obtain a piecewise linear map with finitely many dyadic rational breakpoints of $\mathbb{R}$. It can be shown that this assignment yields a group isomorphism. $\square$

We finish this section recommending the interested reader to check out references [27, 28] for more properties of the Minkowski Question Mark function. If the reader would like to know more about Farey fractions, we encourage to check out the references [1, 14].

5. Conjugacy Invariant in Monod's Group H

In Chapter 3, we presented the Stair Algorithm developed in [18, 21]. In [9], the authors show that Thompson's group F has soluble twisted conjugacy problem, opening the possibility to finding extensions of F with unsoluble conjugacy problem and using ideas similar to those in [18, 21] by finding necessary conditions that a "twisted" conjugator must satisfy. These conditions allow them to create an analogous Stair Algorithm even if the functions involved have no initial slope, since they are periodic. As seen in Chapter 4, Monod's group H contains a copy of Thompson's group F and shares some properties with it, however all Monod groups $H(A)$ are not finitely presented in contrast to the group F. We thus search for a conjugacy invariant for H, as done by Matthew G. Brin and Craig C. Squier in [6] for the group $\mathrm{PL}_+(\mathbb{R})$, which is also not finitely presented.

In the present chapter, we develop our own version of the Stair Algorithm, by generalizing the techniques from [9, 18, 21]. We prove a sequence of lemmas which help us adapt the Stair Algorithm for Monod's group H. The strategy is similar to one seen in Chapter 3, that is, we assume that there exists a conjugator $g \in H$ between two elements $y, z \in H$ and we find necessary conditions which g must to satisfy. Before starting, we remind the reader that the idea of the Stair Algorithm is to build a candidate to conjugator piece-by-piece, by applying the Identification Lemma. We also develop another conjugacy invariant, called *Mather Invariant*, by adapting ideas from [20, 21].

We point out that all results of this chapter are new.

5.1 Stair Algorithm for H

The main goal of this section is an adaptation of the *Stair Algorithm*. If there exists a conjugator, this algorithm allows us to construct it from a given initial germ. First, we need adapt some initial definitions and necessary conditions for conjugacy, as we will see in Lemmas 5.7 and 5.8. The main idea is to reduce our search to some families of maps in H. This reduction arises from the analysis of the set of fixed points of elements from H. Next, we will see the Lemma 5.10, which determines uniquely the first piece of a possible conjugator, given its initial germ. Then we construct a conjugator step-by-step from its initial germ. We will identify y and z inside a box close to the initial box by a suitable conjugator. Then we repeat this process in more boxes until we reach the final box. This algorithm ends after a finite number of steps.

Let us start with some definitions.

Definition 5.1. Let G be any subset of H. We define $G^{>}$ as the subset of G of all maps that lie above the diagonal, that is,

$$G^{>} := \{g \in G \mid g(t) > t,\ t \in \mathbb{R}\}.$$

Similarly, we define $G^{<}$. Given $g \in G$, if either $g \in G^{>}$ or $g \in G^{<}$, then g is said to be an **one-bump function**. Moreover, for every $-\infty \leqslant p < q \leqslant +\infty$, we define $G(p,q)$ as the set of elements of G with support inside (p,q), that is,

$$G(p,q) := \{g \in G \mid g(t) = t,\ t \notin (p,q)\}.$$

We also define the subset

$$G^{>}(p,q) := \{g \in G \mid g(t) = t,\ \forall\, t \notin (p,q),\ \text{and } g(t) > t,\ \forall\, t \in (p,q)\}.$$

Analogously, we define $G^{<}(p,q)$. If g lies either in $G^{>}(p,q)$ or $G^{<}(p,q)$, we say that g is a **one-bump function on** (p,q).

Remark 5.2. If G is a subgroup, then $g \in G^{>}$ if, and only if, $g^{-1} \in G^{<}$. Naturally, we will consider $G = H$.

Example 5.3. The following homeomorphism of $\mathbb{R}$ is an element of $H(0,+\infty)$:

$$b(t) = \begin{cases} t, & \text{if } t \in (-\infty, 0], \\ \dfrac{t}{1-t}, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ 3 - \dfrac{1}{t}, & \text{if } t \in \left[\dfrac{1}{2}, 1\right], \\ t+1, & \text{if } t \in [1, +\infty). \end{cases}$$

Since elements $f \in H$ are defined for all real numbers, we will define our Initial and Final boxes for real numbers close to $\pm\infty$. In order to work with these numbers sufficiently close to $\pm\infty$, we give the next definition.

Definition 5.4. A property $\mathcal{P}$ holds **for t negative sufficiently large** (respectively, **for t positive sufficiently large**) to mean that there exists a real number $L < 0$ such that $\mathcal{P}$ holds for every $t \leqslant L$ (respectively, there is a positive real number R so that $\mathcal{P}$ holds for every $t \geqslant R$).

5.1.1 Some Concepts and Necessary Conditions for Conjugacy

In Chapter 3, we needed to work with the initial and final slopes of elements. If two conjugate elements from $\mathrm{PL}_2(I)$ had the same initial (respectively, final) slope, they coincide on the initial (respectively, final) boxes. Let us define similar concepts for elements from H. Given $y \in H$, let us denote the slope of y for t negative sufficiently large as

$$y'(-\infty) := \lim_{t \to -\infty} y'(t).$$

Similary, we denote the slope of y for g positive sufficiently large as $y'(+\infty)$. However, if two elements from H have the same slopes for t negative sufficiently large, they do not necessarily coincide around $-\infty$. Thus, in order to ensure that two elements coincide for t negative sufficiently large, we give the following definition.

Definition 5.5. We define the **germ of** y **at** $-\infty$ as the pair

$$y_{-\infty} := \left(y'(-\infty), y\left(L\right) - y'(-\infty)L\right),$$

where L is the largest real number for which y is the affine map with slope $y'(-\infty)$ on the interval $(-\infty, L]$. We call $y_{-\infty}$ the **initial germ**. Analogously, we define the **final germ** $y_{+\infty}$.

We remark that the initial and final germs of an element from H lie in the group which we are about to define.

Definition 5.6. Let us denote by $\mathbb{R}_{>0}$ the multiplicative group $(\mathbb{R}_{>0}, \cdot)$ and by $\mathbb{R}$ the additive group $(\mathbb{R}, +)$. We define the **affine group** as the semidirect product group

$$\mathrm{Aff}(\mathbb{R}) := \left\{(a, b) \mid a \in \mathbb{R}_{>0} \text{ and } b \in \mathbb{R}\right\},$$

with operation

$$(a, b)(c, d) := (ac, b + ad).$$

The identity element is $(1, 0)$ and inverses are given by $(a, b)^{-1} = \left(a^{-1}, -a^{-1}b\right)$.

For an element $y \in H(A)$, the germs $y_{-\infty}$ and $y_{+\infty}$ are elements from $\mathrm{Aff}\left(\mathbb{R}\right)$.

Now assume $y, z \in H$ and that there is $g \in H$ such that $g^{-1}yg = z$, which we will denote by $y^g = z$. We have the following result on slopes.

Lemma 5.7. *Let $y, z \in H$ such that $y^g = z$, for some $g \in H$. Then for t negative (respectively, positive) sufficiently large we have $y'(-\infty) = z'(-\infty)$ (respectively, $y'(+\infty) = z'(+\infty)$).*

Proof. Let $y, z \in H$ be as in the hypothesis for some $g \in H$. For t negative sufficiently large we have

$$y\left(t\right) = \frac{a_0 t + b_0}{d_0} \text{ and } g\left(t\right) = \frac{at + b}{d}$$

with $a, b, d \in \mathbb{R}$. Since $ad = 1 = a_0 d_0$, we can rewrite y and g as follows

$$y\left(t\right) = a_0^2 t + a_0 b_0 \text{ and } g\left(t\right) = a^2 t + ab.$$

Since $g^{-1}(t) = a^{-2}t - a^{-1}b$, for t negative sufficiently large we have

$$\begin{aligned} z(t) &= g^{-1}yg(t) = g^{-1}y(a^2t + ab) = g^{-1}\left(a_0^2(a^2t + ab) + a_0b_0\right) \\ &= a^{-2}\left(a_0^2a^2t + a_0^2ab + a_0b_0\right) - a^{-1}b \\ &= a_0^2a^{-2}a^2t + a_0^2a^{-2}ab + a^{-2}a_0b_0 - a^{-1}b \\ &= a_0^2t + a_0^2a^{-1}b + a_0b_0a^{-2} - a^{-1}b \\ &= a_0^2t + a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b. \end{aligned}$$

Thus, we have $y'(-\infty) = z'(-\infty)$. The proof that $y'(+\infty) = z'(+\infty)$ for t positive sufficiently large is the same. □

The previous result gives a first necessary condition in order for two elements from H to be conjugate. If they have different initial (or final) slopes, then they cannot be conjugate. However, as said before, it is not a sufficient condition for conjugacy. Moreover, from the proof of Lemma 5.7, given $y, z \in H$ such that $y^g = z$, at $-\infty$ it follows that the germs of y, g and z are, respectively,

$$y_{-\infty} = \left(a_0^2, a_0b_0\right), \; g_{-\infty} = \left(a^2, ab\right) \text{ and } z_{-\infty} = \left(a_0^2, a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b\right).$$

The conjugacy class of some initial germ $y_{-\infty} = \left(a_0^2, a_0b_0\right)$ in $\mathrm{Aff}(\mathbb{R})$ is

$$y_{-\infty}^{\mathrm{Aff}(\mathbb{R})_{-\infty}} := \left\{\left(a_0^2, a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b\right) \mid g_{-\infty} = (a^2, ab)\right\}.$$

In a similar fashion, we can determine the conjugacy class of a final germ in $\mathrm{Aff}(\mathbb{R})$.

The next result gives another necessary condition for conjugacy between two elements from H by showing the initial and final germs are preserved by conjugacy.

Lemma 5.8. *For any $y, z \in H$ such that $y^g = z$ for some $g \in H$, it holds that $y_{-\infty}^{\mathrm{Aff}(\mathbb{R})_{-\infty}} = z_{-\infty}^{\mathrm{Aff}(\mathbb{R})_{-\infty}}$. Similarly, it holds that $y_{+\infty}^{\mathrm{Aff}(\mathbb{R})_{+\infty}} = z_{+\infty}^{\mathrm{Aff}(\mathbb{R})_{+\infty}}$.*

Proof. Since $g_{-\infty}^{-1} = \left(a^{-2}, -a^{-1}b\right)$, we have

$$\begin{aligned} y_{-\infty}^{g_{-\infty}} &= (a^{-2}, -a^{-1}b) \cdot (a_0^2, a_0b_0) \cdot (a^2, ab) \\ &= (a^{-2}a_0^2, -a^{-1}b + a^{-2}a_0b_0) \cdot (a^2, ab) \\ &= (a_0^2, -a^{-1}b + a^{-2}a_0b_0 + a^{-1}a_0^2b) \\ &= \left(a_0^2, a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b\right). \end{aligned}$$

Moreover, since $y^g = z$, for t negative sufficiently large we have

$$z(t) = a_0^2t + a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b,$$

which implies that the germ of z at $-\infty$ is $z_{-\infty} = \left(a_0^2, a_0b_0a^{-2} + \left(a_0^2 - 1\right)a^{-1}b\right)$. Therefore, $y_{-\infty}^{g_{-\infty}} = z_{-\infty}$. Then $y_{-\infty}^{\mathrm{Aff}(\mathbb{R})_{-\infty}} = z_{-\infty}^{\mathrm{Aff}(\mathbb{R})_{-\infty}}$. Similarly, we prove that $y_{+\infty}^{\mathrm{Aff}(\mathbb{R})_{+\infty}} = z_{+\infty}^{\mathrm{Aff}(\mathbb{R})_{+\infty}}$. □

As a consequence of the previous lemma, if the conjugacy classes of the germs of $y, z \in H$ at $-\infty$, or at $+\infty$, are different, then y and z can not be conjugate. From now on, if $y_{-\infty}$ and $z_{-\infty}$ are conjugate in $\mathrm{Aff}(\mathbb{R})$, we will denote it by $y_{-\infty} \sim_{\mathrm{Aff}(\mathbb{R})} z_{-\infty}$.

5.1.2 Making Fixed Points Coincide

Given a homeomorphism f, we recall that the set of its fixed points is denoted by $\mathrm{Fix}(f)$. We have seen in Section 3.1 how to make the set of fixed points of two homeomorphisms coincide. The ideas we used are general and do not depend on the homeomorphisms being elements from Thompson's group F. Now, if $y, z \in H$ and there exists $g \in H$ so that $y^g = z$, then $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ obtaining a necessary condition for conjugation. Proceeding the same way as in Section 3.1, we conclude that we can restrict our search of conditions to a conjugator for one-bump functions in H. This is possible since we have proved the Lemma 4.10 in Chapter 4. The properties of the fixed points of an homeomorphism are preserved under conjugation. The following is straightforward.

Proposition 5.9. *Given $y, g \in H$, suppose that $t_0 < t_1 < \ldots < t_n$ are fixed points of y. Then $g^{-1}(t_0) < g^{-1}(t_1) < \ldots < g^{-1}(t_n)$ are the fixed points of $g^{-1}yg$.*

5.1.3 Initial and Final Boxes

We intend to deal with the equation $z = g^{-1}yg$ for $y, z \in H^{<}$ and $g \in H$. In Chapter 3, we saw that if $y^g = z$, with $y, z, \in \mathrm{PL}_2^{<}(I)$ and $g \in \mathrm{PL}_2(I)$, then there exists $L > 0$, depending only y and z, so that g is linear inside $[0, L]^2$. In our case, we already have that every element of H is linear[1] for t negative sufficiently large. In this subsection, we aim to show that a possible conjugator between two given elements is determined by its germ inside an initial box $(-\infty, L]^2$ and a final box $[R, +\infty)^2$. The following result is an adaptation of Lemma 3.4 in the group H.

Lemma 5.10 (Initial and Final Boxes)**.** *Let $y, z \in H^{>}(-\infty, p)$ for some $-\infty < p \leqslant +\infty$ and let $g \in H$ such that $y^g = z$. Then there exists a constant $L \in \mathbb{R}$ (depending on y, z and $g_{-\infty}$) such that g is linear on the initial box $(-\infty, L]^2$. An analogous result holds, for $y, z \in H^{>}(p, +\infty)$ for some $-\infty \leqslant p < +\infty$ and a final box $[R, +\infty)$.*

Proof. Since $y^g = z$, by the Lemma 5.7, there exists a constant $L \in \mathbb{R}$ negative sufficiently large such that $y'(t) = z'(t)$ for $t \leqslant L$ negative sufficiently large. Then, let us consider $L \in \mathbb{R}$ negative sufficiently large such that $y'(t) = z'(t)$ for every $t \leqslant L$ and that

$$y(t) = a_0^2 t + a_0 b_0 \text{ and } z(t) = a_0^2 t + a_0 b_0 a^{-2} + a^{-1} b (a_0^2 - 1)$$

for all $t \leqslant L$ and for suitable $a, b \in \mathbb{R}$.

[1] By linear we want to say affine, we are making a common slight abuse of language.

Let us rewrite our goal: if we define

$$\widetilde{L} := \sup\{r \mid g \text{ is linear on } (-\infty, r]\},$$

then $\widetilde{L} \geqslant \min\{L, g^{-1}(L)\}$. Let us assume the opposite, that is, $\widetilde{L} < \min\{L, g^{-1}(L)\}$ and

$$g(t) = \begin{cases} a^2 t + ab, & \text{if } t \in \left(-\infty, \widetilde{L}\right], \\ \dfrac{\bar{a}t + \bar{b}}{\bar{c}t + \bar{d}}, & \text{if } t \in \left[\widetilde{L}, L\right), \end{cases}$$

where $\bar{c} \neq 0$.

Since $\widetilde{L} < L < 0$ and z is an increasing function, let us consider a real number $\sigma > 1$ such that $\sigma\widetilde{L} < \widetilde{L} < L$ and $\widetilde{L} < z(\sigma\widetilde{L}) < L$. Then z is linear around $\sigma\widetilde{L}$ and

$$gz\left(\sigma\widetilde{L}\right) = g\left(z\left(\sigma\widetilde{L}\right)\right). \tag{5.1}$$

On the other hand, $\widetilde{L} < g^{-1}(L)$ and so $\sigma\widetilde{L} < g^{-1}(L)$. Thus we have $g(\sigma\widetilde{L}) < L$, which means that y is linear around $g(\sigma\widetilde{L})$ and

$$\begin{aligned} yg\left(\sigma\widetilde{L}\right) &= y\left(a^2\sigma\widetilde{L} + ab\right) = a^2\left(a_0^2\sigma\widetilde{L}\right) + a_0^2 ab + a_0 b_0 \\ &= a_0^2 a^2 \sigma\widetilde{L} + a_0^2 ab + a_0 b_0 = a^2\left(a_0^2\sigma\widetilde{L} + a^{-2}a_0 b_0 + a^{-1} b a_0^2\right) \\ &= a^2\left(z\left(\sigma\widetilde{L}\right) + a^{-1}b\right) = a^2 z\left(\sigma\widetilde{L}\right) + ab. \end{aligned} \tag{5.2}$$

Since $yg(t) = gz(t)$ for every real number t, we can combine Equations (5.1) and (5.2) and get

$$g\left(z\left(\sigma\widetilde{L}\right)\right) = a^2\left(z\left(\sigma\widetilde{L}\right)\right) + ab,$$

for any real number $\sigma > 1$. But, by definition,

$$g\left(z\left(\sigma\widetilde{L}\right)\right) = \frac{\bar{a}\left(z\left(\sigma\widetilde{L}\right)\right) + \bar{b}}{\bar{c}\left(z\left(\sigma\widetilde{L}\right)\right) + \bar{d}}.$$

Then equating the two equations

$$a^2\left(z\left(\sigma\widetilde{L}\right)\right) + ab = \frac{\bar{a}\left(z\left(\sigma\widetilde{L}\right)\right) + \bar{b}}{\bar{c}\left(z\left(\sigma\widetilde{L}\right)\right) + \bar{d}}. \tag{5.3}$$

As consequence, we get

$$a^2\bar{c}\left(z\left(\sigma\widetilde{L}\right)\right)^2 + \left(ab\bar{c} + a^2\bar{d}\right) z\left(\sigma\widetilde{L}\right) + ab\bar{d} = \bar{a}z\left(\sigma\widetilde{L}\right) + \bar{b}.$$

Since the previous equation holds for suitable interval of values of $\sigma > 1$, then we either have $a^2 = 0$ or $\bar{c} = 0$. If $a^2 = 0$, then g would not be a homeomorphism for $t < L$, which is impossible. If $\bar{c} = 0$, then Equation (5.3) implies that $g(t) = a^2 t + ab$ for $t \in (-\infty, M]$ for some $M > \widetilde{L}$, a contradiction to the definition of $\widetilde{L}$. Hence, in all cases we have a contradiction to the assumption that $\widetilde{L} < \min\{L, g^{-1}(L)\}$ and so we have that $\widetilde{L} \geqslant \min\{L, g^{-1}(L)\}$. The proof for the final box is similar. □

Remark 5.11. We emphasize that the lemma also holds for $y, z \in H^{<}(-\infty, p)$, we just apply the statement to y^{-1} and z^{-1}.

5.1.4 Building a Candidate Conjugator

In this subsection, we describe how to construct a conjugator, if it exists. If this is the case, then we prove that the conjugator must be unique. Given two elements $y, z \in H$, the set of their conjugators is a coset of the centralizer of either y or z. It is important to start by obtaining some properties of centralizers, which we will do next. After that, we will identify y and z inside a box close to the initial box using a suitable conjugator, as mentioned before. Then we repeat this process and build more pieces of this potential conjugator until we reach the final linearity box.

Lemma 5.12. *Let $z \in H$ and suppose that there exist real numbers λ and μ satisfying $-\infty < \lambda \leqslant \mu < +\infty$, $z(t) \leqslant \lambda$, for all $t \in (-\infty, \mu]$ and that there is $g \in H$ so that $g(t) = t$ for all $t \in (-\infty, \lambda]$ and $g^{-1}zg(t) = z(t)$ for each $t \in (-\infty, \mu]$. Then g is the identity map up to μ.*

Proof. Notice that we can rewrite the equation $g^{-1}zg(t) = z(t)$ as $g(t) = z^{-1}gz(t)$, for each $t \in (-\infty, \mu]$. By hypothesis, $z(t) \leqslant \lambda$ for all $t \in (-\infty, \mu]$ and $g(t) = t$ for $t \leqslant \lambda$. Then

$$g(t) = z^{-1}g(z(t)) = z^{-1}z(t), \ \forall \ t \in (-\infty, \mu].$$

Then $g(t) = t$ for each $t \leqslant \mu$, as desired. □

If we consider an element $z \in H^{<}$, the previous lemma yields the following consequence.

Corollary 5.13. *Let $z \in H^{<}$ and $g \in H$ such that $g_{-\infty} = (1, 0)$ and $g^{-1}zg = z$. Then g is the identity map.*

Proof. Since $g_{-\infty} = (1, 0)$, we have there exists a number $L \in \mathbb{R}$ such that $g(t) = t$ for all $t \in (-\infty, L]$. Applying the last Lemma several times, we get $g(t) = t$ for all $t \in \left(-\infty, z^{-k}(L)\right]$. Since $z \in H^{<}$, we have $\lim_{k \to +\infty} z^{-k}(L) = +\infty$. Then g is the identity map. □

The preceding two results allow us to claim that there exists a group monomorphism between the group of centralizers of elements from H, in particular from $H^{<}$, and the group $\mathrm{Aff}(\mathbb{R})$.

Lemma 5.14. *Given $z \in H$, the following map*

$$\begin{aligned} \varphi_z \colon C_H(z) &\longrightarrow \mathrm{Aff}(\mathbb{R}) \\ g &\longmapsto g_{-\infty}, \end{aligned}$$

is a group monomorphism.

Proof. First of all, for each $g_1, g_2 \in C_H(z)$, with $g_{1-\infty} = (a_0^2, a_0 b_0)$ and $g_{2-\infty} = (\bar{a}_0^2, \bar{a}_0 \bar{b}_0)$, there exists $L \in \mathbb{R}$ so that $g_1 g_2\,(t) = a_0^2 \bar{a}_0^2 t + a_0^2 \bar{a}_0 \bar{b}_0 + a_0 b_0$. Then $(g_1 g_2)_{-\infty} = \left(a_0^2 \bar{a}_0^2, a_0 b_0 + a_0^2 \bar{a}_0 \bar{b}_0\right)$ Now, we notice that φ_z is well-defined and

$$\begin{aligned}
\varphi_z\,(g_1 g_2) &= (g_1 g_2)_{-\infty} \\
&= \left(a_0^2 \bar{a}_0^2, a_0 b_0 + a_0^2 \bar{a}_0 \bar{b}_0\right) \\
&= (a_0^2, a_0 b_0)(\bar{a}_0^2, \bar{a}_0 \bar{b}_0) \\
&= g_{1-\infty} g_{2-\infty} \\
&= \varphi_z(g_1)\varphi_z(g_2).
\end{aligned}$$

Then φ_z is a group homomorphism. Now, let us suppose that $\varphi(g_1) = \varphi(g_2)$. Then

$$(a_0^2, a_0 b_0) = (\bar{a}_0^2, \bar{a}_0 \bar{b}_0).$$

Then there exists a number $L \in \mathbb{R}$ such that $g_1\,(t) = g_2\,(t)$ for all $t \in (-\infty, L]$. Let us define $g := g_1 g_2^{-1}$. We have $g\,(t) = t$ for each $t \in (-\infty, L]$. Moreover, we get $g^{-1} z g = z$. By the Lemma 5.12, it follows that $g\,(t) = t$ for all $t \in \mathbb{R}$. Which implies that $g_1\,(t) = g_2\,(t)$ for each $t \in \mathbb{R}$. Then φ_z is a monomorphism. □

Remark 5.15. Naturally, we can consider $z \in H^<$ in the previous lemma. We still get a monomorphism between its centralizers and $\mathrm{Aff}\,(\mathbb{R})$.

Next we prove our version of the Identification Lemma 3.5. It is a tool which identifies the graphs of y and z inside of suitable boxes via some candidate to conjugator.

Lemma 5.16 (Identification Lemma)**.** *Let $y, z \in H^<$ and $L \in \mathbb{R}$ be so that $y\,(t) = z\,(t)$ for all $t \in (-\infty, L]$. Then there exists $g \in H$ such that $z\,(t) = g^{-1} y g\,(t)$ for every $t \in \left(-\infty, z^{-1}\,(L)\right]$ and $g\,(t) = t$ in $(-\infty, L]$. Moreover, this element g is uniquely determined on $\left(L, z^{-1}\,(L)\right]$.*

Proof. We notice that, if such element $g \in H$ exists, for each $t \in \left(-\infty, z^{-1}\,(L)\right]$ we have

$$y\,(g\,(t)) = g\,(z\,(t)) = z\,(t)\,,$$

since $z\,(t) \leqslant L$. Then

$$g\,(t) = y^{-1} z\,(t)\,,$$

for every $t \in \left(-\infty, z^{-1}\,(L)\right]$. Now, let us define

$$g\,(t) := \begin{cases} t, & \text{if } t \in (-\infty, L] \\ y^{-1} z\,(t)\,, & \text{if } t \in \left[L, z^{-1}\,(L)\right] \end{cases}$$

and we extend it to the whole real line from the point $\left(z^{-1}\,(L)\,, y^{-1}\,(L)\right)$. □

Remark 5.17. We can apply the last lemma again and determine g on $\left(-\infty, z^{-2}(L)\right]$. Actually, if we apply it N times, we can construct g on $\left(-\infty, z^{-N}(L)\right]$.

Proposition 5.18 (Uniqueness)**.** *Let $y, z \in H^{<}$ and $g \in H$ be functions such that $y^g = z$. Then the conjugator g is uniquely determined by its initial germ $g_{-\infty}$.*

Proof. Let us assume that there are $g_1, g_2 \in H$ so that $g_1^{-1} y g_1 = z$ and $g_2^{-1} y g_2 = z$ and with the same initial germ. Then we get

$$\left(g_1 g_2^{-1}\right)^{-1} y \left(g_1 g_2^{-1}\right) = y.$$

Then defining $g := g_1 g_2^{-1}$, we get that $g(t) = t$ for all $t \in (-\infty, L]$, which implies that the initial germ of g is $g_{-\infty} = (1, 0)$. By injectivity from the group homomorphism in Lemma 5.14, the unique centralizer of y with this initial germ is the identity map. Then $g(t) = t$ for all $t \in \mathbb{R}$. Therefore, $g_1 = g_2$, which proves the uniqueness of a conjugator with a given initial germ, if it exists. □

5.1.5 The Stair Algorithm for H

In this subsection we prove our main tool and one of the goals of this chapter, the adaptation of the Stair Algorithm for H.

Lemma 5.19 (Conjugator for Powers)**.** *Let $y, z \in H^{<}$. Let us consider $g \in H$ and $n \in \mathbb{Z}_{>0}$. Then $y^g = z$ if, and only if, $(y^n)^g = z^n$.*

Proof. If $y^g = z$, then it follows easily that $(y^n)^g = z^n$ by taking the power n of the first equation. Reciprocally, if we define $f := (y^n)^g = z^n$, we have that y^g and z centralize f. Let us suppose that $(y^g)_{-\infty} \neq z_{-\infty}$. Then either the first entries or the second ones of $(y^g)_{-\infty}$ and $z_{-\infty}$ are different. In both cases, we conclude that $(y^n)^g \neq z^n$, which is a contradiction. Then $(y^g)_{-\infty} = z_{-\infty}$ and, by Lemma 5.14, we have $y^g = z$. □

Theorem 5.20 (Stair Algorithm)**.** *Let $y, z \in H^{<}$ and let $(-\infty, L]^2$ be the initial box. Let us consider $\left(a^2, ab\right) \in \mathrm{Aff}(\mathbb{R})$. Then there exists $N \in \mathbb{Z}_{>0}$ such that the unique candidate conjugator g between y and z with initial germ $g_{-\infty} = \left(a^2, ab\right)$, is given by*

$$g(t) = y^{-N} g_0 z^N (t), \text{ for } t \in \left(-\infty, z^{-N}(L)\right],$$

and linear otherwise, where $g_0 \in H$ is an arbitrary map which is linear on the initial box such that $g_{0_{-\infty}} = \left(a^2, ab\right)$.

Proof. First of all, we notice that we will consider $y, z \in H^{<}$ such that their initial germs are in the same conjugacy class in $\mathrm{Aff}(\mathbb{R})$, otherwise y and z cannot be conjugate to each other by Lemma 5.8. Moreover, we consider $\left(a^2, ab\right) \in \mathrm{Aff}(\mathbb{R})$ so that it conjugates $y_{-\infty}$ to

$z_{-\infty}$ in $\mathrm{Aff}(\mathbb{R})$, there cannot be a conjugator g for y and z with initial germ $g_{-\infty} = \left(a^2, ab\right)$, again by Lemma 5.8. Now, let $[R, +\infty)^2$ be the final box and let $N \in \mathbb{Z}_{>0}$ be sufficiently large so that

$$\min\left\{z^{-N}(L), y^{-N}\left(a^2L + ab\right)\right\} > R.$$

We will build a candidate conjugator g between y^N and z^N, if it exists, as product of two functions g_0 and g_1 and use Conjugator for Powers Lemma 5.19. We point out that the Initial and Final Boxes of y^N and z^N coincide with ones of y and z. Then by Lemma 5.10 g must be linear on $(-\infty, L]^2$. Then, we define an "approximate conjugator" g_0 as

$$g_0(t) := a^2t + ab,$$

on $(-\infty, L]$ and extend it to the whole real line such that $g_0 \in H$. Up to replacing g_0 by g_0^{-1} and switching the role of y and z, we can assume that $a < 1$. Now, we define

$$y_1 := g_0^{-1} y g_0$$

and we look for a conjugator g_1 between y_1^N and z^N. We remark that y_1^N and z^N coincide on $(-\infty, L]$, since

$$y_1^N = g_0^{-1} y^N g_0 = z^N.$$

Making use of the Identification Lemma, we define

$$g_1(t) := \begin{cases} t, & \text{if } t \in (-\infty, L] \\ y_1^{-N} z^N(t), & \text{if } t \in \left[L, z^{-N}(L)\right]. \end{cases}$$

and extend it to the whole real line in order that $g_1 \in H$ and so that

$$g_1^{-1} y_1^N g_1 = z^N,$$

on $\left(-\infty, z^{-N}(L)\right]$. Finally, we build a map $g \in H$ such that

$$g(t) := g_0 g_1(t),$$

for $t \in \left(-\infty, z^{-N}(L)\right]$. We observe that the last part of g is defined inside in the final box since $t = z^{-N}(L) > R$ and

$$\begin{aligned} g\left(z^{-N}(L)\right) &= g_0 g_1\left(z^{-N}(L)\right) \\ &= g_0 y_1^{-N} z^N\left(z^{-N}(L)\right) \\ &= g_0 g_0^{-1} y^{-N} g_0(L) \\ &= y^{-N}\left(g_0(L)\right) \\ &> R. \end{aligned}$$

Moreover, by construction, g is a conjugator for y^N and z^N on $\left(-\infty, z^{-N}(L)\right]$, that is, $g = y^{-N} g z^N$. Therefore,

$$\begin{aligned} g(t) &= y^{-N} g z^N(t) \\ &= y^{-N} g_0 g_1 z^N(t) \\ &= y^{-N} g_0 z^N(t), \end{aligned}$$

since $g_1 z^N(t) = z^N(t)$ for every $t \in \left(-\infty, z^{-N}(L)\right]$.

If g is not a linear Möbius function on $[R, z^{-N}(L)]$, then g cannot be extended to a conjugator of y^N and z^N and the uniqueness of the shape of g (Proposition 5.18) says that continuing the Stair Algorithm will build a function that cannot be a conjugator and therefore, a conjugator with initial germ $\left(a^2, ab\right)$ cannot exist or it would coincide with g on $(-\infty, z^{-N}(L)]$. In the case that g is a linear Möbius function on $[R, z^{-N}(L)]$, we extend g to the whole real line by extending its linear piece on $[R, z^{-N}(L)]$. The map g that we build lies in H.

By Lemma 5.10 and Proposition 5.18, if there exists a conjugator between y^N and z^N, with initial germ $\left(a^2, ab\right)$, it must be equal to g. Then we just check if g conjugates y^N to z^N. If g conjugates y^N to z^N then, by Lemma 5.19, g is a conjugator between y and z, as desired. □

Remark 5.21. Let us suppose that $y, z \in H^< \cup H^>$. In order to be conjugate, the Lemma 5.8 says that their initial must be in the same conjugacy class in Aff $(\mathbb{R})$. Similarly they final germs must be in the same conjugacy class in Aff $(\mathbb{R})$. In other words, either both y and z are in $H^<$ or both are in $H^>$. Furthermore, since $g^{-1} y g = z$ if and only if $g^{-1} y^{-1} g = z^{-1}$, we are able to reduce the study to the case where they are both in $H^<$.

Remark 5.22. The stair algorithm for $H^<$ can be reversed. This means that we can apply it in order to build a candidate for a conjugator between $y, z \in H^>$. In other words, given an element $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$, we can determine whether or not there is a conjugator g with final germ $g_{+\infty} = (a^2, ab)$. The proof is similar. We just begin to construct g from the final box.

We observe that the proof of Stair Algorithm does not depend on the choice of g_0, the only requirements on it are that be linear on the initial box and $g_{0_{-\infty}} = (a^2, ab)$. Moreover, it gives a way to find candidate conjugators, if they exist, and we have chosen an initial germ. The next example shows that, in some cases, the candidate conjugator is, in fact, not a conjugator.

Example 5.23. Let us consider $y(t) = t + 1$ and the following homeomorphism

$$z(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t - 2}, & \text{if } t \in [0, 1]; \\ t + 1, & \text{otherwise.} \end{cases}$$

Notice that $y, z \in H^{>}$. Moreover, the initial and final germs of y and z are the same. Then,

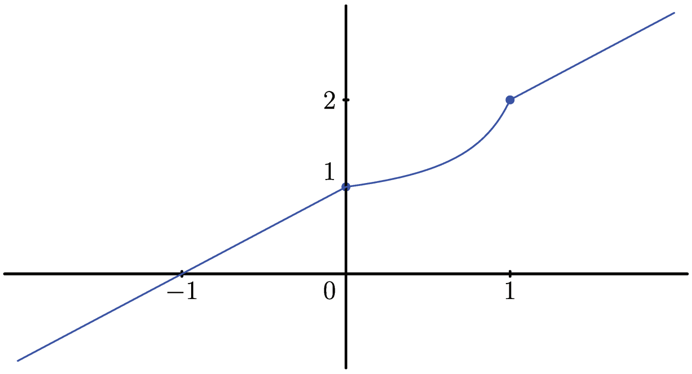

Figure 8 – Graph of z.

neither Lemma 5.7 nor Lemma 5.8 is an obstruction to the conjugacy of y and z. Now, we notice that that $L = 0$ and $R = 1$. Thus, we are able to use the Stair Algorithm in order to construct a potential conjugator g between y and z. If we get $N = 2$, we have

$$z^2(t) = \begin{cases} \dfrac{t-1}{\frac{3}{2}t - \frac{1}{2}}, & \text{if } t \in [-1, 0]; \\ \dfrac{\frac{5}{2}t - 4}{\frac{3}{2}t - 2}, & \text{if } t \in [0, 1]; \\ t + 2, & \text{otherwise.} \end{cases} \quad \text{and } z^{-2}(t) = \begin{cases} \dfrac{-\frac{1}{2}t + 1}{-\frac{3}{2}t + 1}, & \text{if } t \in [1, 2]; \\ \dfrac{-2t + 4}{-\frac{3}{2}t + \frac{5}{2}}, & \text{if } t \in [2, 3]; \\ t - 2, & \text{otherwise.} \end{cases}$$

and $\min\left\{z^2(0), y^2(0)\right\} > 1$. Considering $(1, 0) \in \mathrm{Aff}(\mathbb{R})$, we get the identity map as g_0, that is, $g_0(t) = t$. We observe that $g_0^{-1} y^{-2} g_0(t) = z^{-2}(t)$ for every $t \in (-\infty, 1]$. Now, we define

$$g_1(t) = \begin{cases} t, & \text{if } t \in (-\infty, 0]; \\ y_1^2 z^{-2}(t), & \text{if } t \in \left[0, z^2(0)\right], \end{cases}$$

where $y_1 := g_0^{-1} y g_0$. In other words, since $z^2(0) = 2$,

$$g_1(t) = \begin{cases} t, & \text{if } t \in (-\infty, 1]; \\ \dfrac{-\frac{7}{2}t + 3}{-\frac{3}{2}t + 1}, & \text{if } t \in [1, 2]. \end{cases}$$

We point out that $g_1^{-1} y_1^{-2} g_1$ coincides with z^{-2} on $(-\infty, 2]$. Thus, we extend g_1 for the whole real line as the identity and we define

$$g(t) := g_0 g_1(t) = \begin{cases} \dfrac{-\frac{7}{2}t + 3}{-\frac{3}{2}t + 1}, & \text{if } t \in [1, 2]; \\ t, & \text{otherwise.} \end{cases}$$

Its inverse is

$$g^{-1}(t) := \begin{cases} \dfrac{t - 3}{\frac{3}{2}t - \frac{7}{2}}, & \text{if } t \in [1, 2]; \\ t, & \text{otherwise.} \end{cases}$$

We notice that g is not a linear Möbius function on $[1,2]$. Thus, g cannot be a conjugator between y^2 and z^2. By Lemma 5.19, g cannot be a conjugator between y and z as well. In fact, a direct computation shows that, for every $t \in [1,2]$, we get

$$g^{-1}yg\,(t) = \frac{-5t+4}{-\frac{3}{2}t+1} \neq z\,(t)\,.$$

Now, let us look at an example where the Stair Algorithm build an element from H which in fact is a conjugator.

Example 5.24. Let us consider $y\,(t) = t+1$ and the map

$$z\,(t) = \begin{cases} \dfrac{2t+2}{\frac{3}{2}t+2}, & \text{if } t \in [-1,0]\,; \\ \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]\,; \\ t+1, & \text{otherwise.} \end{cases}$$

Notice that $y, z \in H^{>}$ and that their initial and final germs are the same. As in Example 5.23, neither Lemma 5.7 nor Lemma 5.8 is an obstruction to the conjugacy of y and z. Moreover, $L = -1$ and $R = 1$. Then, as Example 5.23, we are able to apply the Stair Algorithm in order to build a candidate conjugator $g \in H$ between y and z with a given initial germ. Let us consider $(1,1) \in \mathrm{Aff}\,(\mathbb{R})$. We observe that if $N = 3$, then

$$z^3\,(t) = \begin{cases} \dfrac{2t+6}{\frac{3}{2}t+5}, & \text{if } t \in [-3,-2]\,; \\ \dfrac{4t-6}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]\,; \\ t+3, & \text{otherwise.} \end{cases} \quad \text{and } z^{-3}\,(t) = \begin{cases} \dfrac{5t-6}{-\frac{3}{2}t+2}, & \text{if } t \in [0,1]\,; \\ \dfrac{-2t+6}{-\frac{3}{2}t+4}, & \text{if } t \in [3,4]\,; \\ t-3, & \text{otherwise.} \end{cases}$$

$\min\left\{z^3\,(-1)\,, y^3\,(0)\right\} > 1$. Let us start building a candidate conjugator between y^{-3} and z^{-3} as product of two functions. First, we define the map $g_0\,(t) := t+1$ on $(-\infty,-1]$ and extend it to the whole real line so that $g_0 \in H$. Now, define

$$g_1\,(t) := \begin{cases} t, & \text{if } t \in (-\infty,-1]\,; \\ y_1^3 z^{-3}\,(t)\,, & \text{if } t \in \left[-1, z^3\,(-1)\right], \end{cases}$$

where $y_1 := g_0^{-1} y g_0$, and extend it to the whole real line. In other words, after some calculations, we get

$$g_1\,(t) := \begin{cases} \dfrac{\frac{1}{2}t}{-\frac{3}{2}t+2}, & \text{if } t \in [0,1]\,; \\ t, & \text{otherwise.} \end{cases}$$

Thus, we define

$$g\,(t) := g_0 g_1\,(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]\,; \\ t+1, & \text{otherwise.} \end{cases}$$

Its inverse is

$$g^{-1}(t) = \begin{cases} \dfrac{2t-2}{\frac{3}{2}t-1}, & \text{if } t \in [1,2]; \\ t-1, & \text{otherwise.} \end{cases}$$

We point out that g is a Möbius linear function in $\left[1, z^{-3}(-1)\right]$, where $z^{-3}(-1) = 2$. Let us check if $g \in H$ is, in fact, a conjugator between y^{-3} and z^{-3}. If $t \in [0,1]$, then

$$g^{-1}y^{-3}g(t) = g^{-1}\left(\frac{t-2}{\frac{3}{2}t-2} - 3\right).$$

Since $\dfrac{t-2}{\frac{3}{2}t-2} - 3 < 1$, we get

$$g^{-1}y^{-3}g(t) = \left(\frac{t-2}{\frac{3}{2}t-2} - 3\right) - 1 = \frac{-5t+6}{\frac{3}{2}t-2} = z^{-3}(t).$$

Now, if $t \notin [0,1]$, we have

$$g^{-1}y^{-3}g(t) = g^{-1}((t+1)-3).$$

If $1 \leqslant t-2 \leqslant 2$, we get $t \in [3,4]$ and

$$g^{-1}y^{-3}g(t) = \frac{2(t-2)-2}{\frac{3}{2}(t-2)-1} = \frac{2t-6}{\frac{3}{2}t-4} = z^{-3}(t).$$

Otherwise,

$$g^{-1}y^{-3}g(t) = (t-2)-1 = t-3 = z^{-3}(t).$$

Thus, $g \in H$ is a conjugator between y^{-3} and z^{-3}. By Lemma 5.19, it follows that g is a conjugator between y and z.

5.2 Mather Invariants

In this section, we intend to define an invariant of conjugacy for elements from $H^{>}$, called *Mather Invariant*. The original invariant was constructed by John N. Mather in [20] as a conjugacy invariant of one-bump functions f from $\mathrm{Diff}_+(I)$ so that $f'(0) > 1$ and $f'(1) < 1$ and proved that two elements from $\mathrm{Diff}_+(I)$ satisfying these conditions are conjugated if, and only if, they have the same initial and final slopes and the same invariant. We seek to construct a similar type of invariant for H.

In the remainder of this section we assume $y, z \in H^{>}$ such that $y(t) = z(t) = t + b_0$ if $t \in (-\infty, L]$ and $z(t) = y(t) = t + b_1$ if $t \in [R, +\infty)$, for some suitable $b_0, b_1 > 0$, where L and R are real numbers negative and positive, respectively, sufficiently large. Let $N \in \mathbb{Z}_{>0}$ be large enough so that

$$y^N\left(\left(y^{-1}(L), L\right)\right) \subset (R, +\infty) \text{ and } z^N\left(\left(z^{-1}(L), L\right)\right) \subset (R, +\infty).$$

We intend to search for a map $s \in H$ such that

$$s\left(y^k(L)\right) = k,$$

for every $k \in \mathbb{Z}$. Let us define the following maps, which are linear Möbius functions

(i) $s_{-1} : [y^{-1}(L), L] \longrightarrow [-1, 0]$. Let us consider

$$s_{-1}(t) := \frac{\alpha t + \beta}{\gamma}$$

such that $\alpha\gamma = 1$, $s_{-1}(y^{-1}(L)) = -1$ and $s_{-1}(L) = 0$. From the two last conditions,

$$\alpha y^{-1}(L) + \beta = -\gamma \text{ and } \alpha L + \beta = 0.$$

Then $\beta = -\alpha L$ and

$$\begin{aligned} \alpha y^{-1}(L) - \alpha L = -\gamma &\iff \alpha(y^{-1}(L) - L) = -\gamma \\ &\iff \frac{\alpha}{\gamma} = \frac{1}{L - y^{-1}(L)}. \end{aligned}$$

Since $y^{-1}(L) = L - b_0$, we get

$$\frac{\alpha}{\gamma} = \frac{1}{b_0}.$$

Moreover,

$$\frac{\beta}{\gamma} = \frac{-\alpha L}{\gamma} = -\left(\frac{\alpha}{\gamma}\right)L = -\left(\frac{1}{b_0}\right)L = -\frac{L}{b_0}.$$

Then we choose α, β, γ such that

$$\alpha\gamma = 1, \quad \frac{\alpha}{\gamma} = \frac{1}{b_0} \text{ and } \frac{\beta}{\gamma} = -\frac{L}{b_0}.$$

That is

$$\begin{aligned} s_{-1} \colon \left[y^{-1}(L), L\right] &\longrightarrow [-1, 0] \\ t &\longmapsto \frac{t - L}{b_0}. \end{aligned}$$

It is worth pointing out that if $t \in \mathbb{R}$ is so that $\max\{t, y(t)\} \leqslant L$, we have

$$\begin{aligned} s_{-1}(y(t)) &= \frac{y(t) - L}{b_0} \\ &= \frac{t + b_0 - L}{b_0} \\ &= \frac{t - L}{b_0} + \frac{b_0}{b_0} \\ &= s_{-1}(t) + 1. \end{aligned}$$

Then we can extend the definition of s_{-1} to $(-\infty, L]$ so that $s_{-1}(y(t)) = s_{-1}(t) + 1$ when $\max\{t, y(t)\} \leqslant L$ and $t \in (-\infty, L]$.

(ii) $s_{N-1} : [y^{N-1}(L), y^N(L)] \longrightarrow [N-1, N]$. Let us consider

$$s_{N-1}(t) := \frac{\alpha t + \beta}{\gamma}$$

such that $\alpha\gamma = 1$, $s_{N-1}\left(y^{N-1}(L)\right) = N-1$ and $s_{N-1}\left(y^N(L)\right) = N$. From the last two conditions,

$$\alpha y^{N-1}(L) + \beta = \gamma(N-1) \text{ and } \alpha y^N(L) + \beta = \gamma N.$$

Hence,

$$\alpha y^{N-1}(L) + \beta = \alpha y^N(L) + \beta - \gamma \iff \alpha y^{N-1}(L) = \alpha y^N(L) - \gamma$$

Since $y^N(L) = y^{N-1}(L) + b_1$, we get

$$\begin{aligned}
\alpha y^{N-1}(L) = \alpha y^N(L) - \gamma &\iff \alpha y^{N-1}(L) = \alpha y^{N-1}(L) + \alpha b_1 - \gamma \\
&\iff \alpha b_1 - \gamma = 0 \\
&\iff \frac{\alpha}{\gamma} = \frac{1}{b_1}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\alpha}{\gamma} y^N(L) + \frac{\beta}{\gamma} = N &\iff \frac{1}{b_1} y^N(L) + \frac{\beta}{\gamma} = N \\
&\iff \frac{y^{N-1}(L) + b_1}{b_1} + \frac{\beta}{\gamma} = N \\
&\iff \frac{\beta}{\gamma} = N - 1 - \frac{y^{N-1}(L)}{b_1}.
\end{aligned}$$

Then we choose α, β, γ such that $\alpha\gamma = 1$,

$$\frac{\alpha}{\gamma} = \frac{1}{b_1} \text{ and } \frac{\beta}{\gamma} = N - 1 - \frac{y^{N-1}(L)}{b_1}.$$

That is

$$\begin{aligned}
s_{N-1} \colon [y^{N-1}(L), y^N(L)] &\longrightarrow [N-1, N] \\
t &\longmapsto \frac{t - y^{N-1}(L)}{b_1} + N - 1.
\end{aligned}$$

We point out that if $t \in \mathbb{R}$ is such that $\min\{t, y(t)\} \geqslant y^{N-1}(L)$, we have

$$\begin{aligned}
s_{N-1}(y(t)) &= \frac{y(t) - y^{N-1}(L)}{b_1} + N - 1 \\
&= \frac{t + b_1 - y^{N-1}(L)}{b_1} + N - 1 \\
&= \frac{t - y^{N-1}(L)}{b_1} + N - 1 + \frac{b_1}{b_1} \\
&= s_{N-1}(t) + 1.
\end{aligned}$$

Then we can extend the definition of s_{-1} to $\left[y^{N-1}(L), +\infty\right)$ so that $s_{N-1}(y(t)) = s_{N-1}(t) + 1$ when $\min\{t, y(t)\} \geqslant y^{N-1}(L)$ and $t \in \left[y^{N-1}(L), +\infty\right)$.

(iii) For any interval $[y^j(L), y^{j+1}(L)]$, where $j = 0, 1, 2, \ldots, N-3, N-2$, we want to define $s_j\colon [y^j(L), y^{j+1}(L)] \longrightarrow [j, j+1]$ as a Möbius function. Let us consider

$$s_j(t) := \frac{\alpha t + \beta}{\gamma}$$

such that $\alpha\gamma = 1$, $s_j\left(y^j(L)\right) = j$ and $s_j\left(y^{j+1}(L)\right) = j+1$. From the two last conditions,

$$\alpha y^j(L) + \beta = \gamma j \text{ and } \alpha y^{j+1}(L) + \beta = \gamma j + \gamma.$$

So,

$$\begin{aligned}\alpha y^{j+1}(L) + \beta = \alpha y^j(L) + \beta + \gamma &\iff \alpha(y^{j+1}(L) - y^j(L)) = \gamma \\ &\iff \frac{\alpha}{\gamma} = \frac{1}{y^{j+1}(L) - y^j(L)}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\alpha}{\gamma} y^j(L) + \frac{\beta}{\gamma} = j &\iff \frac{y^j(L)}{y^{j+1}(L) - y^j(L)} + \frac{\beta}{\gamma} = j \\ &\iff \frac{\beta}{\gamma} = j - \frac{y^j(L)}{y^{j+1}(L) - y^j(L)}.\end{aligned}$$

Then we choose α, β, γ such that $\alpha\gamma = 1$,

$$\frac{\alpha}{\gamma} = \frac{1}{y^{j+1}(L) - y^j(L)} \text{ and } \frac{\beta}{\gamma} = j - \frac{y^j(L)}{y^{j+1}(L) - y^j(L)}.$$

Then we have

$$\begin{aligned} s_j\colon [y^j(L), y^{j+1}(L)] &\longrightarrow [j, j+1] \\ t &\longmapsto \frac{t - y^j(L)}{y^{j+1}(L) - y^j(L)} + j. \end{aligned}$$

Then we define the map s as

$$\begin{aligned} s\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ t \longmapsto s(t) &:= \begin{cases} s_{-1}(t), & \text{if } t \in (-\infty, L] \\ s_j(t), & \text{if } t \in [y^j(L), y^{j+1}(L)] \\ s_{N-1}(t), & \text{if } t \in [y^{N-1}(L), +\infty) \end{cases} \end{aligned}$$

where $j = 0, 1, 2, \ldots, N-3, N-2$. Notice that, in virtue of Lemma 4.6, $y^j(L)$ is a fixed point of some hyperbolic element from $\mathrm{PSL}_2(\mathbb{R})$. Then $s \in H$. Moreover, it is clear that, for all $k \in \mathbb{Z}$, s satisfies

$$s\left(y^k(L)\right) = k.$$

Now, let us define

$$\overline{y} := sys^{-1} \text{ and } \overline{z} := szs^{-1}.$$

Since $\overline{y}$ and $\overline{z}$ are conjugates of y and z, respectively, by s, we get that both functions are well-defined and lie in H. Note that given $t \in (-\infty, 0]$,

$$\begin{aligned}\overline{y}\,(t) = \overline{z}\,(t) = szs^{-1}\,(t) &= s(y(b_0 t + L)) = s((b_0 t + L) + b_0) \\ &= \frac{[(b_0 t + L) + b_0] - L}{b_0} = \frac{b_0\,(t+1)}{b_0} \\ &= t + 1\end{aligned}$$

and if $t \in [N-1, +\infty)$

$$\begin{aligned}\overline{y}\,(t) = \overline{z}\,(t) = szs^{-1}\,(t) &= sy(b_1 t - b_1 N + b_1 + y^{N-1}\,(L)) \\ &= s((b_1 t - b_1 N + b_1 + y^{N-1}\,(L)) + b_1) \\ &= \frac{[(b_1 t - b_1 N + b_1 + y^{N-1}\,(L)) + b_1] - y^{N-1}\,(L)}{b_1} + N - 1 \\ &= (t - N + 2) + N - 1 \\ &= t + 1.\end{aligned}$$

Also notice that $\overline{y}^N \in H$, since $\overline{y}^N = sy^N s^{-1}$.

Now let us consider the following relations: $t \sim_0 y^u\,(t)\,(= t + ub_0)$ for some $u \in \mathbb{Z}$, if $t \in (-\infty, L]$, and $t \sim_1 y^v\,(t)\,(= t + vb_1)$ for some $v \in \mathbb{Z}$, if $t \in [N-1, +\infty)$. Note that we can create two circles

$$C_0 := (-\infty, L]\,/ \sim_0 \quad \text{and} \quad C_1 := \left[y^{N-1}\,(L)\,, +\infty\right) / \sim_1,$$

which have length $|b_0|$ and $|b_1|$, respectively. Up to replacing y by $\overline{y}$, we get the circles C_0 and C_1 now have length 1:

$$C_0 = (-\infty, 0]/\{t \sim t+1\} \;\text{ and }\; C_1 = [N-1, +\infty)\,/\{t \sim t+1\}.$$

Then let us consider the natural projections

$$p_0 \colon (-\infty, 0] \longrightarrow C_0 \;\text{ and }\; p_1 \colon [N-1, +\infty) \longrightarrow C_1$$

and let us restrict $\overline{y}^N$ to the interval $[-1, 0]$, so that p_0 surjects it onto C_0. Since N is sufficiently large so that $\overline{y}^N((\overline{y}^{-1}\,(L)\,, L)) \subset [R, +\infty)$, then $\overline{y}^N$ maps $[-1, 0]$ to $[R, +\infty)$. Then $\overline{y}^N$ induces the following map by passing to quotients

$$\begin{aligned}\overline{y}^\infty \colon C_0 &\longrightarrow C_1 \\ [t] &\longmapsto \overline{y}^\infty\,([t]) = \left[\overline{y}^N\,(t)\right]\end{aligned}$$

making the following diagram commutes

$$\begin{tikzcd} {[-1,0]} \arrow[r, "\overline{y}^N"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\circlearrowright"] & {[N-1,+\infty)} \arrow[d, "p_1"] \\ C_0 \arrow[r, dashed, "\overline{y}^\infty"'] & C_1 \end{tikzcd}$$

Notice that the map $\overline{y}^{\infty}$ does not depend on the specific chosen value of N. If $m \geqslant N$, we can write $\overline{y}^{m}(t)$ as $\overline{y}^{m-N}\left(\overline{y}^{N}(t)\right)$, where $\overline{y}^{N}(t) \in (R, +\infty)$ and we have

$$\begin{aligned}
\overline{y}^{N}(t) &\sim \overline{y}^{N}(t) + 1 = \overline{y}\left(\overline{y}^{N}(t)\right) \\
&\sim \overline{y}\left(\overline{y}^{N}(t)\right) + 1 = \overline{y}^{2}\left(\overline{y}^{N}(t)\right) \\
&\sim \overline{y}^{2}\left(\overline{y}^{N}(t)\right) + 1 \\
&\quad\vdots \\
&\sim \overline{y}^{m-N}\left(\overline{y}^{N}(t)\right) = \overline{y}^{m}(t)\,.
\end{aligned}$$

Likewise, we define the map $\overline{z}^{\infty}$. Notice that these maps are elements of $H(C_0, C_1)$, which it is the group of all homeomorphisms which are piecewise-Möbius from the circle C_0 to the circle C_1.

Definition 5.25. The maps $\overline{y}^{\infty}$ and $\overline{z}^{\infty}$ are called **Mather Invariants** of y and z, respectively.

Since we are considering $\overline{y}(t) = \overline{z}(t) = t+1$ for $t \in (-\infty, 0) \cup (N-1, +\infty)$, let us consider a conjugator $g \in H$ such that $g(t) = \alpha_0^2 t + \alpha_0\beta_0$ around $-\infty$ and $g(t) = \alpha_1^2 t + \alpha_1\beta_1$ around $+\infty$. Since $gz = yg$, for t negative sufficiently large we get

$$g\overline{z}(t) = g(t+1) = \alpha_0^2(t+1) + \alpha_0\beta_0 = \alpha_0^2 t + \alpha_0^2 + \alpha_0\beta_0$$

and

$$\overline{y}g(t) = \overline{y}(\alpha_0^2 t + \alpha_0\beta_0) = \alpha_0^2 t + \alpha_0\beta_0 + 1.$$

Then

$$\alpha_0^2 t + \alpha_0^2 + \alpha_0\beta_0 = \alpha_0^2 t + \alpha_0\beta_0 + 1$$

which implies $\alpha_0^2 = 1$. Then we get $\alpha_0 = \pm 1$. Similarly, we get $\alpha_1 = \pm 1$. To simplify notations and relate similar results in [9, 21], we call $\ell := \beta_0$ and $m := \beta_1$. Moreover, for $t \in (-\infty, 0) \cup (N-1, +\infty)$ we have $g(t+1) = g(t) + 1$.

Since $z = g^{-1}yg$, we get $z^N = g^{-1}y^N g$ and so $gz^N = y^N g$. Conjugating by s,

$$s(gz^N)s^{-1} = s(y^N g)s^{-1}$$

or

$$(sgs^{-1})(sz^N s^{-1}) = (sy^N s^{-1})(sgs^{-1}).$$

Then we have

$$\overline{g}\,\overline{z}^N = \overline{y}^N \overline{g},$$

where $\overline{g}(t) = g(t)$, if $t \in (-\infty, 0] \cup [N-1, +\infty)$.

Let us keep the following commutative diagram in mind

$$\begin{tikzcd}
& & {[-1,0]} \arrow[r, "\overline{z}^N"] \arrow[dll, "\overline{g}"'] \arrow[dd, "p_0"'] & {[N-1,+\infty)} \arrow[dll, "\overline{g}"] \arrow[dd, "p_1"] \\
{[-1,0]} \arrow[r, "\overline{y}^N"'] \arrow[dd, "p_0"'] & {[N-1,+\infty)} \arrow[dd, "p_1"] & & \\
& & C_0 \arrow[r, "\overline{z}^\infty"] \arrow[dll, "v_{0,\ell}"'] & C_1 \arrow[dll, "v_{1,m}"] \\
C_0 \arrow[r, "\overline{y}^\infty"'] & C_1 & &
\end{tikzcd}$$

We want to show that the bottom square of the diagram holds, knowing that every other side holds and where

$$\begin{aligned} v_{0,\ell} \colon \ C_0 &\longrightarrow C_0 & \qquad v_{1,m} \colon \ C_1 &\longrightarrow C_1 \\ [t] &\longmapsto [t+\ell] = p_0\overline{g}\,(t) & [t] &\longmapsto [t+m] = p_1\overline{g}\,(t) \end{aligned}$$

and where we recall that $\ell := \beta_0$ and $m := \beta_1$. By definition, $v_{0,\ell}$ and $v_{1,m}$ are rotations of the circles C_0 and C_1, respectively.

Since $\overline{gz}^N = \overline{y}^N\overline{g}$, $p_1\overline{y}^N = \overline{y}^\infty p_0$ and $p_1\overline{z}^N = \overline{z}^\infty p_0$, we observe that

$$\overline{y}^\infty v_{0,\ell} p_0 = \overline{y}^\infty p_0 \overline{g} = p_1 \overline{y}^N \overline{g} = p_1 \overline{gz}^N = v_{1,m} p_1 \overline{z}^N = v_{1,m} \overline{z}^\infty p_0.$$

Hence,

$$\overline{y}^\infty v_{0,\ell}\left([t]\right) = \overline{y}^\infty v_{0,\ell} p_0\,(t) = v_{1,m}\overline{z}^\infty p_0\,(t) = v_{1,m}\overline{z}^\infty\left([t]\right),$$

that is,

$$v_{1,m}\overline{z}^\infty = \overline{y}^\infty v_{0,\ell}. \tag{5.4}$$

We have thus proved the forward direction of the following result.

Theorem 5.26. *Let $y, z \in H^>$ be such that $y\,(t) = z\,(t) = t + b_0$ for $t \in (-\infty, L]$ and $y\,(t) = z\,(t) = t + b_1$ for $t \in [R, +\infty)$ and let $\overline{y}^\infty, \overline{z}^\infty \colon C_0 \longrightarrow C_1$ be the corresponding Mather invariants. Then y and z are conjugate in H if and only if $\overline{y}^\infty$ and $\overline{z}^\infty$ differ by rotations $v_{0,\ell}$ and $v_{1,m}$ of the domain and range circles, for some $\ell, m \in \mathbb{R}$.*

$$\begin{tikzcd}
C_0 \arrow[r, "\overline{z}^\infty"] \arrow[d, "v_{0,\ell}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[d, "v_{1,m}"] \\
C_0 \arrow[r, "\overline{y}^\infty"'] & C_1
\end{tikzcd}$$

Definition 5.27. Given a map $f \colon S^1 \longrightarrow S^1$, a **lift** of f is a map $F \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $F\,(t+1) = F\,(t) + 1$ for all $t \in \mathbb{R}$ and F induces f.

Proof. We need to prove the converse statement of the theorem. Let us assume that there are $\ell, m \in \mathbb{R}$ such that

$$\overline{y}^{\infty} v_{0,\ell}\left([t]\right) = v_{1,m}\overline{z}^{\infty}\left([t]\right).$$

Let us choose $g_0 \in H$ which is linear in the initial box with a initial germ $g_{0_{-\infty}} = (1, \ell)$. Then let us define a map g in order it being the pointwise limit

$$g\left(t\right) := \lim_{n \to +\infty} y^n g_0 z^{-n}\left(t\right).$$

By the Stair Algorithm Theorem 5.20, we have $gz = yg$. The Stair Algorithm guarantees that $g \in \mathrm{PPSL}_2(\mathbb{R})$, but we need to show that $g \in H$. By construction, g has finitely many breakpoints in $(-\infty, N-1]$. Conjugating both sides of the equation $gz = yg$ by s, we get

$$s(gz)s^{-1} = s(yg)s^{-1},$$

or

$$(sgs^{-1})(szs^{-1}) = (sys^{-1})(sgs^{-1}),$$

that is

$$\overline{gz} = \overline{yg}.$$

For $t \geqslant N-1$, we have

$$\overline{y}\left(t\right) = \overline{z}\left(t\right) = t + 1.$$

From

$$\overline{gz}\left(t\right) = \overline{g}(\overline{z}\left(t\right)) = \overline{g}\left(t+1\right)$$

and

$$\overline{yg}\left(t\right) = \overline{y}(\overline{g}\left(t\right)) = \overline{g}\left(t\right) + 1$$

we get

$$\overline{g}\left(t+1\right) = \overline{g}\left(t\right) + 1.$$

Analogously, for $t \leqslant 0$ we have $\overline{g}\left(t+1\right) = \overline{g}\left(t\right) + 1$. Then we can pass the equation

$$\overline{gz}^N = \overline{y}^N\overline{g}$$

to quotients and we obtain

$$\overline{g}_{ind}\overline{z}^{\infty}\left([t]\right) = \overline{y}^{\infty} v_{0,\ell}\left([t]\right),$$

where $\overline{g}_{ind}$ exists, and it is well-defined, since $\overline{g}\left(t+1\right) = \overline{g}\left(t\right) + 1$ for every $t \in [N-1, +\infty)$ and so it passes to quotients. By our hypothesis,

$$\overline{g}_{ind}\overline{z}^{\infty}\left([t]\right) = v_{1,m}\overline{z}^{\infty}\left([t]\right),$$

and so, by the cancellation law, we have

$$\overline{g}_{ind} = v_{1,m}.$$

Hence, as before, we have shown that the following commutative diagram holds

$$
\begin{tikzcd}
 & {[-1,0]} \arrow[rr, "\overline{z}^N"] \arrow[dl, "\overline{g}"'] \arrow[dd, "p_0"] & & {[N-1,+\infty)} \arrow[dl, "\overline{g}"] \arrow[dd, "p_1"] \\
{[-1,0]} \arrow[rr, "\overline{y}^N"'] \arrow[dd, "p_0"'] & & {[N-1,+\infty)} \arrow[dd, "p_1"] & \\
 & C_0 \arrow[rr, "\overline{z}^\infty"] \arrow[dl, "v_{0,\ell}"'] & & C_1 \arrow[dl, "\overline{g}_{ind}"] \\
C_0 \arrow[rr, "\overline{y}^\infty"'] & & C_1 &
\end{tikzcd}
$$

and where $\overline{g}_{ind}$ is a rotation by m of the circle C_1 and $\overline{g}$ induces $v_{1,m}$ on the interval $[N-1,+\infty)$ or, in other words, $\overline{g}$ is lift of $v_{1,m}$. To finish the proof, we need to see that $\overline{g}$ is a linear Möbius map on $[N-1,+\infty)$, which will mean that $\overline{g} \in H$.

From the previous commutative diagram we get

$$v_{1,m}p_1 = p_1\overline{g}.$$

and so, for $t \in [N-1,+\infty)$, we have

$$[\overline{g}(t)] = p_1\overline{g}(t) = v_{1,m}p_1(t) = v_{1,m}([t]) = [t+m].$$

By definition of the equivalence relation and the fact that $\overline{g}$ is a continuous function, we have that there exists some $r \in \mathbb{Z}$ such that,

$$\overline{g}(t) := t + m + r,$$

for all $t \in [N-1,+\infty)$. Therefore, $\overline{g} \in H$. $\square$

Remark 5.28. Mather invariants and Theorem 5.26 give a conjugacy invariant for functions $y, z \in H^{>}$ for functions which are translations around $\pm\infty$ and thus they can be used to give an obstruction to the success of the Stair Algorithm which builds candidate conjugators, which actually may never be conjugators in H if y and z are not conjugate.

Example 5.29. Let us consider $y(t) = t + 1$ and the map

$$z(t) = \begin{cases} \dfrac{2t+2}{\frac{3}{2}t+2}, & \text{if } t \in [-1,0]; \\ \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]; \\ t+1, & \text{otherwise} \end{cases}$$

seen in Example 5.24. By Example 5.24, we know that $y, z \in H^{>}$ and that they are conjugate each to other by the following conjugator constructed by the Stair Algorithm

Theorem 5.20

$$g(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]; \\ t+1, & \text{otherwise.} \end{cases}$$

We observe that $L = -1$ and $R = 1$. Moreover, $y^{-1}(-1) = z^{-1}(-1) = -2$. Considering $N = 3$, we have

$$y^3((-2,-1)) \subset (1,+\infty) \text{ and } z^3((-2,-1)) \subset (1,+\infty).$$

We define $s(t) := t+1$ for every $t \in \mathbb{R}$. Thus, $\overline{y}(t) = y(t)$ and

$$\overline{z}(t) = \begin{cases} \dfrac{\frac{7}{2}t+\frac{1}{2}}{\frac{3}{2}t+\frac{1}{2}}, & \text{if } t \in [0,1]; \\ \dfrac{\frac{5}{2}t-\frac{13}{2}}{\frac{3}{2}t-\frac{7}{2}}, & \text{if } t \in [1,2]; \\ t+1, & \text{otherwise.} \end{cases}$$

We stress out that $\overline{y}(t) = \overline{z}(t)$ for each $t \in (-\infty,0] \cup [2,+\infty)$. Defining the circles $C_0 := (-\infty,0]/t \sim t+1$ and $C_0 := [2,+\infty)/t \sim t+1$, the calculations of $\overline{y}^3$ and $\overline{z}^3$ performed in Example 5.24 yield that the Mather invariant of y is

$$\begin{aligned} \overline{y}^\infty \colon C_0 &\longrightarrow C_1 \\ [t] &\longmapsto [t+3], \end{aligned}$$

while the one of z is similar, that is,

$$\begin{aligned} \overline{z}^\infty \colon C_0 &\longrightarrow C_1 \\ [t] &\longmapsto [t+3]. \end{aligned}$$

Since C_1 has length one, we notice that $t+3 \sim t+2 \sim t+1 \sim t$, and so both Mather invariants are the "identity" map, that is, $\overline{y}^\infty([t]) = \overline{z}^\infty([t]) = [t]$. Moreover, we notice that $\overline{g}(t) = t+1$ on $(-\infty,0] \cup [2,+\infty)$. Thus, $\ell = m = 1$ and the following rotations of the circles C_0 and C_1 are

$$v_{0,1}([t]) = [t+1] \text{ and } v_{1,1}([t]) = [t+1],$$

respectively.

The next example show us that Mather invariant and Theorem 5.26 are, in fact, an obstruction to the Stair Algorithm.

Example 5.30. Let us recall that we are not able to build a conjugator with initial germ $(1,0) \in \mathrm{Aff}(\mathbb{R})$ between $y(t) = t+1$ and

$$z(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]; \\ t+1, & \text{otherwise.} \end{cases}$$

by applying the Stair Algorithm Theorem 5.20, as seen in Example 5.23. In fact, in Example 5.23 we found the following map

$$g(t) = \begin{cases} \dfrac{-\frac{7}{2}t+3}{-\frac{3}{2}t+1}, & \text{if } t \in [1,2]; \\ t, & \text{otherwise.} \end{cases}$$

Let us now define Mather invariants to verify that y and z cannot be conjugate. In order to define Mather invariants of both maps y and z, we consider $N = 2$. We have that

$$y^2((-1,0)) \subset (1,+\infty) \text{ and } z^2((-1,0)) \subset (1,+\infty).$$

We point out that we do not need to construct a map $s \in H$, since y and z already coincide on $(-\infty, 0]$ and $[N-1, +\infty)$, where we observe that $N-1 = 1$, and they are translations by 1 on these intervals. Thus, we can define the circles $C_0 := (-\infty, 0]/t \sim t+1$ and $C_1 := [1,+\infty)/t \sim t+1$. Thus, the Mather invariants of y and z are, respectively,

$$\begin{aligned} \overline{y}^\infty \colon\ C_0 &\longrightarrow C_1 \\ [t] &\longmapsto [t+2] \end{aligned} \qquad\qquad \begin{aligned} \overline{z}^\infty \colon\ C_0 &\longrightarrow C_1 \\ [t] &\longmapsto \left[\frac{t-1}{\frac{3}{2}t-\frac{1}{2}}\right]. \end{aligned}$$

Since the circle C_1 has length one, then $t+2 \sim t+1 \sim t$. This implies that the Mather invariant $\overline{y}^\infty$ is the "identity" map $C_0 \to C_1$ given by $\overline{y}^\infty([t]) = [t]$. Now, we notice that we are not able to such map from the Mather invariant of z by rotations. Indeed, if we take the rotation $v_{1,m}([t]) = [t+m]$, we get

$$v_{1,m}\overline{z}^\infty([t]) = \left[\frac{t-1}{\frac{3}{2}t-\frac{1}{2}} + m\right] = \left[\frac{(3m+2)t-(m+2)}{3t-1}\right],$$

which is easily seen to be different from the map $\overline{y}^\infty v_{0,\ell}([t]) = [t+\ell]$, regardless of any choice of ℓ, m.

5.3 Changing from unbounded to bounded intervals

Elements in Monod's group H have a fixed point at ∞. It is sometimes useful to change coordinates and turn ∞ into a point of the real line.

We want to make a change of coordinates which we be able to take an element $g \in H$ such that $g(\infty) = \infty$ and change it in such a way that $g(0) = 0$. This change will be made by conjugation, that is, we find a function $f \in H$ such that $fgf^{-1}(0) = 0$.

Let $g \in H$ be such that $g(0) \neq 0$. Consider the Möbius function

$$f(t) = \frac{-1}{t}$$

and notice that $f \in \mathrm{PPSL}_2(\mathbb{R})$. By construction, $f^{-1}gf \in \mathrm{PPSL}_2(\mathbb{R})$ and, since $f(0) = \infty$, then $f^{-1}gf(0) = 0$ and $f^{-1}gf(\infty) \neq \infty$ as requested. If $g(0) = 0$, choose $\ell \in \mathbb{R}$ such that

$g(\ell) \neq \ell$, consider $f(t) = \dfrac{1}{-t+\ell}$ and construct $f^{-1}gf$ which will satisfy $f^{-1}gf(\ell) = \ell$ and $f^{-1}gf(\infty) \neq \infty$. We can then conjugate $f^{-1}gf$ by a Möbius function h such that $h(0) = \ell$ and $h(\ell) = 0$ to obtain a function fixing 0 but not ∞.

It is interesting to see how the affine group of the initial germ of $g \in H$ (an element of the affine group "at ∞") changes when we conjugate g by f as above. This will become the initial germ at 0.

We have if $g(t) = \dfrac{a}{d}t + \dfrac{b}{d}$ for t negative sufficiently large, then

$$\widetilde{g}(t) := f^{-1}gf(t) = \frac{-dt}{bt-a}.$$

Moreover, we can rewrite as follows

$$\widetilde{g}(t) = \frac{\left(\dfrac{d}{a}\right)t}{\left(\dfrac{-b}{a}\right)t+1}.$$

Notice we still have a positive slope and if we consider two functions $g, h \in H$ and move the fixed point $-\infty$ to 0, we have

$$\widetilde{g}(t) = \frac{\left(\dfrac{d}{a}\right)t}{\left(\dfrac{-b}{a}\right)t+1} \quad \text{and} \quad \widetilde{h}(t) = \frac{\left(\dfrac{z}{u}\right)t}{\left(\dfrac{-v}{u}\right)t+1}.$$

The composition is

$$\widetilde{hg}(t) = \frac{\left(\dfrac{dz}{au}\right)t}{\left(-\dfrac{b}{a} - \dfrac{dv}{au}\right)t+1}.$$

Notice that if we the pairs $\left(\dfrac{d}{a}, \dfrac{-b}{a}\right)$ and $\left(\dfrac{z}{u}, \dfrac{-v}{u}\right)$ are elements of the affine group $\mathbb{R}_{>0} \ltimes \mathbb{R}$. Moreover,

$$\left(\frac{d}{a}, \frac{-b}{a}\right)\left(\frac{z}{u}, \frac{-v}{u}\right) = \left(\frac{dz}{au}, -\frac{b}{a} - \frac{dv}{au}\right),$$

which is exactly the coefficients of the composition. This suggests that a result similar to Lemma 5.14 holds in this case too.

Theorem 5.31. *Let $H|_{[0,1]}$ be the group of all piecewise-Möbius transformations of $[0,1]$ to itself with finitely many breakpoints. There exists a group isomorphism $\nabla : H \to H|_{[0,1]}$.*

Idea of the proof. We only construct the isomorphism. If k is a positive integer, use Lemma 4.10 to construct a function $h_k(t)$ such that $h_k(-k) = \dfrac{1}{k}$ and $h_k(-1) = \dfrac{k-2}{k-1}$. Now consider the map

$$f_k(t) = \begin{cases} \dfrac{-1}{t} & t \in (-\infty, -k] \\ h_k(t) & t \in [-k, -1] \\ \dfrac{t+k-1}{t+k} & t \in [-1, +\infty) \,. \end{cases}$$

Then for each positive integer k, the map $\nabla_k : H \to H|_{[0,1]}$ defined by $\nabla_k(g) = f_k^{-1} g f_k$ is a group isomorphism. $\square$

Remark 5.32. The map ∇_k switches $-\infty$ with 0 and $+\infty$ with 1 and allows us to study maps in Monod's group from a bounded point of view which will be useful in the next chapter.

Remark 5.33. A straightforward calculation shows that, if $y, z \in H$ are such that $y_{-\infty} = z_{-\infty}$ and $y_{+\infty} = z_{+\infty}$, then the initial and final linearity boxes of y and z correspond to initial and final *Möbius boxes* of $\nabla(y)$ and $\nabla(z)$, where the images coincide and are Möbius and a conjugator has to be Möbius.

6. Centralizers

In this chapter, we deal with the centralizers of elements from H as application of the techniques introduced in Chapter 5. We recall that given a group G and an element $g \in G$, the centralizer of g is the set of all elements of G which commute with g, that is,

$$C_H(z) = \{g \in H \mid zg = gz\}.$$

In order to do this, we first perform some calculations for centralizers of elements in $\mathrm{Aff}(\mathbb{R})$. After this, we classify the centralizers of elements from H. All results in this chapter are new.

Given $y, z, g \in H$, we recall the following general facts about centralizers

1. If $y^k = z$ for some $k \in \mathbb{Z}$, then $y \in C_H(z)$;
2. $C_H(z) \cong (C_H(z))^g = C_H(z^g)$.

6.1 Centralizers in $\mathrm{Aff}(\mathbb{R})$

In this section, we deal with centralizers of elements from $\mathrm{Aff}(\mathbb{R})$. Since Lemma 5.14 gives a monomorphism between elements from the centralizer subgroup of $z \in H$ with their initial germs, which are elements from $\mathrm{Aff}(\mathbb{R})$, it makes sense to investigate centralizers in $\mathrm{Aff}(\mathbb{R})$.

Let us consider the monomorphism φ_z defined in Lemma 5.14 and we relabel its image by

$$\mathcal{A}_z := \varphi_z(C_H(z)).$$

By Lemma 4.17, H is torsion-free and so, if $z \in C_H(z)$, we have that $\langle z \rangle$ is an infinite subgroup of $C_H(z)$. By the injectivity of φ_z, we get that $\mathcal{A}_z$ is an infinite group. Moreover, using the injectivity of φ_z, we can define

$$\begin{aligned} \Psi_z\colon \mathcal{A}_z &\longrightarrow C_H(z) \\ (a^2, ab) &\longmapsto \Psi_z\left(a^2, ab\right) := \varphi_z^{-1}\left(\left(a^2, ab\right)\right), \end{aligned}$$

which is a group isomorphism.

Now consider an element $(a, b) \in \mathrm{Aff}(\mathbb{R})$ and its centralizer. If $a \neq 1$ and we consider $(c, d) \in C_{\mathrm{Aff}(\mathbb{R})}(a, b)$, then

$$(c, d) = \left(a^{-1}, -a^{-1}b\right)(c, d)(a, b) = \left(c, -a^{-1}b + a^{-1}d + a^{-1}bc\right)$$

if and only if

$$d = \frac{b(c-1)}{a-1}.$$

So for $a \neq 1$

$$C_{\mathrm{Aff}(\mathbb{R})}(a,b) = \left\{ \left(c, \frac{b(c-1)}{a-1} \right) \in \mathrm{Aff}(\mathbb{R}) \mid c \in (\mathbb{R}_{>0}, \cdot) \right\} \cong (\mathbb{R}_{>0}, \cdot) \cong (\mathbb{R}, +)\,. \tag{6.1}$$

where the last isomorphism is given by the usual exponential map $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$, $\exp(x) = e^x$. If $a = 1$, we get

$$(c,d) = (1,-b)(c,d)(1,b) = (c, b(c-1)+d)$$

which implies that

$$d = b(c-1) + d \Rightarrow b(c-1) = 0 \Rightarrow b = 0 \text{ or } c = 1.$$

If $b = 0$,

$$C_{\mathrm{Aff}(\mathbb{R})}(1,0) = \mathrm{Aff}\,(\mathbb{R})\,.$$

If $b \neq 0$,

$$C_{\mathrm{Aff}(\mathbb{R})}(1,b) = \{(1,d) \mid d \in (\mathbb{R},+)\} \cong (\mathbb{R},+)\,.$$

This information will be useful to us in order to study centralizers of elements from H.

6.2 Centralizers in H

In order to study the centralizers of the elements from H, we will divide it in some cases. We will first consider a $z \in H$ without breakpoints, that is z must be an affine map. Then let us consider that $z\,(t) = a^2t + ab$ for all $t \in \mathbb{R}$. Then we have the following result, in case of $a \neq \pm 1$.

Proposition 6.1. *Let $z \in H$ be so that $z\,(t) = a^2t + ab$, for all $t \in \mathbb{R}$, with $a \neq \pm 1$. Then $C_H\,(z) \cong (\mathbb{R},+)$.*

Proof. First, we notice that in this case, $z_{-\infty} = z_{+\infty} = (a^2, ab)$. From Equation (6.1), we already know that

$$C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab) = \left\{ \left(c, \frac{ab(c-1)}{a^2-1} \right) \mid c \in (\mathbb{R}_{>0}, \cdot) \right\}.$$

For $c > 0$, we observe that the element

$$g\,(t) = ct + \frac{ab(c-1)}{a^2-1} = (\sqrt{c})^2 t + \sqrt{c} \left(\frac{ab(c-1)}{\sqrt{c}(a^2-1)} \right)$$

belongs to H. Moreover, a direct calculation shows that g commutes with z and so $g \in C_H\,(z)$. Considering the map φ_z from Lemma 5.14, we have

$$\varphi_z(C_H\,(z)) \leqslant C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab).$$

Let us consider the subset T from H given by

$$T := \left\{ f \in H \mid f(t) = ct + \frac{ab(c-1)}{a^2-1},\ \forall\, t \in \mathbb{R} \right\}.$$

We notice that T is a subgroup of H. In particular, T is a subgroup of $C_H(z)$. Then by definition from φ_z, we have

$$\varphi_z(T) = \left\{ \left(c, \frac{ab(c-1)}{a^2-1} \right) \mid c \in (\mathbb{R}_{>0}, \cdot) \right\} = C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab).$$

and, since

$$\varphi_z(T) \leqslant \varphi_z(C_H(z)) \leqslant C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab),$$

we get

$$\varphi_z(C_H(z)) = \left\{ \left(c, \frac{ab(c-1)}{a^2-1} \right) \mid c \in (\mathbb{R}_{>0}, \cdot) \right\}.$$

Since φ_z is a group monomorphism, we have

$$C_H(z) \cong \varphi_z(C_H(z)) = \left\{ \left(c, \frac{ab(c-1)}{a^2-1} \right) \mid c \in (\mathbb{R}_{>0}, \cdot) \right\}.$$

Therefore,

$$C_H(z) \cong (\mathbb{R}, +)$$

where the last isomorphism follows from the discussion at the end of Section 6.1. □

As consequence of the preceding proposition, given $y \in H$ so that y is a conjugate map of an affine map, its centralizer is isomorphic to $(\mathbb{R}, +)$.

Corollary 6.2. *Let $y \in H$ be an element such that $y = g^{-1}zg$, where $z, g \in H$ and z is an affine map, with slope different of one. Then $C_H(y) \cong (\mathbb{R}, +)$.*

If an element $z \in H$ does not have breakpoints it is an affine map $z(t) = a^2t + ab$ on the whole real line. Moreover, if $a \neq \pm 1$ for such a $z \in H$, then z crosses the diagonal line given by the identity map $\mathrm{id}(t) = t$. Then in this case, $z \notin H^< \cup H^>$. In the next result, we show which conditions, given any element $z \in H$, must be satisfied by $z'(-\infty)$ and $z'(+\infty)$ in order that $z \in H^<$.

Lemma 6.3. *Let $z \in H^<$ so that around $-\infty$ we have $z(t) = a_0^2 t + a_0 b_0$. Then $a_0^2 \geqslant 1$. Moreover, if $z(t) = a_n^2 t + a_n b_n$ around $+\infty$, then $a_n^2 \leqslant 1$.*

Proof. Let us suppose that $z \in H^<$ so that $z(t) = a_0^2 t + a_0 b_0$ around $-\infty$. Then since $z(t) < t$, we have $a_0^2 t + a_0 b_0 < t$ around $-\infty$. So, let us define

$$h(t) := a_0^2 t + a_0 b_0 - t,$$

for $t \in (-\infty, L]$, where L is a suitable real number so that $z(t) = a_0^2 t + a_0 b_0$. Then we have $h(t) < 0$, that is,

$$a_0^2 t + a_0 b_0 - t < 0.$$

So,

$$t\left(a_0^2 - 1 + \frac{a_0 b_0}{t}\right) < 0,$$

where $t \in (-\infty, L]$. Since t is negative sufficiently large, we get

$$a_0^2 - 1 + \frac{a_0 b_0}{t} > 0.$$

Then we get,

$$a_0^2 - 1 > -\frac{a_0 b_0}{t}.$$

So, passing to limit when $t \to -\infty$,

$$a_0^2 - 1 \geqslant \lim_{t \to -\infty} \frac{a_0 b_0}{t}.$$

Then

$$a_0^2 - 1 \geqslant 0 \iff a_0^2 \geqslant 1.$$

We conclude that $z'(-\infty) \geqslant 1$, if $z \in H^<$. Analogously, we prove that $z'(+\infty) \leqslant 1$, if $z \in H^<$. □

Then as consequence of the previous lemma, we have

Corollary 6.4. *If $z \in H^<$ has no breakpoints, then it is a translation. Moreover, if $g \in C_H(z)$, then g is a periodic element of a suitable period.*

Proof. If $z \in H^<$ has no breakpoints, then Lemma 6.3 implies

$$z'(-\infty) = z'(+\infty) = 1.$$

Then z is a translation, that is,

$$z(t) = t + k,$$

where $k \in \mathbb{R}$ is a fixed number. Moreover, since $z(t) < t$, we have

$$t + k < t \iff k < 0.$$

Then $k \in \mathbb{R}_{<0}$. Notice that, in this case, if $g \in C_H(z)$, thus

$$gz(t) = zg(t) \iff g(t+k) = g(t) + k, \tag{6.2}$$

for all $t \in \mathbb{R}$. Then centralizers of any translation are periodic functions. □

The next result shows that these centralizers are isomorphic to $(\mathbb{R}, +)$.

Proposition 6.5. *If $z \in H^{<}$ is a translation map, then $C_H(z) \cong (\mathbb{R}, +)$.*

Proof. Given $g \in C_H(z)$, we have $zg(t) = gz(t)$ for all $t \in \mathbb{R}$. Since $g \in H$, we have $g(t) = a_0^2 t + a_0 b_0$, for some $a_0, b_0 \in \mathbb{R}$, for $t \in (-\infty, L]$, for suitable $L \in \mathbb{R}$. Since $z(t) = t + k$, for some $k < 0$, then by Equation (6.2), we get

$$\begin{aligned} g(t+k) = g(t) + k &\iff a_0^2(t+k) + a_0 b_0 = a_0^2 t + a_0 b_0 + k \\ &\iff a_0^2 t + a_0^2 k + = a_0^2 t + k \\ &\iff a_0^2 k = k \\ &\iff (a_0^2 - 1)k = 0 \\ &\iff a_0^2 = 1. \end{aligned}$$

This implies that the first piece of g also is a translation by $b_0 \in \mathbb{R}$. Then the initial germ of a suitable centralizer g of z is $g_{-\infty} = (1, b_0)$. If g does not have breakpoints, then g is a translation for all the real line. Otherwise, assume there is an $\varepsilon > 0$, such that for $t \in [L, L+\varepsilon)$, we have

$$g(t) = \frac{a_1 t + b_1}{c_1 t + d_1}.$$

If, for all for all $t \in [L, L+\varepsilon)$, we have $t + k < L$, then Equation (6.2) implies that

$$g(t+k) = g(t) + k \iff (t+k) + b_0 = \frac{a_1 t + b_1}{c_1 t + d_1} + k \iff t + b_0 = \frac{a_1 t + b_1}{c_1 t + d_1}. \tag{6.3}$$

Equivalently we have

$$c_1 t^2 + (b_0 c_1 + d_1)t + b_0 d_1 = a_1 t + b_1,$$

and, from this,

$$a_1 = d_1, \quad b_1 = b_0 a_1 \quad \text{and} \quad c_1 = 0.$$

Since $c_1 = 0$, we must have $a_1 d_1 = 1$. Consequently, as $a_1 = d_1$, we get $a_1 = \pm 1$. Then in this situation, we conclude that $g(t) = t + b_0$ for all $t \in (-\infty, L+\varepsilon)$, for each $\varepsilon > 0$.

Otherwise, if $L < t + k < L + \varepsilon$ for some t in a small subinterval of $[L, L+\varepsilon)$, we have

$$\begin{aligned} g(t+k) = g(t) + k &\iff \frac{a_1(t+k) + b_1}{c_1(t+k) + d_1} = \frac{a_1 t + b_1}{c_1 t + d_1} + k \\ &\iff \frac{a_1 t + a_1 k + b_1}{c_1 t + c_1 k + d_1} = \frac{a_1 t + b_1 + k c_1 t + k d_1}{c_1 t + d_1}, \end{aligned}$$

From this,

$$\begin{aligned} &a_1 c_1 t^2 + (a_1 d_1 + a_1 k + b_1)t + (a_1 k + b_1 d_1) = \\ &\qquad (a_1 c_1 + k c_1)t^2 + (b_1 c_1 + k a_1 c_1 + k^2 c_1^2 + a_1 d_1 + 2k c_1 d_1)t + (k b_1 c_1 + k c_1 d_1 b_1 d_1 + k d_1^2), \end{aligned}$$

which implies that $c_1 = 0$ and so g is an affine map. By Equation (6.3) we then see that again that $g(t) = t + b_0$ on $[L, L + \varepsilon)$ Proceeding this way, we conclude that g must be a translation on all $\mathbb{R}$ and so

$$C_H(z) = \{t + m \mid m \in \mathbb{R}\}.$$

Consider now the map φ_z from the Lemma 5.14. We have

$$\varphi_z(C_H(z)) = \{(1, m) \mid m \in \mathbb{R}\}.$$

Since $C_H(z) \cong \varphi_z(C_H(z))$, we get $C_H(z) \cong \{(1, m) \mid m \in \mathbb{R}\}$. Moreover, defining the following group isomorphism,

$$\begin{aligned} \gamma\colon \varphi_z(C_H(z), \cdot) &\longrightarrow (\mathbb{R}, +) \\ (1, m) &\longmapsto m, \end{aligned}$$

it follows that $\varphi_z(C_H(z)) \cong (\mathbb{R}, +)$. Therefore, $C_H(z) \cong (\mathbb{R}, +)$. □

As a consequence of the last result, given $y \in H$, such that y is a conjugate map of a translation, its centralizer is isomorphic to $(\mathbb{R}, +)$.

Corollary 6.6. *Let $y \in H$ be an element such that $y = g^{-1}zg$, where $z, g \in H$ and z is a translation map. Then $C_H(y) \cong (\mathbb{R}, +)$.*

Remark 6.7. The previous corollary gives a partial answer to the following question: what are the centralizers of elements such that the initial and final germs are exactly equal and of the form $(1, b)$, $b \in \mathbb{R}$? The general answer will be more complicated.

Example 6.8. By Example 5.24, we know that

$$z(t) = \begin{cases} \dfrac{2t+2}{\frac{3}{2}t+2}, & \text{if } t \in [-1, 0], \\ \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0, 1], \\ t+1, & \text{otherwise.} \end{cases}$$

is the result of conjugating $y(t) = t + 1$ by

$$g(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0, 1], \\ t+1, & \text{otherwise.} \end{cases}$$

Therefore $C_H(z) \cong (\mathbb{R}, +)$, by Corollary 6.6.

Now, let us consider $z \in H^{<}$ such that z has breakpoints and $z'(-\infty) \neq z'(+\infty)$. As a consequence, we have $z_{-\infty} \neq z_{+\infty}$. We start with the following lemma.

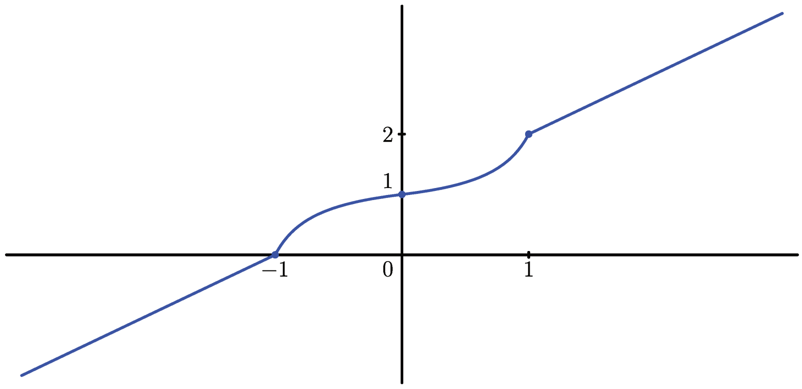

Figure 9 – Graph of z.

Lemma 6.9. *Given $z \in H^<$ such that its initial and final linearity boxes with respect to z and itself are $(-\infty, L]^2$ and $[R, +\infty)^2$, respectively, and so that $z'(-\infty) \neq z'(+\infty)$. Let us consider $s \in \mathbb{Z}_{>0}$ such that $z^s(R) < L$. Then either z^{-s} is not linear on $[z^s(R), L]$ or z^{-2s} is not linear on $[z^{2s}(R), L]$.*

Proof. First of all, since $z \in H^<$, then $z^{-1} \in H^>$. Let us suppose that $z(t) = a_0^2 t + a_0 b_0$ on $(-\infty, L]$ and $z(t) = a_n^2 t + a_n b_n$ on $[R, +\infty)$. Then by hypothesis, $a_0^2 = z'(-\infty) \neq z'(+\infty) = a_n^2$. Moreover, $z^{-1}(t) = a_0^{-2} t - a_0^{-1} b_0$ on $(-\infty, z(L)]$ and $z^{-1}(t) = a_n^{-2} t - a_n^{-1} b_n$ on $[z(R), +\infty)$. Then since $z^{-1} \in H^>$, we have z^{-s} is linear on $(-\infty, z^s(L)]$, with initial germ

$$(z^{-s})_{-\infty} = \left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0 \right).$$

Moreover, z^{-s} is linear on $[z^s(R), +\infty)$, which contains $[R, +\infty)$ with final germ

$$(z^{-s})_{+\infty} = \left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n \right).$$

Let us assume, by contradiction, that both z^{-s} and z^{-2s} are both linear on $[z^s(R), L]$ and $[z^{2s}(R), L]$, respectively, and that their germs on these two intervals are (a, b) and (c, d), respectively. Since $z^{-2s} = z^{-s} \circ z^{-s}$, we get z^{-2s} is linear on $[z^s(R), L]$, because z^{-s} is linear on $[z^s(R), L]$ by initial assumption and z^{-s} is linear on $[R, z^{-s}(L)] \subset [R, +\infty)$, with germ

$$\left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n \right)(a, b).$$

Moreover, z^{-2s} is also linear on $[z^{2s}(R), z^s(L)]$, since z^{-s} is linear on $(-\infty, z^s(L)]$ by linearity of z^{-s} and on $[z^s(R), L]$ by initial assumption, with germ

$$(a, b)\left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0 \right).$$

But, the germ of z^{-2s} on $[z^{2s}(R), L]$ is (c, d). Then we get

$$\left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n\right)(a, b) = (c, d) = (a, b)\left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0\right).$$

In this way,

$$\left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n\right)(a, b) = (a, b)\left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0\right).$$

From this, we must have

$$a_n^{-2s} a = a a_0^{-2s}.$$

Since the group $(\mathbb{R}_{>0}, \cdot)$ is abelian, we have $a_0^{-2s} = a_n^{-2s}$. However, we are considering $z \in H^{<}$ such that the $a_0^2 \neq a_n^2$, so that $a_0^{-2s} \neq a_n^{-2s}$ and we have a contradiction. Therefore, either z^{-s} is not linear on $[z^s(R), L]$ or z^{-2s} is not linear on $[z^{2s}(R), L]$. $\square$

Keeping this information in mind, we have the following Lemma

Lemma 6.10. *Let $z \in H^{<}$ be such that $z(t) = a^2 t + ab$ at $-\infty$ with $a^2 > 1$. Then there exists $\varepsilon > 0$ such that the only $g \in C_H(z)$ with $1 - \varepsilon < \widetilde{g}\,'(0) < 1 + \varepsilon$ and $-\varepsilon < \widetilde{g}\,''(0) < \varepsilon$ is $g = \mathrm{id}$.*

Proof. Let us consider $\widetilde{z}$ the conjugate version of z from Theorem 5.31, that is $\widetilde{z} = \nabla(z)$. Let $[0, \alpha]$ and $[\beta, 1]$ be the initial and final Möbius boxes of $\widetilde{z}$ (see Remark 5.33). By Lemma 6.9 there exists an $N_1 \in \mathbb{Z}_{>0}$ such that $\widetilde{z}^{-N_1}$ has a breakpoint μ_1 on $[\widetilde{z}^{N_1}(\beta), \alpha]$. By considering $\alpha' < \mu_1 < \alpha$ and a new initial (smaller) Möbius box $[0, \alpha']$ for z, we use Lemma 6.9 again and find that there exists $N_2 \in \mathbb{Z}_{>0}$ such that $\widetilde{z}^{-N_2}$ has a breakpoint μ_2 on $[\widetilde{z}^{N_2}(\beta), \alpha']$. Without loss of generality, assume that $\widetilde{z}^{N_2}(\beta) \leqslant \widetilde{z}^{N_1}(\beta)$. Then there exists $\varepsilon > 0$ such that $\{\mu_2 < \mu_1\} \subseteq I_\varepsilon := \left[\widetilde{z}^{N_2}\left(\frac{\beta+\varepsilon}{1+\varepsilon}\right), (1-\varepsilon)\alpha\right]$.

Claim 6.2.1. Let $0 < \varepsilon < 1$ and $g \in C_H(z)$ such that

$$1 - \varepsilon < \widetilde{g}\,'(0) < 1 + \varepsilon \quad \text{and} \quad -\varepsilon < \widetilde{g}\,''(0) < \varepsilon.$$

Then $|\widetilde{g}(t) - \mathrm{id}(t)| < 3\varepsilon + 2\varepsilon^2$, for all $t \in [0, \alpha]$, so g the family of functions $\widetilde{g}$ can be seen as uniformly converging to the identity function id on the interval $[0, \alpha]$.

Proof of Claim 6.2.1. Let $\widetilde{g}(0) = 0$, $\widetilde{g}'(0) := \lambda$ and $g''(0) = \rho$. Then $\widetilde{g}(t) = \dfrac{at+b}{ct+d}$ such that $ad - bc = 1$ on $[0, \alpha]$. Since $0 = g(0) = \dfrac{b}{d}$, then $b = 0$. Knowing that $ad = 1$, we see that

$$\widetilde{g}'(t) = \frac{1}{(ct+d)^2} \quad \text{and} \quad \widetilde{g}''(t) = \frac{-2c}{(ct+d)^3},$$

so that $\lambda = \widetilde{g}(0) = \frac{a}{d} = \frac{1}{d^2}$ and $\rho = \widetilde{g}''(0) = -\frac{2c}{d^3}$. Therefore, $d^2 = \frac{1}{\lambda}$ and $c = \frac{-\rho d^3}{2}$. Observe that we have

$$\widetilde{g}(t) = \frac{at}{ct+d} = \frac{t}{cdt+d^2} = \frac{t}{\frac{-\rho t}{2\lambda^2}+\frac{1}{\lambda}} = \frac{2\lambda^2 t}{-\rho t+2\lambda}$$

and so

$$\begin{aligned}|\widetilde{g}(t)-\mathrm{id}(t)| = \left|\frac{2\lambda^2 t}{-\rho t+2\lambda} - t\right| = \left|\frac{2\lambda^2 t - 2\lambda t + \rho t^2}{-\rho t + 2\lambda}\right| \leqslant \\ \leqslant |2\lambda^2 t - 2\lambda t + \rho t^2| \leqslant 2|\lambda|\cdot|\lambda-1|\cdot|t| + |\rho|\cdot|t| \leqslant \\ \leqslant 2(1+\varepsilon)\varepsilon + \varepsilon = 3\varepsilon + 2\varepsilon^2\end{aligned}$$

where at the various steps we have observed that $|t| \leqslant 1$, $|\lambda| \leqslant 1+\varepsilon$, $|\lambda - 1| \leqslant \varepsilon$ and $1 \leqslant |-\rho t + 2\lambda|$ (since $|\rho| < \varepsilon < 1$). $\square$

Claim 6.2.2. Let $0 < t_0 < 1$ be a point in $\{\mu_2 < \mu_1\}$. Then for any $1-\varepsilon < \widetilde{g}'(0) = \lambda < 1+\varepsilon$, there is at most one $g \in C_H(z)$ such that $-\varepsilon < \widetilde{g}''(0) = \rho < \varepsilon$ and such that $\widetilde{g}^{-1}(t_0) = t_0$.

Proof of Claim 6.2.2. We use the expression calculated in Claim 6.2.1 for $\widetilde{g}$. Assume that $\widetilde{g}(t_0) = t_0$, then

$$t_0 = \frac{2\lambda^2 t_0}{-\rho t_0 + 2\lambda}$$

and so

$$1 = \frac{2\lambda^2}{-\rho t_0 + 2\lambda}$$

and so

$$-\rho t_0 + 2\lambda = 2\lambda^2$$

and so

$$\rho = \frac{2\lambda - 2\lambda^2}{t_0}$$

If we assume that $\lambda = 1 + \tau$ for $0 \leqslant \tau \leqslant \varepsilon$, then

$$\rho = \frac{2(1+\tau) - 2(1+\tau)^2}{t_0} = \frac{-2\tau - 2\tau^2}{t_0}.$$

Now, if $\rho \leqslant -\varepsilon$, then g cannot exist. On the other hand, if $-\varepsilon < \rho < 0$, then the pair (τ, ρ) satisfies the required conditions. $\square$

End of the Proof of Lemma 6.10. Notice that μ_i is a breakpoint for $\widetilde{g}\widetilde{z}^{-N_i}$ and that $g^{-1}(\mu_i)$ is a breakpoint for $\widetilde{z}^{-N_i}\widetilde{g}$. By Claim 6.2.2, there can exist at most one $\mathrm{id} \neq g \in C_H(z)$ such that $\widetilde{g}^{-1}(\mu_i) = \mu_i$, but it is straightforward to verify that such Möbius function cannot fix μ_j, with $j \neq i$, since $\widetilde{g}$ fixes 0 too. However, since $g \in C_H(z)$, we get $\widetilde{g}\widetilde{z}^{N_i}(t) = \widetilde{z}^{N_i}\widetilde{g}(t)$ for every $t \in I_\varepsilon$ and so $\widetilde{g}^{-1}(\mu_i) = \mu_i$ for $i = 1, 2$. Then the only way to avoid a contradiction and have a $g \in C_H(z)$ such that $\widetilde{g}'(0)$ and $\widetilde{g}''(0)$ satisfy the given conditions with respect to the chosen $\varepsilon > 0$ is that $g = \mathrm{id}$. $\square$

We now turn to proving the following result

Proposition 6.11. *Let $z \in H^{<}$ be such that $z\,(t) = a^2 t + ab$ at $-\infty$ and $a^2 > 1$. Then $C_H\,(z)$ is a discrete subgroup of $(\mathbb{R}, +)$ and so it is isomorphic to $(\mathbb{Z}, +)$.*

Proof. By Lemma 6.10, $C_H\,(z)$ is a discrete set. Since $C_H\,(z) \cong \varphi_z(C_H\,(z)) \leqslant C_{\mathrm{Aff}(\mathbb{R})}\,(z) \cong (\mathbb{R}, +)$ and the subgroups of $(\mathbb{R}, +)$ also are subgroups of $(\mathbb{R}, +)$, which are either discrete (then isomorphic to $(\mathbb{Z}, +)$), or dense (Proposition 1.10), we get $C_H\,(z) \cong (\mathbb{Z}, +)$. □

6.3 Mather Invariant and Centralizers

In this section, we will assume that $z \in H^{>}$ such that $z\,(t) = t + b_0$ for $t \in (-\infty, L]$, $z\,(t) = t + b_1$ for $t \in [R, +\infty)$ and $z^N((z^{-1}(L), L)) \subset (R, +\infty)$ for some $N \in \mathbb{Z}_{>0}$ sufficiently large. We prove the following result.

Proposition 6.12. *Let us consider $z \in H^{>}$ such that $z\,(t) = t + b_0$ for $t \in (-\infty, L]$, $z\,(t) = t + b_1$ for $t \in [R, +\infty)$ and $z^N\left(\left(z^{-1}\,(L)\,, L\right)\right) \subset (R, +\infty)$. Then either $C_H\,(z) \cong (\mathbb{Z}, +)$ or $C_H\,(z) \cong (\mathbb{R}, +)$.*

Proof. Up to conjugating z with s, we will work with $z\,(t) = t + 1$, as we have done in the Section 5.2. Then we define the relation $t \sim t+1$ and construct the circles $C_0 := (-\infty, 0]/\sim$ and $C_1 := [N-1, +\infty)/\sim$. By Theorem 5.26, we have $g \in H$ is a centralizer of z if the following equation holds

$$\overline{z}^{\infty} v_{0,\ell} = v_{1,m} \overline{z}^{\infty}. \tag{6.4}$$

Considering the lift $V_0 \colon \mathbb{R} \longrightarrow \mathbb{R}$ of $v_{0,\ell}$ given by $V_0(t) = t + \ell$, so that the the following diagram commutes

$$\begin{tikzcd} \mathbb{R} \arrow[r, "V_0"] \arrow[d, "\tilde{p}_0"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\tilde{p}_0"] \\ C_0 \arrow[r, "v_{0,\ell}"'] & C_0 \end{tikzcd}$$

Similarly, consider the lift $V_1(t) = t + m$ making the following diagram commutative

$$\begin{tikzcd} \mathbb{R} \arrow[r, "V_1"] \arrow[d, "\tilde{p}_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\tilde{p}_1"] \\ C_1 \arrow[r, "v_{1,m}"'] & C_1 \end{tikzcd}$$

Let us consider $Z \in H$ such that it is a lift of $\overline{z}^{\infty}$. Equation (6.4) implies that $ZV_0 = V_1 Z$. Given $t \in \mathbb{R}$, we get

$$ZV_0\,(t) = Z(V_0\,(t)) = Z(t + \ell) \;\text{ and }\; V_1 Z\,(t) = V_1(Z\,(t)) = Z\,(t) + m.$$

Then

$$Z(t+\ell) = Z(t) + m, \tag{6.5}$$

which means that the graph of Z is shifted back to itself. Now, if the lift of $\overline{z}^{\infty}$ does not have breakpoints, then the graph of Z is linear, then there are infinitely many pairs $\ell, m \in \mathbb{R}$ for which the graph can be shifted back to itself and so, for each $\ell \in \mathbb{R}$, there exists an $m \in \mathbb{R}$ so that equation (6.5) holds. Consequently, the image of the map φ_z from Lemma 5.14 is so that $\varphi_z(C_H(z)) \cong (\mathbb{R}, +)$. Otherwise, the lift of $\overline{z}^{\infty}$ has breakpoints and, then, the set of candidates for ℓ forms a discrete subgroup of $(\mathbb{R}, +)$, hence $\varphi_z(C_H(z)) \cong (\mathbb{Z}, +)$. Then we have either $C_H(z) \cong (\mathbb{Z}, +)$ or $C_H(z) \cong (\mathbb{R}, +)$. □

Example 6.13. Let us consider

$$z(t) = \begin{cases} \dfrac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]; \\ t+1, & \text{otherwise.} \end{cases}$$

Notice that $z \in H^{>}$ and that $L = 0$ and $R = 1$. Its inverse is given by

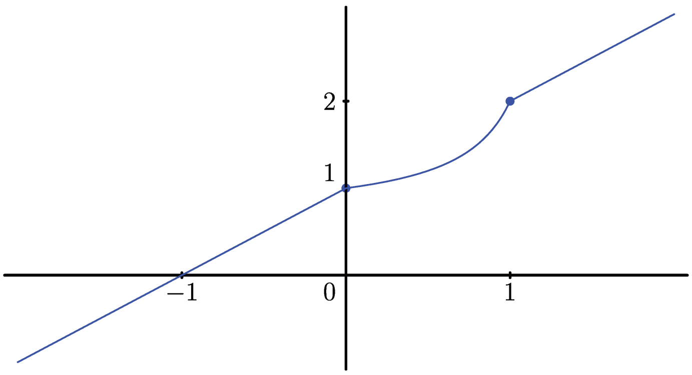

Figure 10 – Graph of z.

$$z^{-1}(t) = \begin{cases} \dfrac{2t-2}{\frac{3}{2}t-1}, & \text{if } t \in [1,2]; \\ t-1, & \text{otherwise.} \end{cases}$$

If $N = 2$, we have

$$z^2((z^{-1}(0), 0)) \subset [1, +\infty).$$

Notice that we do not need of the map s, since both first and final pieces are translations by one unit. Then we have $\overline{z} := z$. Moreover,

$$z^2(t) = \begin{cases} \dfrac{t-1}{\frac{3}{2}t-\frac{1}{2}}, & \text{if } t \in [-1,0]; \\ \dfrac{\frac{5}{2}t-4}{\frac{3}{2}t-2}, & \text{if } t \in [0,1]; \\ t+2, & \text{otherwise.} \end{cases}$$

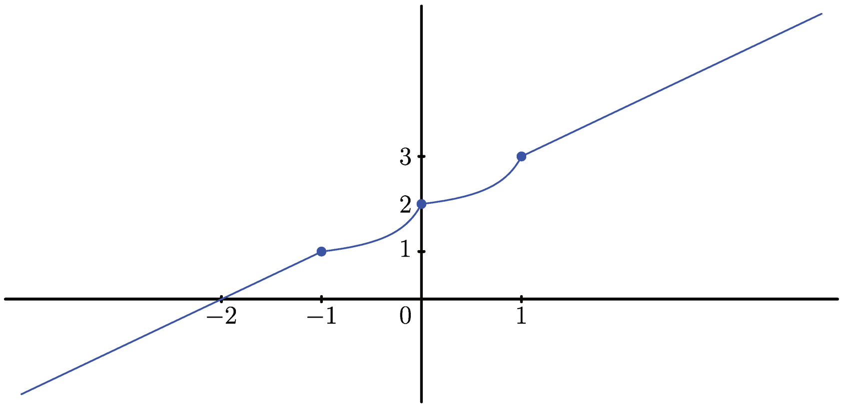

Figure 11 – Graph of z^2.

Considering the relation $t \sim t+1$, we can define the circles

$$C_0 := \frac{(-\infty, 0]}{t \sim t+1} \qquad \text{and} \qquad C_1 := \frac{[1, +\infty)}{t \sim t+1}.$$

Then we get the Mather invariant

$$\begin{aligned} \overline{z}^{\infty} : C_0 &\longrightarrow C_1 \\ [t] &\longmapsto \overline{z}^{\infty}([t]) = \left[z^2(t)\right]. \end{aligned}$$

The lift of this map which makes the following diagram commutes

$$\begin{tikzcd} \mathbb{R} \arrow[r, "Z"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "p_1"] \\ C_0 \arrow[r, "\overline{z}^{\infty}"'] & C_1 \end{tikzcd}$$

is given by the periodic extension of the piece from z^2 defined on $[-1, 0]$ defined as

$$Z(t) = z^2(t - x) + x,$$

if $x - 1 \leqslant t \leqslant x$, where $x \in \mathbb{Z}$. Then the centralizer of Z is $(\mathbb{Z}, +)$. Moreover, notice that $Z \notin H$. On other hand, we saw an element of H in Example 6.8, which we obtained by conjugating the translation $t+1$ by z, and denoted by g in the aforementioned example, such that its centralizer is isomorphic to $(\mathbb{R}, +)$.

6.4 Main Result Concerning Centralizers

Finally, we state the main result of this chapter.

Theorem 6.14. *Given $z \in H$, then*

$$C_H(z) \cong (\mathbb{Z}, +)^n \times (\mathbb{R}, +)^m \times H^k,$$

for suitable $k, m, n \in \mathbb{Z}_{\geqslant 0}$.

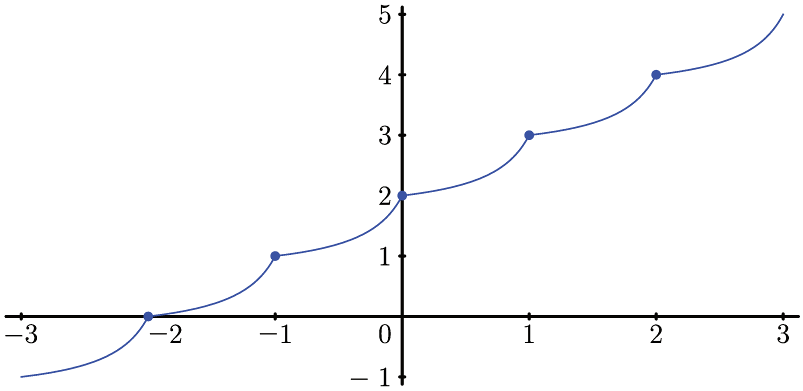

Figure 12 – Graph of the lift Z.

Proof. Let

$$\partial\text{Fix}\,(z) = \{t_0 < t_1 < \ldots < t_n\}\,.$$

Since elements from $\partial\text{Fix}\,(z)$ are fixed by z, then $g \in C_H\,(z)$ fixes $\partial\text{Fix}\,(z)$, that is,

$$g\,(\partial\text{Fix}\,(z)) = \partial\text{Fix}\,(z)\,.$$

Then g must fix elements from $\partial\text{Fix}\,(z)$. As consequence, we can restrict to study centralizers in each of the subgroups $H\,([t_i, t_{i+1}])$ where $i = 0, 1, \ldots, n-1$. If $z\,(t) = t$ on $[t_i, t_{i+1}]$, then it clear that $C_H\,(z) = H$. If $z\,(t) \neq t$ on $[t_i, t_{i+1}]$, by Propositions 6.5, and 6.11, and 6.12, either $C_H\,(z) \cong (\mathbb{R}, +)$ or $C_H\,(z) \cong (\mathbb{Z}, +)$. $\square$

Final Remarks

In this doctoral thesis, we present conjugacy invariants for a piecewise projective homeomorphisms group and calculate its centralizers. We develop the Stair Algorithm and the Mather Invariant for Monod's group H. Let us recall the topics presented in the chapters of this doctoral thesis.

In Chapter 2, we present definitions and properties of Thompson's group F. In Chapter 3, we introduce the Stair Algorithm for $\mathrm{PL}_2(I)$, an overgroup of Thompson's group F, developed by Martin Kassabov and Francesco Matucci in [18] and used to find candidate conjugators. In Chapter 4, we define piecewise projective homeomorphisms of the real line and the groups $\mathrm{PPSL}_2(\mathbb{R})$ and H. Besides, we give the piecewise projective point of view from F. Moreover, we explore some properties shared by F and H, such as k-transitivity and the fact that H is a full group, this last fact is an original result, to the best of our knowledge.

In Chapter 5, we develop our own version of the Stair Algorithm, a machine to build candidate conjugators in H, by generalizing the techniques from [9, 18, 21]. Moreover, we also describe a conjugacy invariant by introducing the Mather invariant for H, which works as an obstruction for the Stair Algorithm. Finally, in Chapter 6, we compute the element centralizers subgroups in H, as applications of the techniques introduced in the preceding chapter.

The main contribution of this work is that we extended the techniques from [9, 18, 20, 21] to a group of piecewise projective homeomorphisms of the projective real line. These techniques appear generalizable to subgroups $H(A)$ of Monod's group H, where A is a subring of $\mathbb{R}$. We already have some partial results concerning this generalizations. Unfortunately, due to time constraints, there are still some cases to be dealt with and we decided not to include them in the present doctoral thesis.

As a perspective of future work, we intend to complete them and work on a solution of the conjugacy problem in $H(A)$ and, also, we intend to investigate the automorphism groups of Monod's groups $H(A)$ by using standard machinery which employs some of the aforementioned partial results for $H(A)$.

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