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Guilherme Kairalla Kolotelo

Output Feedback Control and Filter Design for Continuous-Time Switched Affine Systems

Projeto de Filtro e de Controle via Realimentação de Saída para Sistemas Afins com Comutação a Tempo Contínuo

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Dissertation presented to the School of Mechanical Engineering of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Mechanical Engineering, in the field of Mechatronics.

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Supervisor: Prof. Dr. Grace Silva Deaecto

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Autor: Guilherme Kairalla Kolotelo
Orientadora: Profa. Dra. Grace Silva Deaecto

A Banca Examinadora composta pelos membros abaixo aprovou esta Dissertação:

Profa. Dra. Grace Silva Deaecto, Presidente
Faculdade de Engenharia Mecânica, UNICAMP

Prof. Dr. Niederauer Mastelari
Faculdade de Engenharia Mecânica, UNICAMP

Profa. Dra. Gabriela Werner Gabriel
Divisão de Engenharia Eletrônica, ITA

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Resumo

KOLOTELO, Guilherme Kairalla. Projeto de Filtro e de Controle via Realimentação de Saída para Sistemas Afins com Comutação a Tempo Contínuo, Campinas, Faculdade de Engenharia Mecânica, Universidade Estadual de Campinas, 2018. Dissertação de Mestrado.

Esta dissertação trata de dois problemas de grande interesse no estudo de sistemas dinâmicos, a saber, o projeto de controle via realimentação de saída e o projeto de filtros para sistemas afins com comutação a tempo contínuo. Estes sistemas possuem particularidades interessantes que vêm atraindo a atenção da comunidade científica nas últimas décadas, seja pelos desafios teóricos ou pelo seu grande potencial para aplicações práticas, principalmente na área de eletrônica de potência. Uma das suas características principais é o fato destes sistemas possuírem vários pontos de equilíbrio, o que torna a análise e o projeto de controle e de filtros muito mais abrangentes. Em ambos os problemas mencionados, para assegurar a qualidade do projeto final, índices de desempenho $\mathcal{H}_2$ e $\mathcal{H}_\infty$ são levados em consideração. Basicamente, o texto desta dissertação pode ser dividido em duas partes principais. Na primeira, o objetivo é realizar o projeto simultâneo de um controlador afim de ordem completa e de uma regra de comutação dependentes somente da saída medida, de forma a assegurar estabilidade assintótica e desempenho adequado para o sistema global. Para esta classe de sistemas, a literatura apresenta somente resultados relacionados ao projeto da regra de comutação como única variável de controle. Exemplos acadêmicos colocam em evidência a validade da técnica proposta e a importância da atuação conjunta de ambas as estruturas de controle projetadas. A segunda parte é dedicada ao projeto de filtros afins com comutação. Mais especificamente, o filtro é projetado em conjunto com uma regra de comutação estabilizante dependente da saída medida, assegurando um custo garantido mínimo $\mathcal{H}_2$ ou $\mathcal{H}_\infty$ para o erro de estimação. Além disso, é demonstrado que o filtro que garante o custo ótimo apresenta estrutura de observador e pode ser determinado de maneira independente da regra de comutação, indicando a validade do Princípio da Separação, bastante conhecido em teoria de controle. A efetividade do filtro afim é demonstrada através da sua aplicação em um conversor de potência flyback CC-CC. Segundo o nosso conhecimento, esta é a primeira tentativa de se tratar problemas de filtragem para sistemas afins com comutação. Vale ressaltar que todas as condições obtidas são expressas em termos de desigualdades matriciais lineares e podem ser resolvidas sem grandes dificuldades através de ferramentas computacionais já existentes na literatura.

Palavras-chave: Sistemas Afins com Comutação; Controle $\mathcal{H}_2$ e $\mathcal{H}_\infty$; Filtragem $\mathcal{H}_2$ e $\mathcal{H}_\infty$; Desigualdades Matriciais Lineares.

Abstract

KOLOTELO, Guilherme Kairalla. Output Feedback Control and Filter Design for Continuous-Time Switched Affine Systems, Campinas, School of Mechanical Engineering, University of Campinas, 2018. Master's Thesis.

This dissertation tackles two problems of great interest in the study of dynamical systems, namely, the output feedback control design problem, and the filtering problem in the context of continuous-time switched affine systems. These systems and their unique properties, have been gathering much interest within the scientific community over the past decades, due to the theoretical challenges they pose, as well as their wide scope of practical applications, especially in the field of power electronics. One of their particular characteristics is the existence of several equilibrium points, thus making the analysis and design of controllers and filters a much broader problem. For both cases, to assure a suitable performance of the overall system, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices are considered. In essence, this work can be divided in two main parts. The first deals with the simultaneous design of a full-order affine controller and a switching rule, both dependent only on the measured output, that together assure asymptotic stability and performance of the switched system. For this class of systems, existing results in the literature treat only the problem of designing a switching function that is the single control variable. Academic examples illustrate the validity of the proposed techniques, and make clear the importance of the joint action of both control structures. The second part is dedicated to the design of switched affine filters. More specifically, the filter is designed together with a stabilizing switching function, dependent on the measured output, assuring a minimum $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost for the estimation error. Furthermore, it is proved that the optimal guaranteed cost filter presents an observer-based structure and can be designed independently of the switching function, indicating that the separation principle, well-known in control theory, holds. The effectiveness of the affine filter is demonstrated by means of an application consisting in a flyback DC-DC power converter. To the best of the authors' knowledge, the classical filtering problem in the context of switched affine systems has not been treated in the literature yet. It should be noted that the design conditions are expressed in terms of linear matrix inequalities, which can be solved without difficulty by means of readily available tools.

Keywords: Switched Affine Systems; $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control; $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Filtering; Linear Matrix Inequalities.

List of Figures

Nomenclature

Abbreviations

DC	Direct Current
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant

Latin Letters

$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of non-negative real numbers.
$\mathbb{C}$	Set of complex Numbers.
$\mathbb{R}^{n\times m}$	Set of real matrices of dimension $n \times m$.
$\mathbb{K}$	Set composed of the first N positive natural numbers $\mathbb{K} := \{1, \cdots, N\}$.
$\mathcal{H}$	Set of Hurwitz matrices.
$\mathcal{L}_2$	Set of square-integrable trajectories.
$\mathbf{I}$	Identity matrix.
$\mathcal{C}_\sigma$	Dynamical controller under switching rule σ.
$\mathcal{F}_\sigma$	Dynamical Filter under switching rule σ.

Greek Letters

ω	Angular frequency.
σ	Switching function.
Λ_N	Unit simplex of order N, $\Lambda_N := \left\{\lambda \in \mathbb{R}^N : \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1\right\}$.
λ	Arbitrary vector belonging to Λ_N.

Supercripts

$\mathbf{X}^*$	Conjugate transpose of matrix $\mathbf{X}$.
$\mathbf{X}^T$	Transpose of matrix $\mathbf{X}$.

Subscripts

$\mathbf{x}_i$	i–th element of vector $\mathbf{x}$.
$\mathbf{z}_{e_i}$	i–th element of vector $\mathbf{z}_e$.
$\mathbf{X}_i$	i–th matrix of the set $\{\mathbf{X}_1, \ldots, \mathbf{X}_N\}$.
π_{ij}	Element at row i, column j of matrix $\mathbf{\Pi}$.
$\mathbf{H}_{wz}(s)$	Transfer matrix from input $\mathbf{w}$ to output $\mathbf{z}$.
λ	A given constant vector belonging to Λ_N.
$\mathbf{X}_\lambda$	Convex combination of matrices $\{\mathbf{X}_1, \ldots, \mathbf{X}_N\}$ in λ, $\mathbf{X}_\lambda = \sum_{i=1}^{N} \lambda_i \mathbf{X}_i$.

Symbols

$\bullet$	Symmetric block of a symmetric matrix.
$\mathrm{He}\{\mathbf{X}\}$	Hermitian operator $\mathrm{He}\{\mathbf{X}\} := \mathbf{X} + \mathbf{X}^T$.
$\mathrm{tr}(\mathbf{X})$	Trace of matrix $\mathbf{X}$.
$\mathrm{diag}(\mathbf{X}, \mathbf{Y})$	Block diagonal matrix, whose blocks are $\mathbf{X}$ and $\mathbf{Y}$.
$\|\cdot\|_2$	Euclidean norm operator.
$\|\cdot\|_{\mathcal{L}_2}$	$\mathcal{L}_2$ norm operator.
$\lvert\cdot\rvert$	Absolute value of a scalar.
$\mathbf{X} \succ 0$ ($\mathbf{X} \prec 0$)	Matrix $\mathbf{X}$ is symmetric and positive (negative) definite, such that $\forall \mathbf{v} \neq \mathbf{0}$, $\mathbf{v}^T \mathbf{X} \mathbf{v} > (<) 0$.
$\mathbf{X} \succeq 0$ ($\mathbf{X} \preceq 0$)	Matrix $\mathbf{X}$ is symmetric and positive (negative) semi-definite, such that $\forall \mathbf{v} \neq \mathbf{0}$, $\mathbf{v}^T \mathbf{X} \mathbf{v} \geq (\leq) 0$.
$\min(\cdot)$	Minimum operator.
$\sup(\cdot)$	Supremum operator.
$\arg\ \min_{i \in \mathbb{K}}(\cdot)$	i–th element in $\mathbb{K}$ whose value $(\cdot)$ is minimum.

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Chapter 1
Introduction

1.1 Motivation

From the simplest cellphone charger to the most complex bipedal robot, the use of digital control systems that incorporate decision-making algorithms to act upon continuous-time processes is pervasive in modern times. This interaction between systems of continuous nature at a lower level, and discrete events at a higher level, is subsumed under the term *hybrid systems*. For example, the switching of a transistor in a power converter, an automotive transmission changing gears, a thermostat turning the heat on or off, or an aircraft flight control system alternating between different flight modes are some of the many real-life processes that exhibit an intrinsically hybrid behavior. In many of these applications, the employment of filtering and estimation techniques is also useful to recover valuable information from noisy or corrupted signals. This information sometimes cannot be directly measured from the process, but is essential to successfully implement many widely used control methodologies.

Motivated by the remarkable advancement of embedded systems over these past decades, and its application in control systems and filtering, the study of hybrid systems has never had more relevance than it does now. Although in widespread use today, the history of hybrid systems is fairly recent, stemming from the introduction, and subsequently rapid adoption of the electromagnetic relay in industrial automation by the mid 1900's. Since then, progressively more sophisticated hybrid systems displaying unique and complex behaviors have emerged, and with them, the necessity to develop tools and techniques for the modeling, analysis, control, and filtering of these types of systems, taking into account the intertwined nature of the continuous-time and discrete-time dynamics displayed by these systems. Further details on this topic can be found in the references [1, 2] which discuss at length many of the concepts and challenges presented by hybrid systems.

A particularly important subclass of hybrid systems, known as switched systems, has gathered much attention lately due to its usefulness in modeling a wide range of applications. For instance, active and semi-active automotive suspension control [3, 4], the control of wind turbines [5], power electronics [6, 7], aircraft roll angle control [8], and autonomous robotics [9, 10]. In essence, these systems are comprised of a finite set of subsystems, defining its available modes of operation, and a switching signal that selects which mode will be active at each instant of time. As such, these systems exhibit a complex and nonlinear dynamical behavior, distinct to that of their modes of operation. Furthermore, a suitable switching signal may stabilize the switched system in a situation where all modes of operation are unstable, or conversely, it may destabilize a switched system even in the case where all subsystems are stable. This underscores the importance of the switching signal, which can be an arbitrary time-dependent signal, such as an external input or disturbance, or a control variable to be designed. In both cases, the control problem is centered on developing conditions for stability, while filtering and estimation problems deal with conditions that assure the convergence of an estimated signal. Additionally, assuring certain performance criteria for the design of controllers and filters is often considered, given its relevance for practical applications. The books [11] and [12] as well as the survey paper [13] explore these topics in greater detail.

The subsystems that together constitute the switched system may individually present different kinds of dynamical behavior. In this work, the focus is given to switched systems comprised of affine subsystems, referred to as switched affine systems. This case is more general when compared to a switched system composed of linear subsystems, and it introduces greater difficulty by giving rise to a set of attainable nontrivial equilibrium points. This particular characteristic brings light to the relevance of this category of switched systems, especially given the multitude of applications that can be modeled in this framework. One such application is the previously mentioned switched mode DC-DC power converter, whose switched nature and affine dynamical behavior, make switched affine systems especially suited to model these electronic circuits. These circuits are ubiquitous, and power nearly all electronic devices in everyday life. As such, several publications in the literature treat the control design problem for the *buck*, *boost*, *buck-boost* and *flyback* topologies [14, 15, 16, 17, 18], concerned with devising an appropriate switching function tasked with attaining a desired behavior.

Many results regarding the design of an appropriate switching function, also referred to as a switching rule, in order to guarantee asymptotic stability of switched linear systems are already solidly established [15, 19, 20, 21, 22, 23, 24, 25] some of which also deal with performance criteria, such as the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ indices, and will be presented in greater detail in Chapter 3. With regard to switched affine systems, some references consider exclusively the action of a switching function dependent on state information, capable of stabilizing the system [26, 27, 28], while others deal with the joint action of a switching function, alongside a control law [29], which is based on a set of state feedback matrix gains and a state dependent switching rule assuring asymptotic stability of the closed-loop system, while guaranteeing optimal $\mathcal{H}_2$ or $\mathcal{H}_\infty$ costs.

In practical applications, however, it is not uncommon for some state variables to be unavailable, whether due to the difficulty or expense in effecting these measurements; because of physical constraints in sensor placement or size; or simply due to the unavailability of sensors to measure a certain state variable. As such, control design methodologies that rely on output information in place of state information constitute a very relevant and applicable topic of study. Existing results in the literature consider the case where the only control variable is an output-dependent switching rule [16, 30] implemented by means of a switched dynamical filter, responsible for providing the needed information. On the other hand, as of yet, results in the literature treating the control design problem considering the joint action of an output dependent stabilizing switching function and a dynamical controller, exist only for switched linear systems [24]. To fill this void in the context of switched affine systems, this work proposes design conditions for these control structures in order to assure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, as available in [31]. More specifically, the results of [24] are generalized to take into account the simultaneous design of an output-dependent switching rule and a full-order affine controller that together guarantee global asymptotic stability of a desired equilibrium point, as well as an upper bound for the $\mathcal{H}_\infty$ performance index. Whenever the external input is available for measurement or estimation, less conservative conditions can be obtained by considering a switching rule dependent not only on the measured output, but also on the external input information. The simultaneous design of both control structures assuring an $\mathcal{H}_2$ guaranteed cost and global asymptotic stability of the equilibrium point is also considered in this work. These approaches do not require any stability property of the individual subsystems, and guarantee stability even in the case where the subsystems are not individually controllable, or when a stable convex combination of dynamical matrices cannot be verified. Furthermore, the proposed techniques allow for a wider range of equilibrium points to be attained when compared to existing methods. This reinforces the importance of the joint action of both control structures, and will be discussed at length in Chapter 4.

In a similar vein, these results were generalized to treat the classical filtering problem for switched systems, which thus far had only been treated for the linear case. In this context, the following publications consider either time-dependent switching functions [32, 33, 34], or the joint design of a stabilizing switching function along with a switched filter [35], while taking into account either $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed costs. For the more general case of switched affine systems, only the problem of state estimation, as in references [36] and [37] is tackled, under the simpler switched observer-based structure. The absence of publications in this topic compelled us to develop methodologies for the classical filter design problem for continuous-time switched affine systems. The conditions here introduced are based on a full-order switched affine filter, which is designed in tandem with an output-dependent stabilizing switching function, collectively assuring an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ upper bound for the estimation error. Furthermore, it is proved that the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed cost filters present a simpler observer-based structure, and can be designed independently of the switching rule, indicating the validity of the separation principle, well-known in control theory. Chapter 5 introduces these results, and comments on these findings.

1.2 Publications

This dissertation is based in part on the following papers:

- G. K. Kolotelo and G. S. Deaecto, "Controle $\mathcal{H}_2$ e $\mathcal{H}_\infty$ via Realimentação de Saída de Sistemas Afins com Comutação por Ação Conjunta de Função de Comutação e Entrada de Controle", in *Anais do Congresso Brasileiro de Automática*, 2018. [1]

- G. K. Kolotelo, L. N. Egidio, and G. S. Deaecto, "$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Filtering for Continuous–Time Switched Affine Systems", in Proceedings of the 9$^{\text{th}}$ IFAC Symposium on Robust Control Design (ROCOND), vol. 51(25), pp. 184–189, 2018. [2] **Honorable mention as finalist for the Young Author Award.**

- G. K. Kolotelo, L. N. Egidio, and G. S. Deaecto, "Projeto de Filtros com Comutação $\mathcal{H}_2$ e $\mathcal{H}_\infty$ para Sistemas Afins a Tempo Contínuo", in *Anais do Congresso Brasileiro de Automática*, 2018. [3]

[1] http://www.swge.inf.br/proceedings/paper/?P=CBA2018-0598
[2] https://doi.org/10.1016/j.ifacol.2018.11.102
[3] http://www.swge.inf.br/proceedings/paper/?P=CBA2018-0278

1.3 Outline of Chapters

This work is divided in 5 chapters, explained in brief:

- **Chapter 1: Introduction**

  Presents the motivation and sets the context for the topics that are treated in this work.

- **Chapter 2: Preliminaries**

  Reviews the fundamental concepts of dynamical systems, important for the next chapters. In particular, the stability properties of dynamical systems are studied via Lyapunov's stability theory. Lastly, performance criteria for these systems are defined.

- **Chapter 3: Switched Systems**

  Broaches the subject of switched systems, and discusses in greater detail their unique features. Next, well-known results in the literature are introduced, which present conditions for the stability of switched linear and affine systems. Finally, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices for switched affine systems are defined, important for the subsequent chapters.

- **Chapter 4: Joint Action Output Feedback Control**

  Presents the contributions of this dissertation with regard to the joint design of an output-dependent switching rule alongside a full-order affine controller. More specifically, the methodology for the design of a full-order switched dynamical controller and a switching function that together assure global asymptotic stability of a desired equilibrium point, as well as assuring the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices is introduced. A set of numerical examples are provided to illustrate the theory.

- **Chapter 5: Filtering and Estimation**

  This chapter presents the contributions of this dissertation with regard to the filtering and estimation problem in the scope of switched affine systems. Conditions for the design of a switched dynamical filter along with a stabilizing switching function are introduced, assuring the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed costs for the estimation error. Numerical examples are supplied to validate the theory.

- **Chapter 6: Conclusion**

  Summarizes the topics explored by this dissertation, and examines some prospects for future developments in this subject.

CHAPTER 2

PRELIMINARIES

THE analysis of stability properties in the context of switched systems is largely based on the theory of stability introduced by Lyapunov, due to the nonlinear behavior that is intrinsic to these types of systems. As such, prior to delving into this subject, we review some key ideas and definitions that permeate this work, and constitute the theoretical basis under which its results depend.

The purpose of this chapter is to introduce the underlying concepts concerning the stability analysis and performance indices for Linear Time-Invariant systems, henceforth referred to as LTI systems. Firstly, following a brief discussion on the concepts of equilibrium and stability, Lyapunov's stability theory is introduced. More specifically, the second method of Lyapunov, also known as the direct method, extensively used in the analysis and control of nonlinear systems, will be used to establish the conditions under which stability properties of a dynamical system are verified for a certain equilibrium point. Lastly, the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for LTI systems will be presented. These ideas are extremely important in classical control theory, and will be extended to deal with the stability analysis and the control and filter design problems for switched systems, considered in the next chapters. The books [38, 39, 40] will be used to support the discussions throughout.

2.1 Stability of LTI Systems

Before investigating the stability properties of dynamical systems in the following sections, let us first introduce some basic concepts and notations with regard to linear dynamical systems. The state space representation of a continuous-time LTI system is given by

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{H}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{w}(t) \\
\mathbf{z}(t) &= \mathbf{E}\mathbf{x}(t) + \mathbf{F}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t)
\end{aligned} \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, with the number of states n_x being referred to as the order of the system, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ is the measured output, and $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output. Also $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are constant matrices of appropriate dimensions. In the case of $\mathbf{G} = \mathbf{0}$, the system is known as a strictly proper system. For simplicity, we initiate our discussions considering the following unforced LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.2}$$

which disregards the inputs and outputs of system (2.1), allowing us to analyze its stability properties.

2.1.1 Equilibrium Points

A state vector $\mathbf{x}_e \in \mathbb{R}^{n_x}$ is termed an equilibrium point of the system if once $\mathbf{x}(t) = \mathbf{x}_e$, for some $t = t_0$, the system remains at the equilibrium point from that moment onwards, that is, $\mathbf{x}(t) = \mathbf{x}_e$, $\forall t \geq t_0$, or equivalently, $\dot{\mathbf{x}}(t) = \mathbf{0}$ for $t \geq t_0$. For the case of linear systems, the origin $\mathbf{x}_e = \mathbf{0}$ will always be an equilibrium point. Moreover, it will be the single equilibrium point of the system, unless matrix $\mathbf{A}$ is singular, in which case, there may exist multiple equilibrium points beyond $\mathbf{x}_e = \mathbf{0}$. These equilibrium points are then given by the null space of the matrix $\mathbf{A}$, that is, $\mathbf{x} \in \mathbb{R}^{n_x}$ such that $\mathbf{Ax} = \mathbf{0}$. See the books [38, 39] for more details.

For convenience, any equilibrium point can be shifted to the origin by means of the change of variables $\boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{x}_e$ with no loss of generality, see [39, 40]. This choice will facilitate some of our subsequent analyses.

2.1.2 Stability of Dynamical Systems

The concept of stability for dynamical systems is defined in terms of an equilibrium point. More specifically, [38] defines the equilibrium point $\mathbf{x}_e \in \mathbb{R}^{n_x}$ as stable if whenever the state vector is moved slightly away from that point, it tends to return to it, or at least does not keep moving further away. In other words, the state vector remains within a bounded region of the state space around the equilibrium point. Furthermore, if whenever $t \to \infty$, the state vector $\mathbf{x}(t) \to \mathbf{x}_e$, then the equilibrium point is referred to as asymptotically stable. Whenever the concept of stability only holds for initial conditions within a certain region of the state space, the equilibrium point is said to be locally stable. If, however, it is valid for any given initial condition, , the equilibrium point is said to be globally stable. Moreover, the equilibrium point is known as globally asymptotically stable if $\mathbf{x}(t) \to \mathbf{x}_e$ as $t \to \infty$, for any initial condition in the state space.

For the specific case of LTI systems, necessary and sufficient conditions for the stability of continuous-time systems can be derived by studying the eigenvalues of the matrix $\mathbf{A}$. For these systems, an equilibrium point is globally asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ are located at the open left half of the complex plane, that is, the eigenvalues have strictly negative real part. In this case, the matrix $\mathbf{A}$ is said to be Hurwitz, or $\mathbf{A} \in \mathcal{H}$, where $\mathcal{H}$ is defined as the set of Hurwitz matrices.

2.2 Lyapunov Stability Theory

For many decades, one of the most useful and relevant techniques for the evaluation of stability properties for linear and nonlinear systems has been the theory introduced by Lyapunov in his seminal work *The General Problem of the Stability of Motion*, originally published in 1892, and later translated [41]. When compared to linear systems, nonlinear systems may exhibit new and complex behaviors. This, coupled with the fact that explicit analytical solutions most often cannot be attained for these systems, sheds light upon the importance of Lyapunov's findings. The theory is comprised of two widely used methods, commonly referred to as the linearization or indirect method, and the direct method. The latter will be used extensively throughout this work, and is introduced below in the context of LTI systems, followed by a brief discussion of its applicability on the stability analysis of affine nonlinear systems.

2.2.1 Lyapunov's Direct Method

Lyapunov's direct method, as defined in [38, 39, 40], can be used to evaluate the stability of linear and nonlinear systems in an indirect manner, enabling the characterization of the stability properties of all equilibrium points for a given system, without the need to derive explicit numerical or analytical solutions. The stability analysis is based on a scalar *energy-like* or *summarizing* function, dependent on the state vector. The purpose of this function, known as a Lyapunov function, is to describe the behavior of the entire dynamical system as a function of time, and allow conclusions to be inferred on the stability of the set of its governing differential equations. The search for a suitable Lyapunov function may be a difficult task for more sophisticated systems, since it must respect the requirements introduced below.

2.2.1.1 Lyapunov Function

A scalar function $v(\mathbf{x}) : \mathbb{D} \to \mathbb{R}$, with $\mathbb{D} \subset \mathbb{R}^{n_x}$ being a region of the state space containing $\mathbf{x} = \mathbf{0}$, is a Lyapunov function if it satisfies the following set of conditions [38, 40]:

1. $v(\mathbf{x})$ is continuously differentiable
2. $v(\mathbf{0}) = 0$
3. $v(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in \mathbb{D},\ \mathbf{x} \neq \mathbf{0}$
4. $\dot{v}(\mathbf{x}(t)) \leq 0, \quad \forall \mathbf{x} \in \mathbb{D}$

If, for a given Lyapunov function candidate, these requirements are verified for the system under consideration, then the equilibrium point $\mathbf{x}_e = \mathbf{0}$ is stable. Moreover, whenever

5. $\dot{v}(\mathbf{x}(t)) < 0, \quad \forall \mathbf{x} \in \mathbb{D},\ \mathbf{x} \neq \mathbf{0}$

is satisfied, then $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable. Additionally, if

6. $||\mathbf{x}|| \to \infty \ \Rightarrow \ v(\mathbf{x}) \to \infty, \quad \mathbb{D} = \mathbb{R}^{n_x}$ *radially unbounded*

can also be verified, then the equilibrium point at the origin is globally asymptotically stable.

2.2.1.2 Lyapunov Stability Analysis of LTI Systems

For LTI systems, an adequate choice for the Lyapunov function $v(\mathbf{x}(t))$, or simply $v(\mathbf{x})$, is the quadratic form of a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n_x \times n_x}$, that is

$$v(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x} \tag{2.3}$$

It can be verified that this function meets all the conditions of a Lyapunov function. Indeed, conditions (1) – (3) are evidently verified, as well as condition (6). In addition, consider its time derivative

$$\begin{aligned} \dot{v}(\mathbf{x}) &= \dot{\mathbf{x}}^T \mathbf{P}\mathbf{x} + \mathbf{x}^T \mathbf{P}\dot{\mathbf{x}} \\ &= \mathbf{x}^T \left(\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} \right) \mathbf{x} \\ &= -\mathbf{x}^T \mathbf{Q}\mathbf{x} \end{aligned} \tag{2.4}$$

where $\mathbf{Q} \in \mathbb{R}^{n_x \times n_x}$ is a symmetric matrix defined by the so-called Lyapunov equation

$$\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = \mathbf{0} \tag{2.5}$$

It can be shown that the validity of condition (5) is verified whenever matrix $\mathbf{A}$ is Hurwitz. Indeed, given a positive definite matrix $\mathbf{Q}$ there exists a unique positive definite matrix $\mathbf{P}$ that is a solution to the Lyapunov equation (2.5). Furthermore, the existence of $\mathbf{P} \succ 0$ satisfying the Lyapunov equation is not only a sufficient but also a necessary condition for global asymptotic stability of the equilibrium point $\mathbf{x} = \mathbf{0}$. The proof of sufficiency follows from the fact that the Lyapunov equation is satisfied, and $\dot{v}(\mathbf{x}) < 0$ assures the asymptotic stability of the system. The proof of necessity implies in demonstrating that a unique solution $\mathbf{P}$ always exists. Indeed, take a matrix $\mathbf{P}$ defined by

$$\mathbf{P} = \int_0^\infty \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} \, \mathrm{d}t \tag{2.6}$$

Whenever matrix $\mathbf{A}$ is Hurwitz, and consequently all its eigenvalues have strictly negative real part, this integral exists. Furthermore, it can be verified that the matrix $\mathbf{P}$ is symmetric positive definite, since for any vector $\mathbf{q} \in \mathbb{R}^{n_x}$, such that $\mathbf{q} \neq \mathbf{0}$, we have

$$\begin{aligned} \mathbf{q}^T \mathbf{P}\mathbf{q} &= \int_0^\infty \mathbf{q}^T \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} \mathbf{q} \, \mathrm{d}t \\ &= \int_0^\infty \mathbf{x}^T \mathbf{Q}\mathbf{x} \, \mathrm{d}t \end{aligned} \tag{2.7}$$

where $\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{q}$ is the analytical solution of the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, when assigning $\mathbf{q} = \mathbf{x}(0)$. As such, since $\mathbf{Q} \succ 0$, then $\mathbf{P}$ must be positive definite. Finally, by substituting (2.6) in (2.5) we have

$$\begin{aligned} \mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} &= \int_0^\infty \mathbf{A}^T \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} \, \mathrm{d}t + \int_0^\infty \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} \mathbf{A} \, \mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} \right) \mathrm{d}t \\ &= \lim_{t \to \infty} \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} - \mathbf{Q} \\ &= -\mathbf{Q} \end{aligned} \tag{2.8}$$

which comes from the fact that when matrix $\mathbf{A}$ is Hurwitz, then $\lim_{t \to \infty} \mathrm{e}^{\mathbf{A}^T t} \mathbf{Q} \mathrm{e}^{\mathbf{A}t} = \mathbf{0}$, and thus the Lyapunov equation is verified. In order to demonstrate the uniqueness of the solution $\mathbf{P}$, suppose that there exists another solution $\tilde{\mathbf{P}} \neq \mathbf{P}$ for the same matrix $\mathbf{Q}$, in this case, from (2.5) we have

$$\mathbf{A}^T(\mathbf{P}-\tilde{\mathbf{P}}) + (\mathbf{P}-\tilde{\mathbf{P}})\mathbf{A} = \mathbf{0} \tag{2.9}$$

By multiplying to the left of this equality by $\mathrm{e}^{\mathbf{A}^T t}$ and to the right by its transpose, we have

$$\mathrm{e}^{\mathbf{A}^T t}\left(\mathbf{A}^T(\mathbf{P}-\tilde{\mathbf{P}}) + (\mathbf{P}-\tilde{\mathbf{P}})\mathbf{A}\right)\mathrm{e}^{\mathbf{A}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\mathbf{A}^T t}\left(\mathbf{P}-\tilde{\mathbf{P}}\right)\mathrm{e}^{\mathbf{A}t}\right) = \mathbf{0} \tag{2.10}$$

Thus, $\mathrm{e}^{\mathbf{A}^T t}\left(\mathbf{P}-\tilde{\mathbf{P}}\right)\mathrm{e}^{\mathbf{A}t}$ is constant for all $t \geq 0$. By evaluating this term for $t = 0$ and $t \to \infty$, it becomes clear that $\tilde{\mathbf{P}} = \mathbf{P}$ must hold true, indicating that the solution $\mathbf{P}$ is indeed unique for a given matrix $\mathbf{Q}$.

These ideas give rise to the following theorem, as stated in [40], prescribing global asymptotic stability of the origin for LTI systems, in terms of the solution of the Lyapunov equation

Theorem 2.1. *A matrix* $\mathbf{A}$ *is Hurwitz, that is, the eigenvalues of* $\mathbf{A}$ *have strictly negative real part if and only if, for any given symmetric positive definite matrix* $\mathbf{Q}$*, there exists a unique symmetric positive definite matrix* $\mathbf{P}$ *that satisfies the Lyapunov equation* (2.5)*.*

It is worth noting that although the stability analysis of LTI systems can be easily performed by evaluating the eigenvalues of the dynamical matrix $\mathbf{A}$, this analysis is rendered moot when studying other types of systems, such as switched systems, presented in more detail in the subsequent chapters. This emphasizes the usefulness of Lyapunov's theory, which allows the characterization of the stability properties of more complex nonlinear systems in an indirect manner.

Finally, a related point to consider is that the Lyapunov equation was originally expressed in terms of linear matrix inequalities, in the form known as the Lyapunov inequality

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} \prec 0 \tag{2.11}$$

This inequality was originally solved for $\mathbf{P}$ analytically, via a set of linear equations, by choosing any $\mathbf{Q} \succ 0$ that satisfied (2.5), as discussed in [42]. Only in the 1980's it would become clear that the Lyapunov inequality could be solved numerically by means of convex optimization algorithms, with great efficiency. Throughout this work, we will focus on expressing stability conditions and control design problems in terms of Linear Matrix Inequalities, from this point onwards referred to as LMIs, which can be solved without difficulty by several standard and readily available tools.

2.2.1.3 Lyapunov Stability Analysis of Affine Systems

The analysis of stability in the context of switched affine systems is central to the topics explored in this work. As such, it is important to initially study the stability properties of the affine subclass of nonlinear systems. To this extent, consider the following affine system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.12}$$

where $\mathbf{b} \in \mathbb{R}^{n_x}$ is the affine term. Notice that whenever $\mathbf{b} = \mathbf{0}$, this system reduces to the linear case, previously presented. Also, recall that by the definition of an equilibrium point, we have $\dot{\mathbf{x}}(t) = \mathbf{0}$. Thus, the equilibrium point of the affine system can be calculated as

$$\mathbf{x}_e = -\mathbf{A}^{-1}\mathbf{b} \tag{2.13}$$

As previously mentioned, in order to simplify evaluating the stability properties of this system by means of Lyapunov's direct method, and incurring no loss of generality, the state vector $\xi(t) = \mathbf{x}(t) - \mathbf{x}_e$ is defined, and we now consider the system (2.12) shifted to the origin, as such

$$\dot{\xi}(t) = \mathbf{A}\xi(t), \quad \xi(0) = \xi_0 \tag{2.14}$$

Observe that the problem can now be treated as an LTI system, such that the methodology and conditions for stability introduced in Section 2.2.1.2 remain valid for the analysis of affine time invariant systems. In this case, for a certain affine system under consideration, whenever there exists $\mathbf{P} > 0$ such that the Lyapunov inequality (2.11) is satisfied for this system, then $\mathbf{x}_e$ is a globally asymptotically stable equilibrium point, since for $\xi \to \mathbf{0}$ as $t \to \infty$, we have $\mathbf{x} \to \mathbf{x}_e$.

2.3 Performance Indices for LTI Systems

In this section, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for LTI systems are introduced. These norms are used extensively in control design, see [40, 42, 43, 44], to characterize the effects of a given input signal on the output of the system. They will be expressed both with respect to the transfer function of the system, as well as in terms of its impulse response, where the latter will be essential to allow their generalization in order to deal with switched systems, thus providing an effective measure of performance for this class of systems. But first, some fundamental concepts that are important to the development of the next topics will be presented, namely, Parseval's Theorem for continuous-time LTI systems, and the $\mathcal{L}_2$ space.

2.3.1 Parseval's Theorem

Consider the function $\mathbf{f}(t) : [0, \infty) \to \mathbb{R}^n$ and its Laplace transform $\mathbf{F}(s)$, as well as the conjugate transpose $\mathbf{F}(s)^*$, whose domain dom($\mathbf{F}$), contains the imaginary axis. Parseval's theorem is then defined by [45] as

$$\begin{aligned} \int_0^\infty \mathbf{f}(t)^T \mathbf{f}(t)\, \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{F}(\mathrm{j}\omega)^* \mathbf{F}(\mathrm{j}\omega) \mathrm{d}\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \mathbf{F}(\mathrm{j}\omega)^* \mathbf{F}(\mathrm{j}\omega) \mathrm{d}\omega \end{aligned} \tag{2.15}$$

This will later be employed for the calculation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms in the coming sections. It should be noted that the second equality of (2.15) is valid only when $\mathbf{f}(t) \in \mathbb{R}^n$, since in this case $\mathbf{F}(\mathrm{j}\omega)^* = \mathbf{F}(-\mathrm{j}\omega)^T$.

2.3.2 $\mathcal{L}_2$ Space

The norm $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$, as defined in [40, 46], is a real-valued function satisfying the following four axioms, for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and all $\alpha \in \mathbb{R}$:

- $\|\mathbf{v}\| \geq 0$ *Nonnegativity*
- $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$ *Positivity*
- $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$ *Homogeneity*
- $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ *Triangle Inequality*

One specific example, the Euclidean norm, defined for any vector $\mathbf{x} \in \mathbb{R}^n$

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \left(\mathbf{x}^T \mathbf{x} \right)^{1/2} \tag{2.16}$$

is perhaps the most commonly used norm. The definition of a norm, however, is not exclusive to finite-dimensional vector spaces. It is also useful to define the $\mathcal{L}_2$ norm for continuous real-valued functions of the form $\mathbf{f}(t) \colon [0, \infty) \to \mathbb{R}^n$, as such

$$\|\mathbf{f}\|_{\mathcal{L}_2} = \left(\int_0^\infty \|\mathbf{f}(t)\|_2^2 \, \mathrm{d}t \right)^{1/2} = \left(\int_0^\infty \mathbf{f}(t)^T \mathbf{f}(t) \, \mathrm{d}t \right)^{1/2} \tag{2.17}$$

If the integral amounts to a finite value, the function $\mathbf{f}(t)$ is called a square-integrable function. This characterization of the norm for a function will be helpful to measure the magnitude of the input and output signals of a dynamical system, allowing for the definition of the performance criteria introduced in the next section.

2.3.3 System Definition

The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are introduced considering the following LTI system, defined for $t \geq 0$, with matrix $\mathbf{A}$ Hurwitz.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{H}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{0} \\ \mathbf{z}(t) &= \mathbf{E}\mathbf{x}(t) + \mathbf{G}\mathbf{w}(t) \end{aligned} \tag{2.18}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output, and matrices $\mathbf{A}, \mathbf{H}, \mathbf{E}, \mathbf{G}$ are of appropriate dimensions. Also consider the transfer matrix $\mathbf{H}_{wz}(s) \in \mathbb{R}^{n_z \times n_w}$ of system (2.18), from the input $\mathbf{w}$ to the output $\mathbf{z}$, given by

$$\mathbf{H}_{wz}(s) = \mathbf{E}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{H} + \mathbf{G} \tag{2.19}$$

with $s \in \mathbb{C}$. We can now proceed to the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for continuous-time LTI systems.

2.3.4 $\mathcal{H}_2$ Norm for LTI Systems

The $\mathcal{H}_2$ norm can be interpreted as a measure of the energy of the output signal of a dynamical system, when driven by an impulse. Other interpretations, such as in the context of stochastic systems exist, but will not be discussed in this work. For the continuous-time LTI system (2.18), the $\mathcal{H}_2$ norm may be calculated whenever a strictly proper transfer matrix $\mathbf{H}_{wz}(s)$ is considered, that is, with $\mathbf{G} = \mathbf{0}$. In this case, the $\mathcal{H}_2$ norm is defined by reference [47] as

$$||\mathbf{H}_{wz}(s)||^2_{\mathcal{L}_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(\mathbf{H}_{wz}(\mathrm{j}\omega)^* \mathbf{H}_{wz}(\mathrm{j}\omega)\right) \mathrm{d}\omega \tag{2.20}$$

By applying Parseval's Theorem, introduced in (2.15), the $\mathcal{H}_2$ norm can be expressed in the time domain, as such

$$||\mathbf{H}_{wz}(s)||^2_{\mathcal{L}_2} = \int_{0}^{\infty} \mathrm{tr}\left(\mathbf{h}_{wz}(t)^T \mathbf{h}_{wz}(t)\right)\, \mathrm{d}t \tag{2.21}$$

By realizing that the impulse response $\mathbf{h}_{wz}(t)$ of the system, when initial conditions are set to zero, is

$$\mathbf{h}_{wz}(t) = \begin{cases} \mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}, & t \geq 0 \\ \mathbf{0}, & \text{otherwise} \end{cases} \tag{2.22}$$

and that for multiple-input, multiple-output systems, $\mathbf{h}_{wz}(t)$ is of the form

$$\mathbf{h}_{wz}(t) = \begin{bmatrix} h_{11}(t) & \dots & h_{n_w}(t) \\ \vdots & \ddots & \vdots \\ h_{n_z}(t) & \dots & h_{n_z n_w}(t) \end{bmatrix} \tag{2.23}$$

then, from (2.21) and (2.23), the $\mathcal{H}_2$ norm can be written as

$$||\mathbf{H}_{wz}(s)||^2_{\mathcal{L}_2} = \sum_{k=1}^{n_w}\sum_{i=1}^{n_z}\int_0^{\infty} h^2_{ik}(t)\, \mathrm{d}t \tag{2.24}$$

It is interesting to note that a smaller $\mathcal{H}_2$ norm is generally associated to a faster convergence of the state trajectories to the equilibrium point. Also, for single-input, single-output systems, the $\mathcal{H}_2$ norm becomes simply the $\mathcal{L}_2$ norm of the impulse response for the system in question. This emphasizes the need of considering a strictly proper system in order to obtain a finite $\mathcal{H}_2$ norm. Furthermore, observe that equation (2.24) may alternatively be expressed as

$$||\mathbf{H}_{wz}(s)||^2_{\mathcal{L}_2} = \int_0^{\infty} \mathrm{tr}\left(\mathbf{H}^T \mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}\right)\, \mathrm{d}t = \int_0^{\infty} \mathrm{tr}\left(\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathbf{H}\mathbf{H}^T\mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\right)\mathrm{d}t \tag{2.25}$$

This allows the $\mathcal{H}_2$ norm to be stated either as

$$||\mathbf{H}_{wz}(s)||_{\mathcal{L}_2} = \sqrt{\mathrm{tr}\left(\mathbf{H}^T\mathbf{L}_o\mathbf{H}\right)}, \qquad \text{or} \qquad ||\mathbf{H}_{wz}(s)||_{\mathcal{L}_2} = \sqrt{\mathrm{tr}\left(\mathbf{E}\mathbf{L}_c\mathbf{E}^T\right)} \tag{2.26}$$

where the matrices $\mathbf{L}_o$ and $\mathbf{L}_c$ are respectively referred to as the controllability Gramian and the observability Gramian, given by

$$\mathbf{L}_o = \int_0^\infty \mathrm{e}^{\mathbf{A}^T t}\mathbf{E}^T\mathbf{E}\mathrm{e}^{\mathbf{A}t}\mathrm{d}t, \qquad \text{and} \qquad \mathbf{L}_c = \int_0^\infty \mathrm{e}^{\mathbf{A}t}\mathbf{H}\mathbf{H}^T\mathrm{e}^{\mathbf{A}^T t}\mathrm{d}t \tag{2.27}$$

These are, in turn, the solutions to their associated Lyapunov equations, briefly discussed in Section 2.2.1.2, as follows

$$\mathbf{A}^T\mathbf{L}_o + \mathbf{L}_o\mathbf{A} + \mathbf{E}^T\mathbf{E} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T + \mathbf{H}\mathbf{H}^T = \mathbf{0} \tag{2.28}$$

Observe that the $\mathcal{H}_2$ norm, expressed in this manner, can be easily solved via numerical methods by the following convex optimization problem, subject to LMI constraints, as shown in [42]

$$\begin{aligned} \min \quad & \mathrm{tr}\left(\mathbf{H}^T\mathbf{P}\mathbf{H}\right) \\ \text{subject to:} \quad & \mathbf{P} > 0 \\ & \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} < 0 \end{aligned} \tag{2.29}$$

$$\begin{aligned} \min \quad & \mathrm{tr}\left(\mathbf{E}\mathbf{P}\mathbf{E}^T\right) \\ \text{subject to:} \quad & \mathbf{P} > 0 \\ & \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{H}\mathbf{H}^T < 0 \end{aligned} \tag{2.30}$$

It is important to note that the 'min' and 'inf' terms for optimization problems can be used interchangeably whenever we consider that the non-compact set of constraints is closed by the numerical solver to a known tolerance $\epsilon > 0$. As such, the solution $\mathbf{P}$ obtained is arbitrarily close to the respective solutions of $\mathbf{L}_c$ or $\mathbf{L}_o$ in (2.27). In this case, the $\mathcal{H}_2$ norm is given by $\|\mathbf{H}_{wz}(s)\|^2_{\mathcal{L}_2} = \mathrm{tr}\left(\mathbf{H}^T\mathbf{L}_o\mathbf{H}\right) < \mathrm{tr}\left(\mathbf{H}^T\mathbf{P}\mathbf{H}\right)$, or alternatively $\|\mathbf{H}_{wz}(s)\|^2_{\mathcal{L}_2} = \mathrm{tr}\left(\mathbf{E}\mathbf{L}_c\mathbf{E}^T\right) < \mathrm{tr}\left(\mathbf{E}\mathbf{P}\mathbf{E}^T\right)$.

2.3.5 $\mathcal{H}_\infty$ Norm for LTI Systems

The $\mathcal{H}_\infty$ norm characterizes a measure of the greatest possible $\mathcal{L}_2$ gain of the system, which is the ratio between the $\mathcal{L}_2$ norm of the output signal and the $\mathcal{L}_2$ norm of a square integrable input signal, across all input channels, that maximizes this ratio. It is defined for system (2.18), considering external inputs $\mathbf{w} \in \mathcal{L}_2$, and is defined by [40] as

$$\|\mathbf{H}_{wz}(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(\mathbf{H}_{wz}(\mathrm{j}\omega)\right) \tag{2.31}$$

where $\sigma_{\max}(\mathbf{H}_{wz}(\mathrm{j}\omega)) = \sqrt{\lambda_{\max}\left(\mathbf{H}_{wz}(\mathrm{j}\omega)^*\mathbf{H}_{wz}(\mathrm{j}\omega)\right)}$ is the maximum singular value of $\mathbf{H}_{wz}(\mathrm{j}\omega)$, and $\lambda_{\max}(\cdot)$ is the greatest eigenvalue of a matrix. For single-input, single-output systems, the $\mathcal{H}_\infty$ norm becomes simply the peak gain observed for the frequency response of $\mathbf{H}_{wz}(\mathrm{j}\omega)$, $\omega \in \mathbb{R}$.

The $\mathcal{H}_\infty$ norm of system (2.18) can also be defined in the time domain, as demonstrated in [40, 48], by the following

$$\|\mathbf{H}_{wz}(s)\|_\infty = \sup_{0\neq\mathbf{w}\in\mathcal{L}_2} \frac{\|\mathbf{z}\|_{\mathcal{L}_2}}{\|\mathbf{w}\|_{\mathcal{L}_2}} \tag{2.32}$$

Alternatively, (2.32) can be rewritten as $\|\mathbf{H}_{wz}(s)\|_\infty \leq \gamma$ when considering a scalar $\gamma > 0$ such that

$$\int_0^\infty \mathbf{z}(t)^T\mathbf{z}(t)\,\mathrm{d}t \leq \gamma^2 \int_0^\infty \mathbf{w}(t)^T\mathbf{w}(t)\,\mathrm{d}t, \quad \mathbf{w}(t) \neq 0, \quad \mathbf{w}(t) \in \mathcal{L}_2, \quad t \geq 0 \tag{2.33}$$

This definition allows the $\mathcal{H}_\infty$ norm to be expressed completely in terms of the input and output signals in the time domain. When considering performance indices for switched systems, this becomes especially important, since these systems cannot be expressed in terms of transfer functions, as will become clear in the forthcoming chapter.

As for the $\mathcal{H}_2$ norm, the upper bound for $\mathcal{H}_\infty$ norm can also be calculated by means of a convex optimization problem subject to LMI constraints. By considering the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, with $\mathbf{P} > 0$, we have

$$\begin{aligned}\dot{v}(\mathbf{x}) &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + (\mathbf{z}^T\mathbf{z} - \mathbf{z}^T\mathbf{z}) + (\gamma^2\mathbf{w}^T\mathbf{w} - \gamma^2\mathbf{w}^T\mathbf{w}) \\ &= \begin{bmatrix}\mathbf{x}\\ \mathbf{w}\end{bmatrix}^T \begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} & \mathbf{P}\mathbf{H} + \mathbf{E}^T\mathbf{G} \\ \mathbf{H}^T\mathbf{P} + \mathbf{G}^T\mathbf{E} & \mathbf{G}^T\mathbf{G} - \gamma^2\mathbf{I}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{w}\end{bmatrix} - \mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w} \\ &< -\mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w}\end{aligned} \tag{2.34}$$

where the inequality arises by imposing

$$\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} & \bullet \\ \mathbf{H}^T\mathbf{P} + \mathbf{G}^T\mathbf{E} & \mathbf{G}^T\mathbf{G} - \gamma^2\mathbf{I}\end{bmatrix} < 0 \tag{2.35}$$

Notice that a necessary condition for the feasibility of this inequality is that block $(1,1)$ of this inequality be negative definite, or $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{E}^T\mathbf{E} < 0$, thus implying in $\dot{v}(\mathbf{x}) < 0$, reinforcing the fact that the system is asymptotically stable. Finally, by integrating both sides of (2.34) from $t = 0$ to $t \to \infty$, we have

$$\begin{aligned}\int_0^\infty \dot{v}(\mathbf{x})\,\mathrm{d}t &< \int_0^\infty -\mathbf{z}^T\mathbf{z} + \gamma^2\mathbf{w}^T\mathbf{w}\,\mathrm{d}t \\ \lim_{t\to\infty} v(\mathbf{x}(t)) - v(\mathbf{x}(0)) &< -\|\mathbf{z}\|^2_{\mathcal{L}_2} + \gamma^2\|\mathbf{w}\|^2_{\mathcal{L}_2}\end{aligned} \tag{2.36}$$

which becomes (2.32), since $\lim_{t\to\infty} v(\mathbf{x}(t)) = 0$, as the system is globally asymptotically stable, and $v(\mathbf{x}(0)) = 0$, given that $\mathbf{x}(0) = \mathbf{0}$. As such, whenever (2.35) is satisfied, the inequality (2.34) is guaranteed, and by consequence, so is (2.32). By means of a convex optimization problem, described in terms of LMIs, the $\mathcal{H}_\infty$ norm can be calculated with no difficulty, as follows

$$\begin{aligned}\min \quad & \rho \\ \text{subject to:} \quad & \mathbf{P} > 0,\ \rho > 0 \\ & \begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} & \bullet & \bullet \\ \mathbf{H}^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E} & \mathbf{G} & -\mathbf{I}\end{bmatrix} < 0\end{aligned} \tag{2.37}$$

where the last inequality is equivalent to (2.35), made clear by applying Schur complement with respect to $-\mathbf{I}$. The $\mathcal{H}_\infty$ norm is then given by $\|\mathbf{H}_{wz}(s)\|_\infty < \gamma$, with $\rho = \gamma^2$. Again, as in the $\mathcal{H}_2$ case, the solution ρ is arbitrarily close to the analytical solution, only by the specified tolerance $\epsilon > 0$ of the numerical solver.

Chapter 3
Switched Systems

In this chapter, we introduce the concept of switched systems, followed by the study of the stability properties of switched linear and affine systems, as well as relevant performance criteria, generalized from the concepts established in the previous chapter. The topics presented in this chapter review several foundational results already existent in the literature, such as the papers [13, 19] for switched linear systems, and [15, 29] for switched affine systems. The books [11] and [12] are also important to support many of the ideas presented later in this work.

3.1 Introduction

Switched systems constitute a subclass of hybrid systems, in the sense that these systems are governed by a set of modes of operation, each of which may be represented by a dynamical system, and are coupled with discrete switching events across these modes, thus affecting the trajectory of the overall system. The switching between modes is orchestrated by a switching function, also known as a switching rule, denoted by $\sigma(\cdot)$. It encompasses a decision-making process that selects values within a set $\mathbb{K} := \{1, \ldots, N\}$, at every instant of time, such that each $i \in \mathbb{K}$ corresponds to an individual mode of operation, referred to as a subsystem of the switched system. This chapter will first deal with the so-called continuous-time switched linear systems, which concern the case where all subsystems are governed by linear dynamical systems, and subsequently, we discuss the continuous-time switched affine systems, pertaining to the situation where at least one of the subsystems presents a nonzero affine term, contemplating the main focus of this work.

The effect of switching in switched systems is not trivial, since not only does it establish the nonlinear and time-varying nature of these systems, but also, the stability properties of the switched system are inherently dependent on the switching signal. Indeed, it may give rise to complex and unprecedented behaviors, even when simple subsystems are considered. An example of this is the occurrence of sliding modes, in which the switched system switches infinitely fast. This specific situation, although sometimes undesirable, allows for a behavior significantly different than that of each isolated subsystem, and in the case of switched affine systems, it introduces new attainable equilibrium points that are distinct from those of each subsystem, a topic that will be discussed in greater detail shortly. It is important to note that although the switching function plays an important part in the trajectory of the switched system, the continuous state evolves without discontinuities, that is, the state does not jump impulsively on switching events.

To illustrate how the stability properties of a switched system are intertwined with the switching signal, consider a switched system $\dot{\mathbf{x}} = \mathbf{A}_\sigma \mathbf{x}$, composed of two stable linear subsystems. When subject to a specifically crafted switching signal $\sigma(t)$, the switched system may prove to be unstable, as exemplified in Figure 3.1. This is despite the fact that these two subsystems individually display a monotonically decreasing Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T \mathbf{P}_\sigma \mathbf{x}$, for $\sigma = i$, and for all $t \geq 0$, with $i = \{1, 2\}$. In the figure it can be observed that although $v(\mathbf{x}(t))$ decreases in between switching events, an upward trend for $v(\mathbf{x}(t))$ can be seen, indicating that the switching signal under consideration destabilizes the switched system. Indeed notice that the state trajectories do not converge to the origin. Fortunately, a suitable switching rule can also be used to stabilize the switched system.

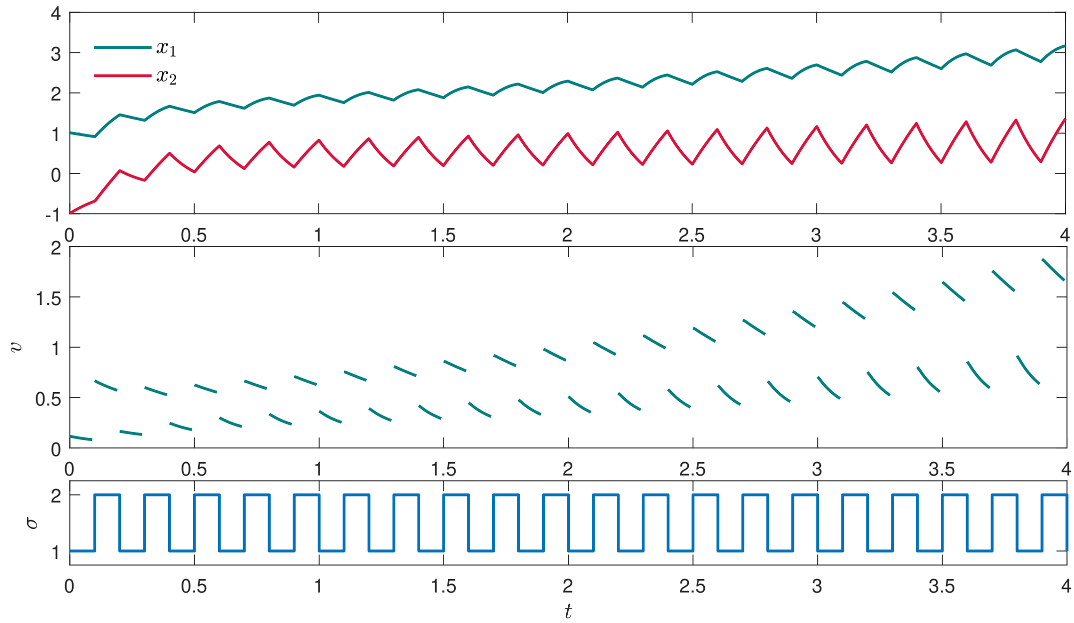


Figure 3.1: Lyapunov function for a destabilizing switching signal.

Given its crucial role on the behavior of switched systems, it is important to characterize the switching function $\sigma(\cdot)$. This function may be either an arbitrary time-dependent function, or a control variable to be designed. In the first case, the central problem is determining conditions to assure stability for some unknown switching signal $\sigma(t)\colon \mathbb{R}_+ \to \mathbb{K}$, such as a disturbance, an assigned external input, or a signal which may model the effects of a component failure. On the other hand, the second case concerns the design of a switching function $\sigma(\cdot) \in \mathbb{K}$, which can be state or output dependent, in order to guarantee stability of the switched system. The survey [13] reviews stability conditions for a variety of switched systems that have been introduced in the literature throughout the past decades.

The design of a switching function $\sigma(\cdot)$ as a control variable attracts much interest, as an appropriate choice for this function may assure stability even in the case where all subsystems are unstable. Furthermore, whenever all subsystems are stable, it may improve the performance of the overall system when compared to that of each isolated subsystem, in this case, the switching function is said to be strictly consistent [25]. This scenario also has a wide scope of applications, such as the automatic transmission of an automobile, the problem of temperature regulation by a thermostat, and several applications on power electronics systems. Given that such systems are intrinsically switched, it makes sense to model these in a switched system framework. Several results in the literature deal with the design of a stabilizing switching rule with applications on switched mode

DC-DC power converters, more specifically the *buck-boost*, see [14, 17], and the *flyback* converters, in [18].

It is also relevant to consider performance metrics when designing switching functions for switched systems. In this work, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices will be presented, as introduced in [23, 24, 25] for the linear case, and in [15, 29] for the affine case. These indices generalize the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms introduced in Chapter 2, as they cannot be directly employed since they have been defined in terms of the transfer function of an LTI system. It should be noted that even though each isolated subsystem may possess a frequency domain representation, the switched system does not, due to its nonlinear and time-varying characteristics that stem from the influence of the switching function. It will become evident, however, that whenever the switching function remains fixed at a certain subsystem $\sigma(t) = i$, $\forall t \geq 0$, these indices are equivalent to the square of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms for the i-th subsystem. These performance criteria will be essential when we treat the problems of output feedback control and filter design in the coming chapters, as we seek to develop techniques that minimize these indices.

3.2 Stability of Switched Linear Systems

Consider the state space representation of an unforced continuous-time switched linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_\sigma \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, and $\sigma(\cdot) \in \mathbb{K}$, for $\mathbb{K} := \{1, \ldots, N\}$, is the switching rule, a piecewise continuous function, which selects one of the N available subsystems as active, at each instant of time. Notice that the origin is the single equilibrium point of the system.

In this case, the problem consists in determining an appropriate switching rule $\sigma(\cdot)$, capable of stabilizing the overall switched system, and making the origin $\mathbf{x}_e = \mathbf{0}$ a globally asymptotically stable equilibrium point. The work of [19] introduces some circumstances which must be satisfied, so that a stabilizing switching rule is guaranteed to exist. These conditions are derived under Lyapunov's direct method, introduced in Section 2.2, by adopting the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, with $\mathbf{P} \succ 0$. Furthermore, these conditions will be expressed in terms of LMIs.

In the context of switched linear systems, several well established results for stabilizing switching rules exist, some of which are based on different Lyapunov functions. This work, however, will mostly deal with the quadratic Lyapunov function, associated with the min-type switching function, for reasons which will be discussed shortly.

3.2.1 Switching Rules for Switched Linear Systems

Over the past decades, several results in the literature have introduced different switching rules for switched linear systems, along with their respective conditions for stability, and with varying degrees of conservativeness.

Switching rules of the form $\sigma(\mathbf{x}(t))$, $\sigma(\mathbf{y}(t))$, and $\sigma(\mathbf{x}(t),\mathbf{w}(t))$ have been proposed, depending whether these measurements are accessible in order to successfully implement the rule. At the moment, we will turn our focus to switching rules dependent on the system state, $\sigma(\mathbf{x}(t))$, and considering the quadratic Lyapunov function, as in [15, 19, 20, 21, 22]. Some other results available in the literature will be shown in brief towards the conclusion of this section.

3.2.1.1 Quadratic Lyapunov Function

The min-type switching rule $\sigma(\mathbf{x}(t))\colon \mathbb{R}^{n_x} \to \mathbb{K}$, introduced by [15, 19, 20, 22], is defined as follows for the switched linear system (3.1)

$$\sigma(\mathbf{x}) = \arg \min_{i\in\mathbb{K}} \mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x} \tag{3.2}$$

where $\mathbf{P} \in \mathbb{R}^{n_x\times n_x}$ is a symmetric positive definite matrix. This switching rule guarantees global asymptotic stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ for the system (3.1), considering the quadratic Lyapunov function $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, whenever there exists a vector $\lambda_0 \in \Lambda_N$, such that $\mathbf{A}_{\lambda_0}$ is Hurwitz. It is interesting to observe that this switching rule is equivalent to

$$\sigma(\mathbf{x}) = \arg \min_{i\in\mathbb{K}} \dot{v}_i(\mathbf{x}) \tag{3.3}$$

Indeed, if we recognize that $\dot{v}_i(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x}$ for the i-th subsystem, then

$$\begin{aligned}\sigma(\mathbf{x}) &= \arg \min_{i\in\mathbb{K}} \ \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} \\ &= \arg \min_{i\in\mathbb{K}} \ 2\,\mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x}\end{aligned} \tag{3.4}$$

which is equivalent to rule (3.2) as stated.

The following theorem, available in [15], gives the conditions under which the min-type switching rule stabilizes the switched linear system (3.1).

Theorem 3.1. *Consider the switched linear system* (3.1) *and a vector* $\lambda_0 \in \Lambda_N$. *If there exists a matrix* $\mathbf{P} > 0$, *such that*

$$\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0 \tag{3.5}$$

then the following switching rule

$$\sigma(\mathbf{x}) = \arg \min_{i\in\mathbb{K}} \mathbf{x}^T\mathbf{P}\mathbf{A}_i\mathbf{x} \tag{3.6}$$

makes the equilibrium point $\mathbf{x}_e = \mathbf{0}$ *globally asymptotically stable.*

Proof. The proof is available on [15], but will be reproduced below for convenience. The stabilizing nature of the min-type switching rule for the system (3.1) is demonstrated via Lyapunov's direct method, by verifying that it indeed ensures that the time derivative of the quadratic Lyapunov function under consideration is strictly negative for any trajectory $\mathbf{x} \neq \mathbf{0}$, as follows

$$\begin{aligned}
\dot{v}(\mathbf{x}) &= \mathbf{x}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\mathbf{x} \\
&= \min_{i\in\mathbb{K}} \mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} && (3.7)\\
&= \min_{\lambda\in\Lambda_N} \mathbf{x}^T\left(\mathbf{A}_\lambda^T\mathbf{P} + \mathbf{P}\mathbf{A}_\lambda\right)\mathbf{x} && (3.8)\\
&\leq \mathbf{x}^T\left(\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0}\right)\mathbf{x} && (3.9)\\
&< 0 && (3.10)
\end{aligned}$$

where equality (3.7) comes from applying the switching rule (3.6); equality (3.8) comes from the fact that the minimum of an objective function that is linear in λ always occurs at one of the vertices of the convex polytope defined by $\lambda \in \Lambda_N$, and thus, selecting the i-th subsystem is equivalent to setting the i-th element of λ to 1, and the remainder to 0; inequality (3.10) stems from the fact that since there exists a vector λ_0, such that a convex combination of the subsystems is Hurwitz, then at any given time, the minimum of the objective function in λ will be always less than or equal to that of the convex combination. Finally, $\dot{v}(\mathbf{x}) < 0$ follows from the fact that $\mathbf{A}_{\lambda_0}$ is Hurwitz, and thus, $\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0$. □

The following simple example, based on [19], provides a valuable discussion on the operation of the min-type switching rule.

Example 3.1

Consider the switched linear system (3.1), comprised of two unstable subsystems, with matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. Also, consider a symmetric positive definite matrix $\mathbf{P} \succ 0$, and suppose that a given $\lambda_0 = [\mu \;\; 1-\mu]^T$, with $\mu \in (0,1)$ exists, such that $\mathbf{A}_{\lambda_0} = \mu\mathbf{A}_1 + (1-\mu)\mathbf{A}_2$ is Hurwitz. Let symmetric matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ defined as

$$\mathbf{Q}_1 = \mathbf{A}_1^T\mathbf{P} + \mathbf{P}\mathbf{A}_1 \qquad \text{and} \qquad \mathbf{Q}_2 = \mathbf{A}_2^T\mathbf{P} + \mathbf{P}\mathbf{A}_2$$

Notice that, since $\mathbf{A}_{\lambda_0} \in \mathcal{H}$, and undertaking the stability analysis by Lyapunov's direct method under the quadratic Lyapunov function, the following inequality

$$\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0$$

is verified, thus implying that $\mathbf{Q}_{\lambda_0} \prec 0$. Indeed, for all $\mathbf{x} \in \mathbb{R}^{n_x}$, $\mathbf{x} \neq \mathbf{0}$, we have

$$\mathbf{x}^T\left(\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0}\right)\mathbf{x} = \mathbf{x}^T\mathbf{Q}_{\lambda_0}\mathbf{x} < 0$$

Expanding this convex combination in λ_0, we have

$$\mu\left(\mathbf{x}^T\mathbf{Q}_1\mathbf{x}\right) + (1-\mu)\left(\mathbf{x}^T\mathbf{Q}_2\mathbf{x}\right) < 0$$

This suggests that either

$$\mathbf{x}^T\mathbf{Q}_1\mathbf{x} < 0 \qquad \text{and/or} \qquad \mathbf{x}^T\mathbf{Q}_2\mathbf{x} < 0$$

However, by our initial hypothesis that both subsystems are not Hurwitz, there does not exist a matrix $\mathbf{P}_1 > 0$ such that $\mathbf{A}_1^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A}_1 < 0$, nor does exist $\mathbf{P}_2 > 0$ satisfying $\mathbf{A}_2^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{A}_2 < 0$. This, coupled with the fact that $\mathbf{Q}_{\lambda_0} < 0$, implies that indeed matrices $\mathbf{Q}_i$, $i \in \{1,2\}$ are sign indefinite, and thus, the sign of $\mathbf{x}^T\mathbf{Q}_i\mathbf{x}$ are dependent of the value of $\mathbf{x}$. As such, it can be inferred that while neither subsystem is stable for all $\mathbf{x}(t)$, as the state trajectory progresses in time, $\mathbf{x}^T\mathbf{Q}_1\mathbf{x}$ and $\mathbf{x}^T\mathbf{Q}_2\mathbf{x}$ take turns in becoming strictly negative, as a consequence of $\mathbf{Q}_i$ being sign indefinite, thus always guaranteeing $\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} < 0$ at any given moment in time.

Notice that $\mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x} = \mathbf{x}^T\mathbf{Q}_i\mathbf{x}$ is precisely the time derivative of the quadratic Lyapunov function for the i-th subsystem. The min-type switching rule, which works by selecting the subsystem whose value of $\mathbf{x}^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\mathbf{x}$ is minimum, thus guarantees that the time derivative of the Lyapunov function is strictly negative for all instants of time. ■

It is important to note that no stability property is imposed on the individual subsystems, however a recurrent condition in the literature, applicable to the results in this chapter, and employed by this example, is that the convex combination of matrices $\mathbf{A}_{\lambda_0}$, for $\lambda_0 \in \Lambda_N$, be Hurwitz, or equivalently $\mathbf{A}_{\lambda_0} \in \mathcal{H}$. This requirement is sufficient to assure that a stabilizing switching rule exists, as discussed in [19, 20].

The following numerical example, where switching across two unstable subsystems is considered, illustrates this case, for the min-type switching rule.

Example 3.2

Consider the switched linear system (3.1) consisting of the following two unstable subsystems

$$\mathbf{A}_1 = \begin{bmatrix} -10 & 3 \\ 8 & -1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$$

Notice that the equilibrium points of both these subsystems are unstable saddle points, as illustrated by the phase portrait in Figure 3.2. In this example, for a value of $\lambda_0 = [0.5 \;\; 0.5]^T$, a Hurwitz convex combination $\mathbf{A}_{\lambda_0}$ is verified.

Figure 3.2 displays the phase portraits of each isolated subsystem, considering trajectories evolving from initial conditions $\mathbf{x}_0$ around a unit circle, indicated by the line '—' centered at the origin '+', that is, $\mathbf{x}_0 = [\cos(\theta) \;\; \sin(\theta)]^T$, $\theta \in [0, 2\pi]$. In order to implement the switching rule (3.6) of Theorem 3.1, first, a matrix $\mathbf{P} > 0$ is calculated satisfying condition (3.5) of Theorem 3.1. For the value of λ_0 given, we have considered the following

$$\mathbf{P} = \begin{bmatrix} 30.2067 & 62.5002 \\ 62.5002 & 135.1940 \end{bmatrix}$$

It can be verified that the switching rule is able to successfully stabilize the switched system, by inspecting

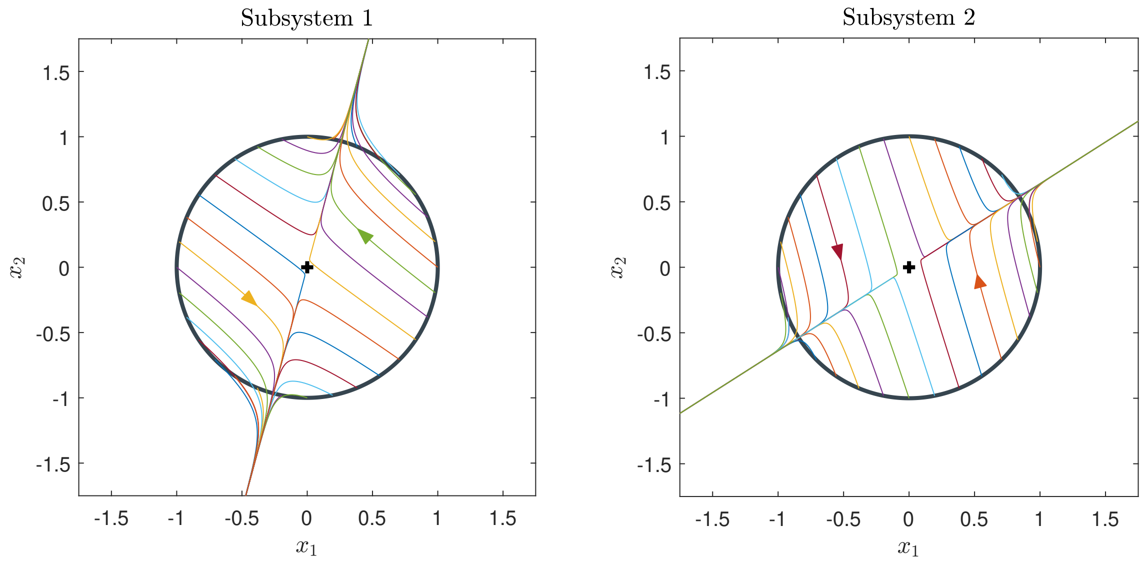


Figure 3.2: Phase portrait for each unforced linear subsystem.

the state trajectories and phase portrait[1] in Figures 3.3 and 3.4, respectively, which asymptotically converge to the origin, as desired.

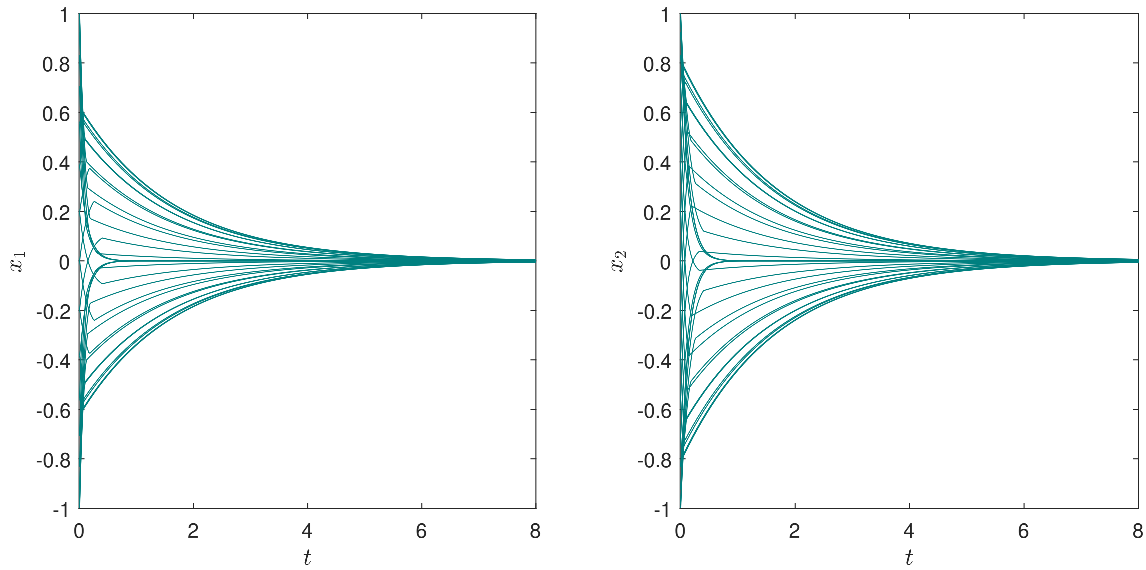


Figure 3.3: Trajectories of each state for the switched linear system under Theorem 3.1.

Also, in Figure 3.4 the boundaries in which switching events occur, referred to as a switching surface, are indicated by the line '—'. This situation arises whenever $\mathbf{x}^T\mathbf{P}\mathbf{A}_1\mathbf{x} = \mathbf{x}^T\mathbf{P}\mathbf{A}_2\mathbf{x}$ which, in this case, produces two straight lines intersecting at the origin. Notice that when at the switching surface, the switching rule may either select another subsystem as active, transitioning from one dynamical behavior to another, or it may also cause the occurrence of sliding modes. In this example, this phenomenon is distinctly visible in Figure 3.4, where the two subsystems switch at an arbitrarily high frequency, resulting in a behavior distinct from that of each

[1]Numerical simulations of switched systems were carried out using the SWSYSToolbox for MATLAB, developed by the author. The toolbox is available at `https://github.com/gkolotelo/SWSYSToolbox`, along with the detailed documentation.

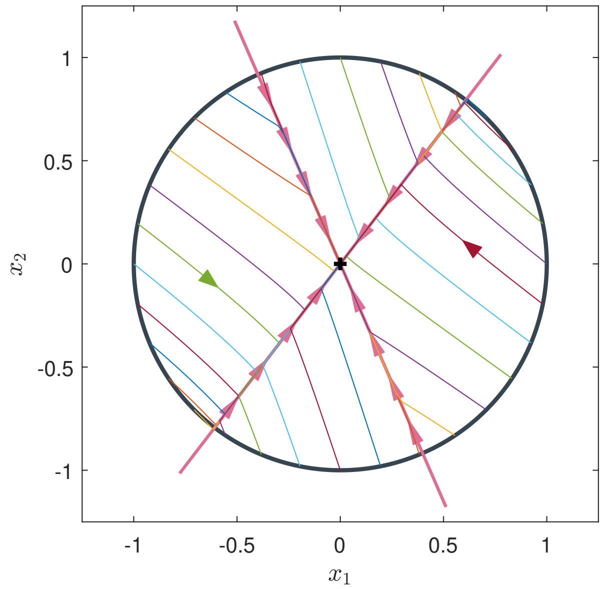


Figure 3.4: Phase portrait for the switched linear system under Theorem 3.1.

isolated subsystem, and cause the state trajectory to evolve along the switching surface towards the equilibrium point, as indicated by the arrows '►' along this surface. ■

3.2.1.2 Min-Type Lyapunov Function

As alluded to earlier, less conservative results when compared to those based on quadratic Lyapunov functions have been obtained, such as those derived from multiple Lyapunov functions, introduced by [2, 49], and from the min-type piecewise quadratic Lyapunov function, as shown in [50, 51]. The latter two adopt the following Lyapunov function

$$v(\mathbf{x}) = \min_{i \in \mathbb{K}} \mathbf{x}^T \mathbf{P}_i \mathbf{x} \tag{3.11}$$

with matrices $\mathbf{P}_i \succ 0$, $\forall i \in \mathbb{K}$, being the solutions of the well-known Lyapunov-Metzler inequalities

$$\mathbf{A}_i^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_i + \sum_{j=1}^{N} \pi_{ji} \mathbf{P}_j + \mathbf{E}_i^T \mathbf{E}_i \prec 0,\ i \in \mathbb{K} \tag{3.12}$$

where $\mathbf{\Pi} = \left\{\pi_{ji}\right\}$ is a subclass of Metzler matrices satisfying the additional property

$$\sum_{j=1}^{N} \pi_{ji} = 0,\ i \in \{1, \ldots, N\} \tag{3.13}$$

Under these conditions, the following switching rule

$$\sigma(\mathbf{x}) = \arg\ \min_{i \in \mathbb{K}} \mathbf{x}^T \mathbf{P}_i \mathbf{x} \tag{3.14}$$

guarantees global asymptotic stability of the origin. An important remark, as discussed in [52], is that the inequalities (3.12) are less conservative than assuring the existence of a Hurwitz convex combination of subsystem matrices, as required in Theorem 3.1. Unfortunately, the generalization of the Lyapunov-Metzler inequalities to cope with switched affine systems is not trivial due to the difficulty of dealing with the affine terms and the different equilibrium points which are introduced. As such, this work pays special attention to conditions based on a quadratic Lyapunov function. In addition, the inequalities in (3.12) are non-convex due to the product of matrix variables, and may be difficult to solve for an arbitrary number of subsystems. Alternative conditions, easier to solve but more conservative, are available in [50].

3.3 Stability of Switched Affine Systems

In this section, we introduce the fundamental concepts of switched affine systems that will be extensively used throughout the remainder of this work. Furthermore, we engage in discussions about the unique characteristics of these types of systems, and how these features are useful for modeling many practical applications.

Consider the following state space representation of an unforced continuous-time switched affine system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_\sigma \mathbf{x}(t) + \mathbf{b}_\sigma, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{z}(t) &= \mathbf{E}_\sigma \mathbf{x}(t) \end{aligned} \tag{3.15}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{b}_i \in \mathbb{R}^{n_x}$ are the affine terms, $\sigma(\cdot) \in \mathbb{K}$ is the switching rule, and $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance output, which will allow for the definition of performance indices for these systems later on. Notice that whenever $\mathbf{b}_i = \mathbf{0}$ for all $i \in \mathbb{K}$, the system (3.15) reduces to a continuous-time switched linear system, whose sole equilibrium point is the origin. However, for the more general case of $\mathbf{b}_i \neq \mathbf{0}$, for at least one $i \in \mathbb{K}$, system (3.15) may exhibit several distinct equilibrium points, constituting a subset of the state space. This imposes greater difficulty in the study of the stability properties of these systems, as will soon become evident.

Definition 1. *The set of all equilibrium points of the system* (3.15) *is given by*

$$\mathbf{X}_e = \left\{ \mathbf{x}_e \in \mathbb{R}^{n_x} \; : \; \mathbf{x}_e = -\mathbf{A}_\lambda^{-1} \mathbf{b}_\lambda, \quad \lambda \in \Lambda_N \right\} \tag{3.16}$$

Notice that this definition requires that $\mathbf{A}_\lambda$ be nonsingular and provides the unique equilibrium point $\mathbf{x}_e = -\mathbf{A}_\lambda^{-1}\mathbf{b}_\lambda$. However, for the case where $\mathbf{A}_\lambda$ is singular, any choice of equilibrium points $\mathbf{x}_e$ satisfying $\mathbf{A}_\lambda \mathbf{x}_e + \mathbf{b}_\lambda = \mathbf{0}$ is possible. With no loss of generality, system (3.15) can be shifted so as to move the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ to the origin by defining the new state vector $\boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{x}_e$, resulting in the equivalent system

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \boldsymbol{\xi}(t) \end{aligned} \tag{3.17}$$

where $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$, for all $i \in \mathbb{K}$, are the new affine terms, and $\mathbf{z}_e(t) = \mathbf{z}(t) - \mathbf{E}_\sigma \mathbf{x}_e$ is the shifted output. It should be noted that for global asymptotic stability, we have $\boldsymbol{\xi}(t) \to 0$ as $t \to \infty$, and in this condition, $\mathbf{x}(t) \to \mathbf{x}_e$ for system (3.15). Furthermore, observe that whenever $\mathbf{x}_e \in \mathbf{X}_e$, with an associated $\lambda_0 \in \Lambda_N$, we have $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. This choice of state variables will simplify our further developments.

3.3.1 Switching Rules for Switched Affine Systems

On the realm of switched affine systems, fewer switching rules have been introduced, when compared to switched linear systems. The authors [26, 27, 28] present state dependent switching rules, whereas the references [16, 30] introduce an output dependent switching function. In this section we first present the min-type switching rule, originally devised in [26], along with conditions for global asymptotic stability of $\mathbf{x}_e \in \mathbf{X}_e$. This is followed by the introduction of the concept of guaranteed cost for switched systems, which will be important for the definition of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices towards the end of this chapter.

3.3.1.1 Min-Type Switching Rule

The following theorem introduces the consolidated results of [15, 26], which present conditions that assure global asymptotic stability of a desired equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$.

Theorem 3.2. *Consider the switched affine system (3.17), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exists a matrix $\mathbf{P} \succ 0$, such that*

$$\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} \prec 0 \tag{3.18}$$

then the following switching rule

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} \ \xi^T\mathbf{P}\mathbf{A}_i\xi + \xi^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.19}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable.

Proof. The proof, available in [26], is presented below for convenience, and explained throughout its unraveling. When adopting the switching strategy (3.19), and realizing that it is equivalent to

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} \ \xi^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.20}$$

then the time derivative of the quadratic Lyapunov function $\dot{v}(\xi(t))$, for any state trajectory $\boldsymbol{\xi} \neq \mathbf{0}$, is given by

$$\begin{aligned}
\dot{v}(\xi) &= \dot{\xi}^T\mathbf{P}\xi + \xi^T\mathbf{P}\dot{\xi} \\
&= \xi^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma \\
&= \min_{i\in\mathbb{K}} \ \xi^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i\right)\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i \\
&= \min_{\lambda\in\Lambda_N} \ \xi^T\left(\mathbf{A}_\lambda^T\mathbf{P} + \mathbf{P}\mathbf{A}_\lambda\right)\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\lambda \\
&\leq \xi^T\left(\mathbf{A}_{\lambda_0}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0}\right)\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_{\lambda_0} \\
&< 0
\end{aligned} \tag{3.21}$$

which unfolds in a similar manner to the proof of the min-type switching rule for switched linear systems, presented in Section 3.2.1.1, when recalling the fact that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$, and that $\mathbf{A}_{\lambda_0}$ is Hurwitz, as ensured by condition (3.18), thus making inequality (3.21) valid. □

Notice that this theorem encompasses Theorem 3.1, since in the event that $\mathbf{b}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, they become equivalent. It is also important to observe that not all equilibrium points $\mathbf{x}_e \in \mathbf{X}_e$ are attainable, but in fact only those for which the vectors $\lambda \in \Lambda_N$ satisfy $\mathbf{A}_\lambda \in \mathcal{H}$. Thus, in the uncommon case where there exists no stable convex combination of dynamical matrices, the system is not stabilizable under Theorem 3.2.

Overall, this result is very attractive for a range of real-life applications, since it allows the switched system to operate at a chosen equilibrium point of interest, different than those of the individual subsystems. The works of [14, 15, 17, 18] apply this particular characteristic of switched affine systems to different topologies of DC-DC power converters with great success. A numerical example is presented below to demonstrate the validity of Theorem 3.2 with respect to the min-type switching rule for switched affine systems.

3.3.1.2 Example

The following numerical example illustrates the peculiarities of switched affine systems and how the switching function plays an important role in guaranteeing asymptotic stability for this class of switched systems.

Example 3.3

Consider the switched affine system (3.15) comprised of a stable and an unstable subsystem, as follows

$$\mathbf{A}_1 = \begin{bmatrix} 8 & 0 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} -2 & -9 \\ 5 & -4 \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

whose respective equilibrium points are

$$\mathbf{x}_{e_1} = \begin{bmatrix} -0.625 \\ -3.1875 \end{bmatrix}, \qquad \mathbf{x}_{e_2} = \begin{bmatrix} 1.0377 \\ 0.5472 \end{bmatrix}$$

For the equilibrium point of interest $\mathbf{x}_e = [0.9158 \;\; 0.8160]^T \in \mathbf{X}_e$, with its associated vector $\lambda_0 = [0.15 \;\; 0.85]^T$, a Hurwitz convex combination $\mathbf{A}_{\lambda_0} \in \mathcal{H}$ is verified. In Figure 3.5, we can observe the dynamical behavior of each isolated subsystem in the $\xi = \mathbf{x} - \mathbf{x}_e$ phase plane, where '✖' denotes the origin of the shifted system $\xi = \mathbf{0}$. For subsystem 1, we have considered initial conditions $\xi_0 = \mathbf{x}_0 + \mathbf{x}_{e_1} - \mathbf{x}_e$ describing points over the line '▬' from $\mathbf{x}_0 = [2 \;\; -2]^T$ to $\mathbf{x}_0 = [-2 \;\; 2]^T$. For subsystem 2 we have considered initial conditions $\xi_0 = \mathbf{x}_0 - \mathbf{x}_e$ with $\mathbf{x}_0 = 3 \times [\cos(\theta) \;\; \sin(\theta)]^T$, $\theta \in [0, 2\pi]$, corresponding to points distributed around the circle '▬'. Notice that the equilibrium point $\mathbf{x}_{e_1}$, marked by '◇', is an unstable node, whereas $\mathbf{x}_{e_2}$, indicated by '○', is a stable focus, also, notice how $\mathbf{x}_e$ differs from the equilibrium points of the isolated subsystems. These different behaviors across subsystems bring about complex dynamical behaviors for the overall switched system, as will become clear.

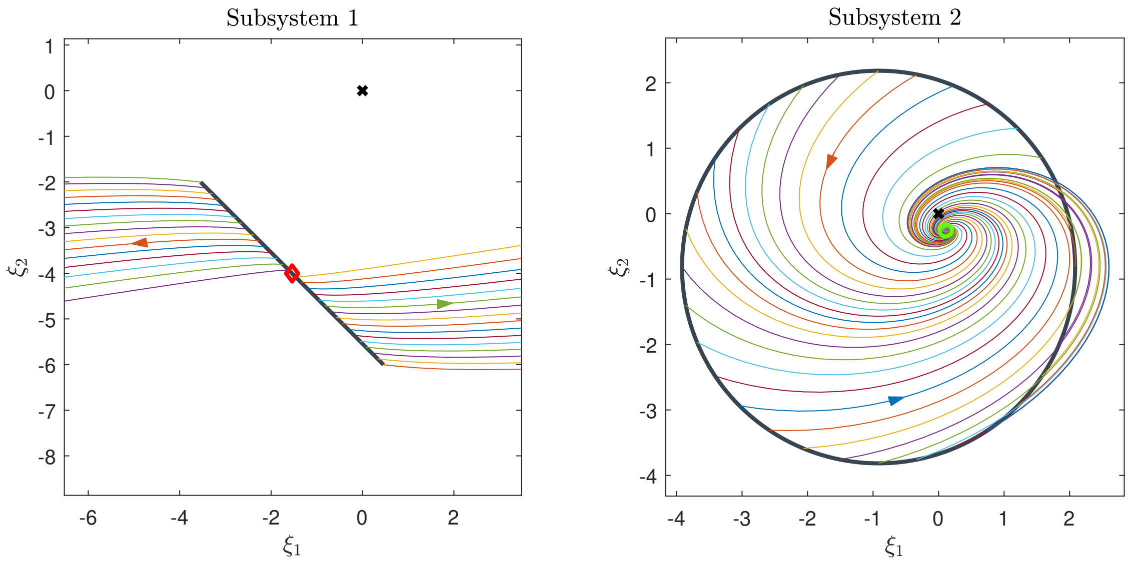


Figure 3.5: Phase portrait for each unforced affine subsystem.

Before proceeding, the vectors $\boldsymbol{\ell}_i = \mathbf{A}_i\mathbf{x}_e + \mathbf{b}_i$, for $i \in \{1,2\}$ are calculated. We have also considered the matrix $\mathbf{P} > 0$ as follows

$$\mathbf{P} = \begin{bmatrix} 0.2491 & -0.0650 \\ -0.0650 & 0.4107 \end{bmatrix}$$

which satisfies the LMI constraint (3.18) of Theorem 3.2. Implementing the switching rule (3.19), and allowing initial conditions distributed around the circle indicated by '—' with $\boldsymbol{\xi}_0 = 2 \times [\cos(\theta) \;\; \sin(\theta)]^T + \boldsymbol{\xi}_\bullet$, $\theta \in [0, 2\pi]$, where $\boldsymbol{\xi}_\bullet = [-0.6079 \;\; -1.0480]^T$, the switched system is successfully stabilized, as observed in the phase portrait shown in Figure 3.6. Notice that $\boldsymbol{\xi}_\bullet$ is the center of the switching surface, indicated in Figure 3.6 by the line '—', taking place when $\boldsymbol{\xi}^T\mathbf{P}\mathbf{A}_1\xi + \boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_1 = \boldsymbol{\xi}^T\mathbf{P}\mathbf{A}_2\xi + \boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_2$, and forming, in this case, an ellipse.

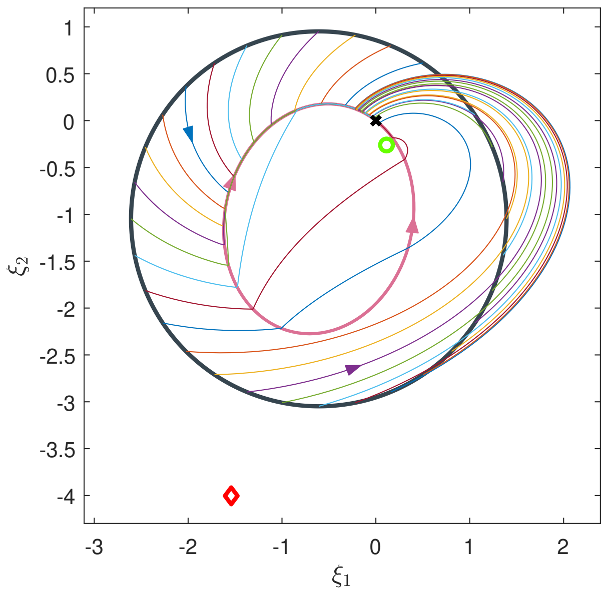


Figure 3.6: Phase portrait for the switched affine system under Theorem 3.2.

Notice that when the trajectory is outside this ellipse, it assumes the behavior of subsystem 2, conversely, when inside the ellipse, the trajectory follows the dynamical behavior of subsystem 1. It is interesting to observe that when at the switching surface, the trajectory may exhibit a particular behavior, characteristic of switched systems, known as sliding mode. Observe in Figure 3.6 that when evolving in sliding modes, the trajectories converge to the equilibrium point '✖', as indicated by the arrows '►' along the switching surface. This phenomenon, resulting from an arbitrarily fast switching between subsystems, or *chattering*, is sometimes an undesirable condition in real-life systems, given the increased equipment wear that may result. However, it may also be a sought after situation, when it allows for the stability of the overall system. This is the case of switched affine systems, where sliding modes are a crucial aspect, making it possible for an equilibrium point different than that of each isolated subsystem to be attained, as evidenced by this example.

Figure 3.7 reveals the states of the switched affine system asymptotically reaching the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$, or equivalently, $\mathbf{x} = \mathbf{x}_e$, as time progresses, for all the initial conditions considered.

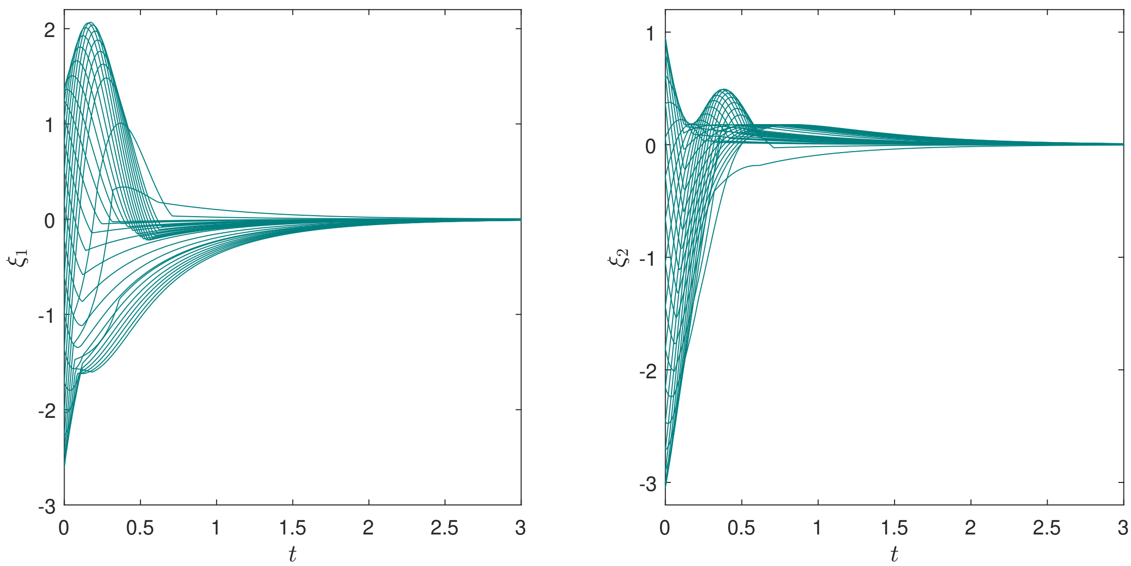


Figure 3.7: Trajectories of each state for the switched affine system under Theorem 3.2.

This example aimed to demonstrate some of the unique and complex behaviors displayed by switched affine systems and to motivate our further discussions and interest in this class of systems.

■

3.3.1.3 Min-Type Switching Rule and Guaranteed Cost

A guaranteed cost is an upper bound for a certain performance criteria of a dynamical system. In this work, we consider as cost function the $\mathcal{L}_2$ norm of the performance output $\|\mathbf{z}_e\|_{\mathcal{L}_2}$ for the switched affine system (3.17). This section will demonstrate how the quadratic Lyapunov function can be used to ensure this upper bound for this class of switched systems, as shown in references [14, 15]. To this end, the following theorem is borrowed from the reference [15].

Theorem 3.3. *Consider the switched affine system (3.17), and a chosen* $\mathbf{x}_e \in \mathbf{X}_e$ *of interest with its associated* $\boldsymbol{\lambda}_0 \in \Lambda_N$. *If there exist a matrix* $\mathbf{P} > 0$, *and symmetric matrices* $\mathbf{Q}_i$, *such that*

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{Q}_i < 0, \quad \forall i \in \mathbb{K} \tag{3.22}$$

$$\mathbf{Q}_{\lambda_0} \geq 0 \tag{3.23}$$

then the following switching rule

$$\sigma(\boldsymbol{\xi}) = \arg\min_{i \in \mathbb{K}} -\boldsymbol{\xi}^T\mathbf{Q}_i\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.24}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ *globally asymptotically stable, and the guaranteed cost*

$$\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 < \boldsymbol{\xi}_0^T\mathbf{P}\boldsymbol{\xi}_0 \tag{3.25}$$

holds.

Proof. The proof is available in [53], and follows from the definition of the guaranteed cost in [15]. Considering the switching strategy (3.24), the time derivative of the quadratic Lyapunov function is given by

$$\begin{aligned}
\dot{v}(\boldsymbol{\xi}) &= \dot{\boldsymbol{\xi}}^T\mathbf{P}\boldsymbol{\xi} + \boldsymbol{\xi}^T\mathbf{P}\dot{\boldsymbol{\xi}} \\
&= \boldsymbol{\xi}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma + \left(\mathbf{z}_e^T\mathbf{z}_e - \mathbf{z}_e^T\mathbf{z}_e\right) \\
&= \boldsymbol{\xi}^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T\mathbf{E}_\sigma\right)\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e \\
&< -\boldsymbol{\xi}^T\mathbf{Q}_\sigma\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e \qquad (3.26)\\
&= \min_{i \in \mathbb{K}} -\boldsymbol{\xi}^T\mathbf{Q}_i\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_i - \mathbf{z}_e^T\mathbf{z}_e \\
&= \min_{\lambda \in \Lambda_N} -\boldsymbol{\xi}^T\mathbf{Q}_\lambda\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\boldsymbol{\ell}_\lambda - \mathbf{z}_e^T\mathbf{z}_e \\
&\leq -\boldsymbol{\xi}^T\mathbf{Q}_{\lambda_0}\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\mathbf{P}\,\boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T\mathbf{z}_e \\
&\leq -\mathbf{z}_e^T\mathbf{z}_e \qquad (3.27)
\end{aligned}$$

for any state trajectory $\boldsymbol{\xi} \neq \mathbf{0}$, where inequality (3.26) comes from the conditions (3.22) of Theorem 3.3, and inequality (3.27) stems from the fact that $\mathbf{Q}_{\lambda_0} \geq 0$ and $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. Thus, global asymptotic stability is guaranteed for the equilibrium point $\mathbf{x}_e$, since $\mathbf{z}_e^T\mathbf{z}_e \geq 0$. Finally, by integrating both sides of (3.27) we have

$$\begin{aligned}
\int_0^\infty \dot{v}(\boldsymbol{\xi})\,\mathrm{d}t &< -\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t \\
\lim_{t\to\infty} v(\boldsymbol{\xi}(t)) - v(\boldsymbol{\xi}(0)) &< -\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t \\
\int_0^\infty \mathbf{z}_e(t)^T\mathbf{z}_e(t)\,\mathrm{d}t &< v(\boldsymbol{\xi}_0) \\
\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 &< \boldsymbol{\xi}_0^T\mathbf{P}\boldsymbol{\xi}_0
\end{aligned} \tag{3.28}$$

where $\boldsymbol{\xi}_0 = \boldsymbol{\xi}(0)$, and $\lim_{t\to\infty} v(\boldsymbol{\xi}(t)) = 0$, since the system is asymptotically stable. This assures the guaranteed cost for the system $\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 < \xi_0^T \mathbf{P} \xi_0$ whenever $\mathbf{z}_e(t)$ is square-integrable. □

It is worth noticing that (3.22) together with (3.23) are equivalent to

$$\mathbf{A}_{\lambda_0}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{\lambda_0} + \sum_{i=0}^{N} \lambda_i \mathbf{E}_i^T \mathbf{E}_i \prec 0 \tag{3.29}$$

as shown in [52], which indicates that the matrices $\mathbf{E}_i$ directly influence matrix $\mathbf{P}$, and consequently, the guaranteed cost for the system, as is to be expected. Furthermore, the above inequality requires that $\mathbf{A}_{\lambda_0}$ be Hurwitz. This, however, is not an onerous imposition, since matrices $\mathbf{Q}_i$, $\forall i \in \mathbb{K}$, are sign indefinite, and thus no stability property is required from the subsystem matrices $\mathbf{A}_i$ themselves, for all $i \in \mathbb{K}$.

To implement Theorem 3.3, matrices $\mathbf{P}$ and $\mathbf{Q}_i$, for $i \in \mathbb{K}$, important for the switching rule (3.24), can be calculated numerically by solving the following convex optimization problem, subject to the LMI constraints (3.22) and (3.23) of Theorem 3.3, that is

$$\begin{aligned} \min \quad & \xi_0^T \mathbf{P} \xi_0 \\ \text{subject to:} \quad & \mathbf{P} \succ 0 \\ & \mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i + \mathbf{Q}_i \prec 0, \quad \forall i \in \mathbb{K} \\ & \mathbf{Q}_{\lambda_0} \succeq 0 \end{aligned} \tag{3.30}$$

This result is key for the definition of an upper bound for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, which will be introduced in the next section.

The following corollary proposes a min-type linear switching rule, possible in the event that all subsystems are individually stable, a situation that often occurs when considering practical applications.

Corollary 3.1. *Consider the switched affine system (3.17), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest. If there exists a matrix $\mathbf{P} \succ 0$, such that*

$$\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{3.31}$$

then the following switching rule

$$\sigma(\xi) = \arg \min_{i \in \mathbb{K}} \ \xi^T \mathbf{P} \mathbf{A}_i \mathbf{x}_e \tag{3.32}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and the guaranteed cost

$$\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 < \xi_0^T \mathbf{P} \xi_0 \tag{3.33}$$

holds.

Proof. The proof is a direct consequence of Theorem 3.3. By imposing $\mathbf{Q}_i \succeq \mathbf{0}$, for all $i \in \mathbb{K}$. □

The conditions in the corollary require that all the subsystems be quadratically stable, that is, $\mathbf{A}_i \in \mathcal{H}$, $\forall i \in \mathbb{K}$ and additionally, they must admit the same matrix solution $\mathbf{P} \succ 0$. Although this result is more conservative when compared to Theorem 3.3, it is useful when considering practical applications, as it guarantees global asymptotic stability for any $\mathbf{x}_e \in \mathbf{X}_e$, by only calculating the matrix $\mathbf{P}$ once, in contrast to Theorem 3.3, where $\mathbf{P}$ depends on the value of λ_0. This allows for the equilibrium point to change at execution time by simply altering $\mathbf{x}_e$ as desired on the switching rule (3.32).

3.4 Performance Indices

This section aims to introduce the concept of performance indices for switched systems, as well as propose suboptimal switching rules that assure an upper bound for these indices. For the $\mathcal{H}_2$ case, the results of Theorem 3.3 will be generalized, where a state dependent switching function is proposed. For the $\mathcal{H}_\infty$ case, two switching rules are introduced, one dependent on the system state, and another, less conservative, that also depends on the external disturbance. These concepts have already been tackled in [23, 24, 25], in the context of switched linear systems, as well as in [16, 29] for the case of switched affine systems.

In this section, the following switched affine system is considered, already in its shifted representation, such that the origin is its unique equilibrium point

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \mathbf{H}_\sigma \mathbf{w}(t) + \boldsymbol{\ell}_\sigma, \quad \boldsymbol{\xi}(0) = \mathbf{0} \\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \boldsymbol{\xi}(t) + \mathbf{G}_\sigma \mathbf{w}(t) \end{aligned} \tag{3.34}$$

For this system, $\boldsymbol{\xi}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{\ell}_i \in \mathbb{R}^{n_x}$ are the affine terms, $\sigma(\cdot) \in \mathbb{K}$ is the switching rule, $\mathbf{z}_e(t) \in \mathbb{R}^{n_z}$ is the performance output, and $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the external disturbance input. The following performance indices will establish a relationship between this disturbance and the performance output.

3.4.1 $\mathcal{H}_2$ Performance Index for Switched Systems

As for the case of the $\mathcal{H}_2$ norm for LTI systems, we must consider $\mathbf{G}_i = \mathbf{0}$ for all $i \in \mathbb{K}$ in order to ensure that the output $\mathbf{z}_e(t)$, when associated to external impulsive disturbances, be square-integrable.

The $\mathcal{H}_2$ performance index is defined as in [25] by

$$J_2(\sigma) = \sum_{k=1}^{n_w} \|\mathbf{z}_{e_k}\|_{\mathcal{L}_2}^2 \tag{3.35}$$

where $\mathbf{z}_{e_k}(t)$ refers to the output of system (3.34) associated to an impulsive disturbance applied to the k-th input, or in other words, $\mathbf{w}(t) = \delta(t)\boldsymbol{\psi}_k$, such that $\boldsymbol{\psi}_k$, for $k \in \{1, \ldots, n_w\}$ form a standard basis. Considering this, observe that (3.35) can also be expressed as

$$J_2(\sigma) = \sum_{k=1}^{n_w} \int_0^\infty \mathbf{z}_{e_k}^T(t)\mathbf{z}_{e_k}(t)\, \mathrm{d}t = \sum_{k=1}^{n_w} \sum_{i=1}^{n_z} \int_0^\infty z_{e_{ik}}(t)^2\, \mathrm{d}t \tag{3.36}$$

where $z_{e_{ik}}(t)$ refers to the i–th output channel with respect to a disturbance applied at the k–th input.

For an LTI system, the $\mathcal{H}_2$ norm introduced in (2.24) can be defined in terms of the frequency response of the system, however this is not the case for switched systems, as they do not admit a transfer matrix due to the action of the switching rule. Nevertheless, notice that whenever the switching rule remains fixed at a certain subsystem, that is $\sigma(t) = i$, for all $t \geq 0$, then the $\mathcal{H}_2$ index is indeed equivalent to the square of the $\mathcal{H}_2$ norm, since in this case, $z_{e_{ik}}(t)$, for an impulsive disturbance, equates to $h_{ik}(t)$ of (2.24).

To deal with switched systems, the upper bound of the $\mathcal{H}_2$ performance index is defined by considering the guaranteed cost introduced in Theorem 3.3, which was obtained with $\mathbf{w}(t) = \mathbf{0}$, and for an arbitrary initial condition. Notice that the system (3.34) subjected to an impulsive disturbance $\mathbf{w}(t) = \delta(t)\boldsymbol{\psi}_k$ can be cast as system (3.17) when $\boldsymbol{\xi}(0) = \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k$ is assigned. This can be verified by integrating (3.34)

$$
\begin{aligned}
\int_0^t \dot{\boldsymbol{\xi}}(t)\,\mathrm{d}t &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t + \int_0^t \mathbf{H}_\sigma \mathbf{w}(t)\,\mathrm{d}t \\
\boldsymbol{\xi}(t) - \boldsymbol{\xi}(0) &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t + \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k \\
\boldsymbol{\xi}(t) - \mathbf{H}_{\sigma(0)}\boldsymbol{\psi}_k &= \int_0^t \mathbf{A}_\sigma \boldsymbol{\xi}(t) + \boldsymbol{\ell}_\sigma\,\mathrm{d}t
\end{aligned}
\tag{3.37}
$$

Hence, the upper bound for the $\mathcal{H}_2$ index can be calculated by using result (3.25) from Theorem 3.3

$$
\begin{aligned}
J_2(\sigma) &= \sum_{k=1}^{n_w} \int_0^\infty \mathbf{z}_{e_k}^T(t)\mathbf{z}_{e_k}(t)\,\mathrm{d}t \\
&< \sum_{k=1}^{n_w} \boldsymbol{\xi}^T(0)\mathbf{P}\boldsymbol{\xi}(0) \\
&= \sum_{k=1}^{n_w} \boldsymbol{\psi}_k^T \mathbf{H}_j^T \mathbf{P}\mathbf{H}_j \boldsymbol{\psi}_k \\
&= \mathrm{tr}\left(\mathbf{H}_j^T \mathbf{P}\mathbf{H}_j\right)
\end{aligned}
\tag{3.38}
$$

where $\mathbf{P}$ satisfies theorem 3.3, and $\sigma(0) = j$ is the initial value of the switching rule, chosen appropriately. Two choices for this value may be of interest: The choice of $j \in \mathbb{K}$ that minimizes an upper bound of J_2 or, alternatively, the worst case choice of j, making the switching function robust with respect to $\sigma(0) \in \mathbb{K}$. More information of this topic can be found in the reference [23].

Finally, it is also worth mentioning that, although this definition has been established for the shifted system (3.17), by considering the performance output $\mathbf{z}_e(t)$, this result equally guarantees an upper bound for the system (3.15) whose performance output is $\mathbf{z}(t)$, since these outputs are shifted by a known amount $\mathbf{z}_e(t) = \mathbf{z}(t) - \mathbf{E}_\sigma \mathbf{x}_e$. With this, the following theorem can be stated.

Theorem 3.4. *Consider the switched affine system (3.34), with $\mathbf{G}_i = 0$ for all $i \in \mathbb{K}$, and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\boldsymbol{\lambda}_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} \succ 0$, and symmetric matrices $\mathbf{Q}_i$, such that*

$$
\mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i + \mathbf{Q}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{3.39}
$$

$$
\mathbf{Q}_{\lambda_0} \succeq 0 \tag{3.40}
$$

then the following switching rule

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} -\xi^T \mathbf{Q}_i \xi + 2\xi^T \mathbf{P} \boldsymbol{\ell}_i \tag{3.41}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assures the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma) < \mathrm{tr}\left(\mathbf{H}_j^T \mathbf{P} \mathbf{H}_j\right) \tag{3.42}$$

with $\sigma(0) = j$ given.

Proof. The proof follows from Theorem 3.3, taking into account the relations (3.37) and (3.38). □

The following convex optimization problem allows for the numerical calculation of matrices $\mathbf{P}$ and $\mathbf{Q}_i$, for $i \in \mathbb{K}$, required to implement the switching rule (3.41) of Theorem 3.4, with $j \in \mathbb{K}$ chosen appropriately.

$$\begin{aligned} \min \quad & \mathrm{tr}\left(\mathbf{H}_j^T \mathbf{P} \mathbf{H}_j\right) \\ \text{subject to:} \quad & \mathbf{P} > 0 \\ & \mathbf{A}_i^T \mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T \mathbf{E}_i + \mathbf{Q}_i \prec 0, \quad \forall i \in \mathbb{K} \\ & \mathbf{Q}_{\lambda_0} \succeq 0 \end{aligned} \tag{3.43}$$

In this manner, the upper bound for the $\mathcal{H}_2$ performance index is given by $J_2(\sigma) < \mathrm{tr}\left(\mathbf{H}_j^T \mathbf{P} \mathbf{H}_j\right)$.

3.4.1.1 Example

A simple numerical example, based on Example 3.3, is provided to illustrate the $\mathcal{H}_2$ state feedback control design technique of Theorem 3.4.

Example 3.4

Consider the switched affine system of Example 3.3, with the additional matrices

$$\mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}, \qquad \mathbf{H}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{H}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

A Hurwitz convex combination $\mathbf{A}_{\lambda_0}$ occurs at $\lambda_0 = [0.2 \;\; 0.8]^T$, with the associated equilibrium point $\mathbf{x}_e = [0.8492 \;\; 0.9167]^T \in \mathbf{X}_e$. In order to implement the switching rule (3.41) of Theorem 3.4, the vectors $\boldsymbol{\ell}_i = \mathbf{A}_i\mathbf{x}_e + \mathbf{b}_i$, for $i = \{1, 2\}$ are first calculated, and the matrices $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are obtained by solving the convex optimization problem (3.43), resulting in the following matrices

$$\mathbf{P} = \begin{bmatrix} 0.3290 & -0.1190 \\ -0.1190 & 0.4847 \end{bmatrix}, \qquad \mathbf{Q}_1 = \begin{bmatrix} -6.0266 & 0.7058 \\ 0.7058 & -2.9389 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 1.5066 & -0.1765 \\ -0.1765 & 0.7347 \end{bmatrix}$$

which are associated to the guaranteed cost $J_2(\sigma) < 0.3290$ for $\sigma(0) = 1$. For comparison, by choosing $\sigma(0) = 2$, the guaranteed cost for the switched system would be $J_2(\sigma) < 1.0518$. This cost would be robust against the choice of $\sigma(0)$. The trajectories in time for each state can be seen in Figure 3.8 for $\mathbf{w}(t) = \delta(t)\boldsymbol{\psi}_1$, which is equivalent to an initial condition $\boldsymbol{\xi}(0) = \mathbf{H}_1\boldsymbol{\psi}_1 = \mathbf{H}_1$, as only one input channel is present, as discussed in Section 3.4.1. The figure reveals that the system state trajectories asymptotically converge to $\boldsymbol{\xi} = \mathbf{0}$, as expected. In

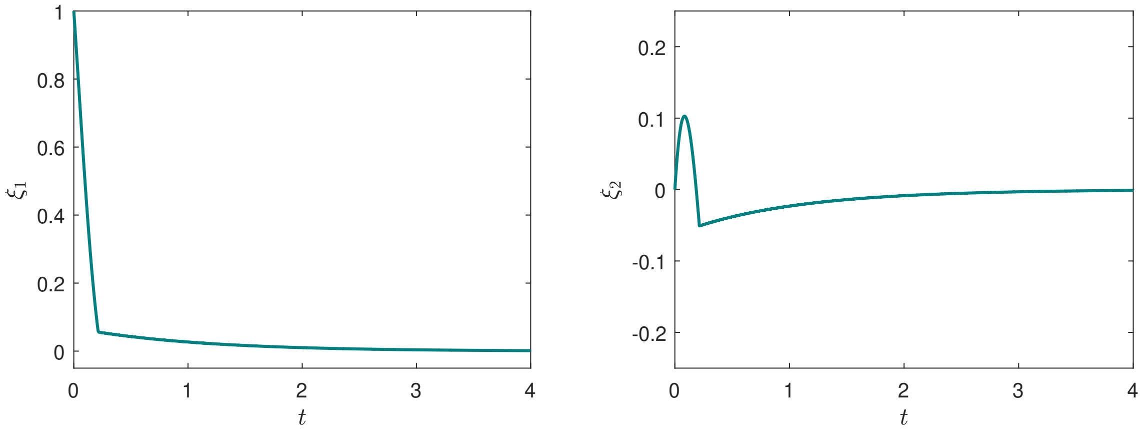


Figure 3.8: Trajectories of each state for the switched affine system under Theorem 3.4.

addition, observe in Figure 3.9 the behavior of the switching rule. It can be seen that the switched system clearly evolves in sliding modes starting at $t \approx 0.21$ seconds, in order to maintain the system at the equilibrium point.

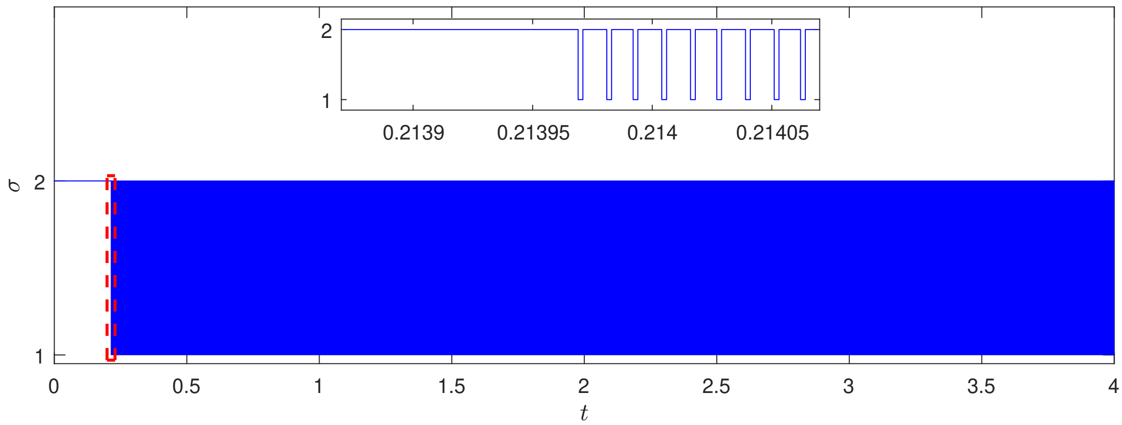


Figure 3.9: Switching rule for the switched affine system under Theorem 3.4.

Finally, by numerical integration of the product $\mathbf{z}_e(t)^T\mathbf{z}_e(t)$ from $t = 0$ to $t \to \infty$, the actual $\mathcal{H}_2$ cost obtained from the implementation of the switching function (3.41) was calculated as being $J_2 = 0.0705 < 0.3290$, within the cost guaranteed by Theorem 3.4. ■

3.4.2 $\mathcal{H}_\infty$ Performance Index for Switched Systems

In a manner analogous to the definition of the $\mathcal{H}_\infty$ norm (2.32) established for LTI systems, the $\mathcal{H}_\infty$ performance index for switched systems is defined as in [25] by

$$J_\infty(\sigma) = \sup_{0 \neq \mathbf{w} \in \mathcal{L}_2} \frac{\|\mathbf{z}_e(t)\|_{\mathcal{L}_2}^2}{\|\mathbf{w}(t)\|_{\mathcal{L}_2}^2} \tag{3.44}$$

or, by considering the scalar $\rho > 0$, we can write $J_\infty(\sigma) < \rho$, with ρ being an upper bound for this index.

Following the same reasoning as in the proof of Theorem 3.3, the upper bound for the $\mathcal{H}_\infty$ index can be inferred from the time derivative of the quadratic Lyapunov function, for state trajectories $\boldsymbol{\xi} \neq \mathbf{0}$, as follows

$$\begin{aligned}
\dot{v}(\xi) &= \dot{\xi}^T\mathbf{P}\xi + \xi^T\mathbf{P}\dot{\xi} \\
&= \xi^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma\right)\xi + \mathbf{w}^T\mathbf{H}_\sigma^T\mathbf{P}\xi + \xi^T\mathbf{P}\mathbf{H}_\sigma\mathbf{w} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma + \left(\mathbf{z}_e^T\mathbf{z}_e - \mathbf{z}_e^T\mathbf{z}_e\right) + \rho\left(\mathbf{w}^T\mathbf{w} - \mathbf{w}^T\mathbf{w}\right) \\
&= \xi^T\left(\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T\mathbf{E}_\sigma\right)\xi + \mathbf{w}^T\left(\mathbf{H}_\sigma^T\mathbf{P} + \mathbf{G}_\sigma^T\mathbf{E}_\sigma\right)\xi + \xi^T\left(\mathbf{P}\mathbf{H}_\sigma + \mathbf{E}_\sigma^T\mathbf{G}_\sigma\right)\mathbf{w} + \\
&\qquad + \mathbf{w}^T\left(\mathbf{G}_\sigma^T\mathbf{G}_\sigma - \rho\mathbf{I}\right)\mathbf{w} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&= \begin{bmatrix}\xi \\ \mathbf{w}\end{bmatrix}^T \begin{bmatrix}\mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma + \mathbf{E}_\sigma^T\mathbf{E}_\sigma & \mathbf{P}\mathbf{H}_\sigma + \mathbf{E}_\sigma^T\mathbf{G}_\sigma \\ \mathbf{H}_\sigma^T\mathbf{P} + \mathbf{G}_\sigma^T\mathbf{E}_\sigma & \mathbf{G}_\sigma^T\mathbf{G}_\sigma - \rho\mathbf{I}\end{bmatrix} \begin{bmatrix}\xi \\ \mathbf{w}\end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w}
\end{aligned} \tag{3.45}$$

Two switching strategies, which provide different $\mathcal{H}_\infty$ guaranteed costs, are introduced along with their respective conditions for stability. The first strategy relies solely on the state information, while the other, a less conservative result, assumes that the external input is either available, or can be measured or estimated. The results of this section are also available in the reference [29].

3.4.2.1 State-Input Dependent Switching Rule

The following theorem, as presented in [29], states conditions for the control design problem of a stabilizing switching function dependent on the system state $\boldsymbol{\xi}(t)$ and also on the external disturbance $\mathbf{w}(t)$, providing an upper bound for the $\mathcal{H}_\infty$ performance index.

Theorem 3.5. *Consider the switched affine system (3.34), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\boldsymbol{\lambda}_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} > 0$, and a scalar $\rho > 0$, such that*

$$\sum_{i\in\mathbb{K}} \lambda_{0_i}\mathcal{L}_i(\rho,\mathbf{P}) < 0 \tag{3.46}$$

with

$$\mathcal{L}_i(\rho,\mathbf{P}) = \begin{bmatrix}\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i & \bullet \\ \mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i & \mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\end{bmatrix} \tag{3.47}$$

then the following switching rule

$$\sigma(\boldsymbol{\xi},\mathbf{w}) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix}\xi \\ \mathbf{w}\end{bmatrix}^T \mathcal{L}_i(\rho,\mathbf{P}) \begin{bmatrix}\xi \\ \mathbf{w}\end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.48}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and guarantees the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$.

Proof. The proof is available in [29], but is demonstrated here for convenience. By adopting the switching strategy in Theorem 3.5, the time derivative of the quadratic Lyapunov function, considered for an arbitrary state trajectory $\xi \neq \mathbf{0}$, is calculated from (3.45) as follows

$$
\begin{aligned}
\dot{v}(\xi) &= \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_\sigma(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\sigma - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&= \min_{i \in \mathbb{K}} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_i - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&= \min_{\lambda \in \Lambda_N} \sum_{i \in \mathbb{K}} \lambda_i \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_\lambda - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&\leq \sum_{i \in \mathbb{K}} \lambda_{0_i} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho, \mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} + 2\xi^T \mathbf{P} \boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
&< -\mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w}
\end{aligned} \tag{3.49}
$$

where inequality (3.49) comes from condition (3.46), and by recalling that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. Notice that, for $\mathbf{w}(t) = \mathbf{0}$, we have $\dot{v}(\xi(t)) < 0$, and as such, the equilibrium point $\xi = \mathbf{0}$ is globally asymptotically stable. This allows for the calculation of the upper bound ρ, by integrating both sides of (3.49), as such

$$
\begin{aligned}
\dot{v}(\xi) &< -\mathbf{z}_e^T \mathbf{z}_e + \rho \mathbf{w}^T \mathbf{w} \\
\int_0^\infty \dot{v}(\xi)\, \mathrm{d}t &< -\int_0^\infty \mathbf{z}_e^T \mathbf{z}_e\, \mathrm{d}t + \rho \int_0^\infty \mathbf{w}^T \mathbf{w}\, \mathrm{d}t \\
\lim_{t \to \infty} v(\xi(t)) - v(\xi(0)) &< -\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 + \rho \|\mathbf{w}\|_{\mathcal{L}_2}^2 \\
\|\mathbf{z}_e\|_{\mathcal{L}_2}^2 &< \rho \|\mathbf{w}\|_{\mathcal{L}_2}^2
\end{aligned} \tag{3.50}
$$

where $\lim_{t\to\infty} v(\xi(t)) = 0$, for an asymptotically stable system, and $v(\xi(0)) = 0$, since $\xi(0) = \mathbf{0}$. For any disturbance $\mathbf{w}(t) \in \mathcal{L}_2$, $\mathbf{w}(t) \neq 0$, ρ gives the upper bound for the $\mathcal{H}_\infty$ performance index

$$
J_\infty(\sigma) = \sup_{0 \neq \mathbf{w} \in \mathcal{L}_2} \frac{\|\mathbf{z}_e\|_{\mathcal{L}_2}^2}{\|\mathbf{w}\|_{\mathcal{L}_2}^2} < \rho \tag{3.51}
$$

as desired. □

Theorem 3.5 may be implemented by solving the following convex optimization problem

$$
\begin{aligned}
\min \quad & \rho \\
\text{subject to:} \quad & \mathbf{P} \succ 0, \quad \rho > 0 \\
& \sum_{i \in \mathbb{K}} \lambda_{0_i} \mathcal{L}_i(\rho, \mathbf{P}) \prec 0
\end{aligned} \tag{3.52}
$$

The solution to this problem provides the matrix $\mathbf{P}$, important for the switching rule (3.48) as well as the upper bound $J_\infty(\sigma) < \rho$ for the $\mathcal{H}_\infty$ performance index.

3.4.2.2 State Dependent Switching Rule

A second switching strategy dependent only on state information $\sigma(\boldsymbol{\xi})$ is relevant whenever the disturbance $\mathbf{w}(t)$ is unavailable for measurement. To accomplish this, we consider a disturbance $\mathbf{w}(t)^\star$, such that it maximizes the product

$$\sup_{\mathrm{w}\in\mathcal{L}_2} \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho,\mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} \tag{3.53}$$

By defining the function $f(\boldsymbol{\xi},\mathbf{w})$ as the matrix product

$$f(\xi,\mathbf{w}) = \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix}^T \mathcal{L}_i(\rho,\mathbf{P}) \begin{bmatrix} \xi \\ \mathbf{w} \end{bmatrix} = \xi^T \left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i\right)\xi + 2\xi^T\left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\mathbf{w} + \mathbf{w}^T\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)\mathbf{w} \tag{3.54}$$

taking the partial derivative of $f(\boldsymbol{\xi},\mathbf{w})$ with respect to $\mathbf{w}(t)$, and subsequently setting it to zero

$$\frac{\partial}{\partial\mathbf{w}} f(\xi,\mathbf{w}) = 2\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi + 2\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)\mathbf{w} = \mathbf{0} \tag{3.55}$$

the disturbance input $\mathbf{w}(t)^\star$ can be calculated

$$\mathbf{w}(t)^\star = -\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi(t) \tag{3.56}$$

Finally, by evaluating the second derivative of $f(\boldsymbol{\xi},\mathbf{w})$ with respect to $\mathbf{w}(t)$

$$\frac{\partial^2}{\partial\mathbf{w}^2} f(\xi,\mathbf{w}) = 2\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right) \tag{3.57}$$

we have that whenever $(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I})$ is negative definite, the input $\mathbf{w}(t)^\star$ maximizes the matrix product $f(\boldsymbol{\xi},\mathbf{w})$.

Substituting $\mathbf{w}(t)^\star = -\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\xi(t)$ for the disturbance $\mathbf{w}(t)$ in $f(\xi,\mathbf{w})$, and thus eliminating its dependency, we obtain

$$f(\xi) = \xi^T\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i - \left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right)\right)\xi \tag{3.58}$$

By defining the matrices $\mathcal{N}_i(\rho,\mathbf{P})$, for all $i \in \mathbb{K}$ as

$$\mathcal{N}_i(\rho,\mathbf{P}) = \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{E}_i^T\mathbf{E}_i - \left(\mathbf{P}\mathbf{H}_i + \mathbf{E}_i^T\mathbf{G}_i\right)\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right)^{-1}\left(\mathbf{H}_i^T\mathbf{P} + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{3.59}$$

and introducing symmetric matrices $\mathbf{Q}_i$, for $i \in \mathbb{K}$, then we have that the following inequalities are satisfied

$$\mathcal{N}_i(\rho,\mathbf{P}) + \mathbf{Q}_i \prec 0 \tag{3.60}$$

$$\left(\mathbf{G}_i^T\mathbf{G}_i - \rho\mathbf{I}\right) \prec 0 \tag{3.61}$$

whenever the condition below is verified

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P}+\mathbf{P}\mathbf{A}_i+\mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{3.62}$$

Indeed, by successively applying Schur complement to (3.62) with respect to $-\mathbf{I}$ and $(\mathbf{G}_i^T\mathbf{G}_i-\rho\mathbf{I})$, we recover the inequalities (3.60) and (3.61). It is based on this condition, that the theorem presented below guarantees global asymptotic stability while assuring the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma)<\rho$.

Theorem 3.6. *Consider the switched affine system (3.34), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist a matrix $\mathbf{P} \succ 0$, symmetric matrices $\mathbf{Q}_i$, and a scalar ρ such that*

$$\mathbf{Q}_{\lambda_0} \succeq 0 \tag{3.63}$$

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P}+\mathbf{P}\mathbf{A}_i+\mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{3.64}$$

then the switching rule

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} \; -\xi^T\mathbf{Q}_i\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i \tag{3.65}$$

makes the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable and guarantees the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma)<\rho$.

Proof. The complete proof can be found in [29], but is discussed here for use in the next chapters. Adopting the switching strategy defined above, the time derivative of the quadratic Lyapunov function, for any arbitrary trajectory $\xi \neq \mathbf{0}$, is calculated from (3.45) as

$$\begin{aligned}
\dot{v}(\xi) &= \begin{bmatrix}\xi\\ \mathbf{w}\end{bmatrix}^T \mathcal{L}_\sigma(\rho,\mathbf{P}) \begin{bmatrix}\xi\\ \mathbf{w}\end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \sup_{\mathbf{w}\in\mathcal{L}_2} \begin{bmatrix}\xi\\ \mathbf{w}\end{bmatrix}^T \mathcal{L}_\sigma(\rho,\mathbf{P}) \begin{bmatrix}\xi\\ \mathbf{w}\end{bmatrix} + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.66)\\
&= \xi^T\mathcal{N}_\sigma(\rho,\mathbf{P})\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.67)\\
&< -\xi^T\mathbf{Q}_\sigma\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\sigma - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.68)\\
&= \min_{i\in\mathbb{K}} \; -\xi^T\mathbf{Q}_i\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_i - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \min_{\lambda\in\Lambda_N} \; -\xi^T\mathbf{Q}_\lambda\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_\lambda - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq \; -\xi^T\mathbf{Q}_{\lambda_0}\xi + 2\xi^T\mathbf{P}\boldsymbol{\ell}_{\lambda_0} - \mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} \\
&\leq -\mathbf{z}_e^T\mathbf{z}_e + \rho\mathbf{w}^T\mathbf{w} && (3.69)
\end{aligned}$$

where equality (3.67) comes from the fact that from definition (3.54) the supremum indicated in (3.66) provides (3.58) with $\mathcal{N}_i(\rho,\mathbf{P})$ defined in (3.59); inequality (3.68) stems from (3.64), which is equivalent to (3.60); and inequality (3.69) arises from condition (3.63), and by realizing that $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$. Once again, we have that for $\mathbf{w}(t) = \mathbf{0}$, $\dot{v}(\xi(t)) < 0$, and consequently, the equilibrium point $\xi = \mathbf{0}$ is globally asymptotically stable, and the upper bound for the $\mathcal{H}_\infty$ performance index $J_\infty(\sigma) < \rho$ is guaranteed, as previously demonstrated in the proof of Theorem 3.5. □

It should be noted that Theorems 3.5 and 3.6 do not impose that $\mathbf{A}_i$ be Hurwitz, for all $i \in \mathbb{K}$, because of the presence of the matrices $\mathbf{Q}_i$, $i \in \mathbb{K}$ as discussed in the previous sections. However, conditions (3.46) and (3.64) do require that $\mathbf{A}_\lambda \in \mathcal{H}$, as has been recurrent thus far.

Furthermore, an important aspect of Theorem 3.6 is that the conditions are based on the fact that $\mathbf{Q}_\lambda \geq 0$, and as such, $\sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho,\mathbf{P}) \prec 0$, leading us to the conclusion that

$$\mathcal{N}_\lambda(\rho,\mathbf{P}) \preceq \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho,\mathbf{P}) \prec 0, \quad \lambda \in \Lambda_N$$

which indicates that a less conservative, and thus more desirable condition would be $\mathcal{N}_\lambda(\rho,\mathbf{P}) \prec 0$. Unfortunately, in most cases, a stability condition assured by this inequality does not exist for the more general class of switched systems. For the special case where the matrices $\mathbf{H}_i$, $\mathbf{E}_i$, and $\mathbf{G}_i$ are constant throughout the subsystems, that is, $\mathbf{H}_i = \mathbf{H}$, $\mathbf{E}_i = \mathbf{E}$ and $\mathbf{G}_i = \mathbf{G}$, $\forall i \in \mathbb{K}$, we have that $\mathcal{N}_\lambda(\rho,\mathbf{P}) = \sum_{i\in\mathbb{K}} \lambda_i \mathcal{N}_i(\rho,\mathbf{P})$, and thus, a stability condition based on the convex combination of subsystem matrices holds true. Notice that, for the conditions of Theorem 3.5, the same conclusion can be drawn without requiring that matrices $\mathbf{H}_i$ be constant across subsystems.

To implement the switching rule (3.65) of Theorem 3.6, the following convex optimization problem, subject to LMI constraints

$$\begin{aligned} \min \quad & \rho \\ \text{subject to:} \quad & \mathbf{P} > 0, \quad \rho > 0 \\ & \mathbf{Q}_{\lambda_0} \geq 0 \\ & \begin{bmatrix} \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{Q}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{P} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \end{aligned} \tag{3.70}$$

provides the matrices $\mathbf{Q}_i$, for $i \in \mathbb{K}$, as well as the matrix $\mathbf{P}$ required by the switching rule, with ρ being the upper bound for the $\mathcal{H}_\infty$ performance index.

3.4.2.3 Example

The following numerical example, based on Example 3.3, compares the upper bounds for $\mathcal{H}_\infty$ performance index guaranteed by Theorems 3.5 and 3.6.

Example 3.5

Consider the switched affine system of Example 3.3, with

$$\mathbf{E}_1 = \mathbf{E}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad \mathbf{H}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{H}_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Before implementing the switching rule (3.48) of Theorem 3.5, matrix $\mathbf{P}$ has been calculated by solving the optimization problem (3.52), which in this case yielded

$$\mathbf{P} = \begin{bmatrix} 0.7339 & -0.2187 \\ -0.2187 & 1.2374 \end{bmatrix}$$

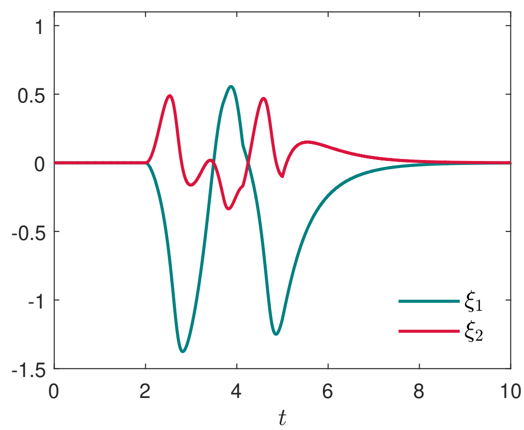
along with the guaranteed cost of $J_\infty(\cdot) < 4.0697$. Similarly, for Theorem 3.6, the matrices $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ were calculated by solving (3.70), resulting in

$$\mathbf{P} = \begin{bmatrix} 0.7669 & -0.2638 \\ -0.2638 & 1.2982 \end{bmatrix}, \qquad \mathbf{Q}_1 = \begin{bmatrix} -13.4815 & 0.5943 \\ 0.5943 & -6.2805 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 3.3704 & -0.1486 \\ -0.1486 & 1.5701 \end{bmatrix}$$

associated to the guaranteed cost $J_\infty(\cdot) < 7.1601$, slightly larger than that guaranteed by Theorem 3.5, which is to be expected given the greater conservativeness of its conditions. By applying the following disturbance to the system

$$\mathbf{w}(t) = \begin{cases} \sin(\pi t), & 2 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

the trajectories in time for each state can be seen in Figure 3.10, for Theorem 3.5, and Figure 3.12, for Theorem 3.6. Notice that, after the disturbance ceases, the respective switching functions are able to successfully stabilize the systems to the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$.



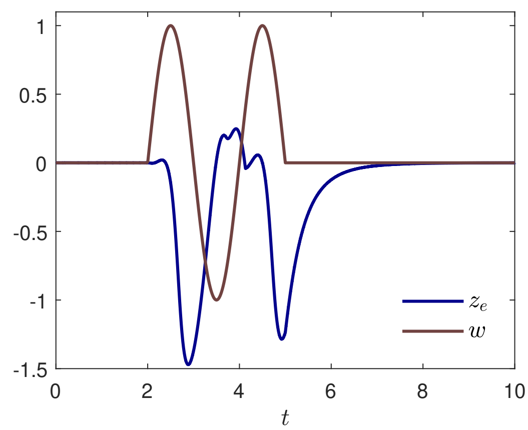


Figure 3.10: Trajectories of each state for the switched affine system under Theorem 3.5.

Observe in Figures 3.11 and 3.13 the behavior of the switching rule for Theorems 3.5 and 3.6, respectively. Although similar at first, the high switching frequencies make it difficult to promptly gauge their effects, notice however, that the dynamical behavior of the switched system under both theorems is in fact distinct.

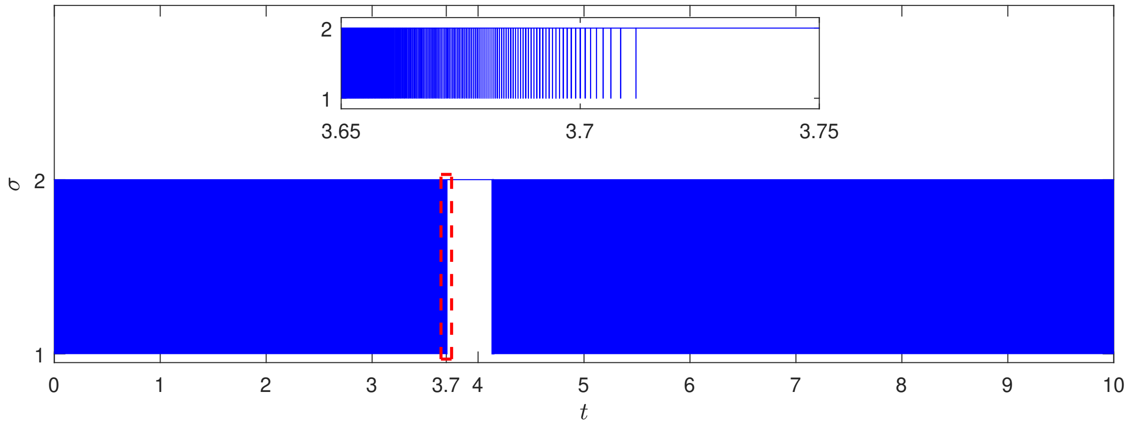


Figure 3.11: Switching rule for the switched affine system under Theorem 3.5.

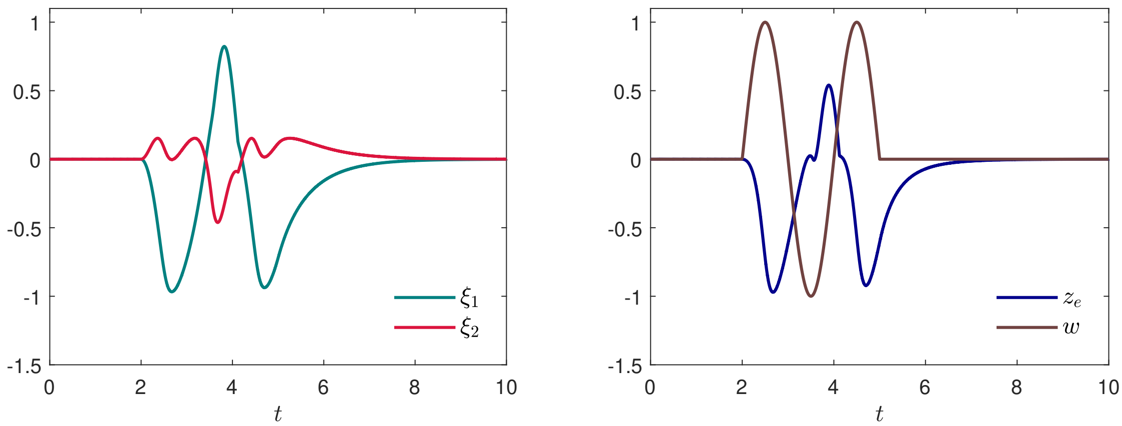


Figure 3.12: Trajectories of each state for the switched affine system under Theorem 3.6.

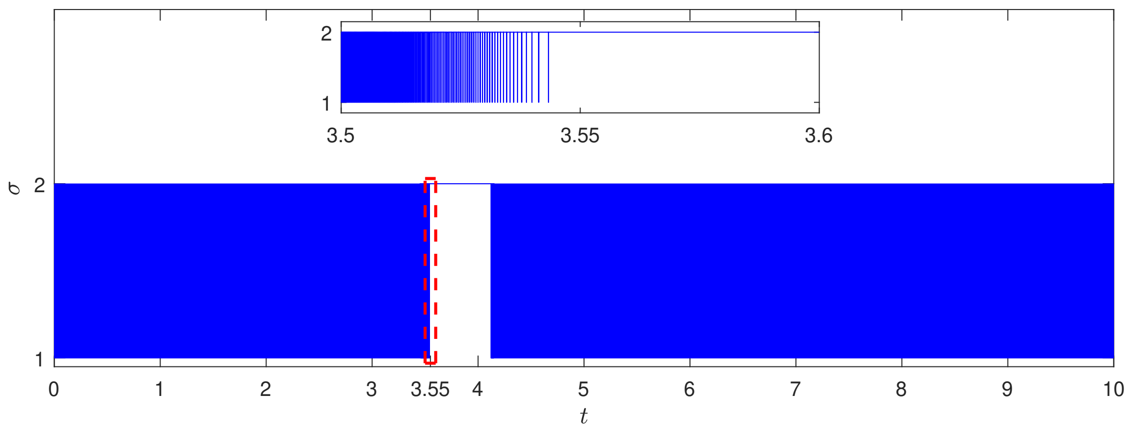


Figure 3.13: Switching rule for the switched affine system under Theorem 3.6.

For the $\mathcal{H}_\infty$ case it is not possible to obtain the actual cost $J_\infty(\sigma)$ by numerical integration, in contrast with the analysis employed for the $\mathcal{H}_2$ case, since the worst case external input $\mathbf{w}(t)$ for switched system is in general extremely difficult to calculate. ■

3.5 Concluding Remarks

Throughout this chapter, the fundamental concepts of switched systems were introduced. First, we discussed the importance of the switching function on the dynamical behavior of a switched system, along with a brief review of existing results in the literature and the many possible applications of switched systems. Then, stabilizing switching rules for switched linear systems were introduced. This was followed by a discussion on switched affine systems, and how this more general class of systems presents a greater theoretical challenge when dealing with the design of stabilizing switching rules. Finally, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, first studied for LTI systems, are presented in the context of switched affine systems, along with different switching rules that guarantee these indices. These topics will be key to the contributions of this work, discussed in the following chapters.

Several examples across the chapter illustrate the unique and complex dynamical behaviors which are imparted by the switching function, helping to demonstrate the theory on switched linear and affine systems.

CHAPTER 4

JOINT ACTION OUTPUT FEEDBACK CONTROL

THE previous chapters introduced underlying concepts that are essential to the main contributions of this work. This chapter is dedicated to one of these contributions, which consists of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ dynamic output feedback control design for continuous-time switched affine systems by considering the simultaneous design of a full-order switched dynamical controller and a switching function. To the best of the author's knowledge, this is the first time that the joint design of two control structures are taken into account in the context of output feedback control of switched affine systems, and have resulted in the following publication [31] in which the contents of this chapter are based upon.

4.1 Introduction

Most results available in the literature treat the control design problem for switched affine systems considering the switching function as the sole control variable to be determined. Some results, which have previously been mentioned, deal with the design of state dependent switching rules, such as in [26, 27, 28], while others consider an output dependent switching rule [16, 30]. In [30], the switching rule is implemented by means of a full-order switched affine observer, while [16] considers the design of a full-order switched dynamical filter to provide the information needed by the switching function. However fewer references treat the control design problem considering the joint action of a switching rule in tandem with a control law $\mathbf{u}(t)$. The following reference [29] approaches this problem by adopting a state dependent switching rule $\sigma(\mathbf{x}(t))$ paired with a control law of the form $\mathbf{u}(t) = \mathbf{K}_\sigma \mathbf{x}(t)$, $\sigma \in \mathbb{K}$. However, in many situations the state vector is not available for measurement, which is the case of several practical applications, and therefore the technique proposed in [29] cannot be applied.

Given the above, the control design problem considering two output-dependent control structures is clearly a relevant topic of research, having only been tackled for the case of switched linear systems in [24] and [54]. As such, the results of this chapter generalize the ideas introduced by [16, 24, 29] to deal with the joint design of two stabilizing control variables in the context of switched affine systems.

More specifically, a full-order switched dynamical controller, together with an output dependent switching rule are proposed in order to guarantee global asymptotic stability of the desired equilibrium point, as well as to assure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs. In the $\mathcal{H}_\infty$ case, a switching function taking into account output and input information is also introduced, which is less conservative than that based only on the measured output, but in turn requires that the external input be available or estimated.

Some important characteristics of this new methodology are that not only no stability property is expected from the individual subsystem matrices $\mathbf{A}_i$, for all $i \in \mathbb{K}$, but also it is no longer required that $\mathbf{A}_\lambda \in \mathcal{H}$, an assumption which has permeated the results of the previous chapters. This characteristic is very compelling since it allows a larger set of equilibrium points $\mathbf{X}_e$ of the switched system to be attainable, instead of only the subset where $\mathbf{A}_\lambda$ is Hurwitz. This is made possible by the joint action of both control structures, and will become clear throughout the course of the chapter.

4.2 Problem Statement

Consider the switched affine system with the following state space representation, already given in its shifted configuration, that is, with $\xi(t) = \mathbf{x} - \mathbf{x}_e$

$$\begin{aligned} \dot{\xi}(t) &= \mathbf{A}_\sigma \xi(t) + \mathbf{B}_\sigma \mathbf{u}(t) + \mathbf{H}_\sigma \mathbf{w}(t) + \boldsymbol{\ell}_\sigma, \quad \xi(0) = \mathbf{0} \\ \mathbf{y}_e(t) &= \mathbf{C}_\sigma \xi(t) + \mathbf{D}_\sigma \mathbf{w}(t) \\ \mathbf{z}_e(t) &= \mathbf{E}_\sigma \xi(t) + \mathbf{F}_\sigma \mathbf{u}(t) + \mathbf{G}_\sigma \mathbf{w}(t) \end{aligned} \tag{4.1}$$

where $\xi(t) \in \mathbb{R}^{n_x}$ is the state vector, considered to be unavailable for measurement, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input to be designed, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the external disturbance, $\mathbf{y}_e(t) \in \mathbb{R}^{n_y}$ is the measured output, $\mathbf{z}_e(t) \in \mathbb{R}^{n_z}$ is the performance output, and $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$ for all $i \in \mathbb{K}$ are the affine terms. Notice that $\mathbf{y}_e(t) = \mathbf{y}(t) - \mathbf{C}_\sigma \mathbf{x}_e$, with $\mathbf{y}(t) = \mathbf{C}_\sigma \mathbf{x}(t) + \mathbf{D}_\sigma \mathbf{w}(t)$, when expressed in terms of $\mathbf{x}(t)$.

Recall from Definition 1 that whenever $\mathbf{b}_i \neq \mathbf{0}$ for some $i \in \mathbb{K}$, the switched system possesses several equilibrium points, characterizing the subset of the state space given by

$$\mathbf{X}_e = \left\{ \mathbf{x}_e \in \mathbb{R}^{n_x} : \mathbf{x}_e = -\mathbf{A}_\lambda^{-1} \mathbf{b}_\lambda, \quad \lambda \in \Lambda_N \right\} \tag{4.2}$$

A given choice of $\mathbf{x}_e \in \mathbf{X}_e$, with its associated vector $\lambda_0 \in \Lambda_N$ completes the definition of system (4.1).

Our main objective is to design a control law $\mathbf{u}(t)$, implemented via a full-order switched dynamical controller, and dependent on the measured output $\mathbf{y}_e(t)$, along with a switching function $\sigma(\mathbf{y}_e(t))\colon \mathbb{R}^{n_y} \to \mathbb{K}$ that together are capable of guaranteeing global asymptotic stability of a chosen equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$. Notice that this is equivalent to a dependency on $\mathbf{y}(t)$, as $\mathbf{y}_e(t)$ is simply shifted by a known amount. These two control structures must also assure upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices.

To this end, the following full-order switched affine controller is proposed, with state space representation given by

$$\mathcal{C}_\sigma : \begin{cases} \dot{\hat{\xi}}(t) = \hat{\mathbf{A}}_\sigma \hat{\xi}(t) + \hat{\mathbf{B}}_\sigma \mathbf{y}_e(t) + \hat{\boldsymbol{\ell}}_\sigma \,, \quad \hat{\xi}(0) = \mathbf{0} \\ \mathbf{u}(t) = \hat{\mathbf{C}}_\sigma \hat{\xi}(t) \end{cases} \tag{4.3}$$

where $\hat{\xi} \in \mathbb{R}^{n_x}$ is the state vector of the controller and $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control signal. Matrices $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$, $\hat{\mathbf{C}}_i$, as well as the affine term $\hat{\boldsymbol{\ell}}_i$, for all $i \in \mathbb{K}$, of appropriate dimensions, are to be determined. The controller state $\hat{\xi}$ will not only be used to provide the control signal $\mathbf{u}(t)$, but will also make it possible to implement the switching function $\sigma(\mathbf{y}_e(t))$.

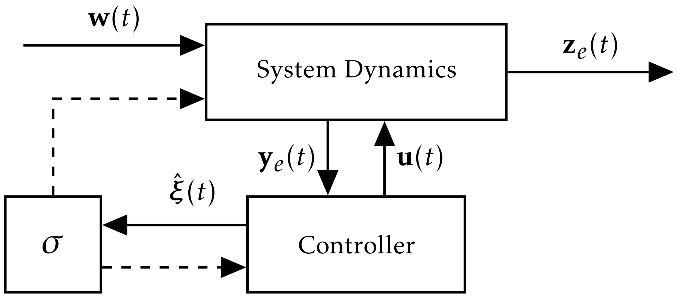


Figure 4.1: Closed-loop system.

By connecting the controller (4.3) to the switched system (4.1), as illustrated in Figure 4.1, and defining the state vector $\tilde{\xi}(t) = \begin{bmatrix} \xi(t)^T & \hat{\xi}(t)^T \end{bmatrix}^T \in \mathbb{R}^{2n_x}$, the following augmented system emerges

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \tilde{\mathbf{A}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{H}}_\sigma \mathbf{w}(t) + \tilde{\boldsymbol{\ell}}_\sigma \,, \quad \tilde{\xi}(0) = \mathbf{0} \\ \mathbf{z}_e(t) &= \tilde{\mathbf{E}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{G}}_\sigma \mathbf{w}(t) \end{aligned} \tag{4.4}$$

with matrices given by

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i\hat{\mathbf{C}}_i \\ \hat{\mathbf{B}}_i\mathbf{C}_i & \hat{\mathbf{A}}_i \end{bmatrix}, \quad \tilde{\boldsymbol{\ell}}_i = \begin{bmatrix} \boldsymbol{\ell}_i \\ \hat{\boldsymbol{\ell}}_i \end{bmatrix}, \quad \tilde{\mathbf{H}}_i = \begin{bmatrix} \mathbf{H}_i \\ \hat{\mathbf{B}}_i\mathbf{D}_i \end{bmatrix}, \quad \tilde{\mathbf{E}}_i = \begin{bmatrix} \mathbf{E}_i & \mathbf{F}_i\hat{\mathbf{C}}_i \end{bmatrix}, \quad \tilde{\mathbf{G}}_i = \mathbf{G}_i \tag{4.5}$$

The control design problem consists in determining appropriate conditions that will satisfy Theorems 3.4, 3.5, and 3.6, introduced in Chapter 3, for the augmented system (4.4), thus guaranteeing global asymptotic stability of the closed-loop system, and assuring an upper bound for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices, as defined in (3.35) and (3.44), respectively.

4.3 Preliminaries

Before proceeding, we first address the problem of obtaining a switching rule that forgoes any dependency on the unknown system state, a result that will be used for the two first theorems, and will be further extended to deal with a switching rule also relying on the external input. To accomplish this, we first define the block symmetric matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}^{-1}$ as follows

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{Y} & \mathbf{V} \\ \mathbf{V}^T & \hat{\mathbf{Y}} \end{bmatrix}, \quad \tilde{\mathbf{P}}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{U} \\ \mathbf{U}^T & \hat{\mathbf{X}} \end{bmatrix} \tag{4.6}$$

such that the product $\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{P}} = \mathbf{I}$ holds. This implies in the following relations

$$\mathbf{X}\mathbf{Y} + \mathbf{U}\mathbf{V}^T = \mathbf{I}, \quad \mathbf{X}\mathbf{V} + \mathbf{U}\hat{\mathbf{Y}} = \mathbf{0}, \quad \mathbf{U}^T\mathbf{Y} + \hat{\mathbf{X}}\mathbf{V}^T = \mathbf{0}, \quad \mathbf{U}^T\mathbf{V} + \hat{\mathbf{X}}\hat{\mathbf{Y}} = \mathbf{I} \tag{4.7}$$

Let $\tilde{\mathbf{Q}}_i$ be the block symmetric matrix

$$\tilde{\mathbf{Q}}_i = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_i \end{bmatrix}, \quad i \in \mathbb{K} \tag{4.8}$$

Now recall the following switching rule, employed in Theorems 3.4 and 3.6, and applied to the augmented system (4.4)

$$\sigma(\tilde{\xi}) = \arg \min_{i \in \mathbb{K}} -\tilde{\xi}^T \tilde{\mathbf{Q}}_i \tilde{\xi} + 2\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i \tag{4.9}$$

Let us first consider the matrix product $\tilde{\xi}^T \tilde{\mathbf{Q}}_i \tilde{\xi}$. Observe that the structure of $\tilde{\mathbf{Q}}_i$ has been defined in order to eliminate any dependency on the unknown system state $\xi(t)$. Also, notice that the choice of a constant matrix $\mathbf{M}$ for block $(1,2)$, and a constant symmetric matrix $\mathbf{N}$ for block $(1,1)$ of $\tilde{\mathbf{Q}}_i$, would also be possible, since although the product $\tilde{\xi}^T \tilde{\mathbf{Q}}_i \tilde{\xi}$ would depend on $\xi(t)$, as such

$$\tilde{\xi}^T \tilde{\mathbf{Q}}_i \tilde{\xi} = \xi^T \mathbf{N} \xi + 2\hat{\xi}^T \mathbf{M}^T \xi + \hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi}, \quad i \in \mathbb{K} \tag{4.10}$$

the terms $\xi^T \mathbf{N} \xi$ and $\hat{\xi}^T \mathbf{M}^T \xi$ are not indexed, and thus, would be constant across $i \in \mathbb{K}$. This would allow for the definition of an equivalent switching function considering only the product $\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi}$, as desired. This approach, however, has not shown any advantage with regard to the optimality of the guaranteed cost over the structure of $\tilde{\mathbf{Q}}_i$ originally defined in (4.8). As such, we base this technique on this simpler choice of $\tilde{\mathbf{Q}}_i$.

Now we consider the product $\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i$, as follows

$$\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i = \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}^T \begin{bmatrix} \mathbf{Y} & \mathbf{V} \\ \mathbf{V}^T & \hat{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_i \\ \hat{\boldsymbol{\ell}}_i \end{bmatrix} \tag{4.11}$$

Notice that by assuring $\mathbf{Y}\boldsymbol{\ell}_i + \mathbf{V}\hat{\boldsymbol{\ell}}_i = \mathbf{0}$, the first n_x rows of term $\tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i$ are null, as such, the dependency on the system state is eliminated by making the appropriate choice of $\hat{\boldsymbol{\ell}}_i$ as

$$\hat{\boldsymbol{\ell}}_i = -\mathbf{V}^{-1} \mathbf{Y} \boldsymbol{\ell}_i, \quad \forall i \in \mathbb{K} \tag{4.12}$$

where $\mathbf{V}$ is such that $\exists \mathbf{V}^{-1}$. Observe that this choice also guarantees the nullity of $\tilde{\boldsymbol{\ell}}_{\lambda_0} = \mathbf{0}$, since $\boldsymbol{\ell}_{\lambda_0} = \mathbf{0}$, as such

$$\tilde{\boldsymbol{\ell}}_{\lambda_0} = \begin{bmatrix} \boldsymbol{\ell}_{\lambda_0} \\ \hat{\boldsymbol{\ell}}_{\lambda_0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_{\lambda_0} \\ -\mathbf{V}^{-1} \mathbf{Y} \boldsymbol{\ell}_{\lambda_0} \end{bmatrix} = \mathbf{0} \tag{4.13}$$

The term $\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i$, given in (4.11), with (4.12) and the relations $\hat{\mathbf{Y}} = -\mathbf{U}^{-1} \mathbf{X} \mathbf{V}$ and $\mathbf{V}^T = \mathbf{U}^{-1} (\mathbf{I} - \mathbf{X}\mathbf{Y})$ from (4.7), can be rewritten as

$$\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i = \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{V}^T \boldsymbol{\ell}_i + \hat{\mathbf{Y}} \hat{\boldsymbol{\ell}}_i \end{bmatrix} = \hat{\xi}^T (\mathbf{V}^T \boldsymbol{\ell}_i + \hat{\mathbf{Y}} \hat{\boldsymbol{\ell}}_i) = \hat{\xi}^T \mathbf{U}^{-1} \boldsymbol{\ell}_i \tag{4.14}$$

Thus, as desired, the switching rule can be expressed with a dependency solely on the state of the controller $\hat{\xi}$, that is

$$\sigma(\tilde{\xi}) = \sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi} + 2\hat{\xi}^T \mathbf{U}^{-1} \boldsymbol{\ell}_i \tag{4.15}$$

The problem now consists in finding conditions that can be expressed in terms of LMIs, allowing us to obtain the matrices $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$, $\hat{\mathbf{C}}_i$, and vectors $\hat{\boldsymbol{\ell}}_i$, for $i \in \mathbb{K}$, needed to implement the dynamical controller, as well as matrices $\hat{\mathbf{Q}}_i$ and $\mathbf{U}^{-1}$ important for the switching rule $\sigma(\cdot)$, that guarantee the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. The following two sections introduce these conditions based on the structures of $\tilde{\mathbf{Q}}_i$ and $\tilde{\boldsymbol{\ell}}_i$ just defined.

4.4 $\mathcal{H}_2$ Control Design

In this section, we will generalize Theorem 3.4 for the augmented system (4.4) in order to deal with the two control structures proposed, thus guaranteeing an upper bound for the $\mathcal{H}_2$ performance index. It is assumed that $\mathbf{G}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, so as to work exclusively with strictly proper subsystems.

Theorem 4.1. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated vector $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, and $\mathbf{S}$, and matrices $\mathbf{L}_i$ and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, such that*

$$\mathbf{R}_{\lambda_0} \succeq 0 \tag{4.16}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.17}$$

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{4.18}$$

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{H}_j & \mathbf{X} & \bullet \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j & \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.19}$$

with $j = \sigma(0)$ given, then the following switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi} + 2\hat{\xi}^T \mathbf{X}^{-1} \boldsymbol{\ell}_i \tag{4.20}$$

along with controller (4.3), whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\left(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}\right), \\ \hat{\mathbf{B}}_i &= -(\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{C}}_i = \mathbf{W}_i\mathbf{X}^{-1}, \\ \hat{\boldsymbol{\ell}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \qquad \hat{\mathbf{Q}}_i = \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \end{aligned} \tag{4.21}$$

make the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assure the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma, \mathcal{C}_\sigma) < \mathrm{tr}(\mathbf{S}) \tag{4.22}$$

for the system.

Proof. The proof consists in demonstrating the validity of Theorem 3.4 whenever the conditions of Theorem 4.1 are satisfied. To this end, consider matrices $\tilde{\mathbf{P}}$, $\tilde{\mathbf{P}}^{-1}$, $\tilde{\mathbf{Q}}_i$, and vector $\hat{\boldsymbol{\ell}}_i$ previously defined, and inequalities (3.39) and (3.40) of Theorem 3.4 expressed in terms of the augmented system (4.4)

$$\tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i + \tilde{\mathbf{Q}}_i < 0, \quad \forall i \in \mathbb{K} \tag{4.23}$$

$$\tilde{\mathbf{Q}}_{\lambda_0} \geq 0 \tag{4.24}$$

It becomes clear that inequality (4.23) and (4.24) are nonlinear with respect to the matrix variables after substituting for the augmented matrices $\tilde{\mathbf{A}}_i$, $\tilde{\mathbf{E}}_i$, $\tilde{\mathbf{P}}$, and $\tilde{\mathbf{Q}}_i$, $i \in \mathbb{K}$. In order to obtain conditions based on LMIs, we define the transformation matrix $\tilde{\boldsymbol{\Gamma}}$ as

$$\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{U}^T & \mathbf{0} \end{bmatrix} \tag{4.25}$$

First, consider the inequality (4.23) multiplied by the transformation matrix $\tilde{\boldsymbol{\Gamma}}$ as follows

$$\mathrm{He}\left\{\tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_i\tilde{\boldsymbol{\Gamma}}\right\} + \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\boldsymbol{\Gamma}} < 0 \tag{4.26}$$

whose intermediary products are given by

$$\begin{gathered}\tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}_i\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{A}_i\mathbf{X} + \mathbf{B}_i\hat{\mathbf{C}}_i\mathbf{U}^T & \mathbf{A}_i \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\hat{\mathbf{C}}_i\mathbf{U}^T + \mathbf{V}\hat{\mathbf{B}}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T & \mathbf{Y}\mathbf{A}_i + \mathbf{V}\hat{\mathbf{B}}_i\mathbf{C}_i \end{bmatrix} \\ \tilde{\mathbf{E}}_i\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\hat{\mathbf{C}}_i\mathbf{U}^T & \mathbf{E}_i \end{bmatrix}, \qquad \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\end{gathered} \tag{4.27}$$

Denoting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i\mathbf{U}^T$, inequality (4.26) becomes

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \end{bmatrix} < 0 \tag{4.28}$$

We can choose $\hat{\mathbf{A}}_i$ in order to make block $(2,1)$ of (4.28) null, thus obtaining simply

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i < 0 \tag{4.29}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & -\mathbf{I} \end{bmatrix} < 0 \tag{4.30}$$

Indeed, (4.29) comes directly from block $(2,2)$ of (4.28), and inequality (4.30) is obtained by applying Schur complement on block $(1,1)$ of (4.28) appropriately. It can be seen that whenever these inequalities are satisfied, condition (4.23) from Theorem 3.4 is verified. Also, observe that (4.16) assures that $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, thus satisfying condition (4.24) of Theorem 3.4 for the augmented system (4.4).

Notice that any arbitrary choice of $\mathbf{U}$ can be made, without loss of generality, so long $\exists\mathbf{U}^{-1}$. The specific choice of $\mathbf{U}=\mathbf{X}$ is considered in throughout work. In this case, from the relations in (4.7), we have that $\mathbf{V}=\mathbf{X}^{-1}-\mathbf{Y}$, and thus we obtain the identities in (4.21). Furthermore, notice that the switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.31}$$

comes directly from (4.15) when considering $\mathbf{U}=\mathbf{X}$, as well as matrix $\hat{\mathbf{Q}}_i$ in (4.21).

Finally, we have that the following inequality

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{H}_j & \mathbf{X} & \bullet \\ \mathbf{Y}\mathbf{H}_j+\mathbf{L}_j\mathbf{D}_j & \mathbf{I} & \mathbf{Y} \end{bmatrix} > 0 \tag{4.32}$$

is equivalent to

$$\begin{bmatrix} \mathbf{S} & \bullet \\ \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j & \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} \end{bmatrix} > 0 \tag{4.33}$$

with intermediary terms given by

$$\tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \tag{4.34}$$

$$\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \mathbf{H}_j^T & \mathbf{H}_j^T\mathbf{Y}+\mathbf{D}_j^T\mathbf{L}_j^T \end{bmatrix} \tag{4.35}$$

By multiplying inequality (4.33) to the left by $\mathrm{diag}(\mathbf{I},(\tilde{\boldsymbol{\Gamma}}^T)^{-1})$, to the right by its transpose, and subsequently applying Schur complement with respect to $\tilde{\mathbf{P}}$, we obtain $\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j < \mathbf{S}$, and thus

$$J_2(\sigma,\mathcal{C}_\sigma) < \mathrm{tr}\left(\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j\right) < \mathrm{tr}(\mathbf{S}) \tag{4.36}$$

with $j=\sigma(0)$, is guaranteed as in Theorem 3.4. This concludes the proof. □

This theorem presents a few compelling characteristics. First, notice that the inequalities in (4.17) do not require that the closed-loop matrices $\mathbf{A}_{cl,i}=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}_{c,i}$, with $\mathbf{K}_{c,i}=\mathbf{W}_i\mathbf{X}^{-1}$, be Hurwitz, since matrices $\mathbf{R}_i$ are sign indefinite. Thus, Theorem 4.1 can assure stability of the switched system even for the case where each individual subsystem is not stabilizable. Also, notice that not only there is no imposition on matrices $\mathbf{A}_i\in\mathcal{H}$, but no convex combination of subsystem matrices $\mathbf{A}_\lambda\in\mathcal{H}$ needs to exist, a recurrent condition throughout the results introduced in chapter 3. Instead, inequalities (4.16) and (4.17) impose that

$$\sum_{i\in\mathbb{K}}\lambda_{0_i}\left(\mathrm{He}\{\mathbf{A}_{cl,i}\mathbf{X}\}+\mathbf{X}\mathbf{E}_{cl,i}^T\mathbf{E}_{cl,i}\mathbf{X}\right) < 0 \tag{4.37}$$

with $\mathbf{E}_{cl,i}=\mathbf{E}_i+\mathbf{F}_i\mathbf{K}_{c,i}$, $i\in\mathbb{K}$. In other words, it is now only necessary that a stable convex combination of the closed loop matrices exists. These considerations bring to light the fact that Theorem 4.1 is able to guarantee global asymptotic stability of the equilibrium point even in the case where a control law $\mathbf{u}(t)$ and a switching rule $\sigma(t)$ are unable to do so independently, allowing for a broader scope of problems to be considered. This interesting scenario is considered in a numerical example presented shortly. It should be noted that although the inequalities in (4.18) impose that the closed loop matrices $\mathbf{A}_i + \mathbf{K}_{o,i}\mathbf{C}_i$, with $\mathbf{K}_{o,i} = \mathbf{Y}^{-1}\mathbf{L}_i$, be quadratically stable with respect to $\mathbf{Y}$, this is not a demanding requirement, since these inequalities are uncoupled from (4.17), through the different matrix variables $\mathbf{Y}$ and $\mathbf{X}$. Furthermore, the matrix gains $\mathbf{K}_{o,i}$ depend on the index i, and thus are independent for each constraint. Finally, notice that as in Theorem 3.4, an appropriate choice of $\sigma(0) \in \mathbb{K}$ can be made to optimize the $\mathcal{H}_2$ performance index, or alternatively, by considering j of worst case, the $\mathcal{H}_2$ control design problem is made robust with respect to $\sigma(0)$, as discussed in the previous chapter.

When compared to existing results in the literature, such as [30] and [16], Theorem 4.1 presents more lenient conditions. The first reference requires that all matrices $\mathbf{A}_i$ be Hurwitz, thus being a more conservative result, while the latter requires $\mathbf{A}_\lambda \in \mathcal{H}$, since a control law $\mathbf{u}(t)$ is not taken into account.

The following optimization problem, subject to the LMI conditions in Theorem 4.1, provides the matrices necessary to implement the dynamical controller and switching rule, through the relations in (4.21).

$$\begin{aligned} \min \quad & \operatorname{tr}(\mathbf{S}) \\ \text{subject to:} \quad & (4.16),\ (4.17),\ (4.18),\ \text{and}\ (4.19) \end{aligned} \tag{4.38}$$

The next examples aim to illustrate the relevance and usefulness of the proposed methodology. The first two reminisce the switched affine system of Example 3.3, while the second is based on the example of [31].

4.4.1 Examples: $\mathcal{H}_2$ Control Design

In the following two examples, the switched affine system of Example 3.3 is considered, with the additional inputs $\mathbf{u}(t)$ and $\mathbf{w}(t)$, and outputs $\mathbf{y}_e(t)$ and $\mathbf{z}_e(t)$, as in (4.1).

Example 4.1

Consider the switched affine system (4.1) with dynamical matrices given in Example 3.3, and with

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{F}_1 = \mathbf{F}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$
$$\mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \mathbf{D}_2 = 1, \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}, \quad \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{0}$$

By choosing the equilibrium point $\mathbf{x}_e = [0.2982 \;\; 1.2586]^T \in \mathbf{X}_e$ of interest, associated to $\lambda_0 = [0.4 \;\; 0.6]^T$, it can be verified that the convex combination $\mathbf{A}_{\lambda_0}$ is in fact not Hurwitz, as such, Theorem 3.4 cannot guarantee global asymptotic stability of this equilibrium point, and thus, another approach must be used. For instance, a state feedback control design technique may be employed if the state is available for measurement, however in many situations, this is not the case. The methodology introduced in this work addresses this scenario, and will be demonstrated in this example.

First, in order to implement the two control structures, the convex optimization problem (4.38) is solved for $\sigma(0) = 2$, obtaining the minimum guaranteed cost $J_2(\sigma, \mathcal{C}_\sigma) < 34.1051$. Via the identities in (4.21), the following matrices, used to implement the dynamical controller $\mathcal{C}_\sigma$, are obtained

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} -4.8637 & -4.8656 \\ 1.1615 & 1.1621 \end{bmatrix} \times 10^4, \quad \hat{\mathbf{A}}_2 = \begin{bmatrix} -17.7251 & -16.3277 \\ 4.1664 & -7.3999 \end{bmatrix}, \quad \hat{\mathbf{B}}_1 = \begin{bmatrix} 4.8646 \\ -1.1617 \end{bmatrix} \times 10^4, \quad \hat{\mathbf{B}}_2 = \begin{bmatrix} 11.0817 \\ -0.5102 \end{bmatrix},$$

$$\hat{\mathbf{C}}_1 = \hat{\mathbf{C}}_2 = \begin{bmatrix} -0.2287 & -0.1773 \end{bmatrix}, \quad \hat{\boldsymbol{\ell}}_1 = \begin{bmatrix} 1.6111 \\ 12.4843 \end{bmatrix}, \quad \hat{\boldsymbol{\ell}}_2 = \begin{bmatrix} -1.0741 \\ -8.3229 \end{bmatrix}$$

as well as the matrices important for the switching rule

$$\hat{\mathbf{Q}}_1 = \begin{bmatrix} -0.2608 & 0.7002 \\ 0.7002 & -1.3657 \end{bmatrix}, \quad \hat{\mathbf{Q}}_2 = \begin{bmatrix} 0.1739 & -0.4668 \\ -0.4668 & 0.9105 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 101.7381 & 10.6627 \\ 10.6627 & 5.5607 \end{bmatrix}$$

For the initial condition $\tilde{\xi}_0 = \tilde{\mathbf{H}}_2 \boldsymbol{\psi}_1 = \tilde{\mathbf{H}}_2$, the trajectories in time for each state can be seen in Figure 4.2. The output $\mathbf{z}_e(t)$ and control signal $\mathbf{u}(t)$ are shown in Figure 4.3. Notice that the joint action of both control inputs, $\sigma(\mathbf{y}_e(t))$ and $\mathbf{u}(t)$, were able to asymptotically stabilize the switched system. Furthermore, by numerical

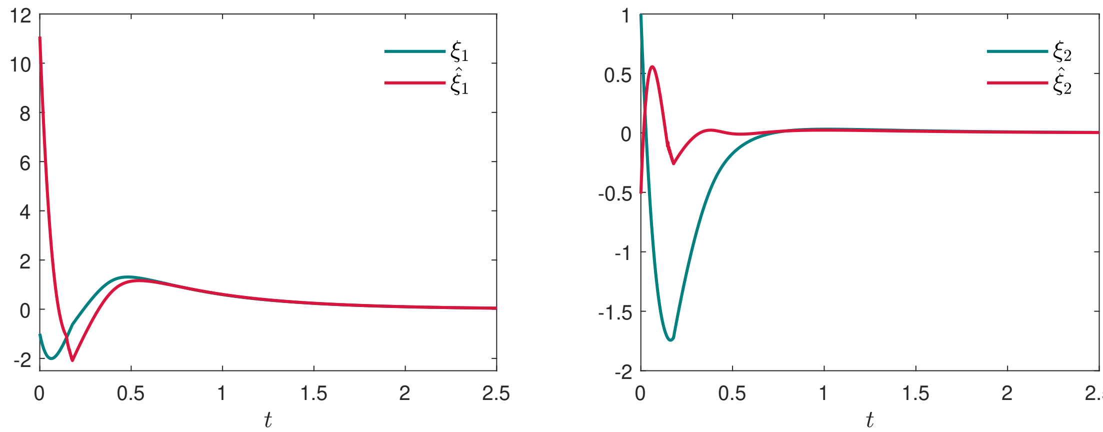


Figure 4.2: Trajectories of each state for the switched affine system under Theorem 4.1.

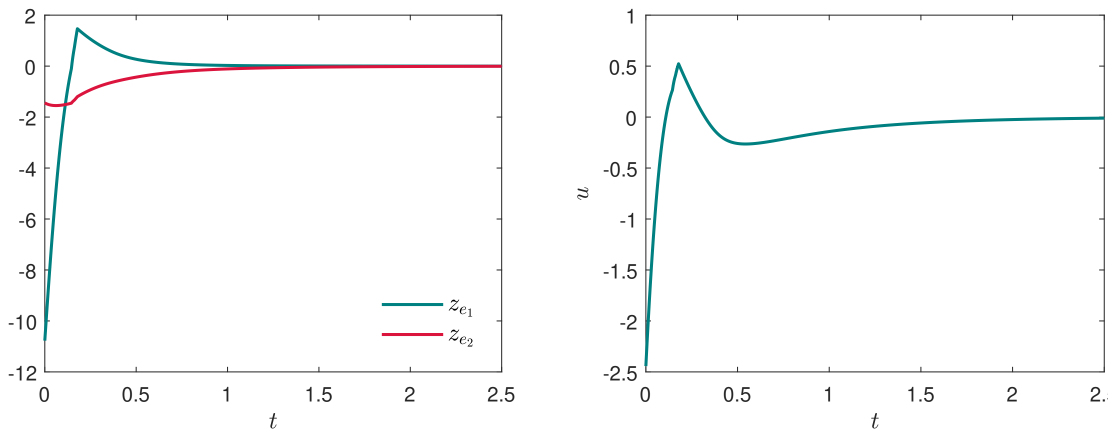


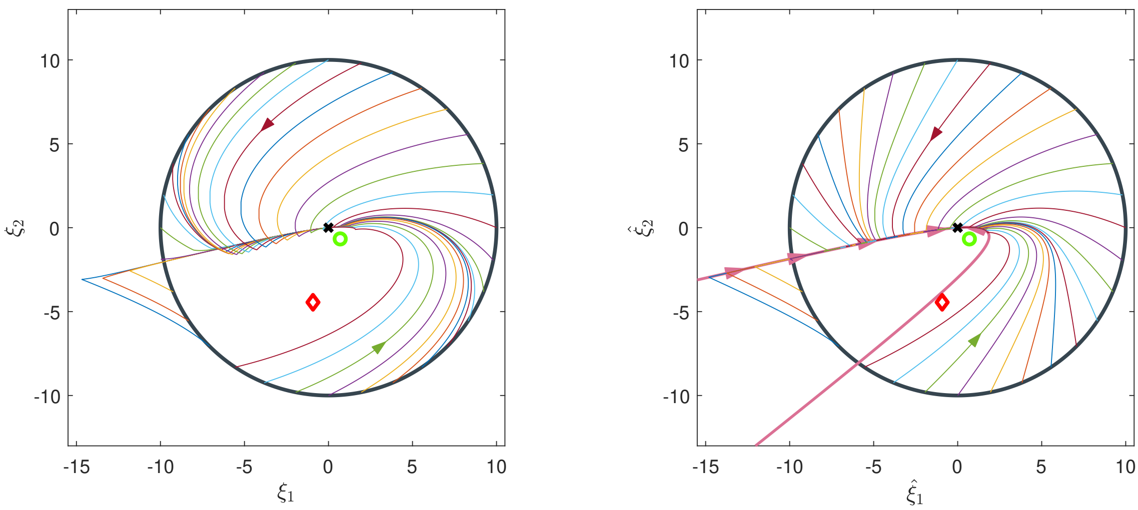
Figure 4.3: Output and control signal for the switched affine system under Theorem 4.1.

integration of the product $\mathbf{z}_e(t)^T\mathbf{z}_e(t)$, the actual $\mathcal{H}_2$ cost of the switched system was calculated as $J_2 = 5.2066 < 34.1051$, within that assured by Theorem 4.1. ■

Example 4.2

Consider again the switched affine system, as well as the controller and switching rule matrices obtained in Example 4.1. We now consider multiple initial conditions distributed around a circle denoted by the line '—' of radius 10 centered at the origin '✖', that is, $\xi_0 = \hat{\xi}_0 = 10 \times [\cos(\theta) \ \ \sin(\theta)]^T$, $\theta \in [0, 2\pi]$ and that no disturbances are being applied to the system.

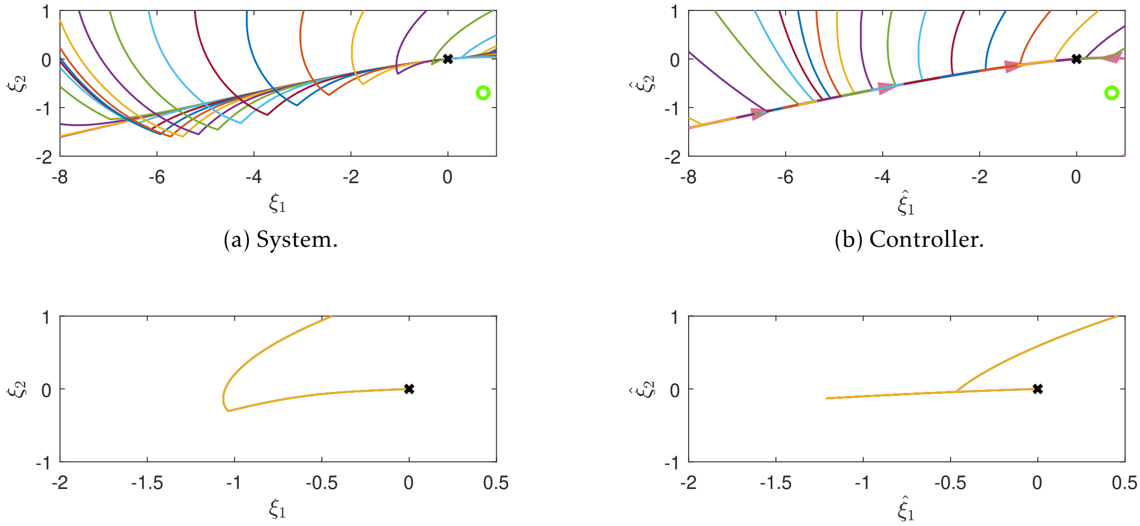
Figure 4.4 shows the phase portrait of the system and controller for this scenario. The equilibrium points of subsystems 1 and 2 are indicated by '◇' and '○', respectively. It can be observed that the switching surface, in this case a hyperbole denoted by the line '—', and given by $-\hat{\xi}^T\hat{\mathbf{Q}}_1\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_1 = -\hat{\xi}^T\hat{\mathbf{Q}}_2\hat{\xi} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_2$, distinguishes only the modes of operation of the controller, as it depends solely on the controller state $\hat{\xi}(t)$. Notice that the switched system is successfully stabilized for all initial conditions considered, even though the two control structures rely exclusively on $\hat{\xi}(t)$.



(a) System. (b) Controller.

Figure 4.4: Phase portrait for the switched affine system under Theorem 4.1.

Figures 4.5a and 4.5b show in greater detail the phase portrait of the system and controller near the equilibrium point. It is interesting to notice that, for some trajectories, when the controller state reaches the switching surface, it evolves in sliding modes in an opposite direction to the equilibrium point until, at a certain instant of time, it begins to slide towards the origin. Nevertheless, both the system state and the controller state asymptotically converge to equilibrium point $\xi = \mathbf{0}$ as $t \to \infty$. This phenomenon can be observed more clearly in Figures 4.5c and 4.5d, which display a single trajectory from the initial condition $\tilde{\xi}_0 = [9.24 \ \ 3.83 \ \ 9.24 \ \ 3.83]^T$. This behavior, although unexpected, results from the fact that the quadratic Lyapunov function is dependent on the augmented system vector, $v(\tilde{\xi}(t))$, and as a result, its time derivative depends both on the system and controller states. Thus, even in this interesting situation, $\dot{v}(\tilde{\xi}(t)) < 0$ occurs as expected, which is evidenced in



(a) System.
(b) Controller.
(c) System. Selected trajectory.
(d) Controller. Selected trajectory.

Figure 4.5: Detailed phase portrait for the switched affine system under Theorem 4.1.

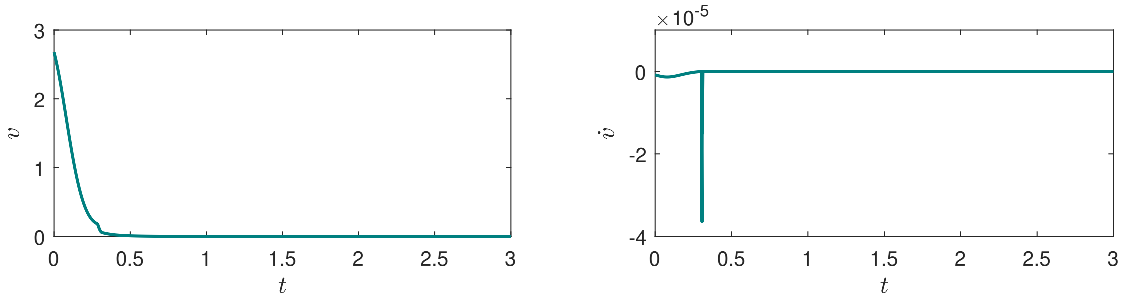


Figure 4.6: Lyapunov function and its time derivative for the switched affine system under Theorem 4.1.

Figure 4.6, for the selected trajectory, indicating that the overall switched system is asymptotically stable. ■

The third example, also available in [31], illustrates a particular case in which the proposed technique can guarantee stability of the switched system and an upper bound for the $\mathcal{H}_2$ performance index, where existing results in the literature, for the situation where the system state is unavailable for measurement, cannot.

Example 4.3

Consider the switched affine system (4.1) comprised of the following three unstable subsystems, for $i \in \{1, 2, 3\}$

$$\mathbf{A}_1 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -9 \end{bmatrix}, \quad \mathbf{b}_1 = \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix},$$

$$\mathbf{B}_i = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}, \quad \mathbf{H}_i = \mathbf{I}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad \mathbf{E}_i = \mathbf{I}, \quad \mathbf{G}_i = \mathbf{0}, \quad \mathbf{D}_i = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{F}_i = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

also available in [31]. It is important to observe that the matrix pairs $(\mathbf{A}_i, \mathbf{B}_i)$ are not individually controllable for the subsystems $i \in \{1,2,3\}$, and that there exists no $\lambda \in \Lambda_3$ such that a convex combination $\mathbf{A}_\lambda$ is Hurwitz. As such, only the joint action of a switching function operating alongside a control law, as proposed in this work, are able to successfully stabilize the switched system.

By choosing the equilibrium point $\mathbf{x}_e = [0.636 \;\; -0.517 \;\; 0.237]^T \in \mathbf{X}_e$ of interest, associated to the vector $\lambda_0 = [0.2 \;\; 0.3 \;\; 0.5]^T$, we proceed with solving the convex optimization problem (4.38). For the present example, a guaranteed cost of $J_2(\sigma, \mathcal{C}_\sigma) < 18.6502$ for $\sigma(0) = 3$ was obtained, along with the following set of matrices

$$\mathbf{Y} = \begin{bmatrix} 116.8807 & 68.5008 & 110.6348 \\ 68.5008 & 50.1472 & 79.2307 \\ 110.6348 & 79.2307 & 129.5470 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 183.9763 & 126.7359 & 115.2653 \\ 126.7359 & 88.5686 & 80.6768 \\ 115.2653 & 80.6768 & 85.1066 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} -145.2052 \\ -101.9092 \\ -93.5959 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -145.2252 \\ -101.9232 \\ -93.6103 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -145.1352 \\ -101.8601 \\ -93.5454 \end{bmatrix}^T,$$

$$\mathbf{L}_1 = \begin{bmatrix} 0.0956 \\ -1.5061 \\ -1.4760 \end{bmatrix} \times 10^3, \quad \mathbf{L}_2 = \begin{bmatrix} -3.6887 \\ -0.0868 \\ -0.1354 \end{bmatrix} \times 10^4, \quad \mathbf{L}_3 = \begin{bmatrix} -104.7484 \\ -73.1233 \\ -115.8267 \end{bmatrix}$$

important to implement the dynamical controller and the switching rule by considering the identities in (4.21).

For the initial condition $\tilde{\xi}_0 = \tilde{\mathbf{H}}_3 \psi_1 = [1 \;\; 0 \;\; 0 \;\; 0.0863 \;\; 3.0471 \;\; -1.0459]^T$, the trajectories in time for the system and controller states can be seen in Figure 4.7. The output $\mathbf{z}_e(t)$ and control signal $\mathbf{u}(t)$ are shown in Figure 4.8. Notice that the effect of the control effort is nontrivial, and plays a crucial role in stabilizing the switched system. In fact, the joint action of both control structures was needed to stabilize the switched system

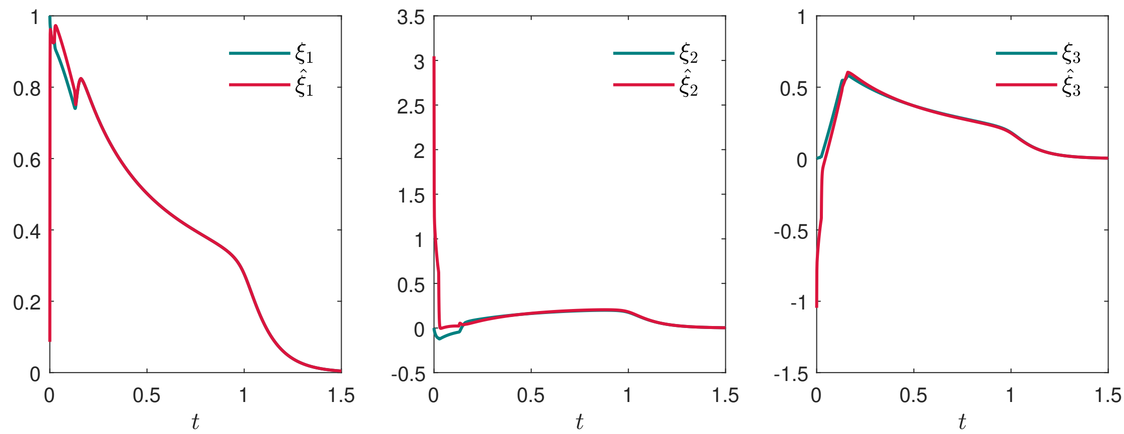


Figure 4.7: Trajectories of each state for the switched affine system under Theorem 4.1.

at the desired equilibrium point $\mathbf{x}_e$. By numerical integration of the product $\mathbf{z}_e(t)^T \mathbf{z}_e(t)$ for all three initial conditions $\tilde{\xi}_0 = \tilde{\mathbf{H}}_3 \psi_i$, for $i \in \{1,2,3\}$, we obtain the actual $\mathcal{H}_2$ cost of the system $J_2 = 2.3490 < 18.6502$.

Finally, Figure 4.9 shows the switching rule $\sigma(\mathbf{y}_e(t))$ for this example. Notice how at $t \approx 0.13$ seconds,

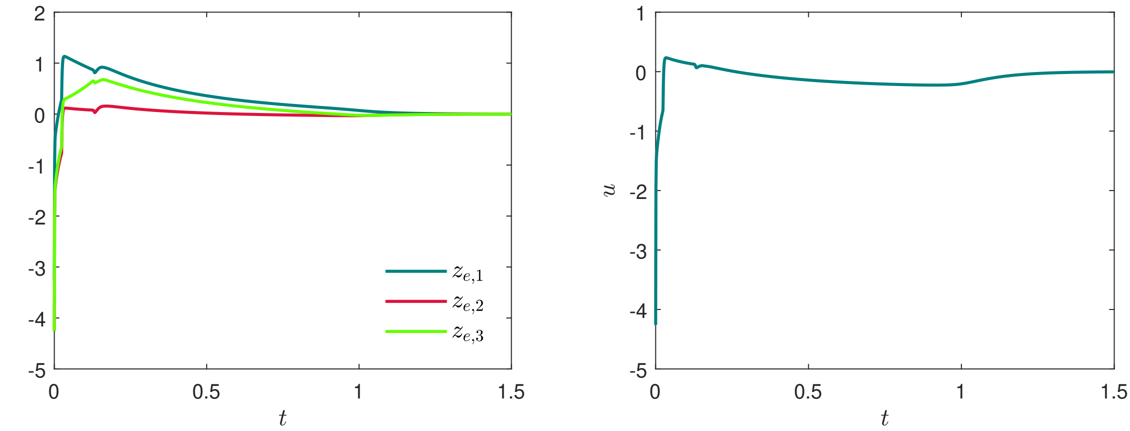


Figure 4.8: Output and control signal for the switched affine system under Theorem 4.1.

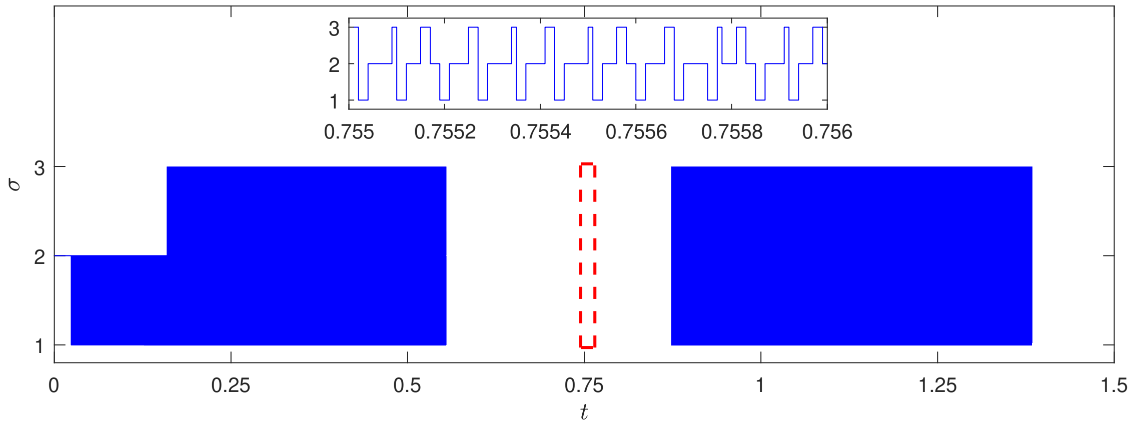


Figure 4.9: Switching rule for the switched affine system under Theorem 4.1.

the system evolves towards the equilibrium point in a sliding mode, indicated by the fast switching across the three subsystems. The zoomed-in interval of time in the figure reveals that the numerical simulation actually progresses over discrete steps of time, in reality, however, the switching events would occur arbitrarily fast. ■

The next section is dedicated the $\mathcal{H}_\infty$ control design problem, where two different approaches for the switching function are proposed.

4.5 $\mathcal{H}_\infty$ Control Design

In this section, in order to treat the simultaneous design of the two control structures proposed, namely the switching function and the full-order switched dynamical controller, so that together they may assure an $\mathcal{H}_\infty$ guaranteed cost, Theorems 3.5 and 3.6 are generalized. Let us recall that for the $\mathcal{H}_\infty$ case we are concerned with disturbances $\mathbf{w}(t) \in \mathcal{L}_2$. First, the $\mathcal{H}_\infty$ control design problem considering a switching function that relies solely on output information is introduced. This technique is of much importance, since in many occasions the disturbance is not available for measurement or it is not known beforehand. Nevertheless, in the event that the disturbance is available, a second, less conservative approach is introduced afterwards, which may provide a significantly improved $\mathcal{H}_\infty$ performance, as it will become clear in the illustrative example.

4.5.1 Output Dependent Switching Rule

The next theorem generalizes the results of Theorem 3.6, and presents LMI conditions for the proposed $\mathcal{H}_\infty$ control design technique, which assures global asymptotic stability of the desired equilibrium point as well as an upper bound for the $\mathcal{H}_\infty$ performance index, for the case where the switching rule depends only on the measured output.

Theorem 4.2. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, matrices $\mathbf{L}_i$ and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, and a scalar ρ, such that*

$$\mathbf{R}_{\lambda_0} \geq 0 \tag{4.39}$$

$$\begin{bmatrix} \mathbf{X} & \bullet \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.40}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.41}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.42}$$

then the following switching rule

$$\sigma(\hat{\xi}) = \arg \min_{i \in \mathbb{K}} -\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi} + 2\hat{\xi}^T \mathbf{X}^{-1} \boldsymbol{\ell}_i \tag{4.43}$$

along with controller (4.3), whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\left(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1} + \boldsymbol{\mathcal{M}}_i(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i)^{-1}\boldsymbol{\mathcal{N}}_i\right), \\ \hat{\mathbf{B}}_i &= -(\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{C}}_i = \mathbf{W}_i\mathbf{X}^{-1}, \\ \hat{\boldsymbol{\ell}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \qquad \hat{\mathbf{Q}}_i = \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \end{aligned} \tag{4.44}$$

with $\boldsymbol{\mathcal{M}}_i = \mathbf{Y}\mathbf{H}_i + \mathbf{L}_i\mathbf{D}_i + \mathbf{E}_i^T\mathbf{G}_i$ and $\boldsymbol{\mathcal{N}}_i = \mathbf{H}_i^T\mathbf{X}^{-1} + \mathbf{G}_i^T\mathbf{E}_i + \mathbf{G}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}$, make the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assure the $\mathcal{H}_\infty$ guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.45}$$

for the system.

Proof. Similarly to Theorem 4.1, this proof consists in demonstrating the validity of Theorem 3.6 whenever the conditions of Theorem 4.2 are verified. For this, consider matrices $\tilde{\mathbf{P}}$, $\tilde{\mathbf{P}}^{-1}$, $\tilde{\mathbf{Q}}_i$, and vector $\hat{\boldsymbol{\ell}}_i$ previously defined, and inequalities (3.63) and (3.64) of Theorem 3.6, applied to the augmented system (4.4)

$$\tilde{\mathbf{Q}}_{\lambda_0} \succeq 0 \tag{4.46}$$

$$\begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \bullet & \bullet \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} & -\rho\mathbf{I} & \bullet \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.47}$$

In order to linearize inequality (4.47), we again consider the transformation matrix $\tilde{\boldsymbol{\Gamma}}$ in (4.25), and multiply to the left of (4.47) by $\mathrm{diag}(\tilde{\boldsymbol{\Gamma}}^T, \mathbf{I}, \mathbf{I})$, and to the right by its transpose, as follows

$$\begin{bmatrix} \tilde{\boldsymbol{\Gamma}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \tilde{\mathbf{P}}\tilde{\mathbf{H}}_i^T & \tilde{\mathbf{E}}_i^T \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} & -\rho\mathbf{I} & \tilde{\mathbf{G}}_i^T \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Gamma}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.48}$$

whose intermediary products were determined in the proof of Theorem 4.1, and are given in (4.27), and (4.35). By adopting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i\mathbf{U}^T$, inequality (4.48) can be expressed as

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T & \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0 \tag{4.49}$$

Applying Schur complement successively with respect to $-\mathbf{I}$, followed by $(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i)$, the following inequality is obtained

$$\begin{bmatrix} \boldsymbol{\Xi}_i & \bullet \\ \boldsymbol{\Omega}_i & \boldsymbol{\Upsilon}_i \end{bmatrix} \prec 0 \tag{4.50}$$

where the intermediary terms are given by

$$\begin{aligned} \boldsymbol{\Xi}_i = {} & \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.51}$$

$$\boldsymbol{\Upsilon}_i = \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{4.52}$$

$$\begin{aligned} \boldsymbol{\Omega}_i = {} & \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ & + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T \left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i\right)^{-1} \left(\mathbf{H}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right) \end{aligned} \tag{4.53}$$

By choosing $\hat{\mathbf{A}}_i$ so as to make $\boldsymbol{\Omega}_i = \mathbf{0}$, and by applying Schur complement appropriately on $\boldsymbol{\Xi}_i$ and $\boldsymbol{\Upsilon}_i$, we arrive at the inequalities (4.41) and (4.42), respectively. As such, whenever these inequalities are verified, that is $\boldsymbol{\Xi}_i < 0$ and $\boldsymbol{\Upsilon}_i < 0$ hold, then (4.49) is satisfied, in turn making condition (4.47) of Theorem 3.6 for the augmented system (4.4) valid.

Also, inequality (4.40) assures $\tilde{\mathbf{P}} > 0$, which can be verified by applying the transformation matrix as such $\tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} > 0$. In addition, we have that inequality $\mathbf{R}_{\lambda_0} \geq 0$ assures $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, verifying condition (4.46) of Theorem 3.6 for the augmented system. Finally, by assigning $\mathbf{U} = \mathbf{X}$ without loss of generality, as discussed in Theorem 4.1, the identities in (4.44) are obtained, and the switching rule (4.43) comes from (4.15). Thus, as in Theorem 3.6, the guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.54}$$

for the augmented system holds, concluding the proof. □

Theorem 4.2 generalizes the results of [16] to cope with the joint design of the two proposed control structures. The same remarks as for the $\mathcal{H}_2$ remain valid, where the requirement for a stable convex combination of matrices $\mathbf{A}_\lambda$ is eschewed, as well as the need for the pairs $(\mathbf{A}_i, \mathbf{B}_i)$, $\forall i \in \mathbb{K}$ of the subsystems to be controllable.

The matrices required to implement the dynamical controller and switching rule proposed in Theorem 4.2, can be calculated numerically via the following convex optimization problem, subject to LMI constraints

$$\begin{aligned} \min \quad & \rho \\ \text{subject to:} \quad & (4.39),\ (4.40),\ (4.41),\ \text{and}\ (4.42) \end{aligned} \tag{4.55}$$

It is important to mention that optimization problems subject to LMI constraints dealing with the minimization of $\mathcal{H}_\infty$ performance may produce ill-conditioned solutions, as discussed in [55]. To overcome this problem, it is proposed that a fixed, suboptimal $\rho > 0$ be supplied, and the objective function $\mathrm{tr}\left(\left(\mathbf{Y} - \mathbf{X}^{-1}\right)^{-1}\right)$ be minimized, as this term multiplies matrices $\hat{\mathbf{A}}_i$, for $i \in \mathbb{K}$. Since the proposed objective function is not linear on the decision variables, consider the following LMI constraint

$$\begin{bmatrix} \boldsymbol{\mathcal{S}} & \bullet & \bullet \\ \mathbf{I} & \mathbf{Y} & \bullet \\ \mathbf{0} & \mathbf{I} & \mathbf{X} \end{bmatrix} > 0 \tag{4.56}$$

Notice that, by applying Schur complement successively with respect to $\mathbf{X}$, and then with respect to $\left(\mathbf{Y} - \mathbf{X}^{-1}\right)$, we obtain $\left(\mathbf{Y} - \mathbf{X}^{-1}\right)^{-1} < \boldsymbol{\mathcal{S}}$. Thus, for a given $\rho > 0$, the following optimization problem

$$\begin{aligned} \min \quad & \mathrm{tr}(\boldsymbol{\mathcal{S}}) \\ \text{subject to:} \quad & (4.39),\ (4.41),\ (4.42),\ \text{and}\ (4.56) \end{aligned} \tag{4.57}$$

yields controller matrices with greater numerical stability overall.

The next section deals with the $\mathcal{H}_\infty$ output feedback control design problem based on less conservative conditions, which consider that the external input $\mathbf{w}(t)$ is either known, or can be measured or estimated.

4.5.2 Output-Input Dependent Switching Rule

The next theorem generalizes the conditions of Theorem 3.5 to deal with the two control structures proposed, whit together assure global asymptotic stability of the equilibrium point, and an $\mathcal{H}_\infty$ guaranteed cost, for the case where the switching rule depends not only on the measured output, but also on the disturbance.

Theorem 4.3. *Consider the switched affine system (4.1), and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ of interest with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}_i$, and $\mathbf{Z}_i$, matrices $\mathbf{J}_i$, $\mathbf{L}_i$, and $\mathbf{W}_i$, for all $i \in \mathbb{K}$, and a scalar $\rho > 0$, such that*

$$\begin{bmatrix} \mathbf{R}_{\lambda_0} & \bullet \\ \mathbf{J}_{\lambda_0}^T & \mathbf{Z}_{\lambda_0} \end{bmatrix} \geq 0 \tag{4.58}$$

$$\begin{bmatrix} \mathbf{X} & \bullet \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{4.59}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T + \mathbf{J}_i^T & -\rho\mathbf{I} + \mathbf{Z}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.60}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} + \mathbf{Z}_i & \bullet \\ \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{4.61}$$

then the following switching rule

$$\sigma(\hat{\xi}, \mathbf{w}) = \arg\min_{i \in \mathbb{K}} -\begin{bmatrix} \hat{\xi} \\ \mathbf{w} \end{bmatrix}^T \mathcal{Q}_i \begin{bmatrix} \hat{\xi} \\ \mathbf{w} \end{bmatrix} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i \tag{4.62}$$

with

$$\mathcal{Q}_i = \begin{bmatrix} \hat{\mathbf{Q}}_i & \bullet \\ \mathbf{J}_i^T\mathbf{X}^{-1} & \mathbf{Z}_i \end{bmatrix} \tag{4.63}$$

along with controller (4.3), whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\left(\mathbf{Y}\mathbf{A}_i + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathbf{L}_i\mathbf{C}_i + \mathbf{A}_i^T\mathbf{X}^{-1} + \mathbf{E}_i^T\mathbf{E}_i + \mathbf{E}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1} + \mathcal{M}_i(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i - \mathbf{Z}_i)^{-1}\mathcal{N}_i\right), \\ \hat{\mathbf{B}}_i &= -(\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{C}}_i = \mathbf{W}_i\mathbf{X}^{-1}, \\ \hat{\boldsymbol{\ell}}_i &= (\mathbf{Y} - \mathbf{X}^{-1})^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \qquad \hat{\mathbf{Q}}_i = \mathbf{X}^{-1}\mathbf{R}_i\mathbf{X}^{-1} \end{aligned} \tag{4.64}$$

with $\mathcal{M}_i = \mathbf{Y}\mathbf{H}_i + \mathbf{L}_i\mathbf{D}_i + \mathbf{E}_i^T\mathbf{G}_i$ and $\mathcal{N}_i = \mathbf{H}_i^T\mathbf{X}^{-1} + \mathbf{J}_i^T\mathbf{X}^{-1} + \mathbf{G}_i^T\mathbf{E}_i + \mathbf{G}_i^T\mathbf{F}_i\mathbf{W}_i\mathbf{X}^{-1}$, make the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ globally asymptotically stable, and assure the $\mathcal{H}_\infty$ guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.65}$$

for the system.

Proof. The proof is based on demonstrating that Theorem 3.5 is valid when the conditions of Theorem 4.3 are satisfied for the augmented system (4.4), considering matrices $\tilde{\mathbf{P}}$, $\tilde{\mathbf{P}}^{-1}$, and vector $\hat{\boldsymbol{\ell}}_i$ previously defined. But first, we define the following structured matrix

$$\tilde{\mathcal{O}}_i = \begin{bmatrix} \tilde{\mathbf{Q}}_i & \tilde{\bar{\mathbf{Q}}}_i \\ \tilde{\bar{\mathbf{Q}}}_i^T & \mathbf{Z}_i \end{bmatrix} = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_i & \bar{\mathbf{Q}}_i \\ \hline \mathbf{0} & \bar{\mathbf{Q}}_i^T & \mathbf{Z}_i \end{array}\right], \quad \forall i \in \mathbb{K} \tag{4.66}$$

where $\hat{\mathbf{Q}}_i \in \mathbb{R}^{n_x \times n_x}$, $\bar{\mathbf{Q}}_i \in \mathbb{R}^{n_x \times n_w}$, and $\mathbf{Z}_i \in \mathbb{R}^{n_w \times n_w}$. It will become clear that this specific choice for the structure of $\tilde{\mathcal{O}}_i$ is important to eliminate the dependency on the unknown system state $\xi(t)$ by the switching rule (3.48) of Theorem 3.5, which when expressed in terms of the augmented system, is as follows

$$\sigma(\tilde{\xi}, \mathbf{w}) = \arg \min_{i \in \mathbb{K}} \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix}^T \tilde{\mathcal{L}}_i(\rho, \tilde{\mathbf{P}}) \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix} + 2\tilde{\xi}^T \tilde{\mathbf{P}} \tilde{\boldsymbol{\ell}}_i \tag{4.67}$$

Observe that condition (3.46) of Theorem 3.5 for the augmented system is equivalent to the following set of inequalities

$$\tilde{\mathcal{L}}_i(\rho, \tilde{\mathbf{P}}) + \tilde{\mathcal{O}}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{4.68}$$

together with $\tilde{\mathcal{O}}_{\lambda_0} \geq 0$, where

$$\tilde{\mathcal{L}}_i(\rho, \tilde{\mathbf{P}}) = \begin{bmatrix} \tilde{\mathbf{A}}_i^T \tilde{\mathbf{P}} + \tilde{\mathbf{P}} \tilde{\mathbf{A}}_i + \tilde{\mathbf{E}}_i^T \tilde{\mathbf{E}}_i & \bullet \\ \tilde{\mathbf{H}}_i^T \tilde{\mathbf{P}} + \tilde{\mathbf{G}}_i^T \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i^T \tilde{\mathbf{G}}_i - \rho \mathbf{I} \end{bmatrix}, \quad i \in \mathbb{K} \tag{4.69}$$

By applying Schur complement in (4.68) with respect to $\left(\tilde{\mathbf{G}}_i^T \tilde{\mathbf{G}}_i - \rho \mathbf{I}\right)$, the equivalent inequality

$$\begin{bmatrix} \tilde{\mathbf{A}}_i^T \tilde{\mathbf{P}} + \tilde{\mathbf{P}} \tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \bullet & \bullet \\ \tilde{\mathbf{H}}_i^T \tilde{\mathbf{P}} + \tilde{\bar{\mathbf{Q}}}_i^T & -\rho \mathbf{I} + \mathbf{Z}_i & \bullet \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -I \end{bmatrix} \prec 0 \tag{4.70}$$

is obtained. Multiplying to the left of this inequality by $\mathrm{diag}(\tilde{\mathbf{\Gamma}}^T, \mathbf{I}, \mathbf{I})$, and to the right by its transpose, in a similar manner to (4.48), then by considering the intermediary products given by (4.27), and (4.35), and further denoting $\mathbf{R}_i = \mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T$, $\tilde{\bar{\mathbf{Q}}}_i^T \tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{J}_i^T & \mathbf{0} \end{bmatrix}$, with $\mathbf{J}_i^T = \bar{\mathbf{Q}}_i^T \mathbf{U}^T$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{W}_i = \hat{\mathbf{C}}_i \mathbf{U}^T$, the following inequality emerges

$$\begin{bmatrix} \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i & \bullet & \bullet & \bullet \\ \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T + \mathbf{J}_i^T & \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} + \mathbf{Z}_i & \bullet \\ \mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i & \mathbf{E}_i & \mathbf{G}_i & -\mathbf{I} \end{bmatrix} \prec 0 \tag{4.71}$$

In a similar fashion to Theorem 4.2, we apply Schur complement successively with respect to $-\mathbf{I}$ followed by $\left(\tilde{\mathbf{G}}_i^T\tilde{\mathbf{G}}_i - \rho\mathbf{I} + \mathbf{Z}_i\right)$, thus obtaining

$$\begin{bmatrix} \Xi_i & \bullet \\ \boldsymbol{\Omega}_i & \boldsymbol{\Upsilon}_i \end{bmatrix} \prec 0 \tag{4.72}$$

where the intermediary terms are given by

$$\begin{aligned}\Xi_i = \mathrm{He}\{\mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{W}_i\} + \mathbf{R}_i + (\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ + \left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)\end{aligned} \tag{4.73}$$

$$\boldsymbol{\Upsilon}_i = \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right) \tag{4.74}$$

$$\begin{aligned}\boldsymbol{\Omega}_i = \mathbf{Y}\mathbf{A}_i\mathbf{X} + \mathbf{Y}\mathbf{B}_i\mathbf{W}_i + \mathbf{L}_i\mathbf{C}_i\mathbf{X} + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T + \mathbf{A}_i^T + \mathbf{E}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i) + \\ + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + \mathbf{G}_i^T\mathbf{E}_i\right)^T\left(\rho\mathbf{I} - \mathbf{G}_i^T\mathbf{G}_i + \mathbf{Z}_i\right)^{-1}\left(\mathbf{H}_i^T + \mathbf{J}_i^T + \mathbf{G}_i^T(\mathbf{E}_i\mathbf{X} + \mathbf{F}_i\mathbf{W}_i)\right)\end{aligned} \tag{4.75}$$

By making the adequate choice of $\hat{\mathbf{A}}_i$ in order to make $\boldsymbol{\Omega}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, and by applying Schur complement as appropriate on Ξ_i and $\boldsymbol{\Upsilon}_i$, the inequalities (4.60) and (4.61) are obtained, respectively. In this manner, whenever these inequalities are satisfied, that is, when $\Xi_i \prec 0$ and $\boldsymbol{\Upsilon}_i \prec 0$ hold, we have that inequality (4.71) is verified, thus making condition (4.68) valid. This, together with (4.58), which guarantees that $\tilde{\mathcal{O}}_{\lambda_0} \geq 0$ is verified, satisfy condition (3.46) of Theorem 3.5 for the augmented system. Once again, inequality (4.40) assures $\tilde{\mathbf{P}} \succ 0$.

Finally, by making the particular choice $\mathbf{U} = \mathbf{X}$, the identities in (4.64) are obtained, and the switching rule (4.62) with (4.63) comes from (4.67) since

$$\begin{aligned}\sigma(\tilde{\xi}, \mathbf{w}) &= \arg\min_{i \in \mathbb{K}} \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix}^T \tilde{\mathcal{L}}_i(\rho, \tilde{\mathbf{P}}) \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \\ &\equiv \arg\min_{i \in \mathbb{K}} - \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix}^T \tilde{\mathcal{O}}_i \begin{bmatrix} \tilde{\xi} \\ \mathbf{w} \end{bmatrix} + 2\tilde{\xi}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \\ &= \arg\min_{i \in \mathbb{K}} - \begin{bmatrix} \hat{\xi} \\ \mathbf{w} \end{bmatrix}^T \mathcal{Q}_i \begin{bmatrix} \hat{\xi} \\ \mathbf{w} \end{bmatrix} + 2\hat{\xi}^T\mathbf{X}^{-1}\boldsymbol{\ell}_i\end{aligned} \tag{4.76}$$

when considering that $\hat{\mathbf{Q}}_i = \mathbf{U}^{-1}\mathbf{R}_i(\mathbf{U}^T)^{-1}$, $\mathbf{J}_i^T = \bar{\mathbf{Q}}_i^T\mathbf{U}^T$, and the choice $\mathbf{U} = \mathbf{X}$, and also recalling that the previously determined $\hat{\boldsymbol{\ell}}_i$ allows for (4.14) which removes any dependency on the system state. In this manner, as in Theorem 3.5, the guaranteed cost

$$J_\infty(\sigma, \mathcal{C}_\sigma) < \rho \tag{4.77}$$

is assured for the augmented system. This concludes the proof. □

This theorem again generalizes the results from [29] in order to deal with the simultaneous design of the two control structures proposed, but considering a switching function dependent also on the external input. As for the previous theorems, Theorem 3.5 does not impose the existence of a stable convex combination of matrices $\mathbf{A}_\lambda$, and the need for the pairs $(\mathbf{A}_i, \mathbf{B}_i)$, $\forall i \in \mathbb{K}$ to be individually controllable, for each subsystem. As such, the proposed conditions for the $\mathcal{H}_\infty$ control design problem assure global asymptotic stability of the equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ of interest even when a control law $\mathbf{u}(t)$ and a switching rule $\sigma(t)$ are ineffective when acting independently. As for the $\mathcal{H}_2$ case, this particular situation is demonstrated in a numerical example.

The solution to the following optimization problem provides the necessary matrices for implementing the dynamical controller and the switching rule proposed in Theorem 4.3, considering the identities in (4.64)

$$\begin{aligned} \min \quad & \rho \\ \text{s. to:} \quad & (4.58),\ (4.59),\ (4.60),\ (4.61),\ \text{and } \rho > 0 \end{aligned} \tag{4.78}$$

Additionally, as discussed in Theorem 4.2, in order to avoid ill-conditioned matrix solutions, by providing a fixed, suboptimal $\rho > 0$, the following optimization problem

$$\begin{aligned} \min \quad & \operatorname{tr}(\mathcal{S}) \\ \text{s. to:} \quad & (4.56),\ (4.58),\ (4.60),\ (4.61),\ \text{and } \rho > 0 \end{aligned} \tag{4.79}$$

results in matrices with greater numerical stability for the dynamical controller.

In the next section, we present two examples on $\mathcal{H}_\infty$ control design to illustrate the two proposed switching rules. Again, these examples are based on [31], and will serve to compare these results.

4.5.3 Example: Output Dependent Switching Rule

This example, also based on that of reference [31], illustrates the $\mathcal{H}_\infty$ control design problem for the case where only the measured output is available. It is important to recall that in this example, only the joint action of the switching function together with a control effort, provided by the switched dynamical controller, is capable of successfully stabilizing the switched system.

Example 4.4

Consider once again the switched affine system of Example 4.3. In order to implement the switching strategy in (4.43) and the affine controller of Theorem 4.2, the matrices $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{W}_i$, and $\mathbf{L}_i$ are obtained as follows

$$\mathbf{Y} = \begin{bmatrix} 5.8486 & 0.0509 & 1.0488 \\ 0.0509 & 6.5174 & 10.0155 \\ 1.0488 & 10.0155 & 18.2095 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 244.9140 & 169.4628 & 159.3359 \\ 169.4628 & 118.5251 & 111.5770 \\ 159.3359 & 111.5770 & 116.9782 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} -194.5703 \\ -136.5212 \\ -129.2966 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -194.5709 \\ -136.5216 \\ -129.2970 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -194.5700 \\ -136.5210 \\ -129.2964 \end{bmatrix}^T,$$

$$\mathbf{L}_1 = \begin{bmatrix} -2.3160 \\ -61.1186 \\ -65.3486 \end{bmatrix}, \qquad \mathbf{L}_2 = \begin{bmatrix} -57.9068 \\ -5.5279 \\ -9.7579 \end{bmatrix}, \qquad \mathbf{L}_3 = \begin{bmatrix} -2.3158 \\ -561.3437 \\ -9.7562 \end{bmatrix}$$

by solving the convex optimization problem in (4.55) followed by (4.57). Implementing the dynamical controller $\mathcal{C}_\sigma$ and the switching rule, by means of the identities in (4.44), Theorem 4.2 assures asymptotic stability of the equilibrium point $\boldsymbol{\xi} = \mathbf{0}$, as well as the upper bound $J_\infty(\sigma, \mathcal{C}_\sigma) < 166.7720$ for the $\mathcal{H}_\infty$ performance index.

By considering the following disturbance

$$\mathbf{w}(t) = \begin{cases} [0 \ \ 5 \ \ 0]^T, & 0.5 \le t \le 1.5 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

the trajectories in time for the system and controller states can be seen in Figure 4.10. The performance output $\mathbf{z}_e(t)$ and the control effort $\mathbf{u}(t)$, as produced by the controller, are shown in Figure 4.11. Also, observe in Figure 4.12 the behavior of the switching rule. Notice how the control effort and the switching rule exhibit a complex

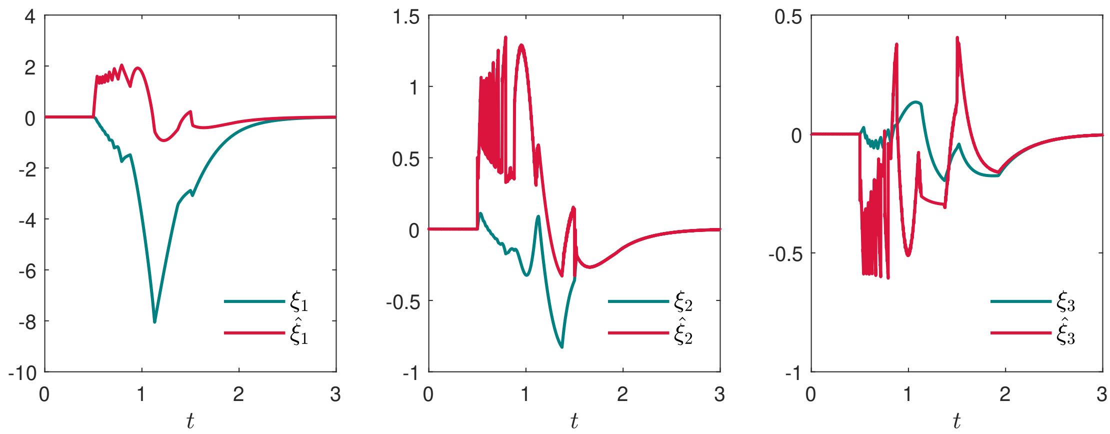


Figure 4.10: Trajectories of each state for the switched affine system under Theorem 4.2.

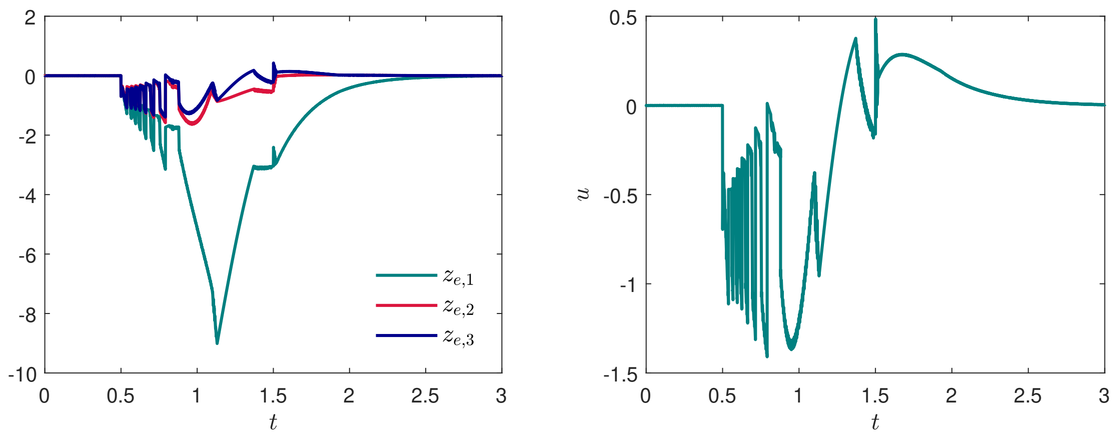


Figure 4.11: Output and control signal for the switched affine system under Theorem 4.2.

behavior while the disturbance is being applied, and how their joint action is able to asymptotically stabilize

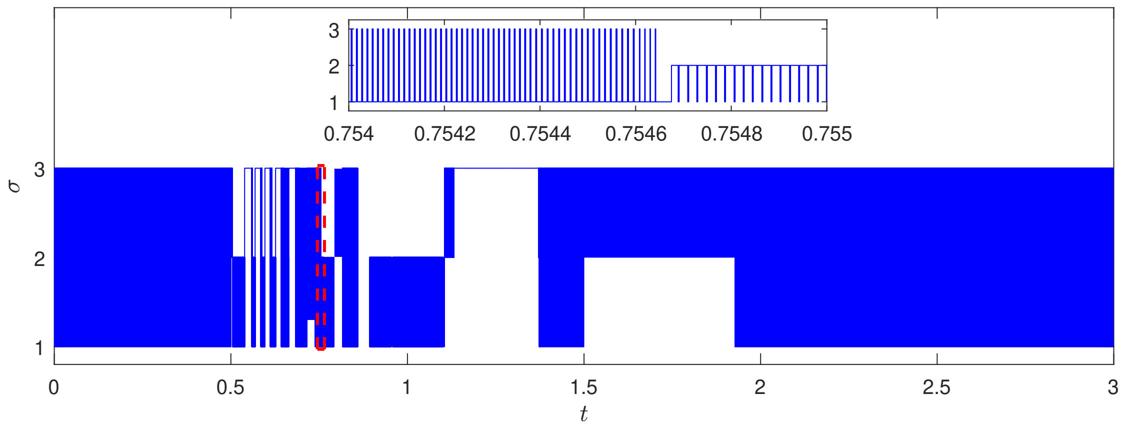


Figure 4.12: Switching rule for the switched affine system under Theorem 4.2.

the switched system to the equilibrium point as the disturbance ceases. ■

The next example compares the results of Example 4.4, obtained for Theorem 4.2, to those of Theorem 4.3, which are based on less conservative conditions.

4.5.4 Example: Output-Input Dependent Switching Rule

This example considers the switched system of Example 4.4, now assuming that the disturbance $\mathbf{w}(t)$ is known, and serves as a comparison between the two switching rules introduced for the $\mathcal{H}_\infty$ control design problem.

Example 4.5

Constructing upon Example 4.4, the following matrices are calculated by solving the optimization problem in (4.78) followed by (4.79), in order to implement the switching rule (4.62) and the controller $\mathcal{C}_\sigma$ of Theorem 4.3, by considering the identities in (4.64)

$$\mathbf{Y} = \begin{bmatrix} 0.4120 & -0.2157 & 0.6139 \\ -0.2157 & 6.5200 & 10.0130 \\ 0.6139 & 10.0130 & 18.2039 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 246.2547 & 170.3845 & 160.2929 \\ 170.3845 & 119.1525 & 112.2339 \\ 160.2929 & 112.2339 & 117.6591 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} -195.6495 \\ -137.2608 \\ -130.0659 \end{bmatrix}^T, \quad \mathbf{W}_2 = \begin{bmatrix} -195.6426 \\ -137.2560 \\ -130.0610 \end{bmatrix}^T, \quad \mathbf{W}_3 = \begin{bmatrix} -195.6394 \\ -137.2537 \\ -130.0587 \end{bmatrix}^T,$$

$$\mathbf{L}_1 = \begin{bmatrix} -0.2434 \\ -61.3656 \\ -65.5285 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -5.0956 \\ -1.6023 \\ -2.9202 \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} -0.4222 \\ -11.1887 \\ -17.6459 \end{bmatrix}$$

For this example, the guaranteed cost $J_\infty(\sigma, \mathcal{C}_\sigma) < 38.6818$ is assured, a cost 76.8% smaller than that assured by Theorem 4.2, an expected result due to the reduced conservativeness of its conditions. As stated, in this example we assume that the disturbance is known, thus allowing the switching rule (4.62) to be implemented. Adopting the same disturbance of Example 4.4, the trajectories in time for each state of the system and controller can be seen in Figure 4.13. The performance output as well as the control effort can be observed in Figure 4.14. The

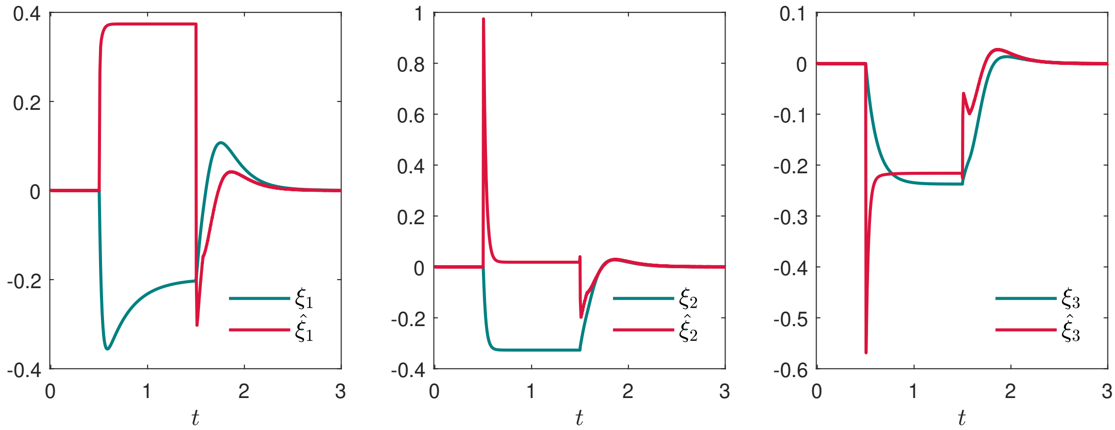


Figure 4.13: Trajectories of each state for the switched affine system under Theorem 4.3.

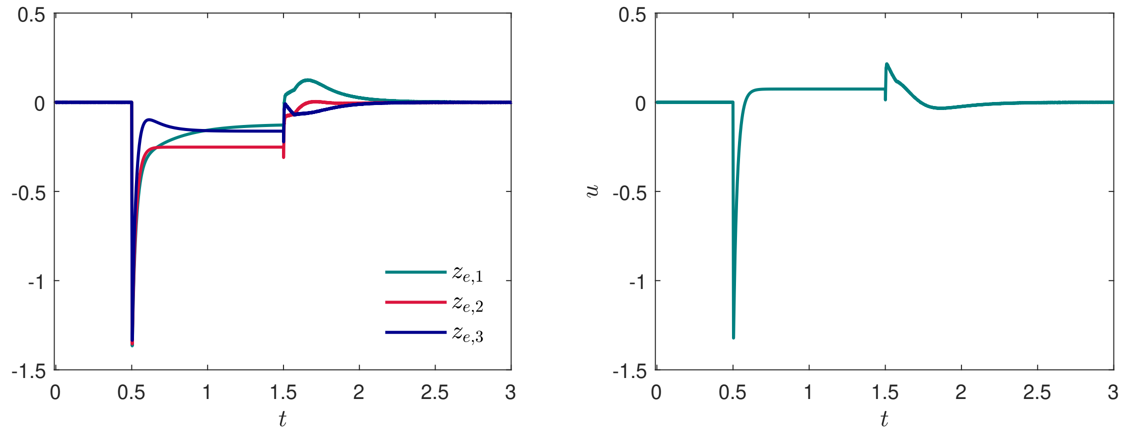


Figure 4.14: Output and control signal for the switched affine system under Theorem 4.3.

switching rule of Theorem 4.3 for this example can be seen on Figure 4.15. Notice the stark difference between

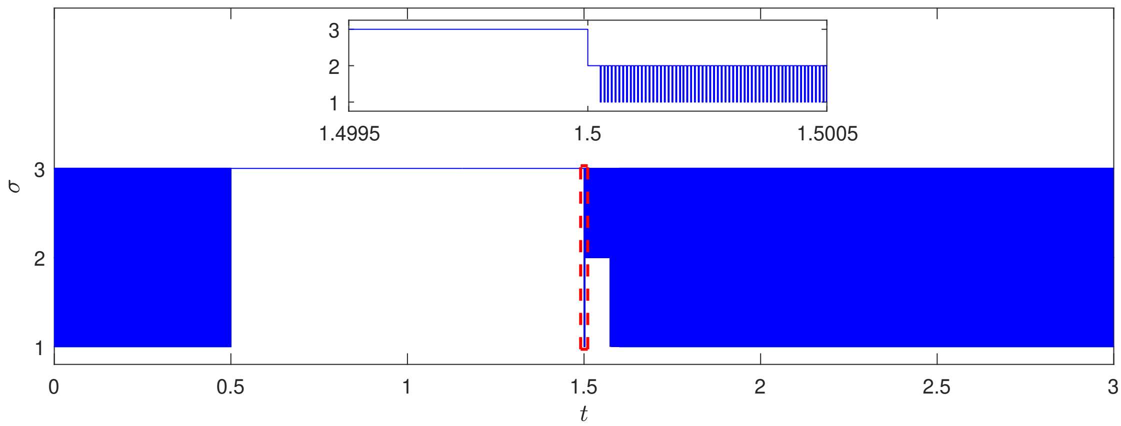


Figure 4.15: Switching rule for the switched affine system under Theorem 4.3.

the behaviors of the switching rules and control efforts of Theorems 4.2 and 4.3. ■

4.6 Concluding Remarks

In this chapter we tackled the output feedback control design problem for continuous-time switched affine systems by considering the joint action of a control law $\mathbf{u}(t)$ and a switching function $\sigma(\mathbf{y}_e(t))$, implemented by means of a switched affine controller. The conditions obtained are expressed in terms of LMIs, and assure the global asymptotic stability of a desired equilibrium point of the switched subsystem, as well as an upper bound for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. The control design methodologies presented generalize existing results in the literature, thus providing less conservative conditions which allow for a wider scope of problems to be considered.

Five examples illustrated the effectiveness of the proposed techniques, and demonstrated the particular case where only the joint action of both control structures is capable of stabilizing the switched system.

Chapter 5

Filtering and Estimation

Several results that deal with state and output feedback control design of a stabilizing switching function already exist in the literature, as discussed thoroughly in the previous chapters. However, less attention has been given to filtering problems for switched systems, while considering the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices. This chapter aims to address the classical filtering problem for continuous-time switched affine systems, considering the joint design of a full-order switched affine filter and an output dependent stabilizing switching rule, together assuring an upper bound for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs for the estimation error. To the best of the author's knowledge, the classical filtering problem in the context of switched affine systems has not been treated in the literature as of yet, except for the following publications [56, 57] which are the basis for this chapter.

5.1 Introduction

Given a dynamical system, the classical filtering problem consists in determining some information of interest from the measurement of a certain output of this system, which may be corrupted by process noise. This information is often internal to the system, and either cannot be directly measured or can only be measured with some degree of uncertainty. The concept of filtering is thus of great importance for practical applications due to the common occurrence of disturbances in measurements, for instance those arising from a noisy sensor, as well as the difficulty in measuring full state information from physical systems. A common example is that of a sensorless DC motor speed controller, where the speed of the DC motor is not available, and must be inferred from the measured armature current and voltage, which may be corrupted by noise. In this case, an appropriate filter can be applied to estimate the DC motor speed from the available measurements, allowing a closed-loop control system to be implemented.

This chapter introduces a methodology for the design of dynamical filters, more specifically a full-order continuous-time switched affine filter is considered, with optimal $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed costs for the estimation error. In this work, the estimation error is defined as the difference between the unavailable signal of interest and the estimated signal, output by the filter. The design of an output dependent stabilizing switching function is considered alongside the switched affine filter. In the literature thus far, references [32], [33], and [34] have considered the design of switched filters for continuous-time switched linear systems, limited to the scope of time-dependent switching functions, considering either $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed costs. However, few references treat the joint design of a switching function along with the switched filter, see [35] as an example. In the mentioned references, only linear systems have been considered. For the more general case of switched affine systems, considered in this work, even fewer references have explored this topic, see [36] and [37]. These, however, consider only the simpler observer-based structure.

The proposed techniques do not require that the individual subsystem matrices $\mathbf{A}_i$, for $i \in \mathbb{K}$, exhibit any stability properties. As is recurrent in the literature, only the convex combination of these matrices must be Hurwitz $\mathbf{A}_\lambda \in \mathcal{H}$, for some vector $\lambda \in \Lambda_N$. Moreover, it is proved that the minimum guaranteed cost filters actually present an observer-based structure and can be designed independently of the switching function, indicating that the well-known separation principle is valid.

5.2 Problem Statement

Consider the following state space representation of a switched affine system, already provided in its shifted arrangement

$$
\begin{aligned}
\dot{\xi}(t) &= \mathbf{A}_\sigma \xi(t) + \mathbf{H}_\sigma \mathbf{w}(t) + \boldsymbol{\ell}_\sigma, \quad \xi(0) = \mathbf{0} \\
\mathbf{y}_e(t) &= \mathbf{C}_\sigma \xi(t) + \mathbf{D}_\sigma \mathbf{w}(t) \\
\mathbf{z}_e(t) &= \mathbf{E}_\sigma \xi(t) + \mathbf{G}_\sigma \mathbf{w}(t)
\end{aligned}
\tag{5.1}
$$

where $\xi(t) \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is an external disturbance, or process noise, $\mathbf{y}_e(t) \in \mathbb{R}^{n_y}$ is the measured output, $\mathbf{z}_e(t) \in \mathbb{R}^{n_z}$ is the performance output, or the unavailable signal of interest, and $\boldsymbol{\ell}_i = \mathbf{A}_i \mathbf{x}_e + \mathbf{b}_i$ for all $i \in \mathbb{K}$ are the affine terms. Again, notice that $\mathbf{y}_e(t) = \mathbf{y}(t) - \mathbf{C}_\sigma \mathbf{x}_e$, with $\mathbf{y}(t) = \mathbf{C}_\sigma \mathbf{x}(t) + \mathbf{D}_\sigma \mathbf{w}(t)$, and $\mathbf{z}_e(t) = \mathbf{z}(t) - \mathbf{E}_\sigma \mathbf{x}_e$, with $\mathbf{z}(t) = \mathbf{E}_\sigma \mathbf{x}(t) + \mathbf{G}_\sigma \mathbf{w}(t)$ when expressed in terms of the state space representation in $\mathbf{x}(t)$.

From Definition 1, whenever $\mathbf{b}_i \neq 0$ for some $i \in \mathbb{K}$, the switched system possesses several nontrivial equilibrium points, which characterize the subset of the state space given by

$$
\mathbf{X}_e = \left\{ \mathbf{x}_e \in \mathbb{R}^{n_x} : \mathbf{x}_e = -\mathbf{A}_\lambda^{-1} \mathbf{b}_\lambda, \quad \lambda \in \Lambda_N \right\}
\tag{5.2}
$$

The definition of system (5.1) is then made complete with $\mathbf{x}_e \in \mathbf{X}_e$, and its associated vector $\lambda_0 \in \Lambda_N$. Differently from Chapter 4, as it will become clear in the following sections, it is now needed that $\mathbf{A}_\lambda \in \mathcal{H}$ be satisfied for a given vector $\lambda \in \Lambda_N$ associated to the desired equilibrium point.

Also, consider a full-order switched affine filter with state space representation given by

$$
\mathcal{F}_\sigma : \begin{cases} \dot{\hat{\xi}}(t) = \hat{\mathbf{A}}_\sigma \hat{\xi}(t) + \hat{\mathbf{B}}_\sigma \mathbf{y}_e(t) + \hat{\boldsymbol{\ell}}_\sigma, \quad \hat{\xi}(0) = \mathbf{0} \\ \hat{\mathbf{z}}_e(t) = \hat{\mathbf{E}}_\sigma \hat{\xi}(t) + \hat{\mathbf{F}}_\sigma \mathbf{y}_e(t) \end{cases}
\tag{5.3}
$$

where $\hat{\xi} \in \mathbb{R}^{n_x}$ is the state vector of the filter, of same dimension as the switched system being considered, which will be available to the switching rule, and $\hat{\mathbf{z}}_e(t)$ is the filter output, or the signal of interest being estimated.

By connecting the filter (5.3) to the switched system (5.1), as shown in Figure 5.1 we define the estimation error as $\mathbf{e}(t) = \mathbf{z}_e(t) - \hat{\mathbf{z}}_e(t)$, or the difference between the unavailable signal of interest and the estimated signal, output by the filter.

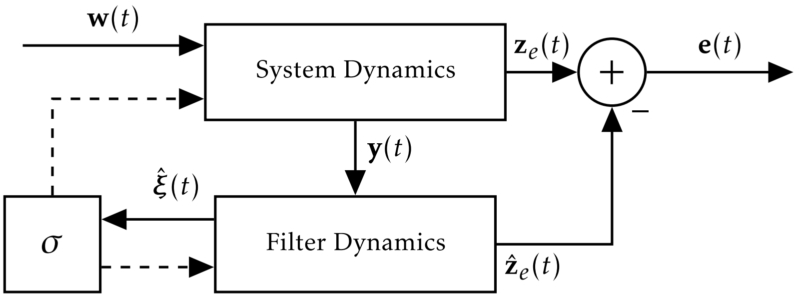


Figure 5.1: System interconnection.

As such, by defining $\tilde{\xi}(t) = \begin{bmatrix} \xi(t)^T & \hat{\xi}(t)^T \end{bmatrix}^T \in \mathbb{R}^{2n_x}$, the following augmented state space representation

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \tilde{\mathbf{A}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{H}}_\sigma \mathbf{w}(t) + \tilde{\boldsymbol{\ell}}_\sigma , \quad \tilde{\xi}(0) = \mathbf{0} \\ \mathbf{e}(t) &= \tilde{\mathbf{E}}_\sigma \tilde{\xi}(t) + \tilde{\mathbf{G}}_\sigma \mathbf{w}(t) \end{aligned} \tag{5.4}$$

is obtained, where $\tilde{\xi}(t)$ is the augmented system state vector, and the augmented system matrices are given by

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \hat{\mathbf{B}}_i \mathbf{C}_i & \hat{\mathbf{A}}_i \end{bmatrix}, \quad \tilde{\boldsymbol{\ell}}_i = \begin{bmatrix} \boldsymbol{\ell}_i \\ \hat{\boldsymbol{\ell}}_i \end{bmatrix}, \quad \tilde{\mathbf{H}}_i = \begin{bmatrix} \mathbf{H}_i \\ \hat{\mathbf{B}}_i \mathbf{D}_i \end{bmatrix}, \quad \tilde{\mathbf{E}}_i = \begin{bmatrix} \mathbf{E}_i - \hat{\mathbf{F}}_i \mathbf{C}_i & -\hat{\mathbf{E}}_i \end{bmatrix}, \quad \tilde{\mathbf{G}}_i = \left(\mathbf{G}_i - \hat{\mathbf{F}}_i \mathbf{D}_i \right) \tag{5.5}$$

The main goal of this chapter consists in determining a dynamical filter $\mathcal{F}_\sigma$ alongside an output-dependent switching function $\sigma(\mathbf{y}_e(t))$ in order to guarantee global asymptotic stability of the equilibrium point $\tilde{\xi} = \mathbf{0}$, thus assuring stability of both the system and filter dynamics, for a given $\mathbf{x}_e \in \mathbf{X}_e$ of interest. Notice that although the switching function is dependent on $\mathbf{y}_e(t)$, this is in fact equivalent to a dependency on $\mathbf{y}(t)$, and will be implemented by means of the filter state $\hat{\xi}(t)$. This is achieved by generalizing the previously introduced Theorems 3.4, and 3.6 for the augmented system (5.4). Furthermore, upper bounds for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices for switched systems, previously introduced, must be assured for the estimation error.

5.3 Preliminaries

In a similar fashion to that undertaken in Chapter 4, we first consider a switching function that eliminates any dependency on the system state, as it is unavailable in filtering and estimation problems. For this, recall the block symmetric matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}^{-1}$

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{Y} & \mathbf{V} \\ \mathbf{V}^T & \hat{\mathbf{Y}} \end{bmatrix}, \quad \tilde{\mathbf{P}}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{U} \\ \mathbf{U}^T & \hat{\mathbf{X}} \end{bmatrix} \tag{5.6}$$

implying in the following relations

$$\mathbf{XY} + \mathbf{UV}^T = \mathbf{I}, \quad \mathbf{XV} + \mathbf{U}\hat{\mathbf{Y}} = \mathbf{0}, \quad \mathbf{U}^T\mathbf{Y} + \hat{\mathbf{X}}\mathbf{V}^T = \mathbf{0}, \quad \mathbf{U}^T\mathbf{V} + \hat{\mathbf{X}}\hat{\mathbf{Y}} = \mathbf{I} \tag{5.7}$$

such that $\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{P}} = \mathbf{I}$ holds. Also recall the block symmetric matrix $\tilde{\mathbf{Q}}_i$

$$\tilde{\mathbf{Q}}_i = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_i \end{bmatrix}, \quad i \in \mathbb{K} \tag{5.8}$$

and the switching rule of Theorems 3.4 and 3.6, for the augmented system (5.4)

$$\sigma(\tilde{\xi}) = \arg\min_{i\in\mathbb{K}} -\tilde{\xi}^T \tilde{\mathbf{Q}}_i \tilde{\xi} + 2\tilde{\xi}^T \tilde{\mathbf{P}}\tilde{\boldsymbol{\ell}}_i \tag{5.9}$$

As previously discussed, the structure adopted in $\tilde{\mathbf{Q}}_i$, together with the specific choices for $\hat{\boldsymbol{\ell}}_i$ given by

$$\hat{\boldsymbol{\ell}}_i = -\mathbf{V}^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \quad \forall i \in \mathbb{K} \tag{5.10}$$

with $\mathbf{V}$ such that $\exists \mathbf{V}^{-1}$, allow for the elimination of the dependency on the system state $\xi(t)$. By employing the relations $\hat{\mathbf{Y}} = -\mathbf{U}^{-1}\mathbf{X}\mathbf{V}$ and $\mathbf{V}^T = \mathbf{U}^{-1}(\mathbf{I} - \mathbf{X}\mathbf{Y})$, from (5.7), the switching function becomes dependent exclusively on the state of the filter $\hat{\xi}(t)$, as such

$$\sigma(\tilde{\xi}) = \sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi} + 2\hat{\xi}^T \mathbf{U}^{-1}\boldsymbol{\ell}_i \tag{5.11}$$

The following sections introduce conditions expressed in terms of LMIs, under which the matrices $\hat{\mathbf{A}}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{E}}_i, \hat{\mathbf{F}}_i$, and the vector $\hat{\boldsymbol{\ell}}_i$, for $i \in \mathbb{K}$ can be obtained, as well as the matrices $\hat{\mathbf{Q}}_i$ and $\mathbf{U}^{-1}$, needed to implement the dynamical filter and the switching rule $\sigma(\cdot)$, and which together guarantee an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance index for the estimation error. In this chapter, the structures of $\tilde{\mathbf{P}}$, $\tilde{\mathbf{P}}^{-1}$, $\tilde{\mathbf{Q}}_i$, and $\hat{\boldsymbol{\ell}}_i$ just introduced will be used throughout.

5.4 $\mathcal{H}_2$ Filter Synthesis

This section deals with the generalization of Theorem 3.4 for the augmented system (5.4). We consider matrices $\tilde{\mathbf{G}}_i = \mathbf{G}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, in order to deal with strictly proper subsystems. The following theorem provides conditions that assure the asymptotic convergence of the estimation error as well as an upper bound for its $\mathcal{H}_2$ performance index.

Theorem 5.1. *Consider the switched affine system (5.4) and a chosen equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{Z}$, $\mathbf{Y}$, $\mathbf{S}$, $\mathbf{R}_i$, and matrices $\mathbf{L}_i$, for all $i \in \mathbb{K}$, such that*

$$\mathbf{R}_{\lambda_0} \geq 0 \tag{5.12}$$

$$\mathbf{A}_i^T\mathbf{Z} + \mathbf{Z}\mathbf{A}_i + \mathbf{R}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{5.13}$$

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{5.14}$$

$$\begin{bmatrix} \mathbf{S} & \bullet & \bullet \\ \mathbf{Z}\mathbf{H}_j & \mathbf{Z} & \bullet \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j & \mathbf{Z} & \mathbf{Y} \end{bmatrix} \succ 0 \tag{5.15}$$

with $j = \sigma(0)$ given, then the following switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{Z}\boldsymbol{\ell}_i \tag{5.16}$$

along with filter (5.3) whose matrices are given by

$$\begin{aligned} \hat{\mathbf{A}}_i &= (\mathbf{Y}-\mathbf{Z})^{-1}\left(\mathbf{A}_i^T\mathbf{Z} + \mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\right), \\ \hat{\mathbf{B}}_i &= (\mathbf{Z}-\mathbf{Y})^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{E}}_i = \mathbf{E}_i, \\ \hat{\mathbf{F}}_i &= \mathbf{0}, \qquad \hat{\boldsymbol{\ell}}_i = (\mathbf{Y}-\mathbf{Z})^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \qquad \hat{\mathbf{Q}}_i = \mathbf{R}_i \end{aligned} \tag{5.17}$$

assure the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma,\mathcal{F}_\sigma) < \mathrm{tr}(\mathbf{S}) \tag{5.18}$$

for the estimation error.

Proof. The proof unfolds by demonstrating the validity of Theorem 3.4 whenever the conditions of Theorem 5.1 are satisfied. For this, consider matrices (5.6), along with the relations in (5.7), matrix (5.8), and the identity (5.10), as well as inequalities (3.39) and (3.40) of Theorem 3.4 for the augmented system (5.4), as such

$$\tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i + \tilde{\mathbf{Q}}_i \prec 0 \tag{5.19}$$

$$\tilde{\mathbf{Q}}_{\lambda_0} \succeq 0 \tag{5.20}$$

By adopting the transformation matrix $\tilde{\mathbf{\Gamma}}$ as

$$\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{U}^T\mathbf{X}^{-1} & \mathbf{0} \end{bmatrix} \tag{5.21}$$

and multiplying (5.19) by $\tilde{\mathbf{\Gamma}}$, as follows

$$\mathrm{He}\left\{\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}}\tilde{\mathbf{\Gamma}}\right\} + \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{E}}_i^T\tilde{\mathbf{E}}_i\tilde{\mathbf{\Gamma}} + \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\mathbf{\Gamma}} \prec 0 \tag{5.22}$$

whose intermediary products are given by

$$\begin{gathered} \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}}\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{A}_i^T\mathbf{X}^{-1} & \mathbf{A}_i^T\mathbf{Y} + \mathbf{C}_i^T\hat{\mathbf{B}}_i^T\mathbf{V}^T + \mathbf{X}^{-1}\mathbf{U}\hat{\mathbf{A}}_i^T\mathbf{V}^T \\ \mathbf{A}_i^T\mathbf{X}^{-1} & \mathbf{A}_i^T\mathbf{Y} + \mathbf{C}_i^T\hat{\mathbf{B}}_i^T\mathbf{V}^T \end{bmatrix} \\ \tilde{\mathbf{E}}_i\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{X}^{-1} & \mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i \end{bmatrix}, \qquad \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{Q}}_i\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{X}^{-1}\mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T\mathbf{X}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{gathered} \tag{5.23}$$

and by denoting $\mathbf{Z} = \mathbf{X}^{-1}$, $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$, and $\mathbf{R}_i = \mathbf{Z}\mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T\mathbf{Z}$, the resulting inequality is obtained

$$\begin{bmatrix} \boldsymbol{\Xi}_i & \bullet \\ \boldsymbol{\Omega}_i & \boldsymbol{\Upsilon}_i \end{bmatrix} \prec 0 \tag{5.24}$$

with

$$\boldsymbol{\Xi}_i = \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i + (\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z})^T(\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}) \tag{5.25}$$

$$\boldsymbol{\Upsilon}_i = \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + (\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i)^T(\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i) \tag{5.26}$$

$$\boldsymbol{\Omega}_i = \mathbf{A}_i^T\mathbf{Z} + \mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T\mathbf{Z} + (\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i)^T(\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}) \tag{5.27}$$

Notice that by making the following choice of $\hat{\mathbf{A}}_i$

$$\hat{\mathbf{A}}_i = -\mathbf{V}^{-1}\left(\mathbf{A}_i^T\mathbf{Z} + \mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i + (\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i)^T(\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z})\right)\mathbf{Z}^{-1}(\mathbf{U}^T)^{-1} \tag{5.28}$$

so as to make $\boldsymbol{\Omega}_i = \mathbf{0}$, we obtain the following two inequalities

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + (\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i)^T(\mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i) \prec 0, \quad \forall i \in \mathbb{K} \tag{5.29}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i & \bullet \\ \mathbf{E}_i - \hat{\mathbf{F}}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z} & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{5.30}$$

Also, by further making the choices of $\hat{\mathbf{F}}_i = \mathbf{0}$, which is a consequence of our assumption of $\mathbf{G}_i = \mathbf{0}$, $\forall i \in \mathbb{K}$, and $\hat{\mathbf{E}}_i = \mathbf{E}_i\mathbf{Z}^{-1}(\mathbf{U}^T)^{-1}$, thus making block $(2,1)$ of (5.30) null, now allows us to obtain inequality (5.13) and (5.14). Thus, it becomes evident that whenever these two inequalities are satisfied, then inequality (5.24) is valid, and consequently, condition (5.19) of Theorem 3.4 for the augmented system is also verified. Furthermore, observe that $\mathbf{R}_{\lambda_0} \geq 0$ is satisfied if and only if $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, and in this case, condition (5.20) of Theorem 3.4 is also verified.

As in Chapter 4, the choice of $\mathbf{U} = \mathbf{X}$ can be made, without incurring loss of generality, and thus, from the relations in (5.7), which imply in $\mathbf{V} = \mathbf{V}^T = \mathbf{Z} - \mathbf{Y}$ and $\hat{\mathbf{Y}} = -\mathbf{V}$ we obtain the identities in (5.17). The switching rule (5.16) is obtained directly from (5.9) by recalling that $\mathbf{U}^{-1} = \mathbf{Z}$, and considering that matrix $\hat{\mathbf{Q}}_i$ is obtained from (5.17), as such

$$\sigma(\hat{\xi}) = \arg\min_{i \in \mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{Z}\boldsymbol{\ell}_i \tag{5.31}$$

Finally, inequality (5.15) is equivalent to

$$\begin{bmatrix} \mathbf{S} & \bullet \\ \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j & \tilde{\boldsymbol{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\boldsymbol{\Gamma}} \end{bmatrix} \succ 0 \tag{5.32}$$

whose intermediary products are given by

$$\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{Z} & \bullet \\ \mathbf{Z} & \mathbf{Y} \end{bmatrix}, \qquad \tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j = \begin{bmatrix} \mathbf{Z}\mathbf{H}_j \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j \end{bmatrix} \tag{5.33}$$

By multiplying inequality (5.32) to the left by $\mathrm{diag}(\mathbf{I}, (\tilde{\mathbf{\Gamma}}^T\tilde{\mathbf{P}})^{-1})$, to the right by its transpose, and applying Schur complement with respect to $\tilde{\mathbf{P}}^{-1}$ in block $(2,2)$ of the ensuing inequality, we obtain

$$\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j \prec \mathbf{S} \tag{5.34}$$

consequently, the upper bound for the guaranteed cost

$$J_2(\sigma, \mathcal{F}_\sigma) < \mathrm{tr}\left(\tilde{\mathbf{H}}_j^T\tilde{\mathbf{P}}\tilde{\mathbf{H}}_j\right) < \mathrm{tr}(\mathbf{S}) \tag{5.35}$$

is assured as in Theorem 3.4, with $j = \sigma(0)$ given. The proof is concluded. □

The following convex optimization problem, subject to the LMI constraints of Theorem 5.1, allows solving for the matrix variables required to implement the proposed filter $\mathcal{F}_\sigma$ and the switching rule

$$\begin{aligned} \min \quad & \mathrm{tr}(\mathbf{S}) \\ \text{s. to:} \quad & (5.12), (5.13), (5.14), \text{and } (5.15) \end{aligned} \tag{5.36}$$

by means of the identities in (5.17).

A few relevant remarks on Theorem 5.1 can be raised. First, notice that the inequalities in (5.13) reveal that there is no imposition on matrices $\mathbf{A}_i$ being Hurwitz, since matrices $\hat{\mathbf{Q}}_i$ are sign indefinite. However, it is required that $\mathbf{A}_{\lambda_0}$ be Hurwitz, for $\lambda_0 \in \Lambda_N$, as is often recurrent in the literature. Also, observe that inequality (5.14) requires the existence of matrices $\mathbf{K}_i = \mathbf{Y}^{-1}\mathbf{L}_i$, such that $\mathbf{A}_i + \mathbf{K}_i\mathbf{C}_i$ be quadratically stable for all $i \in \mathbb{K}$. This, however, is not a severe imposition, because the matrix gains $\mathbf{K}_i$ are index-dependent. Finally, notice that the choice of $\sigma(0) \in \mathbb{K}$ has a direct effect on the performance index, as discussed in Section 3.4. As such, two useful approaches may be considered: firstly, a choice of j such that the guaranteed cost of J_2 is minimized; and secondly, the worst case choice of j, making the filter and switching rule design robust with respect to $\sigma(0) \in \mathbb{K}$.

An important finding is that the optimal guaranteed cost filter actually displays the simpler observer-based structure, as introduced by the next corollary.

Corollary 5.1. *Consider the switched affine system (5.4) and a chosen equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ with its associated $\lambda_0 \in \Lambda_N$. If there exist a symmetric positive definite matrix $\mathbf{Z}$, symmetric matrices $\mathbf{Y}$, $\mathbf{S}$, and matrices $\mathbf{L}_i$, for all $i \in \mathbb{K}$, such that*

$$\mathbf{A}_{\lambda_0}^T\mathbf{Z} + \mathbf{Z}\mathbf{A}_{\lambda_0} \prec 0 \tag{5.37}$$

$$\mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + \mathbf{E}_i^T\mathbf{E}_i \prec 0, \quad \forall i \in \mathbb{K} \tag{5.38}$$

and

$$\begin{bmatrix} \mathbf{S} & \bullet \\ \mathbf{Y}\mathbf{H}_j + \mathbf{L}_j\mathbf{D}_j & \mathbf{Y} \end{bmatrix} \succ 0 \tag{5.39}$$

with $j = \sigma(0)$ given, are satisfied, then switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{Z}\boldsymbol{\ell}_i \tag{5.40}$$

with matrices

$$\hat{\mathbf{Q}}_i = (\mathbf{A}_{\lambda_0} - \mathbf{A}_i)^T\mathbf{Z} + \mathbf{Z}(\mathbf{A}_{\lambda_0} - \mathbf{A}_i) \tag{5.41}$$

along with filter (5.3) whose matrices are given by

$$\begin{aligned} &\hat{\mathbf{A}}_i = \mathbf{A}_i - \hat{\mathbf{B}}_i\mathbf{C}_i, \\ &\hat{\mathbf{B}}_i = -\mathbf{Y}^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{E}}_i = \mathbf{E}_i, \\ &\hat{\mathbf{F}}_i = \mathbf{0}, \qquad \hat{\boldsymbol{\ell}}_i = \boldsymbol{\ell}_i \end{aligned} \tag{5.42}$$

assure the $\mathcal{H}_2$ guaranteed cost

$$J_2(\sigma, \mathcal{F}_\sigma) < \mathrm{tr}(\mathbf{S}) \tag{5.43}$$

for the estimation error.

Proof. For the proof of this corollary, first notice that (5.37) is equivalent to (5.12) together with (5.13). Indeed, by considering matrices $\hat{\mathbf{Q}}_i = \mathbf{R}_i$ as in (5.41), condition (5.13) becomes (5.37), in addition, we have that $\hat{\mathbf{Q}}_{\lambda_0} = \mathbf{0}$, thus inequality (5.12) holds with equality.

Furthermore, notice that by multiplying (5.12) and (5.13) by a scalar $\epsilon > 0$, it becomes evident that the choice $(\epsilon\mathbf{Z}, \epsilon\mathbf{R}_i) \to (\mathbf{Z}, \mathbf{R}_i)$, with $\epsilon \to 0^+$ can be made without imposing conservatism. Thus, the second row and columns of (5.15) can be eliminated, and this inequality is simplified to (5.39).

Notice that the switching rule (5.16) is also not impacted by this choice, as ϵ is a positive scalar, which is constant over $i \in \mathbb{K}$. Finally, the relations in (5.42) follow immediately from (5.17), as $\epsilon \to 0^+$. This concludes the proof. □

Corollary 5.1 makes evident that the filter for which the $\mathcal{H}_2$ guaranteed cost of Theorem 5.1 is optimal presents the simpler observer-based structure. Indeed, by considering the identities in (5.42), and denoting matrices $\mathbf{K}_i = \mathbf{Y}^{-1}\mathbf{L}_i$, for all $i \in \mathbb{K}$, the state space representation of filter (5.3) becomes

$$\dot{\hat{\xi}}(t) = \mathbf{A}_\sigma\hat{\xi}(t) + \mathbf{K}_\sigma\left(\mathbf{C}_\sigma\hat{\xi}(t) - \mathbf{y}_e(t)\right) + \boldsymbol{\ell}_\sigma\,, \quad \hat{\xi}(0) = \mathbf{0} \tag{5.44}$$

In addition, Corollary 5.1 now allows the switching rule design and the observer design to be carried out independently. We can verify this by realizing that the matrix variable $\mathbf{Z}$, which is now the only variable present in the switching function, exerts no influence on the LMIs in (5.38) and (5.39), in which the performance of the filter depends upon. This reveals that the separation principle, well-known in control theory, holds in the more complex case of switched affine systems.

Moreover, Corollary 5.1 encompasses existing results in the literature, where no $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance indices are considered, and in addition, does not require that the subsystem matrices $\mathbf{A}_i$, $\forall i \in \mathbb{K}$ be quadratically stable, as necessary in [30].

To obtain the required matrices to implement the observer and switching rule of Corollary 5.1, the following convex optimization problem, subject to LMI constraints, can be solved

$$\begin{aligned} \min \quad & \mathrm{tr}(\mathbf{S}) \\ \text{s. to:} \quad & (5.37), (5.38), (5.39), \text{and } \mathbf{Z} \succ 0 \end{aligned} \tag{5.45}$$

considering the identities in (5.17) and (5.41).

5.4.1 Examples: $\mathcal{H}_2$ Filter Synthesis

This numerical example, as presented in [56], deals with the design of the optimal $\mathcal{H}_2$ filter for a switched affine system with two unstable subsystems.

Example 5.1

Consider the switched affine system (5.1) composed of two unstable subsystems, as follows

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 2 & -5 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{H}_1 = \mathbf{H}_2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_1 = \mathbf{D}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}$$

Notice that the equilibrium point of $\mathbf{A}_1$ is an unstable focus, while that of $\mathbf{A}_2$ is an unstable saddle. A Hurwitz convex combination $\mathbf{A}_{\lambda_0}$ can be verified at $\lambda_0 = [0.47 \;\; 0.53]^T$, corresponding to the equilibrium point $\mathbf{x}_e = [1.21 \;\; -0.47]^T \in \mathbf{X}_e$. Solving the convex optimization problem in (5.45), under the conditions of Corollary 5.1, and adopting $\sigma(0) = j = 1$, the upper bound for the $\mathcal{H}_2$ performance index $J_2(\sigma, \mathcal{F}_\sigma) < 76.79$ was obtained, along with the matrices $\mathbf{Z}$, $\mathbf{Y}$, $\mathbf{L}_1$, and $\mathbf{L}_2$, as follows

$$\mathbf{Z} = \begin{bmatrix} 1.3702 & 0.3876 \\ 0.3876 & 0.4072 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 2.8488 & -1.0732 \\ -1.0732 & 0.5732 \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} -5.8662 \\ 1.0906 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -7474.3 \\ -8.7258 \end{bmatrix}$$

used to implement the observer and the switching function, through the identities in (5.42) and (5.41).

Considering the disturbance $\mathbf{w}(t) = \delta(t)\psi_2$, which as previously discussed, is equivalent to the initial condition $\tilde{\xi}_0 = \tilde{\mathbf{H}}_1\psi_2 = [0.00 \;\; 5.00 \;\; 4.56 \;\; 6.63]^T$, we obtain the trajectories in time for the system and filter states, as well as the estimation error, presented in Figure 5.2. Notice that the estimation error asymptotically converges to zero, and that the switching rule was capable of stabilizing the switched system, as desired.

Observe that after $t \approx 0.49$ seconds, the switched system begins evolving in sliding modes, as a conse-

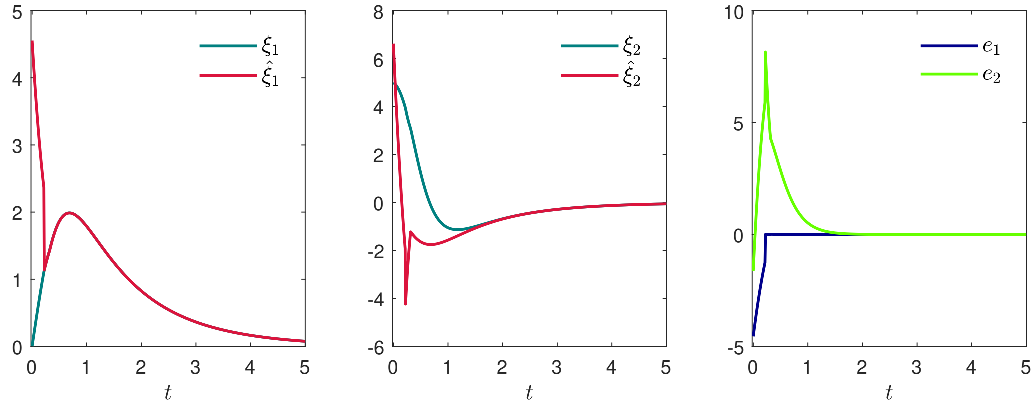


Figure 5.2: Trajectories of system and filter state, and estimation error under Theorem 5.1.

quence of the switching function. Finally, via numerical integration of the product $\mathbf{e}(t)^T\mathbf{e}$, and considering both initial conditions $\tilde{\mathbf{H}}_1\psi_1$ and $\tilde{\mathbf{H}}_1\psi_2$, the actual $\mathcal{H}_2$ cost $J_2 = 31.10 < 76.79$ was obtained, within that assured by Theorem 5.1 for the estimation error.

■

The next example tackles an application for the proposed filtering technique, based on a simplified *flyback* DC-DC power converter subject to a change in its operating point, or setpoint, while in operation. This example is also available in the reference [57].

Example 5.2

This example considers the simplified *flyback* converter topology illustrated in Figure 5.3. The book [58] provides further details on the subject.

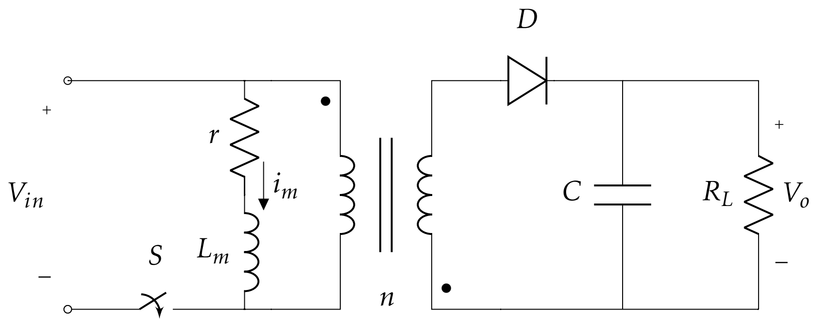


Figure 5.3: *flyback* power converter.

For this example, we consider the following component values: $V_{in} = 12$V; $L_m = 0.848$mH; $r = 1.129\Omega$; $C = 2.2$mF; $R_L = 120\Omega$, and $n = 2$, the transformer turns ratio. Furthermore, we consider that the output voltage is the only available measurement, as is often the case in practical applications, by defining matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ appropriately.

The *flyback* power converter, along with many other topologies for DC-DC power converters, can be readily modeled in a switched affine system framework (5.1). By defining the unshifted state vector as $\mathbf{x}(t) = [i_m(t) \;\; V_o(t)]^T$, the following matrices define the dynamical behavior of the system

$$\mathbf{A}_1 = \begin{bmatrix} -r/L_m & 0 \\ 0 & \frac{-1}{R_L C} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -r/L_m & -n/L_m \\ n/C & \frac{-1}{R_L C} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} \frac{V_{in}}{L_m} \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{H}_1 = \mathbf{H}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$
$$\mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \mathbf{D}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}, \quad \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{0}$$

We consider for this example that a change in the operating point of the *flyback* converter occurs at $t = 0.25$ seconds, from $\mathbf{x}_{e_1} = [1.12\text{A} \;\; 35.03\text{V}]^T \in \mathbf{X}_e$, associated to $\boldsymbol{\lambda}_1 = [0.9187 \;\; 0.0813]^T$, to $\mathbf{x}_{e_2} = [2.56\text{A} \;\; 50.01\text{V}]^T \in \mathbf{X}_e$, associated to $\boldsymbol{\lambda}_2 = [0.8691 \;\; 0.1309]^T$. The following convex optimization problem is considered

$$\begin{aligned} \min \quad & \operatorname{tr}(\mathbf{S}) \\ \text{s. to:} \quad & (5.13), (5.14), (5.15), \\ & \mathbf{R}_{\lambda_1} \geq 0 \\ & \mathbf{R}_{\lambda_2} \geq 0 \end{aligned}$$

which differs from (5.36) by considering the conditions of Theorem 5.1 however imposing both setpoints simultaneously. The upper bound for the $\mathcal{H}_2$ performance index was obtained as $J_2(\sigma, \mathcal{F}_\sigma) < 0.0026$ for $\sigma(0) = j = 2$ along with the following matrices

$$\mathbf{Z} = \begin{bmatrix} 0.3876 & 0.1425 \\ 0.1425 & 1.1628 \end{bmatrix} \times 10^{-3}, \qquad \mathbf{Y} = \begin{bmatrix} 0.5185 & 0.2094 \\ 0.2094 & 1.6526 \end{bmatrix} \times 10^{-3},$$
$$\mathbf{L}_1 = \begin{bmatrix} 0.08492 \\ -268.33 \end{bmatrix}, \qquad \mathbf{L}_2 = \begin{bmatrix} -1.2242 \\ -57.552 \end{bmatrix} \times 10^{-8}, \qquad \mathbf{Q}_1 = \begin{bmatrix} 0.75 & 0.141 \\ 0.141 & -0.011 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} -1.535 & -0.356 \\ -0.356 & 0.502 \end{bmatrix}$$

required to implement the filter $\mathcal{F}_\sigma$ and switching function, via the relations in (5.17).

Initiating the system from null initial conditions $\mathbf{x}_0 = \mathbf{0}$ and $\hat{\xi}_0 = \mathbf{0}$, Figure 5.4 presents the trajectories in time for the current and output voltage for the *flyback* power converter. The filter implementation enabled a rapid convergence of the estimation error to zero, as seen in Figure 5.5. The change in operating points from $\mathbf{x}_{e_1}$ to $\mathbf{x}_{e_2}$ is clearly visible at $t = 0.25$ seconds, and the switching function, shown in Figure 5.6, is effective in stabilizing the system to the new equilibrium point $\mathbf{x}_{e_2}$. Notice in the zoomed-in interval of time the change in its behavior when the change of setpoints occurs.

This example demonstrates how the proposed technique for $\mathcal{H}_2$ filtering may be used in a power electronics application, such as the *flyback* DC-DC power converter, considering the more realistic scenario where the output voltage is the only available measurement, as well as that of dealing with changes in setpoints during operation.

■

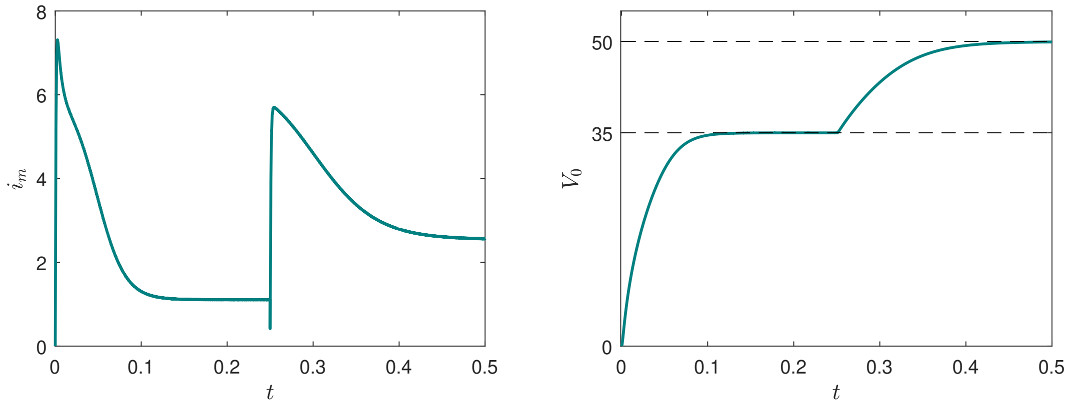


Figure 5.4: Trajectories of each state for the switched affine system under Theorem 5.1.

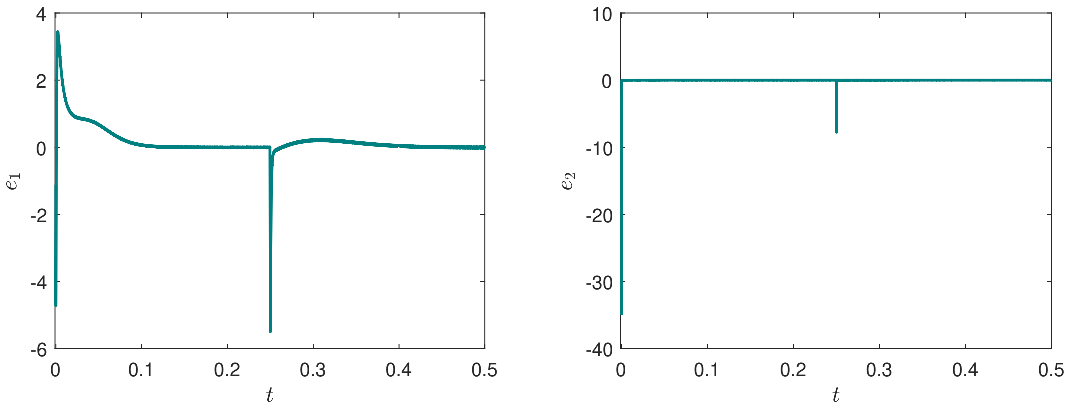


Figure 5.5: Trajectories of the estimation error under Theorem 5.1.

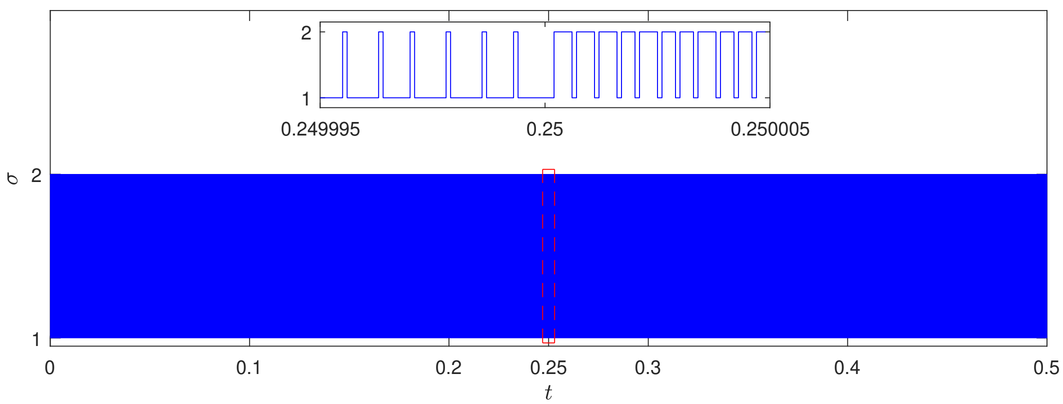


Figure 5.6: Switching rule for the switched affine system under Theorem 5.1.

5.5 $\mathcal{H}_\infty$ Filter Synthesis

This section generalizes Theorem 3.6 to deal with the classical filtering problem assuring the $\mathcal{H}_\infty$ performance index for the estimation error in the context of switched affine systems. Recall that for the $\mathcal{H}_\infty$ case, disturbances

$\mathbf{w}(t) \in \mathcal{L}_2$ are considered. Furthermore, similarly to the proposed $\mathcal{H}_2$ filter, it will be demonstrated that the minimum $\mathcal{H}_\infty$ guaranteed cost filter also presents an observer-based structure, and which can be designed independently of the switching function.

Theorem 5.2. *Consider the switched affine system (5.4) and a chosen equilibrium point $\mathbf{x}_e \in \mathbf{X}_e$ with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{Z}$, $\mathbf{Y}$, and $\mathbf{R}_i$, and matrices $\mathbf{N}_i$ and $\mathbf{L}_i$, for all $i \in \mathbb{K}$, and a scalar ρ, such that*

$$\mathbf{Y} > \mathbf{Z} > 0 \tag{5.46}$$

$$\mathbf{R}_{\lambda_0} \geq 0 \tag{5.47}$$

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{Z} + \mathbf{Z}\mathbf{A}_i + \mathbf{R}_i & \bullet \\ \mathbf{H}_i^T\mathbf{Z} & -\rho\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{5.48}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i & \mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{5.49}$$

then the following switching rule

$$\sigma(\hat{\xi}) = \arg\min_{i\in\mathbb{K}} -\hat{\xi}^T\hat{\mathbf{Q}}_i\hat{\xi} + 2\hat{\xi}^T\mathbf{Z}\boldsymbol{\ell}_i \tag{5.50}$$

along with filter (5.3) whose matrices are given by

$$\begin{aligned} &\hat{\mathbf{A}}_i = (\mathbf{Y}-\mathbf{Z})^{-1}\left(\mathbf{A}_i^T\mathbf{Z} + \mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i + \rho^{-1}(\mathbf{Y}\mathbf{H}_i + \mathbf{L}_i\mathbf{D}_i)\mathbf{H}_i^T\mathbf{Z}\right) \\ &\hat{\mathbf{B}}_i = (\mathbf{Z}-\mathbf{Y})^{-1}\mathbf{L}_i, \qquad \hat{\mathbf{E}}_i = \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i + \rho^{-1}(\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)\mathbf{H}_i^T\mathbf{Z} \\ &\hat{\mathbf{F}}_i = \mathbf{N}_i, \quad \hat{\boldsymbol{\ell}}_i = (\mathbf{Y}-\mathbf{Z})^{-1}\mathbf{Y}\boldsymbol{\ell}_i, \quad \hat{\mathbf{Q}}_i = \mathbf{R}_i \end{aligned} \tag{5.51}$$

assure the $\mathcal{H}_\infty$ guaranteed cost

$$J_\infty(\sigma, \mathcal{F}_\sigma) < \rho \tag{5.52}$$

for the estimation error.

Proof. The proof unfolds by demonstrating the validity of Theorem 3.6 whenever the conditions of Theorem 5.2 are met. Again, we consider matrices (5.6), as well as the identities in (5.7), vectors (5.10), and the matrices $\tilde{\mathbf{Q}}_i$, whose structure is defined in (5.8). First, consider inequalities (3.63) and (3.64) of Theorem 3.6, applied to the augmented system (5.4), as such

$$\tilde{\mathbf{Q}}_{\lambda_0} \geq 0 \tag{5.53}$$

$$\begin{bmatrix} \tilde{\mathbf{A}}_i^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}}_i + \tilde{\mathbf{Q}}_i & \bullet & \bullet \\ \tilde{\mathbf{H}}_i^T\tilde{\mathbf{P}} & -\rho\mathbf{I} & \bullet \\ \tilde{\mathbf{E}}_i & \tilde{\mathbf{G}}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{5.54}$$

By considering again the transformation matrix $\tilde{\boldsymbol{\Gamma}}$ defined in (5.21), multiplying the left hand side of (5.54) by diag($\tilde{\mathbf{\Gamma}}, \mathbf{I}, \mathbf{I}$), the right hand side by its transpose, and proceeding similarly as in Theorem 5.1, the intermediary products are given by (5.23) and (5.33) when denoting $\mathbf{Z} = \mathbf{X}^{-1}$ and $\mathbf{L}_i = \mathbf{V}\hat{\mathbf{B}}_i$. By further denoting $\mathbf{N}_i = \hat{\mathbf{F}}_i$ and $\mathbf{R}_i = \mathbf{X}^{-1}\mathbf{U}\hat{\mathbf{Q}}_i\mathbf{U}^T\mathbf{X}^{-1}$, we obtain the following inequality

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i & \bullet & \bullet & \bullet \\ \mathcal{V}_i & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Z} & \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z} & \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i & \mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{5.55}$$

where $\mathcal{V}_i = \mathbf{A}_i^T\mathbf{Z} + \mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i + \mathbf{V}\hat{\mathbf{A}}_i\mathbf{U}^T\mathbf{Z}$. By performing the Schur complement with respect to the two last rows and columns of (5.55), the following inequality ensues

$$\begin{bmatrix} \mathbf{\Xi}_i & \bullet \\ \mathbf{\Omega}_i & \mathbf{\Upsilon}_i \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{5.56}$$

where, defining $\mathcal{T} = \left(\rho\mathbf{I} - (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T(\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)\right)$, we have

$$\begin{aligned} \mathbf{\Xi}_i = {} & \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i + \left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right)^T\left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right) + \\ & + \left(\mathbf{H}_i^T\mathbf{Z} + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T\left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right)\right)^T\mathcal{T}^{-1}\left(\mathbf{H}_i^T\mathbf{Z} + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T\left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right)\right) \end{aligned} \tag{5.57}$$

$$\begin{aligned} \mathbf{\Upsilon}_i = {} & \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} + (\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i)^T(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i) + \\ & + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i)\right)^T\mathcal{T}^{-1}\left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i)\right) \end{aligned} \tag{5.58}$$

$$\begin{aligned} \mathbf{\Omega}_i = {} & \mathcal{V}_i + (\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i)^T\left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right) + \\ & + \left(\mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i)\right)^T\mathcal{T}^{-1}\left(\mathbf{H}_i^T\mathbf{Z} + (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T\left(\mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z}\right)\right) \end{aligned} \tag{5.59}$$

By choosing $\hat{\mathbf{A}}_i$ so as to make $\mathbf{\Omega}_i = \mathbf{0}$, inequality (5.56) is equivalent to (5.49) together with

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Z} & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z} & \mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i & -\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{5.60}$$

where condition (5.60) was obtained from $\mathbf{\Xi}_i$ and inequality (5.49) was derived from $\mathbf{\Upsilon}_i$, in both cases by performing Schur complement as appropriate. In addition, by swapping the second and third rows and columns of (5.60), as such

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i & \bullet & \bullet \\ \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i - \hat{\mathbf{E}}_i\mathbf{U}^T\mathbf{Z} & -\mathbf{I} & \bullet \\ \mathbf{H}_i^T\mathbf{Z} & (\mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i)^T & -\rho\mathbf{I} \end{bmatrix} \prec 0, \quad \forall i \in \mathbb{K} \tag{5.61}$$

and performing Schur complement with respect to $-\rho\mathbf{I}$, it can be verified that by choosing $\hat{\mathbf{E}}_i$ in order to make block $(2,1)$ of the resulting inequality null, then inequality (5.61) becomes equivalent to condition (5.48), together with the following inequality

$$\begin{bmatrix} -\rho\mathbf{I} & \bullet \\ \mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{5.62}$$

which is also enforced by (5.49). As such, whenever inequalities (5.48) and (5.49) are satisfied, then both main diagonal blocks of (5.56) are negative definite, thus inequality (5.55) is valid, and consequently, so is condition (5.54) of Theorem 3.6, for the augmented system. Furthermore, inequalities (5.46) assure $\tilde{\mathbf{P}} > 0$, and we have that $\mathbf{R}_{\lambda_0} \geq 0$ is satisfied if and only if $\tilde{\mathbf{Q}}_{\lambda_0} \geq 0$, assuring that condition (5.53) of Theorem 3.6 for the augmented system holds.

Finally, by again choosing matrix $\mathbf{U} = \mathbf{X}$, the identities in (5.51) are verified, and the switching rule (5.50) is obtained from (5.9). As in Theorem 3.6, the upper bound for the $\mathcal{H}_\infty$ performance index

$$J_\infty(\sigma, \mathcal{F}_\sigma) < \rho$$

is guaranteed for the estimation error. This concludes the proof. □

The matrices required to implement the filter and the switching rule proposed in Theorem 5.2 can be obtained numerically by solving the following convex optimization problem, subject to LMI constraints

$$\begin{aligned} \min \quad & \rho \\ \text{s. to:} \quad & (5.46), (5.47), (5.48), \text{and } (5.49) \end{aligned} \tag{5.63}$$

by considering the identities in (5.51).

As discussed in Chapter 4, the ill-conditioning of $\mathcal{H}_\infty$ problems is a possible occurrence for convex optimization programs subject to LMI constraints. To circumvent this in the case of Theorem 5.2, we propose that a fixed suboptimal $\rho > 0$ be provided, and the term $(\mathbf{Y} - \mathbf{Z})^{-1}$ be minimized, thus providing greater numerical stability for filter matrices. To obtain a linear objective function to this end, consider a symmetric matrix $\boldsymbol{\mathcal{S}}$, such that

$$\begin{bmatrix} \boldsymbol{\mathcal{S}} & \bullet \\ \mathbf{I} & \mathbf{Y} - \mathbf{Z} \end{bmatrix} > 0 \tag{5.64}$$

Notice that, by Schur complement, this inequality is equivalent to $(\mathbf{Y}-\mathbf{Z})^{-1} < \boldsymbol{\mathcal{S}}$. Thus, the following optimization problem, for a given $\rho > 0$

$$\begin{aligned} \min \quad & \mathrm{tr}\,(\boldsymbol{\mathcal{S}}) \\ \text{s. to:} \quad & (5.47), (5.48), (5.49), (5.64), \text{and } \mathbf{Z} > 0 \end{aligned} \tag{5.65}$$

allows for filter matrices with greater numerical stability.

Similarly to Theorem 5.1, the optimal guaranteed cost filter of Theorem 5.2 also displays an observer-based structure. This result is introduced in the next corollary.

Corollary 5.2. *Consider the switched affine system (5.4) and a chosen $\mathbf{x}_e \in \mathbf{X}_e$ with its associated $\lambda_0 \in \Lambda_N$. If there exist symmetric matrices $\mathbf{Z} > 0$ and $\mathbf{Y} > 0$, matrices $\mathbf{N}_i$ and $\mathbf{L}_i$, for all $i \in \mathbb{K}$, and a scalar ρ, such that*

$$\mathbf{A}_{\lambda_0}^T \mathbf{Z} + \mathbf{Z}\mathbf{A}_{\lambda_0} < 0 \tag{5.66}$$

$$\begin{bmatrix} \mathrm{He}\{\mathbf{Y}\mathbf{A}_i + \mathbf{L}_i\mathbf{C}_i\} & \bullet & \bullet \\ \mathbf{H}_i^T\mathbf{Y} + \mathbf{D}_i^T\mathbf{L}_i^T & -\rho\mathbf{I} & \bullet \\ \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i & \mathbf{G}_i - \mathbf{N}_i\mathbf{D}_i & -\mathbf{I} \end{bmatrix} < 0, \quad \forall i \in \mathbb{K} \tag{5.67}$$

are satisfied, then the switching rule

$$\sigma(\hat{\xi}) = \arg \min_{i \in \mathbb{K}} -\hat{\xi}^T \hat{\mathbf{Q}}_i \hat{\xi} + 2\hat{\xi}^T \mathbf{Z}\boldsymbol{\ell}_i \tag{5.68}$$

with

$$\hat{\mathbf{Q}}_i = (\mathbf{A}_{\lambda_0} - \mathbf{A}_i)^T \mathbf{Z} + \mathbf{Z}(\mathbf{A}_{\lambda_0} - \mathbf{A}_i) \tag{5.69}$$

along with filter (5.3) whose matrices are given by

$$\begin{aligned} &\hat{\mathbf{A}}_i = \mathbf{A}_i - \hat{\mathbf{B}}_i\mathbf{C}_i, \qquad \hat{\mathbf{B}}_i = -\mathbf{Y}^{-1}\mathbf{L}_i \\ &\hat{\mathbf{E}}_i = \mathbf{E}_i - \mathbf{N}_i\mathbf{C}_i, \qquad \hat{\mathbf{F}}_i = \mathbf{N}_i, \qquad \hat{\boldsymbol{\ell}}_i = \boldsymbol{\ell}_i \end{aligned} \tag{5.70}$$

assure the $\mathcal{H}_\infty$ guaranteed cost

$$J_\infty(\sigma, \mathcal{F}_\sigma) < \rho \tag{5.71}$$

for the estimation error.

Proof. In a much similar manner to Corollary 5.1, this result is verified by effecting the replacement $(\epsilon\mathbf{Z}, \epsilon\mathbf{R}_i) \to (\mathbf{Z}, \mathbf{R}_i)$ in Theorem 5.2, with $\epsilon \to 0^+$. Under this choice, inequality (5.67) becomes explicitly decoupled from (5.48). This can be noticed by realizing that (5.48), when considering this replacement, is equivalent to

$$\mathrm{He}\{\mathbf{Z}\mathbf{A}_i\} + \mathbf{R}_i + \epsilon\rho^{-1}\mathbf{Z}\mathbf{H}\mathbf{H}^T\mathbf{Z} < 0 \tag{5.72}$$

and the dependency on ρ is eliminated as $\epsilon \to 0^+$. The proof is concluded. □

It is important to note that the same remarks on Theorem 5.1 remain valid for the $\mathcal{H}_\infty$ filter, particularly that no imposition on individual subsystem matrices $\mathbf{A}_i$ being Hurwitz is made, only on their convex combination $\mathbf{A}_{\lambda_0}$, for $\lambda_0 \in \Lambda_N$, and also, inequality (5.49) requires matrices $\mathbf{K}_i = \mathbf{Y}^{-1}\mathbf{L}_i$, such that $\mathbf{A}_i + \mathbf{K}_i\mathbf{C}_i$ be quadratically stable for all $i \in \mathbb{K}$, which, as already mentioned, is not a severe imposition, given that the matrix gains $\mathbf{K}_i$ are index-dependent. Finally, observe that, as for the $\mathcal{H}_2$ case, the separation principle is again valid, allowing the switching function and the observer to be designed independently.

The following convex optimization problem can be solved to obtain the matrices needed to implement the proposed observer and switching rule of Corollary 5.2

$$\begin{aligned} \min \quad & \rho \\ \text{s. to:} \quad & (5.66),\ (5.67),\ \text{and}\ \mathbf{Z} \succ 0 \end{aligned} \tag{5.73}$$

considering the relations in (5.70) and (5.69). Additionally, the following convex optimization problem

$$\begin{aligned} \min \quad & \mathrm{tr}(\mathcal{S}) \\ \text{s. to:} \quad & (5.66),\ (5.67),\ (5.64),\ \text{and}\ \mathbf{Z} \succ 0 \end{aligned} \tag{5.74}$$

for a given $\rho > 0$, arbitrarily close to the optimal value, provides for observer matrices with greater numerical stability, as previously discussed.

5.5.1 Examples: $\mathcal{H}_\infty$ Filter Synthesis

The following example illustrates the effectiveness of the proposed $\mathcal{H}_\infty$ filter design methodology, even in the case where all subsystems are unstable. This example can also be found in reference [57].

Example 5.3

Consider the switched affine system (5.1) comprised of three unstable subsystems, with matrices

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -10 & -10 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 2 & -1 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{H}_i = \mathbf{I}, \qquad \mathbf{C}_i = [1\ \ 1\ \ 1], \qquad \mathbf{D}_i = [1\ \ 1\ \ 1], \qquad \mathbf{E}_i = \mathbf{I}, \qquad \forall i \in \mathbb{K}$$

Notice that the equilibrium point of subsystem 1 is a node-focus, while that of subsystem 2 is a saddle, and the equilibrium point of subsystem 3 is a saddle-focus. Furthermore, at $\lambda_0 = [0.1822\ \ 0.1022\ \ 0.7156]^T$, the convex combination $\mathbf{A}_{\lambda_0}$ is Hurwitz. For this example, the following disturbance is considered

$$\mathbf{w}(t) = \begin{cases} [0\ \ 3\ \ 0]^T, & 5 \le t \le 10 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

By solving the conditions of Corollary 5.2 by means of the convex optimization problem in (5.73), the upper bound $\rho < 12.9952$ for the $\mathcal{H}_\infty$ performance index was obtained. Then, by solving (5.74) for a value of ρ arbitrarily close to the optimal, the following matrices $\mathbf{Z}$, $\mathbf{Y}$, $\mathbf{L}_i$, and $\mathbf{N}_i$, for $i \in \{1,2,3\}$ are obtained

$$\mathbf{Z} = \begin{bmatrix} 7.2934 & 18.462 & 11.161 \\ 18.462 & 59.968 & 37.788 \\ 11.161 & 37.788 & 24.616 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 6.4159 & 2.9275 & 0.1251 \\ 2.9275 & 5.2256 & 0.0728 \\ 0.1251 & 0.0728 & 0.1044 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} -7.424 \\ -7.035 \\ -4.403 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} -7.498 \\ -7.083 \\ -4.445 \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} -7.581 \\ -7.102 \\ -4.481 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 0.338 \\ 0.337 \\ 0.325 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0.334 \\ 0.333 \\ 0.333 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0.324 \\ 0.338 \\ 0.338 \end{bmatrix}$$

These matrices are needed to implement the observer and the switching function of Corollary 5.2 by means of the identities in (5.70). Figure 5.7 presents the trajectories in time for the state of the system and filter. Notice how the switching function is successfully able to stabilize the switched system, so that it asymptotically converges to the desired equilibrium point once the disturbance is removed. Meanwhile, Figure 5.8 displays the

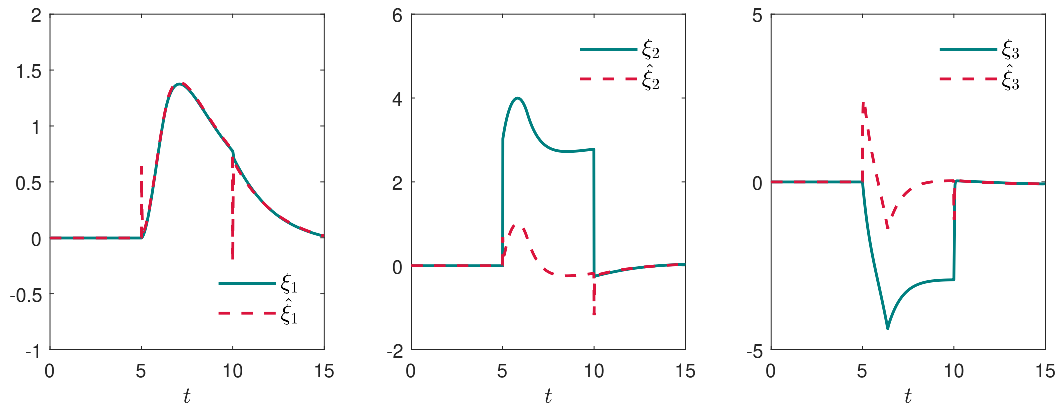


Figure 5.7: Trajectories of system and filter output under Corollary 5.2.

behavior of the switching rule. Observe how its behavior changes once the disturbance is applied. Furthermore,

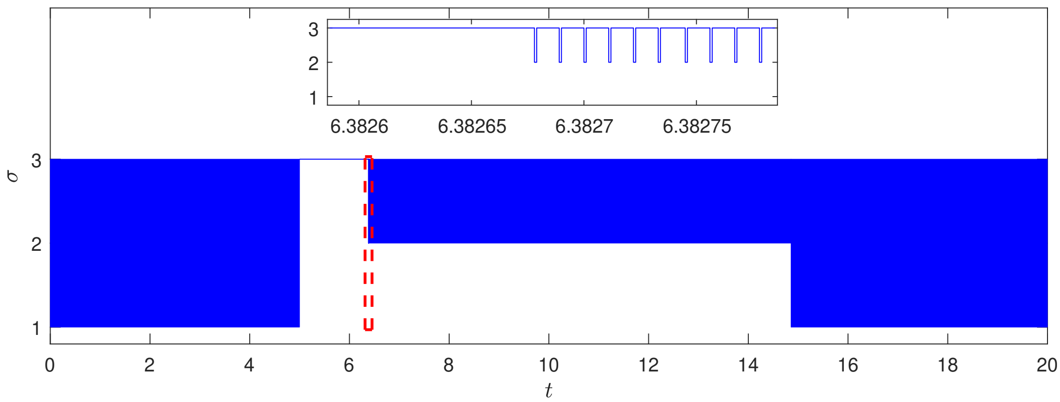


Figure 5.8: Switching rule for the switched affine system under Corollary 5.2.

notice that the observer is successful in estimating the system state, quickly converging once the disturbance ceases. This example demonstrates how the proposed $\mathcal{H}_\infty$ filter design methodology is successful when dealing with switched affine systems composed of unstable subsystems. ■

The next example considers a state estimation problem for the *flyback* power converter of Example 5.2 under Corollary 5.2, where the output voltage is the only available measurement, and subject to process noise. This situation reflects a real-life scenario for DC-DC power converters.

Example 5.4

For this example, also available in the reference [56], recall the the *flyback* power converter of Example 5.2, however with the additional matrix

$$\mathbf{H}_1 = \mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and consider a stable convex combination $\mathbf{A}_{\lambda_0}$ for the vector $\lambda_0 = [0.92 \ \ 0.08]^T$ to which the equilibrium point $\mathbf{x}_e = [3.3187\text{A} \ \ 26.981\text{V}]^T \in \mathbf{X}_e$ is associated. To implement Corollary 5.2, the convex optimization problem (5.73) is solved, through which we obtain the guaranteed cost $J_\infty(\sigma, \mathcal{F}_\sigma) < 0.05920$, associated to the following matrices

$$\mathbf{Z} = \begin{bmatrix} 0.3328 & 0.0833 \\ 0.0833 & 0.0401 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 2.3068 & -0.0001 \\ -0.0001 & 0.0514 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} -2.2984 \\ -0.1533 \end{bmatrix}, \qquad \mathbf{L}_2 = \begin{bmatrix} -2.3073 \\ -0.1133 \end{bmatrix}, \qquad \mathbf{N}_1 = \begin{bmatrix} 18.487 \\ -6.2442 \end{bmatrix} \times 10^{-8}, \qquad \mathbf{N}_2 = \begin{bmatrix} -15.030 \\ -684.90 \end{bmatrix} \times 10^{-8}$$

needed to implement the observer and the switching rule. We have considered the following disturbance

$$\mathbf{w}(t) = \begin{cases} [0 \ \ 5]^T, & 0.2 \leq t \leq 0.7 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

which, due to the choices of $\mathbf{C}_i$ and $\mathbf{H}_i$, $i \in \mathbb{K}$, behaves as a constant noise applied to the voltage measurement, as well as to the dynamics of both states of the switched system. The state trajectories in time for the system and filter states are shown in Figure 5.9, along with the estimation error.

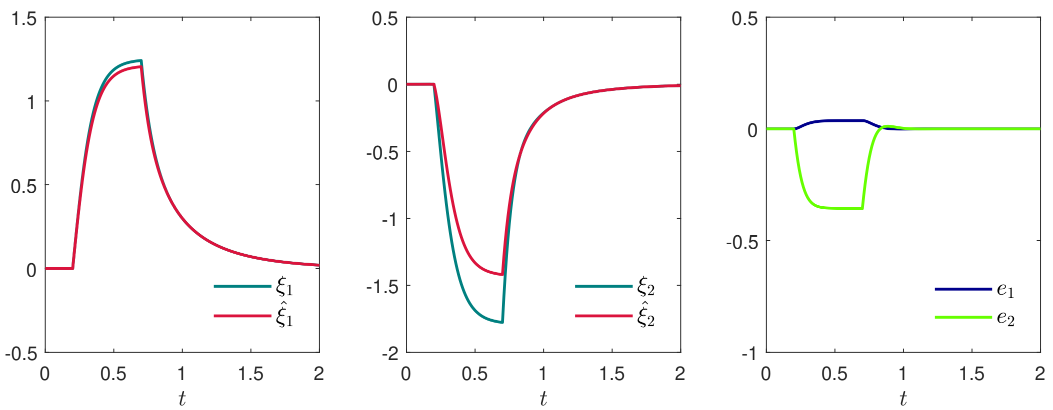


Figure 5.9: Trajectories of system and filter state, and estimation error under Corollary 5.2.

Notice that while the disturbance is being applied, the filter is capable of maintaining the estimation error within −0.3568V, which is small when compared to the operating point of the switched system. Furthermore, after the disturbance ceases, the error rapidly converges to zero and the switching function is able to asymptotically stabilize the switched system. ■

5.6 Concluding Remarks

This chapter introduced a methodology for dealing with the classical filtering problem in the context of switched affine systems. The switched dynamical filter was designed along with an output dependent stabilizing switching function to assure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs for the estimation error. It was proved that the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters actually present the simpler observer-based structure, and that in this case, the switching function and filter design are completely independent, revealing that the separation principle also holds in the more complex case of continuous-time switched affine systems.

Four examples, with two based on a *flyback* power converter, were provided to illustrate the theory and demonstrate the effectiveness of the proposed methodologies in various situations.

Chapter 6
Conclusion

Over the course of this work, we have treated the problems of output feedback control and filter design for continuous-time switched affine systems. First, we presented some important concepts with regard to dynamical systems, focusing on their stability properties, as well as on the definition of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for linear time invariant systems. With these concepts on hand, we then introduced the subclass of hybrid systems known as switched systems, and presented some results already available in the literature on switched linear and affine systems. Examples and discussions throughout Chapter 3 provided valuable insight into several peculiarities and interesting characteristics of this important subclass of systems. For instance, the occurrence of sliding modes, which is a phenomenon characteristic of switched systems that sometimes may be undesirable, due to the high switching frequencies involved, is otherwise crucial when the goal is to attain asymptotic stability of an equilibrium point in a switched affine system, since the desired equilibrium point is in general different than those of the individual subsystem.

For this subclass of hybrid systems, the switching function plays a central role, giving rise to complex and unusual dynamical behaviors. Indeed, a specific choice for this function can guarantee stability even in case where all subsystems are unstable, or in the situation where all subsystems are stable, it may lead to an enhanced performance when compared to that of the individual subsystems. On the other hand, the overall behavior of the switched system becomes more difficult to analyze mathematically, given the nonlinear and time-variant characteristics imparted by the switching function. As such, most of the classical tools and methodologies developed for the analysis and control of dynamical systems, some of which were introduced in Chapter 2, cannot be employed anymore. For instance, the definition of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms can no longer be used, as the switched system cannot be represented by a transfer function. Given this, novel performances indices have been introduced in the literature, and presented with detail towards the end of Chapter 3.

Despite the many mathematical challenges involved in studying these types of systems, the use of a switched system framework for practical applications is especially compelling, since they can directly model a wide range of real-life processes, thus warranting the great interest in this field of research. It is within this scenario that the main contributions of this work are proposed. More specifically, in order to deal with the more realistic case where the state measurements are unavailable, in Chapter 4 we introduced a technique based on the simultaneous design of two control structures, namely an output dependent switching function and a switched dynamical controller, that together guarantee global asymptotic stability as well as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria for switched affine systems. The proposed methodology, to the extent of our knowledge, is first being treated in this work, having already originated the recent publication [31]. The same control structure has already been adopted in the literature, but only for the simpler case of switched linear systems. Compared to the existing approaches on switched affine systems, the results in this work allow for a greater number of equilibrium points to be attained, as the need for a Hurwitz convex combination of dynamical subsystem matrices $\mathbf{A}_\lambda$ is no longer imposed, a recurrent condition in the literature. Instead only the convex combination of closed-loop dynamical matrices must be stable. Furthermore, the proposed techniques are able to guarantee global asymptotic stability of the desired equilibrium point even in the case where all subsystems are not individually controllable. This reveals the importance of the joint action of both these control structures for the stability of switched affine systems.

Also, in Chapter 5, two results on the classical filtering problem in the context of switched affine systems are also presented. This prominent problem, to the best of our knowledge, has so far only been treated for the linear case. For these systems, existing publications consider either time-dependent switching functions, or the joint design of a stabilizing switching function along with a switched filter. However for the more general case of switched affine systems, existing results consider only the state estimation problem under a switched observer structure. The conditions obtained in this work have already been introduced in the recent publications [56] and [57], and are based on a full-order switched affine filter, which is designed together with an output-dependent stabilizing switching function, that collectively assure an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost for the estimation error. Furthermore, it is proved that the optimal guaranteed cost $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters present a simpler observer-based structure, and in this case, it is shown that the filter can be designed independently of the switching function, indicating the validity of the separation principle for the more complex case of continuous-time switched affine systems.

Numerical examples introduced along Chapters 4 and 5 are used to validate the proposed techniques, and showcase the unique features of our contributions, as well as how they contrast with existing results. More specifically, examples considering non-controllable subsystems where no stable convex combination of dynamical matrices exist are considered, as well as examples based on power electronics applications. These examples illustrated that the results obtained in this work successfully accomplished our objectives.

Many topics of research in the context of switched affine systems remain to be explored, some of which stand to have much impact for practical applications. Specifically, the development of robust conditions for control design have much relevance for applications in power electronics, as many electrical component values cannot be precisely identified, or vary during operation. Another interesting topic is the development of conditions that guarantee stability considering only the partial knowledge of the equilibrium point of interest, while assuring global asymptotic stability of the switched system.

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