



UNIVERSIDADE ESTADUAL DE CAMPINAS

INSTITUTO DE FILOSOFIA E CIÊNCIAS HUMANAS

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AN EXPLICATION OF FREGE'S PROCEDURE OF CONTENT  
RECARVING

UMA EXPLICAÇÃO DO PROCEDIMENTO DE RE-MODELIZAÇÃO  
DE CONTEÚDO EM FREGE

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Uma explicação do procedimento de re-modelização de conteúdo  
em Frege

An Explication of Frege's Procedure of Content Recarving

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A Comissão Julgadora dos trabalhos de Defesa de Tese de Doutorado, composta pelos Professores Doutores a seguir descritos, em sessão pública realizada em 25/04/2019, considerou o candidato Vincenzo Ciccarelli aprovado.

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*A Ata de Defesa, assinada pelos membros da Comissão Examinadora, consta no processo de vida acadêmica do aluno.*

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*Bruma de Oro, el Occidente alumbra  
la ventana. El asiduo manuscrito  
aguarda, ya cargado de infinito.  
Alguien construye a Dios en la penumbra.  
Um homem engendra a Dios. Es un judío  
de tristes ojos y de piel cetrina;  
lo lleva el tiempo como lleva el río  
una hoja en el agua que declina.  
No importa. El hechichero insiste y labra  
a Dios con geometría delicada;  
desde su enfermedad, desde su nada,  
sigue erigiendo a Dios con la palabra.  
El más pródigo amor le fue ortogado,  
el amor que no espera ser amado.*

JORGE LUIS BORGES, Baruch Spinoza

## Resumo

Em uma breve e difícil passagem de *Os Fundamentos da Aritmética*, Frege apresenta um novo estilo de definições baseadas no procedimento de *re-modelização do conteúdo*, isto é, um procedimento que atribui um conteúdo para uma sentença por meio de uma re-organização interna do conteúdo expressado por uma sentença de diferente estrutura sintática. O presente trabalho é inspirado por esta passagem e tem o propósito de entender a metáfora da re-modelização de conteúdo no contexto mais moderno da teoria dos mundos possíveis. Em particular, o conteúdo expressado por uma sentença é apresentado como uma entidade estruturada que além de representar as condições de verdade da sentença, diz a respeito dos objetos do mundo que estão envolvidos em todas as formas possíveis de expressar estas condições de verdade. Desenvolvemos um formalismo de acordo com o qual o conteúdo é modelado como um par ordenado de funções, e as operações de modelização e re-modelização do conteúdo são definidas como tipos particulares de decomposição funcional que “respeitam” a estrutura interna do conteúdo. A principal aplicação desta teoria é um novo entendimento das principais questões relacionadas com os princípios de abstração e outros casos paradigmáticos em que supostamente o mesmo conteúdo é expressado em maneiras muito diferentes.

**Palavras chave:** Re-modelização de conteúdo, Princípios de abstração, Conteúdo proposicional

## Abstract

In a short and difficult passage of *The Foundations of Arithmetic*, Frege proposes a new style of definitions based on the procedure of *content recarving*, i.e. a procedure that by means of an internal re-organization of the content expressed by a sentence, allegedly assigns a content – and thus a meaning – to a different sentence having different syntactic structure. This work is inspired by this passage and aims at understanding the metaphor of content recarving outside of Frege's theoretical environment and in the more modern setting of the theory of possible worlds. In particular, the content of a sentence is presented as a structured entity that – besides representing the truth-conditions of the sentence – also encodes all information regarding the object with which every possible way of expressing the same content is concerned. A formalism is developed that models the notion of content as an ordered pair of functions, and the operations of content carving and recarving are defined as special sorts of functional decompositions that “respect” the internal structure of the content. The main application of this theory will be a new understanding of the main issues regarding abstraction principles and other paradigmatic cases in which the same content is expressed in very different ways.

**Keywords:** Content recarving, Abstraction principles, Propositional content

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## Introduction

This work is inspired by a famous passage of Frege's *Foundations of Arithmetic* in which Frege proposes a new procedure of concept formation based on a particular operation on the content expressed by a sentence (or by some of the sentence's constituents): the operation of *carving up the same content in different ways*. The passage of §64 in which this procedure is described is the following:

“ The judgment ‘*line  $a$  is parallel to line  $b$* ’, [...] can be taken as an identity. If we do this, we obtain the concept of direction and say: ‘*The direction of line  $a$  is identical with the direction of line  $b$* ’. Thus we replace the symbol  $\parallel$  by the more generic symbol  $=$ , through removing what is specific in the content of the former and dividing it between  $a$  and  $b$ . We carve up the content in a way different from the original way, and this yields us a new concept. ”(Frege, 1950), pp. 74-75

It has to be admitted: the description of the procedure that justifies the reading of a sentence saying that two lines are parallel as a sentence saying that their directions are identical is as interesting as obscure. Frege is not clear regarding some operations of removing and dividing contents or part of contents that seem to play a fundamental role in the procedure.

Before discussing the details of the passage and the fundamental philosophical problems related to the understanding of Frege's wording, it is worth contextualizing the described procedure in the dialectic of Frege's argument and in the historical developments of the interpretation of his conclusion.

## Frege on Abstract Reference

To understand the role of the passage in §64 of *the Foundations* (henceforth *GLA64*) we have to spell out Frege's argument aimed at showing that it is possible to refer to abstract objects even though these objects are not given to us through "intuition".

Frege's starting point in *the Foundations* is the so-called *context principle* which Frege states in the Introduction: "Never ask for the meaning of a word in isolation, but only in the context of a proposition" ((Frege, 1950), Introduction). The context principle is a methodological guideline and may be understood as saying that if we are to assess whether a certain expression  $\xi$  is meaningful or not, we have to assess whether all sentences in which  $\xi$  appears are meaningful or not. Now it seems clear from Frege's way of applying this principle that if a singular term has a meaning, then it refers to a certain object. Moreover, it seems also clear that for Frege if a sentence  $S$  has meaning, then it expresses a defined content which in turn implies that it has a defined truth-value. Thus we may rephrase the context principle for singular terms as follows:

(CP) To assess whether a singular term  $t$  has reference or not, we have to check whether all sentences in which  $t$  appears express a content or not.

It seems also correct to convert (CP) into the following semantic principle:

(CP') Given a singular term  $t$ , if all sentences in which  $t$  occurs express a content (and thus have a defined truth-value), then  $t$  is referential.

In §66, Frege explicitly says that we may restrict the range of application of (CP'): we don't need to consider all sentences in which the term  $t$  appears, but just all identity sentences. In other words, to show that  $t$  is referential, we need just to show that for every singular term  $\tau$ , the sentence ' $t = \tau$ ' expresses a content and thus has a defined truth-value. The reasons for such a restriction of the range of application of the context principle are related to Frege's notion of *self-subsistent object* and will not interest us in the present discussion.

Hence, the semantic criterion that Frege adopts for singular terms may be expressed as follows:

(CP'') Given a singular term  $t$ , if all *identity sentences* in which  $t$  occurs express a content (and thus have a defined truth-value), then  $t$  is referential.

Frege is interested in applying (CP'') to show that singular terms allegedly standing for cardinal numbers are referential. Given that we are interested in Frege's account of abstract reference in general, I will continue to refer to Frege's example of §64 involving lines and directions.

To show that the singular term 'the direction of line  $a$ ' is referential, we have to show that all identity sentences in which it occurs express a content. We may attempt the following argument:

- (A1) For every line term  $\xi$ , the content of the sentence 'The direction of line  $a$  = the direction of  $\xi$ ' may be seen as obtained by a content preserving operation from the content of the sentence 'line  $a$  is parallel to  $\xi$ ' (*Content recarving*)
- (C0) By (A1), For every line term  $\xi$ , the sentence 'the direction of line  $a$  = the direction of  $\xi$ ' has the same content as the sentence 'line  $a$  is parallel to  $\xi$ ' (*Identity of content*)
- (C1) By (C0), for every line term  $\xi$ , the sentence 'The direction of line  $a$  = the direction of  $\xi$ ' expresses a content (*Content expression*)
- (C2) By (C1), all identity sentences in which the term 'the direction of line  $a$ ' appears express a content (*Identity context*)
- (C3) By (CP''), the singular term 'the direction of line  $a$ ' is referential (*Abstract reference*).

I call this argument *the argument from content recarving*. The argument from content recarving may well represent Frege's first attempt to justify the possibility of abstract reference. Notice that the procedure of content recarving described in *GLA64* is the rationale for the truth of the assumption (A1). However, Frege quickly noticed that the argument is invalid. Indeed, (C2) relies on a false assumption: that all identity sentences in which the term 'the direction of line  $a$ ' occurs have the form 'the direction of line  $a$  = the direction of line  $\xi$ '. As Frege himself suggests, the sentence 'the direction of line  $a$  = England' – though a bit nonsensical – is a grammatical identity sentence containing the singular term 'the direction of line  $a$ ' which has not the form 'the direction of line  $a$  = the direction of line  $\xi$ '. As a consequence, it is not true that by carving up the contents of all sentences of the form 'line  $a$  is parallel to  $\xi$ ' we ensure that all identity sentences in which the term 'the direction of line  $a$ ' occurs express a certain content. In other words, the procedure of content recarving described in *GLA64* is not sufficient to grant what the context principle demands.

This is the reason why Frege abandoned the strategy represented by the argument from content



recarving to justify the possibility of abstract reference. The remaining part of the story is well known: from §66 on, Frege introduces explicit definitions of abstract terms like ‘the direction of line  $a$ ’ in terms of extensions of concepts. However, it is interesting to note that Frege never says that the procedure described in *GLA64* is incorrect, as well as he never says that ‘the direction of line  $a$  = the direction of line  $b$ ’ and the sentence ‘line  $a$  is parallel to line  $b$ ’ express different contents. The argument from content recarving may not be invalid due to a problem concerning the operation of content recarving but just due to an incompleteness of the context of sentences to which this operation is able to assign a content. As a consequence, the possibility of mounting an improved version of the argument from content recarving that may satisfy the requirements of the context principle remains open.

In a certain sense, Crispin Wright’s work – at least in its early stages (Wright, 1983) – may be seen as an attempt to explore this possibility. Without disputing the details of Wright’s proposal, we may say that his work roughly consists in arguing that the identity of content between a sentence expressing the parallelism of two lines and the sentence expressing the identity of their directions is enough to show that all identity sentences in which a direction term occurs have a defined truth-value. Notice that Wright’s strategy is still an attempt to justify the possibility of abstract reference based on the combination of the soundness of the procedure described in *GLA64* and the context principle.

Back to Frege, there is an important fact regarding *GLA64*: the identity of content between ‘line  $a$  is parallel to line  $b$ ’ and ‘the direction of line  $a$  = the direction of line  $b$ ’ might still be a requirement of Frege’s argument even after the introduction of the explicit definition of ‘the direction of line  $a$ ’ in terms of extension of a concept. Suppose that Frege was right in assuming that extensions of concepts are primitive logical objects and that every abstract object may be identified with the extension of a certain concept. Under this assumption, Frege has the problem of determining of which concept the direction of line  $a$  is the extension. Couldn’t he just stipulate that the direction of line  $a$  coincides with e.g. the extension of the concept ‘ $x = \text{line } a$ ’? Clearly not. For if the direction of line  $a$  were the extension of the concept ‘ $x = \text{line } a$ ’, then whenever two distinct lines  $a$  and  $b$  are parallel they would have different directions. Why Frege cannot accept that two distinct parallel lines have different directions? The answer cannot be because it contradicts our pre-theoretical intuitions concerning the concept of direction. Directions, as Frege explicitly remarks, are not given to us through intuition<sup>1</sup>. An answer available to

<sup>1</sup>“But now I ask whether anyone has an intuition of the direction of a straight line. Of a straight line, certainly;

Frege might be: two distinct parallel lines cannot have different directions because the sentence ‘line  $a$  is parallel to line  $b$ ’ has the same content as the sentence ‘the direction of line  $a$  = the direction of line  $b$ ’ and two sentences identical in content cannot differ in their truth-values. As a consequence, the identity of content that follows from the procedure of *GLA64* might still be a needed and valid criterion of soundness for the identification of directions with a special kind of extensions. Frege may identify the direction of line  $a$  with the extension of the concept ‘ $x$  is parallel to line  $a$ ’ because when directions are thus defined, whenever two lines are parallel they have the same direction and vice versa.

One last remark: Frege might have given up the context principle itself, but not the identity of content which follows from *GLA64*. This because when it is assumed as a primitive logical fact that for every concept  $F$  the term ‘the extension of  $F$ ’ is referential, the identification of directions with some sort of extensions ensures the referentiality of direction terms by itself<sup>2</sup>. However, once the referentiality of direction terms is granted by their being extension terms, the problem of understanding which extensions are directions still persists and – as shown – may be dealt with only by invoking the identity of content following from *GLA64*.

Therefore, my conclusion is that the procedure of content recarving described in *GLA64* seems to be crucial for both the formulation of an improved version of the argument from content recarving and Frege’s justification of the possibility of abstract reference.

## Understanding *GLA64*

Given the centrality of the procedure of content recarving described in *GLA64*, it is crucial to understand what kind of operations on content Frege is trying to present.

The description contains some verbs like ‘removing’, ‘dividing’ that to a certain extent are used in a metaphorical sense. Without explaining the metaphor, we may say that the content of the relational expression ‘ $x$  is parallel to  $y$ ’ may be regarded as composed by the content of ‘ $x = y$ ’ and some undefined “specific part” $s$ . Moreover, such a specific part  $s$  may be divided into two “halves” each of which is somehow combined respectively with the content of the terms ‘line

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but do we distinguish in our intuition between this straight line and something else, its direction? That is hardly plausible.” (Frege, 1950), p.75

<sup>2</sup>Perhaps the context principle may still be necessary to ensure the referentiality of extension terms. See (Heck, 2012)

$a'$  and 'line  $b$ '. The result of combining "half" of the content  $s$  with the content of 'line  $a$ ' ('line  $b$ ') is the content of the term 'the direction of line  $a$ ' ('the direction of line  $b$ '). The new carving of the content of 'line  $a$  is parallel to line  $b$ ' is obtained by re-composing the content of ' $x = y$ ' – which is the "remainder" of the content of ' $x$  is parallel to  $y$ ' when  $s$  is removed – and both the contents of 'the direction of line  $a$ ' and 'the direction of line  $b$ '. The identity of content between 'line  $a$  is parallel to line  $b$ ' and 'the direction of line  $a$  = the direction of line  $b$ ' may be justified by the fact that two different carvings may be conceived as different decompositions into "parts" of the same "whole": what is removed from one "part" is added to some others.

Two general operations are presupposed by the description: *content carving and content re-carving*. We may take content carving to be the specification of a particular "structure", "organization", "part composition" of a certain content conceived as a simple entity. The operation of content recarving is performed on a content already carved up in a certain way and returns a different way of carving up the same content. Thus both content carving and recarving are content preserving operations.

The first difficulty that we face in explaining away the metaphoric language of the description is represented by the lack of a rigorous definition of the notion of content. Albeit Frege gives some hints on the matter in the *Begriffsschrift*, no clear account of this notion is provided. Notice that to understand the procedure of *GLA64* we need to define not just the content of an entire sentence, but also the content of conceptual expressions and singular terms. The reasonable assumption that the content of an expression is its semantic value is of little help, for it is not clear what Frege takes to be the semantic value of an expression before the introduction of the sense/reference distinction. We may attempt the preliminary presupposition that the content of a conceptual expression is some sort of function, the content of a singular term is its reference, and the content of an entire sentence is some unspecified entity obtained by combining the content of a conceptual expression with the terms that saturate it. However, if we take the content of a conceptual expression to be a function from e.g. objects to truth-values, the only way of combining this function with an object is by saturating the former with the latter, which amounts to identify the content of an entire sentence with its truth-value. This is highly controversial: at first glance, the sentences considered in *GLA64* seems to have more in common than just their truth-value. Perhaps we should take the content of a conceptual expression as a function from objects to *judgeable contents*, i.e. contents of entire sentences; yet this takes us to a dead end,

for it is not clear what the content of an entire sentence should be.

Even if we attempt a characterization of the notion of content outside Frege's conceptual universe, further difficulties appear. For instance, we may suppose that Frege's notion of judgeable content is some sort of propositional content or proposition. By the fact that a content may be carved up in this way or the other, the notion of proposition to which we should refer seems to be some notion of structured proposition. Structured propositions, under any possible account, are defined in terms of their constituents: however, the two sentences of *GLA64* clearly express structured propositions having different constituents: for instance, one involves lines, while the other directions. Therefore the notion of structured proposition is too fine grained to account for a notion of content that may be carved and recarved. It seems that some notion of plastic structured proposition is required; as far as I know, no one has proposed a notion of proposition of this sort.

The sought notion of content must meet two further important requirements: it must be possible to decompose it in different ways and each decomposition must be re-arrangeable by removing some "part", "adding" it to another, or "distributing it" over some other parts. These two requirements are what makes possible the definition of the operations respectively of carving and recarving. As we will see in the following chapters, Frege gives us a means to decompose a content in different ways: the possibility of reading the syntactic expression of a sentence as obtained by different distinctions between the saturated part and the unsaturated part. In other words, we may be guided by the syntactic structure of the sentence (with all of its possible readings in terms of saturated/unsaturated expressions) to determine the ways in which the content of that sentence may be carved up. However, as *GLA64* shows, not all carvings are determined simply by considering different syntactic parsings of the same sentence. Indeed, there is no distinction between saturated and unsaturated parts in the sentence 'line *a* is parallel to line *b*' that returns a sentence readable as 'the direction of line *a* is identical to the direction of line *b*'. As a consequence, we cannot content ourselves with a syntactic based account of the operations of carving and recarving. According to Frege, it is something characteristic of the meaning of the relation of parallelism that makes us discover that it may be taken as an identity of directions, not something characteristic of the syntactic structure of the sentence 'line *a* is parallel to line *b*'. Hence a general definition of the operations of carving and recarving must be given from a purely semantic perspective.

Another aspect that must be taken into account in formulating an explication of the recarving procedure is its main result. As Frege says at the end of the passage, the procedure of content recarving “yields us a new concept”. Under this respect, the procedure of content recarving may be conceived as the decomposition of a concept (the relational concept of parallelism) in such a way that a new concept is defined, i.e. the concept of direction. Thus content recarving is a procedure of concept formation by decomposition of the initially given concept. This suggests that content recarving is a *special kind of conceptual analysis*. Notice that the formation of the concept of direction by analyzing the relation of parallelism considerably departs from the traditional account of conceptual analysis. According to this latter conception, the decomposition of a concept is represented by the decomposition of its expression by means of boolean operators: for instance, the concept of human may be defined as the composition of the concept of rational with the concept of animal through the conjunction operator. On the contrary, the concept of direction is not formed by forming its defining expression, but it is formed by introducing a criterion of identity between its instances; moreover, the relation of parallelism is not decomposed by combining different concepts through boolean operators, but by dividing its content into parts and re-arranging these parts in a different way. Thus content recarving is a quite unorthodox method of conceptual analysis and a satisfactory explication of the procedure in *GLA64* should provide a characterization of this new kind of conceptual analysis.

We may summarize the main difficulties/requirements in understanding the procedure of *GLA64* in the following points:

- (D.1) A definition of content shall be formulated for conceptual expressions, singular terms, and entire sentences,
- (D.2) If the notion of content is to be understood outside of Frege’s theoretical setting, a notion of *plastic structured proposition* needs to be characterized and made clear,
- (D.3) The content of at least conceptual expressions and entire sentences may admit different decompositions into other contents. The resulting account of content as a complex entity must be provided from a purely semantic perspective,
- (D.4) The operation of recarving must be conceivable as a procedure for concept formations by a new kind of conceptual analysis. This unorthodox method of conceptual analysis must be clarified.

## The Purpose of this Work

This work is entirely concerned with the understanding and the definition of the operations of content carving and recarving described in *GLA64*. Therefore, as clarified in the previous section, this work is aimed at providing solutions to the difficulties described in points (D.1)-(D.4) of the previous section.

It is crucial to understand that this work is not exegetic: as I will argue, given the unsatisfactory treatment of the notion of content in Frege's texts, any attempt to explain the procedure of *GLA64* only by means of Frege's notions and principles cannot result in an acceptable interpretation of the passage. Thus the possibility of understanding *GLA64* outside of Frege's conceptual domain and without any exegetic ambition is more promising.

Albeit this work cannot be considered an interpretation of Frege's relevant texts, it takes inspiration from Frege's wording in *GLA64*. More specifically, basic principles and guidelines that may be extrapolated on Frege's sparse hints on the issue will be considered as basic principles and guidelines also for the present discussion.

The fact that this work is concerned just with the understanding of the operations described in *GLA64* has the immediate consequence that it will not be concerned with the problem of abstract reference. As previously shown, even though the identity of content between parallelism and identity of directions seems to be crucial for any improved version of Frege's argument in favor of abstract reference, it is not sufficient to conclude that abstract reference is possible. This because the sentential contexts to which the procedure of *GLA64* assigns a content are not all the contexts required by the context principle. As a consequence, even if my proposal to understand *GLA64* will result in a satisfactory explanation, no conclusion may be drawn regarding the possibility of abstract reference. In other words, I will not propose any improved version of the argument from content recarving.

In the remaining part of this section I will give the outline of the discussion by listing a resume of each chapter in which this work is divided.

In Chapter 1 I will analyze Frege's passages that are concerned with the notion of content. In particular, I will focus on two aspects: on Frege's view that the content of a sentence may be divided into an unsaturated part and a saturated part in different ways, and on the fact that – as Frege seems to suggest – sentences begin one the result of some unspecified logical analysis of

the other express the same content. This will lead me to consider the problem of specifying the constituents of a content and thus of conceiving the content of a sentence as something close to the notion of structured proposition. The next step will be to consider *the problem of fineness of grain*; I will argue that Frege's notion of content should be neither as fine grained as Russell's notion of structured proposition, nor as coarse grained as Carnap's notion of intension.

In Chapter 2 I will give some theoretical background that is necessary to understand the theoretical setting of my proposal. I will clarify the notion of possible world to which I will refer in the rest of the work, the notion of *partitioning of the logical space* that will be crucial to understand the relation between different carvings of the same content, and I will clarify the distinction between intrinsic and extrinsic properties.

In Chapter 3 I will give a general outline of the theory of content I propose. I will clarify that the content of a sentence may be regarded in different ways according to different distinctions between a functional component and an argument component which mimics Frege's distinction between unsaturated and saturated parts of the content. It is important to understand that the distinction will be traced from a semantic perspective. I will then define the relation of logical analysis in a semantic way, by considering some examples that Frege would have regarded as logical analysis. This will lead me to characterize the content expressed by a sentence as the least analyzed expression of the truth-conditions of a sentence. This result combined with the fact that the notion of content must be of finer grain than that of intension will lead me to conclude that the notion of content may be defined in a semantic of partial worlds rather than total worlds.

In Chapter 4 I will be concerned with the operation of carving up a content. I will propose a preliminary definition of carving in terms of functional decomposition. I will then argue that this definition is unsatisfactory because it does not block the introduction of some arbitrary carvings that lack the required relevance. Thus I will propose an improved definition of carving based on the notion of internal structure of a content. The internal structure of a content will be characterized as a determinable/determinates structure. More specifically, I will argue that the content of a sentence defines determinable properties of partial worlds, and a carving of this content according to its internal structure will amount to a specification of the constituents that make the content true or false by defining properties of partial worlds that determine the determinable properties defined by the content.

In Chapter 5 I will give a definition of the operation of content recarving. By following the same path of Chapter 4, I will preliminary define the operation of recarving as a functional decomposition and then I will give an improved definition based on the internal structure of the content described in Chapter 4. I will finally apply the main point of my proposal to the case of abstraction principles to show how the understanding of the operation of content recarving may shed new light on the debate regarding this sort of definitions.



# Chapter 1

## Frege's Notion of Conceptual Content

### 1.1 Content and logical consequence

The notion of *content* – or more precisely *conceptual content* – is introduced by Frege in the beginning sections of the *Begriffsschrift* as a crucial notion for his formula language. In §3 he writes:

A distinction between subject and predicate does not occur in my way of representing a judgment. In order to justify this I remark that the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, or this is not the case. The two propositions ‘The Greeks defeated the Persians at Plataea’ and ‘The Persians were defeated by the Greeks at Plataea’ differ in the first way. Even if one can detect a slight difference in meaning, the agreement outweighs it. Now I call that part of the content that is the same in both the *conceptual content*. Since it alone is of significance for our ideography, we need not introduce any distinction between propositions having the same conceptual content. [...] Now, all those peculiarities of ordinary language that result only from the interaction between speaker and listener [...] have nothing that answers to them in my formula language, since *in a judgment I consider only that which influences its possible consequences*. (Frege, 1967), p.12

The passage presents one central claim: the content of a sentence  $S$  is what determines all possible consequences of  $S$ . The problem of understanding such a claim is that it is not entirely clear what kind of relation of consequence Frege has in mind. Indeed, there is no formal definition of the relation of consequence in the system of the *Begriffsschrift*. And different interpretations of the relation of consequence produces very different conceptions of content; if we take it to be

logical consequence, then nothing prevents us from identifying the content of a sentence with its truth-value according to a certain interpretation of the language and all logically equivalent sentence will turn out to share the same content; if we think of some stronger relation (e.g. formal consequence, relevant consequence), then the content of a sentence may be identified with some fine-grained proposition it expresses.

Luckily, the passage provide us with some hints on the matter. Consider Frege's example: 'The Greeks defeated the Persians at Plataea' has the same content as 'The Persians were defeated by the Greeks at Plataea'. At first glance, it seems that Frege wants to show with this example that the subject/predicate distinction is irrelevant to the determination of the conceptual content expressed by a sentence. To this end, note that the expression 'The Greeks' appears as grammatical subject only in the first sentence. Yet Frege's example contains more information. If we compare the notion of content with some sort of *Russellian proposition*, we immediately see that content is of a coarser grain than Russellian proposition. Indeed, according to the Russellian account, the two sentences express different propositions: the former is composed by the Greeks, the Persians, and the relation *x defeats y at Plataea*; while the latter is composed by the Persians, the Greeks, and the relation *x were defeated by y at Plataea*<sup>1</sup>. Thus Frege's remark is not just that the content of a sentence is insensitive to certain grammatical distinctions, but also that it is – to a certain extent – insensitive to the all or some of the constituents of the corresponding Russellian proposition. This idea is supported by another example that we find in §9 of the *Begriffsschrift*:

Let us assume that the circumstance that hydrogen is lighter than carbon dioxide is expressed in our formula language; we can then replace the sign for hydrogen by the sign for oxygen or that for nitrogen. This changes the meaning in such a way that "oxygen" or "nitrogen" enters into the relation in which "hydrogen" stood before. If we imagine that an expression can thus be altered, it decomposes into a stable component, resending the totality of relations, and the sign, regarded as replaceable by others, that denotes the object standing in these relations. The former component I call a function, the latter its argument. *The distinction has nothing to do with the conceptual content; it comes about only because we view the expression in a particular way* (Frege, 1967), pp.21–22

Thus the conceptual content of a sentence is also insensitive to the possible distinctions between function and argument we may draw within it. We highlight this fact by a different example:

<sup>1</sup>Russell argues in favour of the fact that a relation and its converse are different constituents of a proposition in (Russell, 1996) §219

given a certain function  $f$  of two arguments, we may regard the complex expression  $f(a, b)$  as composed by  $f$ ,  $a$ , and  $b$ . Yet we may also consider the function of one argument  $f(x, b)$  – say it  $f_b$  – and regard  $f(a, b)$  as composed by the function  $f_b$  and the argument  $a$ . Again, if we interpret the notion of content as something as fine grained as the Russellian propositions, the two different readings should correspond to different contents, for they display different constituents according to Frege’s distinction between concept and object; yet according to the quoted text, they are simply two different ways of regarding the same content. In other words, we cannot see the entities for which  $f$ ,  $a$ ,  $b$  stand as constituents of the content of  $f(a, b)$ , for the same content should also have  $f_b$  and  $a$  as alternative constituents. Thus whatever the conceptual content of a sentence is, it is not something as fine grained as the Russellian proposition.

One may object that the expression ‘the distinction has nothing to do with the conceptual content’ in the quoted passage means that the distinction between function and argument is considered by Frege as merely linguistic – at least at the time the *Begriffsschrift* was written – and thus does not affect the content simply because is ‘external’ or ‘extrinsic’ to the content itself. Thus Frege is not saying that we may trace different distinction between function and argument *within* the content without altering it. I may concede that there is no evidence at the time of the *Begriffsschrift* that Frege thought the linguistic distinction between function and argument as having a semantic correspondent; however, shortly after the publication of the *Begriffsschrift* Frege writes:

If [...] you imagine the 2 in the content of a possible judgment  $2^4 = 16$  to be replaceable by something else, by  $-2$  or by 3 say, which may be indicated by putting an  $x$  in the place of the 2:

$$x^2 = 16$$

the content of a possible judgment is thus split into a constant and a variable part. [...] And so instead of putting a judgment together out of an individual as subject and an already formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of a possible judgment. *Boole’s Logical Calculus and my Concept-script (1880)* (Frege, 1991), p.16-17

It is clear from the passage that in 1880, i.e. just one year after the publication of the *Begriffsschrift*, Frege was thinking the distinction between function and argument as a distinction that may be drawn within the conceptual content and thus not just as a linguistic distinction. As Heck and May (2011) argue, Frege enriched his notion of conceptual content after the *Begriffsschrift* precisely by thinking it as a possible complex notion, something that may be divided into

parts consisting of function and argument in many different ways. Thus, if we don't limit our analysis just to the *Begriffsschrift*, our conclusion seems to be sound: different constituents may determine the same conceptual content. As a consequence, the conceptual content of a sentence is something not as fine grained as the Russellian proposition<sup>2</sup>.

Perhaps the content of a sentence is of a very coarse grain, perhaps the notion of consequence to which Frege is referring in §3 is our familiar notion of logical consequence, and all logically valid sentences end up having the same content. The latter point follows from a criterion for identity of content that may be extrapolated from Frege's dictum in §3:

**(CONTENT=)**

*Two sentences  $\phi$  and  $\psi$  have the same content iff for every sentence  $\delta$  and set of sentences  $\Gamma$ ,  $\delta$  is a consequence of  $\Gamma$  and  $\phi$  iff  $\delta$  is a consequence of  $\Gamma$  and  $\psi$ .*

At a preliminary level of the analysis of Frege's text, it seems plausible to assume that (CONTENT=) represents a correct understanding of the quoted passage of §3.

It is immediate to see that if we intend by 'consequence' the familiar notion of logical consequence, then (CONTENT=) entails that all logically equivalent sentences have the same content. This result has been conceived as undesirable by several scholars, in particular by Kremer (2010). Kremer (2010) maintains that the view that all logically equivalent sentences have the same content is at odds with Frege's logicism; for given that in this case logic and the entire arithmetic will be concerned with one single content, how the informativeness of arithmetic truths may be justified if they express the same content as any other logical truth? For this reason, Kremer argues that the relation of consequence implied by the Frege's formal system has formal properties different from our familiar notion of logical consequence.

I don't question the validity of Kremer's argument, nor I will dispute the details of his proposal; I'd rather question the role his argument plays in the general discussion. For I may agree that not all logically equivalent sentences share the same content and I may also agree that for this fact to hold we may suppose that Frege's notion of consequence is different from the familiar notion of

<sup>2</sup>As briefly mentioned, when I speak of constituents of a Russellian proposition, I speak of Fregean concepts and objects. This clarification is crucial, for the reader may object that besides there being different distinction between function and argument within the same content, the fundamental constituents, i.e. the fundamental relations, properties, or individuals remain unchanged. Nonetheless, if we conceive those constituents from a Fregean standpoint, all possible concepts and objects arising from different distinction between function and argument are to be taken as ontologically significative.

logical consequence. Yet I disagree on the fact that the main motive for making this supposition is the account for the informativeness of analytic truths. For, as Frege himself suggests in §8 of the *Begriffsschrift*, the informativeness of analytic truths may rely precisely in the fact that the same content is presented in different ways. Thus the idea that all logical truths (and thus all arithmetical theorems) express the same content is not necessarily in contrast with the view that there are informative analytic truths.

A deeper understanding of the problem of fineness of grain of the notion of content (and thus of the grade of relevance of the correspondent consequence relation) may be achieved by using notions that do not belong to Frege's conceptual apparatus. Given that two logically equivalent sentences may not share the same content, we have to conclude that the conceptual content of a sentence is of finer grain than its *intension*, i.e. the function that associates to every possible world the truth-value of the sentence in that world. For there may be two sentences having the same intension while differing in content.

Our analysis of the notion of content in Frege based on the relation of consequence showed that content of a sentence  $S$  while being of a coarser grain than the Russellian proposition expressed by  $S$ , is of a finer grain than the intension of  $S$ . However, the notion of consequence alone cannot provide us with further hints on how the notion of conceptual content should be characterized and defined. This because the relation of consequence that we may extrapolate from the *Begriffsschrift* is a syntactic or proof theoretic notion; and, as we will see, an identity of content between two sentences cannot be always determined by the fact that they are mutually derivable. Hence, we need to consider different aspects of the notion of content that go beyond its characterization in terms of a merely syntactic notion of consequence. In other words, we need a semantic characterization of the relation of identity of content. This will be the purpose of the next section.

## 1.2 Content and logical analysis

By taking a stock of what has been considered in the previous section, we may say that the content of a sentence is an intermediate notion between the Russellian proposition it expresses and its intension. As previously shown, two sentences may have the same intension while differing in content; on the other hand, they may express different Russellian propositions while

having the same content. If we accept to partially depart from Frege's account, we may attempt a characterization of the notion of conceptual content by comparing it with other germane notions. At the finest level we have the notion of *cognitive content* which may be roughly identified with Frege's sense. Successively, we may locate the Russellian proposition: two sentences may be associated with the same Russellian proposition while differing in their senses; this because the Russellian proposition depends only upon its fundamental constituents no matter how they are determined or picked out. The notion of conceptual content may be located at the next level: two sentences may express different Russellian propositions while having the same conceptual content; it is the case of the examples taken from the §9 of the *Begriffsschrift*, where Frege says that the distinction between function and argument may not alter the conceptual content of the complex expression. At the bottom of this logical hierarchy we find the intension: two sentences may have the same intensions while differing in their conceptual contents, as may be the case for two logical truths<sup>3</sup>.

Given that both the conceptual content and the intension do not seem to depend upon the Russellian constituents, what is the difference between these two notions? To answer this question we have to regard the issue from a different perspective.

In a letter to Husserl of 1906, Frege writes:

It seems to me that an objective criterion is necessary for recognizing a thought again as the same, for without it logical analysis is impossible. Now it seems to me that the only possible means of deciding whether sentence A expresses the same thought as sentence B is the following, and here I assume that neither of the sentences contains a logically selfevident component part in its sense. If both the assumption that the content of A is false and that of B true and the assumption that the content of A is true and that of B false lead to a logical contradiction, and if this can be established without knowing whether the content of A or B is true or false, and without requiring other than purely logical laws for this purpose, then nothing can belong to the content of A as far as it is capable of being judged true or false, which does not also belong to the content of B ; for there would be no reason at all for any surplus in the content of B , and according to the presupposition above, such a surplus would not be logically self-evident either. In the same way, given our

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<sup>3</sup>A similar hierarchy is presented in (Salmon, 1992)

supposition, nothing can belong to the content of  $B$ , insofar as it is capable of being judged true or false, except what also belongs to the content of  $A$ . Thus what is capable of being judged true or false in the contents of  $A$  and  $B$  is identical, and this alone is of concern to logic, and this is what I call the thought expressed by both  $A$  and  $B$ . (Frege, 1980). p. 70

Frege is proposing a criterion for identity of thought. Given that the passage was written after the introduction of the sense-reference distinction, it may not be correct to take ‘identity of thought’ as ‘identity of conceptual content’. Suppose that Frege’s criterion applies also to conceptual contents. What is interesting is that Frege’s criterion requires that none of the sentences  $A$  and  $B$  contains a logically self-evident component. By this requirement, logically valid sentences fall out of the range of application of the criterion. Note that without this requirement, Frege’s criterion would have implied that all logical truths have the same content. Thus it is reasonable to suppose that Frege introduced the caveat in order to avoid this latter consequence. Besides the formulation of the criterion, there is an interesting aspect in the way Frege presents it. The passage starts with a sort of counter-factual implication: if there were no criterion of identity of thought, logical analysis would not be possible. According to Frege, there seems to be some necessary relation between the criterion for identity of thought and the possibility of logical analysis. Yet in what sense a criterion of identity of content makes logical analysis possible? After all, the criterion proposed by Frege in the passage does not seem to have something to do with logical analysis. We may interpret the counter-factual as saying that without a criterion for identity of thought there would not be any criterion for saying whether a logical analysis is correct or not. In other words, if we analyze some expressions in terms of others the only way to check if the analysis thus carried out is correct is to make sure that no alteration in the thought has been produced. In this sense, logical analysis is conceived as a content preserving operation. And if the performed logical analysis is correct, then the assumption that one sentence is true and the other is false must entail a contradiction either with the salient meaning postulates or with some logical axiom. Here we are assuming that what Frege calls ‘a logical contradiction’ must be taken in a very wide sense, i.e. including what may be called ‘analytic contradictions’.

The first result of the proposed reading of Frege’s letter to Husserl is the following: if a sentence  $\psi$  results from the logical analysis of some expressions contained in a sentence  $\phi$ , then  $\phi$  and  $\psi$  must have the same content. The examples of section 9 of the *Begriffsschrift* confirm this result: for instance, the property of being lighter than oxygen in the sentence ‘Hydrogen is lighter

than oxygen' may be analyzed in terms of the relation ' $x$  is lighter than  $y$ ' and the individual 'Oxygen'. Thus we may suppose that logical analysis is a sufficient condition for identity of content.

Is it also necessary? There are good reasons to offer a negative answer. Consider the sentences  $A$  and  $\neg\neg A$ ; as Frege himself admitted, they express the same content<sup>4</sup>. However, it is hardly plausible that one is the result of a logical analysis of the other, at least if we consider logical analysis as being a sort of decompositional analysis: that would amount to say that the content of the double negation is somehow included in the content of any sentence. Another example: consider Frege's example in §3 of the *Begriffsschrift* according to which the sentences 'The Greeks defeated the Persians at Plataea' and 'The Persians were defeated by the Greeks at Plataea' have the same content: the fact that the relation ' $x$  was defeated by  $y$ ' may be seen as the converse of the relation ' $x$  defeated  $y$ ' does not seem to be a logical analysis, at least if we reasonably suppose that any sort of analysis should be in some sense *decompositional*, i.e. the analysis should break into simpler components a complex component.

It seems that there are cases in which the identity of content is recognized without performing any decompositional analysis. I call such a class of cases the *semantically related* cases. Semantically related cases may be preliminary defined as cases in which the identity of content is recognized in absence of decompositional analysis and in virtue of some necessary relations between the meanings of expressions appearing in the two sentence having the same content. Examples of semantically related cases are: the identity of content between 'The Greeks defeated the Persians at Plataea' and 'The Persians were defeated by the Greeks at Plataea' which holds in virtue of a semantic relation between the relation ' $x$  defeated  $y$ ' and ' $y$  was defeated by  $x$ '; 'John is bachelor' and 'John is an unmarried man', in virtue of the relation of synonymy between 'bachelor' and 'unmarried man'; the case of  $\neg\neg A$  and  $A$  may also be conceived as semantically related, where the relation is between the entire sentences.

Thus we may formulate the following criterion:

**(CONTENT<sub>=</sub>\*)**

*Two sentences  $\phi$  and  $\psi$  have the same content iff either one is the result of a logical analysis of the other or  $\phi$  and  $\psi$  are semantically related.*

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<sup>4</sup>See the preface of the *Begriffsschrift*



Clearly to completely understand this criterion we should have a rigorous definition of both logical analysis and semantically related; for now I will not provide a detailed account. As we will see, that will be the purpose of most parts of this work.

More textual evidences may support the proposal represented by (CONTENT<sub>=</sub>). The most interesting one for our purposes, is the passage taken from *The Foundations of Arithmetic* and quoted in the Introduction. This work precedes the sense/reference distinction, and the main logical notions implied are roughly the same of the *Begriffsschrift*. In §64 Frege writes:

“ The judgment ‘ *line a is parallel to line b*’, [...] can be taken as an identity. If we do this, we obtain the concept of direction and say: ‘*The direction of line a is identical with the direction of line b*’. Thus we replace the symbol  $\parallel$  by the more generic symbol  $=$ , through removing what is specific in the content of the former and dividing it between *a* and *b*. We carve up the content in a way different from the original way, and this yields us a new concept. ”(Frege, 1950), pp. 74-75

Frege is saying that the content of a relation of parallelism may be *recarved* in such a way that an identity of directions obtains. The passage is aimed at providing a description – unfortunately not detailed enough – of the procedure of recarving, according to which the content of the relation is re-organized, re-distributed. I consider this procedure as a form of logical analysis, though an unorthodox and partially obscure one. As I understand the passage, Frege is saying that the relation of parallelism – as well as (all?) other equivalence relations – may be analyzed in terms of an identity between objects of a special sort, i.e. directions. In particular, Frege seems to suggest that the content of the so-called direction operator represents a “part” of the content of the relation of parallelism: precisely the specific part of this latter content that has been removed. This way of describing the recarving procedure seems to be very similar to the traditional metaphoric description of the procedure of compositional analysis: a concept is somehow *decomposed into certain constituents*.

I take this operation to preserve the content of the entire sentence: as in the case of *Begriffsschrift* §9, no new content is being introduced or neglected, yet a content already contained in the content of the entire sentence is arranged (carved up) in a different way. Thus we may assume that Frege took a sentence saying that two lines are parallel to be identical in content to a sentence saying that their directions are identical. Again, we are in presence of an identity of content between two sentences holding in virtue of a certain procedure of logical analysis; and, again, we find the difficulty of understanding – outside of the metaphor of the decomposition –

what should be understood as logical analysis. As in the previous cases, the logical analysis – whatever it may be – may result in a change of the Russellian constituents: one sentence admits straight lines, the other their directions. This aspect is what makes the analysis described in §64 of the *Foundations* an unorthodox one: indeed, it is commonly accepted that logical or conceptual analysis do not affect the list of objects involved in the expression of their truth-conditions. When we say that the concept ‘human’ may be analyzed as the intersection of the concept ‘animal’ with ‘rational’, we are just conceiving one concept as obtained by combining two other concepts according to some Boolean operators. Nevertheless, Frege seems to have a wider conception of logical analysis which probably is the responsible for what he called *the fruitfulness of a definition*<sup>5</sup>.

Does the proposed relation between identity of content and logical analysis help to understand the difference between conceptual content and intension? In spite of not having a definite answer, I may say that the proposed interpretation at least performs a negative theoretical task. When we say that the conceptual content of sentence is of a finer grain than its intension, the immediate intuition is that the conceptual content may be thought as the intension plus the specification of some fundamental constituents of the proposition. For instance, we say that the sentences ‘ $2+2=4$ ’ and ‘ $5 = 5$ ’ differ in content but not in intension simply because one is concerned with the numbers 2 and 4 while the other with the number 5. Yet there seems to be no chance of defining the conceptual content in this way: any specification of a list of constituents makes the notion of content too fine-grained, so that any logical analysis will unavoidably yield a different content. *It is as if the conceptual content of a sentence is concerned with a certain aspect of reality that is crucial for the determination of the truth-value of the sentence, without specifying the entities that constitute this aspect, for in that case it will collapse into a particular analysis of, or way of regarding, or way of carving up the content itself.* We may spell out this fact in more Fregean terms. A content may be divided into a functional part and an argument part and – in some sense – it is the combination of both. Alternative divisions into function and argument whenever combined returns the same content; yet the main problem is to understand in what sense function and argument should be combined. Indeed, the only combination of function and argument that Frege admits is *the saturation*, i.e. the application of the function to the argument, which returns the value of the function for the particular argument and not the content. Whenever we try to specify the constituents of a content we fall into a particular

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<sup>5</sup>(Frege, 1950), §88

distinction into function and argument and we never achieve this unspecified notion of ‘whole’ that should be the content itself.

Any attempt to define the conceptual content of a sentence will encounter a fundamental difficulty represented by what may be called a *principle of indeterminacy of content*: on the one hand, the content depends on the Russellian constituents – or by function and argument – of the expressed proposition for it is the composition of both, on the other hand, it cannot be univocally associated with them, for different constituents may yield the same content. We may find a clue of this difficulty in the following passage from *On Concept and Object*:

When I wrote my *Foundations of Arithmetic*, I had not yet made the distinction between sense and reference; and so, under the expression ‘the content of a possible judgment’ I was combining what I now designate by the distinct words ‘thought’ (*Gedanke*) and ‘truth-value’.(Frege et al., 1951)

To my mind, there are two possible readings of this passage. According to the first one, the notion of content has some aspects in common with the thought and some others with the truth-value. The content may be conceived as structured and in this respect it is close to the thought; on the other hand, the content when considered as a whole, is unstructured as the truth-value: we may highlight a possible internal structure of the content depending of the way we regard it, yet the content remains one and the same. Therefore, under the first reading of the passage we may suppose that Frege is postulating an unspecified notion which lays between the structured thought and the unstructured truth-value. According to the second reading, Frege would be assuming that the indeterminacy of the notion of content produced a confusion in the use of this notion: sometimes it has been regarded as being a structured entity like the thought, sometimes it has been considered unstructured as the truth-value. Thus what Frege would be saying is that he was combining two different notions in the sense that he was overlooking a fundamental distinction that was needed to characterize all the theoretical roles that his notion of content was supposed to play. The interesting fact about this reading is that the presupposed notion of content may not need to be characterized, for it may be explained away by introducing the sense-reference distinction.

If we take the second reading to its extreme consequences a question arises: given that the notion of conceptual content may be explained away by invoking the sense-reference distinction, and thus given the dispensability of this notion, why bother? Why the attempt of characterizing and

determining this notion should worth its grain of salt?

There are at least three reasons to think that the notion of content may not just be indispensable but philosophically interesting and fruitful. The first one is related with the famous puzzle of identity. Frege's theory of sense explains the difference in informativeness between sentences of the form  $a = a$  and sentences of the form  $a = b$  by assuming – against Millianism – that the cognitive value of a proper name is not just its reference but a structured thought linguistically representable as a definite description. From the perspective of contemporary philosophy of language this position is highly controversial. If Frege had still considered the conceptual content as a central logical notion and if he had developed a complete theory of it, he had a notion of logical equivalence, i.e. identity of content, stronger than the one we admit nowadays. As Salmon (1992) points out, if he really had this stronger notion of logical equivalence, he may have argued that  $a = a$  and  $a = b$  have different cognitive value simply because they are logically unequivalent, without making heavy assumptions on the cognitive value of proper names. If one accepts that the cognitive content determines the conceptual content, then it is impossible that two sentences having different conceptual contents have the same cognitive content; thus a response to the puzzle without invoking the notion of sense would be possible.

The second reason, also presented by Salmon (1992), is related to *the paradox of analysis*. This paradox may be described as a conflict between the correctness and the informativeness of analysis. Suppose that a sentence  $B$  is the result of a certain analysis of a sentence  $A$ ; if the analysis is correct, we should infer that the thought expressed by  $A$  is the same as the thought expressed by  $B$ . Yet if as candidate notion of thought we have only a notion close to that of cognitive content – as in the case of the Fregean sense – the identity of cognitive content between  $A$  and  $B$  prevents the analysis to be at the same time correct and informative. Contrariwise, if we have a coarser grained notion of thought – as the notion of conceptual content seems to be – we may maintain that  $A$  and  $B$  are identical in the expressed thought without implying that they must have the same cognitive content. Thus the informativeness of the analysis may be conceived as the possibility of two sentences having the same content while differing in their cognitive value. This simple solution is not available to Frege in its mature period, and this justifies his struggle against the paradox, though Frege never explicitly formulated it<sup>6</sup>.

The third reason is related to Frege's early logicist program and, in particular, to the logi-

<sup>6</sup>For Frege's attempts to overcome the difficulties of the paradox of analysis in terms of the notion of sense see (Horty, 2010), pp.32–49

cal foundation of his definitions by abstraction. It seems evident that the two limbs of a biconditional representing an abstraction principle must have different senses and the same reference. Yet, why are they identical in reference? By invoking §64 of the *Foundations*, we may say that the two limbs of an abstraction principle are logically equivalent because the way the content of an identity of abstracts is carved up may be obtained by a certain content preserving operation applied to the equivalence relation appearing on the left limb. Such an explanation cannot be framed in terms of senses; or, if it might be, it requires the definition of a special relation between senses; and it is hard to imagine how such a relation may be defined without presupposing some commonality of a coarser grained notion of thought.

Thus the notion of conceptual content seems to have a defined theoretical role and its understanding seems to be promising not just as far as definitions by abstractions are concerned, but also from a wider perspective countenancing several profound philosophical issues, such as the puzzle of identity and the paradox of analysis.

On the other hand, I hope to have shown that there are strong limitations in attempting to deepen the characterization of the notion of content within Frege's conceptual universe and by means of the theoretical tools he made available. For Frege does not offer any method of combining function and argument in such a way that the result of such combination does not collapse into the value of the function for the considered argument, i.e. into the reference. For this reason the proposal that I will start outlining from the next chapter will considerably depart from Frege; as a result, the purpose of this work will not be the mere interpretation of the notion of conceptual content according to Frege's text, but rather a new general theory of content.

## **Chapter 2**

# **Interlude: the Theoretical Setting of the Proposal**

### **2.1 Introduction**

As clarified at the end of the previous chapter, there are several difficulties in providing a full characterization of the notion of conceptual content within Frege's conceptual universe. In particular, we have seen that the problem of fineness of grain of the content has an immediate formulation if we appeal to notions such as that of intension. For this reason, the theory of conceptual content that I will propose will be formulated within a very unFregean theoretical setting.

The characterization of the notion of content will be achieved starting from the notion of intension which is defined in terms of possible worlds; the next section is dedicated to the notion of possible world that I will refer to in this work. As it will appear clear from the next chapter I will also need to refer to some other logical and metaphysical notions that are introduced in the following sections. Therefore, the present chapter should be intended not as part of the proposal, but as the theoretical background needed to understand the proposal. The reader who is familiar with the notions herein introduced may skip this chapter.

## 2.2 Possible worlds

### Two conceptions of possible world

It is well known that there is no universal agreement on what a possible world is and in which sense some entities exist according to a certain possible world  $W$ . There are two main conceptions that I will discuss in this section: *the abstractionist account* and *the concretist account*.

According to the abstractionist account<sup>1</sup>, possible worlds are *maximal states of affairs*. That of ‘state of affairs’ and of ‘what it is for a state of affairs to obtain’ are considered as primitive notions. A state of affairs  $A$  is maximal iff for every state of affairs  $s$ , either  $A$  includes  $s$  or  $A$  precludes  $s$ . We say that  $A$  includes  $s$  iff if  $A$  obtains, then necessarily  $s$  must obtain as well; we say that  $A$  precludes  $s$  iff  $A$  and  $s$  cannot obtain together.

The relation of inclusion between a possible world and a certain state of affairs should not be confused with the mereological relation of parthood; worlds are not aggregates of states of affairs. States of affairs are abstract entities and thus worlds – being maximal states of affairs – are abstract too. The same holds when we say that an object  $a$  exists in a world  $W$ : we are not saying that  $a$  is part of  $W$  but just that the state of affairs described by the sentence ‘ $a$  exists’ is included in  $W$ . Abstract possible worlds are not committed to the existence of *possibilia*, i.e. possible entities. When we say that ‘Talking donkeys possibly exist’ we are not saying that there is a possible world in which talking donkeys exist, yet that there is a maximal state of affairs including the state of affairs corresponding to ‘Talking donkeys exist’. Yet such state of affairs does not occur. Only actual entities exist. To better understand this point it is helpful to consider abstract possible worlds as *representational devices*: to say that talking donkeys possibly exist is to say that there is an abstract possible world *representing* the concept ‘ $x$  is a talking donkey’ as instantiated; nevertheless the concept ‘ $x$  is a donkey’ *is not instantiated*. Similarly, when we say that Socrates is possibly German, we are not say that there is a German Socrates in some possible world, yet that there is an abstract possible world that represents the actual Socrates (the only existing Socrates) as German. As a consequence, there is no problem of *transworld identity*: it makes no sense to say that the German Socrates is distinct from the actual Socrates, for there is no German Socrates: the only thing we may say is that the the actual

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<sup>1</sup>(van Inwagen, 2001), p.188

Socrates may be represented as German; and there is no way of reifying the representation of Socrates as German by identifying it with a certain *possibilium*, for abstract possible worlds are not made of possible individuals, but of possible states of affairs.

From the semantic point of view, the abstractionist account of possible worlds is compatible with the idea that there are expressions that *rigidly designate*. The proper name ‘Socrates’ has a unique reference: the actual Socrates. ‘Socrates is German’ is true according to a world  $W$  not because the object to which ‘Socrates’ refers in  $W$  is German; but in virtue of the fact that the actual Socrates (i.e. the only reference of ‘Socrates’) is represented as German by  $W$ .

According to the concretist account – introduced by David Lewis – possible worlds are existing concrete mereological aggregates of individuals. From the ontological point of view, there is no difference between the world we live in and another possible world: both exist in the unique sense of existence. As a consequence, the attribute ‘actual’ is a kind of indexical: when someone living in a possible world  $W$  distinct from ours speaks of the ‘actual world’, she is referring to  $W$ .

The advocate of concretism is committed to the existence of *possibilia*. When we say that talking donkeys possibly exist we are saying that there is a world  $W$  having at least one talking donkey as part; namely, talking donkeys do exist, yet in a world different from ours.

In addition to the concrete parts of a world, the concretist admits as existing also sets having these parts as elements. Thus it is not entirely correct to say that anything that exists in a world  $W$  is part of  $W$ : set theoretic constructions are not parts of  $W$ , yet they exist according to  $W$ .

A crucial notion to understand how modality works in the context of concretism is the notion of *counterpart*. Individuals existing in a world may have counterparts in different worlds, i.e. different individuals that resemble them. Thus when we say that possibly Socrates is German, we are saying that there is a world  $W$  having a part that is a counterpart of the actual Socrates and is German.

Concrete possible worlds are not demarcated by a criterion of maximality. What qualifies a mereological aggregate as a world is a set of external relations (which Lewis tends to identify with spatio-temporal relations<sup>2</sup>). Thus two concrete worlds  $W_1$  and  $W_2$  are isolated from one

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<sup>2</sup>(Lewis, 1986), pp.69-81



another in the sense of being  $\mathcal{R}_1$  and  $\mathcal{R}_2$  the sets of external relations respectively determining the unity of  $W_1$  and  $W_1$ , no part of  $W_1$  stands in a relation of  $\mathcal{R}_1$  with a part of  $W_2$  and no part of  $W_2$  stands in a relation of  $\mathcal{R}_2$  with a part of  $W_1$ .

When concrete worlds are considered, the problem of *transworld identity* has a more interesting formulation and it is immediately “dissolved”: for now it makes sense to say that Socrates of our world may be identical or distinct from Socrates of a different world. Worlds may have parts that are intrinsic duplicates, i.e. parts that are qualitatively identical. We may go further, and admit that worlds may overlap, i.e. may have parts in common; yet this view is more controversial<sup>3</sup>. Anyway, it makes sense to say that worlds may be *partially indistinguishable*. According to the concretist, what does not make sense is to say that Socrates of our world is identical to Socrates of a different world and there is an intrinsic property that one Socrates has and the other lacks<sup>4</sup>. Therefore, the distinction between extrinsic and intrinsic properties is fundamental to understand the problem of transworld identity; for this reason I will dedicate a section in this chapter to this distinction.

From the semantic point of view, the concretist account of possible worlds excludes the view that proper names are rigid designators. For now the proper name ‘Socrates’ may refer to different individuals in different worlds. Clearly, it may happen that the same proper name refers to qualitatively identical individuals across several possible worlds but this occurs by chance. Thus when we say that in a world  $W$  Socrates has six fingers in his right hand, we are saying that the individual to which the proper name ‘Socrates’ refers in  $W$  has six fingers in the right hand; and, necessarily, this individual is different from Socrates of our world, assuming that our Socrates had five fingers in the right hand. The notion of counterpart plays a central role in the doctrine of proper names when concrete worlds are considered: for instance, the reference of the proper name ‘Socrates’ in a possible world  $W$  is the counterpart in  $W$  of the individual to which ‘Socrates’ refers in this world.

I will not dispute the details of the debate between concretists and abstractionists. Roughly speaking, abstractionists are subject to the charge of failing to explain away modal notions, while concretists are targeted for their uneconomical ontology.

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<sup>3</sup>(Lewis, 1986), pp.198-209

<sup>4</sup>(Lewis, 1986), pp.192-198

## The adopted conception

In this work I will favor a particular version of the concretist conception of possible worlds. This choice strongly depends upon some fundamental aspects of the theory of content I will propose from the next chapter on.

In particular, I want to retain an important aspect of the notion of conceptual content as presented by Frege: the content of a sentence may be regarded as “including” a certain unsaturated entity according to the particular distinction between function and argument that we may draw within the sentence. Concepts are functions from the correspondent range of entities to which they are applicable to truth-values. For instance, a first-order concept is a function from individuals to truth-values; the concept function returns the value True for the argument  $a$ , in virtue of the fact that  $a$  *falls under* the considered concept, otherwise it returns the value False. The function associated with the unsaturated expression ‘ $x$  is wise’ has the value True in correspondence of the individual named ‘Socrates’ in virtue of the fact that Socrates falls under the concept ‘wise’, i.e. Socrates is wise.

When the possibility of drawing different distinctions between function and argument is combined with the context of possible worlds, the abstractionist account seems to introduce several complications. The difficulty is the following: we are not able to define a unique truth-function for a given concept. Consider the concept ‘Human’; Socrates falls under this concept. Yet when a non-actual world  $W$  is considered, the very same Socrates may not fall under this concept. Said  $H$  the truth-function associated with the concept ‘Human’,  $H$  must return both the value True and False in correspondence of the argument Socrates. The abstractionist may reply by taking into account the fact that concepts must be well defined: you are confusing different concepts, for to say that according to the world  $W$  Socrates does not fall under the concept ‘Human’ is to say that the actual Socrates (the only possible reference of the name ‘Socrates’) does not fall under the concept ‘Human in  $W$ ’; and ‘Human in  $W$ ’ is a concept different from ‘Human’. Thus the abstractionist and the concretist respectively propose the following analyses in terms of concept and object of the fact that according to  $W$  ‘Socrates is Human’ is true:

- Socrates falls under the concept ‘Human in  $W$ ’ (*Abstractionist reading*)
- Socrates of the world  $W$  falls under the concept ‘Human’ (*Concretist reading*)

Said  $H$  and  $H_W$  the functions respectively representing the concept ‘Human’ and ‘Human in

$W'$ , the distinction between function and argument may be drawn as follows:

- $H_W(\text{Socrates}) = \text{the True}$  (*Abstractionist reading*)
- $H(\text{Socrates of the world } W) = \text{the True}$  (*Concretist reading*)

It is clear that the concretist reading is more simple and elegant, for it proposes a unique function for every unsaturated expression. The uniqueness of the truth-function representing a concept turns out to be important when we want to express relations between concepts, e.g. the conceptual analysis of one concept into others. According to the concretist reading, this fact may be expressed as a certain relation between the concept functions. When the abstractionist reading is considered, a simple relation of conceptual analysis becomes a complex relations between many concepts.

As the exposition of the theory proceeds, more and more advantages of using the concretist conception will become clear.

One may object that to choose concretism as preferred theoretical setting has a high ontological price which may not be worth paying. After all, there is nothing in the notion of content that forces us upon this view. I reply that we may not pay such a high ontological price – i.e. the existence of all *possibilia*. I consider modal realism (which endorses the concretist conception) just as a theoretical setting within which a *model* of the notion of content may be constructed with elegance and simplicity. Yet it is not my intention to dispute ontological issues arising from this or that conception of content. My purpose is to model the notion of content and any conclusion I might draw will always be about the model or within the model. Thus I don't really need to assume that every possible world is an existing concrete entity. The use I make of modal realism may be seen as merely fictional and pragmatic<sup>5</sup>. For instance, whenever I say that there is a function from possible worlds to certain *possibilia*, the statement must be understood with the classical 'in-fiction' prefix 'according to the hypothesis of modal realism'. As far as no ontological conclusions will be drawn, the notion of possible world plays a role in my proposal no more substantial than that of the notion of mass point in classical mechanics.

The theoretical tool of *fictional modal realism* has also another important role in my proposal. As it will become clear, the notion of content we accept and the correspondent relation of identity of content strongly depends upon which truths we are open to accept as necessary

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<sup>5</sup>As in (Rosen, 1990)

and which objects we are open to admit as parts of the ontological inventory of the universe. Any particular view on what there is and which sentences express necessary truths may be represented in terms of the worlds we are open to accept as possible. Fictional modal realism gives us the freedom of introducing possible worlds according to one or another big picture of reality, thus evaluating the impact on the notion of content and on the definition of the relation of identity of content that different metaphysical and logical views might have. This will result in a very interesting analysis of the notion of content.

## 2.3 Intensions and partitionings of the logical space

### Intension

I will make constant use of the notion of *intension of an expression* which dates back to Carnap (1947). Given an expression  $\xi$ , the intension of  $\xi$  – denoted by  $u_\xi$  – is the function that associates to every possible world  $W$  the extension of  $\xi$  in  $W$ . For instance, the intension of the proper name ‘Socrates’ is the function that associates to every possible world  $W$  the reference of ‘Socrates’ in  $W$ . The intension of a predicate  $P$  is a function associating to every possible world  $W$  the extension of  $P$  in  $W$ , i.e. the set of individuals of  $W$  having the property for which  $P$  stands. In the case of entire sentences, the intension of a sentence  $S$  – denoted by  $\mathcal{C}_S$  – is the function that when receives the world  $W$  as argument returns the truth-value of  $S$  in  $W$  as value.

When we distinguish between function and argument in the syntactic expression of a sentence, there is a correspondent semantic distinction which involve the notion of intension. Let  $S$  be the sentence  $\Phi(t)$  where  $t$  is a singular term and  $\Phi$  a predicative expression. Suppose we want to read  $S$  as composed by the function  $\Phi(x)$  and the argument  $t$ . The corresponding semantic reading is defined by introducing the corresponding functions in the expression of the truth-conditions of  $S$ : let  $F_\Phi$  be the truth-function associated to  $\Phi$  and  $u_t$  the intension of the term  $t$ ; it follows that for every possible world  $W$ ,  $S$  is true according to  $W$  iff

$$F_\Phi(u_t(W)) = \text{the True}$$

Whenever a sentence is divided into a predicative expression taken to define the function and a constant expression taken to represent the argument, the truth-conditions of the sentence may be expressed by composing the truth-function associated with the predicative expression with the intension of the argument expression. In the next chapter we will see this in more details.

For reasons that will become clearer in the following section, I have to slightly modify the definition of intension to deal with the case of worlds containing duplicate parts. Consider two worlds  $W$  and  $V$  such that the extension of an expression  $\xi$  in  $W$  is indistinguishable from its extension in  $V$ . According to the given definition of intension, we cannot say that  $u_\xi(W) = u_\xi(V)$ , for we agree that the symbol ‘=’ stands for numerical identity. However, the consequence of this fact is that intensions of sub-sentential expressions are always bijective functions. As we will see in the next section, this will trivialize the notion of partitioning of the logical space introduced by an expression. Thus I introduce the following convention: for every class of qualitatively identical parts of possible worlds, we choose a representative and only these representatives are possible values of intension functions. Thus if the name ‘Socrates’ refers to two qualitatively identical individuals across several possible worlds, the correspondent intension function will return the same value for all these worlds.

## Partitions and subject Matter

A notion that is crucial to my proposal is that of *the partitioning of the logical space defined by an expression  $\xi$  (or by a sequence of expressions  $\xi_1, \dots, \xi_n$ )*.

We may take the intension  $u_\xi$  of an expression  $\xi$  as defining an equivalence relation on worlds that I denote by  $\equiv_\xi$ . Given two worlds  $W$  and  $V$ ,  $W \equiv_\xi V$  iff  $\xi$  has the same extension in both  $W$  and  $V$ . In other words,

$$W \equiv_\xi V \quad \text{iff} \quad u_\xi(W) = u_\xi(V)$$

Similarly, in the case of a sequence of expressions  $\xi_1, \dots, \xi_n$  I define the equivalence relation  $\equiv_{\xi_1 \dots \xi_n}$  as

$$W \equiv_{\xi_1 \dots \xi_n} V \quad \text{iff} \quad \langle u_{\xi_1}(W), \dots, u_{\xi_n}(W) \rangle = \langle u_{\xi_1}(V), \dots, u_{\xi_n}(V) \rangle$$

The partitioning of the logical space defined by the expression  $\xi$  is the set of all equivalence classes of the relation  $\equiv_\xi$ .

An expression  $\xi$  may be seen as dividing the set of all possible worlds into cells; each cell in which  $\xi$  partitions the set of all possible worlds contains all worlds that agree on a particular possible extension of  $\xi$ . For instance, the proper name ‘Socrates’ divides the logical space into cells, and each cell is associated with one possible reference of ‘Socrates’, and all worlds belonging to the same cell agree on which individual is the reference of ‘Socrates’.

For brevity, given a world  $W$ , the cell defined by  $\xi$  to which  $W$  belongs – the equivalence class of  $W$  defined by the relation  $\equiv_\xi$  – will be denoted by  $[W]_{u_\xi}$ .

The method of considering the partitioning of the logical space defined by expressions will turn out to be a general way of characterizing relations of semantic dependence between expressions and metaphysical dependence between entities of specific sorts. For instance, suppose that the meanings of two expressions  $\xi$  and  $\zeta$  are related in the sense that the extension of  $\zeta$  depends – both semantically and metaphysically – on the extension of  $\xi$ . As a consequence, if two worlds agree on the designation of  $\xi$ , then they must agree on the designation of  $\zeta$ . For instance, given a certain world  $W_0$ , for every world  $W$ :

$$W \in [W_0]_{u_\xi} \Rightarrow W \in [W_0]_{u_\zeta} \quad (*)$$

i.e. if two worlds agree on the denotation of  $\xi$ , then – given the semantic dependence of  $\zeta$  upon  $\xi$  – they must agree on the denotation of  $\zeta$ . If we consider all expressions of the form  $(*)$  for every world  $W_0$ , we may characterize the semantic dependence between  $\xi$  and  $\zeta$  in set theoretic terms. Indeed, we may say that every cell defined by  $\xi$  is included in the correspondent cell defined by  $\zeta$ . More precisely, given a possible world  $W$ , the equivalence class  $[W]_{u_\xi}$  – the set of all worlds in which the denotation of  $\xi$  is identical to the denotation of  $\xi$  in  $W$  – is always included in the equivalence class  $[W]_{u_\zeta}$ .

Given two expressions (or series of expressions)  $\xi$  and  $\zeta$ , we say that the partitioning defined by  $\xi$  is a *subpartitioning* of the partitioning defined by  $\zeta$  iff for every world  $W$ ,

$$[W]_{u_\xi} \subseteq [W]_{u_\zeta}$$

We say that the partitioning defined by  $\xi$  is a *proper subpartitioning* of the partitioning defined by  $\zeta$  iff it is a subpartitioning and exists at least one world  $W$  such that:

$$[W]_{u_\xi} \subset [W]_{u_\zeta}$$

.

Partitionings of the logical space have an interesting philosophical interpretation: as observed by Lewis (1988) and Yablo (2014), a certain partitioning of the logical space may be interpreted as a *subject matter*. Indeed, a subject matter may preliminary be taken to be either a specific part or a specific aspect of a possible world. A general method to define a subject matter is by means of the notion of partitioning of the logical space. For instance, the subject matter *the U.S. president in 2018* may be taken to be the set of equivalence classes defined by the equivalence relation that holds between two worlds whenever they agree on which individual is the U.S. President in 2018. Or the subject matter *year 1814* may be taken to be the set of equivalence classes defined by the equivalence relation that holds between two worlds whenever they agree on all events that occurred in 1814.

Modelling subject matters with partitions of the logical space allows us to define relations between subject matters such as ‘smaller than’ or ‘larger than’ or ‘containment’. We say that the subject matter  $m_1$  is larger than (or contains) the subject matter  $m_2$  iff  $m_1$  is a proper subpartitioning of  $m_2$ . To see this point consider the following example: intuitively, the subject matter *year 1814* is contained in the subject matter *the IXX century* for whatever occurred in 1814 occurred in the IXX century while the converse does not hold. Take *year 1814 (the IXX century)* to be the set of equivalence classes defined by the equivalence relation that holds between two worlds whenever they agree on everything that happened in 1814 (the IXX century). It is easy to see that if two worlds belongs to the same *the IXX century*-cell then they must belong to the same *1814*-cell, while the converse does not hold. Thus *the IXX century* partitions the logical space into cells smaller than *year 1814*. This result may be resumed thus: the larger the subject matter, the smaller the cells in which it partitions the logical space (or the finer the partitioning of the logical space). Indeed, the larger the subject matter, the more amount of information it contains and thus the harder for two worlds to agree.

Therefore it has been shown that when we interpret a partitioning of the logical space as a subject matter, we interpret the relation of proper subpartitioning as a relation of inclusion

between subject matters.

The partitioning of the logical space defined by a referential expression  $\xi$  may be identified with a certain subject matter: we may call this subject matter *the reference of  $\xi$*  (or *the extension of  $\xi$* ). Indeed, the intension of  $\xi$  partitions the logical space into cells agreeing on which object is the reference of  $\xi$ . Notice that the subject matter *the reference of ‘Socrates’* and the subject matter *Socrates* may be different subject matters: for instance, if we take *Socrates* to be the subject matter represented by a commonly accepted notion of who is Socrates – e.g. the Greek philosopher who knows just that he doesn’t know anything – it does not follow that two worlds agreeing on who is the Greek philosopher who know just that he doesn’t know anything must also agree on the reference of ‘Socrates’.

As we will see, we may use the relation of subject matter inclusion to define special semantic relations between expressions, relation that will be useful to characterize both the notion of logical analysis and the relation of identity of content. In particular, we may define a special semantic relation between two expressions  $\xi$  and  $\zeta$  that holds whenever the subject matter *the extension of  $\xi$*  is included in the subject matter *the extension of  $\zeta$* , i.e. whenever the partitioning of the logical space defined by  $\zeta$  is a proper subpartitioning of the partitioning defined by  $\xi$ . In the next chapter we will see an interesting application of this relation.

## 2.4 Intrinsic vs extrinsic properties

The distinction between intrinsic and extrinsic properties is crucial for the concretist conception of possible world: as we have previously mentioned, the problem of transworld identity may be understood and solved only if such a distinction may drawn with clarity.

Intuitively, and intrinsic property is a property that an object has in virtue of the way it is, whereas an extrinsic property is a property that an object has in virtue of the way it interacts with the world. As a consequence, one may think that intrinsic properties are genuinely monadic whereas extrinsic properties are always relational; more specifically, one may understand the fact that an object  $a$  has an extrinsic property  $P$  as the fact that  $a$  stands in a certain relation either with the entire world of which  $a$  is part or with some part – not identical with  $a$  – of the world.



These basic intuitions are not uncontroversial: consider the property of ‘having legs longer than arms’; it is reasonably intrinsic but not genuinely monadic. Thus identifying intrinsic with non-relational and extrinsic with relational does not seem to be correct.

A simple characterization of the distinction is given by Lewis:

A sentence or statement or proposition that ascribes intrinsic properties to something is entirely about that thing; whereas an ascription of extrinsic properties to something is not entirely about that thing, though it may well be about some larger whole which includes that thing as part. (Lewis, 1983) pp. 111-2

According to this account, the fundamental feature of intrinsic properties is *locality*. An intrinsic property  $P$  is *local* in the sense that we may decide whether an object  $a$  has  $P$  in a world  $W$  by considering just  $a$  and no other part of  $W$ . Contrariwise, if  $P$  is extrinsic, we are not able to decide whether  $a$  has  $P$  in  $W$  by considering just  $a$ ; we need to take into account a larger whole including  $a$ .

We may characterize the notion of locality – and thus the notion of intrinsic property – in modal terms: given an object  $a$ , a property  $P$  is intrinsic iff all worlds having  $a$  as part agree on whether  $a$  has  $P$  or not. Consider the property of being six feet tall which is reasonably intrinsic; suppose that two worlds  $W_1$  and  $W_2$  are indistinguishable with respect to a individual  $a$  (i.e. there are two copies of  $a$  one in  $W_1$  and the other in  $W_2$ ): it is impossible for  $a$  to be six feet tall in  $W_1$  and five feet tall in  $W_2$ .

Intrinsic properties must not be confused with essential properties: when we say that the property ‘ $x$  is bald’ is intrinsic and that Socrates is bald in a world  $W$  we are not implying that Socrates is bald in every possible world in which he exists. We are only implying that all possible worlds that agree on which individual is the reference of the proper name ‘Socrates’ must agree on whether he is bald or not. Or in other words, it is impossible that the proper name ‘Socrates’ refers to the same individual in two worlds  $W_1$  and  $W_2$  and that he is bald according to  $W_1$  and non-bald according to  $W_2$ . Yet ‘Socrates’ may still refer to different individuals across two possible worlds  $W_3$  and  $W_4$  and thus ‘Socrates is bald’ may be true according to  $W_3$  and false according to  $W_4$ ; thus the property ‘ $x$  is bald’ may be intrinsic and ‘Socrates is bald’ may not be true in every possible world. Notice the difference: a property  $P$  is an essential property of ‘Socrates’ iff for every possible world  $W$ , whatever the reference of ‘Socrates’ in  $W$  might be, it must have  $P$ . Thus the distinction intrinsic vs extrinsic and the distinction essential

vs accidental are independent distinctions.

Concerning extrinsic properties, a consequence of Lewis' view is that to every extrinsic property corresponds an intrinsic property of a larger whole. Consider the sentence 'Adam is Fido's owner'; the property of being Fido's owner is clearly extrinsic. Consider now the mereological sum of Adam and Fido and call this new object  $AF$ . The relational property ' $x$  has a unique part  $y$  such that  $y$  is Fido and  $y$  is owned by the part of  $x$  supplementing  $y$ ' is intrinsic and  $AF$  has this property whenever Adam is Fido's owner.

The distinction between intrinsic and extrinsic properties has a correspondent distinction between relations: using a terminology introduced by Armstrong (1996) we distinguish between *internal and external relations*. Roughly speaking, we say that a relation  $R$  is internal iff  $R$  holds only in virtue of the intrinsic properties of its relata. For instance, the relation ' $x$  is taller than  $y$ ' is internal whereas the relation ' $x$  is within a mile of  $y$ ' is external. Again, we may characterize the distinction internal vs external in terms of the notion of locality: given two objects  $a$  and  $b$  and a relation  $R$ ,  $R$  is internal iff all possible worlds having  $a$  and  $b$  as parts agree on whether  $a$  and  $b$  are  $R$ -related or not. External relations are not local: two worlds may have the parts  $a$  and  $b$  in common and disagree on whether they stand in a certain external relation  $S$  or not.

As we will see in the next chapter, the distinctions spelled out in this section will play a fundamental role in the definition of the operation of dividing a content into function and argument.

## **Chapter 3**

# **Outline of a General Theory of Conceptual Content**

### **3.1 Introduction**

In this chapter I will propose a preliminary exposition of a general theory of conceptual content. It has no ambition of completeness, for many of the raised issues will be analyzed in details in the following chapters.

Our road to content will be traced by means of a quite unorthodox methodology: we will start by analyzing particular ways of regarding a content as complex entities, i.e. as a structured entity made of certain sort of constituents, and, from this standpoint, we will attempt to capture the notion of content as a simpler entity.

As we have seen in chapter 1, Frege suggests that the content expressed by a sentence may be divided into two fundamental categories of constituents: the functional part and the argument part. Such a division represents a natural interpretation of the fact that the content expressed by a sentence is a content of possible judgment: the functional part may be seen as standing for “what the sentence asserts” and the argument part for “the subject matter of the assertion”. The functional part is a function having as co-domain the set of truth-values and as domain the entities of which this part may be true of or false of; in Fregean terms, the functional part is a concept. Possibly, concepts may be of any order. The argument part is the entity to which the functional part applies, or in different terms, the entity that may fall or not under the concept

represented by the functional part. It is important to note that such a division applies to sentences of any logical form and not just to atomic predications; this depends on two facts: firstly, on the fact that – according to Frege’s account – any open formula of any order defines a concept; secondly, on the fact that any sentence may be seen as obtained from an open formula by performing suitable substitutions. A particular division of the content of a sentence into function and argument is a particular way of regarding the content and not the definition of the content itself; different dissections based on different function-argument distinctions should have something in common, something that is precisely the conceptual content expressed by the sentence.

*The purpose of this chapter is to provide a preliminary definition of the notion of content starting from the notion of ways of regarding a content based on the distinction between function and argument. The crucial point of the strategy is the characterization of the relation of logical analysis that – as shown in the first chapter – is a primary importance in the characterization of the relation of identity of content.* The preliminary definition of content will be formulated in three main steps:

1. The notion of way of regarding a content will be characterized in terms of the distinction between function and argument expressed on the semantic level,
2. The relation of logical analysis will be characterized as a semantic relation between different ways of regarding the same content as defined in the previous step,
3. The content of a sentence will be preliminary defined taking into account the fact that a certain way of regarding a content may be conceived as the result of a logical analysis of the content itself. Thus the content of a sentence will be preliminary defined as the “absolute analisandum”, i.e. as the notion that is not the result of any logical analysis of all ways of regarding the content and from which all ways of regarding the content result by logical analysis.

## **3.2 The constituents of the content**

In the very “UnFregean” theoretical setting of modal realism, the distinction between function and argument expressed on the semantic level deserves a bit of explanation. Let the sentence  $S$

be ‘Socrates is wise’; we may divide this sentence into the functor ‘ $x$  is wise’ and the argument expression ‘Socrates’. We want to “translate” this syntactic division into function and argument also on the semantic level; to do this, we have to consider a certain expression of the truth-conditions of the sentence  $S$ . We may try with the following one:

*For every possible world  $W$ ,  $S$  is true in  $W$  iff the extension of the proper name ‘Socrates’ in the world  $W$  satisfies the condition ‘ $x$  is wise’*

An easy way to represent the right-side of the latter bi-conditional as composed of function and argument is to introduce the function  $WISE(...)$  such that for every possible individual  $a$  has the value ‘True’ if  $a$  is wise, the value ‘False’ otherwise. On the other hand, the linguistic argument ‘Socrates’ is rendered as another function, i.e. the function that associates to every possible world  $W$ , the extension of ‘Socrates’ in  $W$ . We denote this function by the expression  $Socrates(W)$ . Therefore, the truth-conditions of  $S$  may be represented as:

For every possible world  $W$ ,

$S$  is true in  $W$  iff  $WISE(Socrates(W)) = True$

Note that the possibility of defining the function  $WISE(...)$  as a function from individuals to truth-values relies on the fact that the property of being wise is intrinsic; indeed, if this were not the case, different truth-values could have been associated to the same individual, thus making impossible to define the function  $WISE(...)$ . This is an important point: suppose that two worlds  $W_1$  and  $W_2$  are indistinguishable with respect to the extension of ‘Socrates’; if the property of being wise were not intrinsic, Socrates could have been wise in  $W_1$  and not wise in  $W_2$ . As a consequence, the truth-function associated with the property of being wise should have returned two different truth-values for the same argument, which is in contradiction with the definition of function.

This fact affects the expression of the truth-conditions according to the function/argument distinction whenever we are in presence of an extrinsic property (or external relation). Let  $P$  be an  $n$ -ary predicate standing for an external  $n$ -ary relation and suppose that  $P$  is applied to the singular terms  $a_1, \dots, a_n$ ; given that  $P$  stands for an external relation, the function  $F_P$  associated with  $P$  must include a world variable, i.e.

$$F_P(W, u_{a_1}(W), \dots, u_{a_n}(W)) = \text{True} \text{ iff } P(a_1, \dots, a_n) \text{ is true in } W$$

Where the functions  $u_{a_1}, \dots, u_{a_n}$  are the intensions respectively of the terms  $a_1, \dots, a_n$ .

The introduction of the world variable in the function associated with an extrinsic property is one of the main novelties that we encounter when we attempt to render the distinction between function and argument within the conceptual content as a distinction within the truth-conditions of a sentence in the theoretical setting of modal realism. Note that without the introduction of the world variable it would be impossible to define a function associated with an extrinsic property.

The fact that an open formula standing for an extrinsic property is rendered as a truth-function having a world argument place is not just a formal expedient. As seen in section 2.4, an object has an extrinsic properties in virtue of the way it interacts with the world. Thus extrinsic properties are *prima facie* relations between an object and the world of which it is a part. In this sense, it is perfectly natural to represent an extrinsic property as a function that receives a world-argument and an object-argument and returns a truth-value. However, one may object that in many cases an object  $a$  has an extrinsic property  $P$  not in virtue of the fact that  $a$  stands in a certain relation with the entire world, but with a specific part of the world; and such a specific part may be another object. So why in these cases don't we represent an extrinsic property as a function of two object-arguments having truth-values as possible values? An example may clarify the issue. Let  $S$  be the sentence 'Adam is Fido's owner' and suppose that we divide the expression of  $S$  into the functor ' $x$  is Fido's owner' and the argument expression 'Adam'. We may formalize the sentence by introducing an individual constant  $a$  for 'Adam' and a primitive non-logical predicate  $P$  for being Fido's owner. The expression of  $S$  in terms of function and argument is  $P(a)$ . According to the suggested way of representing extrinsic properties we should write: for every possible world  $W$ ,

$$S \text{ is true in } W \quad \text{iff} \quad F_P(W, u_a(W)) = \text{True} \quad (1)$$

However, the property of being Fido's owner is extrinsic not in virtue of the fact that Adam stands in a certain relation with the entire world, but in virtue of the fact that he stands in a certain relation (i.e. ownership) just with Fido. Thus, being  $f$  the individual constant 'Fido'

and  $O(x, y)$  the binary predicate ‘ $x$  owns  $y$ ’, the correct expression of the truth-conditions of  $S$  should be:

$$S \text{ is true in } W \quad \text{iff} \quad F_O(u_a(W), u_f(W)) = \text{True} \quad (2)$$

However, to say that (2) is more correct than (1) amounts to overlook the fact that (1) and (2) are different readings of the truth-conditions of  $S$  according to different distinctions between function and argument in the syntactic expression of  $S$ . Once a certain syntactic distinction between function and argument is drawn, we must draw a correspondent distinction on the semantic level. The reading (2) does more justice to our intuitions regarding what makes  $S$  true or false in a certain world: it is not the entire world and Adam, but Adam and Fido. However, if we have decided to read the syntax of  $S$  as  $P(a)$ , (2) is no more the correct reading: for when ways of regarding the content are to be defined, what is at stake is not the best way of expressing the truth-conditions of  $S$  according to what effectively makes  $S$  true or false, but the expression of the truth-conditions of  $S$  that matches with the distinction between function and argument that we have drawn on the syntactic level.

In general, we may characterize the division into function and argument of the truth-conditions of a sentence  $S$  as a functional decomposition of the intension  $S$ ; let  $\mathcal{C}_S$  be the intension of  $S$ , i. e. the function that associates to every possible world  $W$  the truth-value of  $S$  in  $W$ , and let  $\phi(a)$  be the expression of  $S$  with  $\phi(x)$  being an open formula chosen as function and  $a$  a singular term chosen as argument. Assuming that  $\phi(x)$  stands for an intrinsic property, the considered way of regarding the content of  $S$  may be expressed thus: for every possible world  $W$ ,

$$\mathcal{C}_S(W) = F_{\phi(x)}(u_a(W))$$

which is equivalent to:

$$\mathcal{C}_S = F_{\phi(x)} \circ u_a$$

which makes explicit the fact that a particular way of regarding a content of the sentence  $S$  may be represented as a functional decomposition of the intension of  $S$ . The case of  $\phi(x)$

standing for an extrinsic property or the case of functions with multiple arguments may be also considered as functional decompositions of the intension; the treatment of these cases is exposed in Appendix 3.1.

It is helpful to introduce a specific terminology for the components in which the intension of a sentence may be divided: with reference to Frege's distinction between function and argument, I will call *functional component*, the external component of the decomposition (e.g. the function  $F_{\phi(x)}$  of the previous example) and *argument component* the internal one (e.g. the function  $u_a$  of the previous example). A functional component is always a function from a certain set of entities to truth-values; an argument component is always a function from possible worlds to a certain set of entities.

For sake of clarity, I will use some notational conventions; if an expression  $\xi$  (or a group of expressions) appearing in a sentence  $S$  determines the functional component of a certain way of regarding the content of  $S$ , I will denote the functional component defined by  $\xi$  as  $F_\xi$ . On the other hand, if an expression  $\xi$  appearing in  $S$  is used to define the argument component of a particular way of regarding the content of  $S$ , I will denote by  $u_\xi$  the function that associates to each possible world  $W$  the extension of  $\xi$  in  $W$ . Note that the same expression may be used to determine different components of a way of regarding a content; for instance, a predicate  $P$  may be used to define the functional component  $F_P$ , thus indicating that the expression  $P$  is used to make a predication. Alternatively, the expression  $P$  may be read as standing for the entity on which a certain higher-order predication is made; in this case, it will be interpreted as the argument component denoted by  $u_{Px}$  (some examples will follow).

Another important remark is related to the kind of functions that we must admit. I have provided an example in which the functional part of the content mimic the Fregean notion of first-order concept, i.e. a function that assigns to an individual a truth value. However, functions of a different sort are also needed. For instance, we need truth functions associated to the logical connectives. Let  $p$  and  $q$  be two sentences, the conjunction of  $p$  with  $q$  may be expressed as follows: for every possible world  $W$ ,

$$F_\wedge(\mathcal{C}_p(W), \mathcal{C}_q(W)) = \text{True iff } p \wedge q \text{ is true in } W$$

where  $F_\wedge$  is the truth-function of the conjunction, and  $\mathcal{C}_p, \mathcal{C}_q$  are respectively the intensions of  $p$



and  $q$ .

Logical connectives will be implied with two different roles: the first one is when they are used to define functional components applied to truth-values (in the classical way); the second one is when they are used as operators for concept formation, as in the case of the concept ‘Human’ when conceived as formed by the conjunction of the concept ‘Animal’ and ‘Rational’. In the second case, I will use a different notation: instead of using the uppercase  $F$  with the symbol for the logical connective as index, I will use the lower-case  $f$ . For instance, the extension of the formula  $Px \wedge Qx$  in a world  $W$  may be denoted by the expression:

$$u_{Px \wedge Qx}(W) = f_{\wedge}(u_{Px}(W), u_{Qx}(W))$$

where  $f_{\wedge}$  represents nothing but the intersection operator (as well as  $f_{\vee}$  represents the union operator). Particular attention deserves the negation operator, i.e.  $f_{\neg}$ , which corresponds to the operation of relative complement; for instance, the extension of the formula  $\neg Px$  in a world  $W$  may be denoted by:

$$u_{\neg Px}(W) = f_{\neg}(W, u_{Px}(W))$$

where the operator  $f_{\neg}$  represent the operation of complement with respect to the domain of all objects existing in  $W$  (that is the reason why we have to introduce a world argument place).

Truth-functions associated with quantifiers are also needed. Let  $\phi(x)$  be an open formula having only  $x$  as free variable; we may express a particular way of regarding the content of the sentences  $\exists x\phi(x)$  as follows: for every possible world  $W$ ,

$$\exists x\phi(x) \text{ is true in } W \text{ iff } F_{\exists}(u_{\phi(x)}(W)) = \text{True}$$

where  $u_{\phi(x)}$  is the function that given a possible world  $W$ , returns the extension of the formula  $\phi(x)$  in  $W$ ; and  $F_{\exists}$  is the function that given a certain set  $A$  as argument, returns the value *True* if  $A$  is non-empty, the value *False* otherwise.

The case of universal quantification presents some formal differences. Consider the sentence

$\forall x\phi(x)$ ; a way of regarding its content may be expressed as follows: for all possible world  $W$ ,

$$\forall x\phi(x) \text{ is true in } W \text{ iff } F_{\forall}(W, u_{\phi(x)}(W)) = \text{True}$$

where the function of two arguments  $F_{\forall}$  when applied to a world  $W$  and a set  $A$ , returns the value *True* if  $A$  is identical to the domain of all individuals existing in  $W$ , the value *False* otherwise. Note the introduction of the world variable  $W$ ; it is justified by the fact that the property of being universally instantiated is an extrinsic property of a certain extension of  $\phi(x)$ : two worlds may perfectly agree on the extension of  $\phi(x)$  while having different domains, thus associating different truth values to the sentence ‘ $\phi(x)$  is universally instantiated’. The account of quantification may be naturally extended to higher-order cases<sup>1</sup>.

It is important to note that the same sentence may suggest different ways of regarding its content. All the functions I have introduced represent the “fundamental bricks” that we may use to draw different distinctions between the conceptual and the argument component within the truth-conditions of a sentence. For the moment nothing suggests that there is a unique preferred way of decomposition the intension of a sentence; in chapter 6 we will see that it is possible to frame a semantic criterion for determining a preferred way of regarding a content according to the considered sentence that expresses it.

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<sup>1</sup>Quantifiers may also play the role of operators for concept formation. Consider the open formula  $\exists y\phi(x, y)$ ; the existential quantifier may be seen as an operator that receives the extension of the relation  $\phi(x, y)$  and returns the extension of the concept  $\exists y\phi(x, y)$ ; more precisely, the existential quantifier may be seen as an operator that receives a set of ordered pairs and returns the set of elements of the domain that are  $\phi$ -related with any element of the domain. When quantifiers are used in such a way, we will use the lowercase  $f$  to denote the corresponding operator (as in the case of logical connectives implied in concept formation) and we will add an index to the symbol of the quantifier denoting the position of the variable that is being bonded. For instance, we may consider the following way of regarding the content of the sentence  $S$  having expression  $\forall x\exists y\phi(x, y)$ : for every possible world  $W$ ,

$$C_S(W) = F_{\forall}(W, f_{\exists_2}(u_{\phi(x,y)}(W)))$$

By a similar reasoning we may write a way of regarding the content of the sentence  $S$  having expression  $\exists x\forall y\phi(x, y)$  as follows: for every possible world  $W$ ,

$$C_S(W) = F_{\exists}(f_{\forall_2}(W, u_{\phi(x,y)}(W)))$$

Notice that the universal quantifier requires the world variable also in the case in which it is used as a concept formation operator.

Even though there is no absolutely preferred way of regarding the content of a given sentence, we may define for each logical form *a standard reading* that determines a standard way of regarding a content. Consider for instance the atomic sentence ' $Pa$ ' where ' $P$ ' is a first-order monadic predicate and ' $a$ ' is a singular term; it is correct to say that we may read the syntax of ' $Pa$ ' as composed by the higher-order function  $Xa$  (where  $X$  is a second-order variable) and the second-order argument  $Px$ . Yet it is far more natural and more in harmony with the original syntactic role of the expressions that constitute the sentence to read ' $Pa$ ' as composed by the function ' $Px$ ' and the argument  $a$ .

In Appendix 3.2 I propose a standard way of regarding the content firstly for atomic sentences and then recursively for the molecular ones. All the fundamental functions I have defined so far will be implied either to formulate the standard ways of regarding contents or to obtain different ways of regarding contents that in spite of being non-standard may represent some interesting reading of logical forms. What is crucial for our purpose is the fact it is possible to define a standard reading of the logical form of any possible sentence of a formal language in order to define a standard way of regarding a content that provides for all cases. The next step is to discover what this approach has to say concerning the relation between the constituents of two sentences having the same content.

### 3.3 A semantic interpretation of logical analysis

We have seen that given two sentences  $S$  and  $T$  such that  $T$  is the result of some unspecified operation of logical analysis of certain constituents of  $S$ ,  $S$  and  $T$  have the same content. Yet the existence of a logical analysis linking the two sentences cannot be adopted as a criterion for identity of content for we have no idea of what should be considered as logical analysis and how it should be detected looking at the syntactic form of the two sentences. Indeed, there may be many different ways in which one expression may be analyzed in terms others. *In this section, I will show that it is possible to characterize the operation of logical analysis in semantic terms in such a way that a common operational pattern may be defined for all different cases of syntactic logical analysis.*

Given the variety of cases considered under the hat of logical analysis, we will proceed by considering four fundamental examples that Frege would have probably regarded as cases of

genuine logical analysis and we will highlight what features they have in common.

Analysis is an asymmetric relation:  $A$  is the result of the analysis of  $B$  implies that  $B$  is not the result of the analysis of  $A$ . This because analysis has a specific direction: it goes from the complex to the simple, or from the whole to the parts. In this sense, we take the direction of analysis to be determined by a certain sort of operation of decomposition which for the moment will not be defined. The converse operation may be called *logical synthesis*: is the operation of combining two or more constituents of a sentence to obtain a single constituent. We will focus on the following cases:

1. 'Cato killed Cato' results from the logical analysis of 'Cato committed suicide',
2. 'Socrates is a rational animal' results from the logical analysis of 'Socrates is human',
3. 'Hydrogen bears the relation " $x$  is lighter than  $y$ " to Oxygen' results from the logical analysis of 'Hydrogen has the property of being lighter than Oxygen',
4. 'The direction of line  $a$  is identical to the direction of line  $b$ ' results from the logical analysis of 'Line  $a$  is parallel to line  $b$ '

The way I state the relation of analysis may appear to be unclear and to a certain extent controversial, thus some clarifications are needed and will be provided case by case.

Consider (1); firstly, note that the concept of suicide is defined in terms of the relation ' $x$  killed  $y$ ' by applying the condition that the two arguments must be identical, i.e. they must be two occurrences of the same argument. Consider now the operation of logical synthesis that in this case goes from the *definienda* to the *definiendum*: two occurrences of the argument 'Cato' have been "fused" and the resulting functional expression stands for the concept of committing suicide. Thus we may see analysis as the converse operation, i.e. the operation that divides a single occurrence of an argument into two occurrences. We classify this case as a case of *analysis of argument occurrence*.

Consider (2); the conceptual expression ' $x$  is a human' has been decomposed into the conceptual expressions ' $x$  is rational' and ' $x$  is an animal' connected by the conjunction operator. If we see (2) as a definition, again we see that the *definiendum* is analyzed in terms of the *definienda*. We classify this case as *traditional conceptual analysis*, i.e. the decomposition of the concepts into others by means of boolean operators. This is the case of the classical kantian examples.

Consider (3); the predicate ‘ $x$  is lighter than hydrogen’ is decomposed into the relational expression ‘ $x$  is lighter than  $y$ ’ and the term ‘hydrogen’. Considering (3) as a definition, the predicate ‘ $x$  is lighter than hydrogen’ may be seen as defined in terms of the relation ‘ $x$  is lighter than  $y$ ’ and the term ‘hydrogen’ by putting this term into the argument place marked by the variable  $y$  in the relation ‘ $x$  is lighter than hydrogen’. In other words, the operation of logical synthesis behind the definition consists in the partial saturation of a relational expression; conversely, the analysis may be seen as the introduction of an unsaturatedness in an expression that is regarded as partially saturated. For this reason, we classify this case as *an analysis by unsaturation*.

Consider (4); this is surely the weirdest case and the most controversial one under several respects. According to the passage in §64 of *the Foundations* quoted in the Introduction, the content of the relation of parallelism is composed by a “part” represented by the content of the relation of identity and another “part” that when divided between  $a$  and  $b$  results in two occurrences of the direction operator. Hence, we have an unspecified operation of decomposition of the content of the relation of parallelism. To such a decomposition a certain notion of conceptual analysis should correspond. Yet we immediately note that it is a quite unorthodox notion of conceptual analysis: note the difference with (2): in this case a conceptual expression is not decomposed into others connected by boolean operators, yet into another (relational) conceptual expression and two occurrences of a functor. I will classify this case as *unorthodox conceptual analysis*. Notice that there is also another fundamental difference between (4) and the other cases of analysis: if we regard (4) as a definition, we see that the direction operator is defined in terms of the relation of parallelism. However, the conceptual analysis goes in the opposite direction of the relation between the *definiendum* and the *definiens*: parallelism is analyzed in terms of identity of directions, while the direction operator is defined in terms of parallelism.

As we have seen, the four provided examples of analysis have been classified under four different conceptions corresponding to four different syntactic operations: analysis of argument occurrence, traditional conceptual analysis, analysis by unsaturation, and unorthodox conceptual analysis. *Our purpose is to show that there is a unique semantic pattern of analysis that unifies all four different operations; this result will be achieved by considering all standard ways of regarding the contents of the sentences<sup>2</sup> appearing in clauses (1)-(4) and defining the semantic pattern of analysis as a relation between ways of regarding a content.*

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<sup>2</sup>see Appendix 3.2 at the end of the chapter

To express the ways of regarding the contents of the sentences appearing in clauses (1)-(4), it is helpful to formalize these sentences:

1. ‘Cato killed Cato’ and ‘Cato committed suicide’ will be formalized respectively as ‘ $cKc$ ’ and ‘ $Sc$ ’,
2. ‘Socrates is a rational animal’ and ‘Socrates is human’ will be formalized respectively as ‘ $Rs \wedge As$ ’ and ‘ $HS$ ’,
3. ‘Hydrogen bears the relation “ $x$  is lighter than  $y$ ” to Oxygen’ and ‘Hydrogen has the property of being lighter than Oxygen’ will be formalized respectively as ‘ $L(h, o)$ ’ and ‘ $L_o(h)$ ’,
4. ‘The direction of line  $a$  is identical to the direction of line  $b$ ’ and ‘Line  $a$  is parallel to line  $b$ ’ will be formalized respectively as ‘ $d(a) = d(b)$ ’ and ‘ $a \parallel b$ ’.

The standard ways of regarding the contents at issue are listed in the following:

1. Functional component of ‘ $cKc$ ’:

$$F_k$$

argument component of ‘ $cKc$ ’:<sup>3</sup>

$$(u_c ; u_c)$$

Functional component of ‘ $Sc$ ’:

$$F_S$$

argument component of ‘ $Sc$ ’:

$$u_c$$

where we have reasonably assumed that to commit suicide (or to kill himself) is a fact intrinsically about Cato (i.e. Cato of one world may be the intrinsic duplicate of Cato of

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<sup>3</sup>For the reader who decided to skip Appendix 3.1, the used notation is the following. Given two functions  $f_1$  and  $f_2$  having the same domain, the function  $(f_1 ; f_2)$  is defined as:

$$(f_1 ; f_2)(x) = \langle f_1(x), f_2(x) \rangle$$

a different world and both Cato committed suicide)

2. Functional component of ' $Rs \wedge As$ ':

$$F_{\wedge} \circ (F_R ; F_A)$$

argument component of ' $Rs \wedge As$ ':

$$u_s$$

Functional component of ' $HS$ ':

$$F_H$$

argument component of ' $HS$ ':

$$u_s$$

3. Functional component of ' $L(h, o)$ ':

$$F_L$$

argument component of ' $L(h, o)$ ':

$$(u_h ; u_o)$$

Functional component of ' $L_o(h)$ ':

$$F_{L_o}$$

argument component of ' $L_o(h)$ ':

$$(I_{\mathbb{W}} ; u_h)$$

where  $I_{\mathbb{W}}$  stands for the identity function on the set of possible worlds  $\mathbb{W}$ . Notice that we need a world variable in this case (introduced by the identity function  $I_{\mathbb{W}}$ ), for to be lighter than oxygen is not a fact intrinsically about hydrogen.

4. Functional component of ' $d(a) = d(b)$ ':

$$F_{=}$$

argument components of ' $d(a) = d(b)$ ':

$$(u_{d(a)} ; u_{d(b)})$$

Functional component of ' $a \parallel b$ ':

$$F_{\parallel}$$

argument component of ' $a \parallel b$ ':

$$(u_a ; u_b)$$

An interesting characteristic of logical analysis may be determined by comparing the argument components of two sentences being one the result of the analysis of the other.

In the case of (1), the difference between the argument components consists in a multiple occurrence of  $u_c$ ; thus in spite of the fact that the argument components do not coincide, the entities with which the correspondent ways of regarding the content are concerned are the same. In the case of (2) the argument components are identical and the only difference between the ways of regarding the content is just the way the functional components are defined. Cases (3) and (4) present an alteration of the argument component through logical analysis. How this difference may be characterized?

I will show that the relation between the argument component of the way of regarding the content to be analyzed and the argument component of the way of regarding the content resulting from the analysis may be characterized in terms of a relation of *proper subpartitioning* between the partitioning of the logical space defined by the former argument component and the partitioning defined by the latter (see section 2.3.2). Recall that the intension  $u$  of an expression (or a series of intensions of a series of expressions) may be seen as defining an equivalence relation on worlds, and the partitioning of the logical space associated with  $u$  is the set of all equivalence classes determined by such equivalence relation. Moreover, given two intensions (or two series of intensions)  $u$  and  $u^*$ , recall that we say that the partitioning defined by  $u$  is a proper subpartitioning of the partitioning defined by  $u^*$  iff

- (i) For every two possible worlds  $W, V$ , if  $u(W) = u(V)$ , then  $u^*(W) = u^*(V)$ ,
- (ii) There are at least two worlds  $W, V$  such that  $u^*(W) = u^*(V)$  and  $u(W) \neq u(V)$



Consider case (3): the argument component of the way of regarding the content to be analyzed is  $\langle I_{\mathbb{W}}, u_h \rangle$  and that of the way of regarding the content resulting from the proposed analysis is  $\langle u_h, u_o \rangle$ . Suppose that the former argument component has the same value for two possible worlds  $W$  and  $V$ , i.e.

$$\langle I_{\mathbb{W}}(W), u_h(W) \rangle = \langle I_{\mathbb{W}}(V), u_h(V) \rangle$$

Considering that  $I_{\mathbb{W}}$  is the identity function, the argument component  $\langle I_{\mathbb{W}}, u_h \rangle$  has the same value for  $W$  and  $V$  just in case  $W$  is identical to  $V$ . From the fact that  $W$  is identical to  $V$  it follows that:

$$\langle u_h(W), u_o(W) \rangle = \langle u_h(V), u_o(V) \rangle$$

which proves that condition (i) holds by replacing  $u$  with  $\langle I_{\mathbb{W}}, u_h \rangle$  and  $u^*$  with  $\langle u_h, u_o \rangle$ . Given that surely there are two distinct world  $W$  and  $W'$  such that

$$\langle u_h(W), u_o(W) \rangle = \langle u_h(W'), u_o(W') \rangle$$

condition (ii) is also satisfied by the same replacements. Therefore  $\langle I_{\mathbb{W}}, u_h \rangle$  defines a partitioning of the logical space that is a proper subpartitioning of the partitioning defined by  $\langle u_h, u_o \rangle$ .

Case (4) is less straightforward. Suppose that ‘line  $a$ ’ and ‘line  $b$ ’ refer to the same objects across two possible worlds  $W, V$ , i.e. suppose that

$$\langle u_a(W), u_b(W) \rangle = \langle u_a(V), u_b(V) \rangle$$

If, for instance, the same object is the extension of ‘line  $a$ ’ across two possible worlds, and given that the object corresponding to its direction is determined by intrinsic properties of line  $a$ , then there is no reason why the direction of line  $a$  in the world  $W$  should be different from the direction of line  $a$  in  $V$ . In other words, there is nothing over and above the way lines are with respect to their orientational properties that determines which objects their directions are. The same applies to the case of line  $b$ . As a consequence,

$$\langle u_{d(a)}(W), u_{d(b)}(W) \rangle = \langle u_{d(a)}(V), u_{d(b)}(V) \rangle$$

Thus condition (i) holds by replacing  $u$  with  $\langle u_a, u_b \rangle$  and  $u^*$  with  $\langle u_{d(a)}, u_{d(b)} \rangle$ . Notice that the converse implication does not hold; for if two worlds agree on which objects ‘the direction of  $a$ ’ and ‘the direction of  $b$ ’ refer to, by no means they are forced to agree on which objects ‘line  $a$ ’ and ‘line  $b$ ’ refer to: for different pairs of lines may be associated with the same pair of directions. Thus also condition (ii) holds with the same replacements. As a consequence, the partitioning of the logical space defined by  $\langle u_a, u_b \rangle$  – i.e. the argument component of the way of regarding the content to be analyzed – is a proper subpartitioning of the partitioning defined by  $\langle u_{d(a)}, u_{d(b)} \rangle$ , i.e. the argument component of the way of regarding the content resulting from the proposed analysis.

To summarize: in case (1) we have just an alteration of number of occurrences of an argument ; in case (2) there is no alteration of the argument component through logical analysis; thus the way the argument components partition the logical space is unaltered. In cases (3) and (4) there is a relation of proper subpartitioning between the partitioning defined by the argument component of the way of regarding the content to be analyzed and the partitioning of the argument component of the way of regarding the content resulting from the proposed analyses.

*In general, we may say that whenever two ways of regarding the content are in the relation of logical analysis the partitioning of the logical space defined by the argument component of the way of regarding the content to be analyzed is a subpartitioning (proper in (3) and (4), improper in (1) and (2)) of the partitioning defined by the argument component of the way of regarding the content resulting from the analysis.*

The interesting fact is that we have pinpointed a common semantic pattern for four different kinds of logical analysis that also Frege would have acknowledged as such, i.e. analysis of argument occurrence, analysis of saturation, traditional conceptual analysis, and unorthodox conceptual analysis. To the extent that we want our theory of conceptual content to be in line with Frege’s hints on the matter, it does not seem that there are other sorts of analysis that we should consider.

We may preliminary attempt the formulation of a condition characterizing the relation of logical analysis:

### Logical analysis (LA)

Let  $S$  and  $T$  be two sentences, and let the functional compositions  $F_1 \circ u_1$  and  $F_2 \circ u_2$  be two standard ways of regarding the content of respectively  $S$  and  $T$ . The sentence  $T$  is the result of logical analysis of  $S$  iff:

- (i)  $F_1 \circ u_1 = F_2 \circ u_2$  (*Identity of intension*)
- (ii) For every possible world  $W$ ,  $[W]_{u_1} \subseteq [W]_{u_2}$  (*Partitioning inclusion*)

According to the given considerations, the operation of logical analysis may be conceived as an operation that while leaving unaltered the intension of the sentence on which the analysis is performed, either produces an enlargement of the cells in which the logical space is partitioned by the argument component or leaves such cells unaltered. How can we understand this fact?

In section 2.3.2 we have seen that a partitioning of the logical space may be used to represent a certain subject matter, where a subject matter is intended as a specific portion of a possible world. Moreover, we have seen that subject matters are comparable in size: whenever a subject matter  $m_1$  is represented as a subpartitioning of a subject matter  $m_2$ , it results that  $m_2$  is a subject matter included in  $m_1$ , i.e. the portion of world with which  $m_1$  is concerned contains the portion of world with which  $m_2$  is concerned. If we apply these considerations to the present cases, we may conclude that *the operation of logical analysis insofar as it produces an enlargement of the cells in which the argument component is concerned, may be interpreted as a restriction of the subject matter with which the argument component is concerned.*

When we pass from lines to directions, we are no more considering lines in their entirety, but we are rather concerned with a particular aspect of lines: their directions. When we analyse the predicate ‘ $x$  is lighter than oxygen’ into the relation ‘ $x$  is lighter than  $y$ ’ and the term ‘oxygen’, we pass from considering an extrinsic property of hydrogen – thus a property whose assessment is world relative – to an internal relation between oxygen and hydrogen; if in the former case we needed to be concerned with the entire world to determine the truth-value, in the latter one we are concerned just with hydrogen and oxygen.

Why logical analysis should imply a restriction (proper or improper) of the portion of worlds with which the argument component is concerned?

To answer this question, we must regard the considered cases of analysis from a different perspective. In all cases (1)-(4) the operation of analysis presents the common feature of altering the expression of what is regarded as function in the syntax of the sentence. More precisely, a certain conceptual expression is decomposed in a way that may be different case by case. In (1) and (2), the entire operation of analysis amounts to a change in the expression of the concept that is considered: ' $x$  committed suicide' is analyzed as ' $x$  killed  $x$ ', ' $x$  is a human' is analyzed as ' $x$  is a rational animal'. In these cases the argument part is either left unaltered or is altered without producing any change in the portion of world with which the argument components are concerned. The analysis in these cases is merely conceptual. Cases (3) and (4) are different. In cases (3) and (4), the expression representing the function of the sentence to be analyzed is decomposed into several expressions, yet not all of these expressions contribute to define the new function of the analyzed sentence: some of them are used to constitute new arguments. In case (3), ' $x$  is lighter than oxygen' is decomposed into ' $x$  is lighter than  $y$ ' and 'oxygen'; however, 'oxygen' is no more considered as part of the function: it is used to form a new argument. In case (4), ' $x$  is parallel to  $y$ ' is decomposed into ' $\xi$  is identical to  $\zeta$ ' and two occurrences of 'the direction of  $x$ '; however, 'the direction of  $x$ ' is no more part of the function, it is used to define the new argument. In both cases, the expressions resulting from the decomposition of the functional part that are used to form the new arguments are implied as *operators of specification*: the term 'oxygen' that is removed from the predicate ' $x$  is lighter than oxygen' is used to specify the part of a possible world on which the extrinsic property 'lighter than oxygen' depends; the two occurrences of the direction operator that are removed from the relation of parallelism are used to specify the aspect of lines that determines their parallelism. Thus it is in virtue of such an operation of argument specification that is complementary to that of concept decomposition that the portion of world with which the analyzed sentence is concerned is more circumscribed.

It is worth noting that the characterization of logical analysis in terms of the relation of inclusion of the cells of the partitioning between two different argument components has been given with an important premise: the ways of regarding the content of the two sentences standing in the relation of analysis, must be standard. This premise suggests that we may not be able to spot the specified semantic pattern if we consider some ways of regarding the content that are in a certain sense "spurious" or not transparent. Moreover, one may object that the criterion for determining standard ways of regarding a content is to a certain extent conventional and

dependent exclusively on the syntactic structure of the sentence. And the syntactic structure of the sentence may not be entirely transparent with respect to the entities that are fundamental for the determination of the truth-value of the considered sentence. Thus by no means standard ways of regarding a content should be considered as preferred ways for detecting the pattern of analysis, which is purely semantic. What is missing is a semantic criterion for determining preferred ways of regarding a content; in the following chapters we will see how to characterize these preferred ways of regarding a content.

In the next section I will propose a preliminary characterization of the notion of content based on the individuated semantic pattern produced by logical analysis.

### **3.4 From logical analysis to content**

In chapter 1 I have shown that identity of conceptual content may be characterized either as a relation of logical analysis or in terms of a necessary semantic relation between the expressions appearing in the sentences identical in content. For this reason in the present chapter the attention has been turned on the notion of logical analysis; given the varieties of syntactic procedures that may be considered as logical analysis, I have proposed a characterization in semantic terms, which led me to formulate a criterion for logical analysis expressed by (LA). In this section my purpose will be to show how the proposed conception of logical analysis may shed new light on the notion of content itself.

According to the results of the discussion of the previous section, the more a certain content – already presented as divided into function and argument – is analyzed, the coarser the way its argument component partitions the logical space (i.e. the larger the cells composing the partitioning). Moreover, the more analyzed a content is, the more circumscribed the portion of a possible world with which a sentence representing the analysis of the given content is concerned. I have provided a preliminary rationale of this fact by showing that the operation of logical analysis amounts to decompose the functional part of a sentence into a new functional part and an operator of specification that is applied to the former argument to form a new more specific argument.

From this fact we may formulate a preliminary working hypothesis on the notion of content: the conceptual content may be conceived as the least analyzed (or perhaps not analyzed at all)

expression of the truth-conditions of a sentence. As a consequence, *the content of a sentence should be concerned with the largest relevant portion of a possible world; or, alternatively, should determine the finest partitioning of the logical space.*

Taking this view to the extreme consequences, one may think that the content of a sentence is the expression of its truth-conditions in which the entire world appears as value of the argument component. Yet this is in contrast with the results of chapter 1: to say that the content has the entire world as value of the argument component is to say that the content is a function that applies to possible worlds and returns truth-values, i.e. the conceptual content must be identified with the intension; this option has been already ruled out. Given that the reason for ruling out such an identification is grounded on a sound interpretation of Frege's account, one may think that a characterization of the notion of content going beyond the mere exegetic task may well accept to take the content of a sentence to be its intension. Nonetheless, this amounts to throw away the baby with the bathwater: although the purpose of this chapter is not exegetical, some fundamental features of the notion of conceptual content emerging from Frege's account should be preserved, otherwise there would be no reason for using the same terminology and – more significantly – we may lose the theoretical advantages of the notion of content presented in the previous chapter.

Therefore, we should hypothesize that the content of a sentence differs from its intension insofar as the content is concerned with some maximal portion of a possible world but not with a possible world in its entirety, where “maximal” means that the portion of world with which the content is concerned is more comprehensive than any portion of world with which a particular analysis of the same content is concerned. Thus even if there may be two sentences having the same truth-value in all possible worlds, they may have not the same truth-value in the relevant portion of possible worlds; portions of possible worlds large enough to play the role of argument components of the content trace distinctions where entire possible worlds do not. I call these large portions of worlds *partial worlds*. Consider the sentences ‘ $5 = 5$ ’ and ‘ $2 + 2 = 4$ ’; both are necessary truths, i.e. true in all possible worlds. However, they are not true in the same partial worlds: a partial world containing the series of natural numbers up to 4 makes true only the latter sentence. Thus the fundamental semantic unit necessary to characterize the content of a sentence is a certain notion of partial world, rather than a possible world. According to this view, the content of a sentence may be seen as a function that assigns a truth-value to partial worlds. As a consequence, we may write the truth-conditions of a sentence  $p$  as: for every

possible partial world  $s$ ,

$$\mathcal{I}_p^*(s) = \text{True iff } p \text{ is true in the partial world } s$$

where the function  $\mathcal{I}_p^*$  represents the content of  $p$ .

As briefly suggested, such a definition of content is helpful in tracing distinctions between sentences having the same intension and differing in content, at least according to our intuitions: sentences having the same truth-value profile across total worlds may have different truth-value profiles across partial worlds. However, there is a fundamental aspect of the notion of content that is not captured by the definition. We have come to this definition by reasoning on the notion of logical analysis; more precisely, we have characterized logical analysis from a semantic point of view and we have reasonably supposed that the content expressed by a sentence is the least analyzed (or perhaps not analyzed at all) expression of the truth-conditions. So to say, the content is *the absolute analysandum*. A crucial point of this train of thought is represented by the fact that the relation of analysis when defined in a purely semantic way, does not apply to sentences but either to ways of regarding a content or to the content itself. Thus given a certain way of regarding the content of a sentence  $p$  and its content, it must be possible to show that the way the content of  $p$  is regarded stands in the relation of logical analysis with the content of  $p$ , where the relation of logical analysis is defined by (LA). However, (LA) is formulated in terms of the relation of subpartitioning between the way two argument components partition the logical space; yet when we define the content of a sentence as a function from all possible partial worlds to truth-values there is no argument component that may be compared with a particular way of regarding a content, thus (LA) cannot be applied to show that a particular way of regarding a content is the result of an analysis performed on the content itself.

The general idea to overcome this difficulty is that we may represent also the content expressed by a sentence  $p$  as having both a functional component and an argument component. To this end, we may consider instead of the entire function  $\mathcal{I}_p^*$ , a special restriction of it to the set of partial worlds that are *maximally relevant*. More precisely, we may suppose that given a possible world  $W$ , the content of  $p$  is concerned with a special portion of  $W$ , precisely the part of  $W$  containing all constituents with which all ways of regarding the content of  $p$  may be possibly concerned. According to this remark, we may see the content of  $p$  as composed by an argument component that given a possible world  $W$  returns the portion of  $W$  containing all constituents with which

all ways of regarding the content of  $p$  may possibly be concerned, and a functional component that given such a portion of  $W$  returns the value true if all sentences expressing the content of  $p$  are true in this partial world, the value false otherwise. In particular, I will denote by ' $\mathcal{I}_p$ ' the functional component of the content – or *content function* – and by ' $\sigma$ ' the argument component of the content, or *partial world function*. Notice that  $\mathcal{I}_p$  is nothing but the restriction of  $\mathcal{I}_p^*$  to the domain of  $\sigma$ , i.e. to the set of partial worlds containing all constituents with which all ways of regarding the content of  $p$  may possibly be concerned.

According to these preliminary considerations, we may write the truth-conditions of  $p$  as follows: for every possible world  $W$ ,

$$\mathcal{I}_p(\sigma(W)) = \text{True} \text{ iff } p \text{ is true in the } \sigma\text{-portion of the world } W \quad (\text{CC})$$

We may use the function  $\sigma$  to establish an interesting relation between content and intension: for any possible world  $W$ ,

$$\mathcal{I}_p(\sigma(W)) = \mathcal{C}_p(W)$$

where  $\mathcal{C}_p$  denotes the intension of  $p$ .

According to the given considerations, we may define a content as a pair of functions  $\langle \mathcal{I}, \sigma \rangle$  where  $\mathcal{I}$  is the content function and  $\sigma$  is the partial world function. From this standpoint, we may define the condition that a sentence  $p$  must satisfy in order to express the content  $\langle \mathcal{I}, \sigma \rangle$ . Suppose that the expression of  $p$  is such that the following way of regarding the content of  $p$  is determined: for every possible world  $W$ ,

$$\mathcal{C}_p(W) = F(u(W))$$

The reasonable assumption on which we rely is the fact that  $p$  expresses the content  $\langle \mathcal{I}, \sigma \rangle$  iff the way of regarding the content of  $p$  determined by the expression of  $p$  is an analysis of the content  $\langle \mathcal{I}, \sigma \rangle$ . According to the semantic pattern of logical analysis presented in the previous section, we establish the following criterion:

### **Content Expression (CE)**



The sentence  $p$  whose content may be regarded – according to its syntactic expression – as having functional component  $F$  and argument component  $u$ , expresses the content  $\langle \mathcal{I}, \sigma \rangle$  iff

- (i)  $F \circ u = \mathcal{I} \circ \sigma$  (*Identity of Intentions*)
- (ii) For every possible world  $W$ ,  $[W]_\sigma \subseteq [W]_u$  (*Analysis*)

In other words, a sentence  $p$  expresses the content  $\langle \mathcal{I}, \sigma \rangle$  iff the way of regarding the content of  $p$  according to the syntactic expression of  $p$  is the result of the analysis of  $\langle \mathcal{I}, \sigma \rangle$ .

To define the content as the ordered pair composed by the content function and the partial world function seems to satisfy two desiderata of the notion of content:

- The fact that the content of a sentence  $p$  should be more fine grained than the intension of  $p$  and more coarse grained than a particular way of regarding the content of  $p$ ,
- A particular way of regarding the content of  $p$  should be the result of an analysis of the content of  $p$ .

Thus our definition of content (CC) appear at first glance to be satisfactory. The only point that is still unclear is how to define the partial world function, i.e. given a possible world  $W$  how to determine the portion of  $W$  with which the content is concerned. This gap will be filled in the next chapter.

Even without a definition of the partial world function  $\sigma$ , we may prove a fundamental property of this function, i.e. *maximality*. In particular, I will show that two contents associated with the same intension cannot be concerned with portions of the world one included in the other. Or in different words, if a content having an intension  $\mathcal{C}$  is concerned with a portion  $A$  of the world  $W$ , then every proper part of  $A$  is not a partial world with which a content having intension  $\mathcal{C}$  may be concerned. I call this fact *the relative maximality of partial worlds with which the content is concerned*. The adjective ‘relative’ stands for the fact that the maximality is relative to a certain intension, i.e. to a certain truth-value profile or truth-conditions.

Suppose that two sentences  $p$  and  $q$  are logically equivalent (i.e. identical in intension) while differing in content. More precisely, suppose that  $\langle \mathcal{I}_p, \sigma_p \rangle$  is the content expressed by  $p$  and  $\langle \mathcal{I}_q, \sigma_q \rangle$  is the content expressed by  $q$ , with  $\sigma_p \neq \sigma_q$ . Moreover, suppose that for every possible world  $W$ ,  $[W]_{\sigma_p} \subseteq [W]_{\sigma_q}$ , i.e. the partitioning of the logical space defined by  $\sigma_p$  is a subpartitioning of that defined by  $\sigma_q$ . Assume that  $p$  and  $q$  are associated with certain ways of

regarding their contents: suppose that  $F_p$  ( $F_q$ ) is the functional component of  $p$  ( $q$ ) and  $u_p$  ( $u_q$ ) the argument component of  $p$  ( $q$ ).

Given that  $q$  expresses the content  $\langle \mathcal{I}_q, \sigma_q \rangle$ , it follows that the way  $q$  regards its content results from an analysis of the content itself; by applying (LA) it follows that for every possible world  $W$ ,  $[W]_{\sigma_q} \subseteq [W]_{u_p}$ ; combining this fact with the hypothesis on the partial world functions of the contents at issue, it follows that for every possible world  $W$ ,  $[W]_{\sigma_p} \subseteq [W]_{u_q}$ . By the identity of intensions between  $p$  and  $q$ , it follows that  $\mathcal{I}_p \circ \sigma_p = F_q \circ u_q$ . Thus by (CE),  $q$  expresses the content  $\langle \mathcal{I}_p, \sigma_p \rangle$ , i.e.  $p$  and  $q$  have the same content, which contradicts the hypothesis.

To avoid such a contradiction, we cannot admit the fact that two contents associated with the same intension are concerned with two portions of a possible world such that one includes the other. From this conclusion, we derive the *relative maximality of partial worlds with which a content may be concerned*: given a partial world with which a certain content is concerned, no part of it may be a partial world with which a content associated with the same truth-conditions (or intension) is concerned.

### 3.5 Conclusion

In this chapter the fundamental notions of the proposed theory of content has been introduced. We started from the notion of *way of regarding a content* that has been characterized according to Frege's distinction between function and argument. Such a distinction has been rendered on the semantic level using as theoretical setting the concretist conception of possible world. I have shown that the truth-conditions of every sentence may be expressed as a functional decomposition of the intension, and I have called the components of such decomposition respectively *functional component* and *argument component*.

The next step was to characterize the relation of logical analysis as a relation between ways of regarding a content. The definition (LA) represents the characterization of this relation: two ways of regarding a content stand in the relation of logical analysis iff they determine the same intension and the partitionings of the logical space defined by their argument components stand in a relation of subpartitioning.

With a semantic characterization of the relation of logical analysis in hand, I have defined the

notion of content as that particular decomposition of the intension that is the analysandum of any possible way of regarding the content. This leads me to introduce the notion of the partial worlds with which the content is concerned.

It is worth highlighting some important aspects of the resulting notion of content. Firstly, the reader may have noted that we are dealing with a representational notion of content; in other words, the notion of content we are presenting is a metaphysical notion and neither an epistemic nor a purely semantic one.

Now that a preliminary definition of content has been formulated, we may come back to the main purpose of this work, i.e. the understanding of Frege's procedure of content carving both inside and outside the context of Frege's philosophy of mathematics. To achieve this goal there are several open issues that will be addressed in the following chapters:

- The definition of the partial world function  $\sigma$ ,
- The definition of the operations of content carving and recarving,
- The distinction between ways of regarding the content that are transparent with respect to the content itself and ways that are not,
- The formulation of a criterion of identity of content,
- The explication of Frege's description of the procedure of carving up again the same content in a different way.

## Appendix 3.1: Multivariable cases

In this appendix I will give some general guidelines to express the way of regarding the content of a sentence already read as composed by function and argument as a functional decomposition of the intension of the sentence itself. In particular, I will focus on more complex cases, i.e. cases of  $n$ -predication with  $n$  greater than 1 and cases of extrinsic properties/external relations.

Let  $S$  be a sentence and suppose that the expression of  $S$  is regarded as composed by the open formula  $\phi(x_1, \dots, x_n)$  as functional part applied to terms  $a_1, \dots, a_n$  which are considered as arguments. Hence, we may represent the syntactic expression of  $S$  as  $\phi(a_1, \dots, a_n)$ . Suppose

that the open formula  $\phi(x_1, \dots, x_n)$  stands for an internal relation. We may express the truth-conditions of  $S$  according to the considered way of regarding its content as follows: for every possible world  $W$ ,

$$\mathcal{C}_S(W) = F_{\phi(x_1, \dots, x_n)}(u_{a_1}(W), \dots, u_{a_n}(W)) \quad (*)$$

We may see  $(*)$  as a functional decomposition of the intension. To this end, we may define the “vector-like” function  $\vec{u}$  defined as follows. For every possible world  $W$ ,

$$\vec{u}(W) = \langle u_{a_1}(W), \dots, u_{a_n}(W) \rangle$$

We may abbreviate the definition of  $\vec{u}$  by using the following compact notation:

$$\vec{u} = (u_{a_1} ; \dots ; u_{a_n})$$

The function  $\vec{u}$  associates to a possible world  $W$  the  $n$ -ple of extensions respectively of  $a_1, \dots, a_n$  in  $W$ . As a consequence, the intension of  $S$  may be decomposed as follows:

$$\mathcal{C}_S = F_{\phi(x_1, \dots, x_n)} \circ \vec{u}$$

Therefore, in this case the functional component is the function  $F_{\phi(x_1, \dots, x_n)}$  and the argument component is the function  $\vec{u} = (u_{a_1} ; \dots ; u_{a_n})$ .

The case of an external relation may be treated in a similar way. Let  $S$  be a sentence whose expression has been read as  $\phi(a_1, \dots, a_n)$  with  $\phi(x_1, \dots, x_n)$  standing for an external  $n$ -ary relation and  $a_1, \dots, a_n$  singular terms. According to the considerations of section 3.2, the truth-conditions of  $S$  considered as composed by  $\phi(x_1, \dots, x_n)$  and  $a_1, \dots, a_n$  may be written as follows: for every possible world  $W$ ,

$$\mathcal{C}_S(W) = F_{\phi(x_1, \dots, x_n)}(W, u_{a_1}(W), \dots, u_{a_n}(W))$$

We may use the following mathematical trick to write the argument component of the proposed

way of regarding the content of  $S$ . Let  $\mathbb{W}$  be the class of all possible worlds and let  $I_{\mathbb{W}}$  be the identity function on  $\mathbb{W}$ . We define the following vector-like function:

$$\vec{u}^* = (I_{\mathbb{W}}; u_{a_1}, \dots; u_{a_n})$$

The function  $\vec{u}^*$  associates to a world  $W$  the  $(n + 1)$ -ple composed by the world  $W$  and the extensions respectively of  $a_1, \dots, a_n$  in  $W$ . Thus the intension of  $S$  may be expressed as:

$$\mathcal{C}_S = F_{\phi(x_1, \dots, x_n)} \circ \vec{u}^*$$

## Appendix 3.2: Standard ways of regarding a content

In this Appendix standard readings in terms of function and argument of sentences of a formal language are recursively defined.

### Atomic sentences:

Let  $S$  be the sentence  $Pa_1 \dots a_n$ , with  $P$   $n$ -ary predicate and  $a_1, \dots, a_n$  singular terms. The *standard way of regarding the content of  $S$*  is the following:

- (i) As functional component the function  $F_P$  and as argument component the function  $\vec{u} = \langle u_{a_1}, \dots, u_{a_n} \rangle$  (in case of  $P$  standing for an intrinsic property)
- (ii) As functional component the function  $F_P$  and as argument component the function  $\vec{u}' = \langle I_{\mathbb{W}}, u_{a_1}, \dots, u_{a_n} \rangle$  where  $I_{\mathbb{W}}$  is the identity function on the set  $\mathbb{W}$  of worlds in which  $S$  has a defined truth-value (in case of  $P$  standing for an extrinsic property)

Once a standard way of regarding the content is defined for atomic sentences I will pass to define it for molecular sentences obtained by using propositional connectives.

### Molecular sentences:

- (i) Let  $S$  be the sentence  $\phi \wedge \psi$ ; let  $F_{\phi}, F_{\psi}$  be the functional components of the standard ways of regarding the content of respectively  $\phi$  and  $\psi$ . Moreover, let  $\vec{u}_{\phi}$  and  $\vec{u}_{\psi}$  be the

correspondent argument components. The *standard way of regarding the content of  $S$*  admits

$$F_{\wedge} \circ \langle F_{\phi}, F_{\psi} \rangle \text{ as functional component}$$

$$\langle \vec{u}_{\phi}, \vec{u}_{\psi} \rangle \text{ as argument component}$$

(ii) If  $S$  is  $\phi \vee \psi$ , under the previous assumptions the *standard way of regarding the content of  $S$*  admits:

$$F_{\vee} \circ \langle F_{\phi}, F_{\psi} \rangle \text{ as functional component}$$

$$\langle \vec{u}_{\phi}, \vec{u}_{\psi} \rangle \text{ as argument component}$$

(iii) If  $S$  is  $\phi \rightarrow \psi$ , under the previous assumptions the *standard way of regarding the content of  $S$*  admits:

$$F_{\rightarrow} \circ \langle F_{\phi}, F_{\psi} \rangle \text{ as functional component}$$

$$\langle \vec{u}_{\phi}, \vec{u}_{\psi} \rangle \text{ as argument component}$$

(iv) If  $S$  is  $\neg\phi$ , under the previous assumptions the *standard way of regarding the content of  $S$*  admits:

$$F_{\neg} \circ F_{\phi} \text{ as functional component}$$

$$\vec{u}_{\phi} \text{ as argument component}$$

**Example 1.** Let  $S$  be the sentence  $Pa \wedge Rbc$  with  $P$  and  $R$  standing for intrinsic properties/relations. According to the *standard way of regarding the content of  $S$* ,

1. The functional component is  $F_{\wedge} \circ \langle F_P, F_R \rangle$

2. The argument component is  $\langle u_a, u_b, u_c \rangle$

The truth-conditions of  $S$  have the following expression: for every  $W \in \mathbb{W}$ :

$$\mathcal{C}_S(W) = F_{\wedge}(F_P(u_a(W)), F_R(u_b(W), u_c(W)))$$

The case of quantified sentences is still to be defined. As briefly shown in the previous example, in a quantified sentence the argument component is the extension function of an open formula.

### Quantified sentences:

- (i) Let  $S$  be the sentence  $\forall x\phi(x)$  with  $\phi(x)$  being an open formula having only  $x$  as free variable

The standard way of regarding the content of  $S$  admits:

$F_{\forall}$  as functional component

$\langle I_{\mathbb{W}}, u_{\phi(x)} \rangle$  as argument component

- (ii) Let  $S$  be the sentence  $\exists x\phi(x)$  with  $\phi(x)$  being an open formula having only  $x$  as free variable

The standard way of regarding the content of  $S$  admits:

$F_{\exists}$  as functional component

$u_{\phi(x)}$  as argument component

- (iii) Clauses (i) and (ii) are straightforwardly extended to the case of higher-order quantification.

**Example 2.** Let  $S$  be the sentence  $\forall x(Px \vee Qx)$ ; according to the standard way of regarding the content of  $S$ ,

1. The functional component is  $F_{\forall}$
2. The argument component is  $\langle \mathcal{I}_{\mathbb{W}}, f_{\vee} \circ \langle u_{Px}, u_{Qx} \rangle \rangle$

The truth-conditions of  $S$  have the following expression: for every  $W \in \mathbb{W}$ :

$$\mathcal{C}_S(W) = F_{\forall}(W, f_{\forall}(u_{Px}(W), u_{Qx}(W)))$$

**Example 3.** Let  $S$  be the sentence

$$\exists x \forall y (Px \rightarrow Rxy)$$

According to the standard way of regarding the content of  $S$ :

1. The functional component is  $F_{\exists}$
2. The argument component is  $u_{\forall y(Px \rightarrow Rxy)}$  which may be also expressed as:

$$f_{\forall_2} \circ \langle I_{\mathbb{W}}, u_{Px \rightarrow Rxy} \rangle$$

The truth-conditions of  $S$  have the following expression: for every  $W \in \mathbb{W}$ :

$$\mathcal{C}_S(W) = F_{\exists}(f_{\forall_2}(W, u_{Px \rightarrow Rxy}(W)))$$

It is interesting to show that the application of the method for determining the standard ways of regarding a content is straightforward even when we consider a very complex quantified sentence:

**Example 4.** Let  $S$  be the sentence expressing the equinumerosity between two concepts denoted by two primitive monadic predicates  $F$  and  $G$  (i.e. the existence of a one-to-one correspondence between all  $F$ s and all  $G$ s). In second-order logic we may express  $S$  as follows:

$$\exists R \forall x \forall y \forall z$$

$$[(Rxy \wedge Rzy \rightarrow x = z) \wedge (Rxy \wedge Fx \rightarrow Gy) \wedge (Rxy \wedge Gx \rightarrow Fy) \wedge \dots$$

$$\dots(((Rxy \wedge Fx) \wedge (Rzy \wedge Fz)) \rightarrow x = z) \wedge (((Rxy \wedge Gx) \wedge (Rzy \wedge Gz)) \rightarrow x = z)]$$

$S$  says that there is a relation  $R$  which is functional from  $F$  to  $G$  and from  $G$  to  $F$ , is one-to-one from  $F$  to  $G$  and one-to-one from  $G$  to  $F$ .

The sentence  $S$  is obtained by applying the second-order existential  $\exists R$  – with  $R$  second-order



dyadic variable – to the complex open formula in the free variable  $R$  that we will abbreviate by  $\phi(R, F, G)$ . Thus the functional and the argument components of  $S$  are respectively:

$$F_{\exists} \quad ; \quad u_{\phi(R, F, G)}$$

Note that for every possible world  $W$ , the value  $u_{\phi(R, F, G)}(W)$  – i.e the extension of the open formula  $\phi(R, F, G)$  in  $W$  – is the set of all one-to-one correspondences between  $F$  and  $G$  and  $G$  and  $F$  in the world  $W$ .

# Chapter 4

## Carving up a Content

### 4.1 Introduction

In the previous chapter we have introduced two crucial notions: the notion of way of regarding a content and the notion of content. As briefly mentioned, given a certain sentence  $S$ , both a way of regarding the content of  $S$  and the content of  $S$  may be expressed in terms of a functional decomposition of the intension of  $S$ . More precisely, the content of  $S$  may be expressed as:

$$\mathcal{C}_S = \mathcal{I} \circ \sigma$$

where  $\mathcal{I}$  is the content function and  $\sigma$  is the partial world function. A way of regarding the content of  $S$  having  $F$  as functional component and  $u$  as argument component, may also be expressed as a functional decomposition of the intension of  $S$ :

$$\mathcal{C}_S = F \circ u$$

Moreover, we have characterized the relation between these two expressions of the intension in terms of what has been called *the semantic operational pattern of logical analysis*, according to which  $\sigma$  defines a sub-partitioning of the partitioning defined by  $u$ .

In this chapter we will introduce the definition of the operation of *carving up a content*, i.e.

the operation that given a content returns a particular way of regarding it. As we will see, this operation will be preliminary characterized as a functional decomposition of the content function.

We will start by remarking that the fact that  $\sigma$  defines a partitioning of the logical space whose cells are included in the correspondent cells of the partitioning defined by  $u$ , may be expressed in the following way: for every two worlds  $X$  and  $Y$ ,

$$\text{If } \sigma(X) = \sigma(Y), \text{ then } u(X) = u(Y)$$

Which says that if two worlds are indistinguishable with respect to their relevant portion picked out by  $\sigma$  (i.e. the partial world), then they also are indistinguishable with respect to the part picked out by  $u$  of the partial world picked out by  $\sigma$ . It is easy to see that, by definition of function, from this simple fact it follows that always exists a function  $\mu$  from the image of  $\sigma$  to the domain of  $u$  such that for every possible world  $W$ ,

$$\mu(\sigma(W)) = u(W)$$

Indeed, the previous condition ensures that for every two worlds  $X$  and  $Y$ , if  $\sigma$  associates the same value to  $X$  and  $Y$ , then also  $u$  associates the same value to  $X$  and  $Y$ ; in other words, if two worlds agree on the portion with which the content is concerned, then there is always a unique constituent (or a unique series of constituents) of them picked out by  $u$ . As a consequence, the relation between a portion of world with which the content is concerned and the relevant constituent picked out by the argument component is functional, for it never relates two different constituents to the same portion of a possible world.

It follows that the following equalities hold:

$$u = \mu \circ \sigma \text{ and } F \circ (\mu \circ \sigma) = \mathcal{I} \circ \sigma$$

By both the associative property of functional composition and the fact that  $\sigma$  is a surjective function, we derive that:

$$F \circ (\mu \circ \sigma) = (F \circ \mu) \circ \sigma = \mathcal{I} \circ \sigma$$

From which it follows that:

$$u = \mu \circ \sigma \quad ; \quad \mathcal{I} = F \circ \mu$$

which establish the relation between a particular way of regarding a content and the content itself. We call the function  $\mu$  a *selector for the content function*  $\mathcal{I}$ . This because given a partial world  $o$ , the function  $\mu$  selects the part of  $o$  with which the particular way of regarding the content is concerned. We may spell out the relation between a content and a way of regarding it as follows: *any particular way of regarding a content may be seen as obtained by decomposing the content function into the functional component and the selector, in such a way that the argument component is obtained by applying the selector to the partial world with which the content is concerned.* For instance, consider the sentence ‘Socrates is wise’ and consider a way of regarding its content composed by the functional component  $F$  corresponding to the predicate ‘is wise’ and the argument component  $u$  corresponding to the proper name ‘Socrates’. Suppose that the content  $c$  of ‘Socrates is wise’ is defined as the pair  $\langle \mathcal{I}, \sigma \rangle$ . For every possible world  $W$  the function  $\sigma$  picks out the part of  $W$  maximally relevant to ‘Socrates is wise’ (we will see how to define this part later) and the function  $\mathcal{I}$  assigns a truth-value to this part. We may decompose  $\mathcal{I}$  into  $F$  and a selector  $\mu$  such that for every world  $W$ , being  $s = \sigma(W)$ ,  $\mu$  picks out the extension of ‘Socrates’ in  $s$ ; consequently, the value  $u(W)$  may be seen as obtained by applying  $\mu$  to  $s$ , i.e.  $\mu(\sigma(W))$ .

This account also clarifies the relation between a particular way of regarding the content of a sentence  $S$  and the intension of  $S$ . Whenever we say that the intension of  $S$  is decomposed into the functional component and the argument component we must keep in mind that such a decomposition is not performed directly on the intension of  $S$ , but is obtained by decomposing the content function of  $S$  and by applying the associated selector to the partial world function of the content of  $S$ . *Thus the decomposition of the intension is always determined by an intermediate decomposition of the content function.*

In the following sections I will analyze in more details the relation between a way of regarding a content and the content itself in terms of functional decomposition.

## 4.2 A preliminary definition of carving

Our way of modelling notions such as way of regarding a content, content, and intension resulted in an interesting characterization of the relation between them in terms of functional decomposition. This allows us to make the next step and provide a preliminary definition of the operation of *content carving*.

### Content Carving (I)

Let  $\langle \mathcal{I}, \sigma \rangle$  be a content. Let  $F$  be a function from a certain set of entities  $A$  (or set of  $n$ -ples of entities, for some finite  $n \in \omega$ ) to a set of truth-values, and  $u$  a function from a certain set of possible worlds to  $A$ . The pair  $\langle F, u \rangle$  is said to be a carving of  $\langle \mathcal{I}, \sigma \rangle$  iff

$$(i) \quad \mathcal{I} \circ \sigma = F \circ u$$

(ii) Exists a function  $\mu$  from the image of  $\sigma$  to the set  $A$  such that :

$$\mathcal{I} = F \circ \mu \quad \text{and} \quad u = \mu \circ \sigma$$

The definition is interesting insofar as it characterizes the relation between a content and a way of regarding a content in terms of a functional decomposition of the content function and a functional composition of the internal component of the latter decomposition with the partial world function.

The first problem with the given definition is the alleged *preservation of content* that should characterize the carving operation. In other words, the operation of carving is an operation of arrangement of the content that does not alter the content itself. How our definition is expected to ensure such a preservation of content? We are tempted to answer that since content carving amounts just to a re-distribution of the content, in the sense that the same content is divided into “parts” in different ways, the whole content is not affected by the operation. Yet our definition does not say anything about dividing a content function into “parts”; this reading of the definition is flawed due to an illegitimate interpretation of functional decomposition as a sort of mereological decomposition. Indeed, if we consider functional decomposition as a decomposition of a function into “parts” (respectively the internal component and the external

component) we immediately see that the relation between the function that is decomposed and its components does not satisfy the fundamental principles of mereology<sup>1</sup>.

As a consequence, we cannot simply ground the content preserving property of the operation of carving on the operation of functional decomposition. Note that this is a crucial point, especially when the explication of the procedure of content recarving described in section 64 of the *Foundations* will be attempted: for, as we will see, also content recarving may be seen as a content preserving functional decomposition: what is removed from the functional component (e.g. from the relation of parallelism) is applied to the argument component (e.g. uniformly applied to the functions returning the extensions of two straight lines); if there is no ground for considering functional decomposition as a sort of mereological decomposition – at least from a formal point of view – there is no justification for the fact that content recarving is a content preserving operation. And the soundness of the procedure of section 64, i.e. the logical equivalence between the two limbs of an abstraction principle, depends on the fact that the two limbs have the same content, namely on the fact that the functional decomposition associated to the recarving operation is content preserving.

Another problem is related to those ways of regarding a content that require a world variable, e.g. in the case of a functional component associated with an extrinsic property (or an external relation). Let  $F_P$  be the truth-function associated to the extrinsic property  $P$  and let  $u_a$  be the argument component associated to the singular term  $a$  (i.e. the function that for every possible world  $W$  returns the extension of  $a$  in  $W$ ). Consider the sentence  $Pa$  and the following way of regarding its content: for every possible world  $W$ ,

$$\mathcal{C}_{Pa}(W) = F_P(W, u_a(W))$$

where  $\mathcal{C}_{Pa}$  stands for the intension of  $Pa$ . We may attempt to express this way of regarding a content in a partial world semantics. Let  $\mathcal{S}$  be the set of partial worlds relevant to the content of the sentence  $Pa$ ; let  $\mu_a$  be the selector function such that for every  $s \in \mathcal{S}$ ,  $\mu_a(s)$  is the extension of  $a$  in  $s$ ; we may define a selector  $\vec{\mu}$  such that for every  $s \in \mathcal{S}$ ,  $\vec{\mu}(s) = \langle W_s, \mu_a(s) \rangle$  where  $W_s$

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<sup>1</sup>For instance, given four distinct functions  $h, g, l, k$ , both the decompositions  $h = g \circ l$  and  $g = h \circ k$  may be possible, thus making  $g$  part of  $h$  and  $h$  part of  $g$  at the same time, which contradicts the asymmetry of the relation of parthood. Moreover, given a function  $h$ , for every bijection  $g$  having the same domain as  $h$ ,  $g$  is part of  $h$ , for  $h = (g \circ g^{-1}) \circ h$  and by the associative property of functional decomposition  $h = g \circ (g^{-1} \circ h)$ ; and it is at least weird that any function contains as parts all bijections defined over its domain.

stands for the possible world of which  $s$  is a proper part. Thus the considered way of regarding the content of  $Pa$  may be expressed as the pair  $\langle F_P, \vec{\mu} \rangle$ .

Let  $\mathcal{I}$  be the content expressed by the sentence  $Pa$ . It is easy to see that the proposed definition of carving is not immediately applicable to the present case. Indeed, in order to see the decomposition  $\mathcal{I} = F_P \circ \vec{\mu}$  as carving of  $\mathcal{I}$ , there must exist the selector function  $\vec{\mu}$ . Yet this in general cannot be possible: for there might be several possible worlds having  $s$  as common relevant portion, thus the relation that a partial world bears to a possible world of which it is part is not functional. There are two ways of avoiding such a predicament: 1) by assuming that the content is concerned with the entire world and not just with a portion of it, 2) by allowing ways of regarding a content that are not carvings.

The first option does not seem to be always the correct one. Consider the sentence ‘Adam is a car owner’; to be a car owner is reasonably an extrinsic property, yet it does not seem that we need to consider the entire world to assess whether Adam is a car owner; we may need just to consider the set of cars owned by Adam and perhaps some other fundamental entities that represent car ownership (e.g. selling contract, car title etc...). Thus even if there are limit cases in which the content seems to be concerned with the world in its entirety (as for instance the case of the sentence ‘The world is actual’), in the majority of cases the fact that the functional component is associated to an extrinsic property does not require the entire world as relevant part of a possible world with which the content is concerned.

The second option seems to raise an unnecessary and uninteresting terminological issue: after all, what should be the difference between regarding a content in a certain way and carving it up in a certain way? Should not ‘content carving’ mean just to regard a content in a certain way according to the distinction between function and argument? What sort of theoretical advantage such a distinction between carving and regarding a content is expected to provide? Thus also the second option does not seem to be an acceptable way out, and the difficulty persists. As we will see, there are ways of regarding a content that carve up the content in an indirect way; or, in other words, there are carvings of the content that in some sense mask the content they express, i.e. that are not directly obtained by a functional decomposition of the content function. However, I will show that given any sentence, there is always a way of carving up its content that is expressible in terms of a functional decomposition of the content function.

We have seen three main problems with the definitions of the operation of carving:

- Functional decomposition is a too weak operation to justify the required content preservation of the operation of carving. This is due to the fact that functional decomposition is not a mereological decomposition.
- Ways of regarding a content having a world variable do not qualify as carvings.

The first difficulty will be dealt with in the next section. The solution relies on the fundamental idea that the content of a sentence has a defined internal structure and only the functional decompositions of a content that are in accordance with its internal structure may be proven to be content preserving carvings. The internal structure of a content will be characterized in terms of the relation of determination between properties of possible worlds. The second difficulty will be solved in section 4.7 by showing that carvings implying a world variable are acceptable only if they are convertible into carvings respecting the internal structure of the content.

### 4.3 The internal structure of the content

We have seen that a content  $c$  may be defined by the pair of functions  $\langle \mathcal{I}, \sigma \rangle$  where  $\mathcal{I}$  is the content function and  $\sigma$  is the partial world function. The function  $\sigma$  is defined over the set of possible worlds  $\mathbb{W}$  and has a certain set  $\mathcal{S}$  as image, i.e. the set of all partial worlds relevant to the content  $c$ ; the function  $\mathcal{I}$  is defined in  $\mathcal{S}$  and has the set of truth-values as image.

The function  $\mathcal{I}$  divides the set  $\mathcal{S}$  of partial worlds into the set  $\mathcal{S}^T$  of partial worlds to which  $\mathcal{I}$  assigns the value True, and the set  $\mathcal{S}^F$  of partial worlds to which  $\mathcal{I}$  assigns the value False. Clearly,  $\mathcal{S}^T$  and  $\mathcal{S}^F$  are disjoint (no partial world may make both true and false any sentence expressing the content  $c$ ) and their union is the set  $\mathcal{S}$  of partial worlds on which  $\mathcal{I}$  is defined. Therefore, the set  $\{\mathcal{S}^T, \mathcal{S}^F\}$  is a partitioning of the set  $\mathcal{S}$  defined by the content function  $\mathcal{I}$ .

Let the pair of functions  $\langle F, u \rangle$  be a carving of  $c$ , being  $F$  the functional component and  $u$  the argument component. According to **Content Carving (I)**, there is a selector function  $\mu$  such that  $\mathcal{I} = F \circ \mu$  and  $u = \mu \circ \sigma$ . I will show that the selector  $\mu$  defines a subpartitioning of the partitioning defined by  $\mathcal{I}$ . Let  $s_1, s_2$  be two partial worlds in  $\mathcal{S}$ ; by the fact that  $F$  is a function it follows that:

$$\mu(s_1) = \mu(s_2) \Rightarrow F(\mu(s_1)) = F(\mu(s_2))$$



From the fact that  $\mathcal{I} = F \circ \mu$ , the previous expression may be written as:

$$\mu(s_1) = \mu(s_2) \Rightarrow \mathcal{I}(s_1) = \mathcal{I}(s_2)$$

In other words, if the selector function picks out the same entity (or the same series of entities) from  $s_1$  and  $s_2$ , then the content function  $\mathcal{I}$  assigns the same truth-value to  $s_1$  and  $s_2$ . This fact may be rephrased as: whenever  $s_1$  and  $s_2$  belong to the same  $\mu$ -cell, then they belong to the same  $\mathcal{I}$ -cell, i.e. the function  $\mu$  defined a subpartitioning of the partitioning defined by the function  $\mathcal{I}$ .

Hence, given the set of partial worlds  $\mathcal{S}$  with which the content  $c$  is concerned, the following holds:

- The content function  $\mathcal{I}$  partitions  $\mathcal{S}$  into two sets  $\mathcal{S}^T, \mathcal{S}^F$  associated with the truth-value of the content,
- The selector of a certain carving of  $c$  also partitions  $\mathcal{S}$ , and the partitioning defined by the selector is a subpartitioning of the partitioning defined by the content function.

These considerations offer a new perspective on both the notion of carving and the problem of defining non-arbitrary carvings of a content  $c$  (i.e. the problem of defining a content preserving carving operation). *A carving of a content  $c = \langle \mathcal{I}, \sigma \rangle$  may be individuated by a subpartitioning of the way the content function  $\mathcal{I}$  partitions the set of partial worlds  $\mathcal{S}$  with which the content  $c$  is concerned. As a consequence, the problem of defining non-arbitrary carvings of  $c$  may be formulated as the problem of defining special ways of partitioning the two cells in which  $\mathcal{I}$  divides  $\mathcal{S}$ .*

Thus our problem may be spelled out as follows: given the three sets  $\mathcal{S}, \mathcal{S}^T, \mathcal{S}^F$  we have to formulate a criterion to distinguish special subpartitionings of both  $\mathcal{S}^T$  and  $\mathcal{S}^F$  from other partitionings that are the result of mere set theoretic operations to which no special meaning may be ascribed. In other words, we may conceive *the internal structure of a certain content  $c$  as a list of special subpartitionings of the way the content function of  $c$  partitions the set of partial worlds relevant to  $c$ .*

To better understand the proposed way of presenting the problem of determining non-arbitrary carvings, the reader should keep in mind the fact that a subpartitioning of the sets  $\mathcal{S}^T$  and  $\mathcal{S}^F$  is

a way of representing some special world constituents that are crucial to evaluate the truth-value of certain sentences expressing the content  $c$ . Therefore to say that the internal structure of the content may be defined as a list of special subpartitionings amounts to say that the internal structure of a content may be defined as a list of special sorts of world constituents with which the content is concerned.

Consider for instance Frege's example of *GLA64* and suppose that the sentences 'line  $a$  is parallel to line  $b$ ' expresses the same content as the sentence 'the direction of line  $a$  = the direction of line  $b$ '. At the present stage of the discussion, we may assume that there are two non-arbitrary subpartitionings of the common content of the considered sentences: one is the partitioning defined by the intensions of both terms 'line  $a$ ' and 'line  $b$ ', the other is the partitioning defined by the intensions of the terms 'the direction of line  $a$ ' and 'the direction of line  $b$ '. Both partitionings are subpartitionings of the partitioning defined by the correspondent content function: for if two partial worlds agree on which objects are e.g. the extensions respectively of 'line  $a$ ', 'line  $b$ ' ('the direction of line  $a$ ', 'the direction of line  $b$ '), they must agree on whether line  $a$  is parallel to line  $b$  or not (whether the direction of line  $a$  is identical to the direction of line  $b$  or not).

We may formulate the problem of determining non-arbitrary carvings in further details. Given a content  $c$ , two information are also given: the set  $\mathcal{S}$  and the partitioning defined by the content function of  $c$ , i.e. the sets  $\mathcal{S}^T$  and  $\mathcal{S}^F$ . Thus given a content, a partitioning of the set of relevant partial worlds into two cells is also given. This is the largest subpartitioning (i.e. the subpartitioning with the largest cells) for there is no subpartitioning of  $\mathcal{S}$  having less than two cells. Moreover, the subpartitioning  $\{\mathcal{S}^T, \mathcal{S}^F\}$  cannot be arbitrary, for it is determined by the truth-conditions of all sentences expressing the content  $c$ . Hence our problem becomes this: *how it could be possible to determine all non-arbitrary subpartitionings of  $\mathcal{S}$  given its largest non-arbitrary subpartitioning?*

One may be tempted to approach this question along the following lines. Suppose that we are considering a certain language  $\mathcal{L}$ ; given a content  $c$  we may consider the class of sentences of  $\mathcal{L}$  expressing the content  $c$ . Call this class  $S_c$ . For every sentence  $\phi$  in  $S_c$ , we may consider the standard way of regarding the content of  $\phi$ . For each one of those standard carvings, we may consider the subpartitioning of  $\mathcal{S}$  defined by the correspondent selector. The list of all these subpartitioning is the internal structure of the content  $c$ .

The described procedure amounts to define the internal structure of a content according to standard readings of the syntax of all sentences expressing the content at issue. However, as briefly mentioned in the Introduction, the syntax of all sentences expressing a certain content may not help us in determining all non-arbitrary ways of carving up that content. As in the case of Frege's example of lines and directions, there is nothing in the syntax of 'line  $a$  is parallel to line  $b$ ' that tells us that the same content may be carved up as 'the direction of line  $a$  is identical to the direction of line  $b$ '. Syntax is not exhaustive and for a very compelling reason: the notion of content is not language dependent; every language has limited expressive resources and thus may not capture all non-arbitrary ways of carving up a content. A language endowed with line terms but not with direction terms will give us an incomplete picture on the internal structure of the content of 'line  $a$  is parallel to line  $b$ '.

For this reason we will approach our last question from a different perspective. The fundamental idea is to conceive the set  $\mathcal{S}$  and its largest non-arbitrary partitioning  $\{\mathcal{S}^T, \mathcal{S}^F\}$  as extensions of special properties of partial worlds. More specifically, in the case of atomic sentences I will argue that the set  $\mathcal{S}$  may be seen as the extension of a particular *determinable property of partial worlds*  $P_{\mathcal{S}}$  and its largest non-arbitrary subpartitioning will be conceived as *the first level of determination of  $P_{\mathcal{S}}$* ; as a consequence, *the list of all non-arbitrary subpartitionings of  $\mathcal{S}$  will be defined as the list of all further levels of determination of  $P_{\mathcal{S}}$* . And whenever both the extension of a determinable property and the first level of its determinates are given, there is a unique list of all determinates belonging to the following levels. In the case of molecular sentences, the internal structure of the content will be defined in terms of the internal structures of their atomic components.

Nevertheless, before disputing the details of this idea there are several crucial notions that must be clarified. Indeed, it must be clarified how the set of partial worlds relevant to a content is defined and what are these properties that will help us to define the internal structure of the content. This is the purpose of the next section.

## Partial worlds and content sustainment

In the previous section we have seen that given a content  $c = \langle \mathcal{I}, \sigma \rangle$ , the adopted strategy to define the internal structure of  $c$  is to consider the set  $\mathcal{S}$  – i.e. the image of  $\sigma$  – and the cells  $\mathcal{S}^T$ ,  $\mathcal{S}^F$  in which  $\mathcal{I}$  partitions  $\mathcal{S}$  as extensions of some special properties of partial worlds. In this

section we will see in details what kind of properties are these. Therefore, we need to clarify how these sets are defined.

We will start by highlighting some basic assumptions of our theory. The first fundamental assumption is that contents – as propositions – are truth-value bearers. A sentence has a certain truth-value  $\tau$  in virtue of the fact that the content it expresses has the truth-value  $\tau$ , not the contrary. A sentence  $S$  is associated with particular ways of regarding its content  $c$ , thus the picture that  $S$  offers of its content  $c$  is a partial picture. We may spell out this fact by clarifying the difference between what it is required for a content to be true (or false) and what a sentence expressing the same content requires. For a content  $c$  to be true, it is required that all the entities necessary to assess the truth-value of all possible ways of regarding  $c$  satisfy the correspondent requirements expressed by  $c$ . For instance, under the assumption that the identity of content entailed by *GLA64* holds, the content of the sentence ‘line  $a$  is parallel to line  $b$ ’ requires not just that line  $a$  is parallel to line  $b$ , but also that the direction of  $a$  is identical to the direction of  $b$ . Given that a sentence is concerned only with a particular way of regarding a content, what a sentence requires to be true may be less than what its content requires. For what the sentence ‘line  $a$  is parallel to line  $b$ ’ requires to be true, is just the parallelism between line  $a$  and line  $b$ ; we don’t need to consider their directions.

Particular ways of regarding a content are made true or false in virtue of the fact that certain entities existing at certain worlds satisfy or do not satisfy the requirement represented by the functional component. For this reason I say that these entities *witness* the truth-value of the considered way of regarding a content. In a world  $W$ , the individual  $A$  being Socrates at  $W$  witnesses the truth-value of ‘Socrates is wise’: for both the truth and the falsity of ‘Socrates is wise’ depends upon the way  $A$  is. For the same reason, the extension of the concept ‘ $x$  is a zombie’ at a world  $W$  with all of its subsets witnesses the truth-value of ‘There are zombies’. We may say that ways of regarding a content (and thus sentences) are concerned with the entities that witness their truth-value. On the other hand, a content is concerned with all entities that witness the truth-value of all of its possible ways of regarding it. Again, even if the truth-value of ‘line  $a$  is parallel to line  $b$ ’ is witnessed by line  $a$  and line  $b$ , the truth-value of its content is witnessed by line  $a$  and line  $b$ , and by the direction of line  $a$  and the direction of line  $b$ .

I have said that the truth-value of a sentence *depends upon* the way its witnesses are. Such a relation of dependence must be characterized in more details. Given a sentence  $S$  and a series

of entities  $E_1, \dots, E_n$  there seems to be three options:

1. **Entailment:** The entities  $E_1, \dots, E_n$  witness the truth (falsity) of  $S$  iff the existence of  $E_1, \dots, E_n$  *entails* that  $S$  is true (false),
2. **Necessitation:** The entities  $E_1, \dots, E_n$  witness the truth (falsity) of  $S$  iff  $E_1, \dots, E_n$  *necessitate* the truth (falsity) of  $S$ ,
3. **Grounding:** The entities  $E_1, \dots, E_n$  witness the truth (falsity) of  $S$  iff the fact that  $E_1, \dots, E_n$  exist *grounds* the fact that  $S$  is true (false),

Entailment is too weak. The existence of Socrates entails that  $2 + 2 = 4$ , yet we cannot consider Socrates as a witness of the truth of  $2 + 2 = 4$ .

Necessitation has been proposed by Armstrong (2004) as a primitive modal relation, i.e. a modal relation not explainable in terms of possible worlds. Though not uncontroversial, this possibility is interesting to explore. From the fact that necessitation is a primitive modal notion, the only way we have to understand it is by irreducibly using modal vocabulary. My proposal to understand Armstrong's notion of necessitation is the following:

The entities  $E_1, \dots, E_n$  *necessitate* the truth (falsity) of  $S$  iff if  $S$  were false, then  $E_1, \dots, E_n$  would be different under the relevant respect.

For instance, God necessitates Eve's eating the forbidden apple, for if Eve did not eat the forbidden apple, then God would have created Eve in a different way under the relevant respect (e.g. with respect to the capability of resisting temptations). Being  $A$  the individual bearing the name 'Socrates' in a certain world,  $A$  necessitates the truth of 'Socrates is wise' iff if 'Socrates is wise' were false, then  $A$  would be different under the relevant respect (i.e. would not be wise).

The case of grounding is traditionally understood in terms of "in virtue of" clauses:

The fact that the entities  $E_1, \dots, E_n$  exist *ground* the truth (falsity) of  $S$  iff  $S$  is true *in virtue of* the fact that  $E_1, \dots, E_n$  exist.

Grounding presents a difficulty similar to that of necessitation: it is a primitive metaphysical relation which is not explainable in terms of other notion.

For the purposes of the present exposition, the difference between grounding and necessitation is not a crucial point. The present theory of content may be preliminary understood with one or the other relation of ontological dependence.

In the previous chapter we have given two functional representations of a content  $c$ , one using the function  $\mathcal{I}^*$  that applies to all possible partial worlds in which the content has a defined truth-value and its restriction  $\mathcal{I}$  that applies to all partial worlds that are maximally relevant. Intuitively, the domain of the function  $\mathcal{I}^*$  is the set of all partial worlds containing all entities that witness the truth-value of all ways of regarding the content  $c$ , and the function  $\mathcal{I}$  applies to all partial worlds defined as parts of total worlds containing all entities that witness the truth-value of all ways of regarding  $c$  and nothing else.

In order to give a rigorous definition of these sets of partial worlds a clarification is needed. The clarification concerns the fact that a world  $X$  (total or partial) *contains* a certain entity  $E$ .  $X$  may contain an entity  $E$  in two senses: in the sense that  $E$  is a part of  $X$  or in the sense that although  $E$  is not a part of  $X$ ,  $E$  exists according to  $X$ . To understand the difference, consider a partial world  $X$  composed by the mereological fusion of Socrates and Plato. Consider now the individual Socrates: it is a part of  $X$  and, as a consequence, exists according to  $X$ . Consider now the unit set  $U = \{\text{Socrates}\}$ ;  $U$  is not a part of  $X$ , yet exists according to  $X$ , for we need sets of parts of a world to exist in that world. Now it is possible to consider a partial world  $X'$  obtained by considering only the proper part of  $X$  identical to Socrates. The world  $X'$  does not contain just Socrates; it contains also the empty set and the set  $U$ . Conversely, it is impossible to define a partial world containing only  $U$ : this because for  $U$  to exist according to a world, Socrates must exist according to that world. This remark suggests an important caveat when we say that a world  $X$  contains certain entities  $E_1, \dots, E_n$  and nothing else: whenever we say that  $X$  contains  $E_1, \dots, E_n$  *and nothing else* it is implicit that  $X$  must also contain all sets that are *grounded in*  $E_1, \dots, E_n$  and, in case at least one of  $E_1, \dots, E_n$  is a set, all entities in which  $E_1, \dots, E_n$  are grounded.

It is now possible to give the definitions that we have roughly anticipated. We will start with the notion of *content sustainment*:

### **Content Sustainment (CS):**

Let  $c$  be a content and  $s$  a possible partial world. We say that  $s$  *sustains the content*  $c$  iff  $s$

contains all entities that witness the truth-value in  $s$  of all ways of regarding  $c$ .

For instance, suppose that there is only one way of regarding the content of the sentence ‘Socrates is wise’ according to which ‘Socrates’ defines the argument and ‘ $x$  is wise’ the function; a partial world  $s$  containing a possible extension  $a$  of the proper name ‘Socrates’ sustains the content of ‘Socrates is wise’, for  $s$  contains  $a$ ,  $a$  witnesses the truth-value of ‘Socrates is wise’ and there is no other entity witnessing the truth-value of the content. Notice what happens if we take seriously the identity of content between parallelism and identity of direction: a partial world  $s$  containing the extensions of the terms ‘line  $a$ ’ and ‘line  $b$ ’ but not their directions does not sustain the content of ‘ $a$  is parallel to  $b$ ’, for there is a different way of regarding the same content – represented by the sentence ‘the direction of  $a$  is identical to the direction of  $b$ ’ – whose witnesses are not included in  $s$ .

We say that a partial world  $s$  *positively sustains* the content  $c$  iff  $s$  sustains  $c$  and every way of regarding  $c$  is true in  $s$ . Contrariwise, we say that a partial world  $s$  *negatively sustains* the content  $c$  iff  $s$  sustains  $c$  and every way of regarding  $c$  is false in  $s$ .

If we want to define the set  $\mathcal{S}$ , i.e. the image of the partial world function  $\sigma$ , we cannot simply take the set of all partial worlds sustaining the content  $c = \langle \mathcal{I}, \sigma \rangle$ . For partial worlds sustaining  $c$  are too comprehensive: in spite of containing what is relevant to the content at issue, they may contain something else. For instance, a partial world containing possible extensions of ‘Socrates’, ‘Plato’, and ‘Solomon’s Temple’ still sustains the content of ‘Socrates is wise’. We want maximally relevant partial world to contain what is relevant to the content and nothing else. This aspect is captured by the following definition:

### **Content Strict Sustainment (CSS):**

Let  $c$  be a content and  $s$  a partial world. We say that  $s$  *strictly sustains the content*  $c$  iff  $s$  contains all entities  $E_1, \dots, E_n$  that witness the truth-value in  $s$  of all ways of regarding  $c$  and nothing else, with the proviso that  $s$  must also contain all entities that are grounded in  $E_1, \dots, E_n$  and all entities in which  $E_1, \dots, E_n$  are grounded.

Finally, we define the notion of *part of a world maximally relevant to a content*:

### **Maximal Relevant Part (MRP):**

Let  $c$  be a content and  $W$  a total world. The partial world  $s$  is *the part of  $W$  maximally relevant to  $c$*  iff  $s$  is the largest part of  $W$  such that  $s$  strictly sustains  $c$  and  $c$  has the same truth-value in  $s$  and  $W$ .

Thus we define the partial world  $\sigma$  associated with a certain content as the function that receives a total world  $W$  as argument and returns the maximal relevant part of  $W$  (with respect to the considered content).

To help the reader to understand the definition (MRP) we will refer to some examples. Consider the sentence  $S$  ‘There are zombies’ expressing a content  $c$  and let  $W$  be a possible world according to which  $S$  is true. Let  $Z$  be the extension of the concept ‘ $x$  is a zombie’ in  $W$ . We will determine the part of  $W$  that is maximally relevant to  $c$  by using (MRP).

Firstly, we must determine what witnesses the truth of  $S$  in  $W$ . We may reasonably suppose that all entities witnessing the truth of  $S$  in  $W$  are all non-empty sets of zombies existing according to  $W$ . Thus every part  $x$  of  $W$  containing some zombies of  $W$  sustains  $c$ . Let  $s$  be one of these parts: clearly  $s$  may contain entities that are not witnesses of the truth of  $S$ ; as a consequence,  $s$  may not strictly sustain  $c$ . Thus we have to consider all parts of  $W$  containing some zombies of  $W$  and no non-zombies. Among all parts of  $W$  that satisfy this requirement the maximal one is the part  $z$  of  $W$  containing exactly all zombies of  $W$ . Moreover,  $z$  and  $W$  agree on the truth-value of  $c$ . Thus the maximal relevant part of  $W$  with respect to the content  $c$  is the part of  $W$  containing exactly all zombies of  $W$ .

Consider now a possible world  $V$  according to which  $S$  is false, i.e. a world without zombies. Again, assuming that there are no ways of regarding  $c$  differing from the standard way defined by  $S$ , the only entity that witnesses the falsity of  $S$  in  $V$  is the empty set, i.e. the extension of ‘ $x$  is a zombie’ in  $W$ . It is easy to see that every part of  $W$  strictly sustains  $c$ : indeed, every part  $x$  of  $W$  contains the empty set and the entities on which it is grounded (every group of objects grounds the empty set). Thus the largest part of  $W$  that strictly sustains  $c$  and agrees with  $W$  on the truth-value of  $c$  is  $W$  itself. Thus  $W$  is the maximal relevant part of  $W$  with respect to the content  $c$ . This fact has an interesting explanation: to falsify an existential statement is to verify a universal one (i.e. everything is a non-zombie). Given that – as we will see in a while – universal statements are concerned with the entire world, it is not a surprise that the part of a world that falsify an existential is the entire world.



Consider now the case of  $S$  being the sentence ‘Everything is a zombie’ and let  $c$  be the content of  $S$ . Suppose that the standard way of regarding  $c$  defined by  $S$  is the only non-arbitrary way of regarding  $c$ . Let  $W$  be a world according to which  $S$  is true, i.e. a world containing only zombies. By (CSS), every part of  $W$  strictly sustains  $c$ : for every part of  $W$  is composed just by zombies. Moreover, every part of  $W$  strictly sustains  $c$  and assigns to  $c$  the same truth-value as  $W$ . Therefore the maximal part of  $W$  relevant to  $c$  is  $W$  itself. Again this result is not surprising: the content of a universally quantified sentence is concerned with the entire world. Consider now a possible world  $V$  according to which  $S$  is false. Suppose that  $V$  includes some zombies but also non-zombies. Let  $Z$  the extension of the concept ‘ $x$  is a zombie’ in  $V$ . Notice that what witnesses the falsity of a universal is a counter-example: for instance, every set of non-zombies of  $V$  witnesses the falsity of ‘everything is a zombie’. Thus the part  $k$  of  $V$  containing all and only non-zombies of  $V$  strictly sustains  $c$ , for it contains all counter-examples of  $V$  to the statement that everything is a zombie. Moreover, there is no part of  $V$  larger than  $k$  strictly sustaining  $c$  and  $k$  and  $V$  agree on the falsity of ‘everything is a zombie’. Thus  $k$  is the part of  $V$  maximally relevant to  $c$ .

Given a content  $c = \langle \mathcal{I}, \sigma \rangle$ , the definitions (CS), (CSS), and (MRP) allow us to show how the sets  $\mathcal{S}, \mathcal{S}^T, \mathcal{S}^F$  are defined. Specifically,  $\mathcal{S}$  is the set containing for every possible world  $W$  the part of  $W$  maximally relevant to  $c$ ;  $\mathcal{S}^T$  ( $\mathcal{S}^F$ ) is the set of partial worlds in  $\mathcal{S}$  that strictly and positively (strictly and negatively) sustain  $c$ .

The next step to define the internal structure of a content  $c$  of an atomic sentence is to show that the property of sustaining  $c$  is a determinable property immediately determined by the properties of positively and negatively sustaining  $c$ ; these properties are in turn determined by constituent properties (more on this later) that will be associated with the way non-arbitrary argument components partition the set  $\mathcal{S}$ . This will be the purpose of the next section.

## Properties of content sustainment as determinables

Given that the notion of determinable property and the correspondent relation of determination will turn out to be crucial to understand how the internal structure of a content may be defined, it is worth to recall some of its main basic features.

Determinates and determinables are properties that stand in a special kind of specification rela-

tion. Classical examples are color properties, mass properties, shape properties. For instance, color properties such as ‘ $x$  is red’ (determinates) specify the general determinable property ‘ $x$  is colored’. So to say, a determinable property is a property for which a certain specification is required. Indeed, nothing may be colored without being of a certain color, nothing may have mass without having a certain specific mass, nothing may have shape without having a particular shape (e.g. triangular, circular, etc...). In the following I will try to clarify the notions of determinables and determinates by highlighting some basic facts about them.

The first and most simple feature of determinable properties is that they are properties that objects may have in different ways ((Yablo, 1992)). There are many ways in which an object may be colored, many ways in which it may have mass or shape.

A second characteristic is that all determinates of a certain determinable corresponding to the same degree of specification are *mutually exclusive*. No colored object may be red and yellow at the same time, no object having mass may be of 2 and 5 kg of mass at the same time, no object having shape may be triangular and circular at the same time.

A third important feature is represented by the fact that determinates are *non-conjunctive specifiers* of the respective determinable (Körner and Searle, 1959). We may explain this fact as follows. Let  $H$  be the property of being human,  $A$  the property of being animal, and  $R$  the property of being rational. We say that  $A$  is a conjunctive specifier of  $H$  insofar as there is a property not necessarily co-extensional with  $A$  (i.e. the property  $R$ ) such that  $H$  may be defined as the conjunction of  $A$  and  $R$ . In other words, the relation of conjunctive specifier has to do with the Aristotelian distinction between *genus* and *specie*. It is easy to see that color properties do not follow this pattern: there is no property  $P$  such that the conjunction of  $P$  with ‘ $x$  is colored’ defines the property and ‘ $x$  is red’ and  $P$  is not necessarily co-extensional with ‘ $x$  is red’. The specification relation holding between a determinate and a determinable is not reducible to an intersection of the determinable with a distinct property.

The fourth important feature is probably the most interesting one for our purpose: determinables admit *different levels of specification* which makes relative the distinction between determinable and determinates. For instance, ‘ $x$  is red’ is a determinate of ‘ $x$  is colored’; on the other hand, considering a higher level of specification, ‘ $x$  is red’ is a determinable having among its determinates the property ‘ $x$  is scarlet’. Clearly ‘ $x$  is scarlet’ determines both ‘ $x$  is red’ and ‘ $x$  is colored’.

The first step of my argument is to show that the property of sustaining a content  $c$  of an atomic sentence may be classified as a determinable property according to the considered four basic features. Clearly this is not the last word to say on the issue: as we will see, a more profound account of the determination relation is needed to conclude that the property of sustaining  $c$  is a determinable property.

1. **Multiple ways of having the property:** a partial world may sustain a content  $c$  in multiple ways. Obviously, it may sustain  $c$  negatively or positively. Moreover, it may sustain  $c$  (positively or negatively) according to different internal configurations, i.e. according to different constituents that play the role of the entities that witness the truth or the falsity of  $c$ ,
2. **Mutual exclusion of determinates of the same level:** The property of sustaining  $c$  may be immediately specified by saying whether it is a case of positive or negative sustainment. Clearly, no partial world may sustain  $c$  positively and negatively at same time, for no content may be at the same time true and false according to the same world. Moreover, if a partial world sustains  $c$  (positively or negatively) by having as witnesses  $E_1, \dots, E_n$ , it may not sustain  $c$  by having different witnesses, i.e. properties of sustainment in virtue of a certain internal configuration are mutually exclusive (more on this later).
3. **Non-conjunctive specification:** It is hard to imagine a property  $P$  not co-extensional with e.g. ' $x$  positively sustains  $c$ ' such that the conjunction of  $P$  with ' $x$  sustains  $c$ ' defines ' $x$  positively sustains  $c$ '. Notice that the fact that e.g. the property of positive sustainment of  $c$  has been identified with the conjunction between the property of sustaining  $c$  with the property of assigning value true to every ways of regarding  $c$  shall not be understood as a relation of conjunctive specification. Indeed, the property of assigning the value true to all ways of regarding  $c$  is co-extensional to the property of positively sustaining  $c$ . The definition of e.g. positive sustainment has been given in terms of a conjunction just to make explicit in which way  $c$  is sustained by a world that positively sustains it. It is not a definition *per genus et differentiam*.
4. **Multiple levels of specification:** The property of sustaining  $c$  may be specified in a non-conjunctive way by saying whether it is a case of positive or negative sustainment. As briefly suggested, further specifications are possible. Indeed, Yablo (2014) notices that sentences (and thus contents) are true or false in different ways: the sentence 'Socrates is

wise' is true in a certain way in all worlds that, for instance, agree on the fact that Socrates is a certain individual  $a$  having such-and-such intrinsic features. In other words, there is a difference in the way the same truth-value  $\tau$  may be witnessed, namely a difference in intrinsic facts regarding the entities that witness  $\tau$ . Multiple ways of being true (false) corresponds to multiple ways of positively (negatively) sustaining a certain content. And it is easy to verify that two distinct ways of positively sustaining a content are mutually exclusive, for no world may verify at the same time different intrinsic facts regarding the entities that witness a certain truth-value. Moreover, it is hard to imagine that a particular way of e.g. positively sustaining  $c$  individuated by a property  $Q$  may be expressed as a conjunction between the property of positively sustaining  $c$  and a property  $P$  not co-extensive with  $Q$ . This because the only way of defining the property  $P$  is in terms of the list of witnesses, which is exactly the way  $Q$  is defined. Thus multiple ways of making a content true (false) seem to be associated with determinates of the property of positively (negatively) sustaining  $c$ , i.e. there are levels of specification of the property of sustaining a content more determined than the level of the properties of positive and negative sustainment.

May we conclude that the property of sustaining a content is a determinable property determined in the way sketched above? Clearly not. The four features of the relation of determination we have examined represent only a necessary condition that by no means is also sufficient. In other words, if a certain system  $\Pi$  of properties has a determinates/determinable structure, then  $\Pi$  must satisfy the four considered requirements.

There is a very good reason to think that the four requirements are not a sufficient condition. There is a fundamental feature of determinates that is captured by none of them. Consider the case of colors: it is true that an object may not be colored without having a specific color, or that no object may have two different colors, or that e.g. 'red' is a non-conjunctive specifier of 'colored'. Yet there is a fundamental additional fact: two red things are indistinguishable with respect to their color, as well as a red thing and a green thing are distinguishable – inter alia – with respect to their color. *Hence if 'color' is the respect under which instances of different determinates of 'colored' are mutually distinguishable and instances of the same determinate are partially indistinguishable, what is the correspondent respect under which partial worlds having different determinates of 'x sustains c' (being c the content of an atomic sentence) are mutually distinguishable and partial worlds having the same determinate are partially indistin-*

*guishable?*

To answer this question we need a more detailed characterization of the relation of determination. Funkhouser (2006) formulated a proposal that goes precisely in this direction. The model of the relation of determination developed in (Funkhouser, 2006) is based on the following remark: *determinable properties have determination dimensions along which they are determined*. A color property is always determined with respect to what characterizes ‘colorness’ and with respect to no other features: ‘being scarlet and having 2 kg mass’ does not determine ‘red’, for in spite of the fact that ‘scarlet’ determines ‘red’, ‘having 2 kg mass’ is not a feature along which the property ‘red’ may be determined; so to say, it’s irrelevant to “colorness”. The aspect with respect to which a determinable is determined may be individuated by means of what Funkhouser calls *the non-determinable necessities* of a determinable property. Non-determinable necessities represent features that all objects falling under the same determinable share: for instance, ‘having three sides’ is a feature that all triangular shapes have, a certain hue, saturation, and brightness are features that all colors have, having a certain pitch, tone, and timbre is a feature that all sounds have. The non-determinable necessities of a determinable property define *the dimensions along which a determinable is determined*. The expression “dimensions” refers precisely to the fact that a relation of determination may be characterized in a certain *space of properties*. Consider the case of colors: we may construct a three dimensional space having hue, saturation, and brightness as dimensions. Each point of the space corresponds to a fully determined color property (a super-determinate according to Funkhouser’s terminology), i.e. the bottom of the scale of specification. A determinable color property may be represented in the property space as a region defined by certain ranges of hue, saturation, and brightness. It is easy to see that ‘scarlet’ determines ‘red’ insofar as the region of the property space corresponding to ‘scarlet’ (i.e. the range of values of hue, saturation, and brightness corresponding to ‘scarlet’) is strictly included in the region corresponding to ‘red’: indeed there are values of hue, saturation, and brightness that correspond to shades of red that are not scarlet. Thus ‘scarlet’ determines ‘red’ because these two properties are comparable in the same property space and the region corresponding to the latter contains the region corresponding to the former. From this fact, Funkhouser proposes the following sufficient condition for the relation of determination: *the property B determines the property A if*

- (i) *B has the same non-determinable necessities as A,*

- (ii) *B has the same determination dimensions as A,*
- (iii) *The region of the property space of B is strictly included in the region of the property space of A.*

Clearly conditions (i) and (ii) are not independent.

Funkhouser's account of the determination relation is helpful insofar as it defines the respect under which instances of the same (different) determinates are indistinguishable (distinguishable): a certain relation between the determination dimensions. Two objects are both red in virtue of the fact that they have properties that are representable in the same property space and the regions associated to these properties are included in the region associated with 'red'.

Hence, to investigate whether a property *B* determines a property *A* we have to specify the non-determinable necessities and thus the determination dimensions of both *A* and *B*. Thus our question becomes: given a content *c*, what are the non-determinable necessities and the determination dimensions of the property '*x* sustains *c*' and, for instance, '*x* positively sustains *c*'? Note that non-determinable necessities must be features that all instances of a determinable property have in common *qua instances of that determinable property*. Thus what features do all partial worlds sustaining a content *c* have in common *qua instances of the property of sustaining c*?

From the definition of the property of content sustainment, being *c* the content of an atomic sentence, we may infer that every partial world *X* sustaining *c* must have a certain kind of witnesses of either the truth or the falsity in *X* of the content *c*. For instance, every partial world sustaining the content of 'Socrates is wise' must have a constituent bearing the name 'Socrates' that witnesses the truth-value of the content; every partial world *X* sustaining the content of 'line *a* is parallel to line *b*' must contain a pair of objects *l*<sub>1</sub>, *l*<sub>2</sub> being respectively the extensions of 'line *a*' and 'line *b*' in *X*. Thus we may reasonably assume that the non-determinable necessities of the property of sustaining a content *c* are represented by the kind of constituent (or series of constituents) that witnesses the truth-value of *c*: all worlds sustaining the content of 'Socrates is wise' must have a – so to say – "Socrates-part" that witnesses the truth-value of the content according to having or lacking the property of being wise. Different values of the determination dimensions correspond to different objects that play the role of witnesses in different worlds. For instance, being *c* the content of 'Socrates is wise', the non-determinable necessity of '*X* sustains *c*' is represented by the fact of having a constituent for which 'Socrates' stands, and the

“points” of the property space are all properties of the form ‘ $X$  has a constituent  $a$  witnessing the truth-value of  $c$  and  $a$  is the object for which ‘Socrates’ stands in  $X$ ’ for every possible extension of ‘Socrates’  $a$ . The region of the property space corresponding to the property of positively sustaining  $c$  is determined by the list of all extensions of the name ‘Socrates’ that are wise, i.e. that makes the content true.

The problem of defining the determination dimensions of the property ‘ $X$  sustains  $c$ ’ is that a content may admit many non-arbitrary carvings, and thus it may admit different kinds of witnesses. Take the case of *GLA64*: the same content may be witnessed by two objects  $l_1, l_2$  for which the terms ‘line  $a$ ’ and ‘line  $b$ ’ respectively stand, or by two objects  $d_1, d_2$  for which the terms ‘the direction of line  $a$ ’ and ‘the direction of line  $b$ ’ respectively stand. Which pair of objects define the determination dimensions,  $l_1, l_2$  or  $d_1, d_2$ ?

To answer this question we must take in mind that the “points” of the property space represent the super-determinate properties; recall that, according to Funkhouser’s terminology, a super-determinate of a determinable  $P$  is a determinate of  $P$  that is not specifiable any further. Thus we must choose as determination dimensions the property of having certain witnesses that are not determined by other properties. Suppose that we construct a property space having as dimensions the possible extensions of the terms ‘line  $a$ ’ and ‘line  $b$ ’. We have a two dimensional property space in which the property of e.g. positively sustaining  $c$  is represented by the region defined by all possible extensions of ‘line  $a$ ’, ‘line  $b$ ’ that are parallel. If the points of the property space are super-determinates, then every possible property being a determinate of ‘ $x$  sustains  $c$ ’ may be represented as a region in this property space. Consider the property ‘the truth value of  $c$  in  $x$  is witnessed by two constituents  $d_1, d_2$  of  $x$  for which ‘the direction of line  $a$ ’ and ‘the direction of line  $b$ ’ respectively stand’; let’s abbreviate the expression of this property by  $P_{d_1, d_2}(x)$ . Is  $P_{d_1, d_2}(x)$  representable in the property space having extensions of ‘line  $a$ ’ and ‘line  $b$ ’ as dimensions? The answer is positive: indeed we may consider all possible values  $\alpha, \beta$  of ‘line  $a$ ’ and ‘line  $b$ ’ such that the direction of  $\alpha$  is  $d_1$  and the direction of  $\beta$  is  $d_2$ ; these values define the region of the property space corresponding to  $P_{d_1, d_2}(x)$ . Thus under the assumption that identity of directions and parallelism are the only non-arbitrary ways of regarding  $c$ , the determination dimensions of the property ‘ $X$  sustains  $c$ ’ are the extensions of ‘line  $a$ ’ and ‘line  $b$ ’.

We may develop a general strategy to determine the determination dimensions of the property

of sustaining a content  $c$  that admits various ways of being regarded in a non-arbitrary way. Suppose that  $\mathcal{S}_0$  is the set of partial worlds sustaining  $c$ . Any carving of  $c$  may be given through the functional component  $F$  and a selector  $\mu$ . It is easy to see that the non-arbitrary carving that defines the dimensions of the property space is the carving having the selector  $\mu$  that defines the smallest partitioning of  $\mathcal{S}_0$  (i.e. the partitioning having the smallest cells). Indeed, if  $\mu_0$  is the selector of this carving, it is easy to see that the region of the property space defined by the witnesses of any other carving  $\mu$  are representable in the space having as dimension the values of  $\mu_0$ . To this end, note that if  $\mu_0$  defines a subpartitioning of  $\mu$  on the set  $\mathcal{S}_0$ , then it is possible to define a function  $\lambda$  such that  $\mu = \lambda \circ \mu_0$ . As a consequence, the property ‘ $x$  is a partial world such that  $\mu(x) = a$  and  $a$  witnesses the truth-value of  $c$ ’ may be represented in the property space having  $\mu_0$  as dimension as the range of all  $y$  that are values of  $\mu_0$  such that  $\lambda(y) = a$ .

Given a content  $c$ , call *the basic carving of  $c$*  the non-arbitrary way of regarding  $c$  having an argument component that defines the smallest partitioning of the set of partial worlds sustaining  $c$ . Call *the basic witnesses of  $c$*  the possible values of the argument component of the fundamental carving. We may state the following principle:

**Content determination (CD):**

*Given a content  $c$  of an atomic sentence, the property of partial worlds ‘ $x$  sustains  $c$ ’ is a determinable property having the values of the determination dimensions defined by all possible basic witnesses.*

The property of sustaining the content of an atomic sentence is determined by the properties of sustaining it negatively and positively. Note that there cannot be other determinates at the same level of determination. From the fact that the property of sustaining a content has just two determinates at the considered level, it follows that there are no intermediate levels of determination between the properties of negatively (positively) sustaining a content and the property of strictly sustaining a content; indeed, if we suppose that there is an intermediate level of determination, we unavoidably have to admit a determinable having just one determinate which is absurd.

The properties of sustaining the content of an atomic sentence positively and negatively are in turn determined by properties of having certain witnesses as constituents. Depending on how



many non-arbitrary carvings a content admits, there may be different levels of specification associated to different carvings, and the property of containing witnesses of a certain carving may determine the property of containing witnesses of a different carving. Thus each carving of the content  $c$  of an atomic sentence corresponds to a certain level of specification of the determinable property of sustaining  $c$ , and for every level of specification, each value of the argument component (i.e. each list of witnesses at a world) correspond to a particular determinate property of the considered level of specification.

The reader may object that the idea that every content of an atomic sentence has a determinate/determinables structure which implies the existence of a unique non-arbitrary carving for each level of specification is an illegitimate generalization. After all, in principles nothing excludes that different witnesses may partition the set of worlds sustaining a content in the same way. Thus there may be different carvings corresponding to the same level of specification of the property of sustaining the content. For instance, we may assume that there are two terms  $\xi$  and  $\zeta$  such that for every two possible extensions  $A$  and  $B$  respectively of  $\xi$  and  $\zeta$ , the set of worlds agreeing on the fact that  $\xi$  refers to  $A$  coincides with the set of worlds agreeing on the fact that  $\zeta$  refers to  $B$ . Consider now two sentences  $P\xi$  and  $Q\zeta$  with  $P, Q$  primitive monadic predicates, and suppose that  $P\xi$  and  $Q\zeta$  have the same content. In this case, both sentences define co-extensional witnessing properties; thus which are the determinates of the property of sustaining the common content of  $P\xi$  and  $Q\zeta$ ?

We may suppose that a difference in fundamentality may solve the problem. For instance, we may suppose that in every possible (partial or total) world, the extension of  $\xi$  is an entity more fundamental than the extension of  $\zeta$ , i.e. the extension of  $\zeta$  is grounded in the extension of  $\xi$ ; thus we may use as a criterion of choice of the carvings that define the internal structure of the content a criterion of fundamentality, i.e. whenever we have different carvings defining co-extensional witnessing properties, we must choose the carving concerned with the most fundamental entities as the carving that define the correspondent level of specification. Yet this may not be true in all cases. In cases in which there is no difference in fundamentality, we may simply say that the internal structure of the content is not accessible unambiguously, i.e. we have no criterion for saying which are the range of entities that define a certain level of specification of the property of sustaining a content. This may not be a flaw of our proposal, for a criterion for choosing some witnesses and not others should not be provided by our theory of content but by the adopted metaphysics. More precisely, if a certain range of entities is

preferable over a different one as defining the witnesses of the content, the criterion according to which we define such preference may depend on ontological considerations, and thus on the big picture of reality we are open to accept. In this sense, our theory of content must be open to be complemented by our metaphysics rather than offering a big picture of reality.

We may finally answer to the questions left open in the previous section. *The internal structure of a content of an atomic sentence is a determinate/determinable structure of the system of properties composed by the properties of sustaining the content and the properties of containing witnesses of a certain kind of the truth-value of the content.* The super-determinable property is the property of sustaining the content and its super-determinates are all properties of containing particular constituents as basic witnesses. *A carving of the content of an atomic sentence that is in accordance with its internal structure is a carving whose argument component picks out the witnesses of the content defining determinates of a certain level of specification of the property of sustaining the content.* To each level of specification corresponds a certain non-arbitrary carving of the content at issue. In the next section we will see some interesting properties of this class of carvings.

There is a last point that is worth examining. So far I have argued that it is possible to account for the internal structure of the content of an atomic sentence as a determinates/determinable structure according to Funkhouser's model of the property space. Yet in some cases the reader may still find difficult to understand in what sense properties of content sustainment and properties of containing witnesses of the content stand in a relation of determination. In particular, in the case of *GLA64*, it seems unclear why we should consider the property  $P_{d_1, d_2}$ , i.e. the property of having certain objects  $d_1, d_2$  for which the terms 'the direction of line  $a$ ', 'the direction of line  $b$ ' respectively stand as witnesses of the content  $c$  as a determinable property<sup>2</sup> (being  $c$  the content of 'line  $a$  is parallel to line  $b$ '). Indeed, determinable properties are to a certain extent "vague" (i.e. in need of specification): to say that something is red is vague in the sense that it is not specified which shade of red we are referring to. Yet to say that a partial world  $X$  has  $d_1$  and  $d_2$  as witnesses does not seem to be vague. Thus in what sense  $P_{d_1, d_2}$  is a determinable? We may understand this point by considering a fundamental fact regarding determinates and determinable: a certain determinable (determinate) property  $P$  is vague (specific) only when compared with a another property  $Q$  which belongs to a different level of

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<sup>2</sup>Recall that this property is determined by all properties of containing lines  $\alpha, \beta$  such that  $\alpha$  and  $\beta$  are witnesses of the content  $c$  and  $\alpha$  has direction  $d_1$  and  $\beta$  has direction  $d_2$ .

specification. This because the notion of determinable is relative. For instance, ‘red’ is vague when compared to ‘scarlet’, but specific when compared to ‘colored’. And  $P_{d_1, d_2}$  is vague compared to any property that determines it. Let  $P_{l_1, l_2}$  be the property that a partial world has iff  $l_1$  and  $l_2$  are the witnesses of the content  $c$  for which the terms ‘line  $a$ ’ and ‘line  $b$ ’ respectively stand. According to the proposal, if the direction of  $l_1$  is  $d_1$  and the direction of  $l_2$  is  $d_2$ , then  $P_{l_1, l_2}$  determines  $P_{d_1, d_2}$ . Notice that from the fact that a world  $X$  has  $P_{l_1, l_2}$  it follows that  $X$  is composed by  $l_1, l_2, d_1$ , and  $d_2$ ; this because from the fact that  $X$  sustains  $c$  it is required that it contains all kinds of witnesses. On the other hand, from the fact that a world  $X$  has  $P_{d_1, d_2}$  it follows that  $X$  contains  $d_1$  and  $d_2$  and two unspecified objects as lines respectively having  $d_1$  and  $d_2$  as directions. Thus  $P_{d_1, d_2}$  is less specific than  $P_{l_1, l_2}$  regarding the constitution of partial worlds that have  $P_{d_1, d_2}$ . More precisely,  $P_{d_1, d_2}$  is less specific than  $P_{l_1, l_2}$  regarding the way all worlds having  $P_{d_1, d_2}$  sustain  $c$ , i.e. regarding the complete inventory of possible witnesses of  $c$  that a world contains.

We may summarize the hierarchy of levels of specification composing the internal structure of the content of an atomic sentence  $c$  as follows. To sustain  $c$  is vague in the sense that it needs to be specified whether the sustainment is positive or negative. To positively sustain  $c$  is vague in the sense that it must be specified in which way a world  $X$  makes the content true, i.e. which are the constituents of  $X$  that play the role of witnesses of the truth of  $c$  in  $X$ . Suppose now that  $a_1, \dots, a_n$  are the constituents of  $X$  that witnesses the truth of  $c$  according to a certain non-basic determinant carving. The property of sustaining  $c$  that  $X$  has by a witnessing of the truth of  $c$  by means of the constituents  $a_1, \dots, a_n$  is vague in the sense that it needs to be specified which are the remaining constituents of  $X$  that witnesses  $c$  in different ways. When it comes to the property that  $X$  has iff  $X$  contains some objects as witnesses of the basic carving (i.e. the super-determinate), the composition of  $X$  is completely specified: all possible ways in which  $X$  sustains  $c$  are clarified by the property at issue.

## 4.4 Carving up the content of an atomic sentence according to its internal structure

It is time to take a stock. In section 4.2 I have proposed a preliminary definition of content carving (Content Carving (I)) in terms of functional decomposition. This definition has been

subject to the charge of not discriminating between arbitrary or gerrymandered decompositions of the content and decompositions that effectively introduce the entities with which the content is concerned. Such a failure has been characterized as the fact that in general the operation of content carving thus defined may not be proven to be content preserving. Successively, I argued that to trace the distinction between relevant and arbitrary carvings the notion of internal structure of the content is needed. In the case of the content of an atomic sentence, I have proposed to define this structure in terms of a system of determinables and determinates: in other words, I have shown that the property of sustaining a content is a determinable property whose determinates are either properties of positive and negative sustainment or properties that partial worlds have iff they have certain constituents playing the role of witnesses of the truth-value of the content. This view has been based on Funkhouser's account of the relation of determination framed in terms of the notions of determination dimensions and property space. The conclusion to be drawn is that the carvings that are in accordance with the internal structure of the the content are those having argument component composed by a selector function that partitions the space of partial worlds into determinates of the properties of positive and negative sustainment of the content.

In this section I will merge the account of content carving as functional decomposition with the account of relevant carvings as related to determinates of the properties of content sustainment. The result will be an improved definition of content carving in terms of functional decomposition of special sorts of functions. As we will see, the interesting outcome of this new definition is that the operation of content carving may be proven to be content preserving. Such an outcome will be strengthened by proving that when a content is carved up according to its internal structure, the relation between the components of the content (i.e. the functional component and the selector) may be taken – at least from a formal point of view – as a relation of proper parthood. Thus different carvings according to the internal structure of the content of an atomic sentence may be conceived as different decompositions of the same content into parts. The reader interested in this latter result may find the details in the appendix at the end of this work.

## Determinant carvings

My starting point is the fact that given a content  $c$  of an atomic sentence, the internal structure of  $c$  may be used to highlight special features of functions. To this end, I need to introduce several definitions.

The first definition I will give is just an abbreviation and it is not aimed at introducing new notions. Given a content  $c$  and an object  $a$ , the expression *the property of sustaining  $c$  with  $a$*  denotes:

- (i) the property of positively sustaining  $c$ , if  $a$  is the truth-value True,
- (ii) the property of negatively sustaining  $c$ , if  $a$  is the truth-value False,
- (iii) the property of containing  $a$  as witness of the truth-value of  $c$ , in case  $a$  is not a truth-value and is a witness of  $c$  in some partial worlds,

Keeping in mind this abbreviation, I will give the following definition:

**Definition 1.** (*Determination Function*)

Let  $c$  be a content and  $f$  a function having the set  $A$  as domain and the set  $B$  as image. The function  $f$  is a *determination function for  $c$*  iff

- (i) for all  $x \in A$ , there is at least one partial world  $S$  such that  $S$  sustains  $c$  with  $a$ ,
- (ii) for all  $y \in B$ , there is at least one partial world  $S$  such that  $S$  sustains  $c$  with  $y$ ,
- (iv) for all  $x \in A$ , all properties of the form ‘ $S$  sustains  $c$  with  $x$ ’ are determinates of the property of sustaining  $c$  belonging to the same level of specification,
- (v) for all  $y \in B$ , all properties of the form ‘ $S$  sustains  $c$  with  $y$ ’ are determinates of the property of sustaining  $c$  belonging to the same level of specification,
- (vi) For all  $\langle x, y \rangle \in f$ , the property of sustaining  $c$  with  $x$  determines the property of sustaining  $c$  with  $y$ .

In other words, a function is a determination function iff whenever  $f$  associates the value  $b$  to the argument  $a$ , the property of containing  $a$  as witness of  $c$  belongs to a higher level of specification than the property of containing  $b$  as witness of  $c$ . Notice that the adopted terminology may be misleading (I couldn’t find a better one): when we say that  $f$  is a determination function we

intend that the argument of  $f$  define a property that determines the property defined by the value of  $f$  and not the contrary. Perhaps I should have called  $f$  a ‘determinate of’ function.

Consider the function  $F$  defined in the set  $A$  of possible extensions of the name ‘Socrates’ and given  $x \in A$  returns the value **T** (true) if  $x$  is wise and the value **F** (false) if  $x$  is not wise. Being  $c$  the content of ‘Socrates is wise’ and being  $a$  a possible extension of ‘Socrates’ having the property of being wise, it is easy to see that the property of sustaining  $c$  with  $a$  determines the property of sustaining  $c$  with **T** (positive sustainment). Moreover, all possible extensions of the name ‘Socrates’ define properties of witnessing the truth-value of the content of ‘Socrates is wise’ belonging to the same level of specification; similarly, the elements of the image of  $F$  are associated with the properties of positively and negatively sustaining the content of ‘Socrates is wise’ which belong to the same level of specification. Thus the function  $F$  is a determination function for the content of ‘Socrates is wise’.

**Definition 2.** (*Sustainment Function*)

Let  $c$  be a content and  $f$  a function having a certain set of partial worlds as domain and having a set  $B$  as image. The function  $f$  is a *sustainment function* for  $c$  iff

- (i) For all  $\langle x, y \rangle \in f$ ,  $x$  sustains  $c$  with  $y$ ,
- (ii) For all  $y \in B$ , all properties of the form ‘ $S$  sustains  $c$  with  $y$ ’ are determinates of the property of sustaining  $c$  belonging to the same level of specification.

Clearly, the content function  $\mathcal{I}$  of a content  $c$  is a sustainment function for  $c$ , for for every partial world  $s$  in the domain of  $\mathcal{I}$ ,  $\mathcal{I}(s) = \mathbf{T}$  iff  $s$  sustains  $c$  with **T** (i.e. positively sustains  $c$ ). Another example of sustainment function is the selector  $\mu$  of a non-arbitrary carving of  $c$ ; indeed, for every partial world  $s$ ,  $\mu(s) = a$  iff  $s$  contains  $a$  as witness of the truth-value of  $c$ , i.e.  $s$  sustains  $c$  with  $a$ . In both cases it is easy to verify that clause (ii) also holds.

Again, the terminology may engender some misunderstandings. A sustainment function  $f$  such that  $f(s) = a$ , is a function that receives as argument “what sustains the content” (i.e. a partial world) and returns as value the entity representing “the way in which the content is sustained”, i.e. the entity being either a truth-value or a witness of the content. Perhaps a better terminology would have been “mode of sustainment function”; yet also in this case I preferred the less complex terminology.

It is interesting for our purpose to consider special cases of functional decomposition of both

sustainment and determination functions.

**Definition 3.** (*Decomposition by determination of determination functions*)

Let  $f$  be a determination function for a content  $c$ . We say that  $f$  is *decomposed by determination into  $g$  and  $h$*  iff

- (i)  $f = g \circ h$
- (ii) The domain of  $g$  coincides with the image of  $h$ ,
- (iii) Both  $g$  and  $h$  are determination functions for  $c$ .

To understand this definition, consider  $x, y, z$  such that  $f(x) = g(h(x)) = y$  and  $h(x) = z$ . By the fact that  $f$  is a determination function for  $c$  it follows that the property of sustaining  $c$  with  $x$  determines the property of sustaining  $c$  with  $y$ . By the fact that  $g$  and  $h$  also are determination functions for  $c$ , it follows that the property of sustaining  $c$  with  $x$  determines the property of sustaining  $c$  with  $z$  which in turn determines the property of sustaining  $c$  with  $y$ . Hence, the decomposition by determination of a determination function  $f$  amounts to the introduction of an intermediate level of specification in the relation of determination expressed by  $f$ . In the next chapter we will see more details concerning this sort of decomposition that will be essential to the definition of the operation of content recarving.

I will now introduce the notion of decomposition by determination for sustainment functions:

**Definition 4.** (*Decomposition by determination of sustainment functions*)

Let  $f$  be a sustainment function for a content  $c$ . We say that  $f$  is *decomposed by determination into  $g$  and  $h$*  iff

- (i)  $f = g \circ h$ ,
- (ii) The domain of  $g$  coincides with the image of  $h$ ,
- (iii)  $h$  is a sustainment function for  $c$  and  $g$  is a determination function for  $c$ .

This definition may be understood as follows. Let  $x, y, z$  be such that  $f(x) = y$ ,  $h(x) = z$ , and  $g(z) = y$ . From the fact that  $f$  is a sustainment function for  $c$ , it follows that the partial world  $x$  has the property of sustaining  $c$  with  $y$ ; from the fact that  $h$  is a sustainment function for  $c$  it follows that  $x$  has also the property of sustaining  $c$  with  $z$ ; from the fact that  $g$  is a determination function for  $c$  it follows that the property of sustaining  $c$  with  $z$  determines

the property of sustaining  $c$  with  $y$ . Thus a decomposition by determination of a sustainment function  $f$  indicates that the internal component  $h$  is associated with properties of sustainment that determine the properties of sustainment with which  $f$  is associated.

The notion of decomposition by determination is useful to give an improved definition of carving. To this end, I introduce the following notational convention. Being  $f$  a determination (sustainment) function for  $c$ , the expression ' $f = g \bullet h$ ' stands for the fact that  $f$  is decomposed by determination into  $g$  and  $h$ . Clearly the symbol ' $\bullet$ ' does not denote a new operator of functional composition but just the fact that functional composition is performed on functions satisfying the requirements of Definition 3 or 4. The improved definition of carving is the following:

### **Determinant Carving (DC)**

Let  $c = \langle \mathcal{I}, \sigma \rangle$  be a content and  $\chi = \langle F, u \rangle$  a pair of functions. We say that  $\chi$  is a determinant carving of  $c$  iff:

- (i)  $F \circ u = \mathcal{I} \circ \sigma$
- (ii) There is a selector function  $\mu$  such that

$$\mathcal{I} = F \bullet \mu \text{ and } u = \mu \circ \sigma$$

Clause (i) is immediately understandable: it just represent the identity of intension. To understand clause (ii), recall that the function  $\mathcal{I}$  is a sustainment function for  $c$ ; thus the expression  $\mathcal{I} = F \bullet \mu$  indicates that  $F$  must be a determination function for  $c$  and  $\mu$  must be a sustainment function for  $c$  such that the properties of sustainment associated with  $\mu$  are determinates of the properties of sustainment associated with  $\mathcal{I}$ . More precisely, suppose that given a partial world  $s$ ,  $\mathcal{I}(s) = \mathbf{T}$  and  $\mu(s) = a$ . By clause (ii), the property of sustaining  $c$  with  $a$  determines the property of sustaining  $c$  with  $\mathbf{T}$ ; in other words, the property of containing the object  $a$  as witness of the content  $c$  determines the property of positively sustaining  $c$ . Hence the given definition captures the fact that the selector used to define the argument component of a certain carving must pick out only the entities that are witnesses of the truth-value of the content, i.e. entities that define properties of partial worlds that determine the determinable properties of content sustainment. As a consequence, a carving defined according to (DC) divides the



content into function and argument components in accordance with the internal structure of the content.

Clearly a content  $c$  of an atomic sentence may admit multiple determinant carvings; this because the property of sustaining  $c$  may be determined along different levels of specification, to which different kinds of witnesses correspond. Provided the soundness of *GLA64*, the case of lines and directions may be considered as a case of a content that admits at least two different determinant carvings.

In what sense (DC) improves on Content Carving (I)? We have seen that in the case of the latter definition it was not possible to prove that the operation of carving is content preserving. Thus our question becomes: in what sense (DC) makes possible to show that the operation of obtaining determinant carvings is content preserving? A content insofar as it is associated with a determinable property, is in need of specification: the operation of carving up a content may be conceived as an operation of specification, namely the specification of the entities that make the content true or false. To carve up a content without considering its internal structure may consist in an irrelevant specification, i.e. a specification that is not concerned with what the determinable property of sustaining the content demands.

An example may help to clarify the matter. The internal structure of a content has been defined as the internal structure of a determinable property. Suppose that we are considering a set  $A$  of objects having a certain determinable property, e.g. all of them are colored. An operation of specification on  $A$  may be seen as the partitioning of  $A$  into sets of objects having a certain aspect in common. Being colored the determinable property that all objects in  $A$  have, a relevant specification is the partitioning of  $A$  into cells of objects having the same color according to a certain level of specification. Let  $A_p$  be a certain partitioning of  $A$ ; to show that the operation of specification associated with  $A_p$  is concerned with colors we have to show that every cell composing  $A_p$  is a set of objects having the same color according to a certain level of specification. Yet  $A_p$  may partition the set  $A$  according to a different criterion, for instance the cells in  $A_p$  may be the equivalence classes of a certain relation of similarity having nothing to do with colors; as a consequence, the associated operation of specification is altering the aspect according to which we were requested to be specific. Even if all objects belonging to the same cell of  $A_p$  are similar under a fixed respect, this does not grant that the respect under which they are similar is the required one. Thus along the specification, the respect under which we were requested to

specify has been altered.

A set of worlds  $K$  making true (false) a certain content  $c$  of an atomic sentence is like the set  $A$  of the previous example: they all have a determinable property in common, yet a relation of similarity having nothing to do with the sustainment of  $c$  may partition  $K$ ; such a relation of similarity may be to have a certain part in common, for instance the part picked out by an argument component  $u$ . The simple fact that an argument component  $u$  defines a sub-partitioning of  $K$  does not grant that the specification operation is the one required by  $c$ . Even if all worlds belonging to the same  $u$ -cell are similar insofar as they have a part in common, this does not grant that the part they have in common is the part witnessing the truth-value of  $c$ . On the contrary, determinant carvings are the right determinates to be considered when  $c$  is considered: thus they are specifications in the relevant direction.

## 4.5 The internal structure of the content of molecular sentences

So far I have argued that in the case of atomic sentences, the internal structure of the content may be characterized as determinates/determinables structure. In this section we will see that this is not the case of molecular sentence, but that the internal structure of the content may recursively be defined starting from the internal structure of the atomic components. In other words, given the determinant carvings of the atomic sentences composing a certain molecular sentence  $S$ , all non-arbitrary ways of carving up the content of  $S$  may be determined.

Let  $p$  be, for instance, the conjunction of two atomic sentences  $q, r$ , i.e.  $p = q \wedge r$ . An interesting question is whether we may define a determinant non-arbitrary carving of  $p$  in terms of determinant carvings of  $q$  and  $r$ . In other words, we are interested in the possibility of building a recursive theory of non-arbitrary carvings according to which given determinant carvings of atomic sentences, carvings of molecular sentences may be defined.

Even in the case of a molecular sentence  $S$ , the notion of carving according to the internal structure of the content may be defined considering the witnesses of the truth-value of the content of  $S$ .

Suppose that  $W_1$  is a world in which  $q \wedge r$  is true; clearly, the truth of  $q \wedge r$  is witnessed by both

the witnesses of  $q$  and  $r$  associated with their considered determinant carvings. For instance, being  $q$  the sentence ‘Plato is passionate’ and  $r$  the sentence ‘Socrates is wise’, the truth of  $q \wedge r$  is jointly witnessed in  $W_1$  by both Socrates and Plato (by the individuals for which ‘Plato’ and ‘Socrates’ stand in  $W_1$ ).

Suppose now that  $W_2$  is a world in which  $q \wedge r$  is false with  $q$  false and  $p$  true in  $W_2$ . The falsity of the content of  $q \wedge r$  is witnessed just by the witness of the falsity of the content of  $q$  according to the considered determinant carving of  $q$ . Considering the previous example, the individual for which ‘Plato’ stands in  $W_2$  witnesses the falsity of ‘Plato is passionate and Socrates is wise’.

Similarly, in a world  $W_3$  in which  $q \wedge r$  is false with  $q$  true and  $p$  false, is witnessed by the entity witnessing the falsity of  $p$  according to the considered determinant carving of  $p$ .

Finally, a world  $W_4$  in which  $q \wedge r$  is false with  $q$  and  $r$  both false, may be witnessed indifferently by the entity witnessing the falsity of  $q$  and that witnessing the falsity of  $r$ . In this case, there is no unique entity that the argument component of a non-arbitrary carving of  $q \wedge r$  must pick out. Under this respect, this case is similar to the case of witnessing an existential sentence: different entities are equally good at witnessing the truth-value of the content. Thus we may adopt the same solution as in the case of existential: the argument component picks out the set of all witnesses. Considering the previous example, the witness of the falsity of ‘Plato is passionate and Socrates is wise’ in  $W_4$  is the set containing the extensions of ‘Plato’ and ‘Socrates’ in  $W_4$ .

Let  $\chi_q$  a determinant carving of  $q$  having argument component  $u_q$  and functional component  $F_p$  and  $\chi_r$  a determinant carving of  $r$  having argument component  $u_r$  and functional component  $F_r$ . We may resume our analysis as follows: the carving  $\chi_p$  of  $q \wedge r$  according to the internal structure of the content obtained from  $\chi_q, \chi_r$  has as argument component the function  $u_p$  that associates to a world  $w$  the value  $u_q(w)$  if  $q$  is false and  $r$  is true in  $w$ , the value  $u_r(w)$  if  $r$  is false and  $q$  is true in  $w$ , the value  $\langle u_q(w), u_r(w) \rangle$  if  $q$  and  $r$  are both true in  $w$ , the value  $\{u_q(w), u_r(w)\}$  if  $q$  and  $r$  are both false in  $w$ . Moreover, the functional component  $F_p$  of the carving  $\chi_p$  of  $q \wedge r$  will be the function defined in the image of  $u_p$  that associates to an entity  $X$  the value false in two cases: 1) if either  $F_q(X) = \mathbf{F}$  or  $F_r(X) = \mathbf{F}$ , 2) if  $X$  is a set, for every  $y, z \in X$  either  $F_q(y) = F_r(z) = \mathbf{F}$  or  $F_q(z) = F_r(y) = \mathbf{F}$ . In all other cases the value true.

By similar a carving according to the internal structure of the content of a sentence  $qKr$  with  $q, r$  atomic sentences and  $K$  a binary logical connective may be defined.

The case of negation is easy to deal with. Being  $F_p$  and  $u_p$  respectively the functional component and the argument component of a determinant carving of an atomic sentence  $p$ , it is possible to define a carving respecting the internal structure of the content of the sentence  $\neg p$ . It suffices to consider as argument component the function  $u_p$  and as functional component the function  $F_{\neg} \circ F_p$ . Indeed, a sentence and its negation have the same witnesses, the only difference is in the truth-value they witness: verifiers of  $p$  are falsifiers of  $\neg p$  and vice versa.

According to the given considerations, given atomic sentences  $p_1, \dots, p_n$  and a molecular sentence  $\phi(p_1, \dots, p_n)$  obtained by combining  $p_1, \dots, p_n$  by means of propositional connectives, all carvings of  $\phi(p_1, \dots, p_n)$  according to the internal structure of the content may be recursively determined according to all possible combinations of the determinant carvings respectively of  $p_1, \dots, p_n$ . This is an important result, because if all possible carvings according to the internal structure of the content are determined, all possible argument components are also determined, and thus all possible non-arbitrary sub-partitionings of the set of partial worlds sustaining the content of  $\phi(p_1, \dots, p_n)$  are also determined. In this way the internal structure of the content of any sentence may be determined.

Another important consequence of the considerations developed in this section is the fact that the property of sustaining the content of a molecular sentence is not – in general – a determinable property. Consider the case of the conjunction ‘Plato is passionate and Socrates is wise’; given that the constituents of partial worlds witnessing the truth-value of the content of this sentence are in some cases both the extensions of ‘Socrates’ and ‘Plato’, in some cases the extension of ‘Socrates’, in some cases the extension of ‘Plato’ and in some other cases the set of both extensions, it is not possible to define a unique list of kinds of constituents that should represent the determination dimensions.

The issue may be understood from another perspective. Sustainment properties of the content of a molecular sentence are combinations of sustainment properties of the contents of the atomic sentences composing them. For instance, the property of positively sustaining the content of  $q \wedge r$  is the conjunction of the properties of positively sustaining respectively  $q$  and  $r$ ; on the other hand, the property of negatively sustaining the content of  $q \wedge r$  is the disjunction of the properties of negatively sustaining respectively  $q$  and  $r$ . Now it is clear that neither the conjunction nor

the disjunction of determinable properties result in a determinable property. For instance, the property of being colored and having mass is not a determinable, for there is not a unique way of specifying this property.

## 4.6 Spurious carvings

So far I have given two definitions of content carving in the semantics of total worlds, one labeled **Content Carving (I)** and the other **Determinant Carving (DC)**. One difficulty that arises with both is that none of the two definitions classifies as a carving a way of regarding a content requiring a world variable. We have seen that whenever we are dealing with a functional component associated with an extrinsic property or with an external relation a world variable is needed.

In chapter 3 I have argued that ways of regarding a content including a world variable cannot be banished for in many cases these ways of regarding a content reflect the syntactic structure of the sentence. Thus they are carvings according to the syntactic based account: they represent divisions into functional and argument components that correspond to the division into function and argument suggested by the syntactic structure of the sentence. For instance, the sentence  $Pa$  where  $P$  is a first-order predicate standing for an extrinsic property and  $a$  is a singular term, has a standard reading of its logical form consisting in  $P(x)$  as functional part and  $a$  as argument part; from a semantic perspective, the function  $F_P$  associated with  $P(x)$  requires a world variable.

Therefore we must be open to admit cases in which a particular way of regarding a content that is suggested by the syntactic structure of a sentence is neither in accordance with **Content Carving I**, nor with **(DC)**. Clearly, those cases are spurious, in the sense that the syntactic structure of the sentence is – so to say – masking the internal structure of the content it expresses. In other words, *the syntactic structure of the sentence is not always transparent with respect to the internal structure of the content it expresses*. This fact may be spelled out with more accuracy.

Extrinsic properties are properties that an object has in virtue of the way it interacts with the world. Thus extrinsic properties are relational properties. Given an extrinsic property  $P$  and an object  $a$ , there are three possible cases for this fact to take place:

1.  $a$  has  $P$  in virtue of the fact that  $a$  bears an external relation  $R$  to some proper part of the world distinct from  $a$ ,
2.  $a$  has  $P$  in virtue of the fact that  $a$  bears an internal relation  $R$  to some proper part of the world distinct from  $a$ ,
3.  $a$  has  $P$  in virtue of the fact that  $a$  bears an internal relation  $R$  to the entire world.

Case 1 divides into two sub-cases. The property  $P$  may be such that by considering larger and larger wholes, the external relation that  $a$  bears to some part  $b$  of the worlds holds in virtue of an internal relation that  $a$  bears to some part of the world  $c$  larger than  $b$ . Alternatively, it may be the case that the property  $P$  is *irreducibly extrinsic*; in other words, even by considering larger and larger wholes,  $a$  has  $P$  always in virtue of some external relation that  $a$  bears to some part of the world. It should appear clear that this process of considering larger parts of the world with which  $a$  is related ends up by considering an internal relation that  $a$  bears to the entire world. Therefore, case 1 ultimately reduces either to case 2 or to case 3.

Case 3 is unproblematic: if  $a$  has  $P$  in virtue of some internal relation that  $a$  bears to the entire world, then the parts of the world that witness the truth of  $Pa$  are the entire world and  $a$  itself. As a consequence, the maximal relevant part of the world with which the content of  $Pa$  is concerned is the world itself. Being  $\langle \mathcal{I}, \sigma \rangle$  the content of  $Pa$ , the function  $\sigma$  is nothing but the identity function  $I_{\mathbb{W}}$  over the set of all possible worlds  $\mathbb{W}$ . It is easy to see that in this case the way of regarding the content of  $Pa$  that considers  $Px$  as function and  $a$  as argument qualifies as a carving, for it satisfies **Content Carving (I)** and the related distinction between functional and argument component is based on a syntactic distinction. Moreover, this carving may be reasonably considered as a determinant carving. Thus the presence of the world variable does not represent a problem in this case.

Case 2 represents our problem. This case also divides in two sub-cases. In the first sub-case,  $a$  has  $P$  in virtue of the fact that there is a unique part of the world to which  $a$  bears an internal relation  $R$ . Let  $u_b$  be the function that associates to a possible world  $w$  the part of  $w$  to which  $a$  bears the relation  $R$  whenever  $a$  has  $P$ . In this case, a determinant carving of the content of  $Pa$  is the one having as functional component the function  $F_R$  and argument component the vectorial function  $(u_b; u_a)$  such that  $(u_b; u_a)(W) = \langle u_b(W), u_a(W) \rangle$ . Consider now this carving and the way of regarding the same content based on the syntactic structure of the sentence: for every possible world  $w$ ,

$$F_P(w, u_a(w)) = F_R(u_b(w), u_a(w))$$

we may mathematically characterize the relation between these two ways of regarding the content of  $Pa$  by a relation between their functional component. Let  $A$  be the image of  $u_a$ ; for every possible world  $w$  and for every  $x \in A$ , the following holds:

$$F_P(W, x) = F_R(u_b(W), x)$$

Let  $h_b$  be the function having  $\mathbb{W} \times A$  as domain such that for every  $\langle w, x \rangle \in (\mathbb{W} \times A)$ ,  $h_b(w, x) = \langle u_b(w), x \rangle$ ; for every possible world  $w$  and every  $x \in A$ :

$$F_P(w, x) = F_R(h_b(w, x))$$

which is equivalent to:

$$F_P = F_R \circ h_b \quad (*)$$

The identity  $(*)$  may suggest an interesting interpretation of the fact that the way of regarding the content of  $Pa$  implying the world variable masks the internal structure of the content. We may see  $(*)$  as an analysis of the concept  $P(x)$  in terms of the relational concept  $R(x, y)$ ; as a consequence, the relational concept  $R(x, y)$  is a component of the “*analisantes*” into which the concept  $P(x)$  is analyzed. Therefore the content of the functional component of the determinant carving may be seen as included in the content of the functional component of the syntactically based carving requiring the world variable. Hence, we may characterize ways of regarding a content containing a world variable as *unbalanced and thus spurious cases of carvings: the functional component  $F_P$  contains too much content and – as a consequence – the argument component contains not enough content to pick out the entities that effectively witness the truth-value of the content*. For instance, consider the sentence ‘Ludovico Manin is a doge’; the property ‘ $x$  is a doge’ is in fact a relational property: to be a doge just is to be a chief of the former Republic of Venice. We may understand this fact as follows:  $x$  is a doge iff  $x$  bears the (internal?) relation of being a chief to the Republic of Venice. Thus for every possible world

$w$ , there is a unique specific entity to which Ludovico Manin of  $w$  must be related in order to be a doge: the Republic of Venice in  $w$ . Yet for the fact that the predicate ‘is a doge’ is – so to say – overloaded in content, it gives us the false impression that Ludovico Manin is the only part of the world that witnesses the truth-value of ‘Ludovico Manin is a doge’. And such an impression is false because two distinct worlds may perfectly agree on who is the bearer of the name ‘Ludovico Manin’ and at the same time disagree on whether he is a doge or not. The other part of the world we have to look at in order to determine the witnesses of the truth-value of the sentence is the entity that bears the name ‘Republic of Venice’ whose content is somehow included in the predicate ‘is a doge’.

There is another sub-case of case 2 that is to be considered. The case in which  $a$  has  $P$  in virtue of bearing an internal relation  $R$  to some part of the world, but there is no unique specific part to which  $a$  bears  $R$ . In other words, it may be the case that  $Pa$  is true in virtue of the fact that there is some  $x$  such that  $Rax$ . Thus the sentence  $Pa$  expresses the same content as the sentence  $\exists x Rax$ . Also in this case we may express the relation between the functional components in terms of a decompositional analysis. We may suppose that the sentence  $\exists x Rax$  is associated with the determinant carving having  $F_{\exists}$  as functional component and  $u_{Rax}$  as argument component. The two ways of regarding the content of  $Pa$  may be compared by considering both expressions of the truth-conditions of  $Pa$ : for every possible world  $w$ :

$$F_P(w, u_a(w)) = F_{\exists}(u_{Rax}(W))$$

Consider now the function  $f_R$  defined in  $\mathbb{W} \times A$  and such that for every  $\langle w, x \rangle \in (\mathbb{W} \times A)$ ,  $f_R(w, x)$  is the set of objects existing in  $W$  to which  $x$  bears the relation  $R$ . It is easy to see that for every  $w \in \mathbb{W}$ ,

$$f_R(W, u_a(W)) = u_{Rax}(W)$$

From which it follows that for every possible world  $w \in \mathbb{W}$ ,

$$F_P(W, u_a(W)) = F_{\exists}(f_R(W, u_a(W)))$$

which is equivalent to:



$$F_P = F_{\exists} \circ f_R \quad (**)$$

Again,  $(**)$  may be interpreted as a compositional analysis of the concept  $P(x)$  and the fact that the way of regarding the content of  $Pa$  implying a world variable masks the internal structure of the content may be explained by an “excess” of content of the functional component to which corresponds a “dearth” of content of the argument component. An example of this sub-case may be the sentence ‘John is not the tallest man’. This sentence should be analyzed as ‘There is  $x$  such that  $x$  is man and  $x$  is taller than John’. Again, the predicate ‘is not the tallest man’ is overloaded in content and gives us the false impression that John is the only part of the world that witnesses the truth-value of the sentence. On the other hand (and differently from the previous sub-case), the part of the world that we need to look at (besides John) is not a single entity picked out by a fixed law: every man taller than John witnesses the truth of the sentence. The condition that requires that there is at least one of such men is – again – included in the content of the predicate ‘is not the tallest man’.

In the next chapter we will see that when two carvings are related in the ways expressed by  $(*)$  and  $(**)$ , one is said to be a *recarving* of the other.

## 4.7 Conclusion

In this chapter we have started our analysis by giving a preliminary definition of the operation of carving up a content in terms of functional decomposition. We have seen that this definition is problematic insofar as it allows for the formulation of arbitrary or gerrymandered ways of regarding a content. We have interpreted this fact by saying that the given definition of carving does not ensure that the resulting operation is content preserving.

To overcome this difficulty, we have assumed that a content has a defined internal structure. To define this structure we have introduced several notions lying on the boundaries between semantic and metaphysics: the notion of witnessing the truth-value of a content, and the notions of content sustainment. It turned out that in the case of atomic sentence, the property of sustaining a content is a determinable property having as determination dimensions the kinds of witnesses of the truth-value of the content. As a consequence, the internal structure of the

content of an atomic sentence has been defined as a list of sub-partitionings having cells corresponding to determinates of the property of sustaining a content. In other words, different sub-partitionings correspond to different levels of specification of the truth-conditions. Carvings having argument components associated with such sub-partitionings have been labeled determinant carvings.

We have then shown that in the case of molecular sentences it is possible to define all non-arbitrary carvings by combining the components of all possible determinant carvings of the atomic sentence appearing in the molecular sentences. This result allowed us to build a recursive means to define non-arbitrary carvings for sentences of any complexity.

The last part of the chapter was concerned with the problem of ways of regarding the content containing a world variable, i.e. ways of regarding the content that do not qualify as carvings according to none of the two given definitions. We have qualified these cases as cases of spurious carvings in the sense that they are carvings that mask the internal structure of the content. Such a lack of transparency has been spelled out as an unbalanced distribution of the content over the functional and the argument component: while the functional component presents an “excess” of content, the argument component presents a “dearth” of content. For this reason, the argument component does not contain enough content to pick out the entities that effectively witness the truth-value of the content.

One last remark. The reader may wonder whether carvings performed according to the internal structure of a content are all acceptable carvings of a certain content: indeed, if the definition of content carving as simple functional decomposition is to be rejected since it allows for a certain arbitrariness, why do not simply say that (DC) is the definition of the carving operation?

To answer this question we have to rely on our pre-theoretical intuitions regarding the notion of carving up a content. If we look at all examples provided in this work, there seems to be two conceptions of carving, a syntactically based account and a purely semantic account. According to the syntactically based account, a carving is an internal division of a content into a functional and an argument component that reflects a syntactic division of the expression of a sentence into an unsaturated expression and a series of saturated expressions (or a single saturated expression). Clearly, there are syntactically based carvings that in spite of satisfying the definition Content Carving (I) do not qualify as determinant carvings. According to the purely semantic account, a carving of the content of a sentence is the internal division of a

content into an argument component that picks out the entities that are crucial to assess the truth-value of the sentence and a functional component that represents the requirement that those entities must satisfy in order to make the sentence true.

To adopt (DC) as the only definition of carving is to rely just on the purely semantic account. Yet this will be unsatisfactory because our theory of content will not take into account a significant pre-theoretical intuition of what a carving should be. Thus we should allow carvings that are not determinant carvings. Yet if we allow carvings defined according to Content Carving (I) we open the door to arbitrary and gerrymandered cases. Thus we may add to Content Carving (I) the requirement that not all functional decompositions of a content define a carving but just those that are based on a syntactic distinction between function and argument within the sentence, and that such a distinction must be traced according to what I have called *standard ways of regarding a content* in chapter 3. As a consequence we consider as carvings of a certain content either functional decompositions of the content function that are based on syntactic decompositions of the expression of a sentence expressing that content, or carvings that are in accordance with the internal structure of a content.

# Chapter 5

## Content Recarving

### 5.1 Introduction

We are coming closer to the central issue of the present discussion. Thus, it is worth to recall Frege's passage in §64 of the *Foundations* whose understanding and explication is our main purpose:

“ The judgment ‘*line  $a$  is parallel to line  $b$* ’, [...] can be taken as an identity. If we do this, we obtain the concept of direction and say: ‘*The direction of line  $a$  is identical with the direction of line  $b$* ’. Thus we replace the symbol  $\parallel$  by the more generic symbol  $=$ , through removing what is specific in the content of the former and dividing it between  $a$  and  $b$ . We carve up the content in a way different from the original way, and this yields us a new concept. ”(Frege, 1950), pp. 74-75

To my mind, the central claim of the passage is the fact that a statement of parallelism and the corresponding statement of direction identity are associated with different carvings of the same content. For instance, consider the sentences ‘ $a \parallel b$ ’ and ‘ $d(a) = d(b)$ ’ where  $d$  is the direction operator; according to our proposal, the former sentence may be associated with the carving  $\langle F_{\parallel}, (u_a; u_b) \rangle$  and the latter with  $\langle F_{=}, (u_{d(a)}; u_{d(b)}) \rangle$ .

Yet the passage says something more. It does not just say that there are two different carvings of the same content, yet it also says how one carving is obtained from the other. The identity of directions is obtained by removing what is specific in the content of the relation of parallelism and dividing it between  $a$  and  $b$ . Such an operation of removing “part” of a content and redis-

tributing it among the remaining “parts” is crucial for the validity of any abstraction principle: ‘ $a \parallel b$ ’ and ‘ $d(a) = d(b)$ ’ are logically equivalent because they have the same content and they have the same content because the way the latter sentence carves up its content may be obtained by performing the content preserving operation briefly described. And, more importantly, such an operation is content preserving because it amounts just to a redistribution of the same content; in the transformation of one carving into another, no content is allegedly added or neglected, for the procedure Frege describes amounts just to a different decomposition of the same content into parts: what is removed from one part is added to another.

Such an operation that occupies a central role in the justification of the truth of (all?) abstraction principles is commonly referred to as *content recarving*: for a content already carved up in a given way is carved up again in a different one. The understanding of this operation is crucial, for its description surely represents the most obscure portion of §64.

Why bother? There are several interesting reasons. Firstly, as argued in chapter 1, the notion of conceptual content itself is philosophically promising in offering interesting solutions to a battery of issues related to the so called paradox of analysis. And, as it will turn out, the operation of content recarving is a particular characterization of the general operation of analysis. Thus an understanding of the conditions of soundness of this operation together with the provided account of conceptual content may shed new light on the paradox of analysis. Secondly, as remarkably highlighted by Hale and Wright in a vast series of works ((Wright, 1983), (Wright, 1998), (Wright, 2001), (Hale, 1997), (Hale and Wright, 2001)), there could be good reasons to think that Frege gave up too early and thus that there could be a way of showing that arithmetic is analytic; at the heart of the matter – at least in Wright’s early writings – lies the notion of content recarving: to understand the limitations of such operation amounts to understand the limitations of many fundamental points of the Neo-Logicist program. Thirdly, the understanding of the operation of content recarving is connected with one of the most fundamental issues in metaphysics since Plato: the possibility of providing different characterizations of the same content in terms of different distinctions between properties and objects and the conditions that determine to what extent one is preferable to the other is a way of formulating the traditional problem of *carving reality at its joints*. Therefore, both the understanding of the passage and its extensive explication going beyond the mere interpretation have a conceptual import also outside of the context of Frege’s philosophy of mathematics.

In this chapter we will attempt a definition of the operation of content recarving using the conceptual machinery introduced so far. This will help us to draw interesting conclusions on the possible soundness of this operation.

## 5.2 Recarving as functional decomposition

Consider the case of lines and directions. We will start by analyzing the sentence ‘ $a \parallel b$ ’; this sentence is an atomic sentence in which an internal relation is predicated of a pair of lines. Therefore, as suggested in the previous section, it is reasonable to suppose that there is a determinant carving of ‘ $a \parallel b$ ’ having as functional component the function  $F_{\parallel}$  and as argument component  $(u_a; u_b)$  where  $u_a$  ( $u_b$ ) is the function that given a possible world  $W$  returns the extension of the term ‘ $a$ ’ (‘ $b$ ’) in  $W$ <sup>1</sup>. Suppose that the content of ‘ $a \parallel b$ ’ is  $c = \langle \mathcal{I}, \sigma \rangle$ . By definition of determinant carving it follows that:

$$\mathcal{I} = F_{\parallel} \bullet (f_a; f_b) \quad ; \quad (u_a; u_b) = (f_a \circ \sigma; f_b \circ \sigma) \quad (5.1)$$

where  $f_a$  and  $f_b$  are the selectors associated to lines  $a$  and  $b$ .

Consider now the sentence ‘ $d(a) = d(b)$ ’ where  $d$  is the direction operator and suppose that also this sentence expresses the content  $c$ ; moreover, suppose that there is a determinant carving of this content having  $F_{=}$  as argument component and  $(u_{d(a)}; u_{d(b)})$  as argument component. By the fact that this carving is another determinant carving of the content  $c$  it follows that:

$$\mathcal{I} = F_{=} \bullet (f_{d(a)}, f_{d(b)}) \quad ; \quad (u_{d(a)}, u_{d(b)}) = (f_{d(a)} \circ \sigma; f_{d(b)} \circ \sigma) \quad (5.2)$$

where  $f_{d(a)}$  and  $f_{d(b)}$  are the selectors of respectively the terms  $d(a)$ ,  $d(b)$ .

The expression (5.1) says that in every partial world sustaining  $c$ , lines  $a$  and  $b$  witness the truth-value of ‘ $a \parallel b$ ’; similarly (and thus the truth-value of  $c$ ), (5.2) says that the directions of  $a$  and  $b$  are the witnesses of the truth-value of ‘ $d(a) = d(b)$ ’ (and thus of the truth-value of  $c$ ) in the same partial worlds. In the former case, the property of sustaining  $c$  is determined by the determinate property of a partial world of having a certain pair of objects being the extensions

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<sup>1</sup>Recall that given two functions  $f_1, f_2$  having the same domain,  $(f_1; f_2)(x) = \langle f_1(x), f_2(x) \rangle$

respectively of ' $a$ ' and ' $b$ ' as witnesses of the truth-value of  $c$ ; in the latter case, the property of sustaining  $c$  is determined by the determinate property of a partial world of having a certain pair of objects being the extensions respectively of ' $d(a)$ ' and ' $d(b)$ ' as witnesses of the truth-value of  $c$ . As a consequence, (5.1) and (5.2) define different determinates of different levels for the same determinable. For every two partial worlds  $s_1, s_2$  strictly sustaining  $c$ , if

$$\langle f_a(s_1), f_b(s_1) \rangle = \langle f_a(s_2), f_b(s_2) \rangle$$

then

$$\langle f_{d(a)}(s_1), f_{d(b)}(s_1) \rangle = \langle f_{d(a)}(s_2), f_{d(b)}(s_2) \rangle$$

Moreover, by the facts that different pair of lines may have the same pair of directions there are two partial worlds  $s_1, s_2$  strictly sustaining  $c$  such that:

$$\langle f_{d(a)}(s_1), f_{d(b)}(s_1) \rangle = \langle f_{d(a)}(s_2), f_{d(b)}(s_2) \rangle$$

and

$$\langle f_a(s_1), f_b(s_1) \rangle \neq \langle f_a(s_2), f_b(s_2) \rangle$$

Therefore, for every partial world  $s$  strictly sustaining  $c$ ,

$$[s]_{(f_a; f_b)} \subset [s]_{(f_{d(a)}; f_{d(b)})} \quad (5.4)$$

(5.4) says that every determinate of the property of strictly sustaining  $c$  defined by the directions of  $a$  and  $b$  is partitioned by the determinates of the property of strictly sustaining  $c$  defined by lines  $a$  and  $b$ . As a consequence, the determinates of the latter kind determinate the determinates of the former. Now the function  $(f_{d(a)}; f_{d(b)})$  and the function  $(f_a; f_b)$  are both sustainment functions for  $c$ ; moreover, for every partial world  $s$  strictly sustaining  $c$ , the property of sustaining  $c$  with  $f_a(s), f_b(s)$  determines the property of sustaining  $c$  with  $f_{d(a)}(s), f_{d(b)}(s)$ . By this fact

and by (5.4), it is possible to define a determination function  $h$  for  $c$  such that: for every partial world  $s$  strictly sustaining  $c$ ,

$$h(f_a(s), f_b(s)) = \langle f_{d(a)}(s), f_{d(b)}(s) \rangle = (f_{d(a)}; f_{d(b)})(s) \quad (5.5)$$

By definition of decomposition by determination of the sustainment function  $(f_{d(a)}, f_{d(b)})$  for the content  $c$ :

$$(f_{d(a)}; f_{d(b)}) = h \bullet (f_a; f_b) \quad (5.6)$$

It is easy to understand which function is  $h$ : it is a function that receives as arguments a pair of objects for which ‘ $a$ ’ and ‘ $b$ ’ stand and returns the pair of their directions. Let  $\delta$  the function that given a line returns its direction; to express  $h$  in terms of  $\delta$  we adopt the following notational convention: given the  $n$ -ary function  $f_1$  and the  $m$ -ary function  $f_2$  and being  $p = n + m$ ,

$$[f_1; f_2](x_1, \dots, x_p) = \langle f_1(x_1, \dots, x_n), f_2(x_{n+1}, \dots, x_p) \rangle$$

Thus  $h = [\delta; \delta]$  and (5.6) may be expressed as:

$$(f_{d(a)}; f_{d(b)}) = [\delta; \delta] \bullet (f_a; f_b) \quad (5.6')$$

By the identity of content between the sentence of parallelism and that of identity of directions and (5.6’):

$$F_{\parallel} \bullet (f_a; f_b) = F_{=} \bullet (f_{d(a)}; f_{d(b)}) = F_{=} \bullet ([\delta; \delta] \bullet (f_a; f_b))$$

By the fact that  $[\delta; \delta]$  is a determination function for  $c$  and by the transitivity of the relation of determination, we may use the operator of functional decomposition  $\bullet$  distributively:

$$F_{\parallel} \bullet (f_a; f_b) = F_{=} \bullet ([\delta; \delta] \bullet (f_a; f_b)) = (F_{=} \bullet [\delta; \delta]) \bullet (f_a; f_b)$$



From which it follows that:

$$F_{\parallel} = F_{=} \bullet [\delta; \delta] \quad (5.7)$$

And interesting property of decompositions by determination is that it is associated – at least from a formal point of view – with a relation of proper parthood; more precisely, if a function  $f$  may be decomposed as  $f = g \bullet h$ ,  $h$  and  $g$  may be considered as proper parts of  $f$  (see the appendix at the end of this work). Thus (5.7) expresses a fact that is extremely interesting for our purposes: from the identity of content of the determinant carvings associated respectively with the sentence of parallelism and that of identity of directions, it follows that the functional component  $F_{\parallel}$  may be divided into two parts, i.e. into the argument component  $F_{=}$  and the pair of occurrences of the direction operator combined as  $[\delta; \delta]$ . Given that the functional components  $F_{\parallel}$  and  $F_{=}$  may be considered as the contents of the relations  $\parallel$  and  $=$ , we may conclude that (5.7) expresses the fact that two occurrences of the direction operator may be removed from the content of the relation of parallelism and applied to the argument component in which the content was originally divided; moreover, the remaining part of the content of the relation of parallelism is the relation of identity. This procedure is very similar to that described by Frege in §64. As a consequence, *we may preliminary characterize the operation of content recarving as the operation of decomposing by determination the functional component of a certain carving and apply the part corresponding to the internal component of the decomposition to the original argument component so that a new carving obtains. This new carving has as functional component the external part of the original functional component and as argument component the composition of the internal component of the decomposition by determination with the original argument component.*

According to this early account of content recarving, the operation of recarving amounts to a removal of content from a functional component of a given carving followed by the composition of the removed part with the original argument component. The operation described by Frege appears to be very similar, and the conceptual machinery of decompositions by determination and determinant carvings provide us with an explanation of expressions such as ‘removing’ and ‘part of the content’ that were obscure in Frege’s text.

From the given consideration we may attempt the following definition :

### Recarving by determination (RD)

Let  $\langle F, u \rangle$  be a determinant carving.

The carving  $\langle F', u' \rangle$  is a recarving by determination of  $\langle F, u \rangle$ , in symbols:

$$\Theta(\langle F, u \rangle; \langle F', u' \rangle)$$

iff there is a function  $h$  such that:

$$(i) \quad F = F' \bullet h$$

$$(ii) \quad u' = h \circ u$$

Condition (i) is a generalization of (5.7) and represents the fact of dividing the functional component and removing its “internal part” to obtain the functional component of the new carving. Condition (ii) is the generalization in the semantics of total worlds of (5.6') and represents the operation of applying what is removed from the original functional component to the original argument component thus obtaining the argument component of the new carving.

It is easy to show that  $\Theta(\langle F, u \rangle; \langle F', u' \rangle)$  implies that  $\langle F, u \rangle$  and  $\langle F', u' \rangle$  are determinant carvings of the same content. To this end, let  $\langle \mathcal{I}, \sigma \rangle$  be the content carved up as  $\langle F, u \rangle$ . By definition of determinant carving we have that  $\mathcal{I} = F \bullet f$ . By clause (ii) of (RD)  $F = F' \bullet h$ ; thus

$$\mathcal{I} = F \bullet f = (F' \bullet h) \bullet f = F' \bullet (h \bullet f) = F' \bullet (h \circ f)$$

. By clause (ii),

$$u' = h \circ u = h \circ (f \circ \sigma) = (h \circ f) \circ \sigma$$

. From the fact that  $\mathcal{I} = F' \bullet (h \circ f)$  and  $u' = (h \circ f) \circ \sigma$  it follows that  $\langle F', u' \rangle$  is a determinant carving of  $\langle \mathcal{I}, \sigma \rangle$  with the function  $(h \circ f)$  as selector, i.e. it is a determinant carving of the same content as  $\langle F, u \rangle$ . Therefore, the operation defined by (RD) is content preserving.

As in the case of the carving operation, the fact that (RD) defines a content preserving operation depends on the fact that the decomposition of the functional component (of the content function in the case of carving) is not the general operation of functional decomposition, yet is a decomposition by determination, i.e. it respects the internal structure of the content whenever

such a structure is determinates/determinable structure. As in the case of carvings, we may define a more general operation of content recarving that is not based on decomposition by determination, but on functional decomposition:

**General Recarving (GR):**

*Let  $\langle F, u \rangle$  be a carving of a certain content. The carving  $\langle F', u' \rangle$  is a general recarving of  $\langle F, u \rangle$ , in symbols:*

$$\Theta^*(\langle F, u \rangle; \langle F', u' \rangle)$$

*iff there is a function  $h$  such that:*

- (i)  $F = F' \circ h$
- (ii)  $u' = h \circ u$

The main advantage of introducing (GR) is that we may qualify as content recarving some operations on carvings that do not satisfy clauses (i) and (ii) of (RD).

Consider for instance, the case of a sentence  $p$  associated with a carving  $\langle F, \langle I_{\mathbb{W}}, u \rangle \rangle$  requiring a world variable that qualifies as a spurious carving. As we have seen in section 4.6 (formulas  $(*)$  and  $(**)$ ), there is a determinant carving  $\chi = \langle F_0, u_0 \rangle$  of  $p$  and a function  $h$  such that  $F = F_0 \circ h$  and  $u_0 = h \circ u$ . According to our definition of general carving, this fact may be interpreted as the fact that a spurious carving may be converted into a determinant carving by performing a general recarving. In other words, the lack of balance of a spurious carving may be adjusted by removing “a suitable amount content” from the functional component and adding it to the argument component.

The main drawback of the operation of general recarving is that it does not follow from (GR) that the operation is content preserving; more precisely, it is not possible to show that the function  $h$  introduced by decomposing the initial functional component  $F$  is not arbitrary, i.e. that the entities introduced by the recarving operation are relevant to what the content requires to be true or false. Therefore, it may happen that the functional decomposition of the functional component of a certain carving results in an alteration of the content. This may happen due to the possible arbitrariness in the introduction of a function  $h$  as internal component of the

decomposition of the functional component; in this case, nothing ensures that the entities introduced as values of  $h$  play an effective role in the truth-conditions of the considered sentence. In other words, the definition of  $h$  may be gerrymandered and entities that have nothing to do with the determination of the truth-value are arbitrarily introduced. Consider for instance the sentence ‘Socrates is wise’; one may choose a particular set of possible objects  $A$  and say that the elements of  $A$  correspond to all possible extensions of a new term ‘the relevant disposition of Socrates when wisdom is concerned’. Thus one may define the function  $h$  as a function that given a possible extension  $k$  of the proper name ‘Socrates’, returns an element of  $A$  being the main virtue of Socrates. Moreover, one may introduce a functional component  $F'$  corresponding to a new predicate ‘ $x$  is wisdom-like’. Thus one may consider the decomposition  $F = f' \circ h$  as a recarving of the concept ‘ $x$  is wise’ so that the content of the sentence ‘Socrates is wise’ may be recarved in such a way that the sentence ‘Socrates’ disposition when wisdom is concerned wisdom-like’. Clearly, there is no special reason why the objects in  $A$  are to be regarded as particular Socrates’ dispositions, nor there is a reason to think that dispositions may be objectified. The recarving is entirely arbitrary, though not forbidden by (GR).

In the case of content carving we have seen that such an arbitrariness may be avoided by referring to the syntactic structure of the sentence, i.e. by considering a possible division into function and argument of the expressions appearing in the sentence at issue. In the case of the recarving operation this cannot be done; for the recarving operation has no immediate linguistic counterpart and results in the introduction of new expressions that did not appear in the original sentence. Consider the case of an atomic sentence  $Pa$  and suppose that its carving  $\langle F_P, u_a \rangle$  is recarved in such a way that the new carving  $\langle F_0, u_0 \rangle$  obtains. By definition of general recarving, there is a function  $h$  such that  $F_P = F_0 \circ h$  and  $u_0 = h \circ u_a$ . To consider the syntactic counterpart of the resulting carving, we should introduce new expressions (or take into account already existing expressions not appearing in  $Pa$ ), specifically a predicate  $Q$  and a functor  $\phi$  such that  $F_Q = F_0$  and  $u_{\phi(a)} = h \circ u_a$ . In this case, we may conclude that  $Q\phi(a)$  is a recarving of  $Pa$ . Yet nothing in the syntax of  $Pa$  suggested that  $Q$  and  $\phi$  are the suitable expressions to syntactically express the carving  $\langle F_0, u_0 \rangle$ . As a consequence, an operation of general recarving if not guided by additional considerations, may result arbitrary at best, gerrymandered at worst.

Given the general understanding of the operation of recarving as a functional decomposition of the functional component, we may make a further generalization and include as carvings some other cases. Consider the classical case of conceptual analysis, i.e. the case in which a

conceptual expression  $Px$  is decomposed into a series of conceptual expressions  $P_1x, \dots, P_nx$  by means of boolean operators. For instance, being  $c^n$  a  $n$ -ary operator obtained by a combination of  $\neg, \vee, \wedge$ , the following truth may be analytic:

$$\forall x(Px \leftrightarrow c^n(P_1x, \dots, P_nx))$$

As a consequence, the sentence  $Pa$  has the same content as the sentence  $c^n(P_1a, \dots, P_na)$ . Also in this case we may establish the following relation between the functional components of particular ways of regarding the same content according to the syntactic structure of the two sentences:

$$F_P = F_{c^n} \circ (F_{P_1}, \dots, F_{P_n})$$

Note that also in this case the functional component of the initial sentence has been decomposed; yet in this case no part of the decomposition is used to define a new argument component: the decomposition of the functional component  $F_P$  amounts just to a re-organization of the expression of the function  $F_P$ . When a functional component of a sentence is decomposed into several functional components and no part of the decomposition is used to define a new argument component, we say that we are in presence of a *merely conceptual recarving*. Traditional conceptual analysis is a case of merely conceptual recarving.

So much for the technical aspects of the recarving operation. In the following sections we will focus on the main philosophical consequences of admitting a sound operation of content recarving as defined in this section.

### 5.3 Recarving and determination

We have seen that the operation (RD) amounts to decompose by determination the functional component of a carving and apply the internal part of this decomposition to the initial argument component. From the previous chapter we know that the functional component represents the relation of determination between properties that partial worlds have insofar as they have certain constituents as witnesses of the truth-value of the content and the properties of positively or

negatively sustaining the content. Thus to recarve by determination amounts to decompose the determination relation by introducing an intermediate level of determination between the constituents of the initial carving and the truth-values of the content. We have seen how this holds for the case of direction and lines, and I have attempted an explanation for the fact that the property that a partial world has iff contains certain lines as witnesses of the truth value of the content is a specifier of the property of containing their directions as witnesses of the truth-value of the content.

However, there is an apparent conflict between the fact that the operation of content recarving amounts to the introduction of less determinate world properties and the fact that when we recarve a certain content by decomposing the functional component, the resulting carving seems to be more specific under a certain respect. For instance, when we recarve the content of ' $a \parallel b$ ' so that the determinant carving of ' $d(a) = d(b)$ ' obtains, we pass from considering lines to consider directions; more precisely, given that directions are associated with a particular aspect of lines (e.g. their orientational properties) and not with lines in their entirety, it seems that as a result of the operation of recarving by determination we focus on a more restricted topic that is more relevant to the requirement imposed by the functional component. Directions may be seen as associated with the aspect of lines that is crucial to assess their parallelism; on the other hand, when we consider lines in their entirety we are also considering aspects that are not so crucial when parallelism is concerned: for instance, the position of lines with respect to a certain reference frame.

To clarify the generality of the issue and to avoid possible confusions, we may refer to a different example which is not ontologically controversial and less abstract. Suppose that the predicate 'is blackbearded' is true of an individual  $x$  iff  $x$  has a black beard. Consider now the sentences 'Plato is blackbearded' and 'Plato's beard is black': under a particular understanding of the meaning of 'blackbearded' and 'beard' the two sentences express the same content. Moreover, we may suppose that they define two determinant carvings, one having as argument component *Plato* and the other *Plato's beard*. It is evident that the carving concerned with *Plato's beard* is more specific, for when "blackbeardedness" of an individual  $x$  is concerned, the beard of  $x$  matters more than any other part of  $x$ . On the other hand, to carve up the content with *Plato's beard* amounts to introduce less determinate properties of partial worlds and thus less specific properties. How this could be possible?

The answer is that we are talking about two different sorts of specification that are concerned with different respects. The more determinate is a property of containing witnesses for a content, the more accurate is the description of the constitution of a world that the property offers. The more accurate is the description of the constitution of the world, the wider is the scope of how the properties characterizes the world; in this sense the property is more specific, for it says more on how the world is composed. The wider the scope of this characterization of the world, the less specific is the property about which part of the world is crucial (or more relevant) to assess the truth-value of the content. Therefore, in one case we have a specification of how the world is composed, in the other case a specification of which part of the world is more relevant to assess the truth-value. And it should appear clear that there is a trade off between these two specifications: the more accurate is the one, the more general is the other. To say that a certain partial world  $s$  sustaining the content of ‘Plato is blackbearded’ contains a certain object  $b$  satisfying the description ‘Plato’s beard’ as witness, is to be very specific regarding which part of the world matters to assess the truth-value. On the other hand, it results in a less specific description regarding how the world is composed: for we know that  $b$  is Plato’s beard, we know that by the requirement of content sustainment  $s$  must contain an object bearing the name ‘Plato’, yet the property says nothing about who is Plato in  $s$ . It may well be possible that two worlds are indistinguishable with respect to Plato’s beard, and at the same time distinguishable regarding Plato in its entirety.

## 5.4 Recarving and specification

The issue of a trade off between the level of specification of a determinable property of content sustainment and the level of specification of the part of the world that witness the truth-value of the content is related to another trade off, i.e. the trade off between the levels of specification respectively of the functional and argument component of a carving. The more specific an argument component is regarding the portion of the world that witnesses the truth-value of the content, the less specific the condition imposed by the functional part results. We may show this fact through an example. We may linguistically represent different levels of specification of different carvings, by using the notation of lambda calculus. From the perspective of the argument component, the less specific level is the level of the content not carved up in any way, i.e. the level of the general truth-conditions; given a partial world  $s$  sustaining the content of

‘Plato is blackbearded’, we may represent the level of the uncarved content as follows:

$$\lambda X[\text{‘Plato is blackbearded’ is true in } X](s) \quad (C_0)$$

In this case the argument component is maximally general, for it is maximally inclusive: the witnesses of  $s$  of any possible way of regarding the content are contained in  $s$ . On the other hand, the functional component is maximally specific, for the entire sentence – thus the totality of semantic information – is used to define the functional component. The next step is to carve up the content according to the following division into function and argument:

$$\lambda X[\text{‘is blackbearded’ is true of } X](\text{the extension of ‘Plato’ in } s) \quad (C_1)$$

In this case the argument component is more specific, for it clarifies a part of the world with which the content is concerned. Conversely, the functional component is more general, for Plato’s being blackbearded is a particular case of being blackbearded. We may proceed with the specification of the argument component by performing a recarving by determination:

$$\lambda X[\text{‘is black’ is true of } X](\text{the extension of ‘Plato’s beard’ in } s) \quad (C_2)$$

This example is interesting insofar as the decomposition of the functional component has a correspondent syntactic decomposition of the predicate. Clearly it is a toy example, for we have implicitly obtained the word ‘blackbearded’ by merging the word ‘black’ with ‘bearded’; yet it is helpful in providing an intuitive understanding of the operation of recarving by determination. In  $(C_2)$  we see that the argument component is more specific than  $(C_1)$  for it says that to assess the truth-value of the sentence we don’t need to “look at” Plato in its entirety but just to his beard. Conversely, the predicate expresses a more general condition: the fact that a beard is black (required by the predicate in  $(C_1)$ ) is a particular case of blackness. If we made the very controversial assumptions that colors are existing particulars, one may go on and perform a further recarving:

$$\lambda X \lambda Y[\text{‘=’ is true of } X, Y](\text{the ext. of ‘the color of Plato’s beard’ in } s, \text{ the ext. of ‘black’ in } s) \quad (C_3)$$



Clearly, if the semantic of the linguistic expressions appearing in  $(C_3)$  is to be taken at face value, almost nobody will accept the result of the second recarving. Yet this does not interest us now, for we are concerned with the understanding of the trade off between the levels of specification of the functional part and the argument part. In the case of  $(C_3)$ , it seems that we have reached the bedrock of generality of the functional part: indeed, there seems to be no compositional analysis of the identity predicate such that a resulting component of such analysis may be used to make a further specification of the argument part. Even if we are open to accept that identity may be analyzed in terms of Leibniz's law, this amounts to a merely conceptual recarving which leaves unaltered the argument component. And in this case there is no alteration of the level of specification. Thus a recarving that has as outcome a functional component associated with identity corresponds to maximal level of generality of the functional component. Indeed, if we accept  $(C_3)$ , we accept the fact that 'is black' may be analyzed as an identity, i.e. the 'is' should not be understood as a copula, but as an identity. And to be identical to black is a particular case of identity.

We may now provide an explanation for the presented trade off of specificity. *The less specific and less circumscribed is the argument part of a sentence regarding the portion of world that is relevant to truth-value assessment, the more specific must be the functional part, for the task of picking out the most relevant portion of world is partly deferred to the functional component. Conversely, the more specific and circumscribed is the argument part, the more general the functional part.*

## 5.5 The ontological import of a recarving by determination

In the previous section we have seen that a recarving may be performed on the sentence 'Plato is blackbearded' by decomposing the content of the predicate 'blackbearded' and consider a specific proper part of the extension of 'Plato'. The case has been presented as uncontroversial insofar as given the individual for which 'Plato' stands, any of its proper parts is also given. And parts may be considered even as more fundamental than wholes: as a consequence, nobody would qualify this recarving as implying the existence of entities whose ontological status is controversial.

However, we have seen that when we performed the second recarving (from  $(C_2)$  to  $(C_3)$ ), we

introduced colors as existing particulars, at least to the extent that the syntax of the resulting sentence associated with the new carving has been taken at face value. Notice that in this case, the ontological controversy is engendered by the fact that we cannot rely anymore on the relation of proper parthood to characterize the more restricted portion of the world that witnesses the truth-value of the content: under any reasonable account of the relation of proper parthood, colors are not parts of colored things. In the same way, directions are not parts of lines, shapes are not parts of figures, numbers are not parts of concepts. So to say, along the series of re-carvings the level of specification of the argument part increases to the point that the restriction of the topic of the sentence is no more associated with a mereological restriction of the initial whole. As a consequence, new objects are introduced whose existence is no more given as “ontological free lunch” as in the case of parts. To understand this problem, consider the sentence ‘Socrates is wise’. We may meaningfully ask “What is the most relevant respect under which Socrates should be considered in order to assess his wisdom?”. But if we ask “Which part of Socrates determines whether he is wise or not?” our question becomes nonsensical if we take the word ‘part’ with the required seriousness. The difference between the two questions relies on the fact that in our use of ordinary language we introduce nominal expressions with ontological innocence; expressions such as “the respect under which”, “the aspect of” are paradigmatic examples. In other words, in our use of ordinary language there are ways of making the topic of a sentence more restricted – and thus more specific – that present a certain resemblance with the operation of content re-carving: a predicative expression is decomposed and some of its component expressions are used to make the topic more specific. In the majority of cases it appears clear that such specifications are made with ontological innocence; in some others, linguistic operations are confusedly supposed to have in the re-carving operation a semantic counterpart.

On the other hand, I’m not claiming that the only way of performing a re-carving in an uncontroversial way is by considering proper parts of the witnesses given by the initial carving. In the cases of adjustment of a spurious carving, the re-carving operation may be linguistically seen as the decomposition of a monadic predicate into a relational predicate and one or more singular terms. The argument part turns out to be more specific simply because another entity is added to the list of witnesses and not because some part, respect, aspect of the initial entities is pointed out.

Clearly, we cannot expect a general theory of content to provide us with a criterion for saying

when a recarving has a controversial ontological import. For whether a certain ontological commitment is controversial or not is something that is determined by the big picture of reality we are open to endorse, i.e. by metaphysics. However, we may consider a case that is more likely to be suspicious: the case of a recarving of determination having as outcome a functional component associated with the relation of identity.

In the previous section we have seen that the introduction of suspicious entities was made through the second recarving, namely that resulting in the expression  $(C_3)$ ; undoubtedly, the introduction of colors as existing particulars is a highly controversial assumption. As we will see, also abstraction principles are cases of recarving by determination that result in an identity, and the truth of an abstraction principle is a very controversial matter. In some cases, the assumption that a certain functional component may be recarved by determination so that an identity obtains is very strong, for it may imply the reification of a universal. In the case of  $(C_3)$ , we have seen that the property of being black is decomposed into the relation of identity and the reification of this property, i.e. a color particular. In the case of abstraction principles, the recarving of a sentence of the form  $Rab$  with  $R$  equivalence relation, results in an identity of the form  $f(a) = f(b)$  where  $f$  is an unspecified operator. We may interpret also this case as a case of universal reification. The fact that  $Rab$  expresses the same content as  $f(a) = f(b)$  may be interpreted as the fact that there is an object  $k = f(a) = f(b)$  that witnesses the truth of  $Rab$ . The same object witnesses all the truth of the form  $Rxy$  where either  $x$  or  $y$  stands in the  $R$ -relation with  $a$  or  $b$ . As a consequence, each equivalence class of  $R$  is associated with a witness of the truth of sentences of the form  $Rxy$ . Note that such object witnesses the truth of  $Rxy$  not in virtue of some property that it has, but by merely existing. In other words, it is not in virtue of a particular way of being of  $k$  that  $Rab$  is true, but in virtue of the existence of  $k$ , i.e. in virtue of all intrinsic features that  $k$  displays. Given that all objects belonging to the same equivalence class share certain *invariant properties* associated with the equivalence relation  $R$ ,  $k$  may be conceived as the reification of all invariant properties that  $a$  and  $b$  have in common. In the next section we will see in more details the consequences of assuming that an abstraction principle is justified by a recarving by determination.

Another issue that arises from the proposed notion of internal structure of a content is that of fundamentality of witnesses. As we have seen, the property of sustaining the content of an atomic sentence presents different levels of specification. Except for the properties of positively or negatively sustaining a content, each one of these levels correspond to a particular property

of witnessing, i.e. the property that a partial world has iff it contains witnesses picked out by a certain way of carving up the content. Now we may consider two witnessing properties of partial worlds  $P$  and  $Q$  belonging to different levels of specification, such that  $P$  determines  $Q$ . One may interpret this fact as a case of grounding; indeed, there are grounding based account of the relation of determination such that if  $P$  determines  $Q$ , then given an object  $a$ , the fact that  $a$  has  $P$  grounds the fact that  $b$  has  $Q$  (Correia, 2005). As a consequence, the fact that a partial world  $s$  has the witnessing property  $P$  should ground the fact that  $s$  has the witnessing property  $Q$ . One may make a further step and from this fact infer that existence of the witnesses defining the property  $P$  grounds the existence of the witnesses defining the property  $Q$ . An example may help to explain this inference.

Let  $s$  be a partial world positively sustaining the content of ' $a \parallel b$ ' and containing two objects  $l_1, l_2$  as lines. In particular,  $s$  has the property  $P_{l_1, l_2}$ , i.e. the property of containing  $l_1, l_2$  as witnesses of the truth of the content of ' $a \parallel b$ ' with  $l_1$  extension of ' $a$ ' and  $l_2$  extension of ' $b$ '. Suppose that  $s$  also has the property  $P_{d_1, d_2}$ , i.e. the property of containing  $d_1, d_2$  as witnesses of the truth of the content of ' $a \parallel b$ ' with  $d_1$  extension of ' $d(a)$ ' and  $d_2$  extension of ' $d(b)$ '. As shown in the previous chapter, under the given assumption  $P_{l_1, l_2}$  determines  $P_{d_1, d_2}$ ; thus, under the grounding based account of the relation of determination, the fact that  $s$  has  $P_{l_1, l_2}$  grounds the fact that  $s$  has  $P_{d_1, d_2}$ . From this relation, one may be tempted to infer that the existence of  $l_1, l_2$  grounds the existence of  $d_1, d_2$ , i.e. lines are more fundamental than directions. Given the hyperintensionality of the grounding relation, the inference depends upon the particular interpretation of the notion of witness.

As mentioned in section 4.3.1 we may spell out the relation of witnessing in terms of the relation of grounding. More precisely, we may say that  $l_1, l_2$  are witnesses of the truth of the content of ' $a \parallel b$ ' in  $s$  iff the content of ' $a \parallel b$ ' is true in  $s$  in virtue of the fact that  $l_1, l_2$  exist in  $s$ . Similarly,  $d_1, d_2$  are witness of the truth of the content of ' $a \parallel b$ ' in  $s$  iff the content of ' $a \parallel b$ ' is true in  $s$  in virtue of the existence of  $d_1, d_2$ .

At this point, we may make several interesting statements of grounding that we may use as premises to attempt an argument to the effect that the existence of lines  $l_1, l_2$  in a world  $s$  grounds the existence of their directions in the same world. To this end, we will use some useful abbreviations and notations. We may use the symbol  $<$  for the relation of *total grounding* (henceforth simply grounding), recalling that  $<$  may be reasonably taken to be a partial strict

order. To use the relational symbol ‘<’ we must take into account that grounding is a relation between facts; thus given a sentence  $\phi$  we will denote by  $|\phi|$  the fact that  $\phi$ . By  $E_{l_1, l_2}(s)$  we will abbreviate the sentence ‘ $l_1$  and  $l_2$  exist in  $s$ ’ and mutatis mutandis we will also use the abbreviation  $E_{d_1, d_2}(s)$ . Being  $c$  the content of ‘ $a \parallel b$ ’, by  $c^{\mathbf{T}}(s)$  we will abbreviate the sentence ‘the content  $c$  is true in  $s$ ’.

The first point concerns the grounding based account of the relation of determination in the case of lines, that will be expressed as follows:

$$(\text{DET}_1) \quad |P_{l_1, l_2}(s)| < |c^{\mathbf{T}}(s)|$$

Which says that the fact that  $l_1$  and  $l_2$  are witnesses of the truth of  $c$  in  $s$  grounds the fact that  $c$  is true in  $s$ . In the case of directions:

$$(\text{DET}_2) \quad |P_{d_1, d_2}(s)| < |c^{\mathbf{T}}(s)|$$

By the relation of determination between  $P_{l_1, l_2}$  and  $P_{d_1, d_2}$ :

$$(\text{DET}_3) \quad |P_{l_1, l_2}(s)| < |P_{d_1, d_2}(s)|$$

which says that the fact that  $l_1$  and  $l_2$  are witnesses of the truth of  $c$  in  $s$  grounds the fact that  $d_1$  and  $d_2$  are witnesses of the truth of  $c$  in  $s$ .

The next premises we add represent the interpretation of the notion of witness in terms of grounding. In the case of lines:

$$(\text{L-WIT}) \quad |P_{l_1, l_2}(s)| = | |E_{l_1, l_2}(s)| < |c^{\mathbf{T}}(s)| |$$

which says that the fact that  $l_1, l_2$  are witnesses of the truth of  $c$  in  $s$  and the fact that  $c$  is true in  $s$  in virtue of the existence of  $l_1, l_2$  in  $s$  are the same fact. Similarly for the case of directions:

$$(\text{D-WIT}) \quad |P_{d_1, d_2}(s)| = | |E_{d_1, d_2}(s)| < |c^{\mathbf{T}}(s)| |$$

By making the suitable substitutions we obtain the three premises of our argument:

$$(\text{P}_1) \quad | |E_{l_1, l_2}(s)| < |c^{\mathbf{T}}(s)| | < |c^{\mathbf{T}}(s)|$$

$$(\text{P}_2) \quad | |E_{d_1, d_2}(s)| < |c^{\mathbf{T}}(s)| | < |c^{\mathbf{T}}(s)|$$

$$(\text{P}_3) \quad | |E_{l_1, l_2}(s)| < |c^{\mathbf{T}}(s)| | < | |E_{d_1, d_2}(s)| < |c^{\mathbf{T}}(s)| |$$

we are interested in inferring from  $(\text{P}_1)$ ,  $(\text{P}_2)$ ,  $(\text{P}_3)$  the following conclusion:

$$(C) \quad |E_{l_1, l_2}(s)| < |E_{d_1, d_2}(s)|$$

To do this, we need to know how to treat the grounding relational operator when it is included in the “fact” operator  $| \cdot |$ . More precisely, we need to know what grounds grounding facts and what is the relation in the case of a grounding fact  $C = |A < B|$  between  $C$  and  $A$ . There are several proposals – all of them controversial – on whether grounding facts are grounded and what should ground a grounding fact; yet a detailed discussion on this topic will take us too far. I will consider the position that more easily applies to our case. deRosset (2013) offers an account of grounding facts based on the notion of metaphysical explanation, according to which given that a grounding fact is an explanatory fact such that what grounds explains what is grounded, a grounding fact  $X$  should be grounded in *the explanans*, i.e. in the fact that  $X$  presents as more fundamental. What interests us is that according to this view, the fact  $|A < B|$  is grounded in  $A$ . Thus we may add two grounding conditions:

$$(GND_1) \quad |E_{l_1, l_2}(s)| < \quad | |E_{l_1, l_2}(s)| < |c^T(s)| |$$

$$(GND_2) \quad |E_{d_1, d_2}(s)| < \quad | |E_{d_1, d_2}(s)| < |c^T(s)| |$$

At this point all premises are on the table; our question becomes: do  $(P_1)$ ,  $(P_2)$ ,  $(P_3)$  with  $(GND_1)$  and  $(GND_2)$  entail  $(C)$ ? The answer is negative: we are not able to prove that the existence of directions is grounded in the existence of lines. This may be shown by offering a counter-example to a similar case. The relation that a set bears to its elements may be seen as a relation of partial grounding. In the particular case of singletons, this relation may be conceived as a total grounding. We may formulate our premises in a set theoretical model and show that they are compatible with both  $(C)$  and its converse. Given an object  $A$ , the correspondent of our premises may be written as:

$$(P_1) \quad \{A\} < \{\{A\}\}$$

$$(P_2) \quad A < \{\{A\}\}$$

$$(P_3) \quad A < \{A\}$$

$$(GND_1) \quad x < A$$

$$(GND_2) \quad y < \{A\}$$

while the sought conclusion is:

(C)  $x < y$

It should be clear why (C) does not follow. We may build a model  $M$  that makes true all of our premises such that  $A = \{\{B\}\}$ ,  $x = \{B\}$ , and  $y = B$ . Given that  $B < \{B\}$ ,  $M$  falsifies (C).

Even if we were not able to prove that the existence of directions is grounded in the existence of lines, there is still a weak but worth mentioning result that follows from both our metaphysical assumptions and the proposed theory of content. If a fact  $A$  is totally grounded in a fact  $B$  and  $A$  is grounded in a fact  $C$  different from  $B$ , then either  $B$  grounds  $C$  or  $C$  grounds  $B$ . Given that it is a consequence of our assumptions that the fact that the content  $c$  is true in a world  $s$  is grounded in the fact that lines  $l_1, l_2$  exist in  $s$  as well as the fact that  $c$  is true in  $s$  is grounded in the existence of the directions  $d_1, d_2$  in  $s$ , it follows that either the existence of lines  $l_1, l_2$  in  $s$  is grounded in the existence of directions  $d_1, d_2$  in  $s$  or the converse. As a consequence, *under all the considered assumptions regarding the grounding relation and the notion of witness, being  $\chi_1$  the determinant carving of a content  $c$  obtained by performing a recarving by determination of a determinant carving  $\chi_0$  of the same content, the witnesses picked out by  $\chi_1$  cannot be as fundamental as the witnesses picked out by  $\chi_0$* . In other words, a recarving by determination may be seen as an operation that re-organizes the content either by converting a carving expressing a more fundamental fact  $A$  into a carving expressing a derivative fact  $B$  grounded in  $A$ , or – and more interestingly – by converting a carving expressing a derivative fact  $B$  into a carving expressing a more fundamental fact  $A$  that grounds  $B$ . It is crucial to stress that this is not a consequence of the proposed theory of content alone, but also of the afore-mentioned assumptions regarding the relation of total grounding and the notion of witness.

Notice that the fact that we were unable to prove that lines are more fundamental than directions is not an undesirable result. Indeed, we don't want our theory of content to decide what is more fundamental than what; the only requirement we may imagine regarding this issue is that our theory of content should be sensitive to the difference in fundamentality of the facts that are concerned. In other words, for every two facts  $A, B$  such that  $A$  grounds  $B$  or  $B$  grounds  $A$ , we want our theory of content to detect the metaphysical difference between  $A$  and  $B$  by associating them with different determinant carvings of the same content. If directions are less fundamental than lines, it is something that may be decided by additional considerations.

In the next section we will see what are the consequences of applying the proposed theory of content recarving to the case of abstraction principles.

## 5.6 Abstraction Principles

The passage of *GLA64* has been traditionally considered (also by Frege) as a paradigmatic example of what can be considered a new definitional pattern. It is not clear which features of the definition of direction are considered as part of this pattern. Traditionally, definitions following it are called *abstraction principles* or *definitions by abstraction*.

Normally a definition by abstraction includes as definiens an equivalence relation  $R$  and as definiendum a certain *abstraction operator*  $f$ . The definition is the following:

$$\forall x \forall y (Rxy \leftrightarrow f(x) = f(y)) \quad (5.8)$$

What (5.8) has to do with *GLA64*? Firstly, parallelism is an equivalence relation just as  $R$ . Secondly, we may think that (5.8) is obtained by recarving the content of  $R$  in the way described in *GLA64* in each sentence of the form  $Rab$ .

An interesting fact is that abstraction principles may be of any order. Being  $\mathcal{R}$  a second-order relation and  $f$  an operator that form a singular term when receives a conceptual expression as argument, the following is a second-order abstraction principle:

$$\forall X \forall Y (\mathcal{R}_x(Xx, Yx) \leftrightarrow f_x(Xx) = f_x(Yx)) \quad (5.9)$$

Second-order abstraction principles are interesting insofar as Frege's definition of cardinal number – traditionally known as Hume's Principle – may be traditionally considered as a second-order abstraction; being  $Eq_x(Xx, Yx)$  the abbreviation of the expression of the one-to-one correspondence between all  $X$ s and all  $Y$ s, when we see this expression as a function of the arguments  $X, Y$ , an equivalence relation of obtains. Being  $N$  the *cardinality operator*, i.e. the operator that receives a concept as argument and returns its cardinal number, Hume's Principle has the following expression:



$$\forall X \forall Y (Eq_x(Xx, Yx) \leftrightarrow N_x(Xx) = N_x(Yx)) \quad (HP)$$

Why we should consider abstraction principles as true? To answer this question, one should understand what characterizes the definitional pattern of abstraction principles. The fact of  $R$ 's being an equivalence relation, it is commonly considered as an crucial aspect. However, in *GLA64* nothing is explicitly said regarding the fact that parallelism is an equivalence relation. Moreover, the assumption that for every equivalence relation of any order there is an operator that satisfies the related abstraction principle is false. It is well-known that are inconsistent abstraction principles.

To my mind, the procedure of content recarving of *GLA64* is the heart of the matter. If we want to generalize *GLA64*, then we should say that an abstraction principle is true whenever it may be proven to be the result of an operation of content recarving. Notice that we don't even need to assume that  $R$  in (5.8) is an equivalence relation; suppose that  $R$  is not an equivalence relation (in particular, not reflexive) and it is possible to perform a recarving of  $R$  such that:

$$\forall x \forall y (Rxy \leftrightarrow f_1(x) = f_2(y)) \quad (5.8')$$

We may take (5.8') to be the implicit definition of all instances of two different concetps, the concept under which all values of  $f_1$  fall and the concept under which all values of  $f_2$  fall.

Clearly, when we talk of "abstraction" we are intending something more than a recarving of a relation resulting in an identity. I think that the name abstraction principle (which was never used by Frege) comes from the fact that when  $R$  is an equivalence relation, we abstract the common features between two objects that are  $R$ -related and consider the entities that objectify this aspect: directions, numbers, shapes, etc... Notice that Frege explicitly denies that an operation of abstracting some particular feature from an object by disregarding the remaining ones could be consistent (see (Frege, 1950), §34). Thus it is not clear what Frege considers to be part of the definitional pattern, if he really believed that there is a definitional pattern. Probably he considered the fact that  $R$  is an equivalence relation as crucial, but just for the reason that if  $R$  is not an equivalence relation and we want to define an operator  $f$ , then we contradict the laws of identity.

All the same, it seems that the only justification we have for the truth of an abstraction principle is the identity of content between the two limbs of the bi-conditional in any particular instantiation of the principle. And this identity of content holds as a consequence of the procedure of content recarving described in *GLA64*. Thus the operation of content recarving in the way described by Frege should be part of the defintional pattern of abstraction principles.

In the next sections I will consider abstraction principles from the perspective of the theory of content recarving proposed in this chapter.

### ***GLA64* explicated**

At this point we have enough theoretical tool to attempt an explication of the procedure described in *GLA64* in the setting of our proposal. To make the exposition clearer we divide the procedure in the following steps:

- (S<sub>1</sub>) The content of the relation of parallelism may be divided into the content of the relation of identity and another part *D*;
- (S<sub>2</sub>) The content *D* is removed from the content of the relation of parallelism; as a result, the remaining part (i.e. the content of the relation of identity) is considered to be more general than the content of the relation of parallelism;
- (S<sub>3</sub>) The content *D* is divided between the two terms of the relation of parallelism *a* and *b*. I preliminary understand this fact as the fact that the content *D* is *divided into two halves each of which is “combined” respectively with the content of a and the content of b*; as a result, both the contents of the terms ‘*d(a)*’ and ‘*d(b)*’ are formed;
- (S<sub>4</sub>) By recombining the contents of the relation of identity, that of the term ‘*d(a)*’ and that of the term ‘*d(b)*’, the content of ‘*d(a) = d(b)*’ is formed;
- (s<sub>5</sub>) By applying the procedure to all sentences of the form  $\xi \parallel \zeta$  a new concept is formed, i.e. the concept under which all references of the terms *d(ξ)*, *d(ζ)* fall. This concept is the concept of direction.

Consider (S<sub>1</sub>). Firstly we may consider the content of the relation of parallelism in the context of the sentence ‘*a*  $\parallel$  *b*’ as the functional component of the  $F_{\parallel}$  introduced in (5.1): i.e. the function that given two possible extensions *l*, *l*’ respectively of ‘*a*’ and ‘*b*’, returns the value true

if  $l$  is parallel to  $l'$ , the false if this is not the case. The division of the content of  $F_{\parallel}$  may be interpreted by means of the (5.7):

$$F_{\parallel} = F_{=} \bullet [\delta; \delta] \quad (5.7)$$

The idea that a content may be divided is supported by the fact that when a concept function is decomposed by determination, the relation between the components of the decompositions and the concept function satisfies a particular version of the axiom of mereology (see the appendix at the end of the chapter). Notice that there is an aspect of the description of the procedure that does not seem to be captured by (5.7); the content  $D$  that complements the content of the relation of identity in the decomposition of the function  $F_{\parallel}$  appears as already “divided” into two halves, i.e. into two occurrences of the direction operator  $\delta$ . I will show the main problem of modelling  $D$  in the explication of (S<sub>3</sub>).

Consider now (S<sub>2</sub>). The functional component  $F_{=}$  is “the remainder” when the function  $[\delta; \delta]$  is “removed” from the functional component  $F_{\parallel}$ . How may we understand the fact that this content is considered to be more general than that of a relation of parallelism? A preliminary answer is that we may conceive the relation of identity as the most general equivalence relation. Consider an equivalence relation  $R$  distinct from identity; to say that  $a$  and  $b$  stand in the relation  $R$  is to say that  $a$  and  $b$  have the same  $R$ -invariant properties, i.e. the properties that the relation  $R$  preserves. For instance, two parallel lines share all their orientational properties; two similar triangles share all their “shape properties”. When an equivalence relation is understood in this way, we may interpret it as a relation of partial undistinguishability: if  $a$  bears the relation  $R$  to  $b$ , then there is no  $R$ -invariant property that may be used to distinguish  $a$  from  $b$ . Being  $I_R$  the higher-order predicate ‘is an  $R$ -invariant property’, we may express the following:

$$Rab \rightarrow \forall X (I_R(X) \rightarrow (Xa \leftrightarrow Xb)) \quad (5.10)$$

compare now (5.10) with Leibniz’s Law:

$$a = b \rightarrow \forall X (Xa \leftrightarrow Xb) \quad (5.11)$$

Identity is a relation of indistinguishability with respect to all properties, while a distinct equiva-

lence relation  $R$  is a relation of indistinguishability with respect to a specific range of properties; for this reason identity is more general, because it does not have to do with a particular respect, but with the objects considered under all possible respects.

As far as it goes, this explanation of the greater generality of identity compared to other equivalence relations is not bad. Notice that I haven't use any notion of the proposed theory of content to frame this explanation, but just some general facts regarding equivalence relations. But the given account of the recarving by determination has something to say on the matter. As shown in section 5.4, a recarving by determination amounts to an operation that while restricting the portion of the world picked out by the argument component, makes the requirement associated with the functional component more general. This because a less circumscribed – and thus less specific – argument component requires a more specific functional component, for part of the task of picking out the most relevant part of the world for the determination of the truth-value of the content is up to the requirement expressed by the functional component. The more recarvings by determination we perform, the more specific is the argument component regarding the portion of the world that witness the truth-value, the more general is the requirement associated with the functional component. This account is interesting because it explains the greater generality of the functional component resulting from the recarving not just in the case of equivalence relations, but in all cases of recarving by determination. Moreover, in section 5.4, we have seen that the functional component is maximally general when it is associated with the relation of identity: in this case, there is no possible – not even in principles – further decomposition by determination of the functional component. Thus our explanation accounts also for another aspect of Frege's procedure: the part of the content of a concept  $F$  that complements the content of the relation of identity in a recarving by decomposition includes all the specific content of  $F$ . Indeed, given an equivalence relation  $R$ , it may be possible that the content of the sentence  $Rab$  may be recarved by determination in such a way that not all specific content of  $R$  has been removed from the content of  $R$ . Consider an equivalence relation  $R_0$  such that  $\forall x, y (R_0xy \rightarrow Rxy)$  and such that  $\exists x, y (Rxy \wedge \neg R_0xy)$ , i.e.  $R_0$  defines a partitioning of the domain into equivalence classes that is a subpartitioning of the partitioning defined by  $R$ . In principles, there may be the following recarving by determination:

$$R[a, b] \quad ; \quad R_0[f_0(a), f_0(b)] \quad ; \quad [f(a)] = [f(b)]$$

Notice that in the case of the recarving that goes from  $R[a, b]$  to  $R_0[f_0(a), f_0(b)]$ , the content that has been removed from the content of  $R$  is not the whole specific content of  $R$ . For the functional component associated with  $R_0$  may be further decomposed by determination, i.e. the requirement that  $R_0$  imposes may be further generalized.

Consider now (S<sub>3</sub>). Notice that our proposal cannot take into account all details of Frege's description. The problem is the following. Frege is supposing that the content of the relation of parallelism contain – in some unspecified sense – the content of the relation of identity and another content  $D$ . The content  $D$  should be intended as a sort of “compact version” of the content of two occurrences of the direction operator. But how identity is supposed to be combined with  $D$  into the content of the relation of parallelism? Frege offers only one way of combining contents: saturation. All kinds of content combination may be reduced to saturation: combination of the content of a conceptual expression with the content of a term, combination of two or more judgeable contents by means of logical connectives, quantification over a general content are all conceived by Frege as saturation operations.

Yet the content of the relation of identity is an unsaturated content that is saturated only by two saturated contents. We may represent the content of identity as  $x = y$  and the content  $D$  as a function  $D(x, y)$ . Thus  $D$  is saturated by  $a$  and  $b$ , and identity must be saturated by  $D$ . Yet if  $D$  is not divided into two halves, how could  $D$  saturate the content of identity?

I decided not to deal with this difficulty, for I do not consider this part of the procedure as conceptually crucial. I present the part of the content of the relation of parallelism that complement identity as already divided into two occurrences of the direction operator; otherwise, we should suppose either a way of combining contents different from saturation or that “before” dividing  $D$  into two halves, the content of the relation of identity is a content requiring just one saturated expression to be saturated. To my mind, both cases seem incomprehensible.

Consider now (S<sub>5</sub>). In chapter 3, we have seen that the procedure of content recarving does not correspond to any procedure of concept formation that is traditionally accepted. It is time to deepen this analysis. We may consider five main methods of concept formation:

1. **Classification:** a concept may be formed by observing patterns of features in a certain group of particulars. For instance, given a group of animals we may classify them according to a certain genetic pool and separate them into different species,

2. **Boolean combination:** a concept may be formed by combining already defined concepts with one or more nested applications of boolean operators,
3. **Partial saturation:** this method is frequently used by Frege. It consists in considering an already defined conceptual expression having at least two argument places and saturate some of them with saturated expressions. For instance, given a relation  $R(x, y)$  we may form the concepts  $R(a, y)$ ,  $R(x, b)$ ,  $\forall y R(x, y)$ ,
4. **Logical definition:** also this method is one of Frege's resources. We may define a concept starting from some expressions  $\xi_1, \dots, \xi_n$  and building a sentence  $S$  including all of them such that all the other expressions are part of the logical vocabulary. Successively, we may replace  $\xi_1, \dots, \xi_n$  by free variables of the suitable order and obtain an unsaturated expression composed only by logical expressions. For instance, we define the higher-order concept 'X is a subconcept of Y' by considering two conceptual expressions  $F, G$ , constructing the sentence  $\forall x (Fx \rightarrow Gx)$ , and then replace  $F$  by  $X$  and  $G$  by  $Y$ , so that  $\forall x (Xx \rightarrow Yx)$  obtains. In this way Frege forms the relational concept of being a successor in a series and the higher-order relational concept of equinumerosity.
5. **Functionality:** instead of describing a certain pattern of intrinsic features, a concept may be defined in a relational way, i.e. by describing a role/function that the instances of the concept play/have in a given system of objects. In this case we may call the resulting concept a *functional concept*. For instance, the concept of 'piston' may be considered as a concept defined by functionality, for  $a$  is a piston not in virtue of specific intrinsic features, but in virtue of the function  $a$  plays in a given engine. Clearly, the function that a piston plays may foresees certain design parameters that – in turn – may determine certain geometrical characteristics that are common to all pistons. Yet this may not occur with all functional concepts. The concept of 'predator' is not defined in terms of intrinsic properties of the individuals that are predators: we would not say that a lion is a predator in virtue e.g. of the shape of its teeth, but in virtue of the way it behaves in a certain ecosystem with respect to which the concept is defined. Functional concepts may be defined also according to a certain theoretical role: for instance, the concept of 'electromagnetic wave' in physics has been initially defined as the concept whose instances move according to the law of motion resulting from Maxwell's equations. Notice that in this case, no intrinsic feature of electromagnetic waves is specified by the definition of

the concept.

Our problem is to determine which account of the list 1-5 (if any) may be considered as a suitable method of formation of the concept of direction by performing a recarving by determination on each sentence of parallelism.

Classification is out of question, for the recarving by determination of a sentence of parallelism is not performed by inspecting a group of objects and classify some of them as directions in virtue of some pattern of features. Boolean combination, partial saturation, and logical definition are to be rejected as well: the concept of direction is not defined by formulating a general condition that all directions must satisfy, but by introducing its instances as supposed reference of certain singular terms. So we are left with functionality. Is direction a functional concept? Are e.g. shape and number functional concepts?

The option seems to be attractive for one main reason: considering the way directions are defined, no intrinsic characteristic that directions must have is specified. Thus the possibility of the concept of direction being a functional concept should be explored.

Firstly, notice that we have no grasp of what a direction of a straight line is independent from the outcome of the recarving by determination of a sentence of parallelism. As a consequence, we cannot suppose that the expression ‘the direction of’ has a defined meaning “before” we perform the recarving of all sentences of parallelism. The only thing we may say is that directions are those objects whose identity conditions are defined by parallelism conditions between straight lines. We may formulate this within the proposed theory of recarving by determination: being  $c$  the content of ‘ $a \parallel b$ ’,  $s$  a partial world, and  $l, l'$  the extensions of respectively ‘ $a$ ’ and ‘ $b$ ’ in  $s$ , the direction of  $a$  and the direction of  $b$  are those objects  $h, k$  that witness the truth-value of  $c$  in  $s$  in such a way that for  $s$  to sustain  $c$  with  $h, k$  is a determinable property determined by the property that  $s$  has iff  $s$  sustains  $c$  with  $l, l'$ . This is really *all* we can say about the direction of  $a$  and the direction of  $b$ . And, when intrinsic features of directions are concerned, it says nothing.

We may then propose that ‘direction’ is a functional concept. Directions are those entities that play the role of witnessing the truth or the falsity of the content of a sentence of parallelism in the way previously specified. The system within which directions play the defined role may be considered as a certain piece of an interpreted language including ‘ $\parallel$ ’, ‘ $=$ ’, all line terms, and its related semantics.

Note the difference between the case of *GLA64* and the recarving of ‘Plato is blackbearded’ yielding ‘Plato’s beard is black’ as result. In this latter case we are not trying to define the concept of black beard by means of recarvings of all sentences of the form ‘ $\xi$  is blackbearded’: we have an independent grasp of the meaning of ‘Plato’s beard’ that allows us to say that the recarving is correct. It is rather the contrary: it seems that we have no grasp of the meaning of ‘blackbearded’ independent from the meaning of ‘beard’.

It is worth highlighting that the fact that ‘direction’ is a functional concept that does not appeal to any intrinsic feature of directions is a consequence of the fact that the recarving by determination of sentences of parallelism results in an identity. Indeed, a certain object  $k$  plays the role of the common direction of lines  $a$  and  $b$  in virtue of the fact that it witnesses the truth of the content of ‘ $a \parallel b$ ’ in a certain way. However,  $k$  plays this role not in virtue of some ways of being; it is the mere fact that  $k$  exists – whatever  $k$  is – that ensures the truth of ‘ $a \parallel b$ ’.

In the next section we will see some philosophical consequences of the fact that concepts defined by means of a recarving by determination resulting in identities are functional concepts that do not specify any intrinsic feature of their instances.

## Overview of the main philosophical issues

Abstraction principles has been criticized for several reasons. I will consider three main philosophical problems that arise by considering abstraction principles as true:

1. **The Caesar Problem:** This is an objection raised by Frege himself in §66 of *the Foundations*. Given that an abstraction principle (5.8) fix the content of all identity sentences of the form  $f(x) = f(y)$ , it says nothing on identity sentences of the form  $f(x) = t$  where  $t$  is a singular term not given as an application of the operator  $f$ . Thus we are not able to assign a content to all identity sentences containing terms of the form  $f(x)$ . To take Frege’s examples, how could we know that Caesar is not the number 0 or that England is not a direction of a certain line?
2. **Ontological inflationism:** This objection has been formulated in various ways. I will refer to Heck’s formulation in (Heck, 2011). Equivalence relations abound. Given a domain  $A$ , every partitioning of  $A$  into cells corresponds to a particular equivalence relation. Thus we should admit the existence of an enormous population of abstracts, i.e. objects



introduced by the operator defined by an abstraction principle. Yet the fact that a relation  $R$  is an equivalence relation and the fact that all equivalence relations may be recarved as identities seem to be very weak reasons for justifying such a notable ontological commitment.

3. **The Bad Company Problem:** This objection is particularly concerned with Hume's Principle and has been firstly formulated by Dummett (1991). If the reason for accepting Hume's Principle is that the relation appearing in the left limb is an equivalence relation, and equivalence relations may be recarved so that identities obtain, then we should accept the infamous *Basic Law V*, i.e. the following second-order abstraction principle:

$$\forall X, Y (\forall x (Xx \leftrightarrow Yx) \leftrightarrow e_x(Xx) = e_x(Yx)) \quad (BLV)$$

where  $e$  is the extension (or value-range) operator. However, (BLV) is inconsistent due the paradoxes of naïve set theory (e.g. Russell's paradox, Curry's paradox). Thus the simple fact that a certain relation is an equivalence relation does not ensure that the correspondent abstraction principle is true.

It is interesting to note that both 2 and 3 fire at the same target: the fact that a relation is an equivalence relation is not sufficient to ensure that the correspondent abstraction principle is true. In other words, the definitional pattern that we should extrapolate from *GLA64* cannot be limited to the fact that parallelism is an equivalence relation. A stronger requirement is needed, a requirement that should rule out the Basic Law V and all other *arbitrary cases of abstraction*.

Under this respect, notice that the theory of content recarving that has been proposed is general and not framed just for abstraction principles. So our problem becomes: *Given an equivalence relation  $R$ , which requirement  $R$  must satisfy in order for the sentence  $Rab$  to admit a recarving by determination that results in an identity?*

I will deal with the problem of **ontological inflationism** first. When an equivalence relation is given independently of its set theoretic interpretation as partitioning of a domain into equivalence classes, two objects belong to the same equivalence class in virtue of the fact of being related. On the other hand, when our equivalence relation is defined as a set of equivalence classes, two objects are related in virtue of the fact that they belong to the same equivalence

class. For simplicity, we will call such an equivalence relation *an abundant equivalence relation*. Notice that the way we define a certain partitioning of a domain has nothing to do with the intrinsic properties of the objects that we collect. Therefore, when an equivalence relation is abundant, i.e. when it is defined by simply given its equivalence classes which are cells of an arbitrary partitioning of the domain, it cannot be an internal relation. Being  $R$  a relation of this kind, it is evident that two objects  $a$  and  $b$  are  $R$ -related not in virtue of their intrinsic features, but in virtue of the existence of some set to which both of them belong. Thus  $a$  and  $b$  alone cannot witness the truth-value of the content of ' $Rab$ '. This fact may be explained by considering the principle of locality of internal relation: a relation  $K$  is internal iff given every two objects  $a$  and  $b$  and considering them in isolation, we are able to decide whether they are  $K$ -related or not. Now it is evident that given  $a$  and  $b$  and considering them in isolation – i.e. considering a partial world containing just  $a$  and  $b$  – we cannot decide whether they are  $R$ -related or not, for we need the additional information of the equivalence classes of  $R$ . In this case, the carving  $R[a, b]$  is not a determinant carving.

We may define a new relation  $R^*$  in the following way: being  $\pi$  the partitioning in terms of which  $R$  is defined:

$$R^*(\pi, x, y) \leftrightarrow \exists! z(z \in \pi \wedge (x \in z \leftrightarrow y \in z))$$

The relation  $R^*$  is an internal three-places relation defined as follows: a partitioning  $\phi$ , an object  $x$  and an object  $y$  are  $R^*$ -related iff there is a unique element  $z$  of  $\phi$  (i.e. a cell) such that both  $x$  and  $y$  are members of  $z$ . Clearly  $R^*$  need not to be an equivalence relation. Now  $R$  may be defined in terms of  $R^*$  as:

$$Rxy \leftrightarrow R^*(\pi, x, y) \quad (5.12)$$

A determinant carving of the content of  $Rab$  may be defined in terms of  $R^*$ . Now  $R^*(\pi, a, b)$  is an existential sentence with an uniqueness condition. The truth of a unique existential is witnessed by the only object that satisfy the quantified condition; the falsity is witnessed by the set of all objects that satisfy the condition (which may be empty but cannot be a singleton). In the case of  $R^*(\pi, a, b)$  we may pick out the witnesses of the truth-value of the content by means of a fixed rule: in every possible world, we pick out both the equivalence class of  $a$  through

$R$  and the equivalence class of  $b$  through  $R$ ; if they are identical, then the unique witness is determined, if they are different, then both of them witness the falsity of  $R^*(\pi, a, b)$ . Being  $(a)_\pi$  and  $(b)_\pi$  the equivalence classes of  $a$  and  $b$  with respect to  $R$ , the determinant carving of  $R^*(\pi, a, b)$  may be represented as:

$$[(a)_\pi] = [(b)_\pi]$$

Suppose now that there is a determinant carving of  $Rxy$  that has identity as functional component and expressible by means of an operator  $f$  in the following way:  $[f(a)] = [f(b)]$ . Now,  $f$  cannot be function just of  $a$  or  $b$ ; this because whether  $a$  is associated to a certain values of  $f$  or not, depends also on which is the equivalence class in  $\pi$  to which  $a$  belongs. As a consequence, the carving with identity should be expressed as  $f(\pi, a) = f(\pi, b)$ . We list the three carvings of the same content that we have considered:

$$R[a, b] \quad ; \quad [(a)_\pi] = [(b)_\pi] \quad ; \quad [f(\pi, a)] = [f(\pi, b)] \quad (5.13)$$

The recarving operations go from left to right: the first recarving is not by determination, for  $R[a, b]$  is a spurious carving. Consider now the second recarving; it requires that the specific part of the content of '=' is removed and used to define the new arguments  $f(\pi, a)$ ,  $f(\pi, b)$ . However, as we have seen, identity is maximally general: there is no decomposition by determination of the functional component of identity that may be used to define a more specific argument. As a consequence, either the content represented by the two occurrences of  $f$  has been arbitrarily introduced, thus altering the content and making the recarving unsound, or  $f(\pi, a)$  must express the same content as  $(a)_\pi$  and  $f(\pi, b)$  must express the same content as  $(b)_\pi$ . Yet this means that the abstracts introduced by recarving an abundant equivalence relation are nothing but equivalence classes. Thus the identity of content between the two limbs of the abstraction principle associated with  $R$  holds trivially: the equivalence classes of  $a$  and  $b$  are identical iff they are identical. In terms of ontological commitment there is no difference between the two limbs: by the (5.11), the relation  $R$  is already committed to the existence of equivalence classes and thus there is no ontological inflation; or more correctly, the ontological inflation is engendered by the fact that abundant equivalence relations have been considered and not by the abstraction principle.

An interesting consequence of this fact is that abstraction principles should be defined only with respect to sparse equivalence relation. Sparse equivalence relations may be preliminarily identified with internal equivalence relations that are distinct from identity. This because whenever an equivalence relation  $R$  is internal and distinct from identity, the witnesses of the right limb of the abstraction principle are the relata and the witnesses of the left limb (if any) must be different objects. Moreover, when  $R$  is an internal equivalence relation distinct from identity it always has a specific part in its content and such a specific part is determined by the intrinsic features of the relata that counts as  $R$ -invariant properties.

We will now turn our attention to the **Bad Company objection**. As briefly mentioned, if the definitional pattern of abstraction principles requires just that the relation appearing in the left limb is an equivalence relation, then the infamous (BLV) would perfectly match the definition. Therefore, the notion of abstraction would be an inconsistent notion, for in some cases it leads to contradiction. It seems that the only way out for the advocate of abstraction is to point at some bad feature that the (BLV) displays and that other higher-order abstraction principles (such as (HP)) do not display.

In the previous point I have argued that abstraction principles should be defined just for internal equivalence relations that are distinct from identity. This has ruled out all cases of abundant equivalence relations, i.e. equivalence relations defined just in terms of an arbitrary partitioning of the domain of objects. However, this point does not help us insofar as the relation of co-extensionality – i.e. the relation between any two concepts  $P, Q$  defined as  $\forall x(Px \leftrightarrow Qx)$  – is an internal equivalence relation. For given a concept  $P$ , it is intrinsic of  $P$  which objects fall under  $P$ .

To my mind, Frege could not have regarded the Basic Law V as an abstraction principle, under the assumption that he kept his view on content recarving and given the solution to the problem of abstract reference that he gave in *the Foundations*. As I said in chapter 1, Frege never denied that the two limbs of an abstraction principle express the same content. Now, Frege's identification of abstracts with equivalence classes in §66 of *the Foundations* may be taken as a general law regarding the content of the right limbs of abstraction principles: if  $Rab \leftrightarrow f(a) = f(b)$  is a valid abstraction principle, then each object introduced as  $f(x)$  is nothing but the equivalence class of  $x$  with respect to  $R$ , i.e. the extension of the concept 'bearing  $R$ -related to  $x$ '. In other words, the extension operator  $\varepsilon$  – i.e. the operator that applies to an open formula  $Fx$

and returns its extension  $\varepsilon_x(Fx)$  – is a logical operator of “reification” that is used whenever a certain recarving of a content requires to objectify a certain concept. According to this remark, if the Basic Law V is considered as an abstraction principle it must satisfy two requirements: (BLV) must explain the meaning of extension terms; (BLV) must explain this meaning in terms of equivalence classes. It is easy to see that – under the given assumptions – the explanation of the meaning of extension terms would be circular.

I believe that Frege considered (BLV) as a primitive logical law, whose truth is not justifiable by any procedure of content recarving.

In line with this argument, we will show within our theory of content that (BLV) cannot be taken as expressing the fact that the right limb is the result of a recarving by determination of the left limb.

I will abbreviate the relation of co-extensionality with the symbol  $\equiv$ ; thus  $Fx \equiv_x Gx$  iff  $\forall x(Fx \leftrightarrow Gx)$ . Suppose now that there were no Russell’s paradox and that the (BLV) were perfectly consistent. One may still doubt that (BLV) is true. In this case we may opt to check whether (BLV) may be seen as a case of recarving by determination.

An important point in the operation of recarving by determination is that it interprets the fact that the content of an equivalence relation may be divided into a specific part and an the content of identity. Thus to show that an abstraction principle is a case of recarving by determination, we need to show that there is a *surplus of content* in the equivalence relation appearing in the left limb with respect to the content of the relation of identity. In other words, the considered equivalence relation  $R$  must have a specific part that is – so to say – the difference between the content of  $R$  and the content  $=$ ; it is this part that provides a content to the abstraction operator. If there is no difference in content between  $R$  and  $=$ , then the abstraction operator is arbitrarily introduced, thus altering the content of the right limb. In this case,  $Rab$  and  $f(a) = f(b)$  must express different contents.

The point I want to emphasize here is that the relation  $\equiv$  – within Frege’s system of logic – does not express more content than an identity relation and thus the introduction of the extension operator is an arbitrary move that results in an alteration of content. To show this, I will start by considering an example.

Suppose that - for some very compelling reason – one has come to propose an axiomatization

of plane geometry  $\Gamma$  that shall be considered the “true geometry”. Now  $\Gamma$  is very different from the standard axiomatization of Euclidean geometry: in particular, according to  $\Gamma$ , all straight lines pass through a certain point (that we will call  $O$ ). In other words, all straight lines form a bundle of lines intersecting in the same point  $O$ . Notice that in  $\Gamma$  two lines are parallel iff they are identical. Yet  $\Gamma$  still uses the notion of parallelism to avoid some unspecified conceptual confusions. Now I will ask: what does the abstraction principle of directions mean in  $\Gamma$ ? Which part of the content of the relation of “ $\Gamma$ -parallelism” should account for the content of the “ $\Gamma$ -direction operator”? I would say that there is no such part.  $\Gamma$ -parallelism just is identity of  $\Gamma$ -lines. And any argument to the effect that  $\Gamma$ -directions are not  $\Gamma$ -lines in  $\Gamma$ , is probably invoking the difference between parallelism and  $\Gamma$ -parallelism, and thus is arbitrarily introducing new content in the  $\Gamma$ -system. Because there is no available content in  $\Gamma$ -parallelism that may be used to form the content of the  $\Gamma$ -direction operator. Thus  $\Gamma$ -parallelism just is identity of  $\Gamma$ -lines; there is no content recarving at all.

Frege’s mature system of logic  $LF$  (i.e. the one presented in (Frege, 2013)) may be considered as the correlative of the  $\Gamma$  system. I will argue that the relation of co-extensionality in  $LF$  corresponds to the relation of  $\Gamma$ -parallelism, and co-extensionality is not reduced to identity between concepts just to avoid the categorical mistake of confusing concepts with objects. My conclusion will be that there is no available content in the relation of co-extensionality when the content of identity is removed, that may be used to form the content of the extension of operator.

Frege is extremely clear in saying that concepts shall not be identified with their extensions: in §34 of (Frege, 2013), Frege says that concepts “are represented by their value-ranges” but they do not “simply concede their place to them”. However, anything that a concept is over and above its extension is something that is not expressible in  $LF$ . This for a simple reason: Frege’s system (as any other system of higher-order logic with an “extension operator”) allows for a *reduction of orders*. In §34, Frege introduces an apparently innocent definition: being  $F$  a predicate and  $a$  a singular term, the expression ‘ $F(a)$ ’ may be replaced *salva veritate* by the expression  $a \in e_x(Fx)$ . This allows Frege to reduce higher-order predication in the following way: being  $\mathcal{M}$  a higher-order predicate, it is possible to reduce  $\mathcal{M}$  to a first-order predicate  $M$  defined in the following way:

$\mathcal{M}_x(Fx)$  is equivalent to  $\mathcal{M}_x(x \in e_y(Fy))$  is equivalent to  $M(e_y(Fy))$

As a consequence the following principle holds in  $LF$ : for every concepts  $F, G$ , for every higher-order open formula  $\Phi_x(Xx)$  having just the second-order variable  $X$  as free variable,

$$(\Phi_x(Fx) \wedge (Fx \equiv_x Gx)) \rightarrow \Phi_x(Gx) \quad (5.14)$$

This is clearly a law of identity under the traditional account. This point may be further explained. Suppose that we manage to express the intensional difference between two co-extensional concepts  $F$  and  $G$  by the sentence  $\Delta_x(Fx, Gx)$ . Now by (5.14),  $\Delta_x(Fx, Gx)$  is logically false, for by (5.14)  $\Delta_x(Gx, Gx)$  would be true, which amounts to say that the concept  $G$  intensionally differs from itself; under any possible account of intensional difference, this cannot be accepted.

As a consequence, the content of  $\Delta_x(Fx, Gx)$  is a content that is always false, i.e. the content of a contradiction. Yet this is not the content that we wanted to express under our pre-theoretical notion of intensional difference. Hence, in  $LF$  the content of an intensional difference between two co-extensional concepts is not expressible. As a consequence, in  $LF$  any respect under which co-extensionality is not an identity relation is disregarded. This because the axioms and theorems of a system define the contents that the sentences may express within the system.

Yet why co-extensionality is not acknowledged as an identity relation within both  $LF$  and any other system for higher-order logic? The answer is that the relation of identity, according to both Frege's view and contemporary logic, is characteristic of objecthood. Only of two objects we may ask whether they are the same object. Only an object may be characterized in its features by saying whether it is identical or different from other objects. Thus to use just one relation = for both concepts and objects will imply a huge categorical mistake: nonsensical question as whether the concept  $F$  is identical to my left hand will be representable as well formed sentences of our system. Moreover, under the Fregean notion of function, it will be simply false to say that two co-extensional concepts are identical.

The matter may be presented from a different perspective. If we want our system to capture

intensional differences between concepts, we have to change the underlying logic. For now, we will no more say that  $Fa \rightarrow Ga$  just in virtue of the truth-values of  $Fa$  and  $Ga$ . We need a stronger relation of implication. Suppose that we do have this relation and suppose that we denote it by  $\implies$  and we introduce suitable axioms to regulate its behaviour. Suppose also that we may define a stronger bi-conditional  $\iff$  (something of the sort of an *intensional equivalence*). In our new system of logic the following may be possible:

$$(\Phi_x(Fx) \wedge \forall x(Fx \iff Gx)) \wedge \neg\Phi_x(Gx) \quad (5.14')$$

which clearly says that two intensionally equivalent concepts may not be interchangeable *salva veritate*.

Therefore, the content of the relation of co-extensionality does not contain any specific part that may be considered as a *surplus* with respect to the content of the relation of identity. As a consequence, the introduction of the extension operator  $e$  is arbitrary and engenders an alteration of the content in the right limb of the Basic Law V. In other words, even if (BLV) were consistent, we could not be able to show that it is true. Surely not by showing that the two limbs express the same content. Verdict: the Basic Law V is not a bad company for Hume's Principle, because it is not even a company; if in the case of (HP) we may point at a certain surplus of content in the relation of equinumerosity with respect to the content of identity that could justify the recarving, in the case of co-extensionality we cannot do that, at least not by keeping the underlying logic classical.

Finally, we will deal with **the Caesar Problem**. Firstly, it is important to notice that according to Frege, the operation of content recarving is not enough to assign a content to every sentence of identity between directions and, in particular, it does not assign a content to sentences of the form 'the direction of line  $a = t$ '. Thus we don't expect a characterization of the recarving operation to provide us with a solution of the Caesar Problem. Yet our theory of content recarving may help in giving an interesting formulation of the Caesar Problem explaining why even a complete understanding of the recarving operation cannot solve the problem.

It is important to keep in mind that – in the case of abstraction principles – the operation of content recarving is used to formulate a definition of terms whose meaning is not independently given. For instance, the procedure of *GLA64* implicitly defines the concept of direction by fixing



the content of all identities involving two direction terms. And we don't have an independent account of what directions are.

In these cases, we have seen that the concept that the operation of recarving defines is introduced as a *functional concept*. In particular, directions are introduced as those entities that witness the truth-value of the content of a sentence of parallelism in such a way that the property for a partial world to sustain the content with directions is determined by the property of sustaining the same content with lines having those directions. All that we know about directions (and any other abstract) is that they are witnesses of a certain sort; and they witness the truth-value of a content not by having or lacking certain specific features, but by their mere existence. As a consequence, all that we know about directions tells us nothing regarding possible intrinsic features that directions should have or not. And if we know nothing regarding the intrinsic features of a range of objects, we cannot be able to distinguish the objects in the range from the other objects. To say that e.g. the direction of the earth axis is not England, we should highlight some feature that the direction of the earth axis has and England lacks.

A solution to the Caesar's Problem should somehow point at some intrinsic feature that the abstracts introduced by content recarving should have or lack. The fact that the recarving operation does not tell anything about depends on two crucial factors: 1) when a recarving is used to formulate implicit definitions, the formed concept is a functional concept, and functional concepts may be defined without appealing to the common intrinsic features of their instances; 2) as a consequence of the fact that in an abstraction principle the recarving by determination results in an identity, the abstract entities that play the role of witnesses, play this role not in virtue of some feature that they display, but in virtue of the simple fact that they exist. Thus the functional concept defined by content recarving is a quite extreme case of functional concept whose definition does not specify any feature of its instances.

# Chapter 6

## Conclusion

The main purpose of this work was to provide an explication of both operations of carving and recarving a content described in §64 of Frege's *Foundations of Arithmetic*.

In particular, albeit the work is not exegetic, we retained some fundamental aspects of Frege's view on the matter:

1. The fact that the notion of content should be an intermediate notion between that of intension and Russellian proposition. In Fregean terms, we may say that the proposed notion of content is not as coarse as the notion of truth-value, while not as fine as a notion univocally determined by objects and concepts involved in the truth-conditions of a given sentence,
2. The fact that different ways of regarding a content may be conceived as different distinction between argument and function,
3. The fact that the content of a sentence may be recarved, i.e. that part of the content of the functional part may be removed and used to define a new argument part.

Moreover, we have developed our theory of content with the correspondent definitions of carving and recarving in a completely different theoretical setting. Some fundamentally new aspects of our proposal that considerably depart from Frege's account are:

1. The distinction between function and argument has been traced on both the syntactic and the semantic level,

2. The problem of fineness of grain of the notion of content has been solved by introducing the notion of *partial worlds*. If the intension of a sentence is concerned with total worlds and the Russellian proposition with the objects that play the role of constituents, the notion of content is concerned with an intermediate notion, i.e. the partial world containing all entities witnessing the truth-value of all possible ways of regarding the same content,
3. The content of a sentence has been presented as a structured entity and the internal structure of the content has been presented as a list of subpartitionings of the set of all partial worlds containing the relevant entities,
4. In some special cases the internal structure of a content has been presented as a hierarchy of determinate properties of the determinable property of sustaining the content,
5. The method of concept formation associated with the procedure described in §64 of *the Foundations* has been presented as the definition of a functional concept; the main consequence of this fact is that intrinsic features of the concepts introduce in this way may remain beyond the definition.

The proposed theory of content offered a model of the operation of carving and recarving that – at least along general lines – explains away the metaphorical language of Frege’s description. For instance, the operation of removing the specific part of the content of the relation of parallelism has been spelled out as a decomposition by determination of the correspondent functional component.

The resulting understanding of Frege’s procedure helped to conceive abstraction principles from a different perspective. More specifically, the definitional pattern of an abstraction principle is not just the fact that the relation appearing on the left limb of the bi-conditional must be an equivalence; as highlighted at the end of chapter 5, we must also require that such a relation is an internal equivalence relation displaying a certain *surplus of content* when compared with identity. This remark turned out to be crucial in evaluating the main objections to the method of definition by abstraction.

I close this work with a question gesturing at possible future works. What philosophical significance the proposed theory of content may have besides the understanding of Frege’s passage?

The immediate answer would be that our proposal may be contextualized into the general de-

bate regarding the notion of propositional content: for several reasons the Lewisian account of propositions may result too coarse grained, and thus a finer notion of propositional content is required. Clearly, the proposed notion of content is quite unorthodox precisely insofar as it is inspired from Frege's account: it allows sentences with different ontological commitment to express the same content. To my knowledge, nowadays almost nobody would endorse such a notion of content, at least when the notion of content at issue is not a notion of *representational content*.

However, the theory may be adapted so that an account of a fine grained content may be extrapolated. What is considered a content carving may be considered as fine grained content, a recarving may be considered as a special relation between contents; perhaps we may explore the possibility of considering the relation of recarving as the converse of a relation of content containment: for instance, it would be interesting to explore the possibility of considering the content of the sentence 'the direction of  $a$  = the direction of  $b$ ' as containing the content of 'line  $a$  is parallel to line  $b$ '. The interesting outcome of this interpretation of the theory might be a new definition of entailment as content containment and perhaps the development of a system of relevant logic of content.

Therefore, the proposal insofar as it attempts the formulation of general definitions of content and operations on content goes far beyond Frege. And if I cannot claim to have explained "Frege with Frege", I hope to have shown that an understanding of the passage of §64 of *the Foundations* may be achieved in a broader theoretical context; the main consequence is the expected general applicability of the proposal.

# Chapter 7

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## Chapter 8

# Appendix: the mereology of decompositions by determination

In this appendix I will show that it is possible to see the decomposition by determination of a content function (or of the functional component) as a sort of mereological decomposition.

Intuitively, the idea is that when a function  $F$  is decomposed by determination it has two parts of different sorts, the internal part, which is the internal component in the decomposition, and the external part, which is the external component. For instance, given  $F = G \bullet h$ ,  $G$  is an external part of  $F$  and  $h$  is an internal part of  $F$ . Therefore, a function  $F$  that is decomposed by determination is a whole in a very special sense: it is the “sum” (i.e. composition) of functions that bear different kind of parthood relations to  $F$ . In this sense, we may say that  $F$  is a *structured whole*; we may call the structure of  $F$  the list of kinds of parthood relations. For instance, many artifacts are structured whole in this sense: they are made of functional parts and according to different functions there are different kinds of parthood relation (Garbacz, 2007).

We will show that a decomposition by determination may be seen as a mereological decomposition of a structured whole; more precisely, we will show that both relations of internal and external parts satisfy the axioms of mereology, where these axioms have been properly adapted to the case of structured wholes. Thus we need first to understand how a mereological theory of structured wholes should look like and then show that a decomposition by determination is a



mereological decomposition according to this theory.

## 8.1 The mereology of structured wholes

For some  $n \in \mathbb{N}$ , we may consider a mereological structure as an ordered  $n$ -ple of proper parthood relations. Let  $\Sigma = \langle \sqsubset_1, \dots, \sqsubset_n \rangle$  be a mereological structure. An object  $a$  is a  $\Sigma$ -whole iff there is a complete part list of  $a$  having  $n$  items  $a_1, \dots, a_n$  such that

$$a_1 \sqsubset_1 a, \dots, a_n \sqsubset_n a$$

$\Sigma$ -wholes may have other  $\Sigma$ -wholes as parts. In this case the issue of transitivity of parthood is complex. It is reasonable to admit that it makes no sense of speaking of transitivity between different kinds of parthood relations. For instance, the door is a functional part of the house and the door handle is a functional part of the door; however, the door handle is not a functional part of the house. This is because the door handle and the door are functional parts with different functions, and thus stand for different kinds of parthood relations. Parthood relations are strict orders only when we consider the same kind of parthood relation. For this reason we want each relation  $\sqsubset_i$  of  $\Sigma$  to be a strict order.

For the sake of simplicity, I will now refer to a simpler structure  $\Delta = \langle \sqsubset_1, \sqsubset_2 \rangle$  admitting just two kinds of parthood relation; we call these relations respectively ‘part<sub>1</sub>’ and ‘part<sub>2</sub>’. We will define all the fundamental mereological notions for  $\Delta$ -wholes. This means that the quantifiers of our theory will range over a domain of objects that are  $\Delta$ -wholes.

The first notion we have to define is that of *structural overlapping*. Two  $\Delta$ -wholes structurally overlap when either they have a part<sub>1</sub> in common or a part<sub>2</sub>. Being  $O^\Delta$  the relation of structural overlapping for  $\Delta$ -wholes:

$$\forall x, y [O^\Delta(x, y) \Leftrightarrow \exists z (z \sqsubset_1 x \vee z \sqsubset_2 x)] \quad (A.1)$$

Clearly, there may be overlapping that are not structural, i.e. the part that two wholes have in common is part in the general sense but not according to the structure  $\Delta$ . I will not consider these cases.

The notion of mereological sum is also different for structured wholes.  $\Delta$ -wholes are always the result of a mereological sum of two objects, one being the  $\text{part}_1$  and the other the  $\text{part}_2$ . Thus the operator of mereological sum  $\sqcup$ , is a binary operator that always return a  $\Delta$ -whole. In particular, we want  $\sqcup$  to satisfy an intuitive principle regarding structural overlapping:

$$\forall x_1, x_2, y [\sqcup(x_1, x_2) = y \Rightarrow \forall z (O^\Delta(z, y) \Rightarrow O^\Delta(z, x_1) \vee O^\Delta(z, x_2))] \quad (A.2)$$

which says that if  $y$  is the mereological sum of  $x_1, x_2$ , then whatever structurally overlaps with  $y$  must structurally overlap either with  $x_1$  or with  $x_2$ .

Another important principle of standard mereology is weak supplementation. Weak supplementation says that if  $x$  is a proper part of  $y$ , then there is  $z$  such that  $z$  is a proper part of  $y$  that does not overlap with  $x$ . In other words, supplementation says that if  $x$  is a proper part of  $y$ , then there is another part of  $y$  that is part of the complement of  $x$  with respect to  $y$ . Or alternatively, no whole  $y$  may have a unique proper part  $x$ , for there is always another part  $z$  supplementing  $x$  in the mereological decomposition of  $y$ .

The principle of weak supplementation for structured wholes is rendered as a battery of principles. A  $\Delta$ -whole is always composed by a  $\text{part}_1$  and a  $\text{part}_2$ ; in other words, by definition of structured whole, the  $\text{part}_1$  of a  $\Delta$ -whole  $x$  is always supplemented by a  $\text{part}_2$  and vice versa; thus there are two principles of supplementation: one saying that a  $\text{part}_1$  is supplemented by a  $\text{part}_2$ , and another saying that a  $\text{part}_2$  is supplemented by a  $\text{part}_1$ . We will write the two principles in the following:

$$\forall x, y [x \sqsubset_1 y \Rightarrow \exists z (z \sqsubset_2 y \wedge \neg O^\Delta(x, z))] \quad (A.3)$$

$$\forall x, y [x \sqsubset_2 y \Rightarrow \exists z (z \sqsubset_1 y \wedge \neg O^\Delta(x, z))] \quad (A.4)$$

We may summarize axioms and definitions for  $\Delta$ -mereology, i.e. the theory of structured wholes having structure  $\Delta$ , as follows. All relations of proper parthood in  $\Delta$  must be strict partial orders. Structural overlapping is defined according to (A.1). The mereological sum operator must satisfy (A.2). The relations of proper parthood in  $\Delta$  must satisfy the principles of weak supplementation (A.3) and (A.4). In the next section we will see how sustainment and

determination functions for a content may be seen as structured wholes and how all axioms of structured mereology are satisfied.

## 8.2 Content functions as structured wholes

We will start by defining the notions of proper parthood that will characterize the mereological structure for sustainment and determination functions of a certain content.

Let  $c$  be a content. We say that two functions *are of the same  $c$ -type* iff either both are sustainment functions for  $c$  or both are determination functions.

**Definition 1.** (*Internal part*)

*Given a content  $c$ , the function  $f$  is an internal part of the function  $g$  – denoted by ‘ $f \sqsubset_i g$ ’ – iff:*

- (i)  *$f$  and  $g$  are of the same  $c$ -type,*
- (ii) *The domain of  $f$  is identical to the domain of  $g$*
- (iii) *For every  $x$  in the domain of  $f$ , the property of sustaining  $c$  with  $f(x)$  determines the property of sustaining  $c$  with  $g(x)$*

To understand this definition, let  $\mathcal{I}$  be the content function of a content  $c$ , and  $f$  a particular selector of a determinant carving of  $c$ .  $\mathcal{I}$  and  $f$  are of the same  $c$ -type, for both are sustainment functions for  $c$ . Both are defined over the set of worlds strictly sustaining  $c$ . Moreover given a partial world  $s$ , the property of sustaining  $c$  with  $f(s)$  as witness determines the property of sustaining  $c$  with  $\mathcal{I}(s)$ , i.e. with the truth-value determined by  $f(s)$ . The case of an internal part of a determination function may be exemplified by considering a recarving by determination of a functional component  $F$ . In this case, we consider the following decomposition  $F = F_0 \bullet h$ ; the function  $h$  is an internal part of  $F$ , for it associates the witnesses in the domain of  $F$  with witnesses of a less determinate property of content sustainment.

The relation of external part will be defined in different ways for sustainment and determination functions.

**Definition 2.** (*External part for determination functions*)

Let  $F$  be a determination function for a content  $c$ .  $F'$  is an external part of  $F$  – denoted by ' $F' \sqsubseteq_e F$ ' – iff

- (i)  $F'$  is a determination function for  $c$ ,
- (ii) The image of  $F$  is identical to the image of  $F'$
- (iii) For all  $x$  in the domain of  $F$  there is a  $y$  in the domain of  $F'$  such that  $F(x) = F'(y)$  and the property of sustaining  $c$  with  $x$  determines the property of sustaining  $c$  with  $y$ .

Let  $c$  be the content of ' $a$  is parallel to  $b$ '. Consider the recarving by determination of the relation of parallelism  $F_{\parallel}$  associated with the following decomposition  $F_{\parallel} = F_{=} \bullet [\delta; \delta]$  where  $\delta$  is the direction operator. It is easy to see that  $F_{=} \sqsubseteq_e F_{\parallel}$ ; indeed, for every  $\langle u, v \rangle$  in the domain of  $F_{\parallel}$  there are always  $\langle h, k \rangle$  (the directions of  $u$  and  $v$ ) in the domain of  $F_{=}$  such that  $F_{\parallel}(u, v) = F_{=}(h, k)$ ; moreover, the property of witnessing  $c$  with  $u, v$  determines the property of witnessing  $c$  with  $h, k$ .

We will now turn to the notion of external part for sustainment functions:

**Definition 3.** (*External part for sustainment functions*)

Let  $c$  be a content and  $\Gamma$  a sustainment function for  $c$ . The function  $F$  is an external part of  $\Gamma$  – denoted by ' $F \sqsubseteq_e \Gamma$ ' – iff

- (i)  $F$  is a determination function for  $c$ ,
- (ii) The image of  $F$  is identical to the image of  $\Gamma$ ,
- (iii) For all  $s$  in the domain of  $\Gamma$  there is an  $x$  in the domain of  $F$  such that  $\Gamma(s) = F(x)$  and the property of sustaining  $c$  with  $x$  determines the property of sustaining  $c$  with  $\Gamma(s)$ .

It is immediate to see that being  $\mathcal{I}$  the content function of  $c$  and  $F$  the functional component of a determinant carving of  $c$ ,  $F \sqsubseteq_e \mathcal{I}$ .

We may now consider a structure  $\Delta = \langle \sqsubseteq_i, \sqsubseteq_e \rangle$  with an operation of composition by determination  $\bullet$ . We want to show that  $\Delta$  and  $\bullet$  define a structured mereology. To do this, we have to show that both  $\sqsubseteq_i$  and  $\sqsubseteq_e$  define partial strict orders, and that the axioms obtained by replacing  $\sqsubseteq_1$  with  $\sqsubseteq_i$ ,  $\sqsubseteq_2$  with  $\sqsubseteq_e$  and  $\sqcup$  with  $\bullet$  in (A.2), (A.3), (A.4) hold.

**Theorem 1.** (*Internal part is a partial strict order*)

The relation  $\sqsubseteq_i$  defines a partial strict order, i.e.:

$$(a) \forall f \neg(f \sqsubset_i f)$$

$$(b) \forall f, h (f \sqsubset_i h \Rightarrow \neg(h \sqsubset_i f))$$

$$(c) \forall f, g, h [(f \sqsubset_i h \wedge h \sqsubset_i g) \Rightarrow (f \sqsubset_i g)]$$

*Proof.* Proof of (a). Suppose that  $f \sqsubset_i f$ ; by definition 1, for every  $x$  in the domain of  $f$ , the property of sustaining  $c$  with  $f(x)$  should determine itself, which contradicts with the fundamental properties of the relation of determination.

Proof of (b). Assuming that  $f \sqsubset_i h$  and  $h \sqsubset_i f$ , it follows that for every  $x$  in the common domain of  $h$  and  $f$ , the propertie of sustaining  $c$  respectively with  $f(x)$  and  $h(x)$  should determine each other. Yet the determination relation is antisymmetric, thus our assumption lead to an absurdity.

Proof of (c). Assuming  $f \sqsubset_i h$  and  $h \sqsubset_i g$ , the conclusion  $f \sqsubset_i g$  follows from the transitivity of the determination relation.

□

**Theorem 2.** (*External part is a partial strict order*)

*The relation  $\sqsubset_e$  defines a partial strict order, i.e.:*

$$(a) \forall f \neg(f \sqsubset_e f)$$

$$(b) \forall f, h (f \sqsubset_e h \Rightarrow \neg(h \sqsubset_e f))$$

$$(c) \forall f, g, h [(f \sqsubset_e h \wedge h \sqsubset_e g) \Rightarrow (f \sqsubset_e g)]$$

*Proof.* We introduce the following abbreviation: given a content  $c$ , by  $S_x^c$  we denote the property of sustaining  $c$  with  $x$ .

Proof of (a). Given a content  $c$ , suppose that  $f \sqsubset_e f$ . Given the fact that only determination functions may be external parts,  $f$  must be a determination function. Being  $A$  the domain of  $f$ , from the fact that  $f \sqsubset_e f$ , it follows that for every  $x \in A$ , there is  $y \in A$  such that  $f(x) = f(y)$  and  $S_x^c$  determines  $S_y^c$ . But this is absurd, for by the fact that  $f$  is a determination function for  $c$ , it follows that all properties  $S_x^c$  with  $x \in A$  are determinates of the same level of specification, thus none of these properties may determine another.

Proof of (b). Given a content  $c$ , suppose that there are two functions  $f, g$  such that  $f \sqsubset_e g$  and  $g \sqsubset_e f$ . Suppose that the function  $f$  has domain  $A$  and image  $B$ , and that the function  $g$  has domain  $C$ ; by

definition of external part, the image of  $g$  is  $D$ .

Notice that only determination functions for  $c$  may be external parts; thus both  $f$  and  $g$  must be determination functions for  $c$ .

From the fact that  $f \sqsubseteq_e g$ , it follows that for every  $z \in C$  there is an  $x \in A$  such that  $f(x) = g(z)$  and  $S_x^c$  determines  $S_z^c$ . For a certain  $z_0 \in C$ , let  $x_0$  be the element of  $A$  satisfying the previous clause, i.e.  $f(x_0) = g(z_0)$  and  $S_{x_0}^c$  determines  $S_{z_0}^c$ .

From the fact that  $g \sqsubseteq_e f$ , it follows that for every  $x \in A$  there is an  $z \in C$  such that  $f(x) = g(z)$  and  $S_z^c$  determines  $S_x^c$ . Consider the instantiation of this clause for  $x_0 \in A$  and let  $z_1$  be the element of  $C$  such that  $f(x_0) = g(z_1)$  and  $S_{z_1}^c$  determines  $S_{x_0}^c$ . By the transitivity of the relation of determination we have that from the fact that  $S_{z_1}^c$  determines  $S_{x_0}^c$  and  $S_{x_0}^c$  determines  $S_{z_0}^c$ , it follows that there are  $z_0, z_1 \in C$  such that  $S_{z_1}^c$  determines  $S_{z_0}^c$ . Yet, by the fact that  $g$  is a determination function for  $c$ , it follows that all properties of the form  $S_z^c$  for  $z \in C$  are determinates of the same level of specification. Thus the fact that  $S_{z_1}^c$  determines  $S_{z_0}^c$  is absurd.

Proof of (c). Suppose that  $f$  has  $A$  as domain and  $B$  as image,  $g$  has  $C$  as domain and  $B$  as image, and  $h$  has  $D$  as domain and  $B$  as image. Suppose that  $f \sqsubseteq_e g$  and  $g \sqsubseteq_e h$ . We will prove that  $f \sqsubseteq_e h$ .

Notice that, given a content  $c$ ,  $f, g$  must be determination functions for  $c$ , while  $h$  may be a sustainment or a determination function. We will prove the theorem for  $h$  determination function. The proof of the case of  $h$  sustainment function may be developed along the same lines.

From the fact that  $f \sqsubseteq_e g$ , it follows that for every  $y \in C$ , there is  $x \in A$  such that  $f(x) = g(y)$  and  $S_x^c$  determines  $S_y^c$ .

From the fact that  $g \sqsubseteq_e h$ , it follows that for every  $z \in D$ , there is  $y \in C$  such that  $g(y) = h(z)$  and  $S_y^c$  determines  $S_z^c$ . Consider a particular  $z_0 \in D$ ; suppose that  $y_0 \in C$  is such that  $g(y_0) = h(z_0)$  and  $S_{y_0}^c$  determines  $S_{z_0}^c$ . We may now instantiate the condition of  $f \sqsubseteq_e g$  for  $y_0 \in C$ ; thus there is a certain  $x_0 \in A$  such that  $S_{x_0}^c$  determines  $S_{y_0}^c$ . Thus – by transitivity of both the determination and the identity relations – given  $z_0 \in D$ , there is  $x_0 \in A$  such that  $f(x_0) = h(z_0)$  and  $S_{x_0}^c$  determines  $S_{z_0}^c$ . By generalizing this condition, it follows that  $f \sqsubseteq_e h$ .

□

The next step is to show that the operation  $\bullet$  of functional composition by determination satisfies the axiom (A.2) for mereological sum. Firstly, we need to define the relation of overlapping. Given two functions  $f, g$ , we define the overlapping relation as follows:

$$O^\Delta(f, g) \Leftrightarrow \exists h((h \sqsubset_i f \wedge h \sqsubset_i g) \vee (h \sqsubset_e f \wedge h \sqsubset_e g)) \quad (A.1')$$

We can now prove that our structure  $\Delta$  satisfies the axiom (A.2)

**Theorem 4.** *Suppose that for three functions  $f, g, h$  holds that  $f = g \bullet h$ . Then*

$$\forall \phi [O^\Delta(\phi, f) \Rightarrow (O^\Delta(\phi, h) \vee O^\Delta(\phi, g))] \quad (A.2')$$

*Proof.* We will prove the theorem just in case  $f$  is a sustainment function for a certain content  $c$ . The other case may be proved by an analogous reasoning.

Suppose then that  $c$  is a content and  $f$  is a sustainment function for  $c$ . Assume that  $O^\Delta(\phi, f)$ . We have two cases:

**Case 1:** (*internal overlapping*) Suppose that  $O^\Delta(\phi, f)$  holds by existing a function  $\mu$  such that  $\mu \sqsubset_i f$  and  $\mu \sqsubset_i \phi$ . Considering that  $h \sqsubset_i f$  the functions  $\mu, f, \phi$ , and  $h$  have the same domain; let  $A$  be this domain.

By definition of internal part:

$$\mu \sqsubset_i f \Rightarrow \forall x \in A, S_{\mu(x)}^c \text{ determines } S_{f(x)}^c$$

$$h \sqsubset_i f \Rightarrow \forall x \in A, S_{h(x)}^c \text{ determines } S_{f(x)}^c$$

Now two determinates that determine the same determinable either belong to the same level of specification or not. By the fact that  $f$  is a sustainment function and  $\mu$  and  $h$  are both internal parts of  $f$ , it follows that also  $\mu$  and  $h$  are sustainment functions, in particular, selectors. Now two selectors define properties of content sustainment of the same level of specification just if they select the same constituents, i.e. only if for every partial world  $x \in A$ ,  $\mu(x) = h(x)$ . In this case  $h \sqsubset_i \phi$ , thus the theorem holds.

Suppose now that  $\mu$  and  $h$  define determinates of different levels of specification. For all  $x \in A$ , either  $S_{\mu(x)}^c$  determines  $S_{h(x)}^c$  or  $S_{h(x)}^c$  determines  $S_{\mu(x)}^c$ . In the former case  $\mu \sqsubset_i h$ , and thus  $\phi$  and  $h$  overlap in  $\mu$ . In the latter case,  $h \sqsubset_i \mu$ , thus  $h \sqsubset_i \phi$ ; given that any whole overlaps with each one of its parts,  $\phi$  and  $h$  overlap.

**Case 2:** (*external overlapping*) Suppose that  $O^\Delta(f, \phi)$  holds by the fact that there  $F$  such that  $F \sqsubset_e f$

and  $F \sqsubseteq_e \phi$ . Thus  $f, \phi, g$  and  $F$  have the same image. Moreover, given that  $h \sqsubseteq_i f$  and  $f = g \bullet h$ ,  $h$  has the same domain as  $f$  and the image of  $h$  coincides with the domain of  $g$ . Suppose that

$$A = \text{DOM}(f) = \text{DOM}(h) \quad ; \quad B = \text{IM}(f) = \text{IM}(\phi) = \text{IM}(F) = \text{IM}(g)$$

$$K = \text{DOM}(\phi) \quad ; \quad H = \text{DOM}(F) \text{ and } V = \text{IM}(h) = \text{DOM}(g)$$

We will prove that  $O^\Delta(\phi, g)$ .

Suppose that there is a function  $l$  such that  $\phi = F \bullet l$ ; hence,  $l$  has domain  $K$  and image  $H$ .

From the fact that  $F \sqsubseteq_e f$  it follows that there is a function that is the internal part of  $f$  in the decomposition by determination having  $F$  as external part. Call  $\mu$  this function; thus  $f = F \bullet \mu$ .

We have that  $\mu \sqsubseteq_i f$  and  $h \sqsubseteq_i f$ . Thus  $\mu$  and  $h$  are two distinct sustainment functions for  $c$ . As a consequence, they cannot be associated with the same level of specification of the property of sustaining  $c$ . Moreover, given that for any two determinates of different level of specification one must determine the other, it follows that:

$$\forall x \in A \quad \text{either } S_{\mu(x)}^c \text{ determines } S_{h(x)}^c, \text{ or } S_{h(x)}^c \text{ determines } S_{\mu(x)}^c$$

(considering the abbreviation  $S_x^c$  for the property of determining  $c$  with  $x$ ). This equivalent to say that either  $\mu \sqsubseteq_i h$  or  $h \sqsubseteq_i \mu$ .

Suppose that  $h \sqsubseteq_i \mu$ . As a consequence, there is a function  $\alpha : V \rightarrow H$  such that  $\mu = \alpha \bullet h$ . As a consequence:

$$\forall x \in A \quad F(\mu(x)) = F(\alpha(h(x))) = f(x) = g(h(x))$$

Thus  $g = F \circ \alpha$ . We will show that  $F \sqsubseteq_e g$ . We have to show that given  $x \in V$  there is  $z \in H$  such that  $g(x) = F(z)$  and  $S_x^c$  determines  $S_z^c$ . Note that given  $x \in V$  we may consider the element of  $H$  defined as  $\alpha(x)$ . By the fact that  $g = F \circ \alpha$ , it follows that  $g(x) = F(\alpha(x))$ ; by the fact that  $\alpha$  is a determination function for  $c$  (it is the external part of  $\mu$ ), it follows that  $S_x^c$  determines  $S_{\alpha(x)}^c$ . Thus  $F \sqsubseteq_e g$ . From the fact that  $F \sqsubseteq_e g$  and  $F \sqsubseteq_e \phi$ , it follows that  $\phi$  externally overlaps with  $g$ .

Suppose that  $\mu \sqsubseteq_i h$ ; then there is a function  $\alpha'$  having domain  $H$  and image  $V$  such that  $h = \alpha' \bullet \mu$ . It follows that:



$$\forall x \in A \quad g(\alpha'(\mu(x))) = F(\mu(x))$$

thus  $F = g \circ \alpha'$ . We will show that  $g \sqsubseteq_e F$ .

We have to show that given  $x \in H$  there is  $z \in V$  such that  $F(x) = g(z)$  and  $S_x^c$  determines  $S_z^c$ . Note that given  $x \in H$  we may consider the element of  $V$  defined as  $\alpha'(x)$ . By the fact that  $F = g \circ \alpha'$ , it follows that  $g(\alpha'(x)) = F(x)$ ; by the fact that  $\alpha'$  is a determination function for  $c$  (it is the external part of  $h$ ), it follows that  $S_x^c$  determines  $S_{\alpha'(x)}^c$ . Thus  $g \sqsubseteq_e F$ . From the fact that  $g \sqsubseteq_e F$  and  $F \sqsubseteq_e \phi$ , it follows that  $g \sqsubseteq_e \phi$ , thus  $\phi$  and  $g$  externally overlap because  $g$  is an external part of  $\phi$ .

□

The next step is to show that the both relations of internal and external parthood satisfy the correspondent instances of the principle of supplementation as formulated in (A.3), (A.4). This will be the purpose of the following two theorems.

**Theorem 5.** (*Weak supplementation for internal parthood*)

*Given a content  $c$  and two functions  $f$  and  $g$ ,*

$$f \sqsubseteq_i g \Rightarrow \exists \phi (\phi \sqsubseteq_e g \wedge \neg O(\phi, f)) \quad (*)$$

*Proof.* We will prove the theorem by assuming the antecedent of  $(*)$  and the negation of the consequent. Thus suppose that  $f \sqsubseteq_i g$  and

$$\forall \phi (\phi \sqsubseteq_e g \Rightarrow O^i(\phi, f) \vee O^e(\phi, f))$$

Consider a function  $h$  such that  $h \sqsubseteq_e g$  and  $O^i(f, h)$ . Suppose also that:

$$Dom(f) = Dom(g) = A ; Im(f) = B ; Im(g) = C$$

From the fact that  $h \sqsubseteq_e g$  it follows that  $Im(h) = C$ . From the fact that  $O^i(f, h)$  it follows that there is a function  $\psi$  such that  $\psi \sqsubseteq_i f$  and  $\psi \sqsubseteq_i h$ . Thus  $Dom(\psi) = Dom(f)$  and  $Dom(\psi) = Dom(h)$ , i.e.  $Dom(h) = Dom(f) = A$ .

Given that  $h \sqsubseteq_e g$ , there is a function  $\mu$  such that  $g = h \bullet \mu$ . Thus  $Dom(\mu) = Dom(g) = A$  and  $Im(\mu) = Dom(h) = A$ .

From the fact that  $h \sqsubseteq_e g$  it follows that:

$$\forall x \in A \exists y \in A \ g(x) = h(y) \text{ and } S_x^c \text{ determines } S_y^c$$

As a consequence there are at least two elements of  $A$  associated with determinates of the property of sustaining  $c$  belonging to different levels of specification. If  $\mu$  is a sustainment function for  $c$ , this is absurd, for it contradicts clause (ii) of Definition 2 (pag. 94); if  $\mu$  is a determination function, the contradiction arises with both clauses (ii) and (iii) of Definition 1 (pag. 93). Thus if  $f \sqsubseteq_i g$  and  $h \sqsubseteq_e g$ , then  $h$  and  $f$  cannot internally overlap.

We will now show that  $f$  and  $h$  cannot overlap even externally.

Suppose that  $O^e(f, h)$ ; then there is a function  $\phi$  such that  $\phi \sqsubseteq_e f$  and  $\phi \sqsubseteq_e h$ . Considering that  $h \sqsubseteq_e g$ ,  $\phi \sqsubseteq_e h$ , and  $\phi \sqsubseteq_e f$  it follows that:

$$Im(h) = Im(\phi) = Im(f) = Im(g) = C$$

From the fact that  $f \sqsubseteq_i g$ , it follows that there is a function  $\mu$  such that  $g = \mu \bullet f$ . By definition of decomposition by determination,  $Im(\mu) = Im(g) = C$  and  $Dom(\mu) = Im(f) = C$ . Being  $\mu$  an external part, it must be a determination function for  $c$ . As a consequence, for every  $x \in C$ ,  $S_x^c$  determines  $S_{\mu(x)}^c$ . Given that the image of  $\mu$  is also  $C$ , it follows that there are  $x, y$  in the domain (or the image) of  $\mu$  associated with determinates of different levels of specification. Yet this contradicts clause (ii) or (iii) of Definition 1.

□

**Theorem 6.** (*Weak supplementation for external parthood*)

*Proof.* The proof may be carried out using exactly the same strategy used in the proof of the previous theorem.

□

Thus we have proved that the system  $\Delta = \langle \sqsubseteq_i, \sqsubseteq_e \rangle$  represent a system of structured wholes with the operation  $\bullet$  representing the operator of logical sum.

The important consequence of this result is the fact that given a content  $c = \langle \mathcal{I}, \sigma \rangle$ , being every determinant carving of  $c$  a decomposition by determination of  $\mathcal{I}$ , different determinant carvings may be considered as different decompositions into parts of the same structured whole represented by the content function  $\mathcal{I}$ .