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Algebraic methods in G_2-geometry

Métodos algébricos em G_2-geometria

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Supervisor: Henrique Nogueira de Sá Earp

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Someday, everyone will know LINEARALGEBRA,
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and ignorance and superstition will be banished forever.
(Robert Bryant at mathoverflow)

Resumo

Nesta tese estudamos dois tópicos, o espaço de deformação de subvariedades associativas e fluxos de G_2–estruturas co-fechadas invariantes. No primeiro tópico, encontramos uma fórmula de Weitzenböck para o operador de Fueter-Dirac, o qual controla as deformações infinitesimais de uma subvariedade associativa em uma 7–variedade com uma G_2–estrutura. Como aplicações, construímos duas subvariedades associativas rígidas e demos uma prova diferente da rigidez da 3-esfera na 7-esfera redonda, o qual foi feito por Kawai [Kaw13, Kaw17]. No segundo tópico, aplicamos a técnica geral proposta por Lauret [Lau16] para o co-fluxo laplaciano e o co-fluxo laplaciano modificado de G_2-estruturas co-fechadas invariantes em um grupo de Lie. Como resultado, para cada um dos fluxos encontramos um soliton explícito em uma 7-variedade quase abeliana particular.

Palavras-chave: G_2-estrutura, subvariedade associativa, G_2-fluxo, grupo de Lie.

Abstract

In this thesis we deal with two topics, the deformation space of associative submanifolds and flows of invariant co-closed G_2–structures. For the first one, we find a Weitzenböck formula for the Fueter-Dirac operator which controls infinitesimal deformations of an associative submanifold in a 7–manifold with a G_2–structure. As applications, we construct two rigid associative submanifolds and we find a different proof of rigidity for associative 3-sphere in the round 7-sphere from those given by Kawai [Kaw13, Kaw17]. For the second one, we apply the general Ansatz proposed by Lauret [Lau16] for the Laplacian co-flow and the modified Laplacian co-flow of invariant co-closed G_2–structures on a Lie group. As result, for each flow we find an explicit soliton on a particular almost abelian 7–manifold.

Keywords: G_2–structure, associative submanifold, G_2–flow, Lie group.

Contents

Introduction

This thesis is concerned with G_2-geometry, more specifically about associative submanifolds and flows of co-closed G_2–structures.

Associative submanifolds were introduced by Harvey and Lawson [HL82] as particular case of calibrated submanifold. Afterwards, R. McLean in his seminal paper [McL98] addressed the question of deformability of calibrated submanifolds as a generalisation of Kodaira's work on deformation of complex submanifolds [Kod62]. In two particular calibrated geometries, namely, the special Lagrangian and the coassociative geometries, the normal bundles are intrinsic, so, the existence of calibrated deformations of a calibrated submanifold is reduced to topological questions of the submanifold itself. Meanwhile, in the other two calibrated geometries, specifically, the three dimensional associative submanifolds and the four dimensional Cayley submanifolds the normal bundle are not intrinsic, but rather they are twisted spin bundles of extrinsic vector bundles. In this thesis is discussed the case of associative submanifold Y, which only occur when the ambient manifold M has real dimension 7, and the calibration is a 3-form φ. In fact, (M, φ) is a manifold with G_2-structure, in [McL98], McLean proved that a class in the moduli space of associative deformations corresponds to a harmonic spinor of a twisted Dirac operator, under the *torsion-free* hypothesis $T \equiv \nabla\varphi = 0$. Then, Akbulut and Salur [AS08a, AS08b] generalised McLean's theorem for a general G_2-structure identifying the tangent space at an associative submanifold Y^3 in (M^7, φ) with the kernel of

$$\not{D}_{\mathrm{A}} : \Omega^0(Y, NY) \to \Omega^0(Y, NY) \tag{1}$$

where $A = A_0 + a$, for A_0 the induced connection on NY and some $a \in \Omega^1(Y, \mathrm{ad}(NY))$. The first purpose of this thesis is to obtain a Weitzenböck formula for the operator (1), that is, a relation between the second-order elliptic square $\not{D}_{\mathrm{A}}^{\,2}$ and the trace Laplacian $\nabla^*\nabla$ of the induced Levi-Civita connection on NY. Under suitable positivity assumptions on curvature, this implies *rigidity*, i.e., that Y has "essentially" no infinitesimal associative deformations, in the following sense. Denote by $G := \mathrm{Stab}(\varphi) \subset \mathrm{Aut}(M)$ the group of global automorphisms preserving φ. The infinitesimal associative deformations of Y consist of:

(i) *trivial* deformations given by the action of G on Y (see [Kaw17] and [Mor16]);

(ii) *non-trivial* deformations, which depend intrinsically on the geometry of the associative submanifold.

For instance, in [Kaw17], an associative submanifold is considered rigid if all infinitesimal associative deformations are trivial; in the particular case of the homogeneous space

$M = S^7$, the symmetry group of φ is $G = \mathrm{Spin}(7)$. On the other hand, Gayet [Gay14] and McLean [McL98] consider a generic G_2–structure, i.e., without symmetries. So, G is 0–dimensional and Y is rigid if the space of nontrivial infinitesimal deformation vanishes.

The exposition is organised as follows: Chapter 1 is proactive background review in G_2–geometry, in order to fix the notation and the sign convention of some important tensors arising from the G_2–structure. We then deduce Lemma 5, a Leibniz rule for the Levi-Civita connection and the Riemann curvature tensor with respect to the cross product. After that, we collect ε_{ijk}-identities for SU(3)–structure, it will be a key computational tool in Chapter 3. Finally, we concluded by recalling some results from 4-dimensional spin geometry to explain the explicit identification

$$NY \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-,$$

between the normal bundle of Y and a spinor bundle $S = S^+ \oplus S^- \to Y$, in order to describe the Fueter-Dirac operator in detail.

In Chapter 2, we deal with deformation of associative submanifold following the general framework proposed by Akbulut and Salur [AS08a, AS08b]. We then obtain the following Weitzenböck formula, which generalise the previous formula obtained by Gayet [Gay14].

Theorem 1. *The Weitzenböck formula for* (1) *is*

$$\begin{aligned} \not{D}_{\mathrm{A}}{}^2(\sigma) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \pi^\perp\Big(\sum_{i\in\mathbb{Z}_3} e_i \times \mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})\Big) + H \times \mathcal{B}(\sigma) + (\operatorname{tr} S_\sigma)H - \mathcal{A}(\sigma) \\ - \sum_{j=1}^{3} \pi^\perp(T(e_j, \cdot)^\sharp) \times S_\sigma(e_j) + \pi^\perp(T(\mathcal{B}(\sigma), \cdot)^\sharp) + P_1(\sigma) + P_2(\sigma) + P_3(\sigma). \quad (2) \end{aligned}$$

Where P_1, P_2 and P_3 are first order differential operators on NY, involving the torsion of the G_2–structure, $\mathcal{B}$ is a 0^{th}–order operator defined by the shape operator S_σ on the normal section σ

$$\mathcal{B}(\sigma) := \sum_{j=1}^{3} e_j \times S_\sigma(e_j).$$

H is the mean curvature vector field of the immersed associative submanifold, $\mathcal{A}(\sigma) = S^t \circ S(\sigma)$, is a symmetric positive 0^{th}–order operator determined by the shape operator, $\mathcal{R}(\sigma) = \pi^\perp \sum_{i=1}^{3} R(e_i, \sigma)e_i$ is a partial Ricci operator, $\mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})$ is a 0^{th}–order involving the torsion tensor, the Hodge dual 4-form ψ and its covariant derivative, and $\nabla^*\nabla$ is the connection Laplacian

$$\nabla^*\nabla n = -\sum \nabla^\perp_i \nabla^\perp_i n - \nabla^\perp_{\nabla_i e_i} n$$

in a global frame $\{e_i\}$ on the associative submanifold Y.

As application, in Section 2.1, we specialise to the *nearly parallel* case, in which $d\varphi$ and ψ are collinear and the formula (2) simplifies significantly. For a generic nearly parallel G_2–structure, we obtain a vanishing theorem to conclude rigidity under suitable intrinsic geometric conditions on Y.

Theorem 2. *Let (M, φ) be a 7-manifold with a nearly parallel G_2–structure. If $Y \subset M$ is a closed associative submanifold such that the operator $\mathcal{R} - \mathcal{A}$ is non-negative, then Y is rigid.*

As immediate applications, we propose an alternative proof of rigidity for the known case of an associative SU(2)–orbit 3-sphere for Lotay's cocalibrated G_2–structure on S^7 studied by Kawai [Lot12, Kaw13, Kaw17].

Corollary 1. *The 3-sphere in S^7 is rigid as an associative submanifold.*

In sections 2.2 and 2.3, we construct rigid associative submanifolds (Corollaries 7 and 8), respectively. The first one associative submanifold lies in a compact manifold S with locally conformal calibrated G_2–structure obtained from the 3-dimensional complex Heisenberg group by Fernández-Fino-Raffero [FR16] and the second one associative submanifold lies in a seven dimensional nilmanifold with closed G_2–structure obtained from the seven dimensional 2-step nilpotent Lie algebra $\mathfrak{n}_2$ [FR17, Lau17, Nic18]

The second purpose of this thesis is to study the Laplacian co-flow (LC) and the modified Laplacian co-flow (MLC)

$$\text{(LC)} \quad \frac{\partial}{\partial t}\psi_t = -\Delta_\psi \psi, \qquad \text{(MLC)} \quad \frac{\partial}{\partial t}\psi_t = \Delta_\psi \psi + 2d((C - \operatorname{tr} T)\varphi)$$

of co-closed G_2–structures, introduced by Karigiannis et al. [KT12] and Grigorian [Gri13], respectively. The co-closed G_2–structure condition $d\psi = 0$ is weaker than the torsion free condition and even than the closed condition $d\varphi = 0$. Also, any G_2–structure can be deformed to become co-closed, for a closed G_2–structure it does not necessarily true [CN15], thus, in some sense, consider co-closed G_2–structures is more natural than closed ones. However, the Laplacian co-flow does not have a nice behaviour, namely, (LC) is not weakly parabolic, in fact, the symbol of the linearised equation has not sign-definite. For that reason, the modified Laplacian co-flow arises to fixing the non parabolicity of the Laplacian co-flow in the direction of the co-closed forms.

The flows (LC) and (MLC) have been studied in [KT12, Gri16] for two explicit examples of co-closed G_2–structures with symmetry, namely for warped products of an interval, or a circle, with a compact 6-manifold N which is taken to be either a nearly Kähler or a Calabi-Yau manifold and recently, in [BF17] Bagaglini et al. studied both flows for the 7–dimensional Heisenberg group and in [BF18] they showed long time-existence for a class of seven dimensional almost-abelian Lie group for (LC).

In Chapter 3, our main focus is when $M^7 = G$ is a Lie group, we propose to study these flows from the perspective introduced by Lauret [Lau16] in the general context of geometric flows on homogeneous spaces. In section 3.5, we gathered useful identities for co-closed G_2–structures on almost abelian Lie groups, namely, we calculated the remained torsion forms,

Proposition 1. *The torsion forms τ_0 and τ_3 for an almost abelian Lie group (G_A, φ) with co-closed G_2–structure are*

$$\tau_0 = \frac{2}{7}\operatorname{tr}(JA) \quad \text{and} \quad \tau_{27} = \left(\begin{array}{c|c} \frac{1}{14}\operatorname{tr}(JA)I_6 - \frac{1}{2}[J,A] & 0 \\ \hline 0 & -\frac{3}{7}\operatorname{tr}(JA) \end{array}\right)$$

The full torsion tensor,

Corollary 2. *The full torsion tensor T of an almost abelian Lie group (G_A, φ) with an invariant co-closed G_2–structure is*

$$T = \frac{1}{2}\left(\begin{array}{c|c} [J,A] & 0 \\ \hline 0 & \operatorname{tr}(JA) \end{array}\right).$$

And the Laplacian of ψ,

Proposition 2. *If (G_A, φ) is co-closed, we have:*

i) For the Hodge Laplacian of ψ

$$\Delta_\psi \psi = \theta\big(\operatorname{Ric}(g) - \frac{1}{2}T \circ T - (\operatorname{tr} T)T\big) = \theta(Q_A)$$

Furthermore, $Q_A = \operatorname{Ric}(g) - (\operatorname{tr} T)T - \frac{1}{2}T \circ T$ is a symmetric operator and it is given by

$$Q_A = \left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & q \end{array}\right),$$

where

$$Q_1 = \frac{1}{2}[A, A^t] + \frac{1}{2}S_A \circ_6 S_A \quad \text{and} \quad q = -\frac{1}{2}\operatorname{tr}(S_A)^2 - \frac{1}{4}(\operatorname{tr} JA)^2.$$

ii) For the modified Laplacian

$$\Delta_\psi \psi + 2d((C - \operatorname{tr} T)\varphi) = \theta\big(\operatorname{Ric}(g) - \frac{1}{2}T \circ T - (2C - \operatorname{tr} T)T\big) = \theta(P_A)$$

where

$$P_A = \left(\begin{array}{c|c} P_1 & 0 \\ \hline 0 & p \end{array}\right),$$

where $P_1 = \frac{1}{2}[A, A^t] + \frac{1}{2}S_A \circ_6 S_A - \Big(C - \frac{1}{2}\operatorname{tr} JA\Big)[J, A]$ and $p = -\frac{1}{2}\operatorname{tr}(S_A)^2 + \frac{1}{4}(\operatorname{tr} JA)^2 - C\operatorname{tr} JA$.

Where the matrix $A \in \mathfrak{sp}(6, \mathbb{R})$ encode the constant structures of the almost abelian Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

As an application of these formulae, we apply a natural Ansatz to construct examples of invariant self-similar solution, or *soliton*, of both co-flows in the Subsections 3.5.1 and 3.5.2. Solitons are G_2–structures which, under the flow, simply scale monotonically and move by diffeomorphisms. In particular, they provide potential models for singularities of the flow, as well as means for desingularising certain singular G_2–structures, both of which are key aspects of any geometric flow.

In section 3.6, we address a motivational example of a soliton for the Laplacian flow of closed G_2–structures following the framework developed by Lauret [Lau16]. Here, we study the behaviour of the associative submanifold from Example 8 along the Laplacian flow with initial G_2–structure given in (2.28).

Ultimately, we formulate two questions for future work.

1 Preliminary: G_2–Geometry

We first present some algebraic and geometric proprieties of G_2–geometry related with G_2–structures and associative submanifolds, these can be found e.g. in [HL82, Kar09, CP15].

1.1 Linear algebra of dimension $8, 7, 6$

The octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8$ are an 8-dimensional, non-associative, division algebra. For the basis $\{1_{\mathbb{O}} = e_0, e_1, \dots, e_7\}$ we adopt the following convention for the octonionic product:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	e_7	-1	e_1	e_2	$-e_3$
e_5	e_4	e_7	e_6	$-e_1$	-1	$-e_3$	$-e_2$
e_6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	-1	e_1
e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

By the product above follows that $u \in \mathrm{Im}(\mathbb{O})$ if and only $u^2 = u \cdot u$ is real but u in not.

Definition 1. *The group of automorphism of* $\mathbb{O}$ *is* $\mathrm{G}_2 := \mathrm{Aut}(\mathbb{O})$.

For $\gamma \in \mathrm{G}_2$ and $u \in \mathrm{Im}(\mathbb{O})$, $\gamma(u) \notin \mathbb{R}$ and $\gamma(u^2) = \gamma(u)^2$ is real, so $\gamma(u) \in \mathrm{Im}(\mathbb{O})$. Therefore, G_2 is a subgroup of the group of automorphism of $\mathrm{Im}(\mathbb{O})$ preserving the octonionic product on $\mathrm{Im}(\mathbb{O})$. On the imaginary part $\mathrm{Im}(\mathbb{O}) = \mathbb{R}^7$, the cross product is given by (e.g. [HL82, Appendix IV.A])

$$\begin{array}{rcccl} \times & : & \mathbb{R}^7 \times \mathbb{R}^7 & \to & \mathbb{R}^7 \\ & & (u, v) & \mapsto & \dfrac{1}{2}(uv - vu) = \mathrm{Im}(uv). \end{array} \tag{1.1}$$

Notice that, $(u \times v)^2 = -g_0(u,u) g_0(v,v) \in \mathbb{R}$ and $u \times v$ is not real, where g_0 is the standard inner product in $\mathbb{R}^7$. Hence, $\times$ is well defined and also is preserved by the action of G_2 i.e. $\gamma(u \times v) = \gamma(u) \times \gamma(v)$ for all $\gamma \in \mathrm{G}_2$. On the other hand, the inner product in $\mathbb{R}^7$ can be defined in terms of the octonionic product (e.g. [HL82, Appendix IV.A])

$$g_0(u, v) = -\frac{1}{2}(uv + vu) = \mathrm{Re}(uv) \quad \text{for} \quad u, v \in \mathbb{R}^7, \tag{1.2}$$

from the above, follows that G_2 lies in $O(7)$, the orthogonal transformations of $\mathbb{R}^7$. Notice that, the algebra structure of $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{O})$ can be recovered from the vector product (1.1) and the inner product (1.2) by

$$(a, u) \cdot (b, v) = (ab - g_0(u, v), av + bu + u \times v) \quad \text{for} \quad a, b \in \mathbb{R}, u, v \in \mathrm{Im}(\mathbb{O}),$$

So, for $\gamma \in \mathrm{Gl}(7)$ preserving the cross and the inner product, we have that $\gamma(a, u) := (a, \gamma(u))$ lies in $\mathrm{Aut}(\mathbb{O})$. So, we get

$$G_2 = \{\gamma \in \mathrm{Gl}(7) : \ \gamma(u) \times \gamma(v) = u \times v \quad \text{and} \quad g_0(\gamma(u), \gamma(v)) = g_0(u, v)\}. \tag{1.3}$$

From g_0 and $\times$ we can define the trilinear alternating form

$$\varphi_0(u, v, w) = g_0(u \times v, w) \in \Lambda^3(\mathbb{R}^7)^*,$$

choosing the basis $e_1, \dots, e_7$ orthonormal with respect to (1.2) we can write

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{1.4}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Notice that the octonionic multiplication can be recovered from the 3-form φ_0 by

$$e_i \cdot e_j = \varphi_0(e_i, e_j, e_k)e_k,$$

hence, for γ in the stabiliser of φ_0, $\mathrm{Stab}(\varphi_0) \subset \mathrm{Gl}(7)$

$$\gamma(e_i) \cdot \gamma(e_j) = \varphi_0(\gamma(e_i), \gamma(e_j), e_k)e_k = \varphi_0(e_i, e_j, \gamma^{-1}(e_k))\gamma(\gamma^{-1}(e_k)) = \gamma(e_i \cdot e_j).$$

Therefore, we can give a second definition for G_2 following [Joy00, Definition 10.1.1].

Definition 2. *The subgroup of* $\mathrm{Gl}(7)$ *preserving the* 3*-form* φ_0 *is the exceptional Lie group* G_2*. It is compact, connected, simply connected, semisimple and* 14*-dimensional.*

By direct inspection on basis elements of $\mathbb{R}^7$ we get the relation

$$(e_i \lrcorner \varphi_0) \wedge (e_j \lrcorner \varphi_0) \wedge \varphi_0 = 6 g_0(e_i, e_j) e^{1 \cdots 7}, \tag{1.5}$$

notice that, the inner product and the volume form can be recovered from φ_0, so by equation (1.5) the elements of G_2 also preserve the orientation of $\mathbb{R}^7$ and the 4-form

$$\psi_0 = *\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{1.6}$$

We can use ψ_0 and the inner product to obtain an alternating vector valued 3-form $\chi_0 : \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$$\psi_0(u, v, w, z) = *\varphi_0(u, v, w, z) = g_0(\chi_0(u, v, w), z) \quad \text{for} \quad u, v, w, z \in \mathbb{R}^7. \tag{1.7}$$

Notice that, χ_0 is not a triple cross-product since there exist orthonormal triples u, v, w such that $\chi_0(u, v, w) = 0$. Thus $\chi_0 = -\sum_{i=1}^{7} (e_i \lrcorner \psi_0) \otimes e_i$, can be expressed in terms of the cross product (c.f. [HL82]),

$$\chi_0(u, v, w) = -u \times (v \times w) - g_0(u, v)w + g_0(u, w)v, \tag{1.8}$$

Remark 1. *Regarding orientation conventions, some authors adopt the model* 3*-form to be*

$$\phi_0 = e^{567} + e^{125} + e^{136} + e^{246} + e^{147} - e^{345} - e^{237},$$

(cf. [McL98, Chapters 4 and 5]), which relates to (1.4) *by the orientation-reversing automorphism of* $\mathbb{R}^7$

$$\begin{pmatrix} & I_3 \\ \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} & \end{pmatrix}.$$

In this case, relation (1.5) *becomes*

$$(u \lrcorner \phi_0) \wedge (v \lrcorner \phi_0) \wedge \phi_0 = -6 g_0(u, v) \operatorname{vol}_{g_0}. \tag{1.9}$$

And the alternating vector valued 3*-form* (1.8) *by*

$$\chi(u, v, w) = u \times (v \times w) + \langle u, v \rangle w - \langle u, w \rangle v.$$

Unless otherwise stated, we adopt throughout the convention (1.4).

Next, we want to define a G_2–structure on a 7–dimensional real vector space. This arise from the general notion of G–structure which is related with the reduction of the structure group of a principal bundle and the existence of a global section in a specific associated bundle, to more details see [Joy00, Sec. 2.6 and 10.1] and [Hus66, Ch. 6, Sec. 2].

Definition 3. *Let* V *be a* 7*–dimensional real vector space. We call* $\varphi \in \Lambda^3 V^*$ *a* G_2–structure *if there is a linear isomorphism* $V \cong \mathbb{R}^7$ *identifying* φ *with* φ_0*. The* 3*-form with this property is call* positive *and the set o positive* 3*-forms is denoted by* $\Lambda^3_+ V^* \subset \Lambda^3 V^*$*.*

The orbit $\mathrm{Gl}(7) \cdot \varphi_0$ has dimension $35 = \dim \mathrm{Gl}(7) - \dim \mathrm{G}_2$, therefore $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$. Also by Hodge duals of forms, the orbit $\mathrm{Gl}(V) \cdot \psi$ is open in $\Lambda^4 V^*$.

Since the stabiliser of the basis element $e_7 \in S^6 \subset \mathbb{R}^7$ is isomorphic to SU(3) [CP15, Proposition 2.3 (b)], there exist a natural SU(3)–structure arisen from the G_2–structure φ. The orthogonal complement $e_7^\perp$ with respect to the inner product (1.2) can be identified with $\mathbb{C}^3$ by taking a complex basis $w_1 = e_1 - ie_6, w_2 = e_2 + ie_5, w_3 = e_3 + ie_4$. Now, from the G_2–structure (1.4), we have

$$\begin{aligned} -e_7 \lrcorner \varphi_0 &= -e^{16} + e^{25} + e^{34} = \frac{i}{2} \Big(\sum_{k=1}^{3} w^k \wedge \bar{w}^k \Big) = \omega_0 \\ \varphi_0|_{e_7 \perp} &= e^{123} + e^{145} + e^{246} - e^{356} = \rho_+ \\ e_7 \lrcorner \psi_0 &= e^{124} - e^{135} - e^{236} - e^{456} = \rho_- \end{aligned}$$

where $\rho_+ = \frac{1}{2}(\rho + \bar{\rho})$, $\rho_- = -\frac{i}{2}(\rho - \bar{\rho})$ and $\rho = w^1 \wedge w^2 \wedge w^3$ is a decomposable complex 3-form. Notice that the pair (ρ, ω_0) satisfies the relations

$$\omega_0 \wedge \rho_+ = \omega_0 \wedge \rho_- = 0 \quad \text{and} \quad \frac{1}{4}\rho_+ \wedge \rho_- = \frac{\omega_0^3}{3!}.$$

The pair (ρ, ω_0) defines a $\mathrm{SU}(3)$–*structure* on $\mathbb{C}^3$ and notice that $\varphi_0 = -\omega_0 \wedge e^7 + \rho_+$. The following example illustrates a natural construction of G_2–structures on a 7-dimensional Lie algebra, for some key examples, it will be a model to follow.

Example 1. *Consider a* 6*–dimensional real Lie algebra* $\mathfrak{h}$ *endowed with a* $\mathrm{SU}(3)$*–structure* (ρ, ω) *and consider the semi-direct product* $\mathfrak{g} = \mathfrak{h} \times_\nu \mathbb{R}$ *with Lie bracket*

$$[(u, r), (v, s)] = ([u, v]_\mathfrak{h} + \nu(r)v - \nu(s)u, 0)$$

where $\nu : \mathbb{R} \to \mathrm{Der}(\mathfrak{h})$*. Then the induced* G_2*–structure on* $\mathfrak{g}$ *has the form*

$$\varphi = \omega \wedge e^7 + \rho_+.$$

And similarly, the Hodge dual ψ *of* φ *has the form*

$$\psi = \frac{1}{2}\omega^2 + \rho_- \wedge e^7.$$

1.2 Associative 3-planes

Fix $(V^7, \langle \cdot, \cdot \rangle)$ an inner product space. A k-form $\alpha \in \Lambda^k V^*$ is a *calibration* if, for every oriented k-plane π in V, we have $\alpha|_\pi \leq \mathrm{vol}(\pi)$ and when the equality is attained we say that π is *calibrated.*

Lemma 1. *[CP15, Lemma 2.17]*

i) The 3*-form* φ_0 *defined in* (1.4) *is a calibration on* $(\mathbb{R}^7, g_0)$*.*

ii) If u, v, w *is an orthonormal triple of vectors in* $\mathbb{R}^7$*, the* $\varphi_0(u, v, w) = 1$ *if and only if* $w = u \times v$*.*

Definition 4. *An oriented* 3*-plane* π *in* $\mathbb{R}^7$ *calibrated by* φ_0 *is called an* associative *plane.*

It follows from equation (1.8) and Lemma 1 ii), that $\chi_0|_\pi = 0$ for an associative plane. The following example provides a construction of associative planes arisen from other calibrations (see [CP15, Lemma 2.24]).

Example 2. *Let* $(\mathfrak{g}, \varphi)$ *from Example 1:*

1. *Let* $\mathfrak{k} \subset \mathfrak{h}$ *be a* 2*–dimensional Lie subalgebra. Then* $\mathfrak{k} \times_\nu \mathbb{R}$ *is associative in* $\mathfrak{g}$ *if and only if* $\mathfrak{k}$ *is calibrated by* ω*, namely,* $\mathfrak{k}$ *is a complex line for some complex coordinates on* $\mathfrak{h}$*.*

2. *Let* $\mathfrak{m} \subset \mathfrak{h}$ *be a* 3*-dimensional Lie subalgebra. Then* $\mathfrak{m}$ *is associative in* $\mathfrak{g}$ *if and only if* $\mathfrak{m}$ *is calibrated by* ρ_+*, namely,* $\mathfrak{m}$ *is special Lagrangian.*

1.3 G$_2$–manifolds and associative submanifolds

Here the framework are oriented Riemannian manifolds. Particularly, an oriented, spin 7–manifold and an oriented immersed 3–submanifold.

Definition 5. *Let* M *be a smooth oriented* 7*–manifold. A* G$_2$–structure *is a 3–form* $\varphi \in \Omega^3(M)$ *such that, around every* $p \in M$*, there exists a local section* f *of the oriented frame bundle* $P_{\mathrm{SO}}(M)$ *such that*

$$\varphi_p = (f_p)^* \varphi_0.$$

The relation (1.5) holds for a G$_2$–structure from the above definition. Consequently, φ induces a Hodge star operator $*_\varphi$ and the Levi-Civita connection ∇^φ, though for simplicity we omit henceforth the subscripts in $g := g_\varphi$, $* := *_\varphi$ and $\nabla := \nabla^\varphi$.

Definition 6. *A* G$_2$*–structure is* torsion free *if* $\nabla \varphi = 0$.

It follows by the definition that the holonomy group $\mathrm{Hol}(g) \subset \mathrm{G}_2$ for (M, φ, g) if and only if φ is torsion free.

Theorem 3. *[FG82, Férnandez-Gray,1982] A* G$_2$*–structure* φ *is torsion free if and only if* $d\varphi = 0$ *(closed) and* $d\psi = 0$ *(co-closed).*

Moreover, the model cross-product on $\mathbb{R}^7$ induces the bilinear map on vector fields

$$\begin{aligned} P : \Omega^0(TM) \times \Omega^0(TM) &\rightarrow \Omega^0(TM) \\ (u, v) &\mapsto P(u, v) = u \times v. \end{aligned} \tag{1.10}$$

Definition 7. *Let* (M, φ) *be a* 7*–manifold with* G$_2$*–structure. A* 3*–dimensional submanifold* $Y \subset M$ *is called* associative *if* $\varphi|_Y \equiv \mathrm{vol}(Y)$*.*

For an associative subamnifold Y^3 also holds Lemma 1 in the sense that there exist an orthonormal frame e_1, e_2, e_3 of tangent bundle TY satisfying $e_1 \times e_2 = e_3$ for each point of Y. Hence, we have that Y^3 is associative if and only $\chi|_{TY} = 0$, where $\chi \in \Omega^3(M, TM)$ is a section from the vector bundle $\Lambda^3(TM)^* \otimes (TM)$ induced by ψ.

Lemma 2. *If* Y *is an associative submanifold, then there is a natural identification* $TY \cong \Lambda^2_+(NY)$*.*

Proof. Fix local orthonormal frames e_1, e_2, e_3 and $\eta_4, \eta_5, \eta_6, \eta_7$ of TY and NY, respectively, about a point $p \in Y$:

$$\varphi_p = e^{123} + e^1(\eta^{45} + \eta^{67}) + e^2(\eta^{46} + \eta^{75}) - e^3(\eta^{47} + \eta^{56}) \tag{1.11}$$

and

$$\begin{aligned} e_1 \lrcorner \varphi &= e^{23} + \eta^{45} + \eta^{67}, \\ e_2 \lrcorner \varphi &= e^{31} + \eta^{46} + \eta^{75}, \\ e_3 \lrcorner \varphi &= e^{12} - \eta^{47} - \eta^{56}. \end{aligned}$$

Denote $\omega_1 = (e_1 \lrcorner \varphi)|_{N_pY}$, $\omega_2 = (e_2 \lrcorner \varphi)|_{N_pY}$, $\omega_3 = -(e_3 \lrcorner \varphi)|_{N_pY}$ and define on each fibre the isomorphism $e_j \in T_pY \mapsto \omega_j \in \Lambda^2_+(N_pY)$, which obviously varies smoothly with p. □

1.4 G$_2$–decomposition of the space of differential k-forms

We will briefly review the intrinsic torsion forms of a G$_2$–structure and define the full torsion tensor T_{ij}, using local coordinates, following [Kar09, Bry06]. As before, let (M, φ) be a smooth 7–manifold with G$_2$–structure. In a local coordinate system $(x_1, ..., x_7)$, a differential k–form α on M will be written as

$$\alpha = \frac{1}{k!}\alpha_{i_1 \cdots i_k} dx^{i_1 \cdots i_k}$$

where the sum is taken over all ordered subsets $\{i_1 \cdots i_k\} \subset \{1, ..., 7\}$ and $\alpha_{i_1 \cdots i_k}$ is skew-symmetric in all indices, i.e. $\alpha_{i_1 \cdots i_k} = \alpha(e_{i_1}, ..., e_{i_k})$. So, the interior product of a k–form is given by

$$e_j \lrcorner \alpha = \frac{1}{(k-1)!}\alpha_{j i_1 \cdots i_{k-1}} dx^{i_1 \cdots i_{k-1}}.$$

A Riemannian metric g on M induces on $\Omega^k := \Omega^k(M)$ the metric $g(dx^i, dx^j) := g^{ij}$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}), then for decomposable k–forms we have

$$\begin{aligned} g(dx^{i_1 \cdots i_k}, dx^{j_1 \cdots j_k}) =& \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \vdots & \cdots & \vdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix} \\ =& \sum_{\sigma \in S_7} \operatorname{sgn}(\sigma) g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} \end{aligned}$$

With this convention, the inner product of two k–forms $\alpha = \dfrac{1}{k!}\alpha_{i_1 \cdots i_k} dx^{i_1 \cdots i_k}$ and $\beta = \dfrac{1}{k!}\beta_{j_1 \cdots j_k} dx^{j_1 \cdots j_k}$ is given by

$$\begin{aligned} g(\alpha, \beta) =& \frac{1}{(k!)^2}\alpha_{i_1 \cdots i_k}\beta_{j_1 \cdots j_k} \sum_{\sigma \in S_7} \operatorname{sgn}(\sigma) g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}} \\ =& \frac{1}{k!}\alpha_{i_1 \cdots i_k}\beta_{j_1 \cdots j_k} g^{i_1 j_1} \cdots g^{i_k j_k}, \end{aligned}$$

notice that the last equality follows by the skew-symmetry of β, $\beta_{j_{\sigma(1)}\cdots j_{\sigma(k)}} = \mathrm{sgn}(\sigma)\beta_{j_1\cdots j_k}$. A G_2–structure φ splits $\Omega^\bullet$ into orthogonal irreducible G_2 representations, with respect to its G_2–metric g. In particular,

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14} \quad \text{and} \quad \Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}, \tag{1.12}$$

where $\Omega^k_l \subset \Omega^k$ denotes (fibrewise) an irreducible G_2–submodule of dimension l, with an explicit description:

$$\begin{aligned}
\Omega^2_7 &= \{X\lrcorner\varphi; X \in \Omega^0(TM)\} = \{\beta \in \Omega^2; *(\varphi\wedge\beta) = 2\beta\}\\
\Omega^2_{14} &= \{\beta \in \Omega^2; \beta\wedge\psi = 0\} = \{\beta \in \Omega^2; *(\varphi\wedge\beta) = -\beta\}\\
\Omega^3_1 &= \{f\varphi; f \in C^\infty(M)\}\\
\Omega^3_7 &= \{X\lrcorner\psi; X \in \Omega^0(TM)\}\\
\Omega^3_{27} &= \{h_{ij}g^{jl}dx^i \wedge (e_l)\lrcorner\varphi; h_{ij} = h_{ji}, \mathrm{tr}_g(h_{ij}) = g^{ij}h_{ij} = 0\}
\end{aligned} \tag{1.13}$$

Remark 2. *The definitions above for Ω^2_7 and Ω^2_{14} correspond to the convention 1.5. In the convention 1.9, the eigenvalues of the operator $\beta \mapsto *(\varphi\wedge\beta)$ are -2 and 1 instead of $+2$ and -1, respectively.*

The analogous decompositions of Ω^4 and Ω^5 are obtained from the above by the Hodge isomorphism $*_\varphi : \Omega^k \to \Omega^{7-k}$. Studying the symmetries of torsion one finds that $\nabla\varphi \in \Omega^1 \otimes \Omega^3_7$, so that tensor lies in a bundle of rank 49 [Kar09, Lemma 2.24]. Notice also that $\Omega^3_7 \cong \Omega^1$, so, contracting the dual 4-form $\psi = *_\varphi\varphi$ by a frame of TM, then using the Riemannian metric, one has

$$\Omega^2 \oplus \mathrm{S}^2(\mathrm{T}^*\mathrm{M}) = \Omega^1 \otimes \Omega^3_7 \cong \mathrm{End}(TM) = \mathfrak{so}(TM) \oplus \mathrm{sym}(TM).$$

Here $\mathrm{S}^2(\mathrm{T}^*\mathrm{M})$ denotes the symmetric bilinear forms and $\mathrm{sym}(TM)$ the symmetric endomorphisms of TM. Both of the above splittings are G_2–invariant, so, comparing the G_2–irreducible decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus [\mathbb{R}^7]$ and (1.12), we get the following identification between G_2–irreducible summands

$$[\mathbb{R}^7] \cong \Omega^2_7 \quad \text{and} \quad \mathfrak{g}_2 \cong \Omega^2_{14}.$$

For $\mathrm{S}^2(\mathrm{T}^*\mathrm{M}) \cong \mathrm{sym}(TM)$, Bryant defines maps $i : \mathrm{S}^2(\mathrm{T}^*\mathrm{M}) \to \Omega^3$ and $j : \Omega^3 \to \mathrm{S}^2(\mathrm{T}^*\mathrm{M})$ by

$$i(h) = \frac{1}{2}h_{il}g^{lm}\varphi_{mjk}dx^{ijk} \quad \text{and} \quad j(\eta)(u,v) = *((u\lrcorner\varphi)\wedge(v\lrcorner\varphi)\wedge\eta), \tag{1.14}$$

notice that $i(h) = h_{il}g^{lm}dx^i \wedge (e_m\lrcorner\varphi)$ and $i(g) = 3\varphi$. We list the following proprierties (see [Kar09, Propositions 2.14 and 2.17]).

Lemma 3. *Suppose that h is a symmetric tensor then holds:*

$$\begin{aligned}
*i(h) &= \Big(\frac{1}{4}\,\mathrm{tr}_g(h)g_{ij} - h_{ij}\Big)g^{jl}dx^i \wedge (e_l\lrcorner\psi).\\
j(i(h)) &= 2\,\mathrm{tr}_g(h)g + 4h.
\end{aligned}$$

From the above relation follows $j(\varphi) = 6g$, while $j(\Omega^3_7) = 0$. The map i is injective [Kar09, Corollary 2.16] and, by the G_2–decomposition $\mathrm{S}^2(\mathrm{T}^*\mathrm{M}) = \mathbb{R}g_\varphi \oplus \mathrm{S}^2_0(\mathrm{T}^*\mathrm{M})$, it identifies

$$\mathbb{R}g_\varphi \cong \Omega^3_1 \quad \text{and} \quad \mathrm{S}^2_0(\mathrm{T}^*\mathrm{M}) \cong \Omega^3_{27}.$$

Accordingly, we have a decomposition for the torsion components $d\varphi \in \Omega^4$ and $d\psi \in \Omega^5$ given by (see [Bry06, Kar09])

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi = 4\tau_1 \wedge \psi - *\tau_2, \tag{1.15}$$

where $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$ are called the *torsion forms*.

Remark 3. *The constants are chosen for convenience. A slightly different convention for torsion components is used in [Gri13]*

$$d\varphi = 4\tau_1\psi - 3\tau_7 \wedge \varphi - 3 * i(\tau_{27}) \quad \textit{and} \quad d\psi = -4\tau_7 \wedge \psi - 2 * \tau_{14},$$

accordingly with our notation, τ_0 corresponds to $4\tau_1$, τ_1 corresponds to $-\tau_7$, τ_3 corresponds to $-3i(\tau_{27})$ and τ_2 corresponds to $-2\tau_{14}$.

The torsion forms are completely encoded in the *full torsion tensor T*, defined in coordinates by

$$\nabla_l \varphi_{abc} =: T_{lm} g^{mn} \psi_{nabc}, \tag{1.16}$$

which is expressed in terms of the irreducible G_2–decomposition of $\mathrm{End}(TM) = W_0 \oplus W_1 \oplus W_2 \oplus W_3$ where $W_0 \cong \Omega^0$, $W_1 \cong \Omega^3_7$, $W_2 \cong \Omega^2_{14}$ and $W_3 \cong \Omega^3_{27}$.

Proposition 3. *[Kar09, Theorem 2.27] The full torsion tensor $T = T_{lm}$ is*

$$T = \frac{\tau_0}{4} g_\varphi - \tau_{27} - (\tau_1)^\sharp \lrcorner \varphi - \frac{1}{2}\tau_2,$$

where $\tau_3 := i(\tau_{27})$ and $\sharp : \Omega^1 \to \mathcal{X}(M)$ the musical isomorphism induced by the G_2–metric.

Remark 4. *(i) For the G_2–structure convention (1.9), the full torsion tensor is*

$$T = \frac{\tau_0}{4} g_\varphi - \tau_{27} + (\tau_1)^\sharp \lrcorner \varphi - \frac{1}{2}\tau_2,$$

(ii) Notice that, in light of the convention 3, the full torsion tensor is expressed as $T = \tau_1 g + (\tau_7)^\sharp \lrcorner \varphi + \tau_{14} + \tau_{27}$

In [Kar09, Lemmata A.8-A.10], Karigiannis compiles several useful identities among the tensors g, φ and ψ:

$$\varphi_{ijk}\varphi_{abc}g^{kc} = g_{ia}g_{jb} - g_{ib}g_{ja} + \psi_{ijab} \tag{1.17}$$

$$\varphi_{ijk}\psi_{abcd}g^{kd} = - g_{ia}\varphi_{jbc} - g_{ib}\varphi_{ajc} - g_{ic}\varphi_{abj} \tag{1.18}$$

$$+ g_{aj}\varphi_{ibc} + g_{bj}\varphi_{aic} + g_{cj}\varphi_{abi} \tag{1.19}$$

$$\psi_{rstu}\psi_{abcd}g^{ra}g^{sb}g^{tc}g^{ud} = 168 \tag{1.20}$$

$$\psi_{rstu}\psi_{abcd}g^{sb}g^{tc}g^{ud} = 24g_{ra} \tag{1.21}$$

Differentiating (1.20) and (1.21), one obtains

$$\nabla_l\psi_{rstu}\psi_{abcd}g^{ra}g^{sb}g^{tc}g^{ud}=0, \tag{1.22}$$

$$\nabla_l\psi_{rstu}\psi_{abcd}g^{sb}g^{tc}g^{ud}=-\psi_{rstu}\nabla_l\psi_{abcd}g^{sb}g^{tc}g^{ud}. \tag{1.23}$$

Lemma 4. *For any vector field X, the 4-form $\nabla_X\psi$ lies in the subspace Ω^4_7 of Ω^4.*

Proof. It is enough to prove that $\nabla_X\psi\perp\Omega^4_1\oplus\Omega^4_{27}$. Considering $X=e_l$ and applying (1.22), we have

$$g(\nabla_l\psi,\psi)=\frac{1}{24}\nabla_l\psi_{rstu}\psi_{abcd}g^{ra}g^{sb}g^{tc}g^{ud}=0,$$

so $\nabla_l\psi\perp\Omega^4_1$. To see that $\nabla_l\psi\perp\Omega^4_{27}$, consider some $\eta\in\Omega^4_{27}$ in local form,

$$\eta=\frac{1}{3!}\big(\frac{1}{4}\operatorname{tr}_g(h)g_{ij}-h_{ij}\big)g^{jl}\psi_{labc}dx^{iabc},$$

and take the inner product with $\nabla_l\psi$:

$$\begin{aligned}g(\nabla_l\psi,\eta)=&\frac{1}{3!}\nabla_l\psi_{rstu}\big(\frac{1}{4}\operatorname{tr}_g(h)g^l_i-h^l_i\big)\psi_{labc}g^{ri}g^{sa}g^{tb}g^{uc}\\=&\frac{1}{4!}\nabla_l\psi_{rstu}(\operatorname{tr}_g(h)g^{rl}-4h^{rl})\psi_{labc}g^{sa}g^{tb}g^{uc}=0,\end{aligned}$$

using that, $\operatorname{tr}_g(h)g^{rl}-4h^{rl}$ is a symmetric $(0,2)$-tensor, while $\nabla_l\psi_{rstu}\psi_{labc}g^{sa}g^{tb}g^{uc}$ is skew-symmetric in r and l, by (1.23). ☐

Using Lemma 4 above and the identity $*(X\lrcorner\psi)=\varphi\wedge X^\flat$ ($X\in\Omega^0(M)$), where $X^\flat$ is the 1–form defined by $X^\flat(Y)=g(X,Y)$, one has:

Corollary 3. *[Kar09, Remark 2.29] With the above notation,*

$$\nabla_l\psi_{rstu}=-T_{lr}\varphi_{stu}+T_{ls}\varphi_{rtu}-T_{lt}\varphi_{rsu}+T_{lu}\varphi_{rst}.$$

For a torsion-free G$_2$–structure, the cross-product (1.10) is parallel, so it satisfies the Leibniz rule

$$\nabla(u\times v)=\nabla u\times v+u\times\nabla v,\quad\forall u,v\in\Omega^0(TM).$$

In general, the action of ∇ on the cross product can be expressed in terms of the total torsion tensor:

Lemma 5. *For the vector fields $u,v,w,z\in\Omega^0(TM)$, we have*

(i) $\nabla_z(u\times v)=\nabla_z u\times v+u\times\nabla_z v+\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,v).$

(ii) *$R(w,z)(u\times v)=R(w,z)u\times v+u\times R(w,z)v+\mathcal{T}(w,z,u,v)$, where*

$$\begin{aligned}\mathcal{T}(w,z,u,v):=\sum_{m=1}^{7}&T(z,e_m)(\nabla_w\psi)(e_m,u,v,\cdot)^\sharp-T(w,e_m)(\nabla_z\psi)(e_m,u,v,\cdot)^\sharp\\&+\big((\nabla_wT)(z,e_m)-(\nabla_zT)(w,e_m)\big)\chi(e_m,u,v)\end{aligned}\tag{1.24}$$

in an orthonormal local frame $\{e_1,...,e_7\}$ of TM.

(iii) *If Y is an associative submanifold of M, for $u,v,z\in\Omega^0(TY)$ and $\eta\in\Omega^0(NY)$, then*

$$\begin{aligned}\nabla_z^\top(u\times v)=&\nabla_z^\top u\times v+u\times\nabla_z^\top v\\\nabla_z^\perp(u\times\eta)=&\nabla_z^\top u\times\eta+u\times\nabla_z^\perp\eta+\sum_{m=1}^{3}T(z,e_m)\chi(e_m,u,\eta)\end{aligned}$$

where $e_1,e_2,e_3=e_1\times e_2$ is a local frame of TY, $\nabla^\top=\nabla-\nabla^\perp$ is the orthogonal projection of ∇ to TY and $\nabla^\perp$ the normal connection on NY.

Proof. (i) Consider normal coordinates $x_1,...,x_7$ about a given $p\in M$, (i.e. $\nabla_ie_j=0$ at p) and an orthonormal frame $e_1,...,e_7$. At the point p, we have:

$$\begin{aligned}\nabla_z(u\times v)&=\sum_{i=1}^{7}\nabla_z(\langle u\times v,e_i\rangle e_i)=\sum_{i=1}^{7}\nabla_z(\varphi(u,v,e_i)e_i)\\&=\sum_{i=1}^{7}z(\varphi(u,v,e_i))e_i+\varphi(u,v,e_i)\nabla_ze_i\\&=\sum_{i=1}^{7}\big(\varphi(\nabla_zu,v,e_i)+\varphi(u,\nabla_zv,e_i)+\varphi(u,v,\nabla_ze_i)+(\nabla_z\varphi)(u,v,e_i)\big)e_i\\&=\sum_{i=1}^{7}\Big(\varphi(\nabla_zu,v,e_i)+\varphi(u,\nabla_zv,e_i)+\sum_{m=1}^{7}T(z,e_m)\psi(e_m,u,v,e_i)\Big)e_i\\&=\nabla_zu\times v+u\times\nabla_zv+\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,v).\end{aligned}$$

Notice that we used $(\nabla_je_i)_p=0$ in the third and fourth equalities, also the fact that $\nabla_z\varphi=T(z,e_m)e_m\lrcorner\psi\in\Omega^3_7$.

(ii) Using the first part, we have

$$\begin{aligned}\nabla_w\nabla_z(u\times v)\;=\;&\nabla_w\nabla_zu\times v+\nabla_zu\times\nabla_wv+\nabla_wu\times\nabla_zv+u\times\nabla_w\nabla_zv\\&+\sum_{i,m=1}^{7}\Big(T(w,e_m)\big(\psi(e_m,\nabla_zu,v,e_i)+\psi(e_m,u,\nabla_zv,e_i)\big)\\&\quad+\big((\nabla_wT)(z,e_m)+T(\nabla_wz,e_m)\big)\psi(e_m,u,v,e_i)\\&\quad+T(z,e_m)\big(\psi(e_m,\nabla_wu,v,e_i)+\psi(e_m,u,\nabla_wv,e_i)\\&\quad+(\nabla_w\psi)(e_m,u,v,e_i)\big)\Big)e_i.\end{aligned}$$

Using the symmetries of the curvature tensor $R(w,z)=\nabla_w\nabla_z-\nabla_z\nabla_w-\nabla_{[w,z]}$ and the fact that ∇ is torsion-free, one has $[w,z]=\nabla_w z-\nabla_z w$, and we compute

$$\begin{aligned}
R(w,z)(u\times v) \quad = \quad & R(w,z)u\times v+u\times R(w,z)v\\
&+\sum_{i,m=1}^{7}\Big(T(z,e_m)(\nabla_w\psi)(e_m,u,v,e_i)\\
&\quad+\big((\nabla_w T)(z,e_m)-(\nabla_z T)(w,e_m)\big)\psi(e_m,u,v,e_i)\\
&\quad-T(w,e_m)(\nabla_z\psi)(e_m,u,v,e_i)\Big)e_i
\end{aligned}$$

(iii) Now, if u and v are in TY, consider $e_1,e_2,e_3=e_1\times e_2$ an orthonormal frame of TY, then we have

$$(\nabla_z u\times v)^\top=\big(\sum_{i=1}^{7}\varphi(\nabla_z u,v,e_i)e_i\big)^\top=\sum_{i=1}^{3}\varphi(\nabla_z u,v,e_i)e_i=\nabla_z^\top u\times v.$$

Notice that we used the TY-invariance of the $\times$ i.e. $T_pY\times T_pY\subset T_pY$. Then,

$$\begin{aligned}
\nabla_z^\top(u\times v)=&(\nabla_z(u\times v))^\top\\
=&\nabla_z^\top u\times v+u\times\nabla_z^\top v+\big(\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,v)\big)^\top\\
=&\nabla_z^\top u\times v+u\times\nabla_z^\top v+\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,v)^\top
\end{aligned}$$

The first equation follows by the relations $N_pY\times N_pY\subset T_pY$ and $T_pY\times N_pY\subset N_pY$. So, $\chi(e_m,u,v)^\top\in T_pY$ if and only if $m\in\{1,2,3\}$ and by the associative of Y $\chi(e_m,u,v)^\top=0$.
For the second relation we have

$$\begin{aligned}
\nabla_z^\perp(u\times\eta)=&\nabla_z(u\times\eta)-\nabla_z^\top(u\times\eta)=\nabla_z(u\times\eta)-(\nabla_z(u\times\eta))^\top\\
=&\nabla_z^\top u\times\eta+u\times\nabla_z^\perp\eta+\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,\eta)-\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,\eta)^\top\\
=&\nabla_z^\top u\times\eta+u\times\nabla_z^\perp\eta+\sum_{m=1}^{7}T(z,e_m)\chi(e_m,u,\eta)-\sum_{m=4}^{7}T(z,e_m)\chi(e_m,u,\eta).
\end{aligned}$$

□

1.4.1 SU(3)–decompositions of the space of differential k-forms

By the relation between G_2–geometry and SU(3)–geometry mentioned in Section 1.1, in this section we collect some facts about k–differential forms on a 6-manifold. It will be a useful computational tool for the Chapter 3.
Let (N,ω,ρ_+) be an oriented, Riemannian 6-manifold. An SU(3)–structure is a reduction of the oriented frame bundle $P_{SO}(N)$ to an SU(3)-principal subbundle [Joy00, Section 6.1]. The required SU(3) reduction is related to the existence of:

An almost complex structure J, i.e. A smooth map $J : \Omega^0(TN) \to \Omega^0(TN)$ such that $J^2 = -\operatorname{Id}$.

A Riemannian metric h with respect to which J is orthogonal i.e. $h(X, Y) = h(JX, JY)$ for any $X, Y \in \Omega^0(TN)$.

And a nowhere vanishing smooth complex valued 3-form ρ of type $(3, 0)$ i.e. Near to each point of N we can find a local unitary coframe of complex-valued 1-forms (dz^1, dz^2, dz^3) for which $\rho = dz^1 \wedge dz^2 \wedge dz^3$.

From the natural SU(3)–action on $\Omega^\bullet(TN)$ we have the irreducible representation [BV07]

$$\begin{aligned}\Omega^2(TN) =&\Omega^2_1(TN) \oplus \Omega^2_6(TN) \oplus \Omega^2_8(TN)\\ \Omega^3(TN) =&\Omega^3_{\mathrm{Re}}(TN) \oplus \Omega^3_{\mathrm{Im}}(TN) \oplus \Omega^3_6(TN) \oplus \Omega^3_{12}(TN),\end{aligned} \tag{1.25}$$

similar to the G_2–decomposition, $\Omega^k_l(TN) \subset \Omega^k(TN)$ denotes (fibrewise) an irreducible SU(3)–submodule of dimension l, with an explicit description:

- $\Omega^2_1(TN) = \{f\omega; f \in C^\infty(N)\}$.
- $\Omega^2_6(TN) = \{\alpha \in \Omega^2(TN); \quad J^*\alpha = -\alpha\}$.
- $\Omega^2_8(TN) = \{\alpha \in \Omega^2(TN); \quad J^*\alpha = \alpha \quad \text{and} \quad \alpha \wedge \omega^2 = 0\}$.
- $\Omega^3_{\mathrm{Re}}(TN) = \{f\rho_+; f \in C^\infty(N)\}$ and $\Omega^3_{\mathrm{Im}}(TN) = \{f\rho_-; f \in C^\infty(N)\}$.
- $\Omega^3_6(TN) = \{\beta \wedge \omega; \quad \beta \in \Omega^1(TN)\}$.
- $\Omega^3_{12}(TN) = \{\gamma \in \Omega^3(TN); \quad \gamma \wedge \omega = 0, \gamma \wedge \rho_+ = \gamma \wedge \rho_- = 0\}$.

Similarly to the G_2–identities from [Kar09, Appendix A and B], for the SU(3)–structure

$$\omega = \frac{1}{2}\omega_{ij}dx^{ij}, \quad \rho_+ = \rho^+_{ijk}dx^{ijk} \quad \text{and} \quad \rho_- = \rho^-_{ijk}dx^{ijk},$$

the following properties hold [BV07, Section 2.2]

$$\begin{aligned}&\rho^+_{iab}\omega_{ab} = 0, \quad \omega_{ip}\omega_{pj} = -\delta_{ij}, \quad \rho^+_{ijp}\omega_{pk} = \rho^-_{ijk},\\ &\rho^-_{ijp}\omega_{pk} = -\rho^+_{ijk}, \quad \rho^+_{ipq}\rho^-_{jpq} = 4\omega_{ij}, \quad \rho^+_{ipq}\rho^+_{jpq} = 4\delta_{ij} = \rho^-_{ipq}\rho^-_{jpq},\\ &\rho^-_{ijp}\rho^+_{klp} = -\omega_{ik}\delta_{jl} + \omega_{jk}\delta_{il} + \omega_{il}\delta_{jk} - \omega_{jl}\delta_{ik},\\ &\rho^+_{ijp}\rho^+_{klp} = -\omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk} + \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = \rho^-_{ijp}\rho^-_{klp}.\end{aligned} \tag{1.26}$$

1.5 Description of the normal bundle of an associative submanifold

We conclude this chapter applying results from 4-dimensional spin geometry to describe the normal bundle of an associative submanifold in terms of a spinor bundle.

1.5.1 Spin group of 4-dimensional vector space

Here we recall some background and fix the notation, following [Sal00, Chapter 2] and [DK90, Chapter 3].

On an inner product space $(V^n, \langle\cdot,\cdot\rangle)$, the Clifford algebra $\mathrm{Cl}(V)$ is a 2^n-dimensional associative algebra with unit 1, generated by the elements of some orthonormal basis $e_1, ..., e_n$ of V with relations

$$e_i^2 = -1, \quad e_ie_j = -e_je_i \quad \text{for} \quad i \neq j.$$

A basis for $\mathrm{Cl}(V)$ is given by

$$e_0 = 1, \quad e_I = e_{i_1} \cdots e_{i_k}$$

where $I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$ for $i_1 < \cdots < i_k$, and $\mathrm{Cl}(V)$ admits a natural involution

$$\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

defined by $\alpha(x) = \widetilde{x} := \sum_I \epsilon_I x_I e_I$, where $\epsilon_I := (-1)^{k(k+1)/2}$ and $x_I \in \mathbb{R}$ are the components of x in the basis $\{e_I\}$. Denote by $\deg(e_I) := |I|$ the degree of an element $e_I \in \mathrm{Cl}(V)$, by $\mathrm{Cl}_k(V)$ the subset of elements of degree k, and by $\mathrm{Cl}^0(V)$ and $\mathrm{Cl}^1(V)$ the subspaces of elements of even and odd degree, respectively.

Example 3. *On $V = \mathbb{R}^4$ with the Euclidean inner product, we have $\mathrm{Cl}(V) = M_2(\mathbb{H})$, the 2×2 matrices with entries in the quaternions $\mathbb{H} = \langle i, j, k\rangle$. The elements of $\mathrm{Cl}(V)$ are 1, e_i, $\{e_ie_j\}_{i<j}$, $\{e_ie_je_k\}_{i<j<k}$ and $e_1e_2e_3e_4$, with $i, j, k = 1, 2, 3, 4$, with generators*

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad \text{and} \quad e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

and the involution $\alpha(A) = A^*$ is the transpose conjugation.

Denote the set of units of $\mathrm{Cl}(V)$ by $\mathrm{Cl}^\times(V)$. Considering the twisted adjoint representation $\widetilde{Ad} : \mathrm{Cl}^\times(V) \to \mathrm{Gl}(\mathrm{Cl}(V))$ given by

$$\widetilde{Ad}(x)y = ((x)^0 - (x)^1)y\widetilde{x},$$

where $(x)^0 \in \mathrm{Cl}^0(V)$ and $(x)^1 \in \mathrm{Cl}^1(V)$ are the even and odd parts of x, respectively. We define the *Spin group* of V:

$$\mathrm{Spin}(V) := \{x \in \mathrm{Cl}^0(V) | \, \widetilde{Ad}(x)V = V, \, x\widetilde{x} = 1\}.$$

For $\dim V \geq 3$, $\mathrm{Spin}(V)$ is a compact, connected and simply connected Lie group, fitting in a short exact sequence [Sal00, Lemma 4.25]

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1.$$

In particular, the following results hold in dimensions 3 and 4:

Lemma 6. *[Sal00, Lemma 4.4] For every* $x \in \mathrm{Sp}(1)$*, there is a unique orthogonal matrix* $\xi_0(x) \in \mathrm{SO}(3)$*, such that* $\xi_0(x)y = xy\widetilde{x}$*, for all* $y \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$*, and the map* $\xi_0 : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ *is a surjective homomorphism with kernel* $\{\pm 1\}$*, hence*

$$\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\mathbb{Z}_2 \quad \textit{and} \quad \mathrm{Spin}(3) \cong \mathrm{Sp}(1).$$

Lemma 7. *[Sal00, Lemma 4.6] For every* $x, y \in \mathrm{Sp}(1)$*, there is a unique orthogonal matrix* $\eta_0(x,y) \in \mathrm{SO}(4)$*, such that* $\eta_0(x,y)z = xz\widetilde{y}$*, for all* $z \in \mathbb{R}^4 \cong \mathbb{H}$*, and the map* $\eta_0 : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ *is a surjective homomorphism with kernel* $\{\pm(1,1)\}$*, hence*

$$\mathrm{SO}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathbb{Z}_2 \quad \textit{and} \quad \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$

The last lemma provides two natural surjective homomorphisms $\rho^\pm : \mathrm{SO}(4) \to \mathrm{SO}(3)$ and, therefore, two exact sequences

$$1 \to \mathrm{Sp}(1) \xrightarrow{\iota^\pm} \mathrm{SO}(4) \xrightarrow{\rho^\pm} \mathrm{SO}(3) \to 1$$

where $\iota^+(v) = \eta_0([v,1])$ and $\iota^-(v) = \eta_0([1,v])$, interpreting η_0 as the induced homomorphism on the quotient $\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$. Those sequences are related to the SO(4)-action on the spaces of self-dual and anti-self-dual 2-forms of a 4-dimensional inner-product space.

An element $q \in \mathbb{H}$ in the canonical basis $q = t + xi + yj + zk = (t + xi) + (y + zi)j$ can be identified with the 2×2 complex matrix

$$A = \begin{pmatrix} t + xi & -y + zi \\ y + zi & t - xi \end{pmatrix},$$

with

$$\det A = t^2 + x^2 + y^2 + z^2 = |q|^2.$$

Since $A^*A = (\det A)I_2$, every $q \in \mathrm{Sp}(1) \cong S^3$ is identified with a unitary matrix with determinant 1, that is, $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$.

Definition 8. *Let* V *be a real inner product space of dimension* $2n \equiv 2, 4 \mod 8$ *or* $2n + 1 \equiv 3 \mod 8$*. A* Spin structure *on* V *is a quadruple* (S, I, J, Γ)*, where* S *is a* 2^{n+1}*-dimensional real inner product space,* I *and* J *are two anti-commuting orthogonal complex structure*

$$I^{-1} = I^* = -I, \quad J^{-1} = J^* = -J, \quad IJ = -JI,$$

and $\Gamma : V \to \mathrm{End}(S)$ *is a real linear map with the following properties:*

$$\Gamma(v)^* + \Gamma(v) = 0, \quad \Gamma(v)^*\Gamma(v) = |v|^2\mathrm{Id}, \quad \Gamma(v)I = I\Gamma(v), \quad \Gamma(v)J = J\Gamma(v), \quad \forall v \in V.$$

Example 4. *For a vector space* V *of real dimension* 4*, using the identification* $V \cong \mathbb{H}$ *and defining* $S = \mathbb{H} \oplus \mathbb{H}$*, we have the maps* $\Gamma : \mathbb{H} \to \mathrm{End}(\mathbb{H} \oplus \mathbb{H})$*,* $I, J : \mathbb{H} \oplus \mathbb{H} \to \mathbb{H} \oplus \mathbb{H}$ *defined for* $v, x, y \in \mathbb{H}$ *by*

$$\Gamma(v)(x,y) = (vy, -\bar{v}x), \quad I(x,y) = (xi, yi), \quad J(x,y) = (xj, yj).$$

It is interesting to note that

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix},$$

where $\gamma : \mathbb{H} \to \mathrm{End}(\mathbb{H})$ *also satisfies*

$$\gamma(v)^* + \gamma(v) = 0, \quad \gamma(v)^*\gamma(v) = |v|^2\mathrm{Id}, \quad \forall v \in \mathbb{H}.$$

Given a Spin structure on a 4-dimensional space V, consider $S = S^+ \oplus S^-$, where S^+ and S^- are copies of $\mathbb{C}^2$ with standard Hermitian metric $\langle \cdot, \cdot \rangle$. The associated symplectic form compatible with the almost complex structure $I : S^\pm \to S^\pm$ is defined by $\omega(x, y) := \langle x, Iy\rangle$. Now, consider the (real) 4-dimensional space $\mathrm{Hom}_I(S^+, S^-) = \mathrm{Re}(\mathrm{Hom}(S^+, S^-))$ of linear maps over the quaternions, where $\mathrm{Hom}(S^+, S^-)$ are complex linear maps. Unitary elements of $\mathrm{Hom}_I(S^+, S^-)$ preserve the Hermitian and symplectic structures, and $\gamma : V \to \mathrm{Hom}_I(S^+, S^-)$ defined above acts on the standard basis by

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma(e_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma(e_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Up to isomorphism, the above generate $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$, since the symmetry group $\mathrm{SU}(2)^+ \times \mathrm{SU}(2)^-$ of (S^+, S^-) is connected. Thus γ fixes the orientation of V and, using the sympletic form to identify S^+ with its dual, we have

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-. \tag{1.27}$$

Moreover, given $v \in V$, consider the Hermitian adjoint $\gamma(v)^* : S^- \to S^+$ of the map $\gamma(v) : S^+ \to S^-$. Then, for orthonormal vectors $v, v' \in V$, the map $\gamma(v)^*\gamma(v')$ defines an endomorphism of S^+ which satisfies

$$\gamma(v)^*\gamma(v) = 1 \quad \text{and} \quad \gamma(v)^*\gamma(v') + \gamma^*(v')\gamma(v) = 0.$$

In particular, we have a natural action ρ of $\Lambda^2(V)$ on S^+ defined by

$$\rho(v \wedge v')s := -\gamma(v)^*\gamma(v')s \quad \text{for} \quad s \in S^+.$$

Now, with respect to the Euclidean metric, the 2–forms split as $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$, where $\Lambda^2_+(V)$ and $\Lambda^2_-(V)$ denote the self-dual and anti-self-dual forms, respectively:

$$\Lambda^2_\pm(V) := \{\beta \in \Lambda^2(V) \mid *\beta = \pm\beta\}.$$

We observe that $\Lambda^2_-(V)$ acts trivially on S^+, by direct inspection on basis elements:

$$\Lambda^2_-(V) = \mathrm{Span}\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3, e_1 \wedge e_3 - e_4 \wedge e_2\}$$

$$\begin{aligned}\rho(e_1\wedge e_2 - e_3\wedge e_4) &= -\gamma(e_1)^*\gamma(e_2)+\gamma(e_3)^*\gamma(e_4)\\ &= \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}=0,\end{aligned}$$

$$\begin{aligned}\rho(e_1\wedge e_4 - e_2\wedge e_3) &= -\gamma(e_1)^*\gamma(e_4)+\gamma(e_2)^*\gamma(e_3)\\ &= \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}+\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=0,\end{aligned}$$

$$\begin{aligned}\rho(e_1\wedge e_3 - e_4\wedge e_2) &= -\gamma(e_1)^*\gamma(e_3)+\gamma(e_4)^*\gamma(e_2)\\ &= \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}=0.\end{aligned}$$

Thus we get the isomorphisms $\Lambda^2_+(V)\to\mathfrak{su}(S^+)$ and $\Lambda^2_-(V)\to\mathfrak{su}(S^-)$.

1.5.2 The twisted Dirac operator

Let (M,φ) be a smooth 7-manifold with G$_2$–structure and Y an associative submanifold of M. The oriented orthonormal frame of TY has the form $\{e_1,e_2,e_3=e_1\times e_2\}$. So, with respect to the splitting $TM|_Y=TY\oplus NY$, the cross product induces maps

$$\begin{aligned}\Omega^0(TY)\times\Omega^0(TY)&\to\Omega^0(TY),\\ \Omega^0(TY)\times\Omega^0(NY)&\to\Omega^0(NY),\\ \Omega^0(NY)\times\Omega^0(NY)&\to\Omega^0(TY).\end{aligned}\tag{1.28}$$

In particular, the map $\gamma:\Omega^0(TY)\times\Omega^0(NY)\to\Omega^0(NY)$ endows NY with a Clifford bundle structure.

Since the Levi-Civita connection of (M,φ) induces metric connections on the bundles TY and NY, the composition

$$\Omega^0(NY)\xrightarrow{\nabla_{A_0}}\Omega^0(TY)\otimes\Omega^0(NY)\xrightarrow{\gamma}\Omega^0(NY)\tag{1.29}$$

defines a natural *Fueter-Dirac operator* $\not{D}_{A_0}(\sigma):=\gamma(\nabla_{A_0}(\sigma))$, where $A_0\in\Omega^1(Y,\mathfrak{so}(4))$ denotes the connection induced on NY by the Levi-Civita connection ∇^φ of the G$_2$–metric of (M,φ). To simplify the notation, the twisted Dirac operator induced by the normal connection A_0 will be denoted just by $\not{D}$.

The normal bundle NY of an associative submanifold is trivial [CP15, Lemma 5.1, arXiv version: 1207.4470v3]. In particular, the second Stiefel-Whitney class $w_2(NY)$

vanishes, so there exists a spin structure on NY [LM16, Theorem 1.7]. This is equivalent to the existence of a map $\Gamma : NY \to \mathrm{End}(S)$ such that

$$\Gamma(\sigma) + \Gamma(\sigma)^* = 0 \quad \Gamma(\sigma)^*\Gamma(\sigma) = \langle\sigma, \sigma\rangle \,\mathrm{Id} \quad \sigma \in \Omega^0(NY),$$

where S is a vector bundle of (real) rank 8 and it splits into Γ–eigenbundles S^+ and S^- of rank 4. We saw in the last Section that the Spin structure induces an isomorphism

$$\rho_\pm : \Lambda^2_\pm(NY) \to \mathfrak{su}(S^\pm),$$

so, by Lemma 2, the Spin structure $\Gamma_0 : TY \to \mathrm{End}(S^+)$ on TY coincides with the Spin structure on NY via the projection $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Defining the Clifford multiplication

$$\tau := \Gamma_0 \otimes \mathrm{Id}_{S^-} : TY \to \mathrm{End}(S^+ \otimes S^-)$$

and using the Spin connection ∇ on $S^+ \otimes S^-$,

$$\nabla(\sigma \otimes \varepsilon) = \nabla^+\sigma \otimes \varepsilon + \sigma \otimes \nabla^-\varepsilon,$$

we form the Dirac operator $D : \Omega^0(Y, S^+ \otimes S^-) \to \Omega^0(Y, S^+ \otimes S^-)$ by

$$D(\sigma \otimes \varepsilon) := \sum_{i=1}^{3} \tau(e_i)\nabla_i(\sigma \otimes \varepsilon).$$

Proposition 4. *Under the isomorphism (1.27), we have $NY \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-$, the Spin connection ∇ and the Clifford multiplication τ agree with the induced connection $\nabla^\perp$ on NY and γ, respectively.*

Proof. In fact, each section $\sigma\otimes\varepsilon$ of $S^+\otimes_{\mathbb{C}}S^-$ induces a section $\nu = \sigma^*\otimes\varepsilon$ on $\mathrm{Hom}(S^+, S^-) \cong (S^+)^* \otimes S^-$ such that $\nu(\sigma) = \sigma^*(\sigma) \otimes \varepsilon = \varepsilon$, then

$$\begin{aligned}\nabla\nu &= \nabla(\sigma^* \otimes \varepsilon)\\ &= (\nabla^+)^*\sigma^* \otimes \varepsilon + \sigma^* \otimes \nabla^-\varepsilon,\end{aligned}$$

where $\nabla\nu$ is a section on $T^*Y \otimes \mathrm{Hom}(S^+, S^-)$, so, for each σ section on S^+

$$\begin{aligned}(\nabla\nu)(\sigma) &= (\nabla^+)^*\sigma^*(\sigma) \otimes \varepsilon + \sigma^*(\sigma) \otimes \nabla^-\varepsilon\\ &= [d\sigma^*(\sigma) - \sigma^*(\nabla^+\sigma)] \otimes \varepsilon + \sigma^*(\sigma) \otimes \nabla^-\varepsilon\\ &= -\nu(\nabla^+\sigma) + \nabla^-(\nu(\sigma)).\end{aligned}$$

On the other hand, the Spin connection ∇ is compatible with the induced connection $\nabla^\perp$, that is,

$$\nabla^-(\Gamma(n)\sigma) = \Gamma(\nabla^\perp n)\sigma + \Gamma(n)\nabla^+\sigma,$$

where $\Gamma : NY \to \mathrm{Hom}_J(S^+, S^-)$ is the isomorphism induced by (1.27), then for each section n of NY and σ of S^+,

$$\Gamma(\nabla^\perp n) = -\Gamma(n)\nabla^+\sigma + \nabla^-(\Gamma(n)\sigma).$$

Therefore, $\nabla^\perp$ agrees with the Spin connection ∇ via the isomorphism Γ. Finally, with respect to the Clifford multiplications we have

$$\begin{tikzcd}
TY \arrow[r, "\Gamma_0"] \arrow[d, "\gamma"] & \mathrm{End}(S^+) \arrow[r, hook, "\otimes \mathrm{Id}_{\mathrm{End}(S^-)}"] & \mathrm{End}(S^+ \otimes_{\mathbb{C}} S^-) \\
\mathrm{End}(NY \otimes_{\mathbb{R}} \mathbb{C}) \arrow[urr, "\cong"'] & &
\end{tikzcd}$$

and by Schur's lemma γ and τ are the same. $\square$

In conclusion, (1.29) defines a twisted Dirac operator.

2 Deformation of associative submanifolds

We now address the general framework proposed by Akbulut and Salur [AS08a, AS08b], in which the role of torsion in the associative deformation theory is captured by a *twisted* Fueter-Dirac operator. Given an associative submanifold Y^3 in (M, φ), the G_2–structure induces connections on the bundles NY and TY. Moreover, Proposition 4 gives an identification $NY \cong \mathrm{Re}(S^+ \otimes_{\mathbb{C}} S^-)$, with the respective reductions $\Lambda^2_\pm(NY) \cong \mathfrak{su}(S^\pm) = \mathrm{ad}(S^\pm)$. We will refer to elements in the kernel $\ker \not{D}$ of the Dirac operator (1.29) as harmonic spinors twisted by S^-, or simply, *twisted harmonic spinors.*

Denote by $\mathcal{A}(S^\pm)$ the space of connections on each spinor bundle $S^\pm$, and let $A_0 \in \Omega^1(Y, \mathfrak{so}(4))$ be the induced connection on NY, so that the isomorphism $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ gives a decomposition $A_0 = A_0^+ \oplus A_0^-$, with $A_0^\pm \in \mathcal{A}(S^\pm)$. Fixing these reference connections, each $\mathcal{A}(S^\pm)$ is an affine space modelled on $\Omega^1(Y, \mathrm{ad}(S^\pm))$, so a connection $A^\pm \in \mathcal{A}(S^\pm)$ is of the form

$$A^\pm = A_0^\pm + a^\pm \quad \text{for} \quad a^\pm \in \Omega^1(Y, \mathrm{ad}(S^\pm)).$$

Thus a connection on NY has the form

$$A = A_0 + a = (A_0^+ + a^+) \oplus (A_0^- + a^-) \quad \text{for} \quad a \in \Omega^1(Y, \mathrm{ad}(NY)).$$

Now, using the Clifford multiplication (indeed the cross-product), we define the *twisted Dirac operator*

$$\not{D}_{\mathrm{A}} := \sum_{j=1}^{3} e_i \times \nabla_{e_i} \quad : \quad \Omega^0(NY) \to \Omega^0(NY)$$

where $\nabla := \nabla_A$ is given by a connection on NY and the normal sections in $\ker(\not{D}_{\mathrm{A}})$ are called *harmonic spinors twisted by* (S^-, A). The following Definition is adopted from [AS08a]:

Definition 9. *Let Y be an associative submanifold of (M, φ). The* Fueter-Dirac operator *associated with Y is*

$$\not{D}_{\mathrm{A}}\, \sigma := \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} \sigma - e_i \times a(e_i)(\sigma), \tag{2.1}$$

where $a \in \Omega^1(Y, \mathrm{ad}(NY))$ defined by $a(e_i)(\sigma) = (\nabla_\sigma(e_i))^\perp$ is the normal component of $\nabla_\sigma(e_i)$, and ∇ is the Levi-Civita connection on M.

We know from [AS08a, Theorem 6] that the linearisation of the deformation problem for an associative submanifold Y of (M, φ) at Y is identified with $\ker \not{D}_{\mathrm{A}}$, so this space is called the *infinitesimal deformation space* of Y. Our motivation is precisely the expectation that a Weitzenböck formula for (2.1), in favourable cases at least, can give information about the deformation space $\ker \not{D}_{\mathrm{A}}$.

Lemma 8. *Let* $\{e_1, e_2, e_3\}$ *and* $\{\eta_4, ..., \eta_7\}$ *be orthonormal frames of the vector bundles* TY *and* NY*, respectively. Then*

$$\not{D}_{\mathrm{A}}\,\sigma = \sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i}\sigma - \sum_{k=4}^{7}(\nabla_\sigma \psi)(\eta_k, e_1, e_2, e_3)\eta_k. \tag{2.2}$$

Proof. Since A_0 is the connection induced on NY by the Levi-Civita connection on M given by the G_2–metric g_φ, we have $\nabla_{A_0} = \nabla^{\perp}$. Now, for each $\sigma \in \Omega^0(NY)$,

$$\begin{aligned}\sum_{i=1}^{3} e_i \times a(e_i)(\sigma) &= e_1 \times (\nabla_\sigma e_1)^{\perp} + e_2 \times (\nabla_\sigma e_2)^{\perp} + e_3 \times (\nabla_\sigma e_3)^{\perp}\\ &= (e_2 \times e_3) \times (\nabla_\sigma e_1)^{\perp} + (e_3 \times e_1) \times (\nabla_\sigma e_2)^{\perp} + (e_1 \times e_2) \times (\nabla_\sigma e_3)^{\perp}\\ &= \chi((\nabla_\sigma e_1)^{\perp}, e_2, e_3) + \chi((\nabla_\sigma e_2)^{\perp}, e_3, e_1) + \chi((\nabla_\sigma e_3)^{\perp}, e_1, e_2)\\ &= (\diamondsuit).\end{aligned}$$

Since Y is associative exactly when $\chi|_{TY} = 0$, this implies

$$\chi((\nabla_\sigma e_i)^{\perp}, e_j, e_k) = \chi(\nabla_\sigma e_i, e_j, e_k).$$

Furthermore, the section $\chi(\nabla_\sigma(e_i), e_j, e_k)$ lies on the normal component, so

$$\begin{aligned}(\diamondsuit) &= \sum_{k=4}^{7}(\langle \chi(\nabla_\sigma(e_1), e_2, e_3), \eta_k\rangle + \langle \chi(e_1, \nabla_\sigma(e_2), e_3), \eta_k\rangle + \langle \chi(e_1, e_2, \nabla_\sigma(e_3)), \eta_k\rangle)\eta_k\\ &= \sum_{k=4}^{7}(-(\nabla_\sigma\psi)(e_1, e_2, e_3, \eta_k) + \sigma(\psi(e_1, e_2, e_3, \eta_k)) - \psi(e_1, e_2, e_3, \nabla_\sigma(\eta_k)))\eta_k\\ &= \sum_{k=4}^{7}((\nabla_\sigma\psi)(\eta_k, e_1, e_2, e_3)\eta_k.\end{aligned}$$

To obtain the second equality we used the covariant derivative of ψ:

$$(\nabla_\sigma\psi)(e_1, e_2, e_3, \eta_k) = \sigma(\psi(e_1, e_2, e_3, \eta_k)) - \psi(\nabla_\sigma e_1, e_2, e_3, \eta_k) - \cdots - \psi(e_1, e_2, e_3, \nabla_\sigma \eta_k)$$

and equation (1.7), and for the last one we used the skew-symmetry of $\nabla_\sigma\psi$ and the associativity condition $\chi(e_1, e_2, e_3) = 0$. □

Remark 5. *If the* G_2*–structure is choosen with the convention* (1.9)*, then the operator* $\not{D}_{\mathrm{A}}$ *is expressed as*

$$\not{D}_{\mathrm{A}}\,\sigma = -\sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i}\sigma + \sum_{k=4}^{7}(\nabla_\sigma \psi)(\eta_k, e_1, e_2, e_3)\eta_k. \tag{2.3}$$

Fix $p \in Y$ and choose local orthonormal frames $\{e_1, e_2, e_3\}$ and $\{\eta_4, \eta_5, \eta_6, \eta_7\}$ of TY and NY, respectively, such that

$$(\nabla_{e_i} e_j)_p = (\nabla_{e_i} \eta_k)_p = (\nabla_{\eta_l} \eta_k)_p = 0 \tag{2.4}$$

for all $i, j = 1, 2, 3$ and $k, l = 4, 5, 6, 7$. Observe that, for any sections $\sigma, \eta \in \Omega^0(TM|_Y)$, one has

$$\nabla_\sigma(\eta) \in \Omega^0(TM|_Y) = \Omega^0(TY) \oplus \Omega^0(NY), \tag{2.5}$$

so both tangent and normal components of (2.4) vanish at p. Then the following holds at p:

$$\begin{aligned}
\not{D}_{\mathrm{A}}{}^{2}\,\sigma &= \sum_{i,j=1}^{3} e_i \times \nabla_i^\perp (e_j \times \nabla_j^\perp \sigma) - \sum_{i=1}^{3}\sum_{l=4}^{7} e_i \times \nabla_i^\perp \{(\nabla_\sigma \psi)(\eta_l, e_1, e_2, e_3)\eta_l\} \\
&\quad - \sum_{j=1}^{3}\sum_{k=4}^{7} (\nabla_{e_j \times \nabla_j^\perp \sigma} \psi)(\eta_k, e_1, e_2, e_3)\eta_k + \sum_{k,l=4}^{7} (\nabla_{(\nabla_\sigma \psi)(\eta_l, e_1, e_2, e_3)\eta_l} \psi)(\eta_k, e_1, e_2, e_3)\eta_k \\
&= \underbrace{\sum_{i,j=1}^{3} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \sigma)}_{\text{(I)}} + \underbrace{\sum_{i,j,l=1}^{3}\sum_{m=4}^{7} T(e_i, e_l)\psi(e_l, e_j, \nabla_j^\perp \sigma, \eta_m) e_i \times \eta_m}_{\text{(II)}} \\
&\quad \underbrace{- \sum_{j=1}^{3}\sum_{k,n=4}^{7} \varphi(e_j, \nabla_j^\perp \sigma, \eta_n)(\nabla_{\eta_n}\psi)(\eta_k, e_1, e_2, e_3)\eta_k}_{\text{(III)}} \\
&\quad \underbrace{- \sum_{i=1}^{3}\sum_{l=4}^{7} e_i (\nabla_\sigma \psi(\eta_l, e_1, e_2, e_3)) e_i \times \eta_l}_{\text{(IV)}} \\
&\quad \underbrace{+ \sum_{k,l=4}^{7} (\nabla_\sigma \psi)(\eta_l, e_1, e_2, e_3)(\nabla_{\eta_l}\psi)(\eta_k, e_1, e_2, e_3)\eta_k}_{\text{(V)}}\,.
\end{aligned} \tag{2.6}$$

To obtain (I) and (II) we used Lemma 5 (i) and the property $(\nabla_i e_j)_p = 0$, whereas (IV) follows from the Leibniz rule for $\nabla^\perp$ and $(\nabla_i \eta_k)_p = 0$.

Remark 6. *In [Gay14], Gayet obtains a Weitzenböck-type formula when the* G_2*–structure is torsion-free:*

$$\not{D}^2 = \nabla^*\nabla + \mathcal{R} - \mathcal{A}. \tag{2.7}$$

The term $\mathcal{R}(\sigma) = \pi^\perp \sum_{i=1}^{3} R(e_i, \sigma)e_i$ *can be seen as a partial Ricci operator, where* R *is the curvature tensor of* g *on* M *and* $\pi^\perp$ *is the orthogonal projection to* NY*, and*

$$\mathcal{A} : \Omega^0(NY) \to \Omega^0(\mathrm{Sym}(TY)),$$

defined by $\mathcal{A}(\sigma) = S^t \circ S(\sigma)$, *is a symmetric positive* 0^{th}*–order operator determined by the shape operator* $S(\sigma)(X) = -(\nabla_X \sigma)^\top$. *With these data, Gayet formulates a vanishing theorem for a compact associative submanifold* Y *of a* G_2*–manifold and proves that* Y *is rigid when the spectrum of the operator* $\mathcal{R} - \mathcal{A}$ *is positive. The advantage of formula* (2.7) *lies in the relation between the intrinsic and extrinsic geometries of the associative submanifold, because* $\mathcal{R} - \mathcal{A}$ *is obtained from a curvature term*

$$-\sum_{i<j}^{3} (e_i \times e_j) \times R^\perp(e_i, e_j)\sigma. \tag{2.8}$$

While one cannot entirely apply his proof to the general case (because the full torsion tensor is non-zero), we are able to adapt some of its steps.

Given $\sigma \in \Omega^0(NY)$, we define operator $\mathcal{B} : \Omega^0(NY) \to \Omega^0(TY)$ by

$$\mathcal{B}(\sigma) := \sum_{j=1}^{3} e_j \times S_\sigma(e_j). \tag{2.9}$$

We recall the mean curvature vector field H of a immersed submanifold by

$$\begin{aligned}
\sum_{i=1}^{3} (\nabla_i e_i)^\perp &= \sum_{i=1}^{3}\sum_{k=4}^{7} \langle \nabla_i e_i, \eta_k\rangle \eta_k = -\sum_{i=1}^{3}\sum_{k=4}^{7} \langle e_i, \nabla_i \eta_k\rangle \eta_k \\
&= -\sum_{i=1}^{3}\sum_{k=4}^{7} \langle e_i, (\nabla_i \eta_k)^\top\rangle \eta_k = \sum_{i=1}^{3}\sum_{k=4}^{7} \langle e_i, S_{\eta_k}(e_i)\rangle \eta_k \\
&= \sum_{k=4}^{7} \operatorname{tr}(S_{\eta_k})\eta_k = H
\end{aligned}$$

Lemma 9. *Denoting by* $\nabla^*\nabla$ *the Laplacian of the connection* $\nabla^\perp$, *by* $\mathcal{R}$ *the partial Ricci operator* $\mathcal{R}(\sigma) = \pi^\perp \sum_{i=1}^{3} R(e_i, \sigma)e_i$, *and by* $\mathcal{B}$ *the* 0^{th}*–order operator defined in* (2.9), *for a normal vector field* σ *to an associative submanifold* Y *one has*

$$\begin{aligned}
(\mathrm{I}) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \pi^\perp\Big(\sum_{i\in\mathbb{Z}_3} e_i \times \mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})\Big) \\
H \times \mathcal{B}(\sigma) - \sum_{j=1}^{3} \pi^\perp(T(e_j, \cdot)^\sharp) \times S_\sigma(e_j) + (\operatorname{tr} S_\sigma)H - \mathcal{A}(\sigma) + \pi^\perp(T(\mathcal{B}(\sigma), \cdot)^\sharp)
\end{aligned}$$

where $\mathcal{T}$ *is defined in* (1.24) *by*

$$\begin{aligned}
\mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1}) := \sum_{m=1}^{7} T(\sigma, e_m)(\nabla_{i+1}\psi)(e_m, e_i, e_{i+1}, \cdot)^\sharp - T_{i+1m}(\nabla_\sigma\psi)(e_m, e_i, e_{i+1}, \cdot)^\sharp \\
+ \big((\nabla_{i+1}T)(\sigma, e_m) - (\nabla_\sigma T)(e_{i+1}, e_m)\big)\chi(e_m, e_i, e_{i+1}).
\end{aligned}$$

Proof. In terms of an orthonormal frame $\{e_1, e_2, e_3\}$ of TY,

$$(\mathrm{I}) = \sum_{i=1}^{3} e_i \times (e_i \times \nabla_i^\perp \nabla_i^\perp \sigma) + \sum_{\substack{i,j=1\\ i\neq j}}^{3} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \sigma)$$

$$
\begin{aligned}
&= -\sum_i \nabla^\perp_i \nabla^\perp_i \sigma - \sum_{i\neq j}(e_i \times e_j) \times \nabla^\perp_i \nabla^\perp_j \sigma \\
&= -\sum_i \nabla^\perp_i \nabla^\perp_i \sigma - \nabla^\perp_{\nabla^\top_i e_i}\sigma - \sum_{i<j}(e_i \times e_j) \times (\nabla^\perp_i \nabla^\perp_j - \nabla^\perp_j \nabla^\perp_i - \nabla^\perp_{[e_i,e_j]})\sigma \\
&= \nabla^*\nabla\sigma - \sum_{i<j}(e_i \times e_j) \times R^\perp(e_i,e_j)\sigma.
\end{aligned}
$$

Here $R^\perp \in \Omega^0(\Lambda^2 T^*Y \otimes \mathrm{End}(NY))$ is the normal curvature of Y:

$$
R^\perp(e_i,e_j)\sigma = (\nabla^\perp_i \nabla^\perp_j - \nabla^\perp_j \nabla^\perp_i - \nabla^\perp_{[e_i,e_j]})\sigma. \tag{2.10}
$$

To obtain the second equality, we used (1.8) in each term of the form

$$
\begin{aligned}
e_i \times (e_i \times \nabla^\perp_i \nabla^\perp_i \sigma) &= -\chi(e_i, e_i, \nabla^\perp_i \nabla^\perp_i \sigma) - \langle e_i, e_i\rangle \nabla^\perp_i \nabla^\perp_i \sigma + \langle e_i, \nabla^\perp_i \nabla^\perp_i \sigma\rangle e_i \\
&= -\nabla^\perp_i \nabla^\perp_i \sigma.
\end{aligned}
$$

Moreover, for $i \neq j$,

$$
\begin{aligned}
e_i \times (e_j \times \nabla^\perp_i \nabla^\perp_j \sigma) &= -\chi(e_i, e_j, \nabla^\perp_i \nabla^\perp_j \sigma) - \langle e_i, e_j\rangle \nabla^\perp_i \nabla^\perp_j \sigma + \langle e_i, \nabla^\perp_i \nabla^\perp_j \sigma\rangle e_j \\
&= -\chi(\nabla^\perp_i \nabla^\perp_j \sigma, e_i, e_j) = \nabla^\perp_i \nabla^\perp_j \sigma \times (e_i \times e_j) \\
&= -(e_i \times e_j) \times \nabla^\perp_i \nabla^\perp_j \sigma.
\end{aligned}
$$

Now, expanding the summands in the frame $\{\eta_4, \dots, \eta_7\}$ and using anti-symmetry of the mixed product and the Ricci equation, we have

$$
\begin{aligned}
-\sum_{i<j}^{3}(e_i \times e_j) \times R^\perp(e_i,e_j)\sigma &= -\frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle (e_i \times e_j) \times R^\perp(e_i,e_j)\sigma, \eta_k\rangle \eta_k \\
&= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle R^\perp(e_i,e_j)\sigma, (e_i \times e_j) \times \eta_k\rangle \eta_k \\
&= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle R(e_i,e_j)\sigma, (e_i \times e_j) \times \eta_k\rangle \eta_k \\
&\quad + \langle [S_\sigma, S_{(e_i\times e_j)\times\eta_k}]e_i, e_j\rangle \eta_k \\
&= \underbrace{-\frac{1}{2}\pi^\perp \sum_{i,j=1}^{3}(e_i \times e_j) \times R(e_i,e_j)\sigma}_{(\star)} \\
&\quad \underbrace{+ \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle [S_\sigma, S_{(e_i\times e_j)\times\eta_k}]e_i, e_j\rangle \eta_k}_{(\star\star)}.
\end{aligned}
$$

Applying the Bianchi identity $R(e_i,e_j)\sigma = -R(\sigma,e_i)e_j - R(e_j,\sigma)e_i$ to the first term, expanding the sum and using Lemma 5, we have:

$$
(\star) = \pi^\perp \sum_{i,j=1}^{3}(e_i \times e_j) \times R(e_j,\sigma)e_i
$$

$$
\begin{aligned}
&= \pi^{\perp}(e_3 \times R(e_2,\sigma)e_1 - e_2 \times R(e_3,\sigma)e_1 - e_3 \times R(e_1,\sigma)e_2 + e_1 \times R(e_3,\sigma)e_2 \\
&\quad + e_2 \times R(e_1,\sigma)e_3 - e_1 \times R(e_2,\sigma)e_3) \\
&= \pi^{\perp}(\underbrace{-e_1 \times [R(e_2,\sigma)e_1 \times e_2 + e_1 \times R(e_2,\sigma)e_2}_{\text{(I)}} + \mathcal{T}(e_2,\sigma,e_1,e_2)] \\
&\quad \underbrace{-e_2 \times [R(e_3,\sigma)e_2 \times e_3 + e_2 \times R(e_3,\sigma)e_3}_{\text{(II)}} + \mathcal{T}(e_3,\sigma,e_2,e_3)] \\
&\quad - \underbrace{e_3 \times [R(e_1,\sigma)e_3 \times e_1 + e_3 \times R(e_1,\sigma)e_1}_{\text{(III)}} + \mathcal{T}(e_1,\sigma,e_3,e_1)] \\
&\quad + e_3 \times R(e_2,\sigma)e_1 + e_1 \times R(e_3,\sigma)e_2 + e_2 \times R(e_1,\sigma)e_3).
\end{aligned}
$$

Using the identity $u \times (v \times w) + v \times (u \times w) = \langle u, w\rangle v + \langle v, w\rangle u - 2\langle u, v\rangle w$, we check that

$$
\begin{aligned}
\text{(I)} &= -e_3 \times R(e_2,\sigma)e_1 - (e_2,\sigma,e_1,e_2)e_1 + 2(e_2,\sigma,e_1,e_1)e_2 + R(e_2,\sigma)e_2 \\
\text{(II)} &= -e_1 \times R(e_3,\sigma)e_2 - (e_3,\sigma,e_2,e_3)e_2 + 2(e_3,\sigma,e_2,e_2)e_3 + R(e_3,\sigma)e_3 \\
\text{(III)} &= -e_2 \times R(e_1,\sigma)e_3 - (e_1,\sigma,e_3,e_1)e_3 + 2(e_1,\sigma,e_3,e_3)e_1 + R(e_1,\sigma)e_1,
\end{aligned}
$$

where $(e_1,\sigma,e_3,e_1) := \langle R(e_1,\sigma)e_3, e_1\rangle$. Cancelling terms and taking the orthogonal projection on (I) + (II) + (III), we find $(\star) = \mathcal{R}(\sigma) - \pi^{\perp}\big(\sum e_i \times \mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1})\big)$.

Finally, by the symmetry of S_σ and $S_{(e_i \times e_j)\times\eta_k}$, the second term is

$$
\begin{aligned}
(\star\star) &= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\Big(\langle S_{(e_i\times e_j)\times\eta_k}(e_i), S_\sigma(e_j)\rangle - \langle S_\sigma(e_i), S_{(e_i\times e_j)\times\eta_k}(e_j)\rangle\Big)\eta_k \\
&= \sum_{i,j=1}^{3}\sum_{k=4}^{7}\Big(\langle S_{(e_i\times e_j)\times\eta_k}(e_i), S_\sigma(e_j)\rangle\Big)\eta_k = (\star\star\star).
\end{aligned}
$$

Using Lemma 5 i), we compute

$$
\begin{aligned}
S_{(e_i\times e_j)\times\eta_k}(e_i) &= -\Big(\nabla_i(e_i \times e_j) \times \eta_k + (e_i \times e_j) \times \nabla_i\eta_k + \sum_{m=1}^{7} T_{im}\chi(e_m, e_i \times e_j, \eta_k)\Big)^{\top} \\
&= -\Big((\nabla_i e_i \times e_j) \times \eta_k + (e_i \times \nabla_i e_j) \times \eta_k + \sum_{l=1}^{7} T_{il}\chi(e_l, e_i, e_j) \times \eta_k \\
&\quad + (e_i \times e_j) \times \nabla_i\eta_k + \sum_{m=1}^{7} T_{im}\chi(e_m, e_i \times e_j, \eta_k)\Big)^{\top} \\
&= -\big((\nabla_i e_i)^{\perp} \times e_j\big) \times \eta_k - \big(e_i \times (\nabla_i e_j)^{\perp}\big) \times \eta_k \\
&\quad - (e_i \times e_j) \times (\nabla_i \eta_k)^{\top} - \sum_{m=1}^{7} T_{im}(\chi(e_m, e_i, e_j)^{\perp} \times \eta_k + \chi(e_m, e_i \times e_j, \eta_k)^{\top}) \\
&= -\big((\nabla_i e_i)^{\perp} \times e_j\big) \times \eta_k - \big(e_i \times (\nabla_i e_j)^{\perp}\big) \times \eta_k + (e_i \times e_j) \times S_{\eta_k}(e_i) \\
&\quad - \sum_{m=4}^{7} T_{im}(\chi(\eta_m, e_i, e_j)^{\perp} \times \eta_k + \chi(\eta_m, e_i \times e_j, \eta_k)^{\top})
\end{aligned}
$$

Notice that, we used the cross product properties (1.28) in the third line and the associative condition $\chi|_{TY}= 0$ in the last one. Moreover, using the Levi-Civita conection symmetry and the relation $e_3 = e_1 \times e_2$, we have for each $j = 1, 2, 3$

$$\begin{aligned}
\sum_i e_i \times (\nabla_i e_j)^\perp =& \sum_i e_i \times (\nabla_j e_i)^\perp \\
=& e_1 \times (\nabla_j e_1)^\perp + e_2 \times (\nabla_j e_2)^\perp \\
&+ e_3 \times \Big((\nabla_j e_1)^\perp \times e_2 + e_1 \times (\nabla_j e_2)^\perp + \sum_{m=4}^{7} T_{jm} \chi(\eta_m, e_1, e_2) \Big) \\
=& e_1 \times (\nabla_j e_1)^\perp + e_2 \times (\nabla_j e_2)^\perp - (\nabla_j e_1)^\perp (e_3 \times e_2) - (e_3 \times e_1) \times (\nabla_j e_2)^\perp \\
&- \sum_{m=4}^{7} T_{jm} e_3 \times (\eta_m \times (e_1 \times e_2)) = - \sum_{m=4}^{7} T_{jm} \eta_m
\end{aligned}$$

Note that, we used the triality cross product property between e_1, e_2, e_3 and the definition (1.8) of χ.

$$\begin{aligned}
(\star\star\star) =& \sum_{ij=1}^{3} \sum_{k=4}^{7} -\langle \big((\nabla_i e_i)^\perp \times e_j \big) \times \eta_k, S_\sigma(e_j) \rangle \eta_k \\
&+ \sum_{m=4}^{7} \Big(T_{jm} \langle \eta_m \times \eta_k, S_\sigma(e_j) \rangle \eta_k \Big) + \langle (e_i \times e_j) \times S_{\eta_k}(e_i), S_\sigma(e_j) \rangle \eta_k \\
&- \sum_{m=4}^{7} T_{im} \langle \chi(\eta_m, e_i, e_j) \times \eta_k + \chi(\eta_m, e_i \times e_j, \eta_k), S_\sigma(e_j) \rangle \eta_k \\
=& \sum_{ij=1}^{3} \big((\nabla_i e_i)^\perp \times e_j \big) \times S_\sigma(e_j) - \sum_{j=1}^{3} \sum_{m=4}^{7} T_{jm} \eta_m \times S_\sigma(e_j) + \\
&+ \sum_{ij=1}^{3} \sum_{k=4}^{7} \langle S_{\eta_k}(e_i), e_i \rangle \langle e_j, S_\sigma(e_j) \rangle \eta_k - \langle S_{\eta_k}(e_i), e_j \rangle \langle e_i, S_\sigma(e_j) \rangle \eta_k \\
&+ \sum_{i,j=1}^{3} \sum_{m=4}^{7} T_{im} (\chi(\eta_m, e_i, e_j) \times S_\sigma(e_j) + \chi(\eta_m, e_i \times e_j, S_\sigma(e_j))) \\
&= H \times \mathcal{B}(\sigma) - \sum_{j=1}^{3} \pi^\perp (T(e_j, \cdot)^\sharp) \times S_\sigma(e_j) + (\operatorname{tr} S_\sigma) H - \mathcal{A}(\sigma) + \pi^\perp (T(\mathcal{B}(\sigma), \cdot)^\sharp)
\end{aligned}$$

To obtain the last line we computed

$$\begin{aligned}
\sum_j \chi(\eta_m, e_i, e_j) \times S_\sigma(e_j) =& \sum_j -\big(\eta_k \times (e_i \times e_j) \big) \times S_\sigma(e_j) \\
=& \sum_j -\chi(S_\sigma(e_j), \eta_m, e_i \times e_j) + \langle S_\sigma(e_j), e_i \times e_j \rangle \eta_k \\
=& \sum_j \langle e_j \times S_\sigma(e_j), e_i \rangle \eta_k = \langle \mathcal{B}(\sigma), e_i \rangle \eta_k
\end{aligned}$$

□

The correction terms (II),...,(V) can be conveniently organised into three 1^{st} order differential operators P_1, P_2, P_3 on sections of NY.

Lemma 10.

$$(\mathrm{II}) = P_1(\sigma) := \sum_{i,j=1}^{3} T_{ii} e_j \times \nabla^{\perp}_j \sigma - T_{ji} e_j \times \nabla^{\perp}_i \sigma - 2 \sum_{(i,j,k)\in S_3^0}^{3} C_{ij} \nabla^{\perp}_k \sigma,$$

where S_3^0 are the even permutations in S_3, T_{ji} is the full torsion tensor and C_{ij} the anti-symmetric part of T_{ij}.

Proof. By Lemma 1.16, we have

$$(\mathrm{II}) = \sum_{i,j,n=1}^{3} \sum_{k=4}^{7} T(e_i, e_n) \psi(e_n, e_j, \nabla^{\perp}_j \sigma, \eta_k) e_i \times \eta_k = (*).$$

Since $\chi(e_n, e_j, \nabla^{\perp}_j \sigma) \in \Omega^0(NY)$, then using (1.8) we have

$$\begin{aligned}(*) &= \sum_{i,j,n=1}^{3} T(e_i, e_n) e_i \times \chi(\nabla^{\perp}_j \sigma, e_n, e_j) = \sum_{i,j,n=1}^{3} -T(e_i, e_n) e_i \times (\nabla^{\perp}_j \sigma \times (e_n \times e_j)) \\ &= \sum_{i,j,n=1}^{3} T(e_i, e_n) \chi(e_i, \nabla^{\perp}_j \sigma, e_n \times e_j) - \langle e_i, e_n \times e_j \rangle \nabla^{\perp}_j \sigma \\ &= \sum_{i,j,n=1}^{3} T(e_i, e_n)(\nabla^{\perp}_j \sigma \times (e_i \times (e_n \times e_j)) - \varphi(e_i, e_n, e_j) \nabla^{\perp}_j \sigma)\end{aligned}$$

Using relations $e_1 \times e_2 = e_3$ and $e_i \times (e_n \times e_j) = -\chi(e_i, e_n, e_j) - \langle e_i, e_n \rangle e_j + \langle e_i, e_j \rangle e_n$. The first term of the sum is equal to

$$\sum_{i,j=1}^{3} T_{ii} e_j \times \nabla^{\perp}_j \sigma - T_{ji} e_j \times \nabla^{\perp}_i \sigma.$$

Moreover, since $\varphi(e_1, e_2, e_3) = 1$, the second term becomes

$$-2 \sum_{(i,j,k)\in S_3^0}^{3} C_{ij} \nabla^{\perp}_k \sigma. \tag{2.11}$$

where $2C_{ij} = T_{ij} - T_{ji}$. ☐

Lemma 11. *With the above notation*

$$\sum_{k=4}^{7} (\nabla_n \psi)(\eta_k, e_1, e_2, e_3) \eta_k = -\sum_{k=4}^{7} T_{nk} \eta_k. \tag{2.12}$$

Proof. Since Y is associative, Corollary 3 gives $\nabla_n \psi_{k123} = -T_{nk}$. ☐

Denote the following two operators on NY, involving the full torsion tensor

$$P_2(\sigma) = \sum_{i=1}^{3} \sum_{l=4}^{7} ((\nabla_i T)(\sigma, \eta_l) + T(\nabla^{\perp}_i \sigma, \eta_l)) e_i \times \eta_l,$$

$$P_3(\sigma) = \sum_{k,l=4}^{7} \Big(T(\sigma, \eta_l) + \sum_{i=1}^{3} \varphi(e_i, \nabla^{\perp}_i \sigma, \eta_l) \Big) T_{lk} \eta_k.$$

With this notation, we arrive at one of our main theorems:

Theorem 4. *The Weitzenböck formula for* (2.1) *is*

$$\begin{aligned}{\not{D}_{\mathrm{A}}}^2(\sigma) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \pi^{\perp}\Big(\sum_{i\in\mathbb{Z}_3} e_i \times \mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})\Big) + H \times \mathcal{B}(\sigma) + (\operatorname{tr} S_\sigma)H - \mathcal{A}(\sigma) \\ - \sum_{j=1}^{3} \pi^{\perp}(T(e_j, \cdot)^{\sharp}) \times S_\sigma(e_j) + \pi^{\perp}(T(\mathcal{B}(\sigma), \cdot)^{\sharp}) + P_1(\sigma) + P_2(\sigma) + P_3(\sigma) \quad (2.13)\end{aligned}$$

Proof. We examine the five components of ${\not{D}_{\mathrm{A}}}^2$ as on page 36. Components (I) and (II) have been studied in Lemmata 9 and 10. Now, applying Lemma 11, we have

$$(\mathrm{III}) = \sum_{i=1}^{3} \sum_{k,l=4}^{7} \varphi(e_i, \nabla_i^{\perp} \sigma, \eta_l) T_{lk} \eta_k.$$

As to (IV), for each $i = 1, 2, 3$ and $l = 4, 5, 6, 7$, we use Lemma 11 to find

$$e_i((\nabla_\sigma \psi)(\eta_l, e_1, e_2, e_3)) = -e_i(T(\sigma, \eta_l)) = -(\nabla_i T)(\sigma, \eta_l) - T(\nabla_i^{\perp} \sigma, \eta_l).$$

Then, indeed,

$$(\mathrm{IV}) = \sum_{i=1}^{3} \sum_{l=4}^{7} ((\nabla_i T)(\sigma, \eta_l) + T(\nabla_i^{\perp} \sigma, \eta_l)) e_i \times \eta_l = P_2(\sigma).$$

Finally, a simple calculation gives $(\mathrm{V}) = \sum_{k,l=4}^{7} T(\sigma, \eta_l) T_{lk} \eta_k$, and

$$(\mathrm{V}) + (\mathrm{III}) = \sum_{k,l=4}^{7} \Big(T(\sigma, \eta_l) + \sum_{i=1}^{3} \varphi(e_i, \nabla_i^{\perp} \sigma, \eta_l)\Big) T_{lk} \eta_k = P_3(\sigma)$$

□

Notice that for a G_2–manifold the 1st order differential operators P_1, P_2, P_3 vanish because $T = 0$. Also, an associative submanifold is a minimal submanifold hence $H = 0$. Thus, from formula (2.13) we get:

Corollary 4. *Let* (M^7, φ) *be a* G_2*–manifold. Then,*

$${\not{D}_{\mathrm{A}}}^2 = \not{D}^2 = \nabla^*\nabla + \mathcal{R} - \mathcal{A}$$

2.1 The nearly parallel case and applications

The torsion-free condition for a G_2–structure is highly overdetermined, so examples are difficult to construct and seldom known explicitly. In terms of the Fernández-Gray classification recalled in Section 1.4, the next natural ‘least-torsion’ case consists of the so-called nearly parallel structures, for which the torsion forms τ_1, τ_2, τ_3 vanish and the remaining torsion is just a constant:

Definition 10. *Let* (M, φ) *a manifold with a* G_2*–structure,* φ *is called* nearly parallel *if*

$$d\varphi = \tau_0 \psi,$$

with $\tau_0 \neq 0$ *constant.*

Regarding the deformations of associative submanifolds, our approach unifies previously known results by means of a Bochner-type vanishing theorem. This technique requires a certain 'positivity' of curvature, which can in practice be found in cases of interest studied by several authors.

2.1.1 Proof of the vanishing theorem

Following Proposition 3, the full torsion tensor in the nearly parallel case is given by $T_{ij} = \dfrac{\tau_0}{4} g_{ij}$, thus, the covariant derivatives $\nabla\varphi$ and $\nabla\psi$ simplifies.

Lemma 12. *Let* (M, φ) *a manifold with a nearly parallel* G_2*–structure, then we hold the following propierties:*

(i) $\nabla\varphi = \dfrac{\tau_0}{4}\psi.$

(ii) $\nabla_u\psi = -\dfrac{\tau_0}{4} u^\flat \wedge \varphi$ *for any* $u \in \Omega^0(TM)$.

(iii) $u \lrcorner \nabla_u \varphi = 0$ *for any* $u \in \Omega^0(TM)$.

Proof. The propierties (i) and (ii) follow by equations (1.16) and Corollary 3, respectively. And (iii) follows by the skew-symmetry of (i). □

Lemma 13. *Let* Y *an associative submanifold of* (M, φ)*, then* Y *is a minimal submanifold.*

Proof. We will show that the mean vector field curvature H of Y vanishes, for each $p \in Y$

$$H(p) = \sum_{i=1}^{3}\sum_{k=4}^{7} \langle \nabla_i e_i, \eta_k \rangle \eta_k = -\sum_{i=1}^{3}\sum_{k=4}^{7} \langle \nabla_i \eta_k, e_i \rangle \eta_k$$

Using the relation $e_3 = e_1 \times e_2$, for each k we have

$$\begin{aligned}
\sum_{i=1}^{3} \langle \nabla_i \eta_k, e_i \rangle =& \varphi(e_2, e_3, \nabla_1 \eta_k) + \varphi(e_3, e_1, \nabla_2 \eta_k) + \varphi(e_1, e_2, \nabla_3 \eta_k) \\
=& e_1(\varphi(e_2, e_3, \eta_k)) - (\nabla_1 \varphi)(e_2, e_3, \eta_k) - \varphi(\nabla_1 e_2, e_3, \eta_k) - \varphi(e_2, \nabla_1 e_3, \eta_k) \\
&+ e_2(\varphi(e_3, e_1, \eta_k)) - (\nabla_2 \varphi)(e_3, e_1, \eta_k) - \varphi(\nabla_2 e_3, e_1, \eta_k) - \varphi(e_3, \nabla_2 e_1, \eta_k) \\
&+ e_3(\varphi(e_1, e_2, \eta_k)) - (\nabla_3 \varphi)(e_1, e_2, \eta_k) - \varphi(\nabla_3 e_1, e_2, \eta_k) - \varphi(e_1, \nabla_3 e_2, \eta_k) \\
=& -\psi(e_1, e_2, e_3, \eta_k) - \psi(e_2, e_3, e_1, \eta_k) - \psi(e_3, e_1, e_2, \eta_k) \\
=& -\langle \chi(e_1, e_2, e_3) + \chi(e_2, e_3, e_1) + \chi(e_3, e_1, e_2), \eta_k \rangle = 0.
\end{aligned}$$

Notice that in the third equality we used the symmetry of the connection ∇ (i.e. $\nabla_i e_j = \nabla_j e_i$), the orthogonal property $\varphi(e_i, e_j, \eta_k) = 0$ for any $i, j = 1, 2, 3$ and $k = 4, ..., 7$, and Lemma 12 (i). And the last line follows by the associative condition $\chi(e_1, e_2, e_3) = 0$. $\square$

Now , we move on to the Weitzenböck formula (2.13) for the nearly parallel case, we see that (2.13) is drastically simplified:

Proposition 5. *The Weitzenböck formula for the Fueter-Dirac operator* (2.1) *in the nearly parallel case is*

$$\not{D}_{\mathrm{A}}{}^2(\sigma) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \mathcal{A}(\sigma) + \tau_0 \not{D}(\sigma) + \frac{\tau_0^2}{4} \cdot \sigma. \tag{2.14}$$

Proof. By Lemma 13 the terms $H \times \mathcal{B}(\sigma)$ and $(\operatorname{tr} S_\sigma)H$ in (2.13) vanish, as well for $\pi^\perp(T(e_j, \cdot)^\sharp)$, $\pi^\perp(T(\mathcal{B}(\sigma), \cdot)^\sharp)$ since $\{e_1, e_2, e_3, \eta_4, ..., \eta_7\}$ is an orthonormal frame. It suffices to prove that the last three terms in (2.13) satisfy

$$(P_1 + P_2 + P_3)(\sigma) = \tau_0 \not{D}(\sigma) + \frac{\tau_0^2}{16} \cdot \sigma \quad \text{and} \quad \mathcal{T}(\sigma) = -\frac{3}{16}\tau_0^2\sigma$$

At a point $p \in Y$, for P_1, we have $C_{ij} = 0$, because τ_1 and τ_2 are zero, then

$$\begin{aligned}\sum_{i,j=1}^{3} T_{ii} e_j \times \nabla_j^\perp \sigma - T_{ji} e_j \times \nabla_i^\perp \sigma &= \frac{3}{4}\tau_0 \sum_{j=1}^{3} e_j \times \nabla_j^\perp \sigma - \frac{1}{4}\tau_0 \sum_{j=1}^{3} e_j \times \nabla_j^\perp \sigma \\ &= \frac{1}{2}\tau_0 \not{D}(\sigma).\end{aligned}$$

For P_2,

$$\begin{aligned}\sum_{i=1}^{3}\sum_{l=4}^{7}((\nabla_i T)(\sigma, \eta_l) + T(\nabla_i^\perp \sigma, \eta_l))e_i \times \eta_l &= \frac{\tau_0}{4}\sum_{i=1}^{3}\sum_{l=4}^{7} g(\nabla_i^\perp \sigma, \eta_l) e_i \times \eta_l \\ &= \frac{\tau_0}{4}\sum_{i=1}^{3} e_i \times \nabla_i^\perp \sigma = \frac{\tau_0}{4} \not{D}(\sigma).\end{aligned}$$

And, for P_3,

$$\begin{aligned}\sum_{k,l=4}^{7}\Big(T(\sigma, \eta_l) + \sum_{i=1}^{3}\varphi(e_i, \nabla_i^\perp \sigma, \eta_l)\Big)T_{lk}\eta_k &= \frac{\tau_0}{4}\sum_{k,l=4}^{7}\Big(\frac{\tau_0}{4} g(\sigma, \eta_l) + \sum_{i=1}^{3}\varphi(e_i, \nabla_i \sigma, \eta_l)\Big) g(\eta_l, \eta_k)\eta_k \\ &= \frac{\tau_0}{4}\sum_{l=4}^{7}\Big(\frac{\tau_0}{4} g(\sigma, \eta_l) + \sum_{i=1}^{3}\varphi(e_i, \nabla_i \sigma, \eta_l)\Big)\eta_l \\ &= \frac{\tau_0^2}{16} \cdot \sigma + \frac{\tau_0}{4} \not{D}(\sigma).\end{aligned}$$

And finally

$$\begin{aligned}
\mathcal{T}(\sigma) = \sum_{i\in\mathbb{Z}_3} e_i \times \mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1}) = \frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{m,l=1}^{7}\big(&g(\sigma,e_m)(\nabla_{i+1}\psi)(e_m,e_i,e_{i+1},e_l)\\
&- g(e_{i+1},e_m)(\nabla_\sigma\psi)(e_m,e_i,e_{i+1},e_l)\big)e_i\times e_l\\
=&\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}\big((\nabla_{i+1}\psi)(\sigma,e_i,e_{i+1},e_l)\\
&-(\nabla_\sigma\psi)(e_{i+1},e_i,e_{i+1},e_l)\big)e_i\times e_l\\
=&\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}\big((\nabla_{i+1}\psi)(\sigma,e_i,e_{i+1},e_l)\big)e_i\times e_l\\
=&-\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}T(e_{i+1},e_{i+1})\varphi(\sigma,e_i,e_l)e_i\times e_l\\
=&-\frac{\tau_0^2}{16}\sum_{i\in\mathbb{Z}_3}g(e_{i+1},e_{i+1})e_i\times(\sigma\times e_i)\\
=&-\frac{3}{16}\tau_0^2\sigma
\end{aligned}$$

Here we used the skew-symmetry of $\nabla_\sigma\,\psi$ for the third equality and Corollary 3 for the fourth one. ☐

Theorem 5. *Let (M,φ) be a 7-manifold with a nearly parallel G_2–structure. If $Y\subset M$ is a closed associative submanifold such that the operator $\mathcal{R}-\mathcal{A}$ is non-negative, then Y is rigid.*

Proof. Let σ be a section of NY,

$$\begin{aligned}
\Delta|\sigma|^2 &= \sum_i e_ie_i\langle\sigma,\sigma\rangle = 2\sum_i e_i\langle\nabla_i^\perp\,\sigma,\sigma\rangle\\
&= 2\sum_i\langle\nabla_i^\perp\,\nabla_i^\perp\,\sigma,\sigma\rangle + \langle\nabla_i^\perp\,\sigma,\nabla_i^\perp\,\sigma\rangle\\
&= -2\langle\nabla^*\nabla\sigma,\sigma\rangle + 2|\nabla^\perp\,\sigma|^2\\
&= -2\langle{\not{D}_{\mathrm{A}}}^2(\sigma),\sigma\rangle + 2\langle\mathcal{R}(\sigma),\sigma\rangle - 2\langle\mathcal{A}(\sigma),\sigma\rangle + 2\tau_0\langle\not{D}(\sigma),\sigma\rangle + \frac{\tau_0^2}{2}|\sigma|^2 + 2|\nabla^\perp\,\sigma|^2.
\end{aligned}$$

Taking $\sigma\in\ker\not{D}_{\mathrm{A}}$, equation (2.2) gives

$$\langle\not{D}(\sigma),\sigma\rangle = \sum_{k=4}^{7}(\nabla_\sigma\psi)(\eta_k,e_1,e_2,e_3)\langle\eta_k,\sigma\rangle = -\sum_{k=4}^{7}T(\sigma,\eta_k)\langle\eta_k,\sigma\rangle = -\frac{\tau_0}{4}\sum_{k=4}^{7}\langle\sigma,\eta_k\rangle^2.$$

By Stokes' theorem, it follows that

$$\begin{aligned}
0 &= \int_Y\Big(\langle\mathcal{R}(\sigma)-\mathcal{A}(\sigma),\sigma\rangle - \frac{\tau_0^2}{4}\sum_{k=4}^{7}\langle\sigma,\eta_k\rangle^2 + \frac{\tau_0^2}{4}|\sigma|^2 + |\nabla^\perp\,\sigma|^2\Big)d\,\mathrm{vol}_Y\\
&= \int_Y(\Big(\langle\mathcal{R}(\sigma)-\mathcal{A}(\sigma),\sigma\rangle + |\nabla^\perp\,\sigma|^2)d\,\mathrm{vol}_Y\,.
\end{aligned}$$

By assumption, $\langle \mathcal{R}(\sigma) - \mathcal{A}(\sigma), \sigma \rangle \geq 0$, so $\nabla^{\perp}\sigma = 0$ and this implies $\not{D}(\sigma) = 0$. Notice from Lemma 11 that the Fueter-Dirac operator is

$$\not{D}_{\mathrm{A}} = \not{D} + \frac{\tau_0}{4} \quad \text{with} \quad \tau_0 \neq 0.$$

Then, from $\not{D}_{\mathrm{A}}(\sigma) = 0$ it follows that $\sigma = 0$, i.e. $\ker \not{D}_{\mathrm{A}} = \{0\}$. □

2.1.2 An associative submanifold of the 7-sphere

In [Lot12], Lotay defines a natural G_2–structure φ on S^7, writing $\mathbb{R}^8 \setminus \{0\} \cong C(S^7) = \mathbb{R}^+ \times S^7$ where $C(S^7)$ denotes the Riemannian cone and a 4–form

$$\Phi_0|_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 * \varphi|_p,$$

where r the radial coordinate on $\mathbb{R}^+$, $*$ the Hodge star on S^7 induced by the round metric. and Φ_0 is the Spin(7)–structure of $\mathbb{R}^8$, choosing an orthonormal basis of $\mathbb{R}^8$, Φ_0 can be written by

$$\Phi_0 = e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356}$$
$$e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Since Φ_0 is closed, it follows that $d\varphi = 4 * \varphi$ i.e. φ is a nearly parallel G_2–structure.

Consider the totally geodesic submanifold $S^3 \subset S^7$, given by

$$S^3 = S^3 \times \{0\} = \{(x_0, x_1, x_2, x_3, 0, 0, 0, 0) \in \mathbb{R}^8 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

If we think the 7–sphere as the homogeneous space $\mathrm{Spin}(7)/G_2$ and hence Spin(7) as the G_2 frame bundle over S^7. So, the associative submanifold S^3 arise as the SU(2)–orbit through the point $p_0 = (1, 0, 0, 0) \in \mathbb{C}^4$ given by the action

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{C}^4 \cong \mathbb{R}^8 \mapsto \begin{pmatrix} az_1 + bz_2 \\ -\bar{b}z_1 + \bar{a}z_2 \\ az_3 + bz_4 \\ -\bar{b}z_3 + \bar{a}z_4 \end{pmatrix} \in \mathbb{C}^4 \quad \text{for} \quad \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2). \tag{2.15}$$

For the associative submanifold $S^3 \subset S^7$ the Weitzenböck formula 2.14 is

$$\not{D}_{\mathrm{A}}^{\,2}(\sigma) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \mathcal{A}(\sigma) + 4\,\not{D}_{\mathrm{A}}(\sigma),$$

or, in terms of the operator $\not{D}$,

$$\not{D}^2 = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) - \mathcal{A}(\sigma) + 2\,\not{D}(\sigma) + 3\sigma, \tag{2.16}$$

which coincides with the formula given by Kawai [Kaw17]. As the induced metric on S^3, from the round metric on S^7, coincides with the round metric of constant curvature 1, the following results of [Bär96] can be adapted to our case.

Lemma 14. *The normal bundle NS^3 can be trivialized by parallel sections $\sigma_1, \dots, \sigma_4$ of the connection $\nabla^\perp$.*

Proof. It suffices to show that the curvature operator $R^\perp$ vanishes (c.f. (2.10)). Let u, v be tangent vector fields of S^3, and σ a section of NS^3, then the Ricci equation gives

$$\begin{aligned} R^\perp(u,v)\sigma &= \sum_{k=4}^{7} \langle R^\perp(u,v)\sigma, \eta_k\rangle \eta_k \\ &= \sum_{k=4}^{7} (\langle R(u,v)\sigma, \eta_k\rangle + \langle [S_\sigma, S_{\eta_k}]u, v\rangle)\eta_k \\ &= \sum_{k=4}^{7} (\langle u,\sigma\rangle\langle v,\eta_k\rangle - \langle v,\sigma\rangle\langle u,\eta_k\rangle)\eta_k = 0. \end{aligned}$$

At the third equality we used the well-known facts that the metric on S^7 has constant sectional curvature equal to 1 and that $S^3 \subset S^7$ is a totally geodesic immersed submanifold. □

The following Weitzenböck formula relates the operator $D = \not{D} - \mathrm{Id}$ with the Laplacian of the connection $\nabla^\perp$ on NS^3.

Lemma 15. *On the normal bundle NS^3, the following formula holds:*

$$D^2 = \nabla^*\nabla + \mathrm{Id}\,. \tag{2.17}$$

Proof. In a local orthonormal frame e_1, e_2, e_3 around $p \in S^3$, we compute

$$\begin{aligned} D^2(\sigma) &= \not{D}^2(\sigma) - 2\not{D}(\sigma) + \sigma \\ &= \nabla^*\nabla\sigma + \mathcal{R}(\sigma) + 4\sigma \\ &= \nabla^*\nabla\sigma + \Big(\sum_{i=1}^{3} \langle\sigma, e_i\rangle e_i - \langle e_i, e_i\rangle\sigma\Big)^\perp + 4\sigma \\ &= \nabla^*\nabla\sigma + \sigma. \end{aligned}$$
□

Consider a basis $1 = f_0, f_1, f_2, \dots$ of $L^2(S^3, \mathbb{R})$, consisting of eigenfunctions of the Laplace operator:

$$\Delta f_i = \lambda_i f_i.$$

The next lemma describes a natural eigenbasis for the operator D^2 on sections of NS^3.

Lemma 16. $D^2(f_i\sigma_k) = (\lambda_i + 1)(f_i\sigma_k)$.

Proof. This follows directly from Lemma 14 and (2.17). □

Since the metric on S^3 has constant curvature 1, the eigenvalues of the Laplace operator on S^3 are

$$\lambda_k = k(k+2) \quad k \geq 0,$$

with multiplicities $m_k = (k+1)^2$ [SA87, Proposition 22.2 and Corollary 22.1]. Together with Lemma 16, this gives:

Corollary 5. *D^2 has eigenvalues $(k+1)^2$ with multiplicities $4(k+1)^2$, $k \geq 0$.*

In general, for an operator T and a vector u such that $T^2 u = \mu^2 u$, if

$$v^{\pm} := (T \pm \mu)u \neq 0$$

then $v^{\pm}$ is an eigenvector of T with eigenvalue $\pm\mu$. Let us apply this principle to $T = D$, with $\mu_k^2 = (k+1)^2$ and $u_k = f_k \sigma_j$, for $j = 1, \ldots, 4$.

Let us first look at the case $k = 0$, in which $f_0 = 1$ and $\lambda_0 = 0$, so $u_0 = \sigma_j$ and $\mu_0^2 = 1$, i.e.,

$$v^{\pm} = (D \pm \mu_0)\sigma_j = D\sigma_j \pm \sigma_j.$$

Now, $\not{D}\,\sigma_j = 0$ by Lemma 14, so $D\sigma_j = -\sigma_j$ and therefore $v^+ = 0$ and $v^- = -2\sigma_j$. Accordingly, v^- is an eigenvector of D with eigenvalue $-\mu_0 = -1$. Since $v^- = -2\sigma_j$, for $j = 1, \ldots, 4$, the multiplicity of $-\mu_0 = -1$ is at least 4, but the multiplicity of $(-\mu_0)^2 = \mu_0^2 = 1$ is already 4, by Corollary 5, therefore the multiplicity of $-\mu_0 = -1$ is exactly 4.

Now, for $k \geq 1$, we take $u_k = f_k \sigma_j$ and $\mu_k = k+1$, and use the trivial fact that $e_i \times \sigma_j$ and σ_j are linearly independent for all i, j:

$$\begin{aligned} v_k^{\pm} =& (D \pm \mu_k)u_k = \not{D}\, u_k - (1 \mp \mu_k)u_k \\ =& \sum_{i=1}^{3} e_i(f_k) e_i \times \sigma_j - \underbrace{(1 \mp \mu_k)}_{\neq 0} \underbrace{f_k}_{\neq 0} \sigma_j \neq 0. \end{aligned}$$

Thus $v_k^{\pm}$ is an eigenvector of D with eigenvalue $\pm\mu_k$, and it follows that $v^{\pm}$ is an eigenvector of $\not{D}$ with eigenvalue $1 \pm \mu_k$, such that $m(1+\mu_k) + m(1-\mu_k) = 4(k+1)^2$. It remains to determine the multiplicities of the eigenvalues $1 \pm (k+1)$. We introduce the following notation:

$$\mu_0^+ := 1 - \mu_0 = 0, \quad \mu_k^+ := 1 + \mu_k = k+2, \quad \text{and} \quad \mu_{-k}^+ := 1 - \mu_k = -k, \quad k \geq 1.$$

From Corollary 5, multiplicities of opposite index add up as $m(\mu_k^+) + m(\mu_{-k}^+) = 4(k+1)^2$. Alternatively, in the sign convention of Remark 1, we denote the eigenvalues of $\not{D}$ by

$$\mu_0^- = 0, \quad \mu_{-k}^- = -k-2, \quad \text{and} \quad \mu_k^- = k, \quad k \geq 1,$$

and again we know $m(\mu_k^-) + m(\mu_{-k}^-) = 4(k+1)^2$. The multiplicities in both sign conventions satisfy the following relations:

Lemma 17.

$$m(\mu^+_{-k}) = m(\mu^-_k) = 2(k+1)(k+2), \quad k \geq 0.$$

and

$$m(\mu^+_k) = m(\mu^-_{-k}) = 2k(k+1), \quad k \geq 1.$$

Proof. From the above, the operator $\not{D} - \dfrac{3}{2}$ has eigenvalues

$$\alpha^+_0 = -\frac{3}{2}, \quad \alpha^+_k = k + \frac{3}{2} - 1 \quad \text{and} \quad \alpha^+_{-k} = -k - \frac{3}{2}.$$

Let $\alpha^-_k := -\alpha^+_{-k}$. Since $\mu^-_k = -\mu^+_{-k}$, we have $m(\alpha^\pm_k) = m(\mu^\pm_k)$, for all $k \in \mathbb{Z}$, and so

$$m(\alpha^\pm_k) + m(\alpha^\pm_{-k}) = 4(k+1)^2.$$

Now the claim clearly holds for $k = 0$ and, by induction on $k \geq 1$, we have

$$\begin{aligned} m(\mu^+_{-(k+1)}) &= m(\alpha^+_{-(k+1)}) = 4(k+2)^2 - m(\alpha^+_{(k+1)}) \\ &= 4(k+2)^2 - m(\alpha^-_k) = 4(k^2+4k+4) - 2(k+1)(k+2) \\ &= 2(k+2)(k+3). \end{aligned}$$

To obtain the second equality we used the relation

$$\alpha^+_{(k+1)} = (k+1) + \frac{3}{2} - 1 = \alpha^-_k,$$

and for the last one we used the induction hypothesis on α^-_k. ☐

The group $\mathrm{Aut}(S^7, \varphi) = \mathrm{Spin}(7)$ of automorphisms of S^7 which fix the G_2–structure induces trivial associative deformations, and the associative 3–sphere is invariant by the action of the embedded subgroup $K = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \subset \mathrm{Spin}(7)$, where $\mathbb{Z}_2$ is generated by $(-1,-1,-1)$ [HL82, Theorem IV 1.38]. Therefore the space of infinitesimal associative deformations of S^3 has dimension at least $\dim(\mathrm{Spin}(7)/K) = 12$.

Corollary 6. *The* 3*-sphere in* S^7 *is rigid as an associative submanifold.*

Proof. Since μ^+_{-1} is the eigenvalue corresponding to the space of infinitesimal associative deformations, then, by Lemma 17, $\dim(\ker \not{D}_{\mathrm{A}}) = m(\mu^+_{-1}) = 12$. ☐

2.2 Locally conformal calibrated case and applications

As an application of the Fueter-Dirac Weitzenböck formula (2.13), we focus on *locally conformal calibrated* G_2–structures, whose associated metric is (at least locally) conformal to a metric induced by a calibrated G_2–structure. We provide a novel example of a rigid associative submanifold, inside a compact manifold S with a locally conformal calibrated G_2–structure, studied by Fernández, Fino and Raffero [FR16].

Definition 11. *A* G_2*–structure is* locally conformal calibrated *if it has vanishing torsion components* $\tau_0 \equiv 0$ *and* $\tau_3 \equiv 0$*, so*

$$\begin{aligned} d\varphi &= 3\tau_1 \wedge \varphi, \\ d\psi &= 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \end{aligned}$$

A $\mathrm{SU}(3)$–*structure* on a 6-manifold N is a pair $(\omega, \phi_+) \in \Omega^2(N) \times \Omega^3(N)$ such that $\phi_+ = \frac{1}{2}(\Omega + \bar{\Omega})$, where $\Omega \in \Omega^0(\Lambda^3(T^*N \otimes \mathbb{C}))$ is a decomposable complex 3-form and

$$\omega \wedge \phi_+ = 0 \quad \text{and} \quad \frac{\omega^3}{6} = \frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{4}\phi_+ \wedge \phi_- \quad \text{with} \quad \phi_- := \frac{1}{2i}(\Omega - \bar{\Omega}). \tag{2.18}$$

The $\mathrm{SU}(3)$–structure (ω, ϕ_+) is said to be *coupled* if $d\omega = c\phi_+$ with c a non-zero real number. So, the product manifold $N \times S^1$ has a natural locally conformal calibrated G_2–structure defined by

$$\varphi = \omega \wedge dt + \phi_+,$$

with $\tau_0 \equiv 0, \tau_3 \equiv 0$ and $\tau_1 = -\frac{c}{3}dt$.

Example 5. *[FR16, Example 3.3] Consider the* 6*–dimensional Lie algebra* $\mathfrak{n}_{28}$*, and let* $\{e_1, ..., e_6\}$ *be a* $\mathrm{SU}(3)$*–basis. With respect to the dual basis* $\{e^1, ..., e^6\}$*, the structure equations of* $\mathfrak{n}_{28}$ *are*

$$(0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}), \tag{2.19}$$

and we denote its components by $de^i := 0$*, for* $i = 1, \ldots, 4$*,* $de^5 := e^{13} - e^{24}$ *and* $de^6 := e^{14} + e^{23}$*. The pair*

$$\omega = e^{12} + e^{34} - e^{56} \quad \textit{and} \quad \phi_+ = e^{136} - e^{145} - e^{235} - e^{246} \tag{2.20}$$

defines a coupled $SU(3)$*–structure on* $\mathfrak{n}_{28}$ *with* $d\omega = -\phi_+$*. Denote by* G *the* 3*-dimensional complex Heisenberg group with Lie algebra* $\mathrm{Lie}(G) = \mathfrak{n}_{28}$ *given by*

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}; \quad z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

The structure equations (2.19) can be rewritten as

$$dz_1 = e^1 + ie^2, \quad dz_2 = e^3 + ie^4 \quad dz_3 + z_1 dz_2 = e^5 + ie^6.$$

By [Mal49, Theorem 7], G *admits a uniform discrete subgroup* $\Gamma \subset G$*, i.e., a discrete subgroup such that* $\Gamma \backslash G$ *is compact, the elements of which have* $z_1, z_2, z_3 \in \mathbb{Z}[i]$*. The left-invariant forms* ω *and* ϕ_+ *on* G *are well defined in the quotient* $\Gamma \backslash G$*. Consider the automorphism* $\nu : G \to G$ *defined by*

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\nu} \begin{pmatrix} 1 & iz_1 & z_3 \\ 0 & 1 & -iz_2 \\ 0 & 0 & 1 \end{pmatrix},$$

and denote by $\mathrm{Diff}_\nu := \langle (p,t) \mapsto (\nu(p), t+1) \rangle$ *the infinite cyclic subgroup of diffeomorphisms of* $(\Gamma\backslash G) \times \mathbb{R}$. *The manifold*

$$S = \big((\Gamma\backslash G) \times \mathbb{R}\big)/\mathrm{Diff}_\nu$$

is endowed with a locally conformal calibrated G_2*–structure as follows: for the left-invariant coframe given in* (2.19), *we have*

$$\nu^*(e_1) = -e_2,\ \nu^*(e_2) = e_1,\ \nu^*(e_3) = e_4,\ \nu^*(e_4) = -e_3,\ \nu^*(e_5) = e_5,\ \nu^*(e_6) = e_6.$$

Hence $\nu^*\omega = \omega$ *and* $\nu^*\phi_+ = \phi_+$, *for* (ω, ϕ_+) *defined in* (2.20). *Denoting by* $p_1 : (\Gamma\backslash G) \times \mathbb{R} \to \Gamma\backslash G$ *the projection onto the first factor, the forms* $p_1^*\omega \in \Omega^2((\Gamma\backslash G) \times \mathbb{R})$ *and* $p_1^*\phi_+ \in \Omega^3((\Gamma\backslash G) \times \mathbb{R})$ *are invariant under* $\sim_\nu$. *Therefore, we have differential forms* $\widetilde{\omega} \in \Omega^2(S)$ *and* $\widetilde{\phi}_+ \in \Omega^3(S)$ *satisfying the same relations as* (ω, ϕ_+) *from* (2.20). *In this set-up, the* 3*-form*

$$\widetilde{\varphi} = \widetilde{\omega} \wedge e^7 + \widetilde{\phi}_+ \tag{2.21}$$

defines a locally conformal calibrated G_2*–structure on* S. *Here* e^7 *denotes the pullback of the canonical closed* 1*-form on* $\mathbb{R}$ *by the projection* $p_2 : (\Gamma\backslash G) \times \mathbb{R} \to \mathbb{R}$. *The torsion forms of* $\widetilde{\varphi}$ *are*

$$\tau_1 = \frac{1}{3}e^7, \quad \tau_2 = \widetilde{\alpha} \quad \textit{where} \quad \alpha = -\frac{4}{3}\left(e^{12} + e^{34} + 2e^{56}\right)$$

and, by Proposition 3, *the full torsion tensor is*

$$T = \widetilde{\beta}, \quad \textit{with} \quad \beta = e^{12} + e^{34} + e^{56}.$$

The 7-manifold from Example 5 contains an associative submanifold, corresponding to a particular Lie subalgebra:

Example 6. *Consider the abelian subalgebra* $\mathfrak{n}_{28}' = \mathrm{Span}(e_5, e_6) \subset \mathfrak{n}_{28}$ *and its respective Lie group* $G' = [G, G] = \exp(\mathfrak{n}_{28}') \subset G$, *which is generated by the commutator* $[g, h] = ghg^{-1}h^{-1}$. *Since* G' *is obtained as the maximal integral submanifold of* G *given by the left-invariant distribution*

$$\Delta(g) = (dL_g)_1\mathfrak{n}_{28} \quad \textit{for} \quad g \in G,$$

i.e. $(L_h)_*(\Delta(g)) \subset \Delta(hg)$ *(c.f. [SM16, Theorem 6.5]), we get an integral distribution* $\bar{\Delta}$ *on* $\Gamma\backslash G$. *Representing* G' *by*

$$G' = \left\{ \begin{pmatrix} 1 & 0 & z_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad z_3 \in \mathbb{C} \right\},$$

we see that, for each $p = \Gamma g' \in \Gamma\backslash G'$, *we have* $T_p(\Gamma\backslash G') = \bar{\Delta}(\Gamma g')$, *and so* $\Gamma\backslash G'$ *is a compact embedded submanifold of* $\Gamma\backslash G$. *Now* $\nu|_{G'} = Id$ *and the quotient map* $(\Gamma\backslash G) \times \mathbb{R} \to S$ *is a local diffeomorphism, so*

$$Y = \Big((\Gamma\backslash G') \times \mathbb{R}\Big)/\mathrm{Diff}_\nu \cong (\Gamma\backslash G') \times S^1$$

is a compact embedded submanifold of S*. Moreover,*

$$T_{(p,t)}Y = T_p(\Gamma\backslash G') \oplus T_t\mathbb{R} \cong \mathfrak{n}'_{28} \oplus \mathbb{R},$$

and indeed $\widetilde{\varphi}|_{T_pY} \equiv \mathrm{vol}(e_5, e_6, e_7)$*. Hence,* Y *is a closed associative submanifold of* S*.*

Now, we assess formula (2.13) for Example 6. The first correction term is

$$\begin{aligned} P_1(\sigma) &= -T_{56}e_5 \times \nabla^{\perp}_6\sigma - T_{65}e_6 \times \nabla^{\perp}_5\sigma - 2T_{56}\nabla^{\perp}_7\sigma \\ &= -(e_7 \times e_6) \times \nabla^{\perp}_6\sigma - (e_7 \times e_5) \times \nabla^{\perp}_5\sigma - 2\nabla^{\perp}_7\sigma \\ &= e_7 \times \not{D}(\sigma) - \nabla^{\perp}_7\sigma. \end{aligned}$$

Here, to obtain the second equality we used the associative relation $e_5 \times e_6 = -e_7$ and for the last one we used the identity $(u \times v) \times w = -u \times (v \times w)$, for mutually orthonormal u, v, w. To calculate P_2, we need the covariant derivative of the total torsion tensor T

$$\nabla_i T_{kl} = e_i(T_{kl}) - \Gamma^m_{ik}T_{ml} - \Gamma^m_{il}T_{km} = -\Gamma^m_{ik}T_{ml} - \Gamma^m_{il}T_{km}. \tag{2.22}$$

Since S is locally isometric to $G \times \mathbb{R}$, the Christoffel symbols of the G_2–metric on S are defined by the structure constants of the Lie algebra $\mathfrak{n}_{28}$ (cf. [Mil76]):

$$\Gamma^k_{ij} = \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) \quad \text{with} \quad \alpha_{ijk} = \langle [e_i, e_j], e_k\rangle.$$

Applying this to Example 5, we find

$$\begin{aligned} \Gamma^5_{13} = \Gamma^6_{23} = \Gamma^2_{36} = \Gamma^5_{42} = \Gamma^2_{63} = \Gamma^4_{52} = -\frac{1}{2} \\ \Gamma^6_{14} = \Gamma^4_{25} = \Gamma^1_{35} = \Gamma^1_{46} = \Gamma^1_{64} = \Gamma^1_{53} = -\frac{1}{2} \\ \Gamma^4_{16} = \Gamma^5_{24} = \Gamma^5_{31} = \Gamma^6_{41} = \Gamma^4_{61} = \Gamma^3_{51} = +\frac{1}{2} \\ \Gamma^3_{15} = \Gamma^3_{26} = \Gamma^6_{32} = \Gamma^2_{45} = \Gamma^3_{62} = \Gamma^2_{54} = +\frac{1}{2} \end{aligned}$$

$$\Gamma^k_{ij} = 0, \text{ otherwise.}$$

Using the cross product defined by (2.21) and the above Christoffel symbols, we have:

$$\nabla_l e_{i+5} = \nabla_{i+5} e_l = \frac{(-1)^i}{2} e_{6-i} \times e_l \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad l = 1, 2, 3, 4. \tag{2.23}$$

Notice that the full torsion tensor of the G_2–structure (2.21) can be written as

$$T(u, v) = -\langle e_7 \times u^{\top}, v^{\top}\rangle + \langle e_7 \times u^{\perp}, v^{\perp}\rangle \quad \text{for} \quad u, v \in \Omega^0(TS|_Y) = \Omega^0(TY) \oplus \Omega^0(NY), \tag{2.24}$$

where $u^\top$ and $u^\perp$ are the tangent and normal components of u, respectively. Combining these facts with Lemma 5 (i), we have

$$\begin{aligned} \nabla_u(v \times w) &= \nabla_u v \times w + v \times \nabla_u w + \sum_{i=1}^{7} T(u, e_m)\chi(e_m, v, w) \\ &= \nabla_u v \times w + v \times \nabla_u w - \chi(e_7 \times u^\top, v, w) + \chi(e_7 \times u^\perp, v, w). \end{aligned} \tag{2.25}$$

Now, for P_2 we obtain:

$$\begin{aligned} P_2(\sigma) &= \sum_{i=5}^{7}\sum_{k=1}^{4} e_i(T(\sigma, e_k))e_i \times e_k \\ &= \sum_{i=5}^{7}\sum_{k=1}^{4} e_i \times \big(\nabla_i^\perp(T(\sigma, e_k)e_k) - T(\sigma, e_k)\nabla_i^\perp e_k\big) \\ &= \sum_{i=5}^{7} e_i \times \nabla_i^\perp(e_7 \times \sigma) - \sum_{i=0,1}\sum_{k=1}^{4}\langle e_7 \times \sigma, e_k\rangle \frac{(-1)^i}{2} e_{i+5} \times (e_{6-i} \times e_k) \\ &= \sum_{i=5}^{7} e_i \times (e_7 \times \nabla_i^\perp\sigma) - e_i \times \chi(e_7 \times e_i, e_7, \sigma) - \sum_{i=0,1}\frac{(-1)^i}{2} e_{i+5} \times (e_{6-i} \times (e_7 \times \sigma)) \\ &= -2\nabla_7^\perp\sigma + \sum_{i=5}^{7} -e_7 \times (e_i \times \nabla_i^\perp\sigma) \\ &\quad \underbrace{- \sum_{i=0,1} e_{i+5} \times \chi(e_7 \times e_{i+5}, e_7, \sigma) + \frac{(-1)^i}{2}(e_{i+5} \times e_{6-i}) \times (e_7 \times \sigma)}_{(\star)} \\ &= -e_7 \times \not{D}(\sigma) - 2\nabla_7^\perp\sigma - 3\sigma \end{aligned}$$

For the third equality, we used (2.24) in the first term and (2.23) in the second one. The fourth equality follows from (2.25) and, finally, a short calculation gives:

$$\begin{aligned} (\star) &= \sum_{i=0,1} -e_{i+5} \times ((e_7 \times e_{i+5}) \times (e_7 \times \sigma)) + \frac{(-1)^i}{2}(e_{i+5} \times e_{6-i}) \times (e_7 \times \sigma) \\ &= \sum_{i=0,1} -((e_{i+5} \times e_7) \times e_{i+5}) \times (e_7 \times \sigma) + \frac{(-1)^i}{2}(e_{i+5} \times e_{6-i}) \times (e_7 \times \sigma) \\ &= -((e_5 \times e_7) \times e_5) \times (e_7 \times \sigma) + \frac{1}{2}(e_5 \times e_6) \times (e_7 \times \sigma) \\ &\quad - ((e_6 \times e_7) \times e_6) \times (e_7 \times \sigma) - \frac{1}{2}(e_6 \times e_5) \times (e_7 \times \sigma) \\ &= \sigma + \frac{1}{2}\sigma + \sigma + \frac{1}{2}\sigma = 3\sigma. \end{aligned}$$

Finally, for P_3, we have

$$\begin{aligned} P_3(\sigma) &= \sum_{k,l=1}^{4} \big(T(\sigma, e_k) + \sum_{i=5}^{7} \tilde{\varphi}(e_i, \nabla_i^\perp\sigma, e_k)\big)T_{kl}e_l \\ &= \sum_{k=1}^{4} \big(\langle e_7 \times \sigma, e_k\rangle + \sum_{i=5}^{7}\langle e_i \times \nabla_i^\perp\sigma, e_k\rangle\big)e_7 \times e_k \\ &= e_7 \times (e_7 \times \sigma) + e_7 \times \not{D}(\sigma) = -\sigma + e_7 \times \not{D}(\sigma) \end{aligned}$$

Now, writing the curvature tensor as

$$R(e_i,e_j)e_k=\sum_{l,m=1}^{7}\big(\Gamma^l_{jk}\Gamma^m_{il}-\Gamma^l_{ik}\Gamma^m_{jl}-(\Gamma^l_{ij}-\Gamma^l_{ji})\Gamma^m_{lk}\big)e_m$$

and using the last expression, we have

$$\begin{aligned}R(e_5,\sigma)e_5&=\sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\big(\Gamma^l_{j5}\Gamma^m_{5l}-\Gamma^l_{55}\Gamma^m_{jl}-(\Gamma^l_{5j}-\Gamma^l_{j5})\Gamma^m_{l5}\big)e_m\\&=\sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\big(\Gamma^l_{j5}\Gamma^m_{5l}\big)e_m\\&=\sigma^1\Gamma^3_{15}\Gamma^1_{53}e_1+\sigma^2\Gamma^4_{25}\Gamma^2_{54}e_2+\sigma^3\Gamma^1_{35}\Gamma^3_{51}e_3+\sigma^4\Gamma^2_{45}\Gamma^4_{52}e_4=-\frac{\sigma}{4}.\end{aligned}$$

And,

$$\begin{aligned}R(e_6,\sigma)e_6&=\sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\big(\Gamma^l_{j6}\Gamma^m_{6l}-\Gamma^l_{66}\Gamma^m_{jl}-(\Gamma^l_{6j}-\Gamma^l_{j6})\Gamma^m_{l6}\big)e_m\\&=\sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\big(\Gamma^l_{j6}\Gamma^m_{6l}\big)e_m\\&=\sigma^1\Gamma^4_{16}\Gamma^1_{64}e_1+\sigma^2\Gamma^3_{26}\Gamma^2_{63}e_2+\sigma^3\Gamma^2_{36}\Gamma^3_{62}e_3+\sigma^4\Gamma^1_{46}\Gamma^4_{61}e_4=-\frac{\sigma}{4}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{R}(\sigma)&=\big(R(e_5,\sigma)e_5+R(e_6,\sigma)e_6+R(e_7,\sigma)e_7\big)^{\perp}=-\frac{1}{4}\sigma-\frac{1}{4}\sigma+0\\&=-\frac{1}{2}\sigma.\end{aligned}$$

Now, we assess the operator $\mathcal{T}$ defined in equation (1.24) for a pair $e_i,e_j\in\Omega^0(TY)$ and $\sigma\in\Omega^0(NY)$:

$$\begin{aligned}\mathcal{T}(e_j,\sigma,e_i,e_j)=\sum_{m=1}^{7}&\underbrace{T(\sigma,e_m)\nabla_j\psi(e_m,e_i,e_j,\cdot)^\sharp}_{\text{(I)}}-\underbrace{T(e_j,e_m)\nabla_\sigma\psi(e_m,e_i,e_j,\cdot)^\sharp}_{\text{(II)}}\\&+\underbrace{\big(\nabla_jT(\sigma,e_m)-\nabla_\sigma T(e_j,e_m)\big)\chi(e_m,e_i,e_j)}_{\text{(III)}}.\end{aligned}$$

We will use throughout the proof both the expression of $\nabla\psi$ in terms of T and φ from Corollary 3 and the expression for T given in (2.24). For the first term,

$$\begin{aligned}\text{(I)}&=\sum_{m=1}^{7}\langle e_7\times\sigma,e_m\rangle\nabla_j\psi(e_m,e_i,e_j,\cdot)^\sharp=\nabla_j\psi(e_7\times\sigma,e_i,e_j,\cdot)^\sharp\\&=-T(e_j,e_7\times\sigma)\varphi(e_i,e_j,\cdot)^\sharp+T(e_j,e_i)\varphi(e_7\times\sigma,e_j,\cdot)^\sharp-T(e_j,e_j)\varphi(e_7\times\sigma,e_i,\cdot)^\sharp\\&\quad+T(e_j,\cdot)^\sharp\varphi(e_7\times\sigma,e_i,e_j)\\&=-\langle e_7\times e_j,e_i\rangle(e_7\times\sigma)\times e_j=\langle e_7\times e_j,e_i\rangle(e_7\times e_j)\times\sigma.\end{aligned}$$

Here we used the vanishings $T(e_j, e_7 \times \sigma) = 0$, again by (2.24), $T(e_j, e_j) = 0$, by skew-symmetry, and $\varphi(e_7 \times \sigma, e_i, e_j) = \langle e_i \times e_j, e_7 \times \sigma\rangle = 0$, by orthogonality.
For the second term,

$$\begin{aligned}
(\mathrm{II}) &= \sum_{m=1}^{7} \langle e_7 \times e_j, e_m\rangle \nabla_\sigma \psi(e_m, e_i, e_j, \cdot)^\sharp = \nabla_\sigma \psi(e_7 \times e_j, e_i, e_j, \cdot)^\sharp \\
&= -T(\sigma, e_7 \times e_j)\varphi(e_i, e_j, \cdot)^\sharp + T(\sigma, e_i)\varphi(e_7 \times e_j, e_j, \cdot)^\sharp - T(\sigma, e_j)\varphi(e_7 \times e_j, e_i, \cdot)^\sharp \\
&\quad + T(\sigma, \cdot)^\sharp \varphi(e_7 \times e_j, e_i, e_j) \\
&= -\langle e_7 \times \sigma, \cdot\rangle^\sharp \langle (e_7 \times e_j) \times e_i, e_j\rangle = -\langle (e_7 \times e_j) \times e_i, e_j\rangle e_7 \times \sigma.
\end{aligned}$$

Again the vanishings $T(\sigma, e_7 \times e_j) = T(\sigma, e_i) = T(\sigma, e_j) = 0$ follow from (2.24).
For the third term, we use expression (2.22) for the derivatives of the torsion tensor:

$$\begin{aligned}
(\mathrm{III}) &= -\sum_{m=1}^{7} \big(T(\sigma, \nabla_j e_m) - T(e_j, \nabla_\sigma e_m)\big)\chi(e_m, e_i, e_j) \\
&= -\sum_{m=1}^{7} \big(\langle e_7 \times \sigma, \nabla_j e_m\rangle + \langle e_7 \times e_j, \nabla_\sigma e_m\rangle\big)\chi(e_m, e_i, e_j).
\end{aligned}$$

We now apply (I), (II) and (III) for $i = 5$ and $j = 6$:

$$\begin{aligned}
\mathcal{T}(e_6, \sigma, e_5, e_6) &= \langle e_7 \times e_6, e_5\rangle (e_7 \times e_6) \times \sigma + \langle (e_7 \times e_6) \times e_5, e_6\rangle e_7 \times \sigma \\
&\quad - \sum_{m=1}^{7} \big(\langle e_7 \times \sigma, \nabla_6 e_m\rangle + \langle e_7 \times e_6, \nabla_\sigma e_m\rangle\big)\chi(e_m, e_5, e_6) \\
&= e_5 \times \sigma - \sum_{m=1}^{7} \Big(-\frac{1}{2}\langle e_7 \times \sigma, e_5 \times e_m\rangle + \langle e_5, \nabla_\sigma e_m\rangle\Big)\chi(e_m, e_5, e_6) \\
&= e_5 \times \sigma - \sum_{m=1}^{7} \Big(\frac{1}{2}\langle e_5 \times (e_7 \times \sigma), e_m\rangle + \sigma\langle e_5, e_m\rangle - \langle \nabla_\sigma e_5, e_m\rangle\Big)\chi(e_m, e_5, e_6) \\
&= e_5 \times \sigma - \sum_{m=1}^{7} \Big(-\frac{1}{2}\langle e_6 \times \sigma, e_m\rangle - \frac{1}{2}\langle e_6 \times \sigma, e_m\rangle\Big)\chi(e_m, e_5, e_6) \\
&= e_5 \times \sigma + \chi(e_6 \times \sigma, e_5, e_6) = e_5 \times \sigma - (e_6 \times \sigma) \times (e_5 \times e_6) \\
&= e_5 \times \sigma + (e_6 \times \sigma) \times e_7 \\
&= 2e_5 \times \sigma.
\end{aligned}$$

Here we used repeatedly that $e_5 \times e_6 = -e_7$ and $e_i \times (e_j \times \sigma) = -e_j \times (e_i \times \sigma)$ for $i \neq j$. At the second and fourth lines we applied again (2.23), and at the third line we used the compatibility of the Riemannian connection.

For $j = 7$ and $i = 6$, we have trivially

$$\mathcal{T}(e_7, \sigma, e_6, e_7) = 0.$$

Finally, for $j = 5$ and $i = 7$, we have

$$
\begin{aligned}
\mathcal{T}(e_5, \sigma, e_7, e_5) &= \langle e_7 \times e_5, e_7\rangle (e_7 \times e_5) \times \sigma + \langle (e_7 \times e_5) \times e_7, e_5\rangle e_7 \times \sigma \\
&\quad - \sum_{m=1}^{7} \big(\langle e_7 \times \sigma, \nabla_5 e_m\rangle + \langle e_7 \times e_5, \nabla_\sigma e_m\rangle \big) \chi(e_m, e_7, e_5) \\
&= \langle e_6, e_7\rangle e_6 \times \sigma - \langle e_6 \times e_7, e_5\rangle e_7 \times \sigma - \sum_{m=1}^{7} \big(\frac{1}{2}\langle e_7 \times \sigma, e_6 \times e_m\rangle \\
&\quad - \langle e_6, \nabla_\sigma e_m\rangle \big) \chi(e_m, e_7, e_5) \\
&= e_7 \times \sigma - \sum_{m=1}^{7} \big(-\frac{1}{2}\langle e_6 \times (e_7 \times \sigma), e_m\rangle - \sigma\langle e_6, e_m\rangle \\
&\quad + \langle \nabla_\sigma e_6, e_m\rangle \big) \chi(e_m, e_7, e_5) \\
&= e_7 \times \sigma - \sum_{m=1}^{7} \big(-\frac{1}{2}\langle e_5 \times \sigma, e_m\rangle - \frac{1}{2}\langle e_5 \times \sigma, e_m\rangle \big) \chi(e_m, e_7, e_5) \\
&= e_7 \times \sigma + \chi(e_5 \times \sigma, e_7, e_5) = e_7 \times \sigma - (e_5 \times \sigma) \times (e_7 \times e_5) \\
&= e_7 \times \sigma + (e_5 \times \sigma) \times e_6 = 2e_7 \times \sigma.
\end{aligned}
$$

Therefore,

$$
\Big(\sum_{i \in \mathbb{Z}_3} e_{i+5} \times \mathcal{T}(e_{i+6}, \sigma, e_{i+5}, e_{i+6}) \Big)^{\perp} \;=\; -4\sigma.
$$

Following the notation of [CP15, §5.3], we define an operator

$$
\not{D}^c(\sigma) := e_5 \times \nabla_5^{\perp}\sigma + e_6 \times \nabla_6^{\perp}\sigma,
$$

and recall that the cross-product by e_7 defines an almost complex structure on $T(\Gamma\backslash G)$ denoted by $J(\sigma) := e_7 \times \sigma$. Then (2.2) becomes

$$
\not{D}_{\mathrm{A}}(\sigma) = \not{D}^c(\sigma) + J(\dot{\sigma}) + J(\sigma),
$$

where $\dot{\sigma} := \nabla_7^{\perp}\sigma$. To simplify notation, let $\|\cdot\|$ and $\langle\langle \cdot, \cdot \rangle\rangle$ denote the L^2-norm and inner product of sections, respectively (the integral of the corresponding pointwise quantity over the associative submanifold). The next Lemma gathers some relations between the operators $\not{D}$, J and ∇; although some of them will not be used in this article, we state them anyway as a curiosity.

Lemma 18. *With the above notation, we have the following properties:*

(i) $\not{D}^c \circ J(\sigma) = -J \circ \not{D}^c(\sigma) + 2\sigma.$

(ii) $\langle\langle \not{D}^c(\sigma), \eta \rangle\rangle = \langle\langle \sigma, \not{D}^c(\eta) \rangle\rangle + 2\langle\langle \sigma, J(\eta) \rangle\rangle.$

(iii) $\langle\langle \not{D}^c(\sigma), J(\dot{\sigma}) \rangle\rangle = 0.$

(iv) $\langle\langle\dot{\sigma},\sigma\rangle\rangle = 0$ *and* $\langle\langle \not{D}^c(\sigma), J(\sigma)\rangle\rangle \leq 0$.

Proof. (i) Using Lemma 5 (i), we have,

$$\begin{aligned} \not{D}^c \circ J(\sigma) &= -J\circ \not{D}^c(\sigma) - T_{65}e_6\times(e_5\times(e_7\times\sigma)) - T_{56}e_5\times(e_6\times(e_7\times\sigma)) \\ &= -J\circ \not{D}^c(\sigma) + 2T_{56}(e_5\times e_6)\times(e_7\times\sigma) \\ &= -J\circ \not{D}^c(\sigma) + 2\cdot\sigma. \end{aligned}$$

(ii)

$$\begin{aligned} \langle \not{D}^c(\sigma),\eta\rangle_p &= -\sum_{i=5}^{6}\langle\nabla_i^{\perp}\sigma, e_i\times\eta\rangle_p = -\sum_{i=5}^{6}\{e_i\langle\sigma,e_i\times\eta\rangle - \langle\sigma,\nabla_i^{\perp}(e_i\times\eta)\rangle\}_p \\ &= \operatorname{div}(\sigma\times\eta)_p + -\sum_{i=5}^{6}\langle\sigma, e_i\times\nabla_i^{\perp}\eta - \chi(e_7\times e_i, e_i,\eta)\rangle\}_p \\ &= \operatorname{div}(\sigma\times\eta)_p + \langle\sigma,\not{D}^c(\eta)\rangle_p + 2\langle\sigma, e_7\times\eta\rangle_p. \end{aligned}$$

Here we used the Leibniz rule (2.25), then the following trivial calculation:

$$\begin{aligned} \chi(e_7\times e_i, e_i,\eta) =& \chi(\eta, e_7\times e_i, e_i) = -\eta\times\big((e_7\times e_i)\times e_i\big) \\ &= -\eta\times\big(e_i\times(e_i\times e_7)\big) = -e_7\times\eta. \end{aligned}$$

(iii) Using (i) and (ii), one has $\langle\langle \not{D}^c(\sigma), J(\dot{\sigma})\rangle\rangle = \langle\langle J(\sigma), \not{D}^c(\dot{\sigma})\rangle\rangle$, and, by the vanishing of the normal curvature tensor $R^{\perp}(e_i,e_7)\sigma = 0$ for $i=5,6$, we have $\nabla_i^{\perp}\nabla_7^{\perp}\sigma = \nabla_7^{\perp}\nabla_i^{\perp}\sigma$. Using Lemma 5 (i) and the compatibility of $\nabla^{\perp}$ with the induced metric in NY we have

$$\begin{aligned} \langle \not{D}^c(\sigma), J(\dot{\sigma})\rangle_p &= \sum_{i=5}^{7}\langle J(\sigma), e_i\times\nabla_7^{\perp}\nabla_i^{\perp}\sigma\rangle_p \\ &= \sum_{i=5}^{7}\langle J(\sigma), \nabla_7^{\perp}(e_i\times\nabla_i^{\perp}\sigma)\rangle_p \\ &= -\langle\nabla_7^{\perp}(J(\sigma)), \not{D}^c(\sigma)\rangle_p + e_7\langle J(\sigma),\not{D}^c(\sigma)\rangle_p \\ &= -\langle J(\dot{\sigma}), \not{D}^c(\sigma)\rangle_p + \operatorname{div}(\langle J(\sigma),\not{D}^c(\sigma)\rangle e_7)_p. \end{aligned}$$

(iv) Again by compatibility of $\nabla^{\perp}$ with the metric on NY, we have $2\langle\dot{\sigma},\sigma\rangle = 2\langle\nabla_7^{\perp}\sigma,\sigma\rangle = e_7|\sigma|^2$. Now Stokes' Theorem gives

$$\langle\langle\dot{\sigma},\sigma\rangle\rangle = \frac{1}{2}\int_Y e_7|\sigma|^2 d\operatorname{vol}_Y = \frac{1}{2}\int_Y \operatorname{div}(|\sigma|^2 e_7)d\operatorname{vol}_Y = 0. \tag{2.26}$$

Computing the L^2-norm for $\not{D}_{\mathrm{A}}(\sigma)$, we have

$$\left\|\not{D}_{\mathrm{A}}(\sigma)\right\|^2 = \left\|\not{D}^c(\sigma)\right\|^2 + \|\dot{\sigma}\|^2 + \|\sigma\|^2 + 2\langle\langle\not{D}^c(\sigma), J(\dot{\sigma})\rangle\rangle + 2\langle\langle\not{D}^c(\sigma), J(\sigma)\rangle\rangle + 2\langle\langle\dot{\sigma},\sigma\rangle\rangle,$$

and from Lemma 18(iii) and equation (2.26) it follows that

$$\left\|\not{D}_{\mathrm{A}}(\sigma)\right\|^2 = \left\|\not{D}^c(\sigma)\right\|^2 + \|\dot{\sigma}\|^2 + \|\sigma\|^2 + 2\langle\langle\not{D}^c(\sigma), J(\sigma)\rangle\rangle.$$

Therefore, by the triangle inequality,

$$\langle\langle \not{D}^c(\sigma), J(\sigma)\rangle\rangle \leq 0.$$

□

Corollary 7. *The submanifold Y of Example 6 is rigid.*

Proof. We recall the full torsion tensor is $T = e^{12} + e^{34} + e^{56}$, from it follows that $\pi^{\perp}(T(e_j, \cdot)^{\sharp}) = \pi^{\perp}(T(\mathcal{B}(\sigma), \cdot)^{\sharp}) = 0$ for any $j = 5, 6, 7$ and $\sigma \in \Omega^0(NY)$. Now, notice that the operator $\mathcal{A}$ and the mean curvature vector field H vanish on Y, as can be seen from

$$\begin{aligned}
\mathcal{A}(\sigma) &= \sum_{i,j=5}^{7}\sum_{k=1}^{4}\langle S_{e_k}(e_i), e_j\rangle\langle e_i, S_{\sigma}(e_j)\rangle e_k \\
&= -\sum_{i,j=5}^{7}\sum_{k=1}^{4}\langle \nabla_i e_k, e_j\rangle\langle e_i, S_{\sigma}(e_j)\rangle e_k \\
&= -\sum_{i,j=5}^{7}\sum_{k=1}^{4}\Gamma_{ik}^{j}\langle e_i, S_{\sigma}(e_j)\rangle e_k = 0,
\end{aligned}$$

since, $\Gamma_{ik}^{j} = 0$ for $i, j = 5, 6, 7$ and $k = 1, ..., 4$. As well

$$\begin{aligned}
H &= \sum_{i=5}^{7}\sum_{k=1}^{4}\langle S_{e_k}(e_i), e_i\rangle e_k \\
&= -\sum_{i=5}^{7}\sum_{k=1}^{4}\langle \nabla_i e_k, e_i\rangle e_k \\
&= -\sum_{i,j=5}^{7}\sum_{k=1}^{4}\Gamma_{ik}^{i} e_k = 0,
\end{aligned}$$

Applying equation (2.13), Lemma 5 and the previous calculation, we obtain the Weitzenböck formula

$$\not{D}_{\mathrm{A}}{}^{2}(\sigma) = \nabla^*\nabla\sigma + e_7 \times \not{D}(\sigma) - 3\nabla_7^{\perp}\sigma - \frac{1}{2}\sigma.$$

Taking the inner product with σ and integrating over Y,

$$\begin{aligned}
\int_Y \langle \not{D}_{\mathrm{A}}{}^{2}(\sigma), \sigma\rangle d\,\mathrm{vol}_Y &= \int_Y \langle \nabla^*\nabla\sigma, \sigma\rangle d\,\mathrm{vol}_Y + \int_Y \langle e_7 \times \not{D}(\sigma), \sigma\rangle d\,\mathrm{vol}_Y - \int_Y 3\langle \nabla_7^{\perp}\sigma, \sigma\rangle d\,\mathrm{vol}_Y \\
&\quad - \int_Y \frac{1}{2}\langle \sigma, \sigma\rangle d\,\mathrm{vol}_Y \\
&\geq \int_Y \langle e_7 \times \not{D}(\sigma), \sigma\rangle d\,\mathrm{vol}_Y - 3\int_Y \langle \dot{\sigma}, \sigma\rangle d\,\mathrm{vol}_Y - \int_Y \frac{1}{2}\langle \sigma, \sigma\rangle d\,\mathrm{vol}_Y .
\end{aligned}$$

From Lemma 18 (iv), we conclude that

$$\int_Y \langle \not{D}_{\mathrm{A}}{}^{2}(\sigma), \sigma\rangle d\,\mathrm{vol}_Y \geq \int_Y \langle e_7 \times \not{D}(\sigma), \sigma\rangle d\,\mathrm{vol}_Y - \frac{1}{2}\int_Y \langle \sigma, \sigma\rangle d\,\mathrm{vol}_Y . \tag{2.27}$$

So, for $\sigma \in \ker \not{D}_{\mathrm{A}}$, we have $\not{D}(\sigma) = -e_7 \times \sigma$ and, replacing that in (2.27), we get the inequality

$$0 \geq -\int_Y \langle e_7 \times (e_7 \times \sigma), \sigma\rangle d\,\mathrm{vol}_Y - \frac{1}{2}\int_Y \langle \sigma, \sigma\rangle d\,\mathrm{vol}_Y = \frac{1}{2}\int_Y \langle \sigma, \sigma\rangle d\,\mathrm{vol}_Y .$$

Then $\sigma = 0$ and therefore Y is rigid. □

2.3 Calibrated case

Consider a 6-dimensional Lie algebra $\mathfrak{h}$ endowed with a SU(3)-structure $(\omega, \phi_+) \in \Lambda^2(\mathfrak{h})^* \times \Lambda^3(\mathfrak{h})^*$ satisfying the compatibility and normalized condition (2.18) such that both ω and ϕ_+ are closed, in this case the pair (ω, ϕ_+) is a *symplectic half-flat* SU(3)-structure. Thus, for the product Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has a closed G_2–structure given by

$$\varphi = \omega \wedge e^7 + \phi_+,$$

where $\mathbb{R} = \mathrm{Span}(e_7)$.

Example 7. *Consider the nilpotent Lie algebra* $\mathfrak{h}$ *with constant structures given by*

$$\mathfrak{h} = \mathfrak{g}_{5,1} \oplus \mathbb{R} = (0, 0, 0, 0, e^{12}, e^{13}).$$

With respect to the SU(3)*-basis* $\{e_1, \ldots, e_6\}$ *the symplectic half-flat* $SU(3)$*-structure is given by*

$$\omega = e^{14} + e^{26} + e^{35} \quad \textit{and} \quad \phi_+ = e^{123} + e^{156} + e^{245} - e^{346}$$

Hence, the 7*-dimensional Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}_{5,1} \oplus \mathbb{R}^2$ *has a closed* G_2*–structure given by*

$$\varphi = \omega \wedge e^7 + \phi_+ = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}. \tag{2.28}$$

Its dual 4*-form*

$$\psi = \frac{1}{2}\omega^2 + \phi_- \wedge e^7 = e^{2356} - e^{1345} - e^{1246} + e^{4567} + e^{2347} - e^{1367} + e^{1257}.$$

An straightforward calculation shows

$$d\psi = -e^{1246} + e^{1345} \quad \textit{and} \quad \tau_2 = -e^{35} + e^{26} \in \Lambda^2_{14}(\mathfrak{h})^*,$$

therefore, the full torsion tensor is given by

$$T = \frac{1}{2}e^{35} + \frac{1}{2}e^{26}. \tag{2.29}$$

By [Mal49, Theorem 7], the corresponding connected and simply connected nilpotent Lie group G *admits a uniform discrete subgroup* $\Gamma \subset G$ *given by*

$$\Gamma = exp(\mathbb{Z}\langle e_1, ..., e_7\rangle).$$

So, the compact manifold $M = \Gamma \backslash G$ *has a* G*-invariant closed* G_2*–structure*

The 7-manifold from Example 7 contains an associative submanifold corresponding to a particular Lie subalgebra:

Example 8. *Consider the abelian subalgebra* $\mathfrak{a} = \mathrm{Span}(e_1, e_5, e_6)$*, note that the restriction* $\varphi|_{\mathfrak{a}} = e^{156}$*, so* $\mathfrak{a}$ *is an associative* 3*-plane. Since the connected and simply connected Lie subgroup* A *with Lie algebra* $\mathfrak{a}$ *is obtained as integral submanifold of* G *given by the left-invariant distribution*

$$\Delta(g) = (dL_g)_1\mathfrak{a} \quad \textit{for} \quad g \in G,$$

we get an integral distribution $\bar{\Delta}$ *on* $M = \Gamma\backslash G$*. For each* $p = \Gamma a \in (\Gamma\backslash A)$ *we have* $T_p(\Gamma\backslash A) = \bar{\Delta}(\Gamma a)$ *and so* $Y = \Gamma\backslash A$ *is a compact embedded submanifold of* M*. Moreover,*

$$T_pY \cong \mathfrak{a},$$

hence, Y *is an associative submanifold of* M*.*

Fix $e_1, ..., e_7$ an orthonormal frame of TM induced by left invariant vector fields on G, such that the restriction on Y makes e_1, e_5, e_6 an orthonormal frame of TY and e_2, e_3, e_4, e_7 an orthonormal frame of NY. Notice that, the Lie algebra $\mathfrak{g}$ contains an abelian ideal $\mathfrak{u} = \mathrm{Span}(e_2, ..., e_7)$ of codimension 1. Let $L : \mathfrak{u} \to \mathfrak{u}$ be the linear transformation $L(u) = [e_1, u]$. The Riemannian connection ∇ on G is completely determined by L.

Lemma 19. *[Mil76, Lemma 5.5] For each* $u, v \in \mathfrak{u}$*, the covariant derivative satisfies*

$$\nabla_1 e_1 = 0, \qquad \nabla_1 u = \frac{1}{2}(L - L^t)u,$$
$$\nabla_u e_1 = -\frac{1}{2}(L + L^t)u, \qquad \nabla_u v = \langle (L + L^t)u, v\rangle e_1,$$

where L^t *denotes the transpose of* L*.*

Using the above Lemma we have

$$\nabla_1 e_2 = -\nabla_2 e_1 = -\frac{1}{2}e_5 \qquad \nabla_1 e_5 = \nabla_5 e_1 = \frac{1}{2}e_2$$
$$\nabla_1 e_3 = -\nabla_3 e_1 = -\frac{1}{2}e_6 \qquad \nabla_1 e_6 = \nabla_6 e_1 = \frac{1}{2}e_3$$
$$\nabla_2 e_5 = \nabla_5 e_2 = \nabla_3 e_6 = \nabla_6 e_3 = -\frac{1}{2}e_1 \qquad \nabla_i e_j = 0 \text{ otherwise.}$$

Notice that, the normal connection $\nabla^\perp_i e_j = \nabla_i e_j - (\nabla_i e_j)^\top$ vanishes, since $(\nabla_i e_j)^\top = \nabla_i e_j$ for $i = 1, 5, 6$ and $j = 2, 3, 4, 7$.

Lemma 20. *The normal bundle* NY *for the submanifold 8 can be trivialized by parallel sections* e_2, e_3, e_4, e_7 *of the connection* $\nabla^\perp$*.*

Now, from Corollary 3 we have that $\nabla_l \psi_{k156} = -T_{lk}$ for $k, l = 2, 3, 4, 7$, and by equation (2.29) we get $T|_{NY\times NY} = 0$. Therefore, it follows:

Lemma 21. *For the associative submanifold* Y *of Example 8:*

(i) The Fueter operator (2.1) *is*

$$\not{D}_{\mathrm{A}}(\sigma) = \not{D}(\sigma) = e_1 \times \nabla_1^{\perp}\sigma + e_5 \times \nabla_5^{\perp}\sigma + e_6 \times \nabla_6^{\perp}\sigma$$

(ii) The operators P_1, P_2, P_3 defined in Theorem 4 vanishes.

Applying Lemmata 20 and 21 we obtain that $e_2, e_3, e_4, e_7 \in \ker \not{D}_{\mathrm{A}}$. However, each vector field e_k is induced by the one parameter subgroup of diffeomorphism $f_t = R_{\exp(te_k)} \subset \mathrm{Diff}(M)$, indeed, the left-invariant vector field e_k on G is induced by the flow given by the right-translation $R_{\exp(te_k)} : G \to G$. So, define

$$R_{\exp(te_k)} : \Gamma g \in M \mapsto \Gamma(g \exp(te_k)) \in M,$$

notice that this map is well defined, for $\Gamma g_1 = \Gamma g_2$ (i.e. $g_1 g_2^{-1} \in \Gamma$), then

$$R_{\exp(te_k)}(\Gamma g_1) = \Gamma g_1 \exp(te_k) = \Gamma g_1 g_2^{-1} g_2 \exp(te_k) = \Gamma g_2 \exp(te_k) = R_{\exp(te_k)}(\Gamma g_2).$$

Since the Lie group G is nilpotent, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, then, using the Baker-Campbell-Hausdorff formula the structure group of G is

$$gh = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5+\frac{1}{2}(x_2y_1-x_1y_2), x_6+y_6+\frac{1}{2}(x_3y_1-x_1y_3), x_7+y_7),$$

where $g = (x_1, ..., x_7), h = (y_1, ..., y_7) \in G \cong \mathbb{R}^7$, the identity element is the vector 0 and the inverse $g^{-1} = (-x_1, ..., -x_7)$. So, the differential of the left and right-translation are

$$dL_g = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \frac{x_2}{2} & -\frac{x_1}{2} & 0 & 0 & 1 & & \\ \frac{x_3}{2} & 0 & -\frac{x_1}{2} & 0 & & 1 & \\ 0 & 0 & 0 & 0 & & & 1 \end{pmatrix}, \quad dR_g = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ -\frac{x_2}{2} & \frac{x_1}{2} & 0 & 0 & 1 & & \\ -\frac{x_3}{2} & 0 & \frac{x_1}{2} & 0 & & 1 & \\ 0 & 0 & 0 & 0 & & & 1 \end{pmatrix}$$

Notice that $dR_g = dL_{g^{-1}}$, in fact this follows by the fact that A is a normal subgroup of G, since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Thus, the restriction $\{f_t = R_{\exp(te_k)} : Y \to M\}$ induces trivial deformations for each $k = 2, 3, 4, 7$.

Lemma 22. *For the associative submanifold Y of Example 8 we have*

$$\mathcal{R}(\sigma) - \pi^{\perp}\Big(\sum_{i\in\mathbb{Z}_3} e_i \times \mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})\Big) + H \times \mathcal{B}(\sigma) + (\operatorname{tr} S_\sigma)H$$
$$- \mathcal{A}(\sigma) - \sum_{j=1}^{3} \pi^{\perp}(T(e_j, \cdot)^{\sharp}) \times S_\sigma(e_j) + \pi^{\perp}(T(\mathcal{B}(\sigma), \cdot)^{\sharp}) = 0$$

Proof. By Lemma 20 we have that $R^{\perp}=0$ and using the calculation from the proof of Lemma 9

$$-\sum_{i<j}^{3}(e_i\times e_j)\times R^{\perp}(e_i,e_j)\sigma=\mathcal{R}(\sigma)-\pi^{\perp}\Big(\sum_{i\in\mathbb{Z}_3}e_i\times\mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1})\Big)+H\times\mathcal{B}(\sigma)+(\operatorname{tr}S_\sigma)H$$
$$-\mathcal{A}(\sigma)-\sum_{j=1}^{3}\pi^{\perp}(T(e_j,\cdot)^{\sharp})\times S_\sigma(e_j)+\pi^{\perp}(T(\mathcal{B}(\sigma),\cdot)^{\sharp}),$$

the result follows. □

Now, the Weitzenböck formula (2.13) simplify drastically and we obtain the following result.

Corollary 8. *All infinitesimal associative deformation of the associative submanifold Y of Example 8 come from trivial deformations, Y is rigid.*

Proof. Using Lemmata 21 and 22 we have $\not{D}_{\mathrm{A}}{}^2(\sigma)=\nabla^*\nabla(\sigma)$, where

$$\nabla^*\nabla(\sigma)=\begin{pmatrix}\Delta&&&\\&\Delta&&\\&&\Delta&\\&&&\Delta\end{pmatrix}\begin{pmatrix}\sigma^2\\\sigma^3\\\sigma^4\\\sigma^7\end{pmatrix}$$

where $\sigma=\sigma^2e_2+\sigma^3e_3+\sigma^4e_4+\sigma^7e_7\in\Omega^0(NY)$ and $\Delta=-e_1^2-e_5^2-e_6^2$ is the Laplacian of functions on Y. If $\sigma\in\ker\not{D}_{\mathrm{A}}$ then each σ^k is a harmonic function on Y for each $k=2,3,4,7$, hence by the compactness of Y each σ^k is a constant function.

□

3 Co-closed G_2–flows

Geometric flows in G_2–geometry were first outlined by the seminal works of Bryant [Bry06] and Hitchin [Hit08], and have since been studied by several authors, e.g. [Bry11, BF18, Gri13, KT12, Lau16, Lau17]. These so-called G_2*–flows* arise as a tool in the search for ultimately torsion-free G_2–structures, by varying a non-degenerate 3-form on an oriented and spin 7–manifold M towards some $\varphi \in \Omega^3 := \Omega^3(M)$ such that the *torsion* $\nabla^{g_\varphi}\varphi$ vanishes. Such pairs (M^7, φ) solving the non-linear PDE problem $\nabla^{g_\varphi}\varphi \equiv 0$ are called G_2*–manifolds* and are very difficult to construct, especially when M is required to be compact. To this date, all known solutions stem from elaborate constructions in geometric analysis [Joy96, CP15, JK17].

When $M^7 = G$ is a Lie group, we propose to study the Laplacian co-flow [KT12]

$$\frac{\partial \psi_t}{\partial t} = -\Delta_{\psi_t}\psi_t \tag{3.1}$$

and the modified Laplacian co-flow [Gri13]

$$\frac{\partial \psi_t}{\partial t} = \Delta_{\psi_t}\psi_t + 2d((C - \operatorname{tr} T)\varphi_t) \quad \text{for} \quad C \quad \text{a constant}, \tag{3.2}$$

from the perspective introduced by Lauret [Lau16] in the general context of geometric flows on homogeneous spaces. As a proof of principle, we apply a natural Ansatz to construct an example of invariant self-similar solution, or *soliton*, of the Laplacian co-flow.

3.1 Geometric flow of G-invariant structures

Let us briefly survey Lauret's approach to geometric flows on homogeneous spaces [Lau16]. Consider the action of a Lie group G on a manifold M. A (r, s)-tensor γ on M is G-invariant if $g^*\gamma = \gamma$, for each $g \in G$, where

$$g^*\gamma(X_1, ..., X_r, \alpha_1, ..., \alpha_s) := \gamma(g_*X_1, ..., g_*X_r, (g^{-1})^*\alpha_1, ..., (g^{-1})^*\alpha_s),$$

for $X_1, ..., X_r \in \Gamma(TM)$ and $\alpha_1, \ldots, \alpha_s \in \Gamma(T^*M)$. In particular, when $M = G/H$ is a reductive homogeneous space, i.e.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{such that} \quad \operatorname{Ad}(h)\mathfrak{m} \subset \mathfrak{m}, \ \forall h \in H,$$

any G-invariant tensor γ is completely determined by its value γ_{x_0} at the point $x_0 = [1_G] \in G/H$, where γ_{x_0} is an $\operatorname{Ad}(H)$-invariant tensor at $\mathfrak{m} \cong T_{x_0}M$, i.e. $(\operatorname{Ad}(h))^*\gamma_{x_0} = \gamma_{x_0}$ for each $h \in H$. Given $x = [gx_0] \in G/H$, clearly $\gamma_x = (g^{-1})^*\gamma_{x_0}$. Consider now a geometric flow on M of the general form

$$\frac{\partial}{\partial t}\gamma_t = q(\gamma_t), \tag{3.3}$$

where γ_t is one-parameter family of tensor fields attached to a family of geometric structures on M [Hus66, Ch. 6, Sec. 2] and $q : \gamma \mapsto q(\gamma)$ is an assignment of a tensor field on M of the same type of γ such that for any diffeomorphism of M

$$q(f^*\gamma) = f^*q(\gamma) \quad \text{for} \quad f \in \mathrm{Diff}(M). \tag{3.4}$$

Then, if $M = G/H$, requiring G-invariance of γ_t, for all t, the diffeomorphism invariance (3.4) reduces the flow to an ODE for a one-parameter family γ_t of $\mathrm{Ad}(H)$-invariant tensors on the vector space $\mathfrak{m}$:

$$\frac{d}{dt}\gamma_t = q(\gamma_t),$$

thus, short-time existence and uniqueness among the G-invariant solution are guaranteed.

Now, suppose that for a fixed geometric structure, the orbit

$$\mathrm{Gl}(\mathfrak{m}) \cdot \gamma \tag{3.5}$$

is open in the vector space $\mathfrak{T}$ of all tensor of the same type as γ, and it is parametrised by the homogeneous space $\mathrm{Gl}(\mathfrak{m})/G_\gamma$, where

$$G_\gamma := \{h \in \mathrm{Gl}(\mathfrak{m}) \ ; \ h \cdot \gamma = \gamma\}$$

is the stabilizer of γ within $\mathrm{Gl}(\mathfrak{m})$. Consider $\theta : \mathfrak{gl}(\mathfrak{m}) \to \mathrm{End}(\mathfrak{T})$ the infinitesimal representation given by the action (3.5) defined by

$$\theta(A)\gamma := \frac{d}{dt}|_{t=0}(e^{At} \cdot \gamma).$$

Using the reductive decomposition $\mathfrak{gl}(\mathfrak{m}) = \mathfrak{g}_\gamma \oplus \mathfrak{q}_\gamma$ from (3.5), we have

$$\theta(\mathfrak{q}_\gamma)\gamma = \mathfrak{T}. \tag{3.6}$$

In particular, for $q(\gamma)$ there exist a unique linear operator $Q_\gamma \in \mathfrak{q}_\gamma$ such that $q(\gamma) = \theta(Q_\gamma)\gamma$.

3.2 Invariant G_2–structures on Lie groups

At this point, we fix $(M^7 = G, \varphi)$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and φ a left-invariant G_2–structure. We consider $\gamma = \psi$ the dual 4-form of the G_2–structure, which is left-invariant too. Now, we address the geometric flow (3.3) for the cases (3.1) and (3.2), i.e. $q := -\Delta_\psi$ and $q := \Delta_\psi + 2d(C - \operatorname{tr} T)*_\varphi$, respectively. Accordingly with this, we also denote by $\psi \in \Lambda^4(\mathfrak{g})^*$ which lift to G by left-translation. The $\mathrm{Gl}(\mathfrak{g})$-orbit (see Definition 3)

$$\mathrm{Gl}(\mathfrak{g}) \cdot \psi \subset \Lambda^4(\mathfrak{g})^* \tag{3.7}$$

is open under the natural action

$$h\cdot\psi := (h^{-1})^*\psi = \psi(h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot), \quad h\in \mathrm{Gl}(\mathfrak{g}).$$

So, the infinitesimal representation $\theta : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^4(\mathfrak{g})^*)$ at ψ is given by

$$\theta(A)\psi := -\psi(A\cdot,\cdot,\cdot,\cdot) - \cdots - \psi(\cdot,\cdot,\cdot,A\cdot),$$

following (3.6) we have

$$\theta(\mathfrak{gl}(\mathfrak{g}))\psi = \Lambda^4(\mathfrak{g})^*, \tag{3.8}$$

The Lie algebra of the stabilizer subgroup $\mathrm{G}_2(\psi) := \mathrm{Gl}(\mathfrak{g})_\psi \cong \mathrm{G}_2\times\mathbb{Z}_2$ is given by

$$\mathfrak{g}_2(\psi) := \{A\in\mathfrak{gl}(\mathfrak{g}) \ ; \ \theta(A)\psi = 0\} \cong \mathfrak{g}_2.$$

From (1.4) we get the polar decomposition $\mathfrak{gl}(\mathfrak{g}) = \mathfrak{so}(\mathfrak{g}) \oplus \mathrm{sym}(\mathfrak{g})$, we consider the orthogonal complement subspace $\mathfrak{q}_7(\psi) \subset \mathfrak{so}(\mathfrak{g})$ of $\mathfrak{g}_2(\psi)$ relative to the induced inner product from $\mathfrak{gl}(\mathfrak{g})$ (i.e. $\mathrm{tr}(AB^t)$). In the other hand, the G_2–decomposition of $\mathrm{sym}(\mathfrak{g})$ into $\mathfrak{q}_1(\psi) = \mathbb{R}I$, the one dimensional trivial representation and $\mathfrak{q}_{27}(\psi) = \mathrm{sym}_0(\mathfrak{g})$ the fundamental representation of traceless symmetric matrices which has dimension 27. Moreover, by comparing with the reductive decomposition $\mathfrak{gl}(\mathfrak{g}) = \mathfrak{g}_2(\psi)\oplus\mathfrak{q}(\psi)$ it follows the G_2–invariant decomposition

$$\mathfrak{q}(\psi) = \mathfrak{q}_1(\psi)\oplus\mathfrak{q}_7(\psi)\oplus\mathfrak{q}_{27}(\psi),$$

and the faithful representation

$$\theta(\mathfrak{q}(\psi))\psi = \Lambda^4(\mathfrak{g})^*. \tag{3.9}$$

In particular, for the Laplacian $\Delta_\psi\psi$, there exists a unique $Q_\psi\in\mathfrak{q}(\psi)$ such that $\theta(Q_\psi)\psi = \Delta_\psi\psi$. Now, for any other $\phi = h\cdot\psi \in \mathrm{Gl}(\mathfrak{g})\cdot\psi$,

$$\mathrm{Gl}(\mathfrak{g})_\phi = \mathrm{Gl}(\mathfrak{g})_{h\cdot\psi_0} = h^{-1}G_2(\psi)h \quad \text{and} \quad \mathfrak{gl}(\mathfrak{g})_\phi = \mathfrak{gl}(\mathfrak{g})_{h\cdot\psi} = \mathrm{Ad}(h^{-1})\mathfrak{g}_2(\psi),$$

where $\mathrm{Ad} : \mathrm{Gl}(\mathfrak{g}) \to \mathrm{Gl}(\mathfrak{gl}(\mathfrak{g}))$. Moreover, we have the following relations.

Lemma 23. *Let $\bar\psi = h\cdot\psi$ for $h\in\mathrm{Gl}(\mathfrak{g})$, denote $\bar{*}$ the Hodge star and $\bar\Delta$ the Laplacian operator of $\bar\psi$, then*

$$\bar{*} = (h^{-1})^* * h^* \quad \textit{and} \quad h^*\circ\bar\Delta = \Delta\circ h^*,$$

where $*$ and Δ are the Hodge star and the Laplacian operator of ψ, respectively.

Proof. The inner products on $\mathfrak{g}$ and $\mathfrak{g}^*$ induced by a G_2–structure $\bar\varphi = h\cdot\varphi$ are $\bar g = (h^{-1})^*g$ and $\bar g = h^*g$, respectively, where g is the inner product induced by φ. So, for $\alpha\in\Lambda^k(\mathfrak{g})^*$

we have

$$\begin{aligned}\alpha\wedge\bar{*}\,\alpha =&\bar{g}(\alpha,\alpha)\overline{\mathrm{vol}}\\ =&(h^*g)(\alpha,\alpha)(h^{-1})^*\,\mathrm{vol}\\ =&(h^{-1})^*(g(h^*\alpha,h^*\alpha)\,\mathrm{vol})\\ =&\alpha\wedge(h^{-1})^*\ast h^*\alpha,\end{aligned}$$

which gives the first claimed relation. In particular,

$$\bar{*}\,\bar\psi=(h^{-1})^*\ast h^*\bar\psi=(h^{-1})^*\ast\psi=h\cdot\varphi=\bar\varphi.$$

Applying again the first relation to the operator $d^*=(-1)^{7k}\ast d\ast$, we have $d^{\bar *}=(h^{-1})^*\circ d^*\circ h^*$, which yields the claim because d commutes with the pull-back h^*. □

As consequence of the above Lemma, we can relate $Q_{\bar\psi}\in\mathfrak{q}(\bar\psi)$ to $Q_\psi\in\mathfrak{q}(\psi)$:

$$\begin{aligned}\theta(Q_{\bar\psi})\bar\psi =&\Delta_{\bar\psi}\bar\psi=\Delta_{\bar\psi}((h^{-1})^*\psi)=(h^{-1})^*(\Delta_\psi\psi)\\ =&(h^{-1})^*\theta(Q_\psi)\psi=(h^{-1})^*\theta(Q_\psi)h^*\bar\psi\\ =&(h^{-1})^*\frac{d}{dt}(e^{tQ_\psi}\cdot(h^{-1}\cdot\bar\psi))|_{t=0}=\frac{d}{dt}((he^{tQ_\psi}h^{-1})\cdot\bar\psi))|_{t=0}\\ =&\frac{d}{dt}((e^{t\,\mathrm{Ad}(h)Q_\psi})\cdot\bar\psi))|_{t=0}=\theta(\mathrm{Ad}(h)Q_\psi)\bar\psi,\end{aligned}$$

since $\mathfrak{g}_2(\bar\psi)\cap\mathfrak{q}(\bar\psi)=0$. Therefore,

$$Q_{\bar\psi}=\mathrm{Ad}(h)Q_\psi. \tag{3.10}$$

In particular, a G-invariant solution of the Laplacian co-flow (3.1) is given by a 1-parameter family in $\mathfrak{g}$ solving

$$\frac{d}{dt}\psi_t=-\Delta_t\psi_t. \tag{3.11}$$

Writing $\psi_t=:h_t^{-1}\cdot\psi$ for $h_t\in\mathrm{Gl}(\mathfrak{g})$, we have

$$\begin{aligned}\frac{d}{dt}\psi_t =&\psi(h_t'\cdot,h_t\cdot,h_t\cdot,h_t\cdot)+\psi(h_t\cdot,h_t'\cdot,h_t\cdot,h_t\cdot)+\psi(h_t\cdot,h_t\cdot,h_t'\cdot,h_t\cdot)+\psi(h_t\cdot,h_t\cdot,h_t\cdot,h_t'\cdot)\\ =&\psi_t(h_t^{-1}h_t'\cdot,\cdot,\cdot,\cdot)+\psi_t(\cdot,h_t^{-1}h_t'\cdot,\cdot,\cdot)+\psi_t(\cdot,\cdot,h_t^{-1}h_t'\cdot,\cdot)+\psi_t(\cdot,\cdot,\cdot,h_t^{-1}h_t'\cdot)\\ =&-\theta(h_t^{-1}h_t')\psi_t,\end{aligned}$$

thus the evolution of h_t under the flow (3.11) is given by

$$\frac{d}{dt}h_t=h_tQ_t. \tag{3.12}$$

Remark 7. *If we identify* $\mathrm{sym}(\mathfrak{g})$ *with the symmetric* 2*-tensor* $S^2(\mathfrak{g})$ *using the map* $i:\mathrm{sym}(\mathfrak{g})\to\Lambda^3(\mathfrak{g})^*$ *from* (1.14) *and applying Lemma 3 we have*

$$\ast i(Q)=\theta(Q-\frac{1}{4}\,\mathrm{tr}(Q)I)\psi \tag{3.13}$$

We adapt the following proposition to our convention (1.15) instead of the Grigorian convention for the torsion forms (See Remark 3).

Proposition 6. *[Gri13, Proposition 2.3] Suppose we have a co-closed* G_2*-structure on a manifold* M *with* 3*-form* φ*. Let* $\xi = i(h) \in \Omega^3$ *with* h *a symmetric tensor, then the exterior derivative* $d\xi$ *is given by*

$$\begin{aligned} d\xi =& \frac{1}{2}\big(\operatorname{tr} T \operatorname{tr} h - \langle T, h\rangle\big)\psi - (\nabla \operatorname{tr} h - \operatorname{div} h)^\flat \wedge \varphi \\ &+ *i(\operatorname{curl} h_{(ab)} + \frac{1}{2} T \circ h_{ab} + (Th)_{ab} - \frac{1}{2}(\operatorname{tr} h) T_{ab} - \frac{1}{2}(\operatorname{tr} T) h_{ab}) \end{aligned} \tag{3.14}$$

where $(\operatorname{div} h)_a = \nabla^b h_{ba}$ *denotes the* divergence *of a symmetric* 2*-tensor,* $(\operatorname{curl} h)_{(ab)} = (\operatorname{curl} h)_{ab} + (\operatorname{curl} h)_{ba} = (\nabla_m h_{an})\varphi_b^{mn} + (\nabla_m h_{bn})\varphi_a^{mn}$ *is the symmetrized* curl *operator and* $(T \circ h)_{ab} = \varphi_{amn}\varphi_{bpq} T^{mn} T^{pq}$ *a product of* 2*-tensors.*

Lemma 24. *For a co-closed* G_2*–structure* φ *we have:*

(i) For any vector field v *holds* $\theta(A_v)\psi = 3v^\flat \wedge \varphi$ *where* $A_v(w) = v \times w$ *is the skew-symmetric matrix given by the cross product.*

(ii) $d\varphi = -\theta(T)\psi$*, where* T *is the full torsion tensor.*

(iii) $\Delta_\psi \psi = \theta(\frac{10}{21} A_{\operatorname{div} T} - (\operatorname{curl} T)_{(ab)} - \frac{1}{2}(T \circ T)_{ab} - (T^2)_{ab})\psi.$

For a G-invariant solution of the modified Laplacian co-flow (3.2) is given by a one-parameter family in $\mathfrak{g}$ solving

$$\frac{d}{dt}\psi_t = \Delta_t \psi_t + 2(C - \operatorname{tr}(T_t)) d\varphi_t \quad \text{for} \quad C \quad \text{a constant,} \tag{3.15}$$

notice, by the G-invariance of τ_0 for any φ_t then $\operatorname{tr}(T_t)$ is just time-dependent. Thus, writing $\psi_t =: h_t^{-1} \cdot \psi$ for $h_t \in \mathrm{Gl}(\mathfrak{g})$, we have that the evolution of h_t under the flow (3.15) is given by

$$\frac{d}{dt} h_t = -h_t Q_t + 2(C - \operatorname{tr}(T_t)) h_t T_t \quad \text{for} \quad C \quad \text{a constant.} \tag{3.16}$$

3.2.1 Proof of Lemma 24

Before the proof of Lemma 24, we collect the following properties for an invariant co-closed G_2–structure.

Lemma 25. *(i)* $\operatorname{div} \tau_{27} = \frac{1}{7}\nabla(\operatorname{tr} T) - \operatorname{div} T.$

(ii) $(\operatorname{curl} \tau_{27})_{(ab)} = -(\operatorname{curl} T)_{(ab)}$ *and* $\operatorname{tr}((\operatorname{curl} T)_{(ab)}) = 0.$

(iii) $(T \circ \tau_{27}) = \frac{1}{7}((\operatorname{tr} T)^2 g - (\operatorname{tr} T)T) - T \circ T$ *and* $\operatorname{tr}(T \circ T) = (\operatorname{tr} T)^2 - |T|^2.$

Proof. (i) It is enough to apply the div to $\tau_{27} = \frac{1}{7}(\operatorname{tr} T)g - T$.

(ii) Again, we apply curl to τ_{27}, it remains to proof the traceless property

$$(\operatorname{curl} T)_{ab} g^{ab} = (\nabla_m T_{an})\varphi^{mna} = 0.$$

(iii)

$$\begin{aligned}(T \circ \tau_{27})_{ab} = T^{mn}\tau_{27}^{pq}\varphi_{mpa}\varphi_{nqb} =& \frac{1}{7}(\operatorname{tr} T)T^{mn}\varphi_{mpa}\varphi_{nqb}g^{pq} - (T\circ T)_{ab}\\ =& \frac{1}{7}(\operatorname{tr} T)T^{mn}(g_{mn}g_{ab} - g_{mb}g_{an} + \psi_{manb}) - (T\circ T)_{ab}\\ =& \frac{1}{7}(\operatorname{tr} T)^2 g_{ab} - \frac{1}{7}(\operatorname{tr} T)T_{ab} - (T\circ T)_{ab}\end{aligned}$$

For the trace we have

$$\begin{aligned}(T\circ T)_{ab}g^{ab} =& T^{mn}T^{pq}\varphi_{mpa}\varphi_{nqb}g^{ab}\\ =& T^{mn}T^{pq}(g_{mn}g_{pq} - g_{mq}g_{pn} + \psi_{mpnq})\\ =& (\operatorname{tr} T)^2 - T^n_q T^q_n\end{aligned}$$

□

Proof of Lemma 24. (i) Let $v = v^i e_i$ be a vector field, then the skew-symmetric matrix A_v is given by $(A_v)_{jk} = v^i\varphi_{ijk}$, thus we have

$$\begin{aligned}\theta(A_v)\psi =& -\frac{1}{3!}(A_v)^l_a\psi_{bcd}dx^{abcd}\\ =& -\frac{1}{3!}v^i\varphi^l_{ia}\psi_{lbcd}dx^{abcd}\\ =& \frac{1}{3!}v^i(-g_{ib}\varphi_{acd} - g_{ic}\varphi_{bad} - g_{id}\varphi_{bca}\\ & + g_{ab}\varphi_{icd} + g_{ac}\varphi_{bid} + g_{ad}\varphi_{bci})dx^{abcd}\\ =& \frac{3}{3!}v^i g_{ib}\varphi_{acd}dx^{bacd} = 3v^\flat \wedge \varphi.\end{aligned}$$

(ii) Using the equation (3.13) we have

$$\tau_0\psi = *i\Big(\frac{\tau_0}{3}I\Big) = \theta\Big(-\frac{\tau_0}{4}I\Big)\psi \quad \text{and} \quad *\tau_3 = *i(\tau_{27}) = \theta(\tau_{27})\psi.$$

By the co-closed condition the torsion tensor is $T = \frac{\tau_0}{4}I - \tau_{27}$, thus $\tau_0 = \frac{4}{7}\operatorname{tr}(T)$ and $\tau_{27} = \frac{1}{7}\operatorname{tr}(T) - T$, therefore

$$d\varphi = \tau_0\psi + *\tau_3 = \theta\Big(-\frac{\tau_0}{4}I + \tau_{27}\Big)\psi = -\theta(T)\psi.$$

(iii) For a co-closed G_2–structure, the Laplacian of ψ is

$$\Delta\psi = d * d\varphi = d\tau_0 \wedge \varphi + \tau_0^2\psi + \tau_0 * \tau_3 + d\tau_3.$$

Now, we apply Lemma 6 to $d\tau_3 = di(\tau_{27})$, thus, we get

$$\begin{aligned} d\tau_3 =& -\frac{2}{7}\langle T, \tau_{27}\rangle\psi - \frac{1}{2}(\operatorname{div}\tau_{27})^\flat\varphi \\ &+ *i\Big((\operatorname{curl}\tau_{27})_{(ab)} + \frac{1}{2}(T\circ\tau_{27})_{ab} + (T\tau_{27})_{ab} \\ &- \frac{1}{2}(\operatorname{tr}T)(\tau_{27})_{ab} - \frac{1}{14}\langle T, \tau_{27}\rangle g_{ab}\Big). \end{aligned}$$

Thus, the Laplacian of ψ is

$$\begin{aligned} \Delta\psi =&\Big(\frac{4}{7}d(\operatorname{tr}T) - \frac{1}{2}(\operatorname{div}\tau_{27})^\flat\Big)\wedge\varphi + *i\Big((\operatorname{curl}\tau_{27})_{(ab)} + \frac{1}{2}(T\circ\tau_{27})_{ab} + (T\tau_{27})_{ab} \\ &+ \frac{1}{14}(\operatorname{tr}T)(\tau_{27})_{ab} + \frac{16}{147}(\operatorname{tr}T)^2 g_{ab} - \frac{1}{6}\langle T, \tau_{27}\rangle g_{ab}\Big). \end{aligned}$$

Now, replacing $\tau_{27} = \frac{1}{7}(\operatorname{tr}T)g - T$ and using the identity $\operatorname{div}T = \nabla\operatorname{tr}T$, we get

$$\begin{aligned} \Delta\psi =&\frac{10}{7}(\nabla\operatorname{tr}T)^\flat\wedge\varphi + *i\Big(-(\operatorname{curl}T)_{(ab)} - \frac{1}{2}(T\circ T)_{ab} - (T^2)_{ab} \\ &+ \frac{1}{6}(\operatorname{tr}T)^2 g_{ab} + \frac{1}{6}|T|^2 g_{ab}\Big) \\ =&\frac{10}{7}d(\operatorname{tr}T)\wedge\varphi + \theta(-(\operatorname{curl}T)_{(ab)} - \frac{1}{2}(T\circ T)_{ab} - (T^2)_{ab})\psi \end{aligned}$$

Since

$$\operatorname{tr}(-(\operatorname{curl}T)_{(ab)} - \frac{1}{2}(T\circ T)_{ab} - (T^2)_{ab} + \frac{1}{6}(\operatorname{tr}T)^2 g_{ab} + \frac{1}{6}|T|^2 g_{ab}) = \frac{4}{6}((\operatorname{tr}T)^2 + |T|^2)$$

□

3.3 Lie bracket flow

The *Lie bracket flow* is a dynamical system defined on the variety of Lie algebras, corresponding to an invariant geometric flow under a natural change of variables. It is introduced in [Lau16] as a tool for the study of regularity and long-time behaviour of solutions.

For each $h \in \mathrm{Gl}(\mathfrak{g})$, consider the following Lie bracket in $\mathfrak{g}$:

$$\mu = [\cdot,\cdot]_h := h\cdot[\cdot,\cdot] = h[h^{-1}\cdot, h^{-1}\cdot]. \tag{3.17}$$

Indeed, $(\mathfrak{g}, [\cdot,\cdot]) \xrightarrow{h} (\mathfrak{g}, \mu)$ defines a Lie algebra isomorphism, and consequently an equivariant equivalence between invariant structures

$$\eta : (G, \psi_\mu) \to (G_\mu, \psi),$$

where G_μ is the 1-connected Lie group with Lie algebra $(\mathfrak{g},\mu)$, η is an automorphism such that $d\eta_1 = h$ and $\psi_\mu = \eta^*\psi$. In particular, by Lemma 23, $\Delta_\mu\psi_\mu = \eta^*\Delta_\psi\psi$, or, equivalently, $Q_\mu = hQ_\psi h^{-1}$, by equation (3.10).

Lemma 26. *[Lau16, §4.1] Let* $\{h_t\} \subset \mathrm{Gl}(\mathfrak{g})$ *be:*

(i) a solution of (3.12)*, then the bracket* $\mu_t := [\cdot,\cdot]_{h_t}$ *evolves under the flow*

$$\frac{d}{dt}\mu_t = -\delta_{\mu_t}(Q_{\mu_t}). \tag{3.18}$$

(ii) a solution of (3.16)*, then the bracket* $\mu_t := [\cdot,\cdot]_{h_t}$ *evolves under the flow*

$$\frac{d}{dt}\mu_t = \delta_{\mu_t}(Q_{\mu_t} - 2(C - \operatorname{tr} T_t)T_{\mu_t}), \tag{3.19}$$

in which $\delta_\mu : \mathrm{End}(\mathfrak{g}) \to \Lambda^2(\mathfrak{g})^* \otimes \mathfrak{g}$ *is the infinitesimal representation of the* $\mathrm{Gl}(\mathfrak{g})$*-action* (3.17)*, defined by*

$$\delta_\mu(A) := -A\mu(\cdot,\cdot) + \mu(A\cdot,\cdot) + \mu(\cdot,A\cdot).$$

Proof. (i) Setting $Q_{\mu_t} := h_tQ_th_t^{-1}$, we compute:

$$\begin{aligned}\frac{d}{dt}\mu_t =& h_t'[h_t^{-1}\cdot, h_t^{-1}\cdot] + h_t[(h_t^{-1})'\cdot, h_t^{-1}\cdot] + h_t[h_t^{-1}\cdot, (h_t^{-1})'\cdot]\\ =& h_t'h_t^{-1}\mu_t(\cdot,\cdot) - \mu_t(h_t'h_t^{-1}\cdot,\cdot) - \mu_t(\cdot, h_t'h_t^{-1}\cdot)\\ =& -\delta_{\mu_t}(h_t'h_t^{-1}) = -\delta_{\mu_t}(h_tQ_th_t^{-1}) = -\delta_{\mu_t}(Q_{\mu_t}),\end{aligned}$$

since $(h_t^{-1})' = -h_t^{-1}h_t'h_t^{-1}$.

(ii) Similarly, setting $T_{\mu_t} = h_tT_th_t^{-1}$, we compute:

$$\begin{aligned}\frac{d}{dt}\mu_t =& \delta_{\mu_t}(h_t'h_t^{-1})\\ =& \delta_{\mu_t}(h_tQ_th_t^{-1} - 2(C - \operatorname{tr}(T_t))h_tT_th_t^{-1})\\ =& \delta_{\mu_t}(Q_{\mu_t} - 2(C - \operatorname{tr} T_t)T_{\mu_t}),\end{aligned}$$

□

Remark. *Notice that, if* $\{h_t\} \subset \mathrm{Gl}(\mathfrak{g})$ *solves*

$$\frac{d}{dt}h_t = Q_{\mu_t}h_t, \quad \text{or} \quad \frac{d}{dt}h_t = -Q_{\mu_t}h_t + 2(C - \operatorname{tr} T_t)T_{\mu_t}h_t$$

then μ_t *solves the bracket flow* (3.18) *or* (3.19)*.*

3.4 Self Similar Solutions

We say that a 4-form ψ *flows self-similarly* along the flow (3.11) if the solution ψ_t starting at ψ has the form $\psi_t = b_t f_t^* \psi$, for some one-parameter families $\{f_t\} \subset \mathrm{Diff}(G)$ and time-dependent non-vanishing functions $\{b_t\}$. This is equivalent to the relation

$$q(\psi) = \lambda\psi + \mathcal{L}_X \psi,$$

for some constant $\lambda \in \mathbb{R}$, X a complete vector field and q denotes either minus the Hodge Laplace operator Δ_ψ or the modified Laplace operator $\Delta_\psi + 2d(C - \operatorname{tr} T)*_\varphi$. Suppose that the infinitesimal operator defined by $q(\psi) = \theta(Q_\psi)\psi$ had the particular form

$$Q_\psi = cI + D \quad \text{for} \quad c \in \mathbb{R} \quad \text{and} \quad D \in \mathrm{Der}(\mathfrak{g}). \tag{3.20}$$

Then we have

$$\begin{aligned} \theta(Q_\psi)\psi &= -\,4c\psi + \theta(D)\psi = -4c\psi - \frac{d}{dt}\big((e^{tD})^*\psi\big)|_{t=0} \\ &= -\,4c\psi - \mathcal{L}_{X_D}\psi, \end{aligned}$$

where X_D is a vector field on $\mathfrak{g}$ defined by the 1-parameter group of automorphisms $e^{tD} \in \mathrm{Aut}(\mathfrak{g})$.

In that case, (G, ψ) is a soliton for the Laplacian co-flow or for the modified Laplacian co-flow with

$$q(\psi) = -4c\psi - \mathcal{L}_{X_D}\psi,$$

where X_D also denotes the invariant vector field on G defined by the 1-parameter subgroup β_t in $\mathrm{Aut}(G)$ such that $d(\beta_t)_1 = e^{tD} \in \mathrm{Aut}(\mathfrak{g})$.

A G_2–structure whose underlying 4-form ψ satisfies (3.20) is called an *algebraic soliton*, and we say that it is *expanding, steady, or shrinking* if λ is positive, zero, or negative, respectively.

Lemma 27. *Given $\psi_2 = c\psi_1$ with $c \in \mathbb{R}^*$, then:*

(i) The Laplacian operator satisfies the scaling property

$$\Delta_2\psi_2 = c^{1/2}\Delta_1\psi_1. \tag{3.21}$$

(ii) The torsion forms have the scaling property

$$(\tau_0)_2 = c^{-1/4}(\tau_0)_1 \quad \textit{and} \quad (\tau_3)_2 = c^{1/2}(\tau_3)_1.$$

In particular, $\operatorname{tr}_{g_2} T_2 = c^{-1/4} \operatorname{tr}_{g_1} T_1$.

Proof. Notice that $c\psi_1 = (c^{1/4})^4\psi_1$, then $\varphi_2 = c^{3/4}\varphi_1$, $g_2 = c^{1/2}g_1$ and $\mathrm{vol}_2 = c^{7/4}\,\mathrm{vol}_1$. For a k-form α we have

$$\begin{aligned}\alpha\wedge *_2\alpha =& g_2(\alpha,\alpha)\,\mathrm{vol}_2 = \frac{1}{k!}\alpha_{i_1,\dots,i_k}\alpha_{j_1,\dots j_k}(g_2)^{i_1j_1}\cdots(g_2)^{i_kj_k}\,\mathrm{vol}_2\\ =& c^{7/4-k/2}\frac{1}{k!}\alpha_{i_1,\dots,i_k}\alpha_{j_1,\dots j_k}(g_1)^{i_1j_1}\cdots(g_1)^{i_kj_k}\,\mathrm{vol}_1 = c^{7/4-k/2}g_1(\alpha,\alpha)\,\mathrm{vol}_1\\ =& c^{7/4-k/2}\alpha\wedge *_1\alpha.\end{aligned}$$

So, for a k-form $*_2\alpha = c^{\frac{1}{4}(7-2k)} *_1 \alpha$.

(i) For the Hodge Laplacian operator we have

$$\begin{aligned}\Delta_2\psi_2 =& d *_2 d *_2 \psi_2 - *_2 d *_2 d\psi_2 = cd *_2 d *_2 \psi_1 - c *_2 d *_2 d\psi_1\\ =& c^{3/4} d *_2 d *_1 \psi_1 - c^{1/4} *_2 d *_1 d\psi_1\\ =& c^{1/2} d *_1 d *_1 \psi_1 - c^{1/2} *_1 d *_1 d\psi_1 = c^{1/2}\Delta_1\psi_1.\end{aligned}$$

(ii) For the scalar torsion form, we have

$$(\tau_0)_2 = \frac{1}{7} *_2 (\varphi_2\wedge d\varphi_2) = \frac{c^{3/2}}{7} *_2 (\varphi_1\wedge d\varphi_1) = \frac{c^{3/2}c^{-7/4}}{7} *_1 (\varphi_1\wedge d\varphi_1) = c^{-1/4}(\tau_0)_1.$$

Finally, since ψ_2 is co-closed, using the relation $(\tau_3)_2 = *_2 d\varphi_2 - (\tau_0)_2\varphi_2$ the result $(\tau_3)_2 = c^{1/2}(\tau_3)_1$ follows.

□

Lemma 28. *If ψ is an algebraic soliton with $Q_\psi = cI + D$, then $\psi_t = b_t h_t^*\psi$ is a self-similar solution for the Laplacian co-flow (3.11), with*

$$b_t = (2ct+1)^2 \quad \text{and} \quad h_t = e^{s_t D}, \quad \text{for} \quad s_t = -\frac{1}{2c}\log(2ct+1). \tag{3.22}$$

Moreover,

$$Q_t = b_t^{-1/2}Q_\psi.$$

Proof. Applying Lemmata 23 and 27, we have

$$\begin{aligned}\Delta_t\psi_t &= b_t^{1/2}h_t^*\Delta\psi = b_t^{1/2}h_t^*\theta(Q_\psi)\psi\\ &= b_t^{1/2}h_t^*\big(-4c\psi + \theta(D)\psi\big)\\ &= -4cb_t^{1/2}h_t^*\psi + \theta(b_t^{1/2}h_t^{-1}Dh_t)h_t^*\psi.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\psi_t =& b_t' h_t^*\psi + b_t(h_t^*\psi)'\\ =& b_t' h_t^*\psi + b_t\theta(h_t^{-1}h_t')h_t^*\psi.\end{aligned}$$

Replacing the above expressions in (3.11) and comparing terms we obtain the ODE system

$$\begin{cases} b_t' = 4cb_t^{1/2}, & b(0) = 1 \\ b_t h_t' = -b_t^{1/2} D h_t, & h(0) = I \end{cases},$$

the solutions of which are as claimed.

Finally, we have

$$\begin{aligned} \theta(Q_t)\psi_t = \Delta_t \psi_t = b_t^{1/2} h_t^* \Delta\psi &= b_t^{1/2} h_t^* \theta(Q_\psi)\psi \\ &= b_t^{1/2} \theta(h_t^{-1} Q_\psi h_t) h_t^* \psi = \theta(b_t^{-1/2} h_t^{-1} Q_\psi h_t)\psi_t, \end{aligned}$$

so $Q_t = b_t^{-1/2} h_t^{-1} Q_\psi h_t$, which yields the second claim, since $Q_\psi h_t = h_t Q_\psi$. □

In terms of the bracket flow, we have $Q_{\mu_t} = h_t Q_t h_t^{-1} = b_t^{-1/2} Q_\psi$. Then, replacing in (3.18) the Ansatz

$$\mu_t = (\frac{1}{c(t)} I) \cdot [\cdot,\cdot] = c(t)[\cdot,\cdot] \quad \text{for} \quad c(t) \neq 0 \quad \text{and} \quad c(0) = 1, \tag{3.23}$$

we obtain $c_t' = cb_t^{-1/2} c_t$, which has solution $c_t = e^{c.s_t}$, with s_t as above.

Lemma 29. *If ψ is an algebraic soliton with $P_\psi = Q_\psi - 2(C - \operatorname{tr} T)T = cI + D$, then $\psi_t = b_t h_t^* \psi$ is a self-similar solution for the modified Laplacian co-flow (3.15), with*

$$b_t = (-2ct + 1)^2 \tag{3.24}$$

and

$$h_t = e^{s_t(D + 2CT) - 2C r_t T}, \quad \textit{for} \quad s_t = -\frac{1}{2c}\log(-2ct + 1), \quad \textit{and} \quad r_t = \frac{1}{c}(-2ct + 1)^{-1/2} - \frac{1}{c}.$$

Moreover,

$$P_t = b_t^{-1/2} P_\psi - 2C(b_t^{-1/4} - b_t^{-1/2}) \operatorname{Ad}(h_t^{-1}) T.$$

Proof. Applying Lemmata 23 and 27, we have

$$\begin{aligned} \Delta_t \psi_t + 2(C - \operatorname{tr} T_t) d\varphi_t &= b_t^{1/2} h_t^* \Delta\psi + 2(C - \operatorname{tr} T b_t^{-1/4}) b^{3/4} h_t^* d\varphi \\ &= b_t^{1/2} h_t^* \theta(Q_\psi)\psi - 2C b_t^{3/4} h_t^* \theta(T)\psi + 2 \operatorname{tr} T b_t^{1/2} h_t^* \theta(T)\psi \\ &= b_t^{1/2} h_t^* \theta(Q_\psi - 2(C - \operatorname{tr} T)T)\psi - 2C(b_t^{3/4} - b_t^{1/2}) h_t^* \theta(T)\psi \\ &= b_t^{1/2} h_t^* \big(-4c\psi + \theta(D)\psi\big) - 2C(b_t^{3/4} - b_t^{1/2}) h_t^* \theta(T)\psi \\ &= -4cb_t^{1/2} h_t^* \psi + \theta(b_t^{1/2} h_t^{-1}(D + 2CT - 2Cb_t^{1/4} T) h_t) h_t^* \psi. \end{aligned}$$

On the other hand, we know from the proof of Lemma 28 that $\psi_t' = b_t' h_t^* \psi + b_t \theta(h_t^{-1} h_t') h_t^* \psi$, then replacing the above expressions in (3.15) and comparing terms we obtain the ODE system

$$\begin{cases} b_t' = -4cb_t^{1/2}, & b(0) = 1 \\ b_t h_t' = b_t^{1/2}(D + 2CT - 2Cb_t^{1/4} T) h_t, & h(0) = I \end{cases},$$

the solutions of which are as claimed.
Finally, we have

$$\begin{aligned}\theta(P_t)\psi_t &= \Delta_t\psi_t + 2(C - \operatorname{tr}_t T)d\varphi_t\\ &= b_t^{1/2}h_t^*\Delta\psi + 2(C - \operatorname{tr} T b_t^{-1/4})b^{3/4}h_t^* d\varphi\\ &= b_t^{1/2}h_t^*\theta(P_\psi)\psi - 2C(b_t^{3/4} - b_t^{1/2})h_t^*\theta(T)\psi\\ &= \theta(b_t^{-1/2}h_t^{-1}P_\psi h_t - 2C(b_t^{-1/4} - b_t^{-1/2})h_t^{-1}Th_t)\psi_t,\end{aligned}$$

so $P_t = b_t^{-1/2}h_t^{-1}P_\psi h_t - 2C(b_t^{-1/4} - b_t^{-1/2})h_t^{-1}Th_t$, which yields the second claim, since $P_\psi h_t = h_t P_\psi$. □

Indeed, there is an equivalence between the time-dependent Lie bracket given in (3.23) and the corresponding soliton given in Lemma 28:

Theorem 6. *[Lau16, Theorem 6] Let (G, φ) be a 1-connected Lie group with an invariant* G_2*–structure. The following conditions are equivalent:*

(i) *The bracket flow solution starting at $[\cdot,\cdot]$ is given by*

$$\mu_t = (\frac{1}{c(t)}I)\cdot[\cdot,\cdot] \quad \textit{for} \quad c(t) > 0, c(0) = 1.$$

(ii) *The operator $Q_t \in \mathfrak{q}_\psi \subset \operatorname{End}(\mathfrak{g})$, such that $\Delta_\psi\psi = \theta(Q_\psi)\psi$, satisfies*

$$Q_\psi = cI + D, \quad \textit{for} \quad c \in \mathbb{R} \quad \textit{and} \quad D \in \operatorname{Der}(\mathfrak{g}).$$

3.5 Almost abelian Lie groups

In this section we address a class of solvable Lie group named the almost abelian,to exposed some basic notions about this we will follow [Lau17, Section 5]. Let (G, φ) be a connected and simply connected Lie group with an invariant G_2-structure φ, if the corresponding Lie algebra $\mathfrak{g}$ has an abelian ideal $\mathfrak{h}$ of codimension 1, we say that G is an *almost abelian* Lie group and $\mathfrak{g}$ is an almost abelian Lie algebra. For $\dim G = 7$ there exist an orthonormal basis $\{e_1, ..., e_7\}$ of $\mathfrak{g}$ such that $\mathfrak{h} = \operatorname{Span}\{e_1, ..., e_6\}$ and the left invariant G_2–structure is determined by

$$\varphi = \omega\wedge e^7 + \rho^+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{245} - e^{236} \tag{3.25}$$

where

$$\omega = e^{12} + e^{34} + e^{56} \quad \text{and} \quad \rho_+ = e^{135} - e^{146} - e^{245} - e^{236}$$

are the canonical SU(3)–structure of $\mathbb{R}^6 \cong \mathfrak{h}$. an the dual 4-form $\psi = \frac{1}{2}\omega^2 + \rho_-\wedge e^7$ where $\rho_- = J^*\rho_+ = -e^{246} + e^{235} + e^{145} + e^{136}$ and J is the canonical almost structure on $\mathbb{R}^6$

defined by $\omega := \langle J\cdot,\cdot\rangle$. Notice that the Lie algebra structure of $\mathfrak{g}$ is completely determined by a real 6×6 matrix $A := \mathrm{ad}(e_7)|_{\mathfrak{h}}$. So, following the notation of [Lau17], μ_A will denote the Lie bracket and G_A the corresponding connected and simply connected Lie group. In [Fre12] was studied the existence of invariant co-closed G_2–structures on G_A and the condition $d\psi = 0$ is entirely encoded by A.

Proposition 7. *[Fre12] (G_A,φ) is co-closed if and only if $A\in\mathfrak{sp}(6,\mathbb{R})$.*

$$\begin{aligned}\mathfrak{sp}(6,\mathbb{R}) :=& \{A\in\mathfrak{gl}(6,\mathbb{R});\quad A^tJ+JA=0\}\\ =&\left\{A=\left[\begin{array}{c|c} B & C\\ \hline D & -B^t\end{array}\right];\quad C,D\in\mathrm{sym}(3)\right\}\end{aligned}$$

A useful algebraic relations between the geometry of $\mathfrak{g}$, $\mathfrak{h}$ and A are summarised in the following Lemma:

Lemma 30. *Let $*$ and $\star$ the Hodge star operators on $\mathfrak{g}$ and $\mathfrak{h}$, respectively, determined by φ. Also, d_A denote the exterior derivative of left-invariant forms on the G_A, so for $\gamma\in\Lambda^k(\mathfrak{h})^*$ the following properties holds:*

- *[Lau17, Lemma 5.11] $*\gamma = \star\gamma\wedge e^7$, $*(\gamma\wedge e^7) = (-1)^k\star\gamma$ and $\theta(A)\star = -\star\theta(A^t)$ (if $\mathrm{tr}\,A = 0$).*
- *[Lau17, Lemma 5.12] $d_Ae^7 = 0$, $d_A\gamma = (-1)^k\theta(A)\gamma\wedge e^7$ and $d_A(\gamma\wedge e^7) = 0$.*
- *[Lau17, Equation (29)] The Ricci operator Ric_A of G_A is given by*

$$\mathrm{Ric}_A = \left[\begin{array}{c|c} \frac{1}{2}[A,A^t] & 0\\ \hline 0 & -\frac{1}{4}\,\mathrm{tr}(A+A^t)^2\end{array}\right] \tag{3.26}$$

From the above follows that

$$d\varphi = -\theta(A)\varphi\wedge e^7 = -\theta(A)\rho^+\wedge e^7.$$

Lemma 31. *For a matrix $A\in\mathfrak{sp}(6,\mathbb{R})$ holds the following:*

$$\theta(A)\rho_+ = \theta(JA)\rho_-\quad \textit{and}\quad \theta(A)\rho_- = \theta(A^tJ)\rho_+.$$

Proof. Note that $\omega_{ij} = J_i^kh_{kj}$ then

$$\begin{aligned}\theta(A)\rho_+ =& -\frac{1}{2}A_i^l\rho^+_{ljk}dx^{ijk}\\ =& -\frac{1}{2}A_i^l\rho^-_{jkp}\omega_{pl}dx^{ijk}\\ =& -\frac{1}{2}A_i^lJ_l^qh_{qp}\rho^-_{jkp}dx^{ijk}\\ =& -\frac{1}{2}(JA)_i^q\rho^-_{qjk}dx^{ijk} = \theta(JA)\rho_-.\end{aligned}$$

Notice that we used in the second line the identities (1.26). Similarly, $\theta(A)\rho_- = -\theta(JA)\rho_+$ and since $JA = -A^tJ$ the result follows. ☐

For a co-closed G_2–structure on G_A, we want to write the torsion forms in term of the matrix A.

Proposition 8. *The torsion forms* τ_0 *and* τ_3 *for an almost abelian Lie group* (G_A, φ) *with co-closed* G_2*–structure are*

$$\tau_0 = \frac{2}{7}\operatorname{tr}(JA) \quad \textit{and} \quad \tau_{27} = \left(\begin{array}{c|c} \frac{1}{14}\operatorname{tr}(JA)I_6 - \frac{1}{2}[J,A] & 0 \\ \hline 0 & -\frac{3}{7}\operatorname{tr}(JA) \end{array}\right)$$

Proof. Since the G_2–structure (3.25) is co-closed the scalar torsion is given by

$$\begin{aligned} \tau_0 =& \frac{1}{7} * (\varphi \wedge d\varphi) = -\frac{1}{7} * (\rho^+ \wedge \theta(A)\rho^+ \wedge e^7) \\ =& -\frac{1}{7} \star (\rho^+ \wedge \theta(A)\rho^+) = -\frac{1}{7} \star (\rho^+ \wedge \theta(JA)\rho^-) \\ =& \frac{1}{7}\langle \rho_-, \theta(JA)\rho_- \rangle \star (\mathrm{vol}_6) = \frac{2}{7}\operatorname{tr} JA \end{aligned}$$

Here, we used in the second line the Lemma 31 and from the orthogonal SU(3)–decomposition we have

$$\begin{aligned} \langle \rho_-, \theta(JA)\rho_- \rangle =& (JA)^2_2 + (JA)^4_4 + (JA)^6_6 + (JA)^2_2 + (JA)^3_3 + (JA)^5_5 \\ &+ (JA)^1_1 + (JA)^3_3 + (JA)^6_6 + (JA)^1_1 + (JA)^4_4 + (JA)^5_5 \\ =& 2\operatorname{tr} JA. \end{aligned}$$

Now, applying Lemma 30 to $*d\varphi$, we have

$$*d\varphi = - * (\theta(A)\rho^+ \wedge e^7) = \star\theta(A)\rho^+ = -\theta(A^t)\rho^- = -\theta(AJ)\rho^+.$$

Thus, applying j to $*d\varphi$ we get the symmetric bilinear form

$$j(*d\varphi)(u,v) = *(u \lrcorner \varphi \wedge v \lrcorner \varphi \wedge *d\varphi)$$

For $u = e_7$ and $v = e_i$

$$\begin{aligned} e_7 \lrcorner \varphi \wedge e_i \lrcorner \varphi \wedge *d\varphi =& \omega \wedge e_i \lrcorner \omega \wedge e^7 \wedge \star\theta(A)\rho_+ + \delta_{i7}\omega^2 \wedge \star\theta(A)\rho_+ \\ &+ \omega \wedge e_i \lrcorner \rho_+ \wedge \star\theta(A)\rho_+ \\ =& e_i \lrcorner \omega \wedge \omega \wedge \star\theta(A)\rho_+ \wedge e^7 \\ =& h(e_i \lrcorner \omega \wedge \omega, \theta(A)\rho_+)\,\mathrm{vol}_7 \end{aligned}$$

where h is the induced inner product on $\mathfrak{h}$ and notice that

$$\begin{aligned} h(e_i \lrcorner \omega \wedge \omega, \theta(A)\rho_+) &= \frac{1}{4}\omega_{ir}\omega_{st}A^l_a\rho^+_{lbc}h^{ra}h^{sb}h^{tc} \\ &= \frac{1}{4}\omega_{ir}A^{rl}\omega^{bc}\rho^+_{bcl} = 0 \end{aligned}$$

The last result follows by the identities (1.26). So, it is enough to consider $1 \leq i, j \leq 6$, we have:

$$\begin{aligned}
j(*d\varphi)_{ij} &= *\,(e_i \lrcorner \varphi \wedge e_j \lrcorner \varphi \wedge *d\varphi) = *(e_i \lrcorner \varphi \wedge e_j \lrcorner \varphi \wedge \star\theta(A)\rho_+) \\
&= -*(e_i \lrcorner \omega \wedge e_j \lrcorner \rho_+ \wedge *\theta(JA)\rho_- \wedge e^7 + e_j \lrcorner \omega \wedge e_i \lrcorner \rho_+ \wedge \star\theta(JA)\rho_- \wedge e^7) \\
&= -\star(e_i \lrcorner \omega \wedge e_j \lrcorner \rho_+ \wedge \star\theta(JA)\rho_- + e_j \lrcorner \omega \wedge e_i \lrcorner \rho_+ \wedge \star\theta(JA)\rho_-) \\
&= -\big(h(e_i \lrcorner \omega \wedge e_j \lrcorner \rho_+, \theta(JA)\rho_-) + h(e_j \lrcorner \omega \wedge e_i \lrcorner \rho_+, \theta(JA)\rho_-)\big) \star \mathrm{vol}_6 \\
&\quad - h(e_i \lrcorner \omega \wedge e_j \lrcorner \rho_+, \theta(JA)\rho_-) - h(e_j \lrcorner \omega \wedge e_i \lrcorner \rho_+, \theta(JA)\rho_-)
\end{aligned}$$

We compute the first term

$$\begin{aligned}
h(e_i \lrcorner \omega \wedge e_j \lrcorner \rho_+, \theta(JA)\rho_-) &= -\frac{1}{3!}(3\omega_{ir}\rho^+_{jst})((JA)^l_r\rho^+_{lst} - (JA)^l_s\rho^+_{lrt} + (JA)^l_t\rho^+_{lrs}) \\
&= -\frac{1}{2}(\underbrace{\omega_{ir}(JA)^l_r\rho^+_{jst}\rho^+_{lst}}_{\text{(I)}} - \underbrace{\rho^+_{jst}(JA)^l_s\rho^+_{ltr}\omega_{ri}}_{\text{(II)}} + \underbrace{\rho^+_{jst}(JA)^l_t\rho^+_{lsr}\omega_{ri}}_{\text{(III)}}) = \clubsuit
\end{aligned}$$

For each term (I),(II),(III) we apply the SU(3)–identities (1.26)

$$\begin{aligned}
\text{(I)} &= -4\omega_{ir}\omega_{lj}(JA)^l_r = -4J^n_i h_{nr} J^m_l h_{mj}(JA)^l_r \\
&= -4J^n_i h_{nr} h_{mj}(J^2A)^m_r = 4J^n_i A^j_n = 4(AJ)^j_i.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\text{(II)} =& (JA)^l_s\rho^+_{jst}\rho^+_{lit} \\
=& (JA)^l_s(-\omega_{jl}\omega_{si} + \omega_{ji}\omega_{sl} + \delta_{jl}\delta_{si} - \delta_{ji}\delta_{sl}) \\
=& (AJ)^j_i + (JA)^j_i - \mathrm{tr}(JA)\delta_{ji}.
\end{aligned}$$

Notice that, we used the symmetry of JA in the last line. Similarly, for (III) we have

$$\text{(III)} = -(AJ)^j_i - (JA)^j_i + \mathrm{tr}(JA)\delta_{ji}.$$

Summarising, we get

$$\clubsuit = -(AJ)^j_i + (JA)^j_i - \mathrm{tr}(JA)\delta_{ji} = -[A, J]^j_i - \mathrm{tr}(JA)\delta_{ji}.$$

Therefore,

$$j(*d\varphi)_{ij} = [A, J]^j_i + \mathrm{tr}(JA)\delta_{ji} + [A, J]^i_j + \mathrm{tr}(JA)\delta_{ij},$$

since the matrix $[A, J]$ is symmetric we have $j(*d\varphi) = 2\,\mathrm{tr}(JA)I_6 + 2[A, J]$. Finally, by using Lemma 3 we compute

$$\begin{aligned}
i(\tau_{27}) &= *\,d\varphi - \tau_0\varphi \\
4\tau_{27} &= 2\,\mathrm{tr}(JA)I_6 + 2[A, J] - \frac{12}{7}\,\mathrm{tr}(JA)I_7
\end{aligned}$$

$\square$

Corollary 9. *The full torsion tensor* T *of an almost abelian Lie group* (G_A,φ) *with an invariant co-closed* G_2*-structure is*

$$T=\frac{1}{2}\left(\begin{array}{c|c} [J,A] & 0 \\ \hline 0 & \operatorname{tr}(JA) \end{array}\right) \tag{3.27}$$

Remark 8. *Since* G_A *induces diffeomorphism by left translation and* φ *is* G_A*-invariant then* τ_0 *is constant and equal by its value at* $1\in G_A$*. In particular,*

$$\nabla(\operatorname{tr}T)=0.$$

Also, for a co-closed G_2*-structure, the Ricci curvature is given by [Gri13, Eq (4.30)]*

$$\operatorname{Ric}(g)=-\operatorname{curl}(T)-T^2+(\operatorname{tr}T)T$$

Lemma 32. *For the symmetric product of* 2*-tensor defined in Proposition 6 we have*

$$T\circ T=\left(\begin{array}{c|c} -\frac{1}{2}(\operatorname{tr}JA)[J,A]-S_A\circ_6 S_A & 0 \\ \hline 0 & -\operatorname{tr}S_A^2 \end{array}\right), \tag{3.28}$$

where $S_A=\frac{1}{2}(A+A^t)$ *is the symmetric part of* A *and* $(S_A\circ_6 S_A)_{ab}:=S_A^{mn}S_A^{pq}\rho^+_{mpa}\rho^+_{nqb}$*.*

Proof. We are going to calculate the matrix elements $(T\circ T)_{ij}$. So, for $i,j=7$ we have

$$\begin{aligned}(T\circ T)_{77}=&T^{mn}T^{pq}\varphi_{mp7}\varphi_{nq7}=\frac{1}{4}[J,A]^{mn}[J,A]^{pq}\omega_{mp}\omega_{nq}\\ =&\frac{1}{4}(J(A+A^t))^{nm}(J(A+A^t))^{pq}\omega_{mp}\omega_{nq}\\ =&\frac{1}{4}(A+A^t)^{na}J^m_a(A+A^t)^{pb}J^q_b\omega_{mp}\omega_{nq}\\ =&-S_A^{na}S_A^{pb}h_{ap}h_{bn}=-\operatorname{tr}S_A^2.\end{aligned}$$

Notice that we used the relation $AJ=-JA^t$ in the second line and symmetry of $J(A+A^t)$ in the third line. For $j=7$ and $i\neq 7$, we have

$$(T\circ T)_{i7}=T^{mn}T^{pq}\varphi_{mpi}\varphi_{nq7}=T^{mn}T^{pq}\varphi_{mpi}\omega_{nq}=\spadesuit.$$

Since $n,q\in\{1,...,6\}$ by Corollary 9 also $m,p\in\{1,...,6\}$, then

$$\begin{aligned}\spadesuit=&[J,A]^{mn}[J,A]^{pq}\rho^+_{mpi}\omega_{nq}\\ =&4(JS_A)^{mn}(JS_A)^{qp}\rho^+_{mpi}\omega_{nq}\\ =&4(S_A)^{ma}J^n_a(S_A)^{qb}J^p_b\rho^+_{mpi}\omega_{nq}\\ =&-4(S_A)^{ma}(S_A)^{qb}J^p_b\rho^+_{mpi}h_{aq}\\ =&-4(S_A)^m_q(S_A)^{qb}J^p_b\rho^+_{pim}\\ =&4(S_A^2)^m_b\rho^-_{imb}=0\end{aligned}$$

Here we used in the second line the symmetry of $[J, A]$, in the fourth line the relation $J_a^n \omega_{nq} = -h_{aq}$ and in the last one, the symmetry of S_A^2 with the skew-symmetry of ρ_-. Finally, for $i \neq 7$ and $j \neq 7$ we have

$$\begin{aligned}
(T \circ T)_{ij} =& T^{mn} T^{pq} \varphi_{mpi} \varphi_{nqj} \\
=& 2T^{mn} T^{77} \omega_{mi} \omega_{nj} + T^{mn} T^{pq} \rho^+_{mpi} \rho^+_{nqj} \\
=& \frac{1}{2} (\operatorname{tr} JA) [J, A]^{mn} J^a_m h_{ai} J^b_n h_{bj} + \frac{1}{4} [J, A]^{mn} [J, A]^{pq} \rho^+_{mpi} \rho^+_{nqj} \\
=& \frac{1}{2} (\operatorname{tr} JA) (J(A + A^t))^{mn} J^a_m h_{ai} J^b_n h_{bj} + \frac{1}{4} (J(A + A))^{mn} (J(A + A^t))^{pq} \rho^+_{mpi} \rho^+_{nqj} \\
=& \frac{1}{2} (\operatorname{tr} JA) (A + A^t)^m_c J^{cn} J^a_m h_{ai} J^b_n h_{bj} + \frac{1}{4} (A + A^t)^{mc} J^n_c (A + A^t)^{pd} J^q_d \rho^+_{mpi} \rho^+_{nqj} \\
=& \frac{1}{2} (\operatorname{tr} JA) (J(A + A^t))^{ca} h_{ai} (J^2)^{cb} h_{bj} + (S_A)^{mc} (S_A)^{pd} \rho^+_{mpi} \rho^+_{jnq} J^q_d J^n_c \\
=& -\frac{1}{2} (\operatorname{tr} JA) [J, A]^{ca} h_{ai} \delta^{cb} h_{bj} + (S_A)^{mc} (S_A)^{pd} \rho^+_{mpi} \rho^-_{jnd} J^n_c \\
=& -\frac{1}{2} (\operatorname{tr} JA) [J, A]_{ji} - (S_A)^{mc} (S_A)^{pd} \rho^+_{mpi} \rho^+_{djc} \\
=& -\frac{1}{2} (\operatorname{tr} JA) [J, A]_{ij} - (S_A)^{mc} (S_A)^{pd} \rho^+_{mpi} \rho^+_{cdj}
\end{aligned}$$

□

Proposition 9. *If* (G_A, φ) *is co-closed, we have:*

i) For the Hodge Laplacian of ψ

$$\Delta_\psi \psi = \theta\big(\operatorname{Ric}(g) - \frac{1}{2} T \circ T - (\operatorname{tr} T) T\big) = \theta(Q_A) \tag{3.29}$$

Furthermore, $Q_A = \operatorname{Ric}(g) - (\operatorname{tr} T) T - \frac{1}{2} T \circ T$ *is a symmetric operator and it is given by*

$$Q_A = \left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & q \end{array}\right),$$

where

$$Q_1 = \frac{1}{2}[A, A^t] + \frac{1}{2} S_A \circ_6 S_A \quad \text{and} \quad q = -\frac{1}{2} \operatorname{tr}(S_A)^2 - \frac{1}{4} (\operatorname{tr} JA)^2.$$

ii) For the modified Laplacian

$$\Delta_\psi \psi + 2d((C - \operatorname{tr} T)\varphi) = \theta\big(\operatorname{Ric}(g) - \frac{1}{2} T \circ T - (2C - \operatorname{tr} T) T\big) = \theta(P_A)$$

where

$$P_A = \left(\begin{array}{c|c} P_1 & 0 \\ \hline 0 & p \end{array}\right), \tag{3.30}$$

where $P_1 = \frac{1}{2}[A, A^t] + \frac{1}{2} S_A \circ_6 S_A - \Big(C - \frac{1}{2} \operatorname{tr} JA\Big)[J, A]$ *and* $p = -\frac{1}{2} \operatorname{tr}(S_A)^2 + \frac{1}{4} (\operatorname{tr} JA)^2 - C \operatorname{tr} JA$.

Proof. (i) Equation (3.29) follows directly from Lemma 24 (iii) and Remark 8, and the expression for Q_A follows by equation (3.26), Corollary 9 and Lemma 32.

(ii) It follows by a similar reason as above.

□

Lemma 33. *For a symmetric matrix $A \in \mathfrak{sp}(6,\mathbb{R})$ we have $A \circ_6 A \in \mathfrak{sp}(6,\mathbb{R})$, where $(A \circ_6 A)_{ab} = A^{mn}A^{pq}\rho^+_{mpa}\rho^+_{nqb}$.*

Proof. The condition $A \circ_6 A \in \mathfrak{sp}(6,\mathbb{R})$ is equivalent with $\theta(A \circ_6 A)\omega = 0$. So,

$$\theta(A \circ_6 A)\omega = (A \circ_6 A)_{ai}h^{ij}\omega_{jb}dx^{ab} = A^{mn}A^{pq}\rho^+_{mpa}\rho^+_{nqi}h^{ij}\omega_{jb}dx^{ab}.$$

The result follows by the symmetry of $A^{mn}A^{pq}\rho^+_{mpa}\rho^+_{nqi}h^{ij}\omega_{jb}$, in fact

$$\begin{aligned}
A^{mn}A^{pq}\rho^+_{mpa}\rho^+_{nqi}h^{ij}\omega_{jb} =& A^{mn}A^{pq}\rho^+_{mpa}\rho^-_{nqb}\\
=& A^{mn}A^{pq}\rho^+_{mpa}\rho^+_{qbr}h^{rs}\omega_{sn}\\
=& -(AJ)^{mr}A^{pq}\rho^+_{mpa}\rho^+_{qbr}\\
=& (JA)^{mr}A^{pq}\rho^+_{mpa}\rho^+_{qbr}\\
=& h^{mi}\omega_{in}A^{nr}A^{pq}\rho^+_{mpa}\rho^+_{qbr}\\
=& A^{nr}A^{pq}\rho^-_{pan}\rho^+_{qbr}\\
=& A^{nr}A^{pq}\rho^+_{npi}h^{ij}\omega_{ja}\rho^+_{qbr}\\
=& A^{nm}A^{qp}\rho^+_{mpb}\rho^+_{nqi}h^{ij}\omega_{ja}
\end{aligned}$$

Notice that, we had used equation (1.26) time and again, and the symmetry of A. □

The following two propositions involve the evoltion of the matrix A under the flow (3.15). The expectation is that in the future these result allow to inquire about long time existence solution for the modified Laplacian co-flow on almost abelian Lie groups, similar to the Laplacian flow [Lau17] and the Laplacian co-flow [BF17].

Proposition 10. *Let $\mathcal{L}$ be the variety of 7-dimensional Lie algebras. The family $\{\mu_A : A \in \mathfrak{sp}(6,\mathbb{R})\} \subset \mathcal{L}$ of co-closed G$_2$-structures is invariant under the bracket flow $\dot{\mu} = \delta_\mu(P_A)$, which becomes equivalent to the following ODE for a one-parameter family of matrices $A = A(t) \in \mathfrak{sp}(6,\mathbb{R})$:*

$$\begin{aligned}
\frac{d}{dt}A =& \Big(-\frac{1}{2}\operatorname{tr}(S_A)^2 + \frac{1}{4}(\operatorname{tr} JA)^2 - C\operatorname{tr} JA\Big)A + \frac{1}{2}[A,[A,A^t]] + \frac{1}{2}[A, S_A \circ_6 S_A]\\
& - \Big(C - \frac{1}{2}\operatorname{tr} JA\Big)[A,[J,A]]
\end{aligned} \tag{3.31}$$

Proof. Notice that the family $\{\mu_A : A \in \mathfrak{sp}(6,\mathbb{R})\} \subset \mathcal{L}$ is invariant under the bracket flow if and only if $\delta_\mu(P_A) = \mu_B$ for some $B \in \mathfrak{sp}(6,\mathbb{R})$, for any $A \in \mathfrak{sp}(6,\mathbb{R})$. Using (3.30) we have

$$\begin{aligned}\delta_\mu(P_A)(e_7,e_i) =&\mu_A(P_Ae_7,e_i)+\mu_A(e_7,P_Ae_i)-Q_A\mu_A(e_7,e_i)\\ =&p\mu_A(e_7,e_i)+\mu_A(e_7,P_1e_i)-P_1\mu_A(e_7,e_i)\\ =&(pA+AP_1-P_1A)e_i.\end{aligned}$$

Hence, $B = pA + [A, P_1]$, note that $B \in \mathfrak{sp}(6,\mathbb{R})$, indeed

$$[J,A]^tJ+J[J,A]=[J,A^t]J+J[J,A]=JA^tJ+A^t-A-JAJ=0$$

and $S_A \circ_6 S_A \in \mathfrak{sp}(6,\mathbb{R})$ by Lemma 33, thus $P_1 \in \mathfrak{sp}(6,\mathbb{R})$. Therefore, the subset of invariant co-closed G$_2$–structures is invariant under the bracket flow and the matrix A evolves by $\dot{A} = B$. □

Proposition 11. *If $\mu_{A(t)}$ is a bracket flow solution, then the norm of $A(t) \in \mathfrak{sp}(6,\mathbb{R})$ evolves*

$$\begin{aligned}\frac{d}{dt}|A|^2=&\big(-|S_A|^2+\frac{1}{2}(\operatorname{tr}JA)^2-2C\operatorname{tr}JA\big)|A|^2-|[A,A^t]|^2\\ &-\langle S_A\circ_6 S_A,[A,A^t]\rangle+\Big(2C-\operatorname{tr}JA\Big)(\langle[J,A],[A,A^t]\rangle)\end{aligned}$$

Proof. From Proposition 10, we have

$$\begin{aligned}\frac{d}{dt}|A|^2=&2\langle\dot{A},A\rangle=2\operatorname{tr}(\dot{A}A^t)\\ =&\big(-|S_A|^2+\frac{1}{2}(\operatorname{tr}JA)^2-C\operatorname{tr}JA\big)|A|^2+\langle[A,[A,A^t]],A\rangle+\langle[A,S_A\circ_6 S_A],A\rangle\\ &-\Big(2C-\operatorname{tr}JA\Big)\langle[A,[J,A]],A\rangle\\ =&-\big(\frac{1}{4}|S_A|^2+2C(\operatorname{tr}JA)^2\big)|A|^2-|[A,A^t]|-\langle S_A\circ_6 S_A,[A,A^t]\rangle\\ &+\Big(2C-\operatorname{tr}JA\Big)\langle[J,A],[A,A^t]\rangle\end{aligned}$$

□

Similarly to Propositions 10 and 11, we get the following result for the Laplacian co-flow.

Proposition 12. *The bracket flow $\{\mu_A : \ A \in \mathfrak{sp}(6,\mathbb{R})\} \subset \mathcal{L}$ and its norm $|\mu_{A(t)}|^2 = |A|^2$ associated with the Laplacian co-flow (3.11) evolve*

$$\dot{A}=-\big(\frac{1}{2}\operatorname{tr}(S_A)^2+\frac{1}{4}(\operatorname{tr}JA)^2\big)A+\frac{1}{2}[A,[A,A^t]]+\frac{1}{2}[A,S_A\circ_6 S_A] \tag{3.32}$$

$$|\dot{A}|^2=-\big(|S_A|^2+\frac{1}{2}(\operatorname{tr}JA)^2\big)|A|^2-|[A,A^t]|^2-\langle S_A\circ_6 S_A,[A,A^t]\rangle \tag{3.33}$$

In order to proof long time existence solution for (3.11) we need the following identity.

Lemma 34. *For the symmetric part S_A of the matrix $A \in \mathfrak{sp}(6,\mathbb{R})$ holds*

$$|S_A \circ_6 S_A|^2 = 4(|S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2).$$

Proof. This identity is found just by manipulating the SU(3)–representations (1.25) and the contraction identities (1.26) between ω, ρ_+ and ρ_-. □

Now, we are going to study the term $-\langle S_A \circ_6 S_A, [A, A^t]\rangle$ given in the evolution equation (3.33). Using the Cauchy-Schwarz and Peter-Paul inequalities $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ for $a, b \geq 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} -\langle S_A \circ_6 S_A, [A, A^t]\rangle &\leq |S_A \circ_6 S_A||[A, A^t]| \\ &\leq \frac{|S_A \circ_6 S_A|^2}{2\varepsilon} + \frac{\varepsilon |[A, A^t]|^2}{2} \\ &= \frac{2}{\varepsilon}(|S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2) + \frac{\varepsilon}{2}|[A, A^t]|^2 \end{aligned}$$

Taking $\varepsilon = 2$ and replacing the last inequality in the equation 3.33, we have

$$\begin{aligned} \dot{|A|^2} \leq &- \left(|S_A|^2 + \frac{1}{2}(\operatorname{tr} JA)^2\right)|A|^2 - |[A, A^t]|^2 + |S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 + |[A, A^t]|^2 \\ = &- |S_A|^2|S_A|^2 - \frac{1}{2}|S_A|^2|A - A^t|^2 - \frac{1}{2}(\operatorname{tr} JA)^2|A|^2 + |S_A|^2|S_A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 \\ = &- \frac{1}{2}|S_A|^2|A - A^t|^2 - \frac{1}{2}(\operatorname{tr} JA)^2|A|^2 - 2|S_A^2|^2 - \langle JS_A, S_A\rangle^2 \leq 0 \end{aligned}$$

Thus, $|A|^2$ is non-increasing and so long time existence the bracket flow (3.32) follows. In fact, $|A|^2$ is strictly decreasing unless (G_A, φ) is torsion free (that is, $\dot{|A|^2} = 0$ if $A^t = -A$ and $\operatorname{tr} JA = 0$ [Fre13]), and thus $A(t) \equiv A_0$ is constant. In view of the equivalence between the Laplacian co-flow (3.11) and the bracket flow (3.32) (see [Lau16, Theorem 5]), we obtain long time existence for the Laplacian co-flow among this class.

Corollary 10. *The left invariant Laplacian co-flow solutions starting at any co-closed G_2–structure (G_A, φ) is defined for all $t \in (T_-, \infty)$ for some $T_- < 0$.*

Remark 9. *The equations 3.32 and 3.33 correspond to the bracket flow*

$$\dot{\mu}_t = \delta_{\mu_t}(Q_{\mu_t}) \Leftrightarrow \dot{\psi}_t = \Delta\psi_t.$$

However, the results also hold for the co-flow (3.11) and in this case the solution of Corollary 10 are defined for all $t \in (-\infty, T_+)$ for some $0 < T_+$, it as was proved by Bagaglini and Fino [BF18] for a normal matrix $A \in \mathfrak{sp}(6,\mathbb{R})$. Notice that we proved long time existence for (3.11) for any matrix $A \in \mathfrak{sp}(6,\mathbb{R})$.

3.5.1 Example of a co-flow soliton

We now apply the previous theoretical framework to construct an explicit co-flow soliton from a natural Ansatz. Let $\mathfrak{g} = \mathbb{R} \times_\nu \mathbb{R}^6$ be the Lie algebra defined by $\nu(t) = \exp(tA) \in \mathrm{Aut}(\mathfrak{g})$, with

$$A = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & 0 & \\ & & & 1 & & \\ \hline & & 1 & & & \\ & 0 & & & & \\ 1 & & & & & \end{array}\right).$$

The canonical $\mathrm{SU}(3)$–structure on $\mathbb{R}^6$ with respect to the orthonormal basis $\{e_1, e_6, e_2, e_5, e_3, e_4\}$ is

$$\omega = e^{16} + e^{25} + e^{34}, \quad \rho_+ = e^{135} - e^{124} - e^{236} - e^{456}$$

and the standard complex structure of $\mathbb{R}^6$ is

$$J = \left(\begin{array}{ccc|ccc} & & & & & -1 \\ & & & & -1 & \\ & & & -1 & & \\ \hline & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{array}\right)$$

We also have the natural 3-form

$$\rho_- := J \cdot \rho_+ = e^{123} + e^{145} + e^{356} - e^{246}.$$

The structure equations of $\mathfrak{g}^*$ with respect to the dual basis of $\{e_1, e_6, e_2, e_5, e_3, e_4, e_7\}$ are

$$de^1 = e^{67}, \quad de^6 = e^{17}, \quad de^3 = e^{47}, \quad de^4 = e^{37}, \quad de^j = 0 \quad \text{for} \quad j = 2, 5.$$

From the above, we have

$$d\omega = 0, \quad d\rho_+ = -2(e^{2467} + e^{1237}), \quad \text{and} \quad d\rho_- = 2(e^{1357} + e^{4567}).$$

There is a natural co-closed G_2–structure on $\mathfrak{g}$, given by

$$\varphi := \omega \wedge e^7 + \rho_+ = e^{167} + e^{257} + e^{347} + e^{135} - e^{124} - e^{236} - e^{456},$$

with dual 4-form

$$\psi = *\varphi = \frac{\omega^2}{2} + \rho_- \wedge e^7 = e^{1256} + e^{1346} + e^{2345} + e^{1237} + e^{1457} + e^{3567} - e^{2467}.$$

We have $JA = -AJ = \mathrm{diag}(-1,0,-1,1,0,1)$, then by Lemma 8

$$\tau_0 = \mathrm{tr}\, JA = 0 \quad \text{and} \quad \tau_{27} = \mathrm{diag}(1,0,1,-1,0,-1,0).$$

Hence, $T = -\tau_{27} = \mathrm{diag}(-1,0,-1,1,0,1,0)$. To obtain the Laplacian of ψ we apply Proposition 9 (i), notice that $Q_1 = \frac{1}{2}A \circ_6 A$ and $q = -\frac{1}{2}\,\mathrm{tr}\, A^2$ since A is symmetric. By a straightforward computation we have

$$\mathrm{tr}\, A^2 = 4 \quad \text{and} \quad A \circ_6 A = \mathrm{diag}(0,4,0,0,-4,0),$$

So, $\Delta_\psi \psi = \theta(Q_\psi)\psi = 4(e^{1457} + e^{3567})$ where $Q_\psi = \mathrm{diag}(0,2,0,0,-2,0,-2)$. Consider the derivation $D = \mathrm{diag}(a,b,c,c,d,a,0) \in \mathrm{Der}(\mathfrak{g})$, and take the vector field on $\mathfrak{g}$

$$X_D(x) = \frac{d}{dt}(\exp(tD)(x)), \quad \text{for} \quad x \in \mathfrak{g}.$$

Then we have

$$\begin{aligned}\mathcal{L}_{X_D}\psi &= \frac{d}{dt}(\exp(-tD)^*\psi)|_{t=0} = -\theta(D)\psi \\ &= (2a+b+d)e^{1256} + (2a+2c)e^{1346} + (b+2c+d)e^{2345} + (a+b+c)e^{1237} \\ &\quad + (a+c+d)e^{1457} + (a+c+d)e^{3567} - (a+b+c)e^{2467}.\end{aligned}$$

From the soliton equation $-\Delta\psi = \mathcal{L}_{X_D}\psi + \lambda\psi$, we obtain a system of linear equations

$$\begin{cases} 2a+b+d+\lambda &= 0 \\ 2a+2c+\lambda &= 0 \\ a+b+c+\lambda &= 0 \\ a+c+d+\lambda &= -4 \end{cases},$$

which has solution $D = \mathrm{diag}(2,4,2,2,0,2,0)$ and $\lambda = -8$. In particular, for the matrix $Q_\psi = D + \frac{\lambda}{4}I_7$, we have $\Delta\psi = \theta(Q_\psi)\psi$. By Lemma 28, the functions

$$c(t) = (1-4t)^2 \quad \text{and} \quad s(t) = \frac{1}{4}\log(1-4t) \quad \text{for} \quad \frac{1}{4} > t,$$

yield the family of 4-forms $\{\psi_t = c(t)(f(t)^{-1})^*\psi\}$, where

$$\begin{aligned} f(t)^{-1} &= \exp(-s(t)D) \\ &= \mathrm{diag}((1-4t)^{-1/2}, (1-4t)^{-1}, (1-4t)^{-1/2}, (1-4t)^{-1/2}, 1, (1-4t)^{-1/2}, 1). \end{aligned}$$

Hence,

$$\psi_t = e^{1256} + e^{1346} + e^{2345} + e^{1237} + (1-4t)(e^{1457} + e^{3567}) - e^{2467} \tag{3.34}$$

defines a soliton of the Laplacian co-flow:

$$\frac{d\psi_t}{dt} = -4(e^{1457} + e^{3567}) = -c(t)^{1/2}(f(t)^{-1})^*\Delta\psi = -\Delta_t\psi_t.$$

Corollary 11. *The relevant geometric structures associated to the* 4*-form given in* (3.34) *are:*

(i) *the* G$_2$*–structure*

$$\varphi_t = c(t)^{1/4}(e^{167} + e^{257} + e^{347} + e^{135} - e^{456}) - c(t)^{-1/4}(e^{124} + e^{236});$$

(ii) *the* G$_2$*–metric*

$$g_t = (e^1)^2 + (e^3)^2 + (e^4)^2 + (e^6)^2 + c(t)^{-1/2}(e^2)^2 + c(t)^{1/2}((e^5)^2 + (e^7)^2);$$

(iii) *the volume form*

$$\mathrm{vol}_t = c(t)^{1/4}\,\mathrm{vol}_\psi;$$

(iv) *the torsion form and the full torsion tensor*

$$\tau_3(t) = 2(e^{135} + e^{456}) \quad \textit{and} \quad T(t) = c(t)^{-1/4}\big(-(e^1)^2 - (e^3)^2 + (e^4)^2 + (e^6)^2\big);$$

(v) *the Ricci tensor and the scalar curvature*

$$\mathrm{Ric}(g_t) = -4c(t)^{-1/2}(e^7)^2 \quad \textit{and} \quad R_t = -\frac{1}{2}|\tau_3(t)|^2 = -4c(t)^{-1/2};$$

(vi) *the bracket flow solution*

$$\mu_t = c(t)^{-1/4}[\cdot,\cdot].$$

3.5.2 Example of a modified co-flow soliton

We now construct an explicit modified co-flow soliton following the same ideas from the last example. Let $\mathfrak{g} = \mathbb{R} \times_\nu \mathbb{R}^6$ be the Lie algebra defined by $\nu(t) = \exp(tA) \in \mathrm{Aut}(\mathfrak{g})$, with

$$A = \left(\begin{array}{cc|cc|cc} 0 & -1 & & & & \\ 1 & 0 & & & & \\ \hline & & & & 0 & -1 \\ & & & & 1 & 0 \\ \hline & & 0 & -1 & & \\ & & 1 & 0 & & \end{array}\right).$$

The canonical SU(3)–structure on $\mathbb{R}^6$ with respect to the orthonormal basis $\{e_1, ..., e_6\}$ is

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \rho_+ = e^{135} - e^{146} - e^{236} - e^{245}$$

and the standard complex structure of $\mathbb{R}^6$ is

$$J(e_1) = e_2, \quad J(e_3) = e_4, \quad J(e_5) = e_6 \quad \text{and} \quad J^2 = -I$$

We also have the natural 3–form

$$\rho_- := J \cdot \rho_+ = -e^{246} + e^{235} + e^{136} + e^{145}.$$

The natural co-closed G_2–structure on $\mathfrak{g}$ is given by

$$\varphi := \omega \wedge e^7 + \rho_+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with dual 4-form

$$\psi = *\varphi = \frac{\omega^2}{2} + \rho_- \wedge e^7 = e^{1234} + e^{1256} + e^{3456} - e^{2467} + e^{2357} + e^{1367} + e^{1457}.$$

We have

$$JA = AJ = \left(\begin{array}{cc|cc|cc} -1 & 0 & & & & \\ 0 & -1 & & & & \\ \hline & & & & -1 & 0 \\ & & & & 0 & -1 \\ \hline & & -1 & 0 & & \\ & & 0 & -1 & & \end{array}\right).$$

Then, by Proposition 8 we have

$$\tau_0 = -\frac{4}{7}, \quad \tau_{27} = -\frac{1}{7}\operatorname{diag}(1,1,1,1,1,1,-6),$$

and by Corollary 9, $T = \operatorname{diag}(0,0,0,0,0,0,-1)$. Now, we apply Proposition 9 (ii), since A is skew symmetric we have $\Delta_\psi \psi + 2(C - \operatorname{tr} T)d\varphi = \theta(P_A)\psi$ where $P_A = \operatorname{diag}(0,\dots,0,1+2C)$ Now, for $C = 0$ we get

$$P_A = Q_A + 2(\operatorname{tr} T)T = I + D \quad \text{for} \quad D = \operatorname{diag}(-1,-1,-1,-1,-1,-1,0) \in \operatorname{Der}(\mathfrak{g})$$

By Lemma 29, the functions

$$c(t) = (1-2t)^2 \quad \text{and} \quad s(t) = -\frac{1}{2}\log(1-2t) \quad \text{for} \quad \frac{1}{2} > t,$$

yield the family of 4-forms $\{\psi_t = c(t)(f(t)^{-1})^*\psi\}$, where

$$\begin{aligned} f(t)^{-1} &= \exp(-s(t)D) \\ &= (1-2t)^{-1/2}\operatorname{diag}(1,1,1,1,1,1,(1-2t)^{1/2}). \end{aligned}$$

Hence,

$$\psi_t = e^{1234} + e^{1256} + e^{3456} + (1-2t)^{1/2}(e^{1367} + e^{1457} + e^{2357} - e^{2467})$$

defines a soliton of the modified Laplacian co-flow with $C = 0$:

$$\begin{aligned} \Delta_t\psi_t - 2\operatorname{tr}_t T_t d\varphi_t =& c_t^{1/2} f_t \cdot \Delta_\psi \psi - 2c_t^{-1/4}(\operatorname{tr} T)c_t^{3/4} f_t \cdot d\varphi \\ =& -(1-2t)^{-1/2}(e^{1367} + e^{1457} + e^{2357} - e^{2467}) = \frac{d}{dt}\psi_t \end{aligned}$$

3.6 An associative submanifold along the Laplacian flow

Here we pretend to give a connection between the main topics of this work, namely, we consider the deformation of the associative submanifold from Example 8 along the Laplacian flow of closed G_2–structures.

Consider the connected and simply connected nilpotent Lie group G with Lie algebra

$$\mathfrak{g} = (0,0,0,0,e^{12},e^{13},0),$$

from the Example 7. It could be seen as an almost abelian Lie algebra [Lau17] with respect to the orthonormal basis $\mathfrak{g} = \mathrm{Span}(e_1,e_2,e_3,e_4,e_7,e_5,e_6)$, $\mathfrak{h} = \mathrm{Span}(e_2,e_3,e_4,e_7,e_5,e_6)$ and

$$A = \mathrm{ad}(e_1)|_{\mathfrak{h}} = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C}). \tag{3.35}$$

This example corresponds with $\mathfrak{n}_2$ from [Lau17, Example 5.8] under the change of basis

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{G}_2.$$

Thus, the G_2–structure (2.28) is rewritten as

$$\varphi = e^1 \wedge \omega + \rho_+$$

for $\omega = e^{23} + e^{47} + e^{56}$ and $\rho_+ = e^{267} + e^{357} + e^{245} - e^{346}$ a SU(3)-structure on the abelian ideal $\mathfrak{h}$. We calculate the Laplacian of φ by $\Delta_A\varphi = \theta(Q_A)\varphi$ where

$$Q_A = \left(\begin{array}{c|c} q & 0 \\ \hline 0 & Q_1 \end{array}\right),$$

with $Q_1 = \frac{1}{2}[A,A^t] + \frac{1}{12}\,\mathrm{tr}(A+A^t)^2 I - \frac{1}{2}(A+A^t)^2$ and $q = -\frac{1}{6}\,\mathrm{tr}(A+A^t)^2$ (see [Lau17, Proposition 5.15]). Then we have $Q_A = \frac{1}{3}\,\mathrm{diag}(-2,-2,-2,1,1,1,1)$ for the nilpotent matrix A given in (3.35). It can be verified that the matrix A satisfies the relation

$$[A,[A,A^t] - (A+A^t)^2] = \frac{|[A,A^t]|^2}{|A||^2}A,$$

thus, by [Lau17, Proposition 5.22] (G_A,φ) is an algebraic soliton for the Laplacian flow given $D = Q_A - cI$ with

$$c = -\frac{1}{2}\,\mathrm{tr}(A+A^t)^2 - \frac{|[A,A^t]^2}{2|A|^2} = -3,$$

hence, $D = \mathrm{diag}(1,1,1,2,2,2,2) \in \mathrm{Der}(\mathfrak{g})$. Therefore, by [Lau17, Theorem 3.8] we have

$$b(t) = (\frac{10}{3}t+1)^{3/2}, \quad s(t) = \frac{3}{10}\log\left(\frac{10}{3}t+1\right)$$

and

$$\varphi(t) = b(t)(e^{-s(t)D})^*\varphi = (\frac{10}{3}t+1)^{3/5}e^{123} + e^{147} + e^{156} + e^{267} + e^{357} + e^{245} - e^{346}.$$

Notice that $\varphi(t)|_{\mathfrak{a}} = e^{156}$ where $\mathfrak{a} = \mathrm{Span}(e_1, e_5, e_6)$ is the abelian subalgebra. So, the associative submanifold given in the Example 8 remains associative for any $\varphi(t)$.

Concluding Remarks

We would like to conclude with two questions for future work.

1. In view of the equivalence between the bracket flow and the modified Laplacian co-flow given in Lemma 26, it would be interesting to study the evolution of the norm obtained in Proposition 11 to understand the long time behaviour of solutions and thereof give necessary and sufficient conditions on $A \in \mathfrak{sp}(6, \mathbb{R})$ to obtain an algebraic soliton.

2. When the full torsion tensor $T = -\tau_{27}$ is traceless symmetric, the scalar curvature of the corresponding G_2-metric is nonpositive, and it vanishes if, and only if, the structure is torsion-free (c.f. [Bry06, (4.28)] or [Kar09, (4.21)]). This fact was first pointed out by Bryant for a closed G_2-structure, in order to explain the absence of closed Einstein G_2-structures (other than Ricci-flat ones) on compact 7-manifolds, giving rise to the concept of *extremally Ricci-pinched closed* G_2*-structure* [Bry06, Remark 13]. Later on, Fernández et al. showed that a 7-dimensional (non-flat) Einstein solvmanifold (S, g) cannot admit any left-invariant co-closed G_2-structure φ such that $g_\varphi = g$ [FM].

   In that context, it would be interesting to study pinching phenomena for the Ricci curvature of solvmanifolds with a co-closed (non-flat) left-invariant G_2-structure and traceless torsion. In our present construction, for instance, we can see from Corollary 11 that

   $$F(t) = \frac{R_t^2}{|\mathrm{Ric}(g_t)|^2} = 1.$$

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